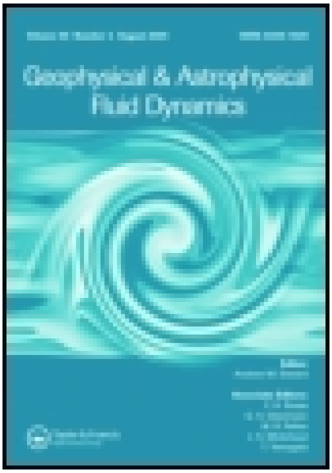
This article was downloaded by: [Universite Laval]
On: 09 July 2014, At: 23:08
Publisher: Taylor & Francis
Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK



Geophysical & Astrophysical Fluid Dynamics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/ggaf20

Equations governing convection in earth's core and the geodynamo

Stanislav I. Braginsky [a] & Paul H. Roberts [a]
[a] Institute of Geophysics and Planetary Physics, University of California , Los Angeles, California, 90024
Published online: 19 Aug 2006.

To cite this article: Stanislav I. Braginsky & Paul H. Roberts (1995) Equations governing convection in earth's core and the geodynamo, Geophysical & Astrophysical Fluid Dynamics, 79:1-4, 1-97, DOI: 10.1080/03091929508228992

To link to this article: http://dx.doi.org/10.1080/03091929508228992

PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis makes every effort to ensure the accuracy of all the information (the "Content") contained in the publications on our platform. However, Taylor & Francis, our agents, and our licensors make no representations or warranties whatsoever as to the accuracy, completeness, or suitability for any purpose of the Content. Any opinions and views expressed in this publication are the opinions and views of the authors, and are not the views of or endorsed by Taylor & Francis. The accuracy of the Content should not be relied upon and should be independently verified with primary sources of information. Taylor and Francis shall not be liable for any losses, actions, claims, proceedings, demands, costs, expenses, damages, and other liabilities whatsoever or howsoever caused arising directly or indirectly in connection with, in relation to or arising out of the use of the Content.

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden. Terms & Conditions of access and use can be found at http://www.tandfonline.com/page/terms-and-conditions

Geophys. Astrophys. Fluid Dynamics, Vol. 79, pp. 1–97
Reprints available directly from the publisher
Photocopying permitted by license only

© 1995 by OPA (Overseas Publishers Association)
Amsterdam B.V. Published under license by
Gordon and Breach Science Publishers SA
Printed in Singapore

EQUATIONS GOVERNING CONVECTION IN EARTH'S CORE AND THE GEODYNAMO

STANISLAV I. BRAGINSKY and PAUL H. ROBERTS

Institute of Geophysics and Planetary Physics, University of California, Los Angeles, California 90024

(Received 27 April 1994; in final form 15 November 1994)

Downloaded by [Universite Laval] at 23:08 09 July 2014

Convection in Earth's fluid core is regarded as a small deviation from a well-mixed adiabatic state of uniform chemical composition. The core is modeled as a binary alloy of iron and some lighter constituent, whose precise chemical composition is unknown but which is here assumed to be FeAd, where Ad = Si, O or S. The turbulent transport of heat and light constituent is considered, and a simple ansatz is proposed in which this is modeled by anisotropic diffusion. On this basis, a closed system of equations and boundary conditions is derived that governs core convection and the geodynamo. The dual (thermal + compositional) nature of core convection is reconsidered. It is concluded that compositional convection may not dominate thermal convection, as had previously been argued by Braginsky (Soviet Phys. Dokl., v. 149, p. 8, 1963; Geomag, and Aeron., v. 4, p. 698, 1964), but that the two mechanisms are most probably comparable in importance. The key parameters leading to this conclusion are isolated and estimated. Their uncertainties, which in some cases are large, are highlighted. The energetics and efficiency of the geodynamo are reconsidered and re-estimated. Arguments are advanced that indicate that the mass fraction of the light constituent in the solid inner core may not be small compared with that in the outer core, e.g. about 60%. This tends to favor silicon or sulfur over oxygen as the principal light alloying constituent.

KEY WORDS: Geomagnetism, core dynamics, compositional convection, thermal convection, geodynamo, dynamo energetics, dynamo efficiency, turbulence.

1. INTRODUCTION

It is sometimes said that the equations governing the geodynamo are "well-known" and that only their solution is difficult. This statement is, however, misleading and unhelpful. It is admittedly true that the geodynamo is governed by the classical equations of fluid mechanics, electrodynamics, thermodynamics and physical chemistry, all of which were well known in the nineteenth century. But they govern physical processes that, in the context of the geodynamo, have significant roles on scales that differ by very many orders of magnitude. Moreover, the classical equations explicitly include only molecular transport processes, and these are so small in Earth's core that diffusive mixing is accomplished not by molecular motion but by fluid motion on a variety of greatly different length scales. The net result is that the classical equations in their original form are suited for neither analytic nor numerical studies of the geodynamo. They must be transformed in such a way that physical processes operating on widely different scales are explicitly separated. To this end, the small parameters relevant to Earth's core must be isolated, and their smallness used to generate a set of reduced equations that, at one and the same time, are not only geophysically sound but also simple enough to be analytically and/or numerically tractable. Our aim in this paper is to develop such a formalism for the study of the geodynamo.

Our theory recognizes at the outset that the core is an iron rich alloy. As Earth cools, iron in the fluid outer core ("FOC") settles onto the surface of the solid inner core ("SIC"), and the gravitational field, pressure, density ... of Earth change. The concomitant release of gravitational energy is a potent source for the geodynamo, one that we explicitly include; see Appendix B. A second crucially significant source is thermal, and arises from the gradual reduction of thermal energy in a cooling Earth. We must therefore describe core convection and the geodynamo against a background of a "reference state", that changes slowly with time, t due to changes in composition and temperature. We find it convenient to introduce a 'slow time' variable, t_a, that changes on the geological time scale, τ_a and a 'fast time' variable, t_c, that changes on the convectional time scale, τ_c.

The reduced equations, which are developed in Sections 2–5, lead us to two simplified models of the geodynamo: an inhomogeneous model (Section 6) and a homogeneous model (Section 8). The former may also be described as a generalized "anelastic model" and the latter as a generalized "Boussinesq model", generalized in each case by some additional and novel features. Many papers (e.g. Backus, 1975; Hewitt *et al.*, 1975; Gubbins, 1977; Gubbins *et al.*, 1979) discuss the energy and entropy balances in the core. They derive these from the primitive equations, without approximating them by forms suitable for studies of core dynamics. Our re-discussion of these balances (Section 7) explicitly separates the effects of the slow evolution of Earth from those associated with convection.

Most discussions of the geodynamo start from systems of equations (almost invariably Boussinesq approximated) that govern core convection and magnetic field generation. The quantities arising in these theories (such as density, pressure, etc.) are in reality very small deviations from the same attributes of the background state on which the magnetoconvection takes place. The separation of the primitive fields into background and convective parts is rarely discussed, but is in fact not a trivial matter, because it demands an understanding of how small additions to a large background behave. And this inevitably introduces the complications of core turbulence. An attempt to derive the equations governing core convection driven by both compositional and thermal buoyancy was previously made by Braginsky (1964b). His procedure was much the same as that adopted here; he too separated the small convective deviations from the background, and recognized that the convective motions have a fluctuating turbulent part that significantly enhances the transport of extensive properties of the mean convective state, such as entropy and composition. His treatment was, however, too incomplete to answer satisfactorily several significant questions, which are addressed in the present paper; see particularly Section 4 and Appendix C.

The Boussinesq approximation is rather obvious in laboratory contexts, where the adiabatic gradient, ∇T_a, is associated with only minute variations across the system, i.e. $T_c \gg L|\nabla T_a|$, where T_c is the departure from the reference state created by convect-

Downloaded by [Universite Laval] at 23:08 09 July 2014

ion and L is a typical length scale of the system. The validity of the Boussinesq approach is far less obvious for Earth's core where the variations in reference state variables with depth greatly exceed those associated with convection ($\rho_c \sim \epsilon_c L|\nabla \rho_a|$ and $\epsilon_c \sim 10^{-8}$; see below). In this respect, the core resembles the convect-ion zones of stars, to study which astrophysicists have developed mixing length theory, in which departures from the adiabatic, such as $T_c = T - T_a$, are more significant in the determining the convective state than T itself (see Jeffreys, 1930; Cox & Giuli, 1968).

We are interested in this paper in making more precise the sense in which the convection is a small deviation from the background state and in discussing in greater detail the assumed transport processes in the core. Small parameters that arise, some of which were referred to above, are:

- ϵ_a. This measures the inhomogeneity of the basic state. It is defined in (3.8) and ~ 0.1;
- ϵ_Ω. This assesses the importance of centrifugal forces in determining the structure of the background state; see (3.9). Its value $\sim 2 \times 10^{-3}$;
- ϵ_c. This measures the relative importance of the forces that control the convection (e.g. the Coriolis force) to the forces determining the background state (e.g. gravity); $\epsilon_c \sim 10^{-8}$. See (3.10);
- ϵ_c^τ. This is the ratio of convective time scale, τ_c, to the time scale τ_a over which the background state evolves (the geological time scale); $\epsilon_c^\tau \sim 10^{-8}$. See (3.11);
- ϵ^R. This is the Rossby number that measures the relative importance of inertial and Coriolis forces on the main scale of core convection; $\epsilon^R \sim 10^{-5}$;
- ϵ_η^Ω. This is the (magnetic) Ekman number, which differs from the usual Ekman number (the ratio of viscous to Coriolis forces) in that the magnetic diffusivity, η, appears in place of the kinematic viscosity, ν, in its definition (8.19); $\epsilon_\eta^\Omega \sim 10^{-9}$.

Even though $\epsilon_\Omega \gg \epsilon_c$, it is with core convection (which is associated with ϵ_c) that we shall be principally concerned in this paper, and not with the asymmetry of the reference state, which is measured by ϵ_Ω and which, as we shall argue in Section 3, is not of prime importance. To keep our model as simple as possible, we set $\epsilon_\Omega = 0$, i.e. we ignore the deviations in the structures of the mantle and core from spherical symmetry, even though it is known that asymmetries do exist. Taking them properly into account is a nontrivial matter that is best left for future development of the present theory.

The core is assumed in this paper to be a binary alloy, consisting mainly of iron and a single light admixture, which we need not specify. This limitation, to one light constituent, almost certainly oversimplifies a complicated core chemistry, but it suffices since it models a process that is vital to core dynamics, namely gravitational stirring by compositional convection. In the absence of detailed knowledge about the core constituents, the complications introduced by the addition of further chemical elements could not be justified even though (see Appendix D) it could easily be accomplished at the expense of introducing further unknown parameters. Our information about the physical properties of the core is far from complete, but not all of those properties

Downloaded by [Universite Laval] at 23:08 09 July 2014

are equally significant for the construction of a geodynamo model. By developing as simple a model of core dynamics as possible, we are able to assess which parameters are crucial for the construction of a geodynamo model and which are less critical. In Appendix E we have tried to estimate as many of the relevant physico–chemical parameters as we could, though we recognize that our values are rather uncertain. All significant geodynamo parameters will be determined in the future only by optimizing the fit of geodynamo models to the observational data. We wish to emphasize that our primary goals in this paper are those of developing a general theory and of establishing, in the simplest and most direct way, the main relations between the relevant physical quantities. The accurate estimation of the key parameters of the theory is of secondary importance to us; it is in any case not achievable at the present time.

The dynamics of the FOC are controlled by the SIC and the mantle. It will be sufficient for our purposes to suppose that

1. Over the geological time scale, τ_a, the mantle flows like a fluid to maintain hydrostatic balance so that, in the limit $\epsilon_\Omega \ll 1$ (which we adopt throughout the paper), the core-mantle boundary ("CMB") is spherical, $r = R_1$;
2. The mantle is rigid on the convectional time scale, τ_c, so that $R_1 = R_1(t_a)$;
3. Neither the iron nor the impurity comprising the core fluid can penetrate the mantle, so implying that the radial component, V_{ar1}, of the fluid velocity, $\mathbf{V}_a$, associated with the slow evolution of Earth coincides at the CMB with the velocity, $\dot{R}_1$, of the CMB and also that the radial flux, I^ξ_{r1}, of light admixture vanishes on the CMB.

It is possible that assumptions (1)–(3) may be lifted in the future, but to do this meaningfully more geophysical information will be required. An improvement to (1) would result from the inclusion of the topography of the CMB when considering core-mantle interaction; insight into mantle rheology on time scales of order 10^4 yr, might lead to the abandonment of (2); information about chemical interactions on, and material exchange across, the CMB would lead to a reconsideration of (3). Modeling the SIC is the main topic of Section 5.

At the present time, the solution to the geodynamo problem is hampered both by a paucity of information about the numerical values of crucial physical parameters (such as the amount of energy available) and perhaps more seriously by a lack of understanding of the principal physical ingredients of the geodynamo mechanism. One might say that the present state of geodynamo theory can be represented by the symbol $\sim$, which we use to signify uncertainty by an "order of magnitude", i.e. by a factor of order 10. The immediate goal of geodynamo theory is to find better values for the key parameters and to improve the understanding of both the main components of the geodynamo mechanism and the structure of the physical fields. When this goal has been reached, one may reasonably hope to be able to replace the symbol $\sim$ by the symbol $\approx$, by which we mean that our answers would then be accurate to about 10%. This, in fact, is precisely the magnitude of ϵ_a. In the intervening period before the goal is attained, one may wonder why one should study geodynamo models at an accuracy better than ϵ_a, and this is the reason why so many studies of magnetoconvection in the core make use of the Boussinesq approximation, in which $\epsilon_a \equiv 0$. It is also the reason why (Section 8) we develop

Downloaded by [Universite Laval] at 23:08 09 July 2014

a generalized Boussinesq model that is especially appropriate for the study of core convection. We also call this the "homogeneous model", since the reference state is independent of position. The many complications of the inhomogeneous (or anelastic) model developed in Sections 4–6 are thereby avoided, at the expense of errors that are no larger than the existing uncertainties in the key parameters describing the core.

Finally, a few words about notation, Earth's core is a complicated chemical-fluid-magnetic system involving widely disparate length and time scales. Its mathematical description, and the reduction of that description to tractable form, raise formidable notational problems. In an effort to minimize these problems, we have developed a consistent and (we hope) transparent, notation. We believe that a concise and self-explanatory system of notation is significant not only because it helps to avoid misunderstandings but also because it provides a convenient language with which to discuss and clarify the subject. Our notational scheme is set out in full in Appendix A, and it is recommended that any reader who wishes to follow our arguments in detail should consult this Appendix at the outset. We point out here only that the suffix a always signifies that the quantity concerned is evaluated in the *a*diabatic reference state, while $_c$ (or occasionally ^{c}) refers to *c*onvective deviations from the reference state; the superscript t refers to the fluctuations that arise from core *t*urbulence. The suffix 1 is attached to quantities evaluated on the CMB; those evaluated on the inner core boundary ("ICB"), carry the suffix 2. In the case of ambiguity, as in the case of the density, ρ, which is discontinuous on the ICB, the suffix 2 refers to values on the upper (fluid) side of the ICB and the suffix N to values on the lower (solid) side.

Downloaded by [Universite Laval] at 23:08 09 July 2014

2. BASIC THEORY

2.1 Governing Equations

The core of Earth is modeled as a binary alloy consisting primarily of iron, but with a light admixture whose composition need not be specified here. This simplification of what is probably a complicated chemical mixture of many elements suffices to characterize its behavior. The governing equations for the motion of the core and the evolution of the magnetic field are, in the frame of reference rotating with the mantle, and in a notation that is set out in Appendix A,

$$\rho d_t \mathbf{V} = -\nabla p + \rho \mathbf{g}_e - 2\rho \times \mathbf{V} + \rho \mathbf{F}^v + \rho \mathbf{F}^B, \tag{2.1}$$

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{V}), \tag{2.2}$$

$$\rho d_t \xi = -\nabla \cdot \mathbf{I}^\xi, \tag{2.3}$$

$$\rho d_t S = -\nabla \cdot \mathbf{I}^S + \sigma^S, \tag{2.4}$$

$$\nabla^2 U = 4\pi k_N \rho, \tag{2.5}$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}), \tag{2.6}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{2.7}$$

Equation (2.1) contains the effective gravitational field, $\mathbf{g}_e$, which includes with the true gravitational field, $\mathbf{g} = -\nabla U$, the centrifugal acceleration, $-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$:

$$\mathbf{g}_e = -\nabla U_e, \quad U_e = U + U^{\Omega}, \quad U^{\Omega} = -\tfrac{1}{2}(\mathbf{\Omega} \times \mathbf{r})^2. \tag{2.8a, b, c}$$

The angular velocity $\mathbf{\Omega}$ of the frame is attached to the mantle, which does not rotate completely uniformly. Strictly the Poincaré force, $-\rho\dot{\mathbf{\Omega}} \times \mathbf{r}$, should therefore also be added to the right-hand side of (2.1). The gravitational field, $\mathbf{g}$, is mainly due to Earth itself, a fact expressed by (2.5) and the condition that $U \to 0$ for $\mathbf{r} \to \infty$. A discussion of the gravitational field and its energy is provided in Appendix B.

Equations (2.1)–(2.6) constitute 10 scalar equations[1] for 11 scalar unknowns, namely ρ, p, S, ξ, U and the components of $\mathbf{V}$ and $\mathbf{B}$. They must be supplemented by boundary conditions and by constitutive laws for the viscous and magnetic forces, $\mathbf{F}^{\nu}$ and $\mathbf{F}^{B}$, and the fluxes of light component and entropy, $\mathbf{I}_{\xi}$ and $\mathbf{I}^{S}$. These constitutive laws introduce a further field, the temperature T. It is therefore necessary to consider the thermodynamics of the fluid. This is specified by two thermodynamic variables, for example ρ and S, and by the mass fraction, ξ, of the light constituent. All other thermodynamic quantities can in principle be derived from these three variables; unfortunately, they are in practice, not well determined. We shall regard the internal energy per unit mass, $\epsilon^{I}(\rho, S, \xi)$, as a given function of ρ, S and ξ. From this, p, T and the chemical potential μ are determined through the relation

$$d\varepsilon^{I} = \frac{p}{\rho^2} d\rho + T dS + \mu d\xi, \tag{2.9}$$

which implies that

$$p = \rho^2 \left(\frac{\partial \varepsilon^{I}}{\partial \rho}\right)_{S,\xi}, \quad T = \left(\frac{\partial \varepsilon^{I}}{\partial S}\right)_{\rho,\xi}, \quad \mu = \left(\frac{\partial \varepsilon^{I}}{\partial \xi}\right)_{\rho,S}. \tag{2.10a, b, c}$$

It is sometimes convenient however to use p, S and ξ or p, T, and ξ in place of ρ, S and ξ as independent variables, in which case the enthalpy, $\varepsilon^{H}(p, S, \xi)$, or the Gibbs free energy (also called the thermodynamic potential), $\varepsilon^{G}(p, T, \xi)$, take over the role of $\varepsilon^{I}(\rho, S, \xi)$, where

$$\varepsilon^{H} = \varepsilon^{I} + \frac{p}{\rho}, \quad \varepsilon^{G} = \varepsilon^{H} - TS = \varepsilon^{I} + \frac{p}{\rho} - TS. \tag{2.9a, b}$$

The relation (2.9) between differentials is then replaced by

$$d\varepsilon^{H} = \frac{1}{\rho} dp + T dS + \mu d\xi, \quad d\varepsilon^{G} = \frac{1}{\rho} dp - S dT + \mu d\xi. \tag{2.11, 2.12}$$

[1] Equation (2.7) is not counted since (2.6) implies $\partial_t \nabla \cdot \mathbf{B} = 0$. Thus (2.7) holds for all t if it holds for any t. It therefore has the status of an initial condition.

Downloaded by [Universite Laval] at 23:08 09 July 2014

Further thermodynamic relations are set out in Appendix D; see also Landau and Lifshitz (1980).

We now summarize the remaining constitutive relations. For $\mathbf{F}^v$ we have

$$\rho F_i^v = \nabla_j \pi_{ji}^v \quad \text{or} \quad \rho \mathbf{F}^v = \nabla \cdot \overleftrightarrow{\pi}^v. \tag{2.13}$$

The double overarrow is here used to signify that the symbol beneath it is a second rank tensor (here the viscous stress tensor) which can be contracted with another vector or tensor from either side. We assume that the fluid is a linear viscous (Newtonian) fluid, for which

$$\pi_{ij}^v = 2\rho\nu(e_{ij} - \tfrac{1}{3}e_{kk}\delta_{ij}) + \rho\nu_b e_{kk}\delta_{ij}, \quad e_{ij} = \tfrac{1}{2}(\nabla_i V_j + \nabla_j V_i). \tag{2.13a, b}$$

Since the kinematic shear viscosity, ν, and the coefficient of second (bulk) viscosity, ν_b, are necessarily non-negative, the same is true of the rate of viscous regeneration of heat, which is

$$Q^v = \pi_{ji}^v e_{ji} = \pi_{ji}^v \nabla_j V_i = 2\rho\nu(e_{ij} + \tfrac{1}{3}\nabla\cdot\mathbf{V}\delta_{ij})(e_{ij} - \tfrac{1}{3}\nabla\cdot\mathbf{V}\delta_{ij}) + \rho\nu_b(\nabla\cdot\mathbf{V})^2. \tag{2.14}$$

According to (2.11) and (2.14), we have

$$\rho\mathbf{V}\cdot\mathbf{F}^v = \nabla\cdot(\overleftrightarrow{\pi}^v\cdot\mathbf{V}) - Q^v, \tag{2.15}$$

a fact we shall need below.

The magnetic force on the fluid depends on both the magnetic field, $\mathbf{B}$, and the electric current density, $\mathbf{J}$:

$$\rho\mathbf{F}^B = \mathbf{J}\times\mathbf{B}, \qquad \mathbf{J} = \nabla\times\mathbf{B}/\mu_0. \tag{2.16, 2.17}$$

Equation (2.6) is a consequence of (2.17), of Faraday's law and of Ohm's law for a dense, isotropic, moving conductor:

$$\partial_t\mathbf{B} = -\nabla\times\mathbf{E}, \qquad \mathbf{J} = \sigma_\eta(\mathbf{E} + \mathbf{V}\times\mathbf{B}). \tag{2.18, 2.19}$$

Here $\mathbf{E}$ is the electric field, $\sigma_\eta = 1/\mu_0\eta \geq 0$ is the electrical conductivity and $\eta \geq 0$ is the magnetic diffusivity. The Joule dissipation of heat is

$$Q^J = J^2/\sigma_\eta = (\eta/\mu_0)(\nabla\times\mathbf{B})^2. \tag{2.20}$$

According to (2.16), (2.19) and (2.20), we have

$$\rho\mathbf{V}\cdot\mathbf{F}^B = \mathbf{E}\cdot\mathbf{J} - Q^J, \tag{2.21}$$

a fact we shall need below.

The entropy source σ^S and the heat flux $\mathbf{I}^q$ are related to the fluxes $\mathbf{I}^S$ and $\mathbf{I}^\xi$ of entropy and light constituent. The form of this relationship follows from energy conservation,

Downloaded by [Universite Laval] at 23:08 09 July 2014

which takes the mathematical form

$$\partial_t u^{\text{total}} + \nabla\cdot\mathbf{I}^{\text{total}} = Q^R, \tag{2.22}$$

where

$$u^{\text{total}} = \rho(\varepsilon^I + \varepsilon^K + U^\Omega) + u^B + u^g, \tag{2.23a}$$

$$\mathbf{I}^{\text{total}} = \rho(\varepsilon^H + \varepsilon^K + U^\Omega)\mathbf{V} - \overleftrightarrow{\pi}^v\cdot\mathbf{V} + \mathbf{I}^B + \mathbf{I}^g + \mathbf{I}^q. \tag{2.23b}$$

Here u^B and u^g are magnetic and gravitational energies per unit volume, with corresponding fluxes $\mathbf{I}^B$ and $\mathbf{I}^g$; also, $\varepsilon^K = \frac{1}{2}V^2$ is the kinetic energy density relative to the rotating frame, $\varepsilon^H = \varepsilon^I + p/\rho$ is the enthalpy per unit mass, and $Q^R \geq 0$ is the volumetric rate of radiogenic heating which (because of convective mixing) will later be assumed to be proportional to ρ.

Consider the time derivatives of the successive terms of (2.23a). According to (2.9) and (2.2)–(2.4), we have

$$\begin{aligned}\partial_t(\rho\varepsilon^I) + \nabla\cdot(\rho\varepsilon^I\mathbf{V}) &= (p/\rho)d_t\rho + \rho T d_t S + \rho\mu d_t\xi \\ &= -p\nabla\cdot\mathbf{V} + T\sigma^S - T\nabla\cdot\mathbf{I}^S - \mu\nabla\cdot\mathbf{I}^\xi.\end{aligned} \tag{2.24}$$

We write this as

$$\partial_t(\rho\varepsilon^I) + \nabla\cdot[\rho\varepsilon^H\mathbf{V} + T\mathbf{I}^S + \mu\mathbf{I}^\xi] = T\sigma^S + \mathbf{V}\cdot\nabla p + \mathbf{I}^S\cdot\nabla T + \mathbf{I}^\xi\cdot\nabla\mu. \tag{2.25}$$

According to (2.1), (2.2), (2.15) and (2.21), we have

$$\partial_t(\rho\varepsilon^K) + \nabla\cdot(\rho\varepsilon^K\mathbf{V} - \overleftrightarrow{\pi}^v\cdot\mathbf{V}) = -\mathbf{V}\cdot\nabla p - \rho\mathbf{V}\cdot\nabla U_e + \mathbf{E}\cdot\mathbf{J} - Q^v - Q^J. \tag{2.26}$$

Since U^Ω is independent of t, (2.2) gives

$$\partial_t(\rho U^\Omega) + \nabla\cdot(\rho U^\Omega\mathbf{V}) = \rho\mathbf{V}\cdot\nabla U^\Omega. \tag{2.27}$$

If we multiply (2.18) scalarly by $\mu_0^{-1}\mathbf{B}$, and apply (2.19), we obtain

$$\partial_t u^B + \nabla\cdot\mathbf{I}^B = -\mathbf{E}\cdot\mathbf{J}, \tag{2.28}$$

where

$$u^B = \frac{B^2}{2\mu_0}, \quad \mathbf{I}^B = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}. \tag{2.29a, b}$$

The gravitational energy balance is formulated in Appendix B as

$$\partial_t u^g + \nabla\cdot\mathbf{I}^g = -\rho\mathbf{V}\cdot\mathbf{g} = \rho\mathbf{V}\cdot\nabla U, \tag{2.30}$$

where

$$u^g = -\frac{g^2}{8\pi k_N}, \quad \mathbf{I}^g = U\left(\rho\mathbf{V} - \frac{1}{4\pi k_N}\partial_t\mathbf{g}\right). \tag{2.31a, b}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

By (2.8b) and (2.25)–(2.31), we now have

$$\partial_t u^{\text{total}} + \nabla\cdot[\rho(\epsilon^H + \epsilon^K + U^\Omega)\mathbf{V} - \overleftrightarrow{\pi}^v\cdot\mathbf{V} + \mathbf{I}^B + \mathbf{I}^g + T\mathbf{I}^S + \mu\mathbf{I}^\xi]$$
$$= T\sigma^S - Q^v - Q^J + \mathbf{I}^S\cdot\nabla T + \mathbf{I}^\xi\cdot\nabla\mu. \tag{2.32}$$

Comparing this with (2.22) and (2.23), we see that

$$\mathbf{I}^q = T\mathbf{I}^S + \mu\mathbf{I}^\xi, \tag{2.33}$$

$$T\sigma^S = Q^v + Q^J + Q^R - \mathbf{I}^S\cdot\nabla T - \mathbf{I}^\xi\cdot\nabla\mu. \tag{2.34}$$

According to (2.34), we may rewrite (2.25) as

$$\partial_t(\rho\varepsilon^I) + \nabla\cdot[\rho\varepsilon^H\mathbf{V} + T\mathbf{I}^S + \mu\mathbf{I}^\xi] = Q^v + Q^J + Q^R + \mathbf{V}\cdot\nabla p. \tag{2.25a}$$

We may recall that we assumed in Section 1 that $\mathbf{I}^\xi\cdot\mathbf{dA} = 0$ on the CMB and it therefore follows from (2.33) that the net flux of entropy from the core is related to $\mathcal{Q}^q_M$, the net heat flux from it, by

$$\oint \mathbf{I}^S\cdot\mathbf{dA} = \frac{\mathcal{Q}^q_M}{T_1}. \tag{2.35}$$

(In anticipation of developments below, we have supposed here that the temperature, T_1, of the CMB is predominantly the basic state temperature which is, with high precision, almost uniform over the CMB.)

Equations (2.33) and (2.34) are basic and transcend in importance any constitutive relation for $\mathbf{I}^S$ and $\mathbf{I}^\xi$. It is nevertheless essential that those relations be such that $\sigma^S \geq 0$, with equality only if the system is source-free ($Q^R = 0$), current-free ($Q^J = 0$), in solid body rotation ($Q^v = 0$) and in thermal equilibrium ($\nabla T = \nabla\mu = 0$). The first three terms on the right-hand side of (2.34) are non-negative; only the last two terms are problematical. The answer, for the case of molecular transport of S and ξ, is summarized in Appendix D. We present here only the conclusions:

$$\mathbf{I}^\xi = -\rho\kappa^\xi\left(\nabla\xi + \frac{k^\xi_T}{T}\nabla T + \frac{k^\xi_p}{p}\nabla p\right), \quad k^\xi_p = \frac{\alpha^\xi_T p}{\rho\mu^\xi_T}, \tag{2.36, 2.36a}$$

$$\mathbf{I}^S = \frac{1}{T}(\mathbf{I}^T + \mu'\mathbf{I}^\xi), \quad \mathbf{I}^T = -K^T\nabla T, \quad K^T = \rho c_p\kappa^T, \tag{2.37, 2.38, 2.38a}$$

$$\mathbf{I}^q = \mathbf{I}^T + (\mu + \mu')\mathbf{I}^\xi, \quad \mu' = \mu^\xi_T k^\xi_T + h^\xi. \tag{2.39, 2.39a}$$

Three independent transport coefficients appear here: the diffusivity of light material, $\kappa^\xi \geq 0$; the thermal conductivity, $K^T \geq 0$ (or equivalently the thermal diffusivity, κ^T), and the Soret coefficient, k^ξ_T. The remaining coefficients, k^ξ_p and $h^\xi = \mu' - \mu^\xi_T k^\xi_T$, are

Downloaded by [Universite Laval] at 23:08 09 July 2014

thermodynamic properties of the fluid. It follows from (2.34) and (2.36)–(2.38) that

$$\sigma^S = \sigma^T + \sigma^\xi + \sigma^v + \sigma^J + \sigma^R. \tag{2.40}$$

The individual sources of entropy comprising σ^S are

$$\sigma^T = K^T\left(\frac{\nabla T}{T}\right)^2, \quad \sigma^\xi = \frac{\mu_T^\xi}{\rho\kappa^\xi T}(\mathbf{I}^\xi)^2, \tag{2.40a, b}$$

$$\sigma^v = \frac{Q^v}{T}, \quad \sigma^J = \frac{Q^J}{T}, \quad \sigma^R = \frac{Q^R}{T}, \tag{2.40c, d, e}$$

which are non-negative since $\kappa^T \geq 0$, $\kappa^\xi \geq 0$, $\mu_T^\xi \geq 0$, $Q^v \geq 0$, $Q^J \geq 0$ and $Q^R \geq 0$. It follows that $\sigma^S \geq 0$.

In subsequent Sections, we shall frequently require integral forms of (2.2)–(2.4) and (2.25). The FOC occupies a volume, $\mathscr{V}_{12}$, that changes with time. With the help of the relation (valid for CMB and ICB moving with velocities $\mathbf{U}_1$ and $\mathbf{U}_2$ and for an arbitrary field Q)

$$d_t\int_{\mathscr{V}_{12}} Q dV = \int_{\mathscr{V}_{12}} \partial_t Q dV + \oint_{A_1} Q\mathbf{U}_1\cdot\mathbf{dA} - \oint_{A_2} Q\mathbf{U}_2\cdot\mathbf{dA}, \tag{2.41}$$

where A_1 and A_2 are the outer and inner boundaries of $\mathscr{V}_{12}$, we deduce from (2.2)–(2.4) and (2.25a) that

$$d_t\int_{\mathscr{V}_{12}} \rho dV = \oint_{A_2} \rho(\mathbf{V}-\mathbf{U}_2)\cdot\mathbf{dA} = -d_t\int_{\mathscr{V}_2} \rho dV, \tag{2.42}$$

$$d_t\int_{\mathscr{V}_{12}} \rho\xi dV = \oint_{A_2} [\mathbf{I}^\xi + \rho\xi(\mathbf{V}-\mathbf{U}_2)]\cdot\mathbf{dA} = -d_t\int_{\mathscr{V}_2} \rho\xi dV, \tag{2.43}$$

$$d_t\int_{\mathscr{V}_{12}} \rho S dV = \int_{\mathscr{V}_{12}} \sigma^S dV - \oint_{A_1} \mathbf{I}^S\cdot\mathbf{dA} + \oint_{A_2} [\mathbf{I}^S + \rho S(\mathbf{V}-\mathbf{U}_2)]\cdot\mathbf{dA}, \tag{2.44}$$

$$\begin{aligned} d_t\int_{\mathscr{V}_{12}} \rho\epsilon^I dV = &\int_{\mathscr{V}_{12}} (Q^v + Q^J + Q^R + \mathbf{V}\cdot\nabla p) dV - \oint_{A_1} (\mathbf{I}^q + p\mathbf{U}_1)\cdot\mathbf{dA} \\ &+ \oint_{A_2} [\mathbf{I}^q + p\mathbf{U}_2 + \rho\varepsilon^H(\mathbf{V}-\mathbf{U}_2)]\cdot\mathbf{dA}, \end{aligned} \tag{2.45}$$

where $\mathscr{V}_2$ is the volume occupied by the SIC. We postpone discussion of the SIC until Section 5. Equations (2.42) and (2.43) contain the statements that the total mass of each constituent of the alloy in the entire core (FOC + SIC) is conserved.

2.2 *Continuity Conditions*

Corresponding to the balance laws set out in Subsection 2.1, there are continuity demands on the ICB. These can be obtained by using a pill box argument in the usual way. The conditions are simplified by the absence of surface currents and masses. It

Downloaded by [Universite Laval] at 23:08 09 July 2014

follows that the magnetic field must be continuous everywhere:

$$[\![\mathbf{B}]\!] = \mathbf{0}, \quad \text{on the ICB and CMB,} \tag{2.46}$$

and that the same is true of U_e and $\mathbf{g}_e$:

$$[\![U_e]\!] = 0, \qquad \text{and} \qquad [\![\nabla U_e]\!] = \mathbf{0}, \qquad \text{on the ICB and CMB;} \tag{2.47a, b}$$

see Appendix B. (Here $[\![Q]\!]$ denotes the jump in a quantity Q across the boundary concerned.) We also have

$$[\![T]\!] = 0, \qquad \text{on the ICB and CMB.} \tag{2.48}$$

Let $\mathbf{n}$ denote the unit outward normal for both the ICB and the CMB. Consider first the CMB. According to our model of the mantle (Section 1),

$$\mathbf{V} = \mathbf{U}_1, \quad \mathbf{n}\cdot\mathbf{I}^{\xi} = 0, \qquad \text{on the CMB.} \tag{2.49a, b}$$

Energy balance requires that

$$[\![\mathbf{n}\cdot\mathbf{I}^{q}]\!] = 0, \qquad \text{on the CMB.} \tag{2.50}$$

In the mantle, $\mathbf{B}$ obeys[2]

$$\partial_t \mathbf{B} = -\nabla \times (\eta_M \nabla \times \mathbf{B}). \tag{2.51}$$

The electrical conductivity of the mantle, $1/\mu_0\eta_M$, is concentrated near the CMB but even there it is much smaller than the core conductivity, i.e. $\eta_M \gg \eta$. The magnetic field in the mantle must obey (2.46) and must join continuously to a source-free potential field in the "vacuum" surrounding Earth. Similarly the U obeying (2.47) must match smoothly to a source-free potential outside Earth.

Consider next the ICB. Corresponding to (2.2), (2.1), (2.3) and (2.22) there are continuity conditions corresponding to conservation of mass, momentum, light constituent and energy. These are[3]

$$[\![\rho\mathbf{n}\cdot(\mathbf{V} - \mathbf{U}_2)]\!] = 0, \qquad \text{on the ICB,} \tag{2.52}$$

$$[\![p - \mathbf{n}\cdot \overleftrightarrow{\boldsymbol{\pi}}^{v}\cdot\mathbf{n} + \rho\mathbf{n}\cdot(\mathbf{V} - \mathbf{U}_2)\mathbf{n}\cdot\mathbf{V}]\!] = 0, \qquad \text{on the ICB,} \tag{2.53}$$

$$[\![\mathbf{n}\cdot\{\mathbf{I}^{\xi} + \rho\xi(\mathbf{V} - \mathbf{U}_2)\}]\!] = 0, \qquad \text{on the ICB,} \tag{2.54}$$

$$[\![\mathbf{n}\cdot\{\mathbf{I}^{q} - \overleftrightarrow{\boldsymbol{\pi}}^{v}\cdot\mathbf{V} - p\mathbf{U}_2 + \rho(\epsilon^H + \epsilon^K)(\mathbf{V} - \mathbf{U}_2)\}]\!] = 0, \qquad \text{on the ICB.} \tag{2.55}$$

[2] The relative motions in the mantle are too small to have any inductive effect, and are omitted in (2.51). Our frame of reference is attached to the mantle so that its velocity of rotation as a whole is zero by definition.

[3] Because of the simplified model we adopt for the inner core (see Section 5), we do not need to impose continuity of the tangential stress, either on the ICB or on the CMB. We have therefore excluded this from (2.53).

Downloaded by [Universite Laval] at 23:08 09 July 2014

We shall make use of the smallness of the inertial and viscous forces in our application, as compared with the pressure gradient, to replace (2.53) and (2.55) by

$$[\![p]\!]=0, \qquad \text{on the ICB,} \tag{2.56}$$

$$[\![\mathbf{n}\cdot\{\mathbf{I}^q-\rho\varepsilon^H(\mathbf{V}-\mathbf{U}_2)\}]\!]=0, \qquad \text{on the ICB.} \tag{2.57}$$

By (2.33), we may write the last of these as

$$[\![(\varepsilon^G-\mu\xi)\rho\mathbf{n}\cdot(\mathbf{V}-\mathbf{U}_2)]\!]+[\![\mu\mathbf{n}\cdot\{\mathbf{I}^\xi+\rho\xi(\mathbf{V}-\mathbf{U}_2)\}]\!] + [\![T\mathbf{n}\cdot\{\mathbf{I}^S+\rho S(\mathbf{V}-\mathbf{U}_2)\}]\!]=0, \qquad \text{on the ICB.} \tag{2.58}$$

The ICB is a surface in phase equilibrium, at which therefore[4]

$$[\![\mu]\!]=0, \qquad [\![\varepsilon^G-\mu^\xi]\!]=0. \qquad \text{on the ICB.} \tag{2.59, 2.60}$$

Applying (2.48), (2.52), (2.54), (2.59) and (2.60) to (2.58), we obtain

$$[\![\mathbf{n}\cdot\{\mathbf{I}^S+\rho S(\mathbf{V}-\mathbf{U}_2)\}]\!]=0, \qquad \text{on the ICB.} \tag{2.61}$$

This shows that entropy is conserved at the ICB; there is no surface source of entropy corresponding to a breakdown in the continuum approximation there. (Velocities are so small that inertia is negligible, and there is no shock at the surface of discontinuity.) Conditions (2.57) and (2.61) are now seen to be equivalent, and we need use only the more convenient, which is usually (2.61).

The ICB is a no-slip surface, so that

$$[\![\mathbf{n}\times(\mathbf{V}-\mathbf{U}_2)]\!]=\mathbf{0}, \qquad \text{on the ICB.} \tag{2.62}$$

3. THE REFERENCE STATE

It is extremely convenient to describe magnetoconvection in the core as a departure of core conditions from a basic *reference state*. The most convenient reference state is a hydrostatic, well-mixed, non-magnetic state. It is therefore governed by (2.1) with $\mathbf{V}$ and $\mathbf{B}$ set zero, by (2.5), and by statements that the state is isentropic and chemically homogeneous:

$$\rho_a^{-1}\nabla p_a=-\nabla U_a\equiv\mathbf{g}_a, \tag{3.1}$$

$$\nabla\cdot(\rho_a\mathbf{V}_a)=-\dot\rho_a, \tag{3.2}$$

$$\nabla\xi_a=\mathbf{0}, \tag{3.3}$$

$$\nabla S_a=\mathbf{0}, \tag{3.4}$$

$$\nabla^2U_a=4\pi k_N\rho_a-2\Omega^2. \tag{3.5}$$

[4] See for example Loper and Roberts (1978). Alternatively, we may recall (see Appendix D) that $\mu=\mu_L-\mu_H$ and $\varepsilon^G=\mu_L\xi_L+\mu_H\xi_H$, where $\xi_L=\xi$, $\xi_H=1-\xi$ and the suffices L and H refer to the light and heavy constituents of the alloy. Thermodynamic equilibrium requires that $[\![\mu_L]\!]=[\![\mu_H]\!]=0$, and these imply (2.59) and (2.60). We shall later use the fact that $\varepsilon^G-\mu\xi=\mu_H$; see footnote 5 below.

Downloaded by [Universite Laval] at 23:08 09 July 2014

The suffix a is used to distinguish variables in this adiabatic state. Since the gravitational field appearing in (3.1) and (3.5) is created by ρ_a and not the full ρ, we have replaced the effective field, $\mathbf{g}_e$, and effective potential, U_e, of Section 2 by $\mathbf{g}_a$ and U_a rather than by the more cumbersome $\mathbf{g}_{ea}$ and U_{ea}.

A few comments should be made about (3.1)–(3.5) as applied to the FOC. First, although the state was described as hydrostatic, it is important to incorporate the fact that it is slowly evolving on the geological time scale: the inner core grows secularly and the concentration of light constituent in the core fluid increases as it does so. It is particularly necessary to recognize that fact in equation (3.2) of mass conservation. This accounts for the presence of the term involving $\dot{\rho}_a$ on the right-hand side of (3.2) and the term involving $\mathbf{V}_a$ on its left-hand side. Both these terms would be absent in a truly hydrostatic state, and in the geophysical context they are minute, but necessary in order to incorporate evolution on the geological time scale. Their effect on convective and magnetohydrodynamic processes is negligible. Because $\mathbf{V}_a$ and $d_t\mathbf{V}_a$ are so small, they can be (and have been) discarded in (3.1), resulting in the hydrostatic equation shown.

Second, since ξ_a and S_a are, by (3.3) and (3.4), functions of t alone, it follows that spatial variations in

$$\rho_a = \rho(p_a, S_a, \xi_a), \qquad T_a = T(p_a, S_a, \xi_a), \qquad \mu_a = \mu(p_a, S_a, \xi_a) \qquad (3.6a, b, c)$$

arise only because of variations in p_a, the gradient of which is determined by (3.1). In this way we find that in the basic state

$$\rho_a^{-1}\nabla\rho_a = \mathbf{g}_a/u_S^2, \qquad \nabla T_a = \alpha^S\mathbf{g}_a, \qquad \nabla\mu_a = \alpha^\xi\mathbf{g}_a. \qquad (3.7a, b, c)$$

We have here introduced the *entropy coefficient of volume expansion*, α^S, which plays a larger role in our work than the more familiar isothermal coefficient of volume expansion, α, to which it is related by

$$\alpha^S = -\rho^{-1}(\partial\rho/\partial S)_{p,\xi} = \alpha T/c_p; \qquad (3.7d)$$

see (D18). A more familiar form of (3.7b) is

$$T_a^{-1}\nabla T_a = \gamma\mathbf{g}_a/u_S^2, \qquad \gamma = \alpha u_S^2/c_p, \qquad (3.7e, f)$$

where γ is the Grüneisen parameter. The gradients (3.7) are generally called "adiabatic gradients".

Because of centrifugal forces, the surfaces of constant U_a are not spherical, but it is clear that p_a, ρ_a and T_a, are constant on surfaces of constant U_a, and that they can all be labeled uniquely by that value of U_a. This is also true of all other thermodynamic parameters, such as α, γ, u_S, c_p,. They too are, through U_a, functions of position; for notational simplicity, we have not added the suffix a to these variables in (3.7) and shall not do so below. Since S_a and ξ_a are constants, the density is, according to (3.3), (3.4) and (3.6a), a function of pressure alone: $\rho_a = \rho(p_a)$. When equations (3.1) and (3.5) are solved,

Downloaded by [Universite Laval] at 23:08 09 July 2014

subject to suitable boundary conditions for U_a and with the density a given fnction of the pressure, solutions are obtained in which the surfaces of constant U_a coincide with those of constant p_a and ρ_a. This is required for self-consistency. The problem of determining such solutions is known as the problem of determining the equilibrium figure of Earth, and (3.1) and (3.5) define the part of this problem that pertains to the core. In reality the figure of Earth deviates slightly from the hydrostatic equilibrium figure. It may be noted that, if (3.1) is supplemented by the equation of heat conduction rather than (3.4), it is no longer true that $\rho_a = \rho(p_a)$ and, for a general distribution of heat sources, there is no solution in which the surfaces of constant ρ and p coincide. The hydrostatic problem then has no solution, as can be seen immediately by taking the curl of (3.1). This means that an imbalance of forces exists which results in some circulations if the gravitating body is fluid. In a fluid–like inner core this would result in meridional motions but, since the effective viscosity of the inner core is so high, these would not be significant in the leading approximation (3.1). In an elastic inner core, the imbalance of forces would create deviations in the surfaces of constant ρ and p, but these would again be so small that they would easily be balanced by elastic stresses. In either case, we may safely neglect the imbalance of forces in the SIC and use (3.1)–(3.5) there, as in the FOC. See also Section 5.

Equations (3.3) and (3.4) express the fact that we are interested only in convection that is so intense that all extensive properties of the basic state are well-mixed by the convection that is superimposed on it. Of course, this will not be true in the boundary layers at the ICB and the CMB, where the fluid moves with the adjacent solid, and vertical mixing by convection is small or absent. In the bulk of the core however, where the approximation is a good one, the thermodynamic variables in the convective state differ from their values in the basic state by so little that the difference can be treated as a perturbation; see Section 4. In fact, the more vigorous the convection, the better the perturbation treatment works[5]. The choice of an adiabatic hydrostatic reference state is usually made, though in disguised form, when modeling laboratory systems. The variation in pressure across these is so slight that the assumption of constant entropy differs little from the assumption of constant temperature, T_0. The difference, ΔT_a, in the adiabatic temperature across the system is small compared with the typical temperature differences, T_c, associated with the convection, in sharp contrast to Earth's core where $T_c \sim 10^{-6}\Delta T_a$.

We noted above that the adiabatic well-mixed reference state is close to being realized throughout the entire volume of the fluid core apart from thin nonadiabatic boundary layers of thickness δ_{na} (say). If nothing special occurs near the ICB and CMB, these layers will be very thin, as the following argument shows. Suppose that the

[5] The system is very far from being in the steady state that might exist were convection weak (or absent) and the core close to (or precisely in) "sedimentation equilibrium", where T, μ and $\varepsilon^G - \xi\mu + U_a = \mu_H + U_a$ are constant. The composition and entropy are not uniform in such a state, and it cannot be regarded as a perturbation of a well–mixed adiabatic state. Such a quiescent 'steady state' is unrealistic for the core. Even if the core were isolated, such a state would arise only after a time of order $\tau_{sed} = L^2/\kappa^\xi \sim 10^{12}$ yr, which is much greater than the age of Earth. (Here κ^ξ is the molecular diffusivity.) In the well–mixed state considered in this paper, (2.11), (3.3) and (3.4) show that $\nabla\varepsilon_a^H = \rho_a^{-1}\nabla p_a$. It therefore follows from (3.1) that $\varepsilon_a^H + U_a$ is constant.

Downloaded by [Universite Laval] at 23:08 09 July 2014

temperature gradient deviates significantly from its adiabatic value, $\nabla_r T_a$, over the length δ_{na}. Then a temperature perturbation $\delta T \sim \delta_{na}\nabla_r T_a$ will arise that results in a fractional density perturbation of $\delta C \sim \alpha\delta_{na}\nabla_r T_a$. Equating this to $C_0 \sim 10^{-8}$ (a value we later show is characteristic of the FOC), we obtain $\delta_{na} \sim C_0/\alpha\nabla_r T_a \sim 1$ m, where we have assumed that $\nabla_r T_a \sim 1\,°\text{K km}^{-1} = 10^{-3}\,°\text{Km}^{-1}$ and $\alpha \sim 10^{-5}\,°\text{K}^{-1}$. In fact, however, special physical processes may become significant near the ICB (e.g. Loper & Roberts, 1981, 1983) and near the CMB (Braginsky, 1993) which invalidate these estimates of δ_{na} and δC, which are found to be much too small. These special boundary layers will not be considered in the present paper.

Let us now consider the different types of inhomogeneities that arise in the core. The greatest inhomogeneity in the reference state is connected with the variation across the core of variables such as ρ_a and T_a; see (3.7). This variation can be measured by a small parameter, ϵ_a, where

$$\epsilon_a = L|\nabla\rho_a|/\rho_a \sim Lg_1/u_S^2. \tag{3.8}$$

Taking the characteristic length over which the density changes to be $L \sim 10^6$ m, the gravitational acceleration at the CMB to be $g_1 = 10.68\,\text{m s}^{-2}$, and the velocity of longitudinal sound waves to be $u_S \sim 10^4\,\text{m s}^{-1}$, we obtain $\epsilon_a \sim 0.1$.

The appropriate dimensionless parameter with which to assess the importance of centrifugal forces on the structure of the reference state is

$$\epsilon_\Omega = 4\Omega^2 L/g_1. \tag{3.9}$$

Using the values given above, we find that $\epsilon_\Omega \approx 2 \times 10^{-3}$. Because of centrifugal forces, the surfaces of constant U_a are not quite spherical. They resemble more oblate spheroids with an ellipticity of order ϵ_Ω, which is approximately 1/299.8 at the surface of Earth, and varies across the fluid core from $e_1 = 1/393.0$ on the CMB to $e_2 = 1/414.9$ on the ICB; see Mathews *et al.* (1991), a paper from which we also took the abbreviations FOC and SIC. This ellipticity is very significant for core motions that are driven by the precession and nutation of the Earth's rotation axis. It is, however, insignificant for the slow convection that drives the geodynamo. We shall therefore neglect it here and assume that *all thermodynamic variables are functions of r alone in the reference state.*

A further significant dimensionless parameter is

$$\epsilon_c = 2\Omega V/g, \tag{3.10}$$

where V is a typical convective flow speed. This measures the relative importance of Coriolis forces associated with the convection and gravitational forces acting on the basic state. It is a more significant quantity in more dynamics than V^2/gL, which is the ratio of inertial forces on the convection to the basic gravitational force; inertial forces play a negligible role in core convection. Traditional estimates of V are of order $3 \times 10^{-4}\,\text{m s}^{-1}$, based on the velocity of westward drift; integrations of model-Z (Braginsky, 1978) lead to values 10 times greater. If we take $V = 10^{-3}\,\text{m s}^{-1}$ as a compromise, we obtain $\epsilon_c \sim 10^{-8}$. The smallness of ϵ_c is extremely significant from

Downloaded by [Universite Laval] at 23:08 09 July 2014

a dynamical point of view: ϵ_c provides an estimate of the error made in supposing that the reference state is in hydrostatic equilibrium (3.1) and in adopting (3.3) and (3.4). Its smallness justifies the omission of Coriolis, magnetic and buoyancy forces in modeling the reference state.

The parameter ϵ_c may be related to a further small parameter which compares the time scale τ_a over which the basic state evolves and the time scale τ_c of core convection:

$$\epsilon_c^\tau = \tau_c/\tau_a. \tag{3.11}$$

Depending on the physical process considered, these times span wide ranges. The time scale τ_a is sometimes called the "geological time scale", although this may be somewhat inappropriate since it suggests that τ_a, is of the order of 4.5 10^9 yr, this being the age of the Earth. In reality, the temperature of the core has diminished only to a small degree during its history, and τ_a, defined as $T_a/\dot{T}_a$, greatly exceeds 4.5×10^9 yr. We expect in fact that that $\tau_a \sim 10^{10}\,\text{yr} - 10^{11}\,\text{yr}$. The time scale τ_c is even more uncertain. At one extreme, convection associated with turbulence operates on time scales of a few years; at the other extreme, large-scale MAC waves typically vary on periods of 10^3 yr, which also characteristic of convective overturning; the time scale of the general circulation of the core is $\sim 10^4$ yr. If as a compromise we take $\tau_a \sim 4 \times 10^{10}$ yr and $\tau_c \sim 400$ yr, we obtain $\epsilon_c^\tau \sim 10^{-8}$, but this value is extremely uncertain. In what follows, we shall usually not distinguish between ϵ_c^τ and ϵ_c, writing either as ϵ_c.

Finally we list in Table 1 some properties of the core that are well determined. They are mostly taken from the PREM model of Dziewonski & Anderson (1981). For a discussion of these and other core parameters, see Appendix E.

One quantity in Table 1 deserves special comment, namely $\Delta\rho$, the density jump at the inner core boundary. This plays a very significant role in our theory. According to

Downloaded by [Universite Laval] at 23:08 09 July 2014

Table 1 Well-determined Parameters

$R_E = 6.371 \times 10^6$ m	average radius of Earth,
$R_1 = 3.480 \times 10^6$ m	radius of the fluid outer core (FOC),
$R_2 = 1.2215 \times 10^6$ m	radius of the solid inner core (SIC),
$\rho_0 = 10.9 \times 10^3\,\text{kg}\,\text{m}^{-3}$	mean density of the FOC,
$\rho_1 = 9.9035 \times 10^3\,\text{kg}\,\text{m}^{-3}$	density of the FOC at the CMB,
$\rho_2 = 12.166 \times 10^3\,\text{kg}\,\text{m}^{-3}$	density of the FOC at the ICB,
$\rho_N = 12.764 \times 10^3\,\text{kg}\,\text{m}^{-3}$	density of the SIC at the ICB,
$\rho(0) = 13.088 \times 10^3\,\text{kg}\,\text{m}^{-3}$	density at the geocenter,
$\Delta\rho = \rho_N - \rho_2 = 0.6 \times 10^3\,\text{kg}\,\text{m}^{-3}$	density jump at the ICB (relatively poorly known),
$g_1 = 10.68\,\text{m}\,\text{s}^{-2}$	acceleration due to gravity at the CMB,
$g_2 = 4.40\,\text{m}\,\text{s}^{-2}$	acceleration due to gravity at the ICB,
$p_1 = 135.75$ GPa	pressure at the CMB,
$p_2 = 328.85$ GPa	pressure at the ICB,
$p(0) = 363.85$ GPa	pressure at the geocenter,
$u_{S1} = 8.065 \times 10^3\,\text{m}\,\text{s}^{-1}$	longitudinal seismic velocity in the FOC at the CMB,
$u_{S2} = 10.356 \times 10^3\,\text{m}\,\text{s}^{-1}$	longitudinal seismic velocity in the FOC at the ICB.

Don Anderson (private communication), the error in the value shown can be no more than 20%, i.e. $0.5 \times 10^3\,\mathrm{kg\,m^{-3}} < \Delta\rho < 0.7 \times 10^3\,\mathrm{kg\,m^{-3}}$. See also Jephcoat & Olson (1987) and Shearer & Masters (1991).

4. THE NATURE OF CORE CONVECTION

4.1 *The Anelastic Approximation*

We stressed in Section 3 that our basic reference state depended on the presence of convection sufficiently vigorous to homogenize the entropy and chemical composition of the fluid core; see (3.3) and (3.4). In this section we study this convection explicitly.

We noted in Section 1 the existence of two distinct time scales: the slow evolutionary time scale, τ_a, of the reference state, and the much shorter time scale, τ_c, associated with convection. It is often convenient to employ a two time scale procedure in which t_a denotes slow time and t_c fast time. The reference state depends on t_a alone; the superimposed convection depends on both t_c and t_a, and $\partial_t = \partial_t^a + \partial_t^c$. Wherever it cannot lead to confusion, we replace ∂_t^a by an overdot, and omit the superfix c on ∂_t^c and d_t^c.

We decompose all quantities into basic and convectional parts, writing

$$\begin{array}{lllll} p = p_a + p_c, & T = T_a + T_c, & \xi = \xi_a + \xi_c, & S = S_a + S_c, & \\ \rho = \rho_a + \rho_c, & U = U_a + U_c, & \mathbf{g}_e = \mathbf{g}_a + \mathbf{g}_c, & \mathbf{V} = \mathbf{V}_a + \mathbf{V}_c, & \text{etc.,} \end{array} \tag{4.0}$$

where the subscript c on a quantity shows that it is associated with the convection. On substituting (4.0) into (2.1)–(2.5) and making use of (3.1)–(3.5), we obtain

$$\rho_a d_t \mathbf{V} = -\nabla P_c + \rho_a \mathbf{g}_c - \rho_c \mathbf{g}_a - 2\rho_a \mathbf{\Omega} \times \mathbf{V} + \rho_a \mathbf{F}^{v} + \rho_a \mathbf{F}^{B}, \tag{4.1}$$

$$\nabla \cdot (\rho_a \mathbf{V}) = 0, \tag{4.2}$$

$$\rho_a d_t \xi_c + \nabla \cdot \mathbf{I}^{\xi} = -\rho_a \dot{\xi}_a, \tag{4.3}$$

$$\rho_a d_t S_c + \nabla \cdot \mathbf{I}^{S} = \rho_a \dot{S}_a + \sigma^{S}, \tag{4.4}$$

$$\nabla^2 U_c = 4\pi k_N \rho_c, \tag{4.5}$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}), \tag{4.6}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4.7}$$

Here and in what follows, we for brevity omit the suffix a on thermodynamic functions evaluated in the reference state, while retaining them on ρ_a, S_a and ξ_a. For example, u_S appears below in place of u_{Sa}. The superfix ^{c} has been omitted from d_t^c and ∂_t^c.

In deriving (4.1)–(4.4), all terms of order ϵ_c times the corresponding terms in the reference state have been retained; those of order ϵ_c^2 have been discarded. The fact that $p_c/p_a = O(\epsilon_c) \ll 1$, $S_c/S_a = O(\epsilon_c) \ll 1$, $\xi_c/\xi_a = O(\epsilon_c) \ll 1$, allowed us to replace ρ in many

Downloaded by [Universite Laval] at 23:08 09 July 2014

terms of (2.1)–(2.7) by ρ_a, a simplification also made frequently below. Consider for example (4.1). The quadratic term $\rho_c \mathbf{g}_c$ has been discarded. In the term $\rho d_t \mathbf{V} = (\rho_a + \rho_c)(d_t^a + d_t^c)(\mathbf{V}_a + \mathbf{V}_c)$, we recognize that d_t^a/d_t^c and $\mathbf{V}_a/\mathbf{V}_c$ are both $O(\epsilon_c)$, and that therefore the dominant part of $\rho d_t \mathbf{V}$ is $\rho_a d_t^c \mathbf{V}_c$. The same process of linearization allows us to replace (2.16) by

$$\rho_a \mathbf{F}^B = \mathbf{J} \times \mathbf{B}. \tag{4.8}$$

We shall also replace (2.11)–(2.13) by the single equation

$$\rho \mathbf{F}^\nu = \rho_a \nu \nabla^2 \mathbf{V}, \tag{4.9}$$

an approximation we discuss further below.

The constitutive relations (2.36)–(2.40) for $\mathbf{I}^\xi$, $\mathbf{I}^S$ and σ^S may be simplified similarly. For example, $\rho\kappa^\xi$, k_T^ξ/T and k_p^ξ/p in (2.36) may be evaluated in the reference state, and may therefore be written as $\rho_a\kappa^\xi$, k_T^ξ/T_a and k_p^ξ/p_a where (see above) the suffix a on κ^ξ, k_T^ξ and k_p^ξ is implied but suppressed. It would however be incorrect to replace $\nabla\xi$, ∇T and ∇p in (2.36) by $\nabla\xi_a = 0$, ∇T_a and ∇p_a. Although $|\xi_c| \ll |\xi_a|$ and $|T_c| \ll |T_a|$, an important component of the convective motions is on small length scales associated with turbulence, and it is *not* necessarily true that $|\nabla\xi_c| \ll |\nabla\xi_a|$ and $|\nabla T_c| \ll |\nabla T_a|$. Similar remarks apply to all of equations (2.36)–(2.40).

The smallness of p_c, S_c, ξ_c, ρ_c, etc. allows "thermodynamic linearization", by which we mean that we may, with an error only of order ϵ_c, inter-relate the deviations, ρ_c, T_c, μ_c, etc., created by the convection in ρ, T, μ, etc., by applying relations such as (D5)–(D7) or (D13)–(D15), treating p_c, S_c, ξ_c, ρ_c etc. as the infinitesimals dp, dS, $d\xi$, $d\rho$, etc. [An example is given in (4.16) below, which follows from (D13).] The resulting simplifications are very significant, but in Section 8 we shall reduce the complications still further by introducing what we shall call "the homogeneous model" or "the modified Boussinesq model". In the homogeneous model, we approximate all thermodynamic coefficients, such as α, u_S etc. by constants, while in the present inhomogeneous model they are functions of r.

Equation (4.2) goes beyond thermodynamic linearization; it incorporates what is generally called "the anelastic approximation". It is justified by noting that

$$\frac{\partial_t \rho_c}{\nabla\cdot(\rho_a \mathbf{V})} \sim \frac{\rho_c/\tau_c}{\rho_a V/L} \sim \frac{\rho_c}{\rho_a} \cdot \frac{L}{\tau_c V} \sim \frac{\rho_c}{\rho_a} \sim \varepsilon_c. \tag{4.10}$$

The replacement of $\rho\mathbf{V} = (\rho_a + \rho_c)\mathbf{V}$ in (2.2) by $\rho_a\mathbf{V}$ in (4.2) follows as before from the smallness of ρ_c. The absence of the time derivative of the density in (4.2) *excludes elastic waves* from the solution of system (4.1)–(4.9) and explains why the approximation is termed "anelastic". The slow motions, characterized by (4.10), are included, while uninteresting high frequency oscillations associated with sound (seismic) waves are filtered out. If, instead of (4.2), $\nabla\cdot\mathbf{V} = 0$, (4.9) would, for constant $\rho\nu$, be an exact consequence of (2.11)–(2.13); since $\nabla\rho_a \neq 0$, (4.9) is not precisely correct. It should be

Downloaded by [Universite Laval] at 23:08 09 July 2014

borne in mind however that we are concerned with small viscosity flows in which $\mathbf{F}^\nu$ is significant only in thin boundary or shear layers across which ρ_a and ν vary little. The expression (4.9) therefore holds with high accuracy wherever $\mathbf{F}^\nu$ is non-negligible.

4.2 *A Significant Simplification*

We devote this subsection to a remarkable simplification of (4.1). We introduce

$$P = \frac{p_e}{\rho_a}, \qquad C = -\alpha^S S_c - \alpha^\xi \xi_c, \tag{4.11,4.12}$$

$$p_e = p_c + \rho_a U_c, \qquad \text{and} \qquad \rho_e = \rho_c + \frac{\rho_a}{u_S^2} U_c, \tag{4.13,4.14}$$

the last two of which are a new "effective" pressure and density; we call P "the reduced pressure". The quantity C plays such a central role in the theory that, in our opinion, it merits a name. We propose to call it *the codensity*. It determines the buoyancy force due to the deviation of the density from the well-mixed basic state of constant S_a and ξ_a. Note that C is independent of p_c.

The first three terms on the right-hand side of (4.1) may, with the help of (4.13), (4.14) and (3.7a), be written as

$$-\nabla p_c - \rho_a \nabla U_c + \rho_c \mathbf{g}_a = -\nabla p_e + U_c \nabla \rho_a + \rho_c \mathbf{g}_a$$
$$= -\nabla p_e + \frac{U_c \rho_a}{u_S^2} \mathbf{g}_a + \rho_c \mathbf{g}_a = -\nabla p_e + \rho_e \mathbf{g}_a. \tag{4.15}$$

The expression for ρ_c that follows from (D13), namely

$$\frac{\rho_c}{\rho_a} = \frac{1}{\rho_a u_S^2} p_c - \alpha^S S_c - \alpha^\xi \xi_c, \tag{4.16}$$

may be written in terms of C and the effective variables (4.13) and (4.14) as

$$C = \frac{\rho_c}{\rho_a} - \frac{p_c}{\rho_a u_S^2}, \qquad \text{or} \qquad C = \frac{\rho_e}{\rho_a} - \frac{p_e}{\rho_a u_S^2}. \tag{4.17a, b}$$

This leads to further simplifications in (4.15). We have, by (3.7a) and (4.11),

$$-\nabla p_e = -\rho_a \nabla\left(\frac{p_e}{\rho_a}\right) - \frac{p_e}{\rho_a} \nabla \rho_a = -\rho_a \nabla P - \frac{p_e}{u_S^2} \mathbf{g}_a,$$

so that by (4.17b)

$$-\nabla p_e + \rho_e \mathbf{g}_a = -\rho_a \nabla P + \rho_a C \mathbf{g}_a. \tag{4.18}$$

It follows from (4.15) and (4.18) that (4.1) may be written in the very simple form

$$d_t \mathbf{V} = -\nabla P + C \mathbf{g}_a - 2\boldsymbol{\Omega} \times \mathbf{V} + \mathbf{F}^\nu + \mathbf{F}^B. \tag{4.19}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

The resemblance of (4.19) to the Boussinesq momentum equation is so striking that it is worth re-iterating here that (4.19) is a consequence of the assumption of an adiabatic, well-mixed reference state. Its precision is of order $\epsilon_c \sim 10^{-8}$. It should be stressed that the density inhomogeneity is taken into account in (4.19) through (3.7a). The elastic part, p_e/u_S^2, of the density perturbation has not been neglected but has been absorbed into P. Equation (4.19) shows clearly that the buoyancy force associated with deviations of order ϵ_c from a well-mixed adiabatic state is created *only* by the codensity through variations in entropy and composition; the buoyancy force associated with pressure variations, though it may be equally large, does not contribute because it is conservative and can be absorbed into the effective pressure to create the potential term, $-\nabla P$, in (4.19). It does not contribute to the generally non-potential term, $C\mathbf{g}_a$. This is the basic reason why the codensity plays such a central role in the theory and why it deserves a special name.

Through the reductions made here, the unpleasant necessity of computing U_c during the process of solution is evaded; $\mathbf{g}_c = -\nabla U_c$ has been eliminated from (4.1), though it has not been neglected. After the solution has been completed, U_c can, if desired, be evaluated by solving (4.5). Though (4.19) resembles the momentum equation for Boussinesq theory, the anelastic continuity equation (4.2) is unchanged and is very different from the corresponding equation ($\nabla\cdot\mathbf{V}=0$) of Boussinesq theory. Thus, our simplification is *not* tantamount to a reduction to Boussinesq theory.

4.3 *Core Turbulence: General Considerations*

We now discuss a very significant component of the convective motions: turbulence. It is hard to doubt that the core is mixed far more effectively than molecular diffusion coefficients such as $\kappa^\xi \sim 10^{-8}\eta$ would suggest and that this is due to turbulence. Because of the notorious difficulties of turbulence theory, and because it would in any case be impractical to add such difficulties to the already formidable geophysical complexities, only a simple "engineering" approach to core turbulence has so far been contemplated. In this approach, one writes

$$\mathbf{V} = \langle \mathbf{V} \rangle^t + \mathbf{V}^\dagger, \qquad \xi_c = \langle \xi_c \rangle^t + \xi_c^\dagger, \qquad \text{etc.,} \tag{4.20}$$

where $\langle \mathbf{V}^\dagger \rangle^t$, $\langle \xi^\dagger \rangle^t$, etc. are zero. The averages are over an ensemble of realizations of the turbulence. More practically, they are taken over the short length and/or time scales of the turbulent components. One seeks to determine the evolution of the average fields $\langle \mathbf{V}\rangle^t$, $\langle \mathbf{B}\rangle^t$, $(\xi)^t$, $\langle S\rangle^t,\ldots$, and to replace (4.1)–(4.9) and (2.36)–(2.40) by equations governing those averages. The effects of turbulence are supposed to be local so that, as for molecular transport processes, all turbulent transport fluxes at a point are proportional to gradients at that point. For example, in the simplest ansatz, the flux $\mathbf{I}^{\xi t}$ of mean composition and the flux $\mathbf{I}^{St}$ of mean entropy due to turbulence are proportional to the local gradients of $\langle \xi_c \rangle^t$ and $\langle S_c \rangle^t$:

$$\mathbf{I}^{\xi t} = -\rho_a \overleftrightarrow{K}^t\cdot\nabla\langle \xi_c \rangle^t, \qquad \mathbf{I}^{St} = -\rho_a \overleftrightarrow{K}^t\cdot\nabla\langle S_c \rangle^t. \tag{4.21, 4.22}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

The fluxes are thus not parallel to the gradients but are linearly related to them by the tensor $\vec{\kappa}^t$ and, more importantly, the significant turbulent transport coefficients contained in $\vec{\kappa}^t$ greatly exceed the two molecular scalars.

Not surprisingly, $\mathbf{I}^{\xi t}$ and $\mathbf{I}^{St}$ are large compared with the molecular contributions $\mathbf{I}^{\xi m}$ and $\mathbf{I}^{Sm}$ to $\mathbf{I}^{\xi}_c$ and $\mathbf{I}^{S}_c$ obtained by averaging (2.36)–(2.40).[6] Since the turbulence transports ξ and S in the same way, the same tensor, $\vec{\kappa}^t$, arises in both (4.21) and (4.22). Double diffusion processes in the core therefore differ greatly from double diffusion processes in the laboratory of the type investigated by Cardin & Olson (1992). The transport of $\langle \xi_c \rangle^t$ and $\langle S_c \rangle^t$ in the core differ *not* because these quantities diffuse differently (that happens only in laboratory conditions) but because their sources are of a different nature. The light component is injected from the ICB but thermal convection is principally determined by the ability of the core to transmit heat to the mantle; the latent heat emitted during freezing at the ICB may be secondary, though it is not very small.

It is hard to avoid parametrizing turbulence in this way. Anyone who prefers instead to employ the primitive variables, $\mathbf{V}, \mathbf{B}, \xi_c, S_c, \ldots$, and the corresponding forms (2.36)–(2.40) for $\mathbf{I}^{\xi}$, $\mathbf{I}^{S}$ and σ^S is free to do so, but he must then use values of κ^T and κ^{ξ} of at most $10^{-5}\eta$ and $10^{-8}\eta$ respectively, and therefore must contend with enormous Rayleigh numbers and other dimensionless parameters. The resulting flows would be turbulent and would require him to strive for impossibly high numerical resolutions. Sooner or later he would be forced to accept an engineering approximation, probably of the type we seek to develop here. The turbulent transport coefficients that then arise are of order $\ell V^t \sim 1\,\mathrm{m^2 s^{-1}}$, which is many orders of magnitude greater than κ^{ξ} and is even much larger than κ^T. (In making this estimate, we have taken $\ell \sim 10^4$m and have assumed the moderate value $10^{-4}\mathrm{m\,s^{-1}}$ for the rms turbulent velocity, V^t.)

Even within the engineering approximation, several different scenarios have been proposed. Braginsky (1964b) and Braginsky & Meytlis (1990) supposed that motions in the core exist on essentially only two, widely disparate, scales, the macroscale L and associated time scale τ_c, and the microscale ℓ and related time scale τ^t. According to their theory, local turbulence consists of an ensemble of plate–like cells having thicknesses, $\ell_\perp$, in the s–direction much less than their other two (z and ϕ) dimensions, both of which are of the order of the microscale ℓ. They argue that, because of the smallness of $\ell_\perp$, the turbulent microscale magnetic Reynolds number is very small, so that microscale induction does not seriously modify Ohm's law for the macroscale. The mean field, $\langle \mathbf{B} \rangle^t$, is therefore governed by (4.6) with the mean velocity $\langle \mathbf{V} \rangle^t$ replacing $\mathbf{V}$ but with the same *molecular* value of η. They derive expressions for $\vec{\kappa}^t$ that are of order η, but they do not derive an approximate form for the Reynolds tensor, $\vec{\pi}^t$.

The Braginsky–Meytlis picture is not the only possibility. There is a second scenario that is theoretically extremely complicated: large-scale turbulence arising from the instability of MAC waves of planetary scale. Such a turbulence

[6] The adiabatic gradient (3.7b) makes, however, a contribution to $\mathbf{I}^{S}_a$ which should not be neglected; see below.

Downloaded by [Universite Laval] at 23:08 09 July 2014

would be of the classical type, involving "cascade" from macroscale to microscale, i.e. the microscale envisaged by Braginsky & Meytlis (1990) would overlap with a macroscale, and their estimates of $\vec{\kappa}^t$ would be invalid; perhaps even the forms (4.21) and (4.22) themselves would be inadmissible. Possibly a turbulent Ohm's law (including a turbulent α-effect) would also be required, as in mean field electrodynamics. One way of investigating whether this second scenario is plausible or not would be first to solve the large-scale convection problem on the assumption that the turbulence is of Braginsky–Meytlis type, i.e. $\mathbf{I}^{\xi t}$ and $\mathbf{I}^{St}$ from (4.21) and (4.22) would be used rather than the corresponding molecular expressions given by (2.36)–(2.40). Second, the instability of that state would be sought. If it were unstable, transition to cascading turbulence would be anticipated, i.e. the second scenario would be plausible. It seems likely, however, that the enhanced diffusion associated with $\vec{\kappa}^t$ would help to stabilize large-scale motions. If this were the case, the second scenario would not be plausible. Stevenson (1979) developed a heuristic theory of core turbulence, based on the assumption that all three characteristic dimensions of the cells are of the same order, L, as that of the core. His results may be relevant to the second scenario but should be treated with caution because of the possible influence of smaller scales of turbulence on the larger scales.

A third scenario has been proposed by Moffatt (1989) and Moffatt and Loper (1994). They imagine that the light material emerging from the ICB during freezing rises in discrete blobs of dimensions ℓ between 10^2 m and 10^4 m and perhaps most typically 10^3m. They suppose that these blobs preserve their identity as they ascend from ICB to CMB. [To the contrary, the simulations of St. Pierre (1995) suggest that the blobs will be enormously distorted after rising only a few hundred km from the ICB.] Moffatt (1989) and Moffatt and Loper (1994) argue that, as they rise, the blobs induce helicity sufficient to self-excite a magnetic field. A full statistical theory of blob motion has not yet been developed. One may imagine that at one extreme, where the blobs interact strongly with one another, such a theory would have strong points of similarity with that of Braginsky & Meytlis (1990). At the other extreme, in which the blobs interact weakly, it may be possible to develop a theory based on a rarefied "gas" of blobs. Further investigations will be required before the role of blobs in core MHD can be properly assessed, but one may again anticipate that, from a statistical mechanics of blobs, a transport theory will emerge that fits into the general framework we have developed below, albeit with a different form for $\vec{\kappa}^t$.

Which of the three scenarios is geophysically the most realistic is unknown. The Braginsky–Meytlis scenario is, at the present time, the most highly developed and (we believe) the most plausible. It does, however, rest on uncertain ground. In the absence of magnetic field, Coriolis forces impart a columnar structure to convective motions; see for example the theoretical studies of Roberts (1968), Busse (1970, 1994), and Glatzmaier & Olson (1993), and the experimental investigations of Busse and Carrigan (1974, 1976) and Boubnov & Golitsyn (1986). It is plausible that Lorentz forces will stretch these structures in the direction of the magnetic field (i.e. primarily longitudinally, the ϕ–direction) and that the convective cells of core turbulence will therefore be plate–like. The uncertainties were highlighted by Braginsky (1964b), who

Downloaded by [Universite Laval] at 23:08 09 July 2014

concluded that additional analysis was necessary before answers could be given to crucial questions such as: 'How long are the cells in the z-direction (i.e. parallel to $\mathbf{\Omega}$)?' 'What is the mechnism that limits their length $\ell(\ll L)$ in the z-direction?' The lengthening of the plate–like cells depends on the diffusivities operating on the instabilities that produce those cells. In Earth's core, the most significant compositional diffusivity is nevertheless small. It is difficult to believe that cells of a thickness, $\ell_\perp$, of only about 1 km extend across the FOC from one hemisphere of the CMB to the other. But what is the mechanism that "breaks-up" these cells? At the present time this question has no satisfactory answer. Braginsky & Meytlis (1990) suggested a heuristic approach which predicts that the plate-like cells have comparable dimensions in the z- and ϕ-directions, and about 20 times smaller in the s-direction (i.e. in the direction away from the rotation axis). A complete theory of turbulence in the presence of Coriolis, Lorentz and buoyancy forces is for the present no more than a dream.

Downloaded by [Universite Laval] at 23:08 09 July 2014

4.4 *Core Turbulence: Averaged Equations*

Let us now proceed more formally to derive the average forms of (4.1)–(4.7). Most discussion centers on (4.3) and (4.4), which by (4.2) may be written as

$$\rho_a \partial_t \xi_c + \nabla \cdot (\rho_a \xi_c \mathbf{V}) + \nabla \cdot \mathbf{I}^\xi = -\rho_a \dot{\xi}_a, \tag{4.23}$$

$$\rho_a \partial_t S_c + \nabla \cdot (\rho_a S_c \mathbf{V}) + \nabla \cdot \mathbf{I}^S = -\rho_a \dot{S}_a + \sigma^S. \tag{4.24}$$

Let us focus first on (4.23). Its average is

$$\rho_a \partial_t \langle \xi_c \rangle^t + \nabla \cdot (\rho_a \langle \xi_c \rangle^t \langle \mathbf{V} \rangle^t) + \nabla \cdot \mathbf{I}^{\xi t} + \nabla \cdot \mathbf{I}^{\xi m} = -\rho_a \dot{\xi}_a, \tag{4.25}$$

where $\mathbf{I}^{\xi m} = \langle \mathbf{I}^\xi \rangle^t$ is the molecular flux due to the average gradients, and

$$\mathbf{I}^{\xi t} = \rho_a \langle \xi_c^\dagger \mathbf{V}^\dagger \rangle^t \tag{4.26}$$

is the turbulent flux of light component. From the average of (4.2), we see that (4.25) may be written as

$$\rho_a d_t \langle \xi_c \rangle^t + \nabla \cdot \mathbf{I}^\xi_{\text{total}} = -\rho_a \dot{\xi}_a, \tag{4.27}$$

where $\mathbf{I}^\xi_{\text{total}} = \mathbf{I}^{\xi m} + \mathbf{I}^{\xi t}$ is the total irreversible flux of admixture due to molecular diffusion and turbulent mixing, and $d_t = \partial_t^c + \langle \mathbf{V} \rangle^t \cdot \nabla$ is the derivative following the mean convective motion. This has exactly the same form as (4.3) but with ξ_c replaced by $\langle \xi_c \rangle^t$, and $\mathbf{I}^\xi$ replaced by $\mathbf{I}^\xi_{\text{total}}$. The term $\mathbf{I}^{\xi m} = \mathbf{I}^{\xi m}_a + \mathbf{I}^{\xi m}_c$ consists of two parts, the first being the result of substituting $\nabla \xi_a$ $(= \mathbf{0})$, ∇T_a and ∇p_a into (2.36); the second arises similarly from the averaged gradients $\nabla \langle \xi_c \rangle^t$, $\nabla \langle T_c \rangle^t$ and $\nabla \langle p_c \rangle^t$ and is extremely small. Even the first term, $\mathbf{I}^{\xi m}_a$, is minute because of the smallness of the molecular diffusion coefficient: $\kappa^\xi \sim 10^{-8}\eta$. From now on, we shall recognize that $\mathbf{I}^{\xi t}$ is the dominating part of $\mathbf{I}^{\xi}_{\text{total}}$ and shall write

$$\mathbf{I}^{\xi}_{\text{total}} = \mathbf{I}^{\xi m} + \mathbf{I}^{\xi t} \simeq \mathbf{I}^{\xi t}. \tag{4.28}$$

This approximation was suggested by Braginsky (1964b). The final form of (4.27) is now

$$\rho_a d_t \langle \xi_a \rangle^t + \nabla \cdot \mathbf{I}^{\xi t} = -\rho_a \dot{\xi}_a. \tag{4.29}$$

The consequences of (4.4) follow similar but more complicated lines, because of the necessity of obtaining an expression for $\langle \sigma^S \rangle^t$. As in (4.26) and (4.28), a total irreversible entropy flux, $\mathbf{I}^S_{\text{total}} = \mathbf{I}^{Sn} + \mathbf{I}^{St}$, replaces the molecular flux, and

$$\mathbf{I}^{St} = \rho_a \langle S_c^{\dagger} \mathbf{V}^{\dagger} \rangle^t, \tag{4.30}$$

but the molecular diffusion term $\mathbf{I}^{Sm} = \mathbf{I}^{Sm}_a + \mathbf{I}^{Sm}_c$ cannot here be omitted. The thermal diffusivity, $\kappa^T \sim 10^{-5}\eta$, greatly exceeds the compositional diffusivity, $\kappa^{\xi} \sim 10^{-8}\eta$, and we must retain $\mathbf{I}^{Sm}_a$ in (4.30) to allow for the molecular diffusion of heat down the adiabat. On neglecting $\mathbf{I}^{\xi m}$ and using (2.37) and (2.38), we find that the molecular flux is approximately

$$\mathbf{I}^{Sm}_a = \left\langle \frac{1}{T}(\mathbf{I}^T + \mu' \mathbf{I}^{\xi})^m \right\rangle^t \simeq \frac{1}{T_a}\mathbf{I}^{Tm}_a \simeq -\frac{K^T}{T_a}\nabla T_a. \tag{4.31}$$

The total entropy flux is approximately

$$\mathbf{I}^S_{\text{total}} = T_a^{-1}\mathbf{I}^T + \mathbf{I}^{St}, \qquad \mathbf{I}^T = -K^T \nabla T_a, \tag{4.32, 4.32a}$$

where, for brevity, $\mathbf{I}^T$ has replaced $\mathbf{I}^{Tm}_a$. The total heat flux, $\mathbf{I}^q_{\text{total}}$, may now be obtained from (2.33) by making use of (4.28) and (4.32)

$$\mathbf{I}^q_{\text{total}} = T_a \mathbf{I}^S_{\text{total}} + \mu_a \mathbf{I}^{\xi}_{\text{total}} = \mathbf{I}^T + T_a \mathbf{I}^{St} + \mu_a \mathbf{I}^{\xi t}. \tag{4.32b}$$

We may write the average of (4.4) as

$$\rho_a d_t \langle S_c \rangle^t + \nabla \cdot \mathbf{I}^{St} = -\rho_a \dot{S}_a + \nabla \cdot [(K^T/T_a)\nabla T_a] + \langle \sigma^S \rangle^t, \tag{4.33}$$

but an expression for $\langle \sigma^S \rangle^t$ is still lacking. Before deriving it, we raise and dismiss an apparent inconsistency that arises when we compare (4.26) and (4.30) with (4.21) and (4.22). The former expressions for $\mathbf{I}^{\xi t}$ and $\mathbf{I}^{St}$ vanish on the walls where $\mathbf{V}^{\dagger} = \mathbf{0}$, but there is no reason why the latter should; indeed, for the success of our later considerations, there is every reason why they should not! The paradox evaporates when we recognize the existence of boundary layers on the ICB and CMB. At the edges of these layers, turbulence is strong; the fluxes are nonzero and are given by (4.21) and (4.22). Within a boundary layer, the turbulent fluxes diminish to zero with the vigor of the turbulent motions as the wall is approached, but this is

Downloaded by [Universite Laval] at 23:08 09 July 2014

simultaneously compensated by an increase in the molecular fluxes, the gradients in $\langle \xi \rangle^t$ and $\langle T \rangle^t$ growing to make that possible. We are not concerned here with the detailed structures of the boundary layers, but we have to appeal to their existence in order to justify the application of (4.21) and (4.22) even "at the walls", by which we mean "at the edges of the boundary layers attached to the walls".

Returning now to the evaluation of $\langle \sigma^S \rangle^t$, we adopt the Reynolds analogy, in which transport through the random motion of the turbulent eddies is likened to that of the random molecular motions, although with much larger diffusion coefficients. In (2.34) we see included, within the rate of entropy production σ^S, contributions made by the molecular fluxes, $\mathbf{I}^\xi$ and $\mathbf{I}^S$. In analogy, we use the same expression for $\langle \sigma^S \rangle^t$, the rate of increase of entropy created by both molecular *and* turbulent diffusion, i.e. we replace $\mathbf{I}^\xi$ and $\mathbf{I}^S$ in (2.34) by $\mathbf{I}^\xi_{\text{total}} = \mathbf{I}^{\xi t}$ and $\mathbf{I}^S_{\text{total}} = \mathbf{I}^{Sm}_a + \mathbf{I}^{St}$. It should be particularly noticed that, according to the Reynolds analogy, *the gradient operators scalarly multiplying these diffusive fluxes now act not on* $\xi_c = \langle \xi_c \rangle^t + \xi_c^\dagger$ *and* $S_c = \langle S_c \rangle^t + S_c^\dagger$ *but on* $\langle \xi_c \rangle^t$ *and* $\langle S_c \rangle^t$. Thus ∇T and $\nabla \mu$ in (2.34) are *not* $\nabla(T_a + T_c)$ and $\nabla(\mu_a + \mu_c)$ as in the molecular case *but* are $\nabla(T_a + \langle T_c \rangle^t)$ and $\nabla(\mu_a + \langle \mu_c \rangle^t)$. And $\langle T_c \rangle^t$ and $\langle \mu_c \rangle^t$ vary on the same length scale as T_a and μ_a, namely the macroscale L. Thus, while it was incorrect, because of the small length scale, $\ell_\perp$, over which $T_c^\dagger$ and $\mu_c^\dagger$ vary, to ignore ∇T_c and $\nabla \mu_c$ in comparison with ∇T_a and $\nabla \mu_a$ (see above), we may make use of the smallness of $|\langle T_c \rangle^t|/|T_a|$ and $|\langle \mu_c \rangle^t|/|\mu_a|$ to replace ∇T and $\nabla \mu$ in (2.34) by ∇T_a and $\nabla \mu_a$, with an error only of order ϵ_c. We may therefore write

$$\langle \sigma^S \rangle^t = \langle \sigma^v \rangle^t + \langle \sigma^J \rangle^t + \sigma^R + \sigma^T - \frac{1}{T_a}(\mathbf{I}^{St} \cdot \nabla T_a + \mathbf{I}^{\xi t} \cdot \nabla \mu_a), \tag{4.34}$$

where by (4.32)

$$\sigma^T = -\frac{1}{T_a} \mathbf{I}^{Sm}_a \cdot \nabla T_a = K^T \left(\frac{\nabla T_a}{T_a} \right)^2; \tag{4.35}$$

cf. (2.40a). By (3.7b, c) and (4.34), we have

$$\langle \sigma^S \rangle^t = \langle \sigma^v \rangle^t + \langle \sigma^J \rangle^t + \sigma^R + \sigma^T + \sigma^t, \tag{4.36}$$

where

$$\sigma^t = -\frac{1}{T_a} \mathbf{g}_a \cdot (\alpha^S \mathbf{I}^{St} + \alpha^\xi \mathbf{I}^{\xi t}). \tag{4.37}$$

According to (4.21) and (4.22), we may rewrite (4.37) as

$$\sigma^t = \frac{\rho_a}{T_a} \mathbf{g}_a \cdot \overleftrightarrow{\boldsymbol{\kappa}}^t \cdot \langle \alpha^S \nabla S_c + \alpha^\xi \nabla \xi_c \rangle^t. \tag{4.38}$$

To avoid violating the positivity of entropy production, we must set $\overleftrightarrow{\boldsymbol{\kappa}}^t$ zero at all points at which σ^t, as calculated from (4.38), is negative. We demonstrate in

Downloaded by [Universite Laval] at 23:08 09 July 2014

Downloaded by [Universite Laval] at 23:08 09 July 2014

Appendix C that, to a satisfactory degree of approximation, the regions where $\sigma^t > 0$ are those that are gravitationally unstable and, as a result, are sources of turbulence. Regions in which (4.38) gives $\sigma^t < 0$ are locally stable; turbulence is then absent and the turbulent fluxes $\mathbf{I}^{\xi t}$ and $\mathbf{I}^{St}$ are zero, as is σ^t.

Most of the remaining terms in (4.1)–(4.7) are linear and easily averaged. Indeed, (4.2) and (4.5)–(4.7) are unchanged on averaging. Two further issues concerning turbulent transport arise in connection with the equation of motion (4.19). All but two terms of (4.19) are linear and easily averaged. Using (4.2), we may write the inertial term with an error of order ϵ_c as $\rho d_t\mathbf{V} = \rho_a \partial_t^c \mathbf{V} + \nabla\cdot(\rho_a \mathbf{V}\mathbf{V})$. When we average we obtain

$$\langle \rho d_t \mathbf{V}\rangle^t = \rho_a d_t^c \langle \mathbf{V}\rangle^t - \rho_a \mathbf{F}^{Vt}, \qquad \rho_a \mathbf{F}^{Vt} = \nabla\cdot\overleftrightarrow{\pi}^{Vt}, \tag{4.39, 4.40}$$

where

$$\overleftrightarrow{\pi}^{Vt} = -\rho_a \langle \mathbf{V}^\dagger \mathbf{V}^\dagger \rangle^t \tag{4.41}$$

is the Reynolds stress tensor. Again using the Reynolds analogy, we may expect that

$$\pi_{ij}^{Vt} = \rho_a v_{ijkl}^{Vt} \nabla_k \langle V_l \rangle^t, \tag{4.42}$$

where $\overleftrightarrow{v}^{Vt}$ is the (fourth order) turbulent viscosity tensor, anisotropic because of the effects of Coriolis and Lorentz forces on the turbulence. It has 36 independent components, since without loss of generality $v_{jikl}^{Vt} = v_{ijlk}^{Vt} = v_{ijkl}^{Vt}$. It is also necessary that the associated entropy production, $\langle e_{ij}\rangle^t \pi_{ij}^{Vt} = \rho_a v_{ijkl}^{Vt} \nabla_i \langle V_j \rangle^t \nabla_k \langle V_l \rangle^t$, is nonnegative.

The other nonlinear term arising in (4.19) is the Lorentz force. This is similar to the inertial force and may be treated in a similar way:

$$\rho_a \langle \mathbf{F}^B \rangle^t = \langle \mathbf{J}\rangle^t \times \langle \mathbf{B}\rangle^t + \langle \mathbf{J}^\dagger \times \mathbf{B}^\dagger \rangle^t. \tag{4.43}$$

Because the magnetic pressure can be absorbed into p_c, (4.43) is effectively equivalent to

$$\rho_a \langle \mathbf{F}^B \rangle^t = \frac{1}{\mu_0} \nabla\cdot[\langle \mathbf{B}\rangle^t \langle \mathbf{B}\rangle^t] + \nabla\cdot\overleftrightarrow{\pi}^{Bt}, \qquad \text{where} \qquad \overleftrightarrow{\pi}^{Bt} = \frac{1}{\mu_0}\langle \mathbf{B}^\dagger \mathbf{B}^\dagger \rangle^t. \tag{4.44, 4.45}$$

According to the local turbulence theory of Braginsky & Meytlis (1990), the magnetic Reynolds number of the microscale motions is very small, and a linear relationship therefore subsists between $\mathbf{B}^\dagger$ and $\mathbf{V}^\dagger$:

$$B_i^\dagger = m_{ij} V_j^\dagger. \tag{4.46}$$

According to their estimates $B^\dagger \sim V^\dagger$; see Appendix C. It follows from (4.46) that (4.45) can be written as

$$\pi_{ij}^{Bt} = v_{ijkl}^{Bt} \nabla_k \langle V_l \rangle^t, \tag{4.47}$$

where

$$\nu_{ijkl}^{Bt} = -\frac{1}{\mu_0} m_{in} m_{jp} \nu_{npkl}^{Vt}, \tag{4.47a}$$

has the same symmetries as ν_{ijkl}^{Vt}.

The similarity of (4.42) and (4.47) suggests that the last term of (4.43) should be transferred to the viscous stress. We therefore write

$$\rho_a \mathbf{F}^B = \langle \mathbf{J} \rangle^t \times \langle \mathbf{B} \rangle^t, \qquad \rho_a \mathbf{F}^{\nu t} = \nabla \cdot \overleftrightarrow{\pi}^t, \tag{4.48, 4.49}$$

where

$$\pi_{ij}^t = \pi_{ij}^{Vt} + \pi_{ij}^{Bt} = \rho_a \nu_{ijkl}^t \nabla_k \langle V_l^\dagger \rangle^t, \qquad \overleftrightarrow{\upsilon}^t = \overleftrightarrow{\upsilon}^{Vt} + \overleftrightarrow{\upsilon}^{Bt}. \tag{4.49a, b}$$

Since $B^\dagger \sim V^\dagger$, the two contributions to $\overleftrightarrow{\pi}^t$ are of the same order of magnitude. We shall assume that positivity of the entropy production, $e_{ij}\pi_{ij}^{Vt} \geq 0$, is maintained even after the addition of $\overleftrightarrow{\pi}^{Bt}$ to $\overleftrightarrow{\pi}^{Vt}$; i.e. we shall suppose that $\rho_a \nu_{ijkl}^t \nabla_i \langle V_j \rangle^t \nabla_k \langle V_l \rangle^t \geq 0$.

The total viscous force is $\mathbf{F}_e^\nu = \mathbf{F}^{\nu m} + \mathbf{F}^{\nu t}$, where $\rho \mathbf{F}^{\nu m} = \rho_a \nu \nabla^2 \langle \mathbf{V}^\dagger \rangle^t$ is the mean force produced by the molecular viscosity ν and the mean velocity gradients; see (4.9). Turbulent mixing is greatly reduced near solid boundaries, and it is therefore unclear whether ν_{ijkl}^t is significantly greater than the molecular viscosity in the bounary layers on the CMB and ICB. Elsewhere even the smallest of the turbulent viscosities, ν_{ijkl}^t, is of order $\nu_\perp^t \sim \ell_\perp V^\dagger \sim (\ell_\perp/\ell)\eta \sim \eta/25 \sim 0.1\mathrm{m}^2\mathrm{s}^{-1}$, according to the estimates of Braginsky and Meytlis (1990). This is of order 10^5 times greater than the molecular viscosity, if estimated as $\nu = 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$. Thus, in the main body of the core, $\mathbf{F}^{\nu m}$ is negligible and $\mathbf{F}_e^\nu \approx \mathbf{F}^{\nu t}$. Despite their much greater size, the effect of the turbulent viscous stresses is scarcely more significant than that of the molecular viscous stresses. This can be seen from the minute size of the turbulent Ekman number, $\epsilon^{\Omega t} \sim \epsilon_\eta^\Omega/25 \sim 4 \times 10^{-11}$, where $\epsilon_\eta^\Omega \sim 10^{-9}$ according to Section 1. The significant viscous stresses within the shear layer surrounding the tangent cylinder, $s = R_2$, are turbulent ones: $\pi_{s\phi} = \pi_{\phi s} \sim \rho_a \nu_\perp^t \nabla_s V_\phi$. In Sections 6–8, we shall absorb the molecular viscosity into the turbulent viscosity, make use of

$$\nu_{ijkl} = \nu_{ijkl}^t + \tfrac{1}{2}\nu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{4.50a}$$

and suppose (see above) that

$$Q^\nu \equiv \rho_a \nu_{ijkl} \nabla_i \langle V_j \rangle^t \nabla_k \langle V_l \rangle^t \geq 0. \tag{4.50}$$

It should be noted that, even though $|\mathbf{B}^\dagger| \ll |\langle \mathbf{B} \rangle^t|$, it is not true that $|\mathbf{J}^\dagger| \sim |\mathbf{B}^\dagger|/\mu_0 \ell$ is much less than $|\langle \mathbf{J} \rangle^t| \sim |\langle \mathbf{B} \rangle^t|/\mu_0 L$. Thus, even though $|\mathbf{B}^\dagger| \ll |\langle \mathbf{B} \rangle^t|$, the contribution, $Q^j = \mu_0 \eta \langle (\mathbf{J}^\dagger)^2 \rangle^t$, made by $\mathbf{J}^\dagger$ to the total ohmic dissipation, $\mu_0 \eta \langle \mathbf{J}^2 \rangle^t = Q^J + Q^j$ is not negligible, where $Q^J = \mu_0 \eta (\langle \mathbf{J} \rangle^t)^2$ is the macroscale ohmic dissipation. In fact, we show in Appendix C that, on Braginsky–Meytlis theory, $Q^j = T_a \sigma^t$ accounts for all the entropy production by the turbulence.

Downloaded by [Universite Laval] at 23:08 09 July 2014

5. THE INNER CORE

5.1 General Properties and Long Time Behavior

Earth's core consists of two parts, the fluid outer core (FOC) and the solid inner core (SIC). We shall sometimes call the SIC "Earth's nucleus". The SIC occupies approximately 35% of the radius, 5% of the mass, and 4% of the volume of the core. It plays, however, a crucial part in convection by providing a source of light fluid at the ICB during the freezing of the FOC and the growth of the SIC. It is essential that this process is properly accounted for in modeling the geodynamo. It is the objective of this Section to consider the role of the SIC and to obtain the boundary conditions on the ICB necessary for the analysis of convection in the FOC.

As far as seismic waves and bodily tides are concerned, and indeed for all phenomena on time scales of seconds to days, the SIC responds as a solid elastic body. Over geological times however, it behaves as a fluid. This can be seen from the fact that its oblateness due to centrifugal forces associated with Earth's rotation is close to that of a body in hydrostatic equilibrium. (Indeed, this statement is true for the entire Earth.) The geodynamo mechanism involves characteristic times ranging from about one year to $\sim 10^4$ yr and maybe more. The rheological properties of the SIC over these time intervals is poorly known. It is even uncertain whether it consists of a single phase. For instance, Fearn *et al.* (1981) argued that a significant fraction of the SIC consists of a matrix of iron dendritic crystals filled with liquid, i.e. that the SIC is in a mixed phase state. Fortunately, a detailed knowledge of the rheology of the SIC is not required for the goals of the present paper to be attained. The bulk of the SIC plays a somewhat passive role in our considerations.

Although the SIC is a body that has a complicated viscoelastic rheology, it behaves, for our purposes, much as a rigid solid on the short (convectional) time scale and behaves as a highly viscous fluid on the long (geological) time scale. It moves with velocity $\mathbf{V} = \mathbf{V}_c + \mathbf{V}_a$ where $\mathbf{V}_a$ is the slow velocity with which the basic state of the SIC adjusts to changing conditions on the geological time scale and $\mathbf{V}_c$ is a solid body rotation with angular velocity $\mathbf{\Omega}_N$ (say):

$$\mathbf{V}_c \approx \mathbf{\Omega}_N \times \mathbf{r}. \tag{5.1}$$

Because of stresses exerted by the FOC across the ICB, this angular velocity may even change on the short (geodynamo) time scale.

Concerning $\mathbf{\Omega}_N$, we recall that throughout Section 4 we have relied on the smallness of ϵ_Ω to neglect the oblateness of the reference state, and in particular the flattening of the ICB. Since however, $\varepsilon_\Omega \gg \varepsilon_c$, the oblateness of the ICB is sufficient to strongly inhibit the rotation of the SIC about any axis perpendicular to $\mathbf{\Omega} = \Omega \mathbf{l}_z$. Rotation of the SIC about the z-axis is however possible, i.e. $\mathbf{\Omega}_N = \Omega_N \mathbf{l}_z$. And Ω_N is determined by the state of convection in the FOC and the nature of the interaction between the core fluid and the SIC. For motions on the characteristic time scale $\sim (10^3$–$10^4)$yr, the inertia of the SIC may be neglected, so that the SIC is in equilibrium under the action of the sum, $\mathscr{L}_2$, of

Downloaded by [Universite Laval] at 23:08 09 July 2014

the z-components of all couples exerted by the core fluid on the SIC:

$$\mathscr{L}_2 = 0. \tag{5.2}$$

These couples consist of the couple created by magnetic forces,

$$\mathscr{L}_2^B = \frac{1}{\mu_0} \oint_{A_2} s B_r B_\phi dA, \tag{5.3}$$

the viscous couple and the topographic couple. We combine the latter two together as

$$\mathscr{L}_2^V = \oint_{A_2} s K_{\mathrm{fr}} (V_{N\phi} - V_{2\phi}) dA. \tag{5.4}$$

To compute $\mathscr{L}_2^B$ from (5.3), we must solve (4.6) and (4.7), which by (5.1) require that

$$\partial_t \mathbf{B} + \Omega_N \partial_\phi \mathbf{B} = -\nabla \times (\eta_N \nabla \times \mathbf{B}), \tag{5.5}$$

where η_N is the magnetic diffusivity of the Nucleus. Solutions to (5.5) must be continuous at the ICB.

To compute $\mathscr{L}_2^V$ from (5.4), we require the coefficient of friction K_{fr} but this is hard to estimate, especially because little is known about the topography of the ICB. If K_{fr} is sufficiently large, (5.2) will require that $\mathscr{L}_2^V = 0$. If K_{fr} is sufficiently small, (5.2) will demand that $\mathscr{L}_2^B = 0$. This may result in a significant change in the behavior of $\mathbf{B}$ in the core. For example, Braginsky (1964a) found, in a kinematic geodynamo model where the condition $\mathscr{L}_2^B = 0$ was imposed, that the magnetic field was much changed; in particular, B_ϕ was small in the SIC. The Joule dissipation was also markedly increased. A similar effect was recently reported for the dynamo model of Hollerbach & Jones (1993). It was demonstrated in that paper that the axisymmetric zonal field was expelled from the interior of the "tangent cylinder", i.e. the whole region $s < R_2$ that includes not only the SIC ($r < R_2$) but also the adjacent parts of the FOC to the North and South of it. In contrast, the influence of the SIC was found to be small for the model-Z dynamo of Braginsky (1989). It appears that the importance of the SIC and the tangent cylinder in the MHD of the core is still uncertain.

In addition to the solid body rotation (5.1), there must, as Earth evolves and the force balance slowly changes, be some slow relative motion, $\mathbf{V}_a$, of adjustment within the inner core. The magnitude of the radial velocity due to thermal expansion, $\mathbf{V}_a$, can be estimated as $V_a \sim \frac{1}{3}\alpha \dot{T}_2 R_2 \sim R_2 \alpha \Delta T / 3t_2$ where ΔT is the change in T_2 during the time, $t_2 \sim R_2 / 3\dot{R}_2$, over which the inner core grows. Thus $V_a / \dot{R}_2 \sim \alpha \Delta T$. Substituting $\alpha \sim 10^{-5}$°K and $\Delta T \sim 100$°K, we obtain $V_a / \dot{R}_2 \sim 10^{-3}$. In short, mass is added at the top of the SIC by freezing (and is perhaps sometimes removed by hot descending streams in the FOC) causing the ICB to advance (and maybe occasionally retreat) with velocities a thousand times larger than the internal relative motions of adjustment within the SIC. The velocity, $\sim \kappa^\xi / L$, with which the light material diffuses in the SIC is of the same order as $\mathbf{V}_a$; taking $\kappa^\xi \sim 10^{-8}\,\mathrm{m^2\,s^{-1}}$ and

Downloaded by [Universite Laval] at 23:08 09 July 2014

Downloaded by [Universite Laval] at 23:08 09 July 2014

a characteristic length $L_N \sim R_2/3$, we obtain $\kappa^\xi/L_N \sim \dot{R}_2\tau_2\kappa^\xi/R_2^2 \sim 10^{-3}\dot{R}_2$, where $\tau_2 \sim 4 \times 10^9$ yr $\sim 10^{17}$ s. Thus, the ICB moves on the τ_a and τ_c timescales and is not precisely spherical. We denote its position by $r = R_N(t, \theta, \phi)$ and the mass of the nucleus by $\mathscr{M}_N(t)$. We write

$$R_N = R_2(t_a) + R_{2c}(t_c, \theta, \phi), \quad \mathscr{M}_N = \mathscr{M}_2(t_a) + \mathscr{M}_{2c}(t_c), \tag{5.6, 5.7}$$

where R_2 and $\mathscr{M}_2$ vary only on the evolutionary (geological) time scale; we consider the derivatives of R_{2c} and $\mathscr{M}_{2c}$ with respect to t_a to be negligibly small. According to the PREM model of Dziewonski & Anderson (1981), $R_2 = 1221.5\,\text{km} = 0.351\,R_1$ at the present time; $R_{2c}(t, \theta, \phi)$ and $\mathscr{M}_{2c}(t)$ are created by the convection and are of order ϵ_c. Differentiating (5.6), we obtain

$$d_t R_N = \dot{R}_2 + \partial_t R_{2c}, \tag{5.8}$$

where d_t is not the material derivative but stands for $\partial_t^a + \partial_t^c$. Although R_{2c} is of order ϵ_c, it varies on the τ_c time scale, so that $\partial_t R_{2c}$ may be or order $\dot{R}_2$, and is therefore not necessarily small. To extract R_{2c} from R_N, and to do likewise for other variables in Section 6, it is convenient to introduce an average over the t_c time scale. We shall denote the *convective average* of a quantity, $Q(t_a, t_c)$ by $\bar{Q}$. Clearly its time dependence is limited to t_a alone ($\partial_t^c \bar{Q} \equiv 0$). Also

$$\overline{\partial_t^c Q} = 0. \tag{5.9}$$

It may be seen from (5.8) and (5.9) that

$$\overline{d_t R_N} = \dot{R}_2. \tag{5.10}$$

The inertial forces associated with $\mathbf{V}_a$ are completely negligible; the equation (3.1) of hydrostatic equilibrium applies also in the SIC. Deviations from hydrostatic equilibrium must, of course, exist in the SIC, but it does not matter to us whether they are equilibrated by elastic stresses or by the stresses associated with small shears in a large viscosity medium, or by some combination of these. For simplicity, we may adopt (2.1) for the SIC with the understanding that the viscosity of the SIC is so large that the convective velocities are negligible. This makes it possible for us to use the same governing equations (2.1)–(2.7) for all time scales and for the entire core.

Since the advection of material within the SIC is so insignificant, and the diffusion of the light constituent is so slight, both may be safely ignored:

$$\mathbf{V}_a = \mathbf{0}, \qquad \mathbf{I}_a^\xi = \mathbf{0}. \tag{5.11, 5.12}$$

It might be imagined that the part of the radial flux, I_{2c}^ξ, that changes on the convective timescale might cause a layer of horizontally varying ξ_c to be deposited on the ICB. During the time $\tau_c \sim 10^4$ yr, the thickness, δ_c, of such a layer would be of order $\delta_c \sim R_2\tau_c/\tau_2 \sim 3$ m. Even though the diffusivity κ^ξ is so small, such structures would be smoothed out very quickly, in a time of no more than $\delta_c^2/\kappa^\xi \sim 30$ yr. The possibility of such layered structures can therefore be ignored. The composition inside the SIC is practically unchanging. It is, at any depth, the same as it was when the ICB passed through that level earlier in Earth's history, and when new material, with the composition appropriate to core conditions at that time, was deposited onto the ICB. During gradual freezing of the SIC, the concentration of admixture in the FOC gradually increases, which implies that ξ_a in the SIC increases outwards. This stable stratification of the SIC makes the possibility of (slow creeping) overturning in the SIC seem quite implausible (Stacey, 1994). We shall suppose for simplicity that ξ_a is spherically symmetric in the SIC; this symmetry could be at least partially brought about by horizontal motions in a boundary layer near the top of the SIC. In this context we may recall the suggestion that the SIC is anisotropic (Morelli *et al.*, 1986), and may also be inhomogeneous. The magnitude of such deviations from spherical symmetry inferred from the observations appears, however, to be small; see also Dziewonski and Woodhouse (1987).

It follows from (5.11) and (5.12) that[7]

$$V_N \equiv V_r(R_2-) = 0, \qquad \mathbf{I}_N^\xi = \mathbf{I}_r^\xi(R_2-) = 0. \tag{5.11a, 5.12a}$$

We may now appeal to (2.52) and (2.54). As in (5.8), we write

$$V_r(R_N+) = V_2 + V_{2c}, \qquad \mathbf{I}_r^{\xi t}(R_N+) = I_2 + I_{2c}. \tag{5.13a, b}$$

Since $U_{2r} = d_t R_N$, we have

$$V_2 \equiv \overline{V_r(R_N+)} = -(\Delta\rho/\rho_2)\dot{R}_2, \qquad V_{2c} = -(\Delta\rho/\rho_2)\partial_t R_{2c}, \tag{5.14a, b}$$

$$I_2^\xi \equiv \overline{I_r^{\xi t}(R_N+)} = \rho_N \xi_{2N} \dot{R}_2, \qquad I_{2c}^\xi = \rho_N \xi_{2N} \partial_t R_{2c}, \tag{5.14c, d}$$

where $\Delta\rho = \rho_N - \rho_2$ is the discontinuity in ρ_a at the ICB, and

$$\xi_{2N} \equiv \xi_a - \xi_N = \xi_a r_{FS} \tag{5.15}$$

is the mass fraction of light constituent that is rejected from the solid and is added to the FOC when core fluid freezes onto the ICB. We call r_{FS} the *rejection factor*; the suffices $_{FS}$ stand for Fluid and Solid. It is determined by the form of the phase diagram of the alloy; see Appendix E.

The velocities (5.14a, b) are of order 10^6 times smaller than the characteristic poloidal convectional velocity, $\mathbf{V}_c$, in the FOC, and when we apply (5.14a, b) as boundary conditions on the ICB, we make negligible error if we replace both of

[7] By our notational convention, the subscript N distinguishes values of variables at the top of the SIC in the basic state (adiabatic reference state), while $_2$, and not the more cumbersome $_{a2}$, denotes the values of the same variables at the bottom of the FOC. An exception is made in the case of R_N and R_2; see (5.8) above. Since neither the ICB nor the CMB are precisely spherical, conditions (5.11a) and (5.12a) are slightly inaccurate, but similar simplifications are frequently made in this paper. The concomitant errors are negligible to the order to which we are working, as are the errors we make when, as we shall, we set $\mathbf{n} = \mathbf{1}_r$, the unit vector in the radial direction.

Downloaded by [Universite Laval] at 23:08 09 July 2014

them simply by

$$V_r = 0, \qquad \text{on} \qquad r = R_2. \tag{5.16}$$

In contrast, the typical magnitude of $\mathbf{I}^\xi$ in the FOC is of the same order as (5.14d) and it would be incorrrect to replace (5.14d) by $\mathbf{I}^\xi_{2c} = 0$.

5.2 *Heat conduction in the SIC*

Any definition of a basic state for the SIC is to some extent arbitrary. Unlike the FOC, nothing changes in the SIC on a fast time scale (apart from changes imposed on it by the FOC) and there is therefore no unique way of extracting a reference state for the SIC. As there is no vigorous mixing in the SIC, its temperature is determined by heat conduction, and we have no strict foundation for assuming (3.4). We may nevertheless use (3.4) to define a reference state and, because the temperature varies little within the SIC, this should differ only slightly from the actual temperature of the SIC. We believe that assumption (3.4) is adequate and more practical than alternatives. Moreover, the negative slope of the adiabat ($\nabla_r T_a < 0$) necessarily agrees a little better with the negative slope of the actual temperature distribution than a constant reference temperature would. It is certainly true however that S deviates from uniformity in the SIC far more strongly than it does in the FOC. It is possible, as we shall now describe, to take into account the heat sources and heat flux in the SIC and to derive corrections, $T_c \equiv T - T_a$ and $S_c \equiv S - S_a$, to the reference temperature and entropy. (In this Subsection, the subscript c will be used to denote the deviation from adiabaticity created by *conduction*. The small amendments ρ_c and $\mathbf{g}_c$ to the density and gravitational field will be ignored.) If the SIC is a mixed phase region, then some small scale convection is also possible within the solid matrix. These might convect heat, and so markedly increase the effective thermal conductivity of the SIC, at the same time reducing the temperature gradient within it. Large scale circulations within the SIC are, however, strongly impeded by having to take place through a porous matrix. In any case, the diffusivity of heat greatly exceeds that of composition, and we cannot ignore heat conduction in the SIC.

We have argued above that the density and composition of the SIC can only change on the geological time scale. We therefore have

$$\xi_c = 0, \tag{5.17}$$

and only two thermodynamic parameters are therefore required to describe the thermal state of the SIC. By (D5) and (D13) we have

$$\frac{\rho_c}{\rho_a} = \frac{1}{\rho_a u_T^2} p_c - \alpha T_c = \frac{1}{\rho_a u_S^2} p_c - \alpha^S S_c. \tag{5.18}$$

No significant relative movement can take place on the convective time scale of 10^4 yr or less, i.e. (5.11) holds. Equation (4.4) therefore gives

$$\rho_a T_a d_t S_c = -\nabla \cdot \mathbf{I}^T - \rho_a T_a \dot{S}_a + Q^R + Q^J, \tag{5.19}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

where only thermal conduction transports heat:

$$\mathbf{I}^T = -K^T\nabla(T_a + T_c). \tag{5.20}$$

Equation (5.19) can be transformed into an alternative, and more convenient, form by using (D6):

$$S_c = \frac{c_p}{T_a} T_c - \frac{\alpha}{\rho_a} p_c. \tag{5.21}$$

To determine the pressure variation, p_c, we should specify the SIC model more precisely. Fortunately, this complicated task can be side-stepped because the last term in (5.21) is much smaller than the others and can be neglected. In order of magnitude, $p_c/R_2 \sim g\rho_c$ and $\rho_c/\rho_a \sim \alpha T_c$, so that the ratio of the two terms on the right-hand side of (5.21) is

$$\frac{\alpha p_c/\rho_a}{c_p T_c/T_a} \sim \frac{\alpha^2 T_a g R_2}{c_p} \sim \frac{\gamma \alpha T_a g R_2}{u_S^2} \sim 10^{-3}. \tag{5.22}$$

We may therefore write

$$S_c = (c_p/T_a)T_c, \qquad \text{and similarly} \qquad \dot{S}_a = (c_p/T_a)\dot{T}_a. \tag{5.21a,b}$$

Neglecting also the variation in c_p/T_a across the SIC, we obtain from (5.19) and (5.21a, b)

$$\rho_a c_p d_t T_c = -\nabla\cdot\mathbf{I}^T - \rho_a c_p \dot{T}_a + Q^R + Q^J. \tag{5.23}$$

The basic (adiabatic) temperature contrast is of order $\Delta T_a \sim 100°$K, which is an order of magnitude smaller than that across the FOC. Because there is no turbulent transport of heat in the SIC, the deviation, T_c, of the basic temperature from that of the basic state is much greater in the SIC than in the FOC. Let us for example estimate the contribution made to T_c by Joule heating, using $\rho_a c_p T_c \sim Q^J \tau_{\kappa N}$, where $\tau_{\kappa N} = L_N^2/\kappa_N^T$ is the thermal time constant of the nucleus. We take $\kappa_N^T \sim 10^{-5}\,\mathrm{m^2\,s^{-1}}$ to be the thermal diffusivity of the nucleus and $L_N = R_2/\pi$ as its characteristic length scale, so obtaining $\tau_{\kappa N} \sim 5\times 10^8$ yr, a very long time. The estimate $Q^J \sim \eta_N B^2/\mu_0 L_N^2$ leads to $T_c \sim (\eta_N/\kappa_N^T)B^2/\mu_0\rho_0 c_p$ which, for $B = 50$ G, gives $T_c \sim 1°$K. Comparing this with a typical value, $T_c \sim 10^{-3}$ °K for the FOC, we see that the FOC provides an almost isothermal environment for the SIC.

Other contributions to T_c in the nucleus are Q^R and $\dot{T}_a$; see (5.23). These are an order of magnitude greater than Q^J, and are spherically symmetric and stationary (on the t_c timescale). The T_c due to Q^J could depend on t, but only very weakly. If Joule heating has a component, Q^J_ω, varying with frequency ω, then a time varying temperature component is generated of order $T_{c\omega} \sim Q^J_\omega/\rho_a c_p \omega \sim (Q^J_\omega/Q^J)T_c/\omega\tau_{\kappa N}$. If $Q^J_\omega \sim Q^J$, then the ratio of the varying component, $T_{c\omega}$, of T_c to the stationary one is very small: $T_{c\omega}/T_c \sim (\omega\tau_{\kappa N})^{-1} \sim 3\times 10^{-5}$. This means

Downloaded by [Universite Laval] at 23:08 09 July 2014

that T_c in the nucleus can be considered to be stationary on the t_c time scale, and (5.23) may therefore be replaced by

$$\rho_a c_p \dot{T}_a = -\nabla\cdot\mathbf{I}^T + Q^R + \bar{Q}^J, \qquad \text{where} \qquad \mathbf{I}^T = -\rho_a c_p \kappa^T \nabla(T_a + T_c). \tag{5.24, 5.25}$$

Solutions to (5.24) must satisfy a boundary condition on $r = R_2$. Since it is convenient to match the basic adiabatic temperatures on the ICB, continuity of T implies continuity of T_c. But T_c is very small in the FOC ($T_c \sim 10^{-3}\,°\mathrm{K}$), so that the temperature differences over the ICB are about 4 orders of magnitude smaller than elsewhere in the SIC. In effect, the FOC provides a uniform temperature "heat bath" in which the SIC lies. And (5.24) must therefore be solved subject to the spherically–symmetric boundary condition

$$T_c = 0. \tag{5.26}$$

Such a solution provides the thermal flux $\mathbf{I}^T$ in the SIC. Because $\bar{Q}^J$ is relatively small, $\mathbf{I}^T$ is nearly spherically symmetric and depends only on t_a, this despite the fact that the state on the fluid side of the ICB is neither spherically symmetric nor independent of t_c. We have

$$I_r^T(R_N-) = I_N^T = -(K_N^T \nabla_r [T_a + T_c])_{R_2-}, \tag{5.27}$$

which is nearly independent of θ, ϕ and t_c.

6. THE CONVECTIVE STATE: INHOMOGENEOUS MODEL (ANELASTIC THEORY)

In this section we use the theory developed in Section 4 to formulate a model of core convection.

We start by summarizing the basic equations derived in Section 4. The angle brackets $\langle$ and $\rangle^t$ of Section 4 will be omitted wherever feasible, as will the superscript c from d_t^c and ∂_t^c, but it should be understood that we are now dealing with turbulently averaged quantities. We write (4.19), (4.2), (4.29), (4.33) and (4.5)–(4.7) as

$$d_t\mathbf{V} = -\nabla P + C\mathbf{g}_a - 2\mathbf{\Omega}\times\mathbf{V} + \mathbf{F}^v + \mathbf{F}^B, \tag{6.1}$$

$$\nabla\cdot(\rho_a\mathbf{V}) = 0, \tag{6.2}$$

$$\rho_a d_t\xi_c + \nabla\cdot\mathbf{I}^{\xi t} = \sigma_e^\xi, \tag{6.3}$$

$$\rho_a d_t S_c + \nabla\cdot\mathbf{I}^{St} = \sigma_e^S, \tag{6.4}$$

$$\nabla^2 U_c = 4\pi k_N \rho_c, \tag{6.5}$$

$$\partial_t\mathbf{B} = \nabla\times(\mathbf{V}\times\mathbf{B}) - \nabla\times(\eta\nabla\times\mathbf{B}), \tag{6.6}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{6.7}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

where

$$C = -\alpha^S S_c - \alpha^\xi \xi_c, \tag{6.1a}$$

is the codensity. By (4.48) and (4.49), we have

$$\rho_a F_i^v = \nabla_j(\rho_a v_{ijkl} \nabla_k V_l), \qquad \rho_a \mathbf{F}^B = \mathbf{J} \times \mathbf{B}, \tag{6.8, 6.9}$$

where v_{ijkl} is the total viscosity, given by (4.50a). The effective sources appearing in (6.3) and (6.4) are seen from (4.29), (4.33) and (4.37) to be

$$\sigma_e^\xi = -\rho_a \dot{\xi}_a, \qquad \sigma_e^S = -\rho_a \dot{S}_a + \sigma_a^S + \sigma_c^S, \tag{6.10, 6.11}$$

where

$$\sigma_a^S = \sigma^R + \sigma_-^T, \qquad \sigma_c^S = T_a^{-1} Q^D, \tag{6.12, 6.13}$$

and

$$\sigma^R = T_a^{-1} Q^R, \quad \sigma_-^T = -T_a^{-1} \nabla \cdot \mathbf{I}^T, \quad \mathbf{I}^T = -\mathbf{K}^T \nabla T_a, \tag{6.12a,b,c}$$

$$Q^D = Q^v + Q^J + Q^t, \tag{6.13a}$$

$$Q^t = -(\mathbf{I}^{St} \cdot \nabla T_a + \mathbf{I}^{\xi t} \cdot \nabla \mu_a) = -\mathbf{g}_a \cdot (\alpha^S \mathbf{I}^{St} + \alpha^\xi \mathbf{I}^{\xi t}). \tag{6.14a, b}$$

For brevity, we here and below replace the $\mathbf{I}_a^{Tm}$ of Section 4 by $\mathbf{I}^T$. Corresponding to (6.8) and (6.9),

$$Q^v = \rho_a v_{ijkl} (\nabla_i V_j)(\nabla_k V_l) \geq 0, \qquad Q^J = \mu_0 \eta J^2 \geq 0. \tag{6.15a, b}$$

Also, by (4.21) and (4.22), we have

$$\mathbf{I}^{\xi t} = -\rho_a \overleftrightarrow{\kappa}^t \cdot \nabla \xi_c, \qquad \mathbf{I}^{St} = -\rho_a \overleftrightarrow{\kappa}^t \cdot \nabla S_c. \tag{6.16a, b}$$

It may be noted that σ_-^T is not the rate of entropy production by conduction down the adiabatic gradient, which is $\sigma^T = -T_a^{-2} \mathbf{I}^T \cdot \nabla T_a = \mathrm{K}^T (T_a^{-1} \nabla T_a)^2 \geq 0$; see (4.35). It is, in fact, a combination of that term and the divergence of the entropy flux down the adiabatic gradient, i.e. $\nabla \cdot \mathbf{I}_a^{Sm} = \nabla \cdot (T_a^{-1} \mathbf{I}^T)$; see (4.32). It therefore need not be positive and, in the geophysical context is, in fact, negative. In (6.12) we recognize two well-known effects acting on the reference state: radioactive heating, Q^R, which tends to promote convection and $-\nabla \cdot \mathbf{I}^T$ which, by diminishing the effectiveness of Q^R, tends to suppress convection. In (6.13) we see sources that arise from convection alone; they cannot therefore be a primary cause of convection.

An equation governing the evolution of the codensity, C, can be obtained by multiplying (6.3) and (6.4) by $-\alpha^\xi$ and $-\alpha^S$ respectively, and by adding corresponding sides:

$$\rho_a d_t C + \nabla \cdot \mathbf{I}^C = \sigma_e^C + \sigma_i^C, \tag{6.17}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

where

$$\mathbf{I}^C = -\alpha^\xi \mathbf{I}^{\xi t} - \alpha^S \mathbf{I}^{St}, \qquad \sigma_e^C = -\alpha^\xi \sigma_e^\xi - \alpha^S \sigma_e^S, \tag{6.18a, b}$$

$$\sigma_i^C = -(\xi_c \rho_a \mathbf{V} + \mathbf{I}^{\xi t})\cdot\nabla\alpha^\xi - (S_c \rho_a \mathbf{V} + \mathbf{I}^{St})\cdot\nabla\alpha^S. \tag{6.18c}$$

The source σ_i^C arises because of inhomogeneities in α^ξ and/or α^S.

Equations (6.1)–(6.16), together with boundary conditions on the ICB and CMB, define the inhomogeneous model. On solid boundaries, $\mathbf{V}$ must obey the no–slip conditions:

$$\mathbf{V}(R_1) = 0, \qquad \mathbf{V}(R_2) = \mathbf{V}_N, \qquad \mathbf{V}_N = \mathbf{\Omega}_N \times \mathbf{r}, \tag{6.19a, b, c}$$

where $\mathbf{\Omega}_N$ is the angular velocity of solid-body rotation of the Nucleus (SIC). The magnetic and gravitational fields are continuous; see (2.46) and (2.47).

Conditions (2.49b), (2.33) and (2.50) give

$$\mathbf{I}_r^{\xi t}(R_1) = 0, \qquad I_r^{St}(R_1) = \frac{1}{T_1}(I_M^q - I_1^T), \tag{6.20a, b}$$

where $I_1^T \equiv I_r^T(R_1)$ is the heat flux down the adiabat at the CMB, and $I_M^q \equiv I_r^q(R_1)$ is the heat flux from the core to the mantle. These are determined by conditions in the mantle, namely by the temperature distribution and the state of convection there. They change on the slow geological time scale, and are regarded here as being prescribed quantities. In a broader statement of the problem, the core and mantle should be considered together in determining the thermal history of Earth. The flux of heat from the core, I_M^q, is primarily determined by convection in the mantle, and in its turn that determines the intensity of all dynamical processes in the core. These two subsystems, the core and the mantle, are separated by the D″ layer, where a rather large decrease in temperature (about 1000 °K) occurs. This decrease is possible because the thermal conductivity of the mantle is about ten times smaller than that of the core. Each subsystem adjusts to the other, and each evolves in its own (but mutually coupled) way. Even the characteristic time, $\tau'' \sim L_D^2/\kappa_M$, of the D″–layer is much longer than the magnetic diffusion time, τ_η, of the core. If we take $L_D \sim 10^5$m as the characteristic scale of the layer and $\kappa_M \sim 10^{-6}\,\mathrm{m^2\,s^{-1}}$ as the thermal diffusivity of the mantle, we obtain $\tau'' \sim 3 \times 10^8$ yr $\sim 3 \times 10^4 \tau_\eta$. Nevertheless, τ'' is much less than the diffusive time scale L^2/κ_M of the mantle as a whole.

The conditions on the composition and temperature (entropy) at $r = R_2$ are more complicated than (6.20). Let us consider first composition. To solve (6.3) we need an expression for $\dot{\xi}_a$; see (6.10). This can be obtained by conservation of light component in the FOC. Integrating (6.3) over the FOC, we obtain

$$\partial_t \int_{\mathscr{V}_{12}} \xi_c \rho_a dV + \oint_{A_{12}} \xi_c \rho_a \mathbf{V}\cdot\mathbf{dA} + \oint_{A_{12}} \mathbf{I}^{\xi t}\cdot\mathbf{dA} = -\int_{\mathscr{V}_{12}} \dot{\xi}_a \rho_a dV. \tag{6.21}$$

We average this integral balance over the convective time scale. The first term on the left-hand side disappears identically by (5.9), and the second term vanishes because V_r is zero on the CMB and ICB by (6.19). Because of (6.20a), the third term is

Downloaded by [Universite Laval] at 23:08 09 July 2014

integrated only over A_2, and to perform this integration we apply the boundary conditions (5.14c, d) on

$$I_r^{\xi t}(R_2+) = I_2^\xi + I_{2c}^\xi, \tag{6.22}$$

namely

$$I_2^\xi = \rho_N \xi_{2N} \dot{R}_2, \qquad I_{2c}^\xi = \rho_N \xi_{2N} \partial_t R_{2c}. \tag{6.22a, b}$$

Noting that the convectional average of I_{2c}^ξ is zero and that I_2 is independent of θ and ϕ, we see from (6.21) that

$$\dot{\xi}_a = \frac{A_2}{\mathcal{M}_{12}} I_2^\xi, \qquad \text{or} \qquad \dot{\xi}_a = \frac{A_2}{\mathcal{M}_{12}} \xi_{2N} \rho_N \dot{R}_2, \tag{6.23a, b}$$

where $\mathcal{M}_{12} = \int_{\mathscr{V}_{12}} \rho_a dV$ is the mass of the FOC and $A_2 = 4\pi R_2^2$ is the area of the ICB. It follows from (6.10) and (6.22) that

$$\sigma_e^\xi = \sigma_2^\xi, \tag{6.24}$$

where

$$\sigma_2^\xi = -\frac{\rho_a A_2}{\mathcal{M}_{12}} I_2^\xi, \qquad \text{or} \qquad \sigma_2^\xi = -\tilde{\rho} \frac{\dot{\mathscr{V}}_2}{\mathscr{V}_{12}} \xi_{2N} \rho_N, \tag{6.24a, b}$$

and

$$\tilde{\rho} = \rho_a \mathscr{V}_{12} / \mathcal{M}_{12} = \rho_a / \rho_0 \tag{6.25}$$

is a non-dimensional function that describes the form of the density distribution in the FOC. It was here convenient to introduce a mean density, ρ_0; later we shall need a mean temperature, T_0, also. These are defined by

$$\rho_0 = \langle \rho_a \rangle^V, \qquad T_0 = \langle T_a \rangle^V, \tag{6.25a, b}$$

where the volumetric average $\langle Q \rangle^V$ of a quantity Q is given by

$$\langle Q \rangle^V = \frac{1}{\mathscr{V}_{12}} \int_{\mathscr{V}_{12}} Q \, dV. \tag{6.26}$$

Condition (6.23) can be obtained more easily if we recall that the total mass of light constituent,

$$\mathcal{M}^\xi = \xi_a \mathcal{M}_{12} + \int_{\mathscr{V}_2} \xi \rho_a dV, \tag{6.27a}$$

and the total mass,

$$\mathcal{M}_1 = \mathcal{M}_2 + \mathcal{M}_{12}, \tag{6.27b}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

of the core are constant. Operating with ∂_t^a therefore gives $\partial_t^a \mathscr{M}^\xi \equiv \dot{\xi}_a \mathscr{M}_{12} + \xi_a \dot{\mathscr{M}}_{12} + \xi_N \dot{\mathscr{M}}_2 = 0$ or, since $\dot{\mathscr{M}}_{12} = -\dot{\mathscr{M}}_2$,

$$\dot{\xi}_a \mathscr{M}_{12} = \xi_{2N} \dot{\mathscr{M}}_2, \tag{6.28a}$$

and, substituting

$$\dot{\mathscr{M}}_2 = \rho_N A_2 \dot{R}_2, \tag{6.28b}$$

we obtain (6.23b). It may be noted that, although $I_r^{\xi t}(R_2+)$ varies on the t_c time scale, σ_2^ξ does not, because $\sigma_2^\xi = -\rho_a \dot{\xi}_a$; see (6.10). The slowly varying surface flux corresponding to σ_2^ξ is I_2^ξ; see (5.14c). In essence, the volumetric source σ_e^ξ of ξ_c arises from a surface source of ξ_c on the ICB, and this is the reason we have introduced the suffix 2 on σ_2^ξ in (6.24) and (6.24a). The light fluid, released when the heavy constituent of core fluid freezes onto the ICB, rises and distributes itself homogeneously throughout the volume $\mathscr{V}_{12}$, so producing an effective sink within it represented by $\sigma_e^\xi = \sigma_2^\xi < 0$. The thermal parallel is a fluid core, heated from below, and cooled homogeneously within, both factors being favorable for convective instability. We could also easily allow for sources at the top of the layer. For example, a flux of iron into the core from the mantle would introduce a contribution, σ_1^ξ, to σ_e^ξ, but we do not consider such effects in this paper.

Consider next entropy. Condition (2.61) of entropy conservation may be written as

$$[\![I_{\text{total}}^S]\!] - \rho_N U_2 (S_2 - S_N) = 0, \qquad \text{at} \qquad r = R_2,$$

where (2.52) has been used and V_N has been neglected. Using (4.32) and $U_2 = \dot{R}_2 + \partial_t R_{2c}$, we separate

$$I_r^{St}(R_2+) \equiv I_2^S + I_{2c}^S, \tag{6.29}$$

into

$$I_2^S \equiv \overline{I_r^{St}(R_2+)} = \frac{1}{T_2}[I_N^T - I_2^T + h_N \rho_N \dot{R}_2], \qquad I_{2c}^S = \frac{1}{T_2} h_N \rho_N \partial_t R_{2c}. \tag{6.29a, b}$$

In (6.29a) and (6.29b), h_N is the heat released at the ICB due to the freezing of the ICB. In view of the presence of the light element, this differs from the latent heat of melting, h_L, as it is usually defined:

$$h_N = h_L - \mu_2 \xi_{2N}, \qquad h_L = \varepsilon_2^H - \varepsilon_N^H, \qquad h_N = T_2 (S_2 - S_N); \tag{6.30a, b, c}$$

see Appendix D. We could also use the continuity condition (2.57) for the total heat flux (4.32b) instead of (2.61). The results of doing so are identical because of (6.30a, b) and the equation (2.54) expressing conservation of admixture.

The arguments leading to (6.24) and (6.24a) show that the effective source, σ_e^ξ, of ξ_c arises from the need to conserve light material, which requires a nonzero

Downloaded by [Universite Laval] at 23:08 09 July 2014

flux, I_2^ξ, at the ICB. A similar, but more complex, connection can, and must now, be established between the volumetric and surficial sources of S_c. The derivation of (6.23b) and (6.24) is a good preparation for the corresponding derivation of $\dot{S}_a$ and σ_e^S.

Integrate (6.4) over $\mathscr{V}_{12}$ and average over t_c. As before, the first term on the left-hand side vanishes by (5.9). We therefore obtain

$$\rho_0 \dot{S}_a = -\frac{1}{\mathscr{V}_{12}}(I_1^S A_1 - I_2^S A_2) + \langle \sigma_a^S + \overline{\sigma_c^S} \rangle^V, \tag{6.31}$$

where

$$I_1^S = I_r^{St}(R_1) = \frac{1}{T_1}(I_M^q - I_1^T), \tag{6.31a}$$

$$I_2^S = \overline{I_r^{St}(R_2)} = \frac{1}{T_2}(I_N^T - I_2^T + h_N \rho_N \dot{R}_2); \tag{6.31b}$$

cf. (6.29a). The convective average of the flux (6.29b) is zero, and it therefore does not appear in (6.31). The expressions (6.29b) and (6.31a, b) provide boundary conditions for equation (6.4).

We now substitute expression (6.31) into (6.11), thereby eliminating $\dot{S}_a$. Some insight is gained by dividing the resulting "effective" entropy source, σ_e^S into three parts:

$$\sigma_e^S = \sigma_1^S + \sigma_2^S + \sigma_{12}^S, \tag{6.32}$$

where by (6.11), (6.29b) and (6.31a, b)

$$\sigma_1^S = \tilde{\rho}\frac{A_1}{\mathscr{V}_{12}} I_1^S = \tilde{\rho}\frac{A_1}{\mathscr{V}_{12}}\frac{1}{T_1}[I_M^q - I_1^T], \tag{6.32a}$$

$$\sigma_2^S = -\tilde{\rho}\frac{A_2}{\mathscr{V}_{12}} I_2^S = -\tilde{\rho}\frac{A_2}{\mathscr{V}_{12}}\frac{1}{T_2}[I_N^T - I_2^T + h_N \rho_N \dot{R}_2], \tag{6.32b}$$

$$\sigma_{12}^S = \sigma_a^S + \sigma_c^S - \tilde{\rho}\langle \sigma_a^S + \overline{\sigma_c^S} \rangle^V. \tag{6.32c}$$

The second forms in (6.32a, b) were obtained from (6.31a, b).

The three volumetric sources appearing in (6.32) arise respectively from the CMB, the ICB and the bulk of the FOC. It may be particularly noted that, according to (6.32a), only the excess of I_M^q over the adiabatic heat flux, I_1^T, enters σ_1^S. If the flux of heat down the adiabat is too great, that $I_M^q < I_1^T$. Then $\sigma_1^S < 0$, and the situation resembles that arising in thermal convection when a layer is heated from above and cooled from within. Compositional buoyancy arising from the light fluid source σ_2^ξ will, if large enough, drive convection even if σ_1^S is negative, although the magnitude of C will be reduced (Loper, 1978). The heat flux from the ICB is associated with a negative σ_2^S and a positive I_2^S; see (6.31b). It promotes convection. The terms σ_1^S and σ_2^S do not vary on the t_c time scale.

Consider next σ_{12}^S given by (6.32c). The two contributions, σ^R and σ_-^T, to σ_a^S given by (6.12a, b) act in opposite directions, σ^R to assist convection and σ_-^T to oppose it, but a net effect can arise only through the radial inhomogeneity of their sum. To see this,

Downloaded by [Universite Laval] at 23:08 09 July 2014

note that, if either σ_a^S or σ_c^S were proportional to ρ_a, it would make no contribution to σ_{12}^S. For this reason, we expect that σ^R will play a small part: convective mixing makes Q^R/ρ_a uniform and, if the core were uniform in temperature, σ^R would be proportional to ρ_a. Similarly, bearing in mind that g_r is approximately proportional to r, we see that, in a uniform core ($\tilde{\rho} = 1$), $T_a\sigma_-^T$ would, like Q^R, be constant by (3.7b) and (6.12b, c). Again, σ_a^S would not contribute to σ_{12}^S. Only through the inhomogeneity of T_a can σ^R and σ_-^T contribute terms to σ_{12}^S, but even then these terms are only of order ϵ_a.

The term $\dot{R}_2$ appears in (5.14c), (6.23b), (6.31b) and (6.32b); it should now be evaluated. The ICB is a surface in phase equilibrium, and must therefore be at the melting temperature, corresponding to the liquidus 'curve' (really a surface in $p\xi T-$space), $T_m(p, \xi)$, for the pressure $p = p_a + p_c$ and composition $\xi = \xi_a + \xi_c$ on the fluid side of the ICB:

$$T(R_N) = T_m(p(R_N), \xi). \tag{6.33}$$

The corresponding composition for the SIC is given by the solidus 'curve'. Substituting $R_N = R_2 + R_{2c}$ into (6.33) and writing $T = T_a + T_c$, $p = p_a + p_c$ and $\xi = \xi_a + \xi_c$, we obtain in the zeroth approximation

$$T_2 \equiv T_a(R_2, t_a) = T_m(p_a(R_2, t_a), \xi_a(t_a)). \tag{6.34}$$

We consider (6.34) to be the definition of R_2. Differentiating (6.34) with respect to t_a, we find that

$$d_t^a T_2 \equiv \dot{R}_2\nabla_r T_2 + \dot{T}_2 = \frac{\partial T_m}{\partial p_a}(\dot{R}_2\nabla_r p_2 + \dot{p}_2) + \frac{\partial T_m}{\partial \xi_m}\dot{\xi}_a, \tag{6.35}$$

where $\nabla_r T_2 = (\nabla_r T_a)_2$ and $\nabla_r p_2 = (\nabla_r p_a)_2$. Differentiating $T_a = T(p_a, S_a, \xi_a)$ with respect to t_a, we obtain

$$\dot{R}_2\nabla_r T_2 + \dot{T}_2 = \left(\frac{\partial T_a}{\partial p_a}\right)_2 (\dot{R}_2\nabla_r p_2 + \dot{p}_2) + \left(\frac{\partial T_a}{\partial S_a}\right)_2 \dot{S}_a + \left(\frac{\partial T_a}{\partial \xi_a}\right)_2 \dot{\xi}_a. \tag{6.36}$$

Using this to eliminate T_2 from (6.35) and recognizing that $\nabla_r p_2 = -\rho_2 g_2$, we see that

$$\dot{R}_2/R_2 = r_{2p}\dot{p}_2 - r_{2S}\dot{S}_a - r_{2\xi}\dot{\xi}_a, \tag{6.37}$$

where, by (D14), (D18) and (3.7e).

$$r_{2p} = \frac{1}{\rho_2 g_2 R_2}, \qquad r_{2S} = \frac{1}{\Delta_{ma} c_p}, \qquad r_{2\xi} = -\frac{1}{\Delta_{ma}}\left(\frac{h^\xi}{c_p T_a} + \frac{\partial \ln T_m}{\partial \xi_a}\right), \tag{6.37a, b, c}$$

$$\Delta_{ma} = \frac{\gamma_2 g_2 R_2}{u_S^2}\left(\frac{\partial T_m/\partial p_2}{\partial T_a/\partial p_2} - 1\right). \tag{6.37d}$$

Here γ_2 is the Grüneisen parameter (3.7f) evaluated at $r = R_2$; the dimensionless parameter Δ_{ma} is proportional to the difference between the melting point gradient and

Downloaded by [Universite Laval] at 23:08 09 July 2014

the adiabatic gradient at $r = R_2$. We note that $\partial T_a/\partial p_2 = \alpha^S/\rho_2$, according to (D18). Working to the next approximation, we obtain from (6.33) a form for R_{2c} similar to (6.37):

$$\partial_t R_{2c}/R_2 = r_{2p} p_c - r_{2S} S_c - r_{2\xi}\xi_c. \tag{6.38}$$

The hydrostatic pressure, p_a, changes because of mass redistribution. That due to processes outside the core was considered by Gubbins (1983), who found it too small to affect significantly either convection in the bulk of the FOC or freezing of the SIC. As mentioned in Section 1, we exclude from consideration all such processes, except the one responsible for the heat flux I_M^q is emanating from the core. The change in p_a due to the redistribution of mass within the core is small, being proportional to the density drop, $\Delta\rho$, at the ICB; in fact, $\dot{p}_2 \sim g_2 \dot{R}_2 \Delta\rho$.[8] The term $r_{2p}\dot{p}_2$ in (6.37) is therefore of order $\Delta\rho/\rho_0 \sim 6 \times 10^{-2}$ times smaller than the left-hand side of (6.37). Since $\Delta\rho/\rho_0 \sim 6 \times 10^{-2}$, we may neglect $r_{2p}\dot{p}_2$. According to (6.22), $\dot{\xi}_a$ is proportional to $\dot{R}_2$, and (6.37) can be written in the simple form

$$\dot{S}_a = -c_p \Delta_2 \frac{\dot{R}_2}{R_2}. \tag{6.39}$$

Here Δ_2 is a new dimensionless parameter defined by

$$\Delta_2 = \Delta_{ma} + \Delta_{m\xi}, \tag{6.40}$$

where by (6.37c) we have

$$\Delta_{m\xi} = -\frac{3\mathscr{V}_2}{\mathscr{V}_{12}} \cdot \frac{\rho_N}{\rho_0}\left[\frac{\xi_{2N} h^\xi}{c_p T_2} + \frac{\xi_{2N}}{T_2}\left(\frac{\partial T_m}{\partial \xi_a}\right)_p\right]. \tag{6.41}$$

If p_c in (6.38) is similarly negligible, we also have

$$\partial_t S_c(R_2) = -c_p \Delta_2 \frac{\partial_t R_{2c}}{R_2}. \tag{6.42}$$

An argument that leads to the estimate $\Delta_2 \approx 0.05$ is presented in Appendix E. It should be stressed that this value is very uncertain.

Equation (6.42) shows that too strong a heating of the core results in a negative $\partial_t R_{2c}$ and hence in reductions in the fluxes $I_2^{\xi t}$ and I_2^{St}, according to (5.14c) and (6.29a) or (6.31b); similarly, too weak a heating leads to growth of these fluxes. This favors the establishment of some 'average' level in the intensity of convection. Especially the term $\sigma_c^S - \langle \overline{\sigma_c^S} \rangle^V$ in (6.40c) produces such a stabilizing effect: if convection is too intense, dissipation rises and enhances S_c, resulting in a diminishment in the sources of convection. This stabilizing effect was noted by Braginsky (1964b).

[8] See Appendix B. It should perhaps be noted that $\dot{p}_2$ is the Eulerian derivative of p_a at $r = R_2$ and not the rate of change of p_2 following the motion of the ICB, which is of order $g_2 \dot{R}_2 \rho_2$, i.e. is much larger.

Downloaded by [Universite Laval] at 23:08 09 July 2014

Downloaded by [Universite Laval] at 23:08 09 July 2014

Through (6.39) we can write σ_e^S in a third form which is particularly convenient if Q^R is known. While it is true that cooling of the core by the heat I_M^q lost to the mantle is the primary cause of both thermal and gravitational stirring of the core, the value of I_M^q is not directly observable and is poorly known. It can be obtained only indirectly through investigations of the thermal evolution of the coupled core–mantle system. The age, t_2, of the inner core is comparatively better known: on the one hand, t_2 cannot be very small because the birth and growth of the inner core would then be evident from paleomagnetic data; on the other hand, the creation of the inner core could easily be missed if it occurred in the remote geological past, for which paleomagnetic information is comparatively scant. We shall estimate t_2 to be 4×10^9 yr, but we recognize that this may be too large by a factor of order 2. If we assume that the age of the core is t_2 and that it has been growing at an approximately uniform rate ever since, we obtain $\dot{\mathcal{M}}_2 \sim \mathcal{M}_2/t_2$ and an estimate of $\dot{R}_2$ which, though rough, is probably more reliable than any estimate of I_M^q. Moreover, it may in principle be made more precise by detailed investigation of the evolution of Earth. It seems therefore reasonable to use $\dot{R}_2$ as the main parameter determining the amplitude of the power source fueling the geodynamo, in the case of dominating compositional convection, when the thermal source can be neglected. In this way, if we still wish to derive S_c, we obtain from (6.11) and (6.39) a new expression for σ_e^S, namely

$$\sigma_e^S = \rho_a c_p \Delta_2 \dot{R}_2 / R_2 + \sigma_a^S + \sigma_c^S. \tag{6.43}$$

To use this, we need σ_a^S, which requires knowledge of the magnitude of Q^R.

The expression for the boundary condition on $I_1^{St} = I_r^{St}(R_1)$ in terms of $\dot{R}_2$ is a complicated matter. This is because I_1^q can be linked to S_a only through the entropy balance (6.11) and (6.31). This gives

$$I_1^S = \frac{A_2}{A_1} I_2^S + \frac{\mathcal{V}_{12}}{A_1} \langle \overline{\sigma_e^S} \rangle^V. \tag{6.44}$$

The flux I_2^S is given by (6.29a) in terms of $\dot{R}_2$ and, provided we know Q^R, we can evaluate the final term in (6.44) by averaging (6.43).

In summary, the intensity of convection is determined by (in addition to the physical properties of the core) just two parameters: $\dot{R}_2$ and $I_M^q - I_1^T$, the former of which can be roughly estimated. The radioactive heating, Q^R, can be expressed in terms of the other two parameters, and (supposing that $\dot{R}_2 \geq 0$) $Q^R \mathcal{V}_1 \lesssim A_1 I_M^q$. According to (6.32c), Q^R is relatively ineffective as a source of entropy but it does influence the size of $I_M^q - I_1^T$.

7. DYNAMO ENERGETICS AND EFFICIENCY

7.1 *Energetics of a Heat Engine*

There are significant points of difference between the geodynamo system and a common heat engine. To understand the former better, let us start from a heat engine operating steadily in the well-known Carnot cycle. In this classical device, the heat input $\mathcal{Q}_+$, is provided at a higher temperature, T_+, than the temperature, T_-, at which heat ($\mathcal{Q}_-$, say) is extracted. Averaged over the working cycle, the rate at which the machine does useful work is

$$\mathcal{A} = \mathcal{Q}_+ - \mathcal{Q}_-. \tag{7.1}$$

In the absence of any losses due to imperfections in the engine, the entropy input and output are equal in the steady state:

$$\frac{\mathcal{Q}_+}{T_+} = \frac{\mathcal{Q}_-}{T_-}. \tag{7.2}$$

The efficiency of a perfect engine is therefore

$$\eta_E \equiv \frac{\mathcal{A}}{\mathcal{Q}_+} = 1 - \frac{\mathcal{Q}_-}{\mathcal{Q}_+} \equiv \eta_C, \quad \text{say}, \tag{7.3a}$$

where

$$\eta_C = 1 - \frac{T_-}{T_+} \tag{7.3b}$$

is the "Carnot Efficiency".

Applying similar ideas to Earth's core, considered as a system that is stationary on average, Backus (1975) and Hewitt *et al.* (1975) noted significant differences. First, it is no longer clear what should be classed as the "useful work" done by the engine. They defined it, as we shall, to be the rate, $\mathcal{Q}^J$, of production of large-scale magnetic energy by large-scale fluid motions. This energy is ohmically degraded into heat. The engine must make good not only this energy loss but also the energy, uselessly dissipated at the rate $\mathcal{Q}^F$ (say), by internal friction. It follows that

$$\mathcal{A} = \mathcal{Q}^D, \tag{7.4a}$$

where

$$\mathcal{Q}^D = \mathcal{Q}^J + \mathcal{Q}^F \tag{7.4b}$$

is the total dissipation. Second, both the Joule and frictional heat reappear *within* the fluid; they must be regarded as part of the energy source driving the engine. The energy balance is therefore

$$\mathcal{Q}_+ + \mathcal{Q}^D = \mathcal{Q}_- + \mathcal{A}, \tag{7.5}$$

or, by (7.4a),

$$\mathcal{Q}_+ = \mathcal{Q}_-. \tag{7.5a}$$

The entropy balance is expressed by

$$\frac{\mathcal{Q}_+}{T_+} + \frac{\mathcal{Q}^D}{T_D} = \frac{\mathcal{Q}_-}{T_-}, \tag{7.6}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

where T_D is the effective temperature at which $\mathcal{Q}^D$ is produced. This is related to the effective temperatures, T_J and T_F, at which $\mathcal{Q}^J$ and $\mathcal{Q}^F$ are dissipated by

$$\frac{\mathcal{Q}^D}{T_D}=\frac{\mathcal{Q}^J}{T_J}+\frac{\mathcal{Q}^F}{T_F}. \tag{7.6a}$$

According to (7.5a), we may rewrite (7.6) as

$$\mathcal{Q}^D=\mathcal{Q}_+T_D\left(\frac{1}{T_-}-\frac{1}{T_+}\right). \tag{7.6b}$$

It may be seen from (7.4a) and (7.6) that heat is needed *not* to maintain the energy balance [note particularly that (7.5a) does not contain $\mathcal{Q}^J$ or $\mathcal{Q}^F$ at all] but to preserve the entropy balance.

Regarding the magnetic energy production to be the only work that the engine does usefully, we may write the dynamo efficiency in the form

$$\eta_E=\frac{\mathcal{Q}^J}{\mathcal{Q}_+}=\frac{\mathcal{Q}^J}{\mathcal{Q}^D}\cdot\frac{\mathcal{Q}^D}{\mathcal{Q}_+}=\eta_F\eta_I, \tag{7.7a}$$

where the "frictional factor", η_F gives the fraction of the energy dissipation that is "useful dissipation":

$$\eta_F=\frac{\mathcal{Q}^J}{\mathcal{Q}^J+\mathcal{Q}^F}, \tag{7.7b}$$

while the factor

$$\eta_I=T_D\left(\frac{1}{T_-}-\frac{1}{T_+}\right), \tag{7.7c}$$

represents an "ideal efficiency" which cannot be exceeded, even if there is no source of internal friction. Since $T_-\le T_D\le T_+$, it follows from (7.7c) that

$$\eta_C\le\eta_I\le\eta_B,\qquad\text{where}\qquad \eta_B=\frac{T_+}{T_-}-1; \tag{7.8a,b}$$

see Backus (1975).

This example demonstrates clearly that the necessity of preserving the entropy balance limits the efficiency of the device in producing magnetic power. It suggests that, in analogy with the oft encountered phrase 'the available energy', it is useful to call a quantity such as $\mathcal{Q}^D$ "the available dissipation", of which only the fraction η_F goes into useful work, $\mathcal{Q}^J$, while the remainder, $\mathcal{Q}^F$, represents work wasted by the engine. The crucial importance of "the entropy balance in relating magnetic field to the energy sources" was stressed by Gubbins *et al.* (1979).

Downloaded by [Universite Laval] at 23:08 09 July 2014

Downloaded by [Universite Laval] at 23:08 09 July 2014

7.2 *Energetics of the Inhomogeneous Model*

Let us now turn to the core. This differs from the example just considered in an essential way: it is an *evolving* system that receives its energy from that evolution, namely through its cooling and the resulting gravitational settling. We shall completely confine this non-stationarity to the reference state, and shall consider the superimposed convection as cyclic, i.e. one that, when averaged over a time τ_c equal to the period of the convection cycle, varies only on the τ_a time scale. To provide a suitable definition of geodynamo efficiency, we must consider the balances of energy and entropy. The total kinetic and magnetic energies associated with the macroscale are

$$\mathscr{E}^K = \frac{1}{2}\int_{\mathscr{V}_{12}} \rho_a V^2 dV, \qquad \mathscr{E}^B = \frac{1}{2\mu_0}\int_{\mathscr{V}_\infty} B^2 dV. \tag{7.9, 7.10}$$

The viscous and Joule dissipations are[9]

$$\mathscr{Q}^\nu = \int_{\mathscr{V}_{12}} \rho_a \nu_{ijkl} \nabla_i V_j \nabla_k V_l dV, \qquad \mathscr{Q}^J = \int_{\mathscr{V}_1} \mu_0^{-1}\eta(\nabla \times \mathbf{B})^2 dV, \tag{7.11, 7.12}$$

and the rate of working of the fluid on the large scale magnetic field is

$$\mathscr{A}^B = -\int_{\mathscr{V}_{12}} \rho_a \mathbf{V}\cdot\mathbf{F}^B dV. \tag{7.13}$$

By multiplying (6.1) scalarly by $\rho_a\mathbf{V}$, applying (6.2) and integrating over $\mathscr{V}_{12}$, we obtain the macroscale kinetic energy balance

$$\partial_t \mathscr{E}^K = \mathscr{A}^C - \mathscr{Q}^\nu - \mathscr{A}^B, \tag{7.14}$$

which ∂_t replaces ∂_t^c and $\mathscr{A}^C$ is the rate of working of the buoyancy force (the Archimedean power):

$$\mathscr{A}^C = \int_{\mathscr{V}_{12}} \rho_a C\mathbf{V}\cdot\mathbf{g}_a dV. \tag{7.15}$$

By multiplying (6.6) scalarly by $\mu_0^{-1}\mathbf{B}$ an integrating over all space, we obtain the macroscale magnetic energy balance:

$$\partial_t \mathscr{E}^B = \mathscr{A}^B - \mathscr{Q}^J. \tag{7.16}$$

[9] Strictly, (7.12) is correct only if the mantle is an insulator. In the more realistic case of an electrically conducting mantle, an additional term, $\mathscr{Q}^{JM}$, arises. This is the Joule dissipation associated with electric currents that either leak from core to mantle or that are induced in the mantle by time varying fields in the core. For brevity, we shall not write down an explicit expression for $\mathscr{Q}^{JM}$, but shall consider that it is implicitly included in the viscous term, $\mathscr{Q}^\nu$, a procedure that is possible since $\mathscr{Q}^\nu$ and $\mathscr{Q}^{JM}$ never appear separately in the energy balance, but only as a sum $\mathscr{Q}^\nu + \mathscr{Q}^{JM}$.

Averaging this over the convection cycle, we obtain

$$\bar{\mathscr{A}}^B = \bar{\mathscr{Q}}^J. \tag{7.16a}$$

It follows from (7.14) and (7.16) that

$$\partial_t(\mathscr{E}^K + \mathscr{E}^B) = \mathscr{A}^C - \mathscr{Q}^J - \mathscr{Q}^v, \tag{7.17}$$

and after averaging over the convective cycle we obtain

$$\bar{\mathscr{A}}^C = \bar{\mathscr{Q}}^J + \bar{\mathscr{Q}}^v. \tag{7.17a}$$

This is the analogue of (7.4a, b), in which the quantities shown are in fact also averages over the working cycle. A result similar to (7.17a) holds for the microscale. Recalling results from Appendix C, we see that

$$\bar{\mathscr{Q}}^t \equiv \bar{\mathscr{Q}}^j, \tag{7.18}$$

where

$$\mathscr{Q}^t = \int_{\mathscr{V}_{12}} \langle \rho_a C^\dagger \mathbf{V}^\dagger \rangle^t \cdot \mathbf{g}_a dV = \int_{\mathscr{V}_{12}} \mathbf{I}^C \cdot \mathbf{g}_a dV, \qquad \mathscr{Q}^j = \int_{\mathscr{V}_{12}} \mu_0^{-1} \langle \eta (\nabla \times \mathbf{B}^\dagger)^2 \rangle^t dV, \tag{7.18a,b}$$

are respectively the rate of working of the buoyancy force on the microscale motions and the Joule dissipation of the microscale currents.

An energy balance for the core similar to (7.5) and (7.5a) can be obtained by applying (6.3), (6.4) and (7.17a). We multiply (6.3) and (6.4) by μ_a and T_a respectively, sum the two resulting expressions, and simplify using

$$\mu_a \rho_a \mathbf{V} \cdot \nabla \xi_c + T_a \rho_a \mathbf{V} \cdot \nabla S_c = \nabla \cdot [\rho_a(\mu_a \xi_c + T_a S_c)\mathbf{V}] + \rho_a C \mathbf{V} \cdot \mathbf{g}_a,$$

which follows from (6.1a), (6.2) and (3.7b,c); the last term here is the volumetric rate of working of the buoyancy force. In this way we obtain, using (6.10)–(6.14),

$$\begin{aligned} &\rho_a(\mu_a \partial_t \xi_c + T_a \partial_t S_c) + \nabla \cdot [(\mu_a \xi_c + T_a S_c)\rho_a \mathbf{V} + \mu_a \mathbf{I}^{\xi t} + T_a \mathbf{I}^{St} + \mathbf{I}^T] \\ &\quad = -\rho_a(\mu_a \dot{\xi}_a + T_a \dot{S}_a) + Q^R + Q^v + Q^J - \rho_a C \mathbf{V} \cdot \mathbf{g}_a. \end{aligned} \tag{7.19}$$

Integrate this over $\mathscr{V}_{12}$, take the convective average, and use (6.19a, b, c), (6.20a, b), (6.22a), (6.31a, b) and (7.17a) to obtain

$$\int_{\mathscr{V}_{12}} (\mu_a \dot{\xi}_a + T_a \dot{S}_a)\rho_a dV = -I^q_M A_1 + (I^T_N + I^L_2)A_2 - \bar{\mathscr{Q}}^J_2 + \mathscr{Q}^R_{12}, \tag{7.20}$$

where [see (5.14c) and (6.30a)] I^L_2 is the flux of latent heat from the SIC,

$$I^L_2 = h_L \rho_N \dot{R}_2 = h_N \rho_N \dot{R}_2 + \mu_2 I^\xi_2, \tag{7.21}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

$\bar{\mathcal{Q}}^J_2 = \bar{\mathcal{Q}}^J - \bar{\mathcal{Q}}^J_{12}$ and

$$\mathcal{Q}^R_{12} = \int_{\mathcal{V}_{12}} Q^R dV, \qquad \bar{\mathcal{Q}}^J_{12} = \int_{\mathcal{V}_{12}} \bar{Q}^J dV, \qquad \bar{\mathcal{Q}}^J_2 = \int_{\mathcal{V}_2} \bar{Q}^J dV. \tag{7.20a,b,c}$$

We obtain an expression for the heat balance for the SIC from (5.19):

$$\int_{\mathcal{V}_2} T_a \dot{S}_a \rho_a dV = -I^T_N A_2 + \bar{\mathcal{Q}}^J_2 + \mathcal{Q}^R_2. \tag{7.22}$$

Adding (7.20) and (7.22), we obtain the heat balance for the entire core:

$$\int_{\mathcal{V}_1} (\mu_a \dot{\xi}_a + T_a \dot{S}_a)\rho_a dV = \mathcal{Q}^R + \bar{\mathcal{Q}}^L - \bar{\mathcal{Q}}^q_M, \tag{7.23}$$

where $\bar{\mathcal{Q}}^L = I^L_2 A_2$ is due to the latent heat of SIC crystallization and

$$\mathcal{Q}^R = \mathcal{Q}^R_{12} + \mathcal{Q}^R_2, \qquad \bar{\mathcal{Q}}^L = h_L \dot{\mathcal{M}}_2, \qquad \bar{\mathcal{Q}}^q_M = I^q_M A_1. \tag{7.23a,b,c}$$

Equation (7.23) can be obtained directly by integrating (7.19) over the entire core; the term $\bar{\mathcal{Q}}^L$ then arises as an internal heat source. From now onwards in this Section and throughout Section 8, we shall consider exclusively convectively averaged balances, and without risk of ambiguity we may (and shall) omit the overbar on $\bar{\mathcal{Q}}^J$, $\bar{\mathcal{Q}}^D$, etc.

Like the earlier simple example (7.5a), the balance law (7.23) does not involve the Joule dissipation, but there is a crucial difference between (7.5a) and (7.23): the heat engines most commonly considered are on average in a steady state but, because Earth's core is evolving, (7.23)—unlike (7.5a)—involves time derivatives, namely $\dot{S}_a$ and $\dot{\xi}_a$. The geodynamo is fed not merely by the heat source Q^R but also by the changing state of the core. The left-hand side of (7.23) is in fact a potent source of energy for the geodynamo.

The heat balance (7.23) can be transformed into a more familiar statement expressing internal energy balance. The time derivative of the internal energy of the basic state is

$$\dot{\mathcal{E}}^I_a = d^a_t \int_{\mathcal{V}_1} \varepsilon^I_a \rho_a dV = \int_{\mathcal{V}_1} (\dot{\varepsilon}^I_a \rho_a + \varepsilon^I_a \dot{\rho}_a) dV + \varepsilon^I_1 \rho_1 A_1 \dot{R}_1 + (\varepsilon^I_N \rho_N - \varepsilon^I_2 \rho_2) A_2 \dot{R}_2. \tag{7.24a}$$

Substituting the expression for $\dot{\varepsilon}^I_a$ implied by (2.9), and introducing the enthalpy $\epsilon^H = \epsilon^I + p/\rho$, we may write this as

$$\dot{\mathcal{E}}^I_a = \int_{\mathcal{V}_1} (\mu_a \dot{\xi}_a + T_a \dot{S}_a)\rho_a dV + \int_{\mathcal{V}_1} \varepsilon^H_a \dot{\rho}_a dV + \varepsilon^I_1 \rho_1 A_1 \dot{R}_1 + (\varepsilon^H_N \rho_N - \varepsilon^H_2 \rho_2) A_2 \dot{R}_2. \tag{7.24b}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

We have here used $\rho\varepsilon^H - \rho\varepsilon^I = p$, so that $[\![\rho\varepsilon^H - \rho\varepsilon^I]\!] = [\![p]\!] = 0$ at the ICB. Using (3.2) and $\nabla\varepsilon_a^H = \rho_a^{-1}\nabla p_a$ which follows from (2.11) and $\nabla S_a = \nabla\xi_a = 0$, we obtain

$$\int_{\mathscr{V}_1} \varepsilon_a^H \dot{\rho}_a dV = -\int_{\mathscr{V}_1} \varepsilon_a^H \nabla\cdot(\rho_a \mathbf{V}_a) dV$$

$$= \int_{\mathscr{V}_1} \rho_a \mathbf{V}_a \cdot \nabla\varepsilon_a^H dV - \varepsilon_1^H \rho_1 A_1 V_1 - A_2(\varepsilon_N^H \rho_N V_N - \varepsilon_2^H \rho_2 V_2)$$

$$= \int_{\mathscr{V}_1} \mathbf{V}_a \cdot \nabla p_a dV - \varepsilon_1^H \rho_1 A_1 V_1 - A_2(\varepsilon_N^H \rho_N V_N - \varepsilon_2^H \rho_2 V_2), \quad (7.24c)$$

where we have applied the divergence theorem; V denotes the radial component of $\mathbf{V}$. Conservation of mass at the ICB requires that $\rho_N(\dot{R}_2 - V_N) = \rho_2(\dot{R}_2 - V_2)$ and, since $V_1 = \dot{R}_1$, we may combine (7.24b) and (7.24c) to give

$$\dot{\mathscr{E}}_a^I = \int_{\mathscr{V}_1} (\mu_a \dot{\xi}_a + T_a \dot{S}_a)\rho_a dV + \mathscr{A}_a^g + \mathscr{A}_1^p - \mathscr{Q}^L. \quad (7.25)$$

We have here ignored V_N in comparison with $\dot{R}_2$ (see Section 5) and have recognized that $h_L \rho_N A_2 \dot{R}_2 = \mathscr{Q}^L$. We have also defined

$$\mathscr{A}_a^g = \int_{\mathscr{V}_1} \mathbf{V}_a \cdot \nabla p_a dV = \int_{\mathscr{V}_1} \rho_a \mathbf{V}_a \cdot \mathbf{g}_a dV, \qquad \mathscr{A}_1^p = -\oint_{A_1} p_a \mathbf{V}_a \cdot \mathbf{dA}. \quad (7.25a,b)$$

The two expressions for $\mathscr{A}_a^g$ are equal in virtue of the equation (3.1) of hydrostatic equilibrium. Equations (3.1) and (7.25a) show that $\mathscr{A}_a^g$ is the work done by the gravitational field through the geologically averaged motion of the core.

Using (7.25) we may now write (7.23) in the form

$$\dot{\mathscr{E}}_a^I = \mathscr{A}_a^g + \mathscr{A}_1^p + \mathscr{Q}^R - \mathscr{Q}_M^q. \quad (7.26)$$

This may be compared term by term with Eqn. (5) of Gubbins *et al.* (1979). A significant difference is that we have separated the slow evolutionary effects of the evolving background from short time scale processes, whereas only one time scale is explicitly included in their analysis. The changes occurring on the convective time scale were filtered out by them from those occurring on the geological time scale in their subsequent discussion. It may be noted that the first two terms in (7.26) may be combined into one term describing the rate of working of the pressure on the fluid:

$$\mathscr{A}^p \equiv \mathscr{A}_a^g + \mathscr{A}_1^p = -\int_{\mathscr{V}_1} p_a \nabla\cdot\mathbf{V}_a dV. \quad (7.25c)$$

To derive the entropy balance analogous to (7.6), we integrate (6.4) over $\mathscr{V}_1$ and take the convective average of the result, so obtaining

$$\int_{\mathscr{V}_1} \dot{S}_a \rho_a dV = \frac{\mathscr{Q}^R}{T_R} + \Sigma^T + \frac{\mathscr{Q}^D}{T_D} + \frac{\mathscr{Q}^N}{T_2} - \frac{\mathscr{Q}_M^q}{T_1}, \quad (7.27)$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

where

$$\mathscr{Q}^N = \mathscr{Q}^L - \mu_2 \xi_{2N} \dot{\mathscr{M}}_2 = h_N \dot{\mathscr{M}}_2, \tag{7.28}$$

$$\frac{\mathscr{Q}^R}{T_R} = \int_{\mathscr{V}_1} \frac{Q^R}{T_a} dV, \qquad \mathscr{Q}^R = \int_{\mathscr{V}_1} Q^R dV, \tag{7.29a, b}$$

$$\frac{\mathscr{Q}^D}{T_D} = \int_{\mathscr{V}_1} \frac{Q^D}{T_a} dV, \qquad \mathscr{Q}^D = \int_{\mathscr{V}_1} Q^D dV, \tag{7.30a, b}$$

$$\Sigma^T = \int_{\mathscr{V}_1} \sigma^T dV = \int_{\mathscr{V}_1} K^T \frac{(\nabla T_a)^2}{T_a^2} dV = -\int_{\mathscr{V}_1} \frac{\mathbf{I}^T \cdot \nabla T_a}{T_a^2} dV. \tag{7.31}$$

In the derivation of (7.27) we have substituted $\sigma_-^T = \sigma^T - \nabla \cdot (T_a^{-1} \mathbf{I}^T)$ and have used the boundary conditions (6.31a, b) along with (7.23b, c) and the expression

$$\sigma^T = \mathbf{I}^T \cdot \nabla T_a^{-1} = K^T (\nabla T_a / T_a)^2, \tag{7.31a}$$

which follows from (6.12b, c). Because ∇T_c is not necessarily negligibly small in the SIC, the slightly different expression

$$\sigma^T = K^T \overline{(\nabla T)^2} / T_a^2, \tag{7.31b}$$

where $T = T_a + T_c$, should be used in preference to (7.31a) in the SIC. In deriving (7.28) we have used $h_N = h_L - \mu_2 \xi_{2N}$ where $\xi_{2N} = \xi_a - \xi_N$; see (6.30a).

The heat balance (7.23) can be given a different form in which the heat flux to the mantle, $\mathscr{Q}_M^q$, is expressed as a sum of sources arising from gravitational differentiation, $\mathscr{A}^\xi$, the decreasing entropy of the core, $\mathscr{Q}^S$, the radiogenic heating, $\mathscr{Q}^R$, and the growth of the nucleus, $\mathscr{Q}^N$:

$$\mathscr{Q}_M^q = \mathscr{A}^\xi + \mathscr{Q}^S + \mathscr{Q}^R + \mathscr{Q}^N. \tag{7.32}$$

Here

$$\mathscr{A}^\xi = -\int_{\mathscr{V}_{12}} (\mu_a - \mu_2) \dot{\xi}_a \rho_a dV, \tag{7.33}$$

$$\mathscr{Q}^S = -\int_{\mathscr{V}_1} T_a \dot{S}_a \rho_a dV. \tag{7.34}$$

To derive (7.32) from (7.23), we have used (6.28a) and (7.28).

By multiplying the entropy balance equation (7.27) by T_1 and subtracting the result from (7.32), we may eliminate the unknown $\mathscr{Q}_M^q$ and obtain an expression for $\mathscr{Q}^D$. This quantity, representing the 'available dissipation' is proportional to the sum of the compositional and thermal terms:

$$(T_1/T_D)\mathscr{Q}^D = \mathscr{A}^\xi + \mathscr{Q}^H, \tag{7.35}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

where

$$\mathcal{Q}^H = \mathcal{Q}_1^S + \mathcal{Q}^R\left(1 - \frac{T_1}{T_0}\right) - T_1\Sigma^T + \mathcal{Q}^N\left(1 - \frac{T_1}{T_2}\right), \tag{7.36}$$

$$\mathcal{Q}_1^S = -\int_{\mathscr{V}_1} (T_a - T_1)\dot{S}_a\rho_a dV. \tag{7.37}$$

All terms except $\mathcal{Q}^R$ on the right-hand side of (7.32) and (7.36) may be estimated by writing $\dot{\mathscr{M}}_2 = \mathscr{M}_2/t_2$ and by assuming a reasonable value for t_2. Comparison of (7.34) and (7.37) shows that $\mathcal{Q}_1^S/\mathcal{Q}^S \sim \Delta T_a/T_a$. The effect of core cooling is therefore more significant in the heat balance (7.32) than in the dissipation (7.35).

Expression (7.36) may be rewritten as

$$\mathcal{Q}^H = \mathcal{Q}^S\frac{\Delta T_S}{T_0} + \mathcal{Q}^R\frac{\Delta T_{01}}{T_0} - \mathcal{Q}_1^T\frac{\Delta T_\Sigma}{T_0} + \mathcal{Q}^N\frac{\Delta T_{21}}{T_2}, \tag{7.38}$$

where the following temperature differences have been introduced:

$$\Delta T_{01} = T_0 - T_1, \qquad \Delta T_{21} = T_2 - T_1, \tag{7.39a, b}$$

$$\frac{\Delta T_S}{T_0} = \frac{\int_{\mathscr{V}_1}(T_a - T_1)\rho_a dV}{\int_{\mathscr{V}_1} T_a\rho_a dV}, \qquad \frac{\Delta T_\Sigma}{T_0} = \frac{T_1\Sigma_T}{\mathcal{Q}_1^T}, \tag{7.39c, d}$$

where Σ_T is given by (7.31) and $\mathcal{Q}_1^T = A_1 I_1^T$.

By (6.31a), the total rate, $\mathcal{Q}^D$, of energy dissipation associated with core convection is the sum of contributions $\mathcal{Q}^v$ from viscosity, $\mathcal{Q}^J$ from magnetic field generation, and $\mathcal{Q}^t$ from turbulence, the last of which is mainly due to magnetic friction. It is also the total rate of working of the buoyancy forces, averaged over the convection cycle. This is the sum of the averaged contributions, $\mathscr{A}^C$ and $\mathcal{Q}^t$, from macroscale and microscale, respectively:

$$\mathcal{Q}^D = \mathscr{A}^C + \mathcal{Q}^t = \mathcal{Q}^v + \mathcal{Q}^J + \mathcal{Q}^t. \tag{7.40}$$

It does not contain the term $T\Sigma^T$, corresponding to the entropy sources (7.31) appearing in (7.27), a term independent of the convection; only the Q^J part of Q^D is used to power the geodynamo while $\mathcal{Q}^v$ and $\mathcal{Q}^t$ represent energy wasted in useless 'friction'.

We may define dynamo efficiency to be

$$\eta_D = \frac{\mathcal{Q}^J}{\mathcal{Q}_M^q}, \tag{7.41}$$

which relates the total 'effective' energy supply (7.32) to the useful work $\mathscr{A}^B$ $(= \mathcal{Q}^J)$, as given by (7.16a). It should be stressed that η_D is defined using the most significant *quantity*, Q_M^q, the total heat flux from core to mantle. It may be compared with $\mathcal{Q}_-$ in the example of Subsection 7.1, which is there (but not here) equal to $\mathcal{Q}_+$. We may

Downloaded by [Universite Laval] at 23:08 09 July 2014

rewrite (7.41) as

$$\eta_D = \eta_F \eta_G, \tag{7.42}$$

where the frictional factor is

$$\eta_F = \frac{\mathscr{Q}^J}{\mathscr{Q}^D} = \frac{\mathscr{Q}^J}{\mathscr{Q}^J + \mathscr{Q}^\nu + \mathscr{Q}^t}. \tag{7.43}$$

This represents the attenuation in efficiency arising from friction, both macroscopic (because of viscosity) and microscopic (because of turbulence). By (7.32) and (7.35), the 'ideal geophysical efficiency', analogous to the ideal efficiency of Subsection 7.1, is

$$\eta_G = \frac{\mathscr{Q}^D}{\mathscr{Q}_M^q} = \frac{T_D}{T_1} \cdot \frac{\mathscr{A}^\xi + \mathscr{Q}^H}{\mathscr{A}^\xi + \mathscr{Q}^S + \mathscr{Q}^R + \mathscr{Q}^N}. \tag{7.44}$$

This expresses the way that the efficiency depends on which of the energy sources dominates. It is difficult to evaluate η_G accurately at present. If compositional convection dominates thermal convection then η_G will be close to unity (Braginsky, 1964b) but in the reverse case $\eta_G \sim \Delta T_a/T_0$. More generally $\Delta T_a/T_0 \lesssim \eta_G \lesssim 1$. Even if the molecular diffusivity is negligible and compositional convection dominates, the total efficiency, η_D, cannot be close to unity, because the turbulent losses, $\mathscr{Q}^t$, are not small. Perhaps $\eta_F \sim \frac{1}{3}$ is a reasonable guestimate.

The relative importance of the compositional and thermal contributions to $\mathscr{Q}^D$ is assessed in Section 8, and it is concluded that $\mathscr{A}^\xi \sim \mathscr{Q}^H$. We infer that, in all probability, both driving mechanisms are significant in Earth's core but that this cannot be established with certainty until the values of key parameters are known with greater precision.

In all our calculations of the sources of convection, such as (6.22) and (6.31b), we considered $\dot{R}_2$ to be a prescribed quantity. Speaking more physically, it would be natural to take the heat flux, $\mathscr{Q}_M^q$, as the prescribed quantity, determined by the way that the mantle extracts heat from the core. This flux is, however, poorly known at present. It is related to $\dot{R}_2$ by the condition (7.32) of heat balance, in which the terms $\mathscr{A}^\xi$, $\mathscr{Q}^S$ and $\mathscr{Q}^N$ are proportional to $\dot{R}_2$, according to (7.28), (7.33), (7.34) and (6.39). We shall write their sum as

$$\mathscr{A}^\xi + \mathscr{Q}^S + \mathscr{Q}^N = \mathscr{Q}_*(3\dot{R}_2 t_{20}/R_2), \tag{7.45}$$

where $\mathscr{Q}_*$ is a convenient constant with which to measure power and, in anticipation of Section 8, we have arbitrarily introduced a nominal magnitude, t_{20}, for the age of the SIC. After this has been selected (e.g. $t_{20} = 4 \times 10^9$ yr), $\mathscr{Q}_*$ becomes unique; it is a very convenient parameter with which to assess the importance of the terms on the left of (7.45). We may rewrite (7.32) as

$$\dot{R}_2 \equiv \frac{dR_2}{dt_a} = \frac{\mathscr{Q}_M^q - \mathscr{Q}^R}{\mathscr{Q}_*} \cdot \frac{R_2}{3t_{20}}, \tag{7.46}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

a differential equation determining $R_2(t_a)$ from $\mathcal{Q}_M^q - \mathcal{Q}^R$. We do not attempt to solve this; we simply assume that $\dot{\mathscr{V}}_2$ is constant and therefore replace $\dot{R}_2$ by $R_2/3t_2$, so that

$$t_{20}/t_2 = (\mathcal{Q}_M^q - \mathcal{Q}^R)/\mathcal{Q}_*. \tag{7.47}$$

The considerable uncertainty in t_2 is matched by a like uncertainty in $\mathcal{Q}_M^q - \mathcal{Q}^R$. The quantities $\dot{R}_2$, $\mathcal{Q}_M^q$ and $\mathcal{Q}^R$ are related by (7.46), so that the thermal input into the core is determined by only two independent parameters. Numerical values in this relation are considered in Section 8. While $\mathcal{Q}_M^q - \mathcal{Q}^R$ decides the thermal balance and the rate of cooling of the core, as (7.47) illustrates, it is $\mathcal{Q}_M^q - \mathcal{Q}_1^T$ that determines how strongly the thermal sources drive core convection.

In conclusion, we reiterate that all forms of core convection are essentially due to thermal effects, both

(1) directly, through the thermal codensity, $\alpha^S S_c$, and
(2) indirectly, through the general cooling of the core and the concomitant growth of the nucleus by freezing, thus producing the computational codensity $\alpha^\xi \xi_c$.

8. THE CONVECTIVE STATE: HOMOGENEOUS MODEL (MODIFIED BOUSSINESQ THEORY)

8.1 Basis of the Homogeneous Model

In Section 6 we constructed a rather general model of core convection and the geodynamo. Because the values of key parameters in Earth's core are so uncertain, this "inhomogeneous model" is perhaps too sophisticated for use in numerical geodynamo calculations. In this section, a simpler, and perhaps even simplistic, model is developed that is hopefully of some practical utility. It may also be the simplest possible model that retains all the main features of the geodynamo mechanism.

Three small parameters were introduced in Section 3: $\epsilon_a \sim 10^{-1}$, $\epsilon_\Omega \sim 2 \times 10^{-3}$ and $\epsilon_c \sim 10^{-8}$. The smallest of these determines how far convection causes the configuration of the core to differ from the basic reference state; the smallness of ϵ_c was exploited in Sections 4 and 5. The parameters ϵ_Ω and ϵ_a measure inhomogeneities of the reference state. The asphericity of that state created by centrifugal forces is of order ϵ_Ω. Asphericity has a very small effect on the convective motions on the time scale of hundreds of years and longer, and we will continue to neglect it. The parameter ϵ_a measures the radial gradients in quantities such as p_a and ρ_a arising from the gravitational compression of the core. It is not very small ($\epsilon_a \sim 0.1$), but in this section we exploit its supposed smallness in order to simplify the model introduced in Section 6. In other words, we develop a "homogeneous model" of core convection. More precisely, since small variations in density are essential in order to retain the buoyancy forces driving the geodynamo, we construct a Boussinesq model of core convection.

Downloaded by [Universite Laval] at 23:08 09 July 2014

In non-dimensional units, the governing system of equations of our model is

$$\epsilon_\eta^\Omega(d_t\mathbf{V}-\mathbf{F}^v)=-\nabla P-C\mathbf{r}-\mathbf{1}_z\times\mathbf{V}+\mathbf{F}^B, \tag{8.1}$$

$$\mathbf{F}^v=(\nu/\eta_0)\nabla^2\mathbf{V}, \qquad \mathbf{F}^B=\mathbf{J}\times\mathbf{B}=\mathbf{B}\cdot\nabla\mathbf{B}-\tfrac{1}{2}\nabla B^2, \tag{8.1a, b}$$

$$C=-X-Y, \tag{8.1c}$$

$$\nabla\cdot\mathbf{V}=0, \tag{8.2}$$

$$d_tX+\nabla\cdot\mathbf{I}^X=\sigma^X, \quad \text{where} \quad \mathbf{I}^X=-\overleftrightarrow{\mathbf{D}}\cdot\nabla X, \tag{8.3, 8.3a}$$

$$d_tY+\nabla\cdot\mathbf{I}^Y=\sigma^Y, \quad \text{where} \quad \mathbf{I}^Y=-\overleftrightarrow{\mathbf{D}}\cdot\nabla Y, \tag{8.4, 8.4a}$$

$$\nabla^2U_c=3\rho_c, \tag{8.5}$$

$$d_t\mathbf{B}=\mathbf{B}\cdot\nabla\mathbf{V}+\nabla^2\mathbf{B}, \tag{8.6}$$

$$\nabla\cdot\mathbf{B}=0. \tag{8.7}$$

As for any Boussinesq model, the essence of (8.1)–(8.7) is that all basic variables for which it is meaningful to do so are assumed to be constant. We have replaced ρ_a by $\rho_0=\mathcal{M}_{12}/\mathcal{V}_{12}$ (i.e. we have taken $\tilde{\rho}=1$) and have replaced T_a by T_0 everywhere *except* in places where the inhomogeneity of T_a enters the theory directly, as in the expression for $\mathbf{I}^T$ and in expression (8.27a) below for σ_a^Y. The coefficients, α^ξ and α^S, are defined on the reference state and are no longer functions of r; they are constants. We used $\Delta\rho$ as a surrogate for α^ξ in the following way. The density discontinuity is the sum of $\Delta_s\rho$, the change in the density of pure iron on solidification, and $\Delta_\xi\rho$ the change in density due to the discontinuity in composition at the ICB; see (D48). We suppose (see Appendix E) that $\Delta_s\rho$ is negligible. Then (D48c) gives

$$\alpha^\xi=\Delta\rho/\rho_N\xi_{2N}. \tag{8.8}$$

A representative average value, α_0^S and α^S was chosen by integrating relation (3.7b) approximately using (8.42) below. This led to a value of α^S intermediate between α_1^S and α_2^S:

$$\alpha_0^S=\frac{2(T_2-T_1)}{g_1R_1(1-r_2^2)}=7.98\times10^{-5}\,\mathrm{s^2\,m^{-2}\,{}^\circ K}. \tag{8.9}$$

We accept the assumed constancy of the coefficients as providing a useful but somewhat crude approximation. It now follows that we can write

$$\alpha^\xi d_t\xi_c=-d_tC^\xi, \qquad \text{where} \qquad C^\xi=-\alpha^\xi\xi_c, \tag{8.10, 8.10a}$$

$$\alpha_0^S d_tS_c=-d_tC^S, \qquad \text{where} \qquad C^S=-\alpha_0^S S_c. \tag{8.11, 8.11a}$$

Neither of these simplifications is precisely correct, the second being more in error than the first since α^S varies by about 30% across the core whereas α^ξ changes perhaps by only a few percent. The variables X and Y are defined by

$$X\equiv-\frac{C^\xi}{C_I}=\frac{1}{C_I}\alpha^\xi\xi_c, \qquad Y\equiv-\frac{C^S}{C_I}=\frac{1}{C_I}\alpha_0^S S_c. \tag{8.12, 8.13}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

To make the system (8.1)–(8.7) non-dimensional, we have introduced a characteristic magnetic field and corresponding Alfvén velocity by

$$V_* = (2\Omega\eta_0)^{1/2}, \quad B_* = (\mu_0\rho_0)^{1/2}V_*. \tag{8.14a, b}$$

We have then taken $L_I = R_1$ as unit of length, $t_I = R_1^2/\eta_0$ as unit of time and $V_I = L_I/t_I = \eta_0/R_1$ as unit of velocity, $B_I = B_*$ as unit of field, $J_I = B_*/\mu_0 R_1$ as unit of electric current density, and $\rho_0 C_I$ as unit of density perturbation (ρ_c), where $C_I = 2\Omega V_I/g_1 = V_*^2/g_1 R_1$ is the unit of codensity; $g_1 = 10.68\,\mathrm{m\,s^{-2}}$. Taking $\eta_0 = 2$ $\mathrm{m^2 s^{-3}}$ and $\rho_0 = 10.9 \times 10^3\,\mathrm{kg\,m^{-3}}$, we find that $V_* = 1.71 \times 10^{-2}\,\mathrm{m\,s^{-1}}$, $B_* = 20\,\mathrm{G}$, $t_I = 6.055 \times 10^{12}$ s $= 1.92 \times 10^5$ yr, $V_I = 5.74 \times 10^{-7}\,\mathrm{m\,s^{-1}}$ and $C_I = 0.785 \times 10^{-11}$. Note that $V_* \gg V_I$. Energy densities per unit mass are measured in units of V_*^2; this is also the unit of P and U_c. Power density is measured in units of Q_I where

$$Q_I = \frac{\rho_0 g_1 C_I R_1}{t_I} = \frac{\rho_0 V_*^2}{t_I} = 5.235 \times 10^{-13}\,\frac{\mathrm{W}}{\mathrm{m}^3}. \tag{8.14c}$$

This unit also appears in a combination that is often used in the non-dimensionalization:

$$\frac{t_I}{\rho_0 C_I} = \frac{g_1 R_1}{Q_I}; \tag{8.14d}$$

$\mathbf{I}^T$ is measured in units of $Q_I R_1$.

The fluxes are made dimensionless in a similar way:

$$\mathbf{I}^X = \frac{g_1}{Q_I}\alpha^\xi \mathbf{I}^{\xi t}, \quad \sigma^X = \frac{g_1 R_1}{Q_I}\alpha^\xi \sigma_e^\xi, \tag{8.15, 8.16}$$

$$\mathbf{I}^Y = \frac{g_1}{Q_I}\alpha_0^S \mathbf{I}^{St}, \quad \sigma^Y = \frac{g_1 R_1}{Q_I}\alpha_0^S \sigma_e^S. \tag{8.17, 8.18}$$

Comparing (8.1)–(8.7) with (6.1)–(6.7), we see that the dimensionless fluxes and sources labeled with superscripts X and Y correspond simply and directly with the corresponding variables labeled with superscripts ξ and S. This correspondence between Sections 6 and 8 will arise many times below.

The "Ekman number" used in (8.1) is based on magnetic diffusivity rather than viscosity:

$$\epsilon_\eta^\Omega = \eta_0/2\Omega R_1^2. \tag{8.19}$$

It is very small ($\epsilon_\eta^\Omega \approx 10^{-9}$), and the more usual (viscous) Ekman number, $\nu_0/2\Omega R_1^2$, is even smaller. The Ekman layers at $r = R_1$ and $r = R_2$ can be described using the simple isotropic expression (4.9) for the viscous force, with appropriate molecular viscosities, ν_1 and ν_2. A turbulent viscosity should be used to describe internal shear layers inside the main volume of the core (see Appendix C). In the bulk of the core, $\mathbf{F}^\nu$ is insignificant. The dimensionless turbulent diffusivity tensor,

$$\overleftrightarrow{\mathbf{D}} = \overleftrightarrow{\boldsymbol{\kappa}}^t/\eta_0, \tag{8.14e}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

is poorly known at present. It is hoped that future studies of the theory of local turbulence will eventually rectify this. At the present time it is necessary to make assumptions that, though unsubstantiated, are plausible. Numerical dynamo integrations with different choices of $\vec{\mathbf{D}}$ will hopefully, when compared with the observed geophysical data, provide information about both $\vec{\mathbf{D}}$ and local turbulence in the core.

We shall find that consequences of the theory rely particularly heavily on five poorly known parameters: the rate of evolution of the core, expressed through the time scale $t_2 = \mathscr{V}_2/\dot{\mathscr{V}}_2$, the heat released at the ICB during freezing of the formation of the SIC, h_N, the heat flux, $\mathscr{Q}^q_M$, from the core into the mantle and the parameters $\alpha^\xi \xi_{2N}$ and Δ_2; see (8.8) and (6.40). To expose the effects of the uncertainties more clearly, we introduce four "nominal values"

$$t_{20} = 4 \times 10^9\,\text{yr}, \quad h_{N0} = 10^6\,\text{J kg}^{-1}, \quad \Delta\rho_0 = 0.6 \times 10^3\,\text{kg m}^{-3}, \quad \Delta_{20} = 0.05. \tag{8.20a, b, c, d}$$

The value of $\Delta\rho_0$ given in (8.20c) is the seismically inferred value of Dziewonski & Anderson (1981). If $\Delta_s\rho$ and $\Delta_\xi\rho$ were comparable, we would have to assume smaller $\Delta\rho$. For h_{N0} in (8.20b) we have taken the value of h_N given in Appendix E; it seems that this estimate may be uncertain by a factor of 2 either way. Our value (8.20a) of t_{20} is comparable with the age of Earth. We cannot, however, rule out the possibility that the SIC is significantly younger than this, and the ratio t_2/t_{20} through which we express our uncertainty might well be 0.5 rather than our preferred value of 1. The parameter Δ_2 is the worst determined of all. The value shown in (8.20d) is defended in Appendix E. Taken with the estimate (8.20a) of t_{20}, it implies a geophysically acceptable estimate of the core cooling rate; see (8.41a) below.

Whenever t_2, h_N, $\Delta\rho$ and Δ_2 arise in the theory, we give them the values t_{20}, h_{N0}, $\Delta\rho_0$ and Δ_{20} shown in (8.20a, b, c, d), but we also include ratios t_2/t_{20}, h_N/h_{N0}, $\Delta\rho/\Delta\rho_0$ and Δ_2/Δ_{20}, so that the cause of uncertainty can be readily identified. We refer to a unit value for any ratio as its "preferred value" (by which we mean, of course, only "preferred by us here").

The heat flux from the core, $\mathscr{Q}^q_M$, is replaced in this section by the Nusselt number, $Nu = \mathscr{Q}^q_M/\mathscr{Q}^T_1$, where $\mathscr{Q}^T_1$ is the heat flux along the adiabat at the CMB, which can be estimated easily using numerical values listed in Appendix E where it was found that $K^T \sim 40\,\text{W m}^{-1}\,{}^\circ\text{K}^{-1}$ and $\nabla_r T_a \sim -0.89\,{}^\circ\text{K km}^{-1}$, so that $I^T_1 = -K^T \nabla_r T_a \sim 0.0356\,\text{W m}^{-2}$ and $\mathscr{Q}^T_1 = 4\pi R_1^2 I^T_1 = 5.42 \times 10^{12}\,\text{W}$. Nevertheless, Nu, though very significant for the present theory, is largely unknown. Its uncertainty is not related to the uncertainty of t_2, which depends on $\mathscr{Q}^q_M - \mathscr{Q}^R$. Even if the estimate $t_2 = 4 \times 10^9$ yr is accepted, the uncertainty in the radioactive power supply, $\mathscr{Q}^R$, translates this into a corresponding uncertainty in Nu. It should be emphasized that in the core, unlike the laboratory, a small value of $Nu - 1$ does *not* mean that convection is weak. This is because conduction of heat is driven by a temperature contrast across the core of order $T_2 - T_1 = 1300\,{}^\circ$K, while convection of heat, through vigorous turbulent motions, is driven by temperature differences of order only $T_c \sim 10^{-3}\,{}^\circ$K. In the laboratory but not in the core, $|Nu - 1| \ll 1$ means that convection is weak.

Downloaded by [Universite Laval] at 23:08 09 July 2014

Equations (8.1)–(8.7), supplemented by appropriate boundary conditions, define the "homogeneous model" of the geodynamo. It may be noted that the factor 3 in (8.5), which arises from taking $g_1 = 4\pi k_N \rho_0 R_1/3$ and $\mathbf{g} = -g_1\mathbf{r}/R_1$ for this model, is slightly inaccurate—it should be multiplied by $1-(R_2/R_1)^3(\mathcal{M}_2/\rho_0\mathcal{V}_2 - 1)$. This factor differs from unity by only approximately 10^{-3}, which we ignore. It may be recalled from Section 4 that we need the functions ρ_c and U_c only if we wish to evaluate the pressure perturbation $p_c = \rho_a(P - U_c)$. It should be noted that the magnetic Reynolds number does not appear in (8.6); it has been absorbed into the magnitude of $\mathbf{V}$, which may be large in amplitude when the unit of velocity is chosen as we have done. We continue to ignore any changes in the mass distribution of the mantle, and assume that $R_{1c} = 0$. We retain, however, the crucial change in radius of the ICB: $\dot{R}_2 \neq 0$.

The non-dimensional expressions for the sources σ^X and σ^Y can be obtained from the the σ_e^ξ and σ_e^S derived in Section 6. In the homogeneous model, $\sigma^X = \sigma_2^X$ is a constant given by (6.24b), (8.8) and (8.16) as:

$$\sigma_2^X = -\frac{g_1 R_1 \Delta\rho}{t_2 Q_1}\cdot\frac{\mathcal{V}_2}{\mathcal{V}_{12}} = -\sigma_{20}^X \frac{\Delta\rho}{\Delta\rho_0}\cdot\frac{t_{20}}{t_2}, \tag{8.21}$$

where

$$\sigma_{20}^X = \frac{g_1 R_1 \Delta\rho_0}{t_{20} Q_I}\cdot\frac{\mathcal{V}_2}{\mathcal{V}_{12}} \approx 1.53\times 10^4. \tag{8.21a}$$

Here $\mathcal{V}_2/\mathcal{V}_{12} = r_2^3/(1-r_2^3) = 0.0452$, $r_2 = R_2/R_1 = 0.351$. We have also taken $A_2\dot{R}_2 = \dot{\mathcal{V}}_2 \approx \mathcal{V}_2/t_2$ with $t_{20} = 4\times 10^9$ yr. We have (see above) cast (8.21) into a form where the dependence of σ_2^X on poorly known parameters like $\Delta\rho$ and t_2 is explicitly shown, while its dependence on better known parameters is implicitly contained in σ_{20}^X. Other expressions, such as σ_1^Y and σ_2^Y, are treated similarly below.

By (6.24a) and (8.15), the volume source of light fluid, σ_2^X, is associated with a flux at the ICB of

$$I_2^X = \frac{g_1 R_1 \Delta\rho}{t_2 Q_I}\cdot\frac{r_2}{3} = \frac{\mathcal{V}_{12} r_2}{3\mathcal{V}_2}\cdot\sigma_{20}^X\cdot\frac{\Delta\rho}{\Delta\rho_0}\cdot\frac{t_{20}}{t_2} \approx 3.93\times 10^4 \frac{\Delta\rho}{\Delta\rho_0}\cdot\frac{t_{20}}{t_2}. \tag{8.22}$$

A transparent relation similar to (6.24a) follows from (8.21) and (8.22):

$$I_2^X = -\frac{\mathcal{V}_{12} r_2}{3\mathcal{V}_2}\sigma_2^X \quad \text{or} \quad A_2 I_2^X = \mathcal{V}_{12}\sigma_2^X, \tag{8.23a, b}$$

where $A_2 = 3\mathcal{V}_2/r_2$ (the dimensionless form of $A_2 = 3\mathcal{V}_2/R_2$). The large constants appearing in (8.21) and (8.22) betoken a plentiful source of light fluid that drives core convection powerfully. They also strongly suggest that the convection is rather far beyond threshold, a fact that was noted by Braginsky (1991).

We may use (6.32) and (6.32a, b, c) to obtain the non-dimensional entropy source as

$$\sigma^Y = \sigma_1^Y + \sigma_2^Y + \sigma_{12}^Y, \tag{8.24}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

the suffices being in 1–1 correspondence with those appearing in (6.32). According to (6.32a), the first term on the right-hand side of (8.24) depends on the difference between I^q_M and I^T_1, the former of which is unknown, while (see above) $I^T_1 = 0.0356$ W m^{-2}. According to (6.32b), the second term on the right-hand side of (8.24) involves the difference between I^T_N and I^T_2 which we neglect, and a term proportional to $\dot{R}_2$ that can be conveniently expressed in terms of σ^X_2 and hence evaluated with the help of the estimate $h_N \approx h_{N0} \sim 10^6$ J kg^{-1} [see (8.20b)]. In this way, using also (8.18), we obtain

$$\sigma^Y_1 = \sigma^Y_{10}(Nu - 1), \quad \sigma^Y_2 = -\sigma^Y_{20}\frac{h_N}{h_{N0}}\cdot\frac{t_{20}}{t_2}, \tag{8.25, 8.26}$$

$$\sigma^Y_{10} = \frac{A_1}{\mathscr{V}_{12}}\cdot\frac{t_I}{\rho_0 C_I}\alpha^S_1\frac{I^T_1}{T_1} \approx 4.53\times 10^4, \tag{8.25a}$$

$$\sigma^Y_{20} = \frac{g_1 R_1 \rho_N}{t_{20} Q_I}\cdot\frac{\mathscr{V}_2}{\mathscr{V}_{12}}\cdot\frac{\alpha^S_0 h_{N0}}{T_2} = \frac{\alpha^S_0 h_{N0}}{T_2}\cdot\frac{\rho_N}{\Delta\rho_0}\cdot\sigma^X_{20} \approx 0.32\sigma^X_{20} = 0.49\times 10^4. \tag{8.26a}$$

In making these estimates, the values given in Appendix E were adopted.

The parameters σ^X_2, σ^Y_1 and σ^Y_2 are the dimensionless numbers that characterize the nature of the convection; they play roles similar to that of the Rayleigh number[10] in classical thermal convection theory . Note that σ^Y_1 is not small in comparison with σ^X_2; on the contrary, $\sigma^Y_{10} \sim 3\sigma^X_{20}$. The magnitude of the 'source' σ^Y_1 depends on the factor $Nu - 1$, which is poorly known. We are not even certain of its sign; σ^Y_{10} might be negative, i.e. a 'sink'! On the one hand, σ^X_2 and I^X_2 are proportional to $\Delta\rho/t_2$ and may be significantly changed if these poorly known values are re-estimated; on the other hand, σ^Y_1 will be markedly altered if the poorly known ratio $I^q_M/I^T_1 = Nu$ is changed.

The term σ^Y_{12} originates from σ^S_{12} in (6.32c), that is composed of two very different parts σ^S_a and σ^S_c. The former is proportional to the inhomogeneous part of the basic quantity $(Q^R - \nabla\cdot\mathbf{I}^T)/T_a$, while the latter depends on convective quantities. Using also (8.18), we have, in non-dimensional terms,

$$\sigma^Y_{12} = \sigma^Y_a + \sigma^Y_c, \tag{8.27}$$

[10] The Rayleigh number as usually defined can be written in the form $Ra = \omega_\alpha^2\tau_\nu\tau_\kappa$, where $\tau_\nu = L^2/\nu$ and $\tau_\kappa = L^2/\kappa^T$ are the viscous and thermal diffusion time scales and $\omega_\alpha^2 = g\alpha\Delta T/L$ is the square of the buoyancy frequency. Since magnetic diffusion is more significant to us than viscous or thermal diffusion, we may replace τ_ν and τ_κ and $\tau_\eta = L^2/\eta$ and Ra and $Ra_\eta = \omega_\alpha^2\tau_\eta^2$. We have taken $L = R_1$ above. Our non-dimensional parameters $\sigma^X_1, \sigma^Y_1, \ldots$ are proportional to the contributions they make to ω_α^2, but they are less than the corresponding Rayleigh numbers by the factor $2\Omega\tau_\eta \sim 10^9$. Let us, for example, take $C \sim \alpha^\xi\sigma^\xi_e\tau_\eta/\rho_0$; see (6.3). Then $\omega_\alpha^2 \sim gC/R_1 \sim \alpha^\xi\sigma^\xi_e g\tau_\eta/\rho_0 R_1$ and hence $Ra_\eta \sim \alpha^\xi\sigma^\xi_e g\tau_\eta^3/\rho_0 R_1$. According to (8.16), we have $\sigma^\xi_e = \sigma^X\rho_0 C_1/\alpha^\xi\tau_\eta$ and $C_I = 2\Omega V_1/g_1$. It follows that $Ra_\eta \sim \sigma^X g C_I\tau_\eta^2/R_1 \sim 2\Omega\sigma^X\tau_\eta$. Our non-dimensional numbers σ^X_1, $\sigma^Y_1, \ldots$ are therefore similar to the so-called 'modified Rayleigh number', $Ra_{\text{mod}} = \omega_\alpha^2\tau_\eta/2\Omega$. After reducing them by about two orders of magnitude (numerical factors of order 10^2 arise if we take $L \sim R_1/\pi \sim 10^6$ m instead of $L = R_1$), they provide measures of how far core convection is operating beyond critical.

Downloaded by [Universite Laval] at 23:08 09 July 2014

where[11]

$$\sigma_a^Y = \sigma_{10}^Y\left[\frac{\mathcal{Q}_{12}^R}{\mathcal{Q}_1^T}\left(\frac{\tilde{\rho}}{\tilde{T}} - \left\langle\frac{\tilde{\rho}}{\tilde{T}}\right\rangle^V\right) - \frac{\mathscr{V}_{12}}{3\mathscr{V}_1 I_1^T}\left(\frac{\nabla\cdot\mathbf{I}^T}{\tilde{T}} - \left\langle\frac{\nabla\cdot\mathbf{I}^T}{\tilde{T}}\right\rangle^V\right)\right], \tag{8.27a}$$

$$\sigma_c^Y = [Q^D - \langle Q^D\rangle^V]\frac{R_1\partial_r T_1}{Q_I T_1}. \tag{8.27b}$$

Here $\tilde{\rho} = \rho_a/\rho_0$ and $1/\tilde{T} = T_1/T_a$; (3.7b) was used to obtain (8.27b). The radioactive source, Q^R, is proportional to ρ_a.

If we adopt the Boussinesq approximation, all thermodynamic quantities are nearly uniform ($\varepsilon_a \ll 1$) so that $|1 - \tilde{T}| \ll 1$, and the terms in round brackets in (8.27a) are small compared with unity, so that σ_a^Y may be neglected in comparison with σ_1^Y. This greatly simplifies the theory. Some contributions to the dissipation Q^D are relatively concentrated, and their averages may then to a good approximation be omitted in (8.27b). In the resulting simplified model, the sources σ_1^Y and σ_2^Y of thermal codensity, Y, are constants. The source, σ_2^X, of compositional codensity X, always promotes convection, as does σ_2^Y. The term σ_1^Y may assist convection or oppose it, depending on whether $Nu > 1$ or $Nu < 1$. The role of σ_c^Y is stabilizing, as was discussed in Section 6.

Boundary conditions for the system (8.1)–(8.7) may be derived from the corresponding conditions obtained in Section 6. Those applying to $\mathbf{V}$, $\mathbf{B}$ and U follow from (2.46), (2.47) and (6.19a, b, c); those required of the fluxes at $r = 1$ follow from (6.20a, b) and are

$$I_r^X(1) = 0, \quad I_r^Y(1) = \sigma_{10}^Y(\mathscr{V}_{12}/A_1R_1)(Nu - 1). \tag{8.28, 8.29}$$

More complicated conditions arise at $r = r_2$. Corresponding to (6.22), the compositional flux on the ICB can be written as $I_N^X = I_2^X + I_{2c}^X$. The boundary condition (8.23a, b) on I_2^X may be written as

$$I_2^X = -(\mathscr{V}_{12}/A_2R_1)\sigma_2^X = (\mathscr{V}_{12}/A_2R_1)|\sigma_2^X|; \tag{8.30}$$

here $\sigma_2^X < 0$; see (8.21). Similarly, by (6.32b), the averaged flux of entropy corresponding to the source σ_2^Y is

$$I_2^Y = -(\mathscr{V}_{12}/A_2R_1)\sigma_2^Y = (\mathscr{V}_{12}/A_2R_1)|\sigma_2^Y|. \tag{8.31}$$

Expressions for σ_2^X and σ_2^Y are given by (8.21) and (8.26).

Assuming that $\kappa^t \sim \eta$, we estimate the diffusional operator to be $\nabla\cdot\overleftrightarrow{\mathbf{D}}\cdot\nabla \sim (\kappa^t/\eta)\nabla^2 \sim 10$–$30$ (a few multiples of π^2), and recalling that the non-dimensional sources of X and Y are of order 10^4, we may expect from (8.3) and (8.4) that

[11] In (8.27a) we have restored $\tilde{\rho}$, tand $\tilde{T}$, despite having stated earlier that we would set these to unity in this section. This is because differences in $\tilde{\rho}/\tilde{T}$ from its average enter this formula, and not $\tilde{\rho}/\tilde{T}$ itself.

Downloaded by [Universite Laval] at 23:08 09 July 2014

$X \sim Y \sim 300$–1000. Taking $C_I = 0.785 \times 10^{-11}$, we estimate the codensity from (8.12) and (8.13) as $C \sim 3 \times 10^{-9} - 10^{-8}$. The same value is obtained by comparing the buoyancy and Coriolis forces: $C \sim 2\Omega V/g_1 \sim 3 \times 10^{-9} - 10^{-8}$ for $V \sim 3 \times 10^{-4} - 10^{-3}\,\mathrm{ms}^{-1}$. This provides some qualitative support for the heuristic theory of Braginsky & Meytlis (1990), on which the assumption, $\kappa^t \sim \eta$, is based.

Let us now consider the oscillating fluxes of codensity. By (6.22b) and (6.29b), these may be written as

$$I^X_{2c} = I^X_2 \frac{\partial_t R_{2c}}{\dot{R}_2}, \qquad I^Y_{2c} = I^Y_2 \frac{\partial_t R_{2c}}{\dot{R}_2}. \tag{8.32, 8.33}$$

Using (6.42) and (8.13), we see that

$$\partial_t R_{2c}/\dot{R}_2 = -r_{2Y}\partial_t Y_2, \tag{8.34}$$

where $Y_2 = Y(r_2, t)$ and

$$r_{2Y} = \frac{3t_2}{t_I} \cdot \frac{C_I}{c_p \alpha^S_0 \Delta_2} \sim 1.9 \times 10^{-4} \frac{t_2}{t_{20}} \cdot \frac{\Delta_{20}}{\Delta_2}. \tag{8.34a}$$

Here we have taken $\Delta_{20} = 0.05$ and $c_p\alpha^S = \alpha_2 T_2 = 5.3 \times 10^{-2}$; see Appendix E. Let us write $\partial_t Y_2 = (\omega_0 t_I)\tilde{Y}_2$, where $\tilde{Y}_2$ is the amplitude of the Y_2 oscillation and $\omega_0 = 2\pi/(8 \times 10^3\,\mathrm{yr})$ is the fundamental frequency of the geomagnetic field; then $\omega_0 t_I \approx 1.5 \times 10^2$ and $r_{2Y}\partial_t Y_2 \sim 10^{-2}\,\tilde{Y}_2$. According to (8.34), $\partial_t R_{2c}/\dot{R}_2$ is of order unity when $\tilde{Y}_2 \sim 30$, and such a value is quite probable since, according to our estimate, $Y \sim 3 \times 10^2$.

8.2 *Energetics of the Homogeneous Model*

The principal integral relations expressing the energetics of the geodynamo were obtained in Section 7 but the calculation of specific numerical coefficients was postponed until the present Section. To avoid unnecessary complications, we derive here the energy balance for the homogeneous model. We also calculate the coefficient $\mathcal{Q}_*$, which determines the rate, $\dot{R}_2$, at which the inner core grows, and which is used to estimate the sum, $\mathcal{A}^\xi + \mathcal{Q}^S + \mathcal{Q}^N$, of three significant terms of the energy balance; see (7.45).

To estimate $\mathcal{A}^\xi$, we note that, for the homogeneous model,

$$\mu_a - \mu_2 = \alpha^\xi(U_2 - U_a), \quad T_a - T_1 = -\alpha^S(U_a - U_1), \tag{8.35, 8.36}$$

can be obtained by integrating (3.7b, c). Also, expressing $\dot{\mathcal{M}}_2$ in terms of $\dot{R}_2$ and using (6.28a) and (8.8), we obtain from (7.33)

$$\mathcal{A}^\xi = \Delta\rho(\langle U_a\rangle^V - U_2)\mathcal{V}_2/t_2 = 0.250\,\Delta\rho g_1 R_1 \mathcal{V}_2/t_2. \tag{8.37}$$

We here used the fact that, apart from an irrelevant additive constant, $U_a = g_1 r^2/2R_1$ so that we may replace $\langle U_a\rangle^V - U_2$ by $0.250\,g_1 R_1$. Substituting our

Downloaded by [Universite Laval] at 23:08 09 July 2014

preferred values (8.20), we find that

$$\mathscr{A}^{\xi} = 0.34\frac{\Delta\rho}{\Delta\rho_0}\frac{t_{20}}{t_2}10^{12}\,\mathrm{W}. \tag{8.37a}$$

It may be seen that, although α^{ξ}, ξ_a and ξ_N are poorly known, we are fortunately able, with the help of the simplified model of Appendix B, to express $\mathscr{A}^{\xi}$ directly in terms of the much better determined quantity $\Delta\rho$.

Using the relation (6.39) for $\dot{S}_a$, replacing T_a by T_0, and ignoring the difference between $\dot{S}_a$ in the SIC and in the FOC, we may derive

$$\mathscr{Q}^S \approx \Delta_2 c_p T_0 \mathscr{M}_1/3t_2 \tag{8.38}$$

from (7.34). In similar fashion, we find from (7.28) that

$$\mathscr{Q}^N = \dot{\mathscr{M}}_2 h_N = h_N \rho_N \mathscr{V}_2/t_2. \tag{8.39}$$

Substituting our preferred values (8.20) into (8.38) and (8.39), we find

$$\mathscr{Q}^S = 1.0\frac{\Delta_2}{\Delta_{20}}\cdot\frac{t_{20}}{t_2}10^{12}\,W, \quad \mathscr{Q}^N = 0.77\frac{h_N}{h_{N0}}\cdot\frac{t_{20}}{t_2}10^{12}\,\mathrm{W}. \tag{8.38a, 8.39a}$$

Collecting together (8.37a), (8.38a) and (8.39a), and substituting into (7.45), we obtain

$$\mathscr{Q}_* = \left(0.34\frac{\Delta\rho}{\Delta\rho_0} + 0.77\frac{h_N}{h_{N0}} + 1.0\frac{\Delta_2}{\Delta_{20}}\right)\cdot\frac{t_{20}}{t_2}10^{12}\,\mathrm{W}. \tag{8.40}$$

The origin of each term in (8.40) should be obvious from the corresponding poorly known ratio attached to it. With the preferred unit values of the ratios, we have

$$\mathscr{Q}_* = 2.1 \times 10^{12}\,\mathrm{W}. \tag{8.40a}$$

It may be clearly seen that, according to the $\mathscr{Q}_*$ given in (8.40), thermal effects dominate in determining the rate of growth of the inner core. We will find below, however, that the compositional part of $\mathscr{Q}^D$ is, in order of magnitude, as potent as the thermal part in powering the geodynamo.

The rate of cooling of the core can easily be estimated from (6.39) and the approximation $\dot{T}_a \approx (T_a/c_p)\dot{S}_a$, which neglects a term proportional to $\dot{\xi}_a$. This gives

$$\dot{T}_a \approx -T_a\Delta_2/3t_2, \tag{8.41}$$

and for the preferred values of the parameters this gives

$$\dot{T}_a \approx = -\frac{80°\,\mathrm{K}}{t_2} = -\frac{20\,°\mathrm{K}}{10^9\,\mathrm{yr}}. \tag{8.41a}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

It is interesting to see that the characteristic time of cooling of the core is not t_2 but is of order $T_a/\dot{T}_a \sim 3t_2/\Delta_2 \sim 60\,t_2$.

Now we turn to the energy balance. Equations (7.35) and (7.32), which express the buoyant power balance and the heat balance, can be rewritten in simplified forms appropriate to the homogeneous model. At the expense of a slight loss of precision, the expression (7.44) for the geophysically perfect efficiency can be cast into a simple and transparent form. To do this, we roughly estimate the different averaged temperatures in (7.38), by assuming that the temperature, T_a, follows a simple parabolic law which is a consequence of the approximations $\alpha^S = \alpha_0^S$ and $g_r = -g_1 r$:

$$T_a = T_1 + T_d(1 - r^2), \quad T_d = \tfrac{1}{2}\alpha_0^S g_1 R_1, \tag{8.42, 8.42a}$$

where $T_d = (T_2 - T_1)/(1 - r_2^2) = 1483°\text{K}$ and $r_2 = R_2/R_1 = 0.351$.

Using (8.42), we find that $T_0 = 4558°\text{K}$ and

$$\Delta T_a = \langle T_a - T_1 \rangle^V = \Delta T_{01} = 558\,°\text{K}. \tag{8.43}$$

This value of ΔT_a differs by less than 1% from that implied by Table E2. The ratio $\Delta T_{01}/T_0 \approx 0.122$ nearly coincides with $\frac{1}{2}\Delta T_{21}/T_2 \approx 0.123$. Supposing, in the spirit of the Boussinesq approximation, that $\Delta T_{21} = T_2 - T_1 \ll T_1$ is infinitesimally small (although in reality $\Delta T_{21}/T_1 = 0.325$), the average of every ΔT appearing in (7.38), no matter how weighted (provided that the weighting factor is close to unity) is unique and equal to ΔT_a. Calculations made using (8.42) and (8.42a) give approximately the same results. Equation (7.38) then takes the form (with all terms having a precision of 5% or better)

$$\mathcal{Q}^H = (\Delta T_a/T_0)(\mathcal{Q}^R + \mathcal{Q}^S - \mathcal{Q}_1^T + 2\mathcal{Q}^N), \tag{8.44}$$

and, by substituting $\mathcal{Q}^S + \mathcal{Q}^R$ from (7.32), we obtain

$$\mathcal{Q}^H = (\Delta T_a/T_0)(\mathcal{Q}_M^q - \mathcal{Q}_1^T + \mathcal{Q}^N - \mathcal{A}^\xi). \tag{8.44a}$$

Substituting $T_0 = T_1 + \Delta T_a$ here, we find from (7.35) that

$$\mathcal{Q}^D = \frac{T_D}{T_0}\left[\mathcal{A}^\xi + \frac{\Delta T_a}{T_1}(\mathcal{Q}_M^q - \mathcal{Q}_1^T + \mathcal{Q}^N)\right]. \tag{8.45}$$

This provides the simple expression for the geophysically perfect efficiency referred to above:

$$\eta_G = \frac{T_D}{T_0}\left[\frac{\mathcal{A}^\xi + (\Delta T_a/T_1)(\mathcal{Q}_M^q - \mathcal{Q}_1^T + \mathcal{Q}^N)}{\mathcal{Q}_M^q}\right], \tag{8.46}$$

where [see (7.32)]

$$\mathcal{Q}_M^q = \mathcal{A}^\xi + \mathcal{Q}^S + \mathcal{Q}^R + \mathcal{Q}^N. \tag{8.47}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

This may also be written as

$$\mathscr{Q}_M^q = \mathscr{Q}^R + \mathscr{Q}_* t_{20}/t_2. \tag{8.47a}$$

The relative importance of compositional and thermal driving is clearly exhibited in (8.46); it depends crucially on the difference $\mathscr{Q}_M^q - \mathscr{Q}_1^T = \mathscr{Q}_1^T(Nu - 1)$, and on the ratio of $\mathscr{A}^\xi$ and $\mathscr{Q}_M^q - \mathscr{Q}_1^T$, the latter being reduced by the factor $\Delta T_a/T_1 \approx 0.14$. One of these magnitudes can be easily estimated: $\mathscr{Q}_1^T = 5.42 \times 10^{12}$ W; see Appendix E. It follows that

$$(\Delta T_a/T_1)(\mathscr{Q}_M^q - \mathscr{Q}_1^T) = 0.76 \times 10^{12}(Nu - 1)\,\mathrm{W}. \tag{8.46a}$$

Similarly, taking $h_N = h_{N0}$ and $t_2 = t_{20}$, we obtain from (8.39a)

$$(\Delta T_a/T_1)\mathscr{Q}^N = 0.11 \times 10^{12}\,\mathrm{W}. \tag{8.46b}$$

The term $\mathscr{Q}^N$ in (8.46) is small compared with $\mathscr{Q}_1^T$. Let us temporarily ignore it. Comparing (8.46a) and (8.37a), we observe that, despite the small factor $\Delta T_a/T_1$, the thermal terms (8.46a) in (8.45) is about twice the compositional term (8.37a). The role of the factor Nu is crucial. If $|Nu - 1| \gtrsim \frac{1}{2}$, the thermal driving dominates compositional driving and either assists ($Nu > 1$) or impedes ($Nu < 1$) dynamo action.

It is appropriate here to make the following point. Our Boussinesq approximation does not correspond to the limit $\varepsilon_a \to 0$, with heat sources and sinks, like Q_1^T, Q_M^q and so on, held fixed—this is the appropriate for the usual 'laboratory case'. Rather, we consider situations in which $\varepsilon_a Q_1^T$, $\varepsilon_a Q_M^q$ and so on are held constant as $\varepsilon_a \to 0$—this is a Boussinesq approximation tailored to the case of Earth's core.

We have just obtained the Boussinesq form for $\mathscr{Q}^D$ by approximating the corresponding expression from Section 6. It is possible also to extract it directly from the equations governing the homogeneous model in the following way.

We multiply (8.3) by $U_2 - U_a(r)$ and (8.4) by $U_1 - U_a(r)$, average the results over a cycle of the convection, average them over $\mathscr{V}_{12}$, and add them together. The non-dimensional gravitational field in the core is approximated by $\mathbf{g}_a = -\mathbf{r}$, so that the gravitational potential is, apart from an irrelevant constant, $U_a = \frac{1}{2}r^2$. Hence

$$U_2 - U_a = \tfrac{1}{2}(r_2^2 - r^2), \quad U_1 - U_a = \tfrac{1}{2}(1 - r^2). \tag{8.48a, b}$$

Recalling (6.26) and integrating by parts, we obtain on the left-hand side

$$\mathscr{A}_{\mathrm{nd}}^C + \mathscr{Q}_{\mathrm{nd}}^t - \tfrac{1}{2}(1 - r_2^2)\frac{I_2^Y A_2 R_1}{\mathscr{V}_1} = \mathscr{Q}_{\mathrm{nd}}^D + \tfrac{1}{2}(1 - r_2^2)\frac{\mathscr{V}_{12}}{\mathscr{V}_1}\sigma_2^Y, \tag{8.49}$$

where we have used (7.15), (7.18a), (7.35), (8.31), and

$$\mathscr{A}_{\mathrm{nd}}^C = -\int_{\mathscr{V}_{12}} (X + Y)\mathbf{V}\cdot\mathbf{r}\,dV, \quad \mathscr{Q}_{\mathrm{nd}}^t = -\int_{\mathscr{V}_{12}} (\mathbf{I}^X + \mathbf{I}^Y)\cdot\mathbf{r}\,dV. \tag{8.49a, b}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

Here the subscripts nd stands for "non-dimensional". To obtain dimensional values, it is necessary to multiply by $Q_I \mathscr{V}_{12} = 0.884 \times 10^8$ W; see (8.14c).

On the right-hand sides of (8.3) and (8.4) we have constant terms σ_2^X, $\sigma_1^Y + \sigma_2^Y$ and the variable term $\sigma_{12}^Y = \sigma_a^Y + \sigma_c^Y$. The latter consists of two parts, both of which are of order $\Delta T/T_a$, and we neglect them here. This is a somewhat crude simplification, but significant magnitudes are not dangerously distorted, and the calculations are greatly simplified; the result is

$$\mathscr{Q}_{\rm nd}^D = \mathscr{Q}_2^X + \mathscr{Q}_2^Y + \mathscr{Q}_1^Y. \tag{8.50}$$

Here

$$\mathscr{Q}_2^X = \tfrac{1}{2}\langle r^2 - r_2^2\rangle^V |\sigma_2^X| = 0.250 \frac{g_1 R_1 \Delta\rho}{Q_1 \mathscr{V}_{12}} \cdot \frac{\mathscr{V}_2}{t_2} \approx 0.38 \times 10^4 \frac{\Delta\rho}{\Delta\rho_0} \cdot \frac{t_{20}}{t_2}, \tag{8.50a}$$

$$\mathscr{Q}_2^Y = \tfrac{1}{2}\langle r^2 - r_2^2\rangle^V |\sigma_2^Y| = \frac{\Delta T_a}{T_1} \cdot \frac{\mathscr{Q}^N}{Q_1 \mathscr{V}_{12}} \approx 0.12 \times 10^4 \frac{h_N}{h_{N0}} \cdot \frac{t_{20}}{t_2}, \tag{8.50b}$$

$$\mathscr{Q}_1^Y = \tfrac{1}{2}\langle 1 - r^2\rangle^V \sigma_1^Y = \frac{\Delta T_a}{T_1} \cdot \frac{\mathscr{Q}_1^T}{Q_I \mathscr{V}_{12}} (Nu - 1) \approx 0.86 \times 10^4 (Nu - 1). \tag{8.50c}$$

To obtain these results, the following relations were used [see (8.42) and (8.42a)]:

$$\tfrac{1}{2}\langle r^2 - r_2^2\rangle^V = \tfrac{3}{10} \cdot \frac{1 - r_2^5}{1 - r_2^2} - \tfrac{1}{2} r_2^2 = 0.250, \tag{8.51a}$$

$$\tfrac{1}{2}\langle r^2 - r_2^2\rangle^V \alpha_0^S g_1 R_1 = \Delta T_{20} \approx \Delta T_a (T_2/T_1), \tag{8.51b}$$

$$\tfrac{1}{2}\langle 1 - r^2\rangle^V \alpha_0^S g_1 R_1 = \Delta T_{01} = \Delta T_a, \tag{8.51c}$$

together with the approximate but very accurate relation $\Delta T_{20}/T_2 = \Delta T_a/T_1$.

If we multiply (8.50) by $Q_1 \mathscr{V}_{12}$, it coincides term by term with (8.45), where $T_D = T_0$ is assumed. Expression (8.50a), when multiplied by $Q_I \mathscr{V}_{12}$ coincides exactly with (8.37). After multiplying by $Q_I \mathscr{V}_{12}$ the expressions (8.50b) and (8.50c) coincide with $\mathscr{Q}_N$ and $\mathscr{Q}_1^T(Nu - 1)$ respectively, when multiplied by $\Delta T_a/T_1$.

The term $\mathscr{Q}_2^Y$ is associated with the latent heat released on the ICB through the crystallization of the SIC; the existence of this source of thermal forcing was first pointed out by Verhoogen (1961). The term $\mathscr{Q}_2^X$ is associated with the gravitational energy release due to the flux of light admixture from the ICB during the same crystallization process; the existence of this source of compositional forcing was first pointed out by Braginsky (1963). The term $\mathscr{Q}_1^Y$ is associated with the cooling of the core through heat conduction to the mantle from the superadiabatic temperature gradient alone.

Equation (8.50) clearly demonstrates the relative significance of the compositional and thermal sources of convection in Earth's core. Values of the key parameters are given in (8.50a)–(8.50c), but these are poorly known. If $\Delta\rho = \Delta\rho_0$, $t_2 = t_{20}$ and $h_N = h_{N0}$, the contributions made by compositional and thermal convection to $\mathscr{Q}^D$ are approximately equal if $Nu \approx 1.3$.

Downloaded by [Universite Laval] at 23:08 09 July 2014

The homogeneous (Boussinesq) model governed by (8.1)–(8.7) can be simplified even further by excluding (8.4). Though the source of compositional codensity is a consequence of thermal processes that results in the freezing of the inner core, the source terms of the model are expressed explicitly through $d_t R_N = \dot{R}_2 + \delta_t R_{2c}$, the growth rate of the SIC. It is possible therefore, by prescribing $\dot{R}_2$ and ignoring $\delta_t R_{2c}$, to separate completely the compositional effect, X, from the thermal effect, Y. Then, setting $Y = 0$ in (8.1c) and omitting (8.4), we obtain the simiplest possible self-sustained dynamical model of core convection and the geodynamo, which we may call "the compositional geodynamo". This model, which has only one source of buoyancy, namely the compositional codensity, $C = -X$, was considered by Braginsky (1991).

Geodynamo theory is very complicated mathematically, and the compositional model, though possibly over-simplified, recommends itself through its comparative simplicity. It is physically sound, but is it at least qualitatively correct? The answer depends on the numerical values of the parameters relevant to Earth's core, and on the sensitivity of the features of the geodynamo to details of the convective sources. It is difficult at the present time to be dogmatic. There is no doubt, however, that, if t_2 is smaller than our nominal value, the model is qualitatively correct. For example, if $t_2 = 2 \times 10^9$ yr or less, and $\Delta\rho > \Delta\rho_0$, compositional driving makes a greater contribution to $\mathcal{Q}^D$ than thermal driving; the simplest compositional model would then become qualitatively realistic. If the values we have taken are considered more plausible, however, the compositional and thermal codensities are comparable, and their interplay can generate interesting effects that are absent from the compositional model.

This interplay requires special study, and it depends of course on unknown details of the geodynamo process and of the mechanism of heat transfer through the D″ layer and mantle. The conducting and convecting D″ layer is some kind of complicated thermal valve. Here, in the low viscosity region of the mantle on the side of the D″ layer adjacent to the core, mantle plumes originate, according to Stacey and Loper (1983) and Loper and Stacey (1983), who stressed the crucial role of plumes in cooling the core. A stably stratified layer may also exist at the top of the core that plays a significant role in the exchange of heat between mantle and core (Braginsky, 1993). A complete understanding of the thermal coupling of core and mantle is still lacking. The following speculations may, however, be of some interest. The coefficient appearing in (8.25) is a few times greater than the coefficients appearing in (8.21) and (8.26). The rather large prominence of the thermal terms is the basic reason why both the intensity of convection and the field generation mechanism depend sensitively on $Nu - 1$. This dependence is particularly strong when $Nu < 1$. The compositional source of convection is then partially spent in overcoming the negative (stabilizing) influence of the thermal sources, as was pointed out by Loper (1978). When $Nu - 1 \sim -\frac{1}{2}$, as happens when $\mathcal{Q}_1^T \sim \frac{3}{2}\mathcal{Q}_M^q$ the effective 'heating from the top' is so great that it may even stifle core convection and magnetic field generation completely. This indicates that core convection depends sensitively on heat transport through the mantle. One may therefore suspect that the factor $Nu - 1$ establishes itself at rather a small value, though the mechanism through which this adjustment is

Downloaded by [Universite Laval] at 23:08 09 July 2014

effected is unknown. A sensitivity of core processes to $Nu-1$ might explain the observed variation in both the geomagnetic field intensity and the frequency of reversals over the geologically long period, $\tau_G \sim 2 \times 10^8$ yr.

A little support for these speculations is provided by the fact that the thermal time constant, $t'' \sim L''^2/\kappa_M$ of the D″ layer is of the same order as τ_G. Here L'' is a characteristic dimension of the layer and $\kappa_M \sim 10^{-6}$ m^2 s^{-1} is the thermal diffusivity of the mantle. It follows that $t'' \sim \tau_G$ for $L'' \approx 80$ km, which is comparable with the thickness of the D″ layer. To establish the plausibility or implausibility of the ideas advanced here, it would be necessary to treat the core and mantle as a coupled system—the mechanism does not operate when, as in this paper, $\mathcal{Q}_M^a$ is specified and the core alone is considered.

If the value of Nu were known, we would be able to estimate the radioactive heat production in the core by using (7.47), which can be written as

$$\mathcal{Q}^R = \mathcal{Q}_1^T\, Nu - \frac{t_{20}}{t_2}\,\mathcal{Q}_*. \tag{8.52}$$

With $Nu = 1$ and our previous estimates of $\mathcal{Q}_* = 2.1 \times 10^{12}$ W, we obtain $\mathcal{Q}^R = 3.3 \times 10^{12}$ W for $t_2 = t_{20}$, $\mathcal{Q}^R = 1.2 \times 10^{12}$ W for $t_2 = \frac{1}{2}t_{20}$, and $\mathcal{Q}^R = 0$ for $t_2 = 0.4t_{20}$. It $t_2 = 2 \times 10^9$ yr instead of 4×10^9 yr, then $\mathcal{Q}^R = 0$ is attained if $Nu \approx 0.8$. Then $|Nu - 1| \approx 0.2$, which is rather small. Lacking precise values for the crucial parameters, we may only suggest that $\mathcal{Q}^R$ is not significantly greater than about 3×10^{12}W, but may also be much smaller (including zero).

CONCLUSIONS

A proper foundation has been laid in this paper for studying core convection and the geodynamo; a complete set of workable equations has been consistently derived from first principles. We have formulated the MHD theory for the motion of a binary alloy of iron and some light admixture, in which the momentum balance is simplified by an anelastic approximation. The dual (thermal + compositional) character of core convection has been properly recognized. Although compositional and thermal driving depend significantly on the thermal interaction of core and mantle and on the thermal history of Earth, neither of which are yet known with any precision, we can (and have) introduced the dimensionless parameters that appropriately measure the relative importance of the key physical mechanisms. We have argued that the geomagnetic field is a bye-product of large-scale magnetoconvection in the core, but the important role of small-scale motions has also been recognized through the introduction of a local turbulence model. We have seen that the existence of turbulent mixing is essential for the existence of the basic state of uniform entropy and composition.

Amongst the novelties and achievements of the paper, we wish to draw particular attention to the following:

1. A significant simplification of the anelastic equations has been established in Subsection 4.2, where the momentum equation was transformed *without approximation* into Boussinesq form. We there introduced a new quantity,

Downloaded by [Universite Laval] at 23:08 09 July 2014

which we have christened 'the codensity', and which determines the non-conservative buoyancy force resulting from small perturbations of the well-mixed adiabatic state associated with the convection. This allows the irrelevant complications, created both by the pressure variations, and by the changes in gravity associated with the convective motions, to be filtered out, leaving behind only the crucial compositional and thermal buoyancy forces. This simplifies the re-assessment of the relative importance of these forces;

2. Emphasis has been placed on the probable dominance of core turbulence in the transport of mean large-scale fields such as entropy and chemical composition. This emphasis is not new, but goes beyond the early ideas adumbrated by Braginsky (1964b). The formalism developed here does not require one particular description of core turbulence rather than another; it *does* however suppose that the turbulence can be adequately described by a *local* theory, i.e. that, to a first approximation, the fluxes of mean fields (such as entropy and composition) can, at every point, be expressed as a linear combination of the gradients in those fields at that point. The coefficients in these relationships are the turbulent transport coefficients, which are expected to be very much larger than the corresponding molecular coefficients. The theory that is developed here has some points of similarity with the mixing-length theory used by astrophysicists in studying the convection zones of stars. See, for example, Ch. 14 of Cox and Giuli (1968);
3. We have stressed that, because the microscale magnetic Reynolds number is very probably small, turbulence in the core is likely to be quite different from 'classical' turbulence, in which inertial forces are all important. We have argued, however, that Coriolis and Lorentz forces are so potent that the turbulent cells in the core have a plate-like strucutre, so that the associated turbulent transport of macroscopic quantities by the turbulence is strongly anisotropic. We have made use of the turbulence model of Braginsky and Meytlis (1990) to estimate that transport. We have also emphasized that in the core, unlike the laboratory, the (tensor) turbulent diffusivities of entropy and composition are identical to one another;
4. We have provided an expression for the entropy production rate, σ^t, due to the turbulence. We have shown that this is simply $-\mathbf{g}\cdot\mathbf{I}^C$, where $\mathbf{I}^C = -\alpha^\xi\mathbf{I}^\xi - \alpha^S\mathbf{I}^S$ is the flux of codensity. Translated into simple terms, the source of turbulent energy is not inertial cascade from the macroscale, but is the gravitational instability associated with mean gradients of composition and entropy. According to all estimates, the largest of the molecular diffusivities in the core is the magnetic diffusivity. Assuming that this is the principal diffusivity that affects core turbulence, it is shown that the entropy production by the turbulence arises entirely from the Joule dissipation of the microscale electric currents, Q^j, i.e. $Q^t = Q^j$. The rate of mean field entropy production due to turbulent processes must be positive definite (or, more precisely, it must be non-negative); in local turbulence theory it must be non-negative, at every point in space. Consequently, as is shown here, the notion, that a simple enhancement of the transport coefficients is all that is required to incorporate the effects of turbulent diffusion, is incorrect. Such

Downloaded by [Universite Laval] at 23:08 09 July 2014

an idea would lead to locally increasing entropy everywhere, *including* regions that are locally stable. At a point of local stability, turbulent diffusion will be absent, according to any local theory of turbulence, and only molecular transport can take place. This creates a positive but very small entropy production. It should be stressed that, unlike the corresponding astrophysical application mentioned above, the boundaries separating unstable regions of the core from stable regions is not known, and may not even be spherical.

5. The relative importance of thermal buoyancy (from the cooling of the core and the release of latent heat at the inner core boundary) and compositional buoyancy (from the release of the light constituent of the alloy at the inner core boundary) in driving core convection has been estimated, using modern geophysical data. Braginsky (1963, 1964b) argued that compositional convection dominates thermal convection in Earth's core. His arguments were subsequently examined by a number of authors (e.g. Gubbins, 1977; Loper 1978; Gubbins *et al.*, 1979; Loper & Roberts, 1983) who confirmed that compositional driving was an effective mechanism for stirring the core. In Section 8 we have found that, most plausibly, the contributions made by compositional and thermal sources to the codensity are comparable and that some interplay between these two mechanisms must be expected. This opinion depends on the spatial distribution and magnitude of the two sources and on the sizes of various parameters that are poorly known today, and it is therefore impossible to be dogmatic about this matter. The flux of heat from core to mantle is crucially important for both convection mechanisms, and the connection between convection processes in mantle and core significantly influences the geodynamo. Nevertheless, we argue that compositional buoyancy is especially significant since it admits the possibility that heat is pumped downwards, against the adiabat (see point 7);
6. In Sections 7 and 8, we have given new expressions for the efficiency, η_D, of the geodynamo, considered as a heat engine. These differ from earlier derivations in that the effect on the efficiency of the slow evolution of the core is explicitly separated from contributions made on the convective time scale. We expressed η_D as a product, $\eta_D = \eta_F \eta_G$, where η_F is the frictional factor (7.43) and is the ratio of the Joule dissipation of the geodynamo to the total dissipation of the core, arising from Joule dissipation and all forms of friction, and η_G is the geophysically ideal efficiency (7.44), which replaces the Carnot efficiency of a traditional heat engine, and which represents the maximum attainable efficiency of the geodynamo. We have also presented new arguments that relate the magnitude of the geomagnetic field to the available power. (Unfortunately, it is again hard to apply these arguments with confidence to the core because of the uncertainty with which some key parameter are known).
7. Many parameters important for the description of core convection are poorly known. Even the heat flux from the core, which is central to the character and vigor of core convection, is so badly determined that it is even uncertain whether it exceeds, or is less than, the heat conducted down the adiabat. The rate at which the inner core grows through freezing, i.e. R_2, is also not reliably known but (we argue) it can be more accurately estimated than the heat flux. We have therefore,

Downloaded by [Universite Laval] at 23:08 09 July 2014

where possible, cast the theory into a form in which badly determined parameters are removed in favor of quantities dependent on $\dot{R}_2$. Where this was not possible, we expressed a poorly determined quantity as the product of a nominal value for that quantity (which we used to evaluate our formulae) and the ratio of the quantity to that nominal value. For example, where the age, t_2 of the solid inner core appears, we took $t_{20} = 4 \times 10^9$ yr as the nominal value, used it to evaluate expressions in which t_2 appears, but retained the ratio t_2/t_{20} in those expressions. In this way not only were the consequences of the uncertainties of t_2 made evident, but also anyone who prefers to take some value of t_2 other than 4×10^9yr can easily see the implications of his choice. This illustrates what we have striven to do in this paper: we have tried to describe clearly what is a very complex physical situation;

8. The mass fraction, ξ_N, of light constituent, Ad, in the inner core has been estimated using a simple model of a binary alloy. We have defined a rejection coefficient, $r_{FS} = (\xi_a - \xi_N)/\xi_a$, and have derived a simple expression for that coefficient that relates it to the well determined density jump, $\Delta\rho$, at the inner core boundary. Adopting the PREM value of $0.6 \times 10^3\,\mathrm{kg\,m^{-3}}$ for $\Delta\rho$, we find that $r_{FS} \approx 0.4$. Thus, most of the light material is retained by core fluid when it solidifies. Once the phase diagrams of the relevant alloys under high pressure become better known, it should be possible to use this value of r_{FS} as a means of determining which alloying element is most abundant in the core. For the present, we prefer models that take Ad = S or Si rather than Ad = O.

Downloaded by [Universite Laval] at 23:08 09 July 2014

Let us suppose that the density discontinuity, $\Delta\rho$, at the inner core boundary is $0.6 \times 10^3\,\mathrm{kg\,m^{-3}}$ and that the age of the inner core is 4×10^9 yr. Let us further suppose that the heat flux, $\mathscr{Q}^q_M \mathscr{Q}^q_M$, from the core is that conducted down the adibat, $\mathscr{Q}^T_1$. (In reality, it is not known which is the larger.) Then according to (8.50) – (8.50c),

$$\mathscr{Q}^D = 0.5 \times 10^4\ \mathscr{Q}_I \approx 4.4 \times 10^{11}\ \mathrm{W}. \tag{9.1}$$

Here $\mathscr{Q}_I = Q_I \mathscr{V}_{12} \approx 0.88 \times 10^8$ W, where Q_I is the basic unit of dissipation per unit volume in the core, defined in (8.14c). The dissipation rate, $\mathscr{Q}^D = \mathscr{A}^C + \mathscr{Q}^t$, given in (9.1) includes both Joule and viscous losses (including friction between core and mantle) from both the large-scale ($\mathscr{A}^C$) and turbulent ($\mathscr{Q}^t$) fields and motions. The macroscale parts $\mathscr{Q}^v + \mathscr{Q}^J$, are provided by the Archimedean power, $\mathscr{A}^C$, driving the dynamo, and can be estimated from integrations of kinematic or intermediate dynamo models:

$$\mathscr{A}^C = (\gamma_A \mathscr{V}_1/\mu_0 t_I)\, B^2_{\mathrm{av}}, \tag{9.2}$$

where γ_A is a dimensionless constant that is model dependent, B_{av} is the rms toroidal field, averaged over the volume of the core, $t_I = R_1^2/\eta \approx 1.92 \times 10^5$ yr is the electromagnetic time constant of the core, and η is its magnetic diffusivity, which we assume is $\eta \approx 2\,\mathrm{m}^2\,\mathrm{s}^{-1}$. With these values (9.2) may be written as

$$\mathscr{A}^C = 2.32 \times 10^5 \gamma_A\, B_{\mathrm{av}}^2\,\mathrm{W}. \tag{9.2a}$$

It is here supposed that B_{av} is measured in Gauss so that the constant multiplying B_{av}^2 in (9.2) has units of WG^{-2}. Suppose that $B_{\mathrm{av}} = 100\,\mathrm{G}$ and $\mathscr{A}^C \approx 0.7\,\mathscr{Q}^D \approx 3 \times 10^{11}\,\mathrm{W}$, so that $0.3\,\mathscr{Q}^D$ remains to supply the turbulent dissipation, $\mathscr{Q}^t$. Then we find from (9.2a) that $\gamma_A = 130$, which happens to be close to the value given by the Kumar–Roberts dynamo model; see the final column of Table 6 of Kumar and Roberts (1975). Model–Z dynamos require γ_A of order twice as large; γ_A tends to be rather smaller than 130 for Taylor–type models. It is much smaller for the free decay of either the poloidal dipole ($\gamma_A = \pi^2$) or the toroidal quadrupole ($\gamma_A \approx 20$), both of which are often used in similar calculations. Despite the uncertainty in the way that γ_A varies from one model to another, we may say with some confidence that a geodynamo in which $B_{\mathrm{av}} \sim 100\,\mathrm{G}$ can be maintained in Earth's core. This is a typical magnitude for the toroidal field in the so-called "strong field dynamo"; although the field is not extremely strong, it is much greater than the poloidal field, which is the only magnetic field seen at Earth's surface.

Theories of the geodynamo should rest on equations that are both geophysically realistic and sufficiently tractable for theoretical progress to be possible. The conflict between these two desiderata has led us to develop models at different degrees of complexity. These are roughly of three types: in the order of increasing simplicity but decreasing realism, they are

(I) The inhomogeneous model (Section 6);
(II) The homogeneous model (Section 8);
(III) The compositional model (Section 8).

Model I seems to provide a rather satisfactory basis for the study of core magnetoconvection, a framework on which further improvements can be constructed. Model II is much simpler than model I but it employs the rough Boussinesq approximation ($\epsilon_a \to 0$). It is just this model that has been used here to provide numerical estimates with a minimum of complications. It is worth remarking that it is possible to define models that are intermediate between models I and II and that these models are almost as easy to employ in massive numerical computations as model II. For example, one could incorporate the inhomogeneities of ρ_a and T_a, as given for example by the PREM model, but continue to suppose that α^ξ and α^S are constants. Then instead of (8.2), i.e. $\nabla \cdot \mathbf{V} = 0$, one would use the anelastic approximation (6.2), i.e. $\nabla \cdot (\rho_0 \mathbf{V}) = 0$. The effect of the small spherically-symmetric deviation from incompressibility on field generation could then be investigated, and the consequences of the limit set by the Carnot factor, $\Delta T/T$, could also be studied directly. This could be done with little added computational effort.

Model III is obtained by omitting thermal forcing that, though poorly known, may be as large as compositional forcing; it therefore also rules out all effects that arise through the interplay between these two mechanisms. It is however much simpler than models I and II because it uses only one equation governing the codensity together with a simple boundary condition. It evades most of the uncertain-

Downloaded by [Universite Laval] at 23:08 09 July 2014

ties in the physical chemistry of the core. Moreover, the number of parameters that must be assigned is at a minimum: $d_t R_2$, κ^t and the strength of core-mantle friction. (The interaction between solid inner core and fluid outer core across the inner core boundary can be approximately incorporated without the addition of another parameter—see Braginsky, 1989.) This simplest model could, as suggested by Braginsky (1991), be the best weapon to wield at this time in the formidable battle of finding a self-consistent geodynamo.

It is possible, within the framework of model III, to investigate in a self-consistent way all the main elements of the geodynamo mechanism, namely:

(A) magnetic field generation resulting in the mutual excitation of the mean (ϕ-averaged) poloidal and toroidal field components;
(B) generation of MAC waves, that account for the existence of the asymmetric fields and velocities which allow the dynamo to evade the restrictions of Cowling's theorem;
(C) the advection and turbulent diffusion of the mean codensity;
(D) the local turbulence mechanism which creates the diffusional transport of mean quantities.

All these four processes are nonlinear and interact with one another, thus turning the geodynamo into an auto-oscillating system. Such a system holds promise of exciting developments in the future. Perhaps a little hopefully, one may imagine that geodynamo theory will throw light on the composition of the core, and in particular on which alloying element is its principal light constituent. This would be achieved by solving the geodynamo equations for many choices of the key, but poorly determined, parameters we have isolated above, and by deciding which model fits best the geomagnetic observations and all other relevant geophysical data. These 'best values' for the poorly determined parameters would provide information, unavailable from any other source at present, about the composition of the core.

Downloaded by [Universite Laval] at 23:08 09 July 2014

Acknowledgements

We thank NASA for grant NAGW/2546, during the tenure of which this research was initiated and completed. One of us (SIB) is also very grateful to the Alexander von Humboldt Foundation for the award of a Fellowship that enabled him to benefit from the hospitality of Professor Freidrich Busse at Bayreuth University, and to bring this paper to fruition. We wish also to acknowledge helpful discussions with Professors Orson Anderson, Bill McDonough and Frank Stacey. We are grateful to David Stevenson and the other (anonymous) referee for their critical comments on the first submission of this paper. David Stevenson's remarks led to improvements in the presentations of Section 9. The main contents of this paper were presented at the SEDI meeting at Whistler in August 1994.

References

Anderson, O.L., "Physics of iron: showdown in Colorado Springs," *EOS, trans.* AGU, 237–238 (1994).
Anderson, O.L., "Mineral physics of iron and the core," submitted to *Rev. Geophys*, (1995).
Anderson, O.L. and Young, D.A., "Crystallization of the Earth's inner core," in: *Structure and Dynamics of Earth's Deep Interior*, (Eds. D.E. Smylie and R. Hide). AGU Geophysical Monograph **46**, pp. 83–89 (1988).
Backus, G.E., "Gross thermodynamics of heat engines in the deep interior of the Earth, *Proc. Nat. Acad. Sci. Wash.* **72**, 1555–1558 (1975).

Boehler, R., "Temperatures in the Earth's core from melting-point measurements of iron at high static pressures," *Nature* **363**, 534–536 (1993).

Boubnov, B.M. and Golitsyn, G.S., "Experimental study of convective structures in rotating fluids," *J. Fluid Mech.* **167**, 503–531 (1986).

Boness, D.M. and Brown J.M., "The electronic band structure of Iron, Sulfur and Oxygen at high pressures and the Earth's core," *J. Geophys. Res.* **95**, 21721–21730 (1990).

Braginsky, S.I., "Structure of the F layer and reasons for convection in the Earth's core," *Soviet Phys. Dokl.* **149**, 8–10 (1963).

Braginsky, S.I., "Kinematic models of the Earth's hydromagnetic dynamo," *Geomag. Aeron.* **4**, 572–583 (1964a).

Braginsky, S.I., "Magnetohydrodynamics of Earth's core," *Geomag. Aeron.* **4**, 698–712 (1964b).

Braginsky, S.I., "Nearly axially symmetric model of the hydromagnetic dynamo of the Earth," *Geomag. Aeron* **18**, 225–231 (1978).

Braginsky, S.I., "The Z model of the geodynamo with an inner core and the oscillations of the geomagnetic dipole," *Geomag. Aeron. 29*, 121–126 (1989).

Braginsky, S.I., "Towards a realistic theory of the geodynamo", *Geophys. Astrophys. Fluid Dynam.* **60**, 89–134 (1991).

Braginsky, S.I., "MAC-oscillations of the hidden ocean of the core," *J. Geomag, Geoelectr.* **45**, 1517–1538 (1993).

Braginsky, S.I. and Meytlis, V.P., "Local turbulence in the Earth's core," *Geophys, Astrophys. Fluid Dynam.* **55**, 71–87 (1990).

Brown, J.M. and McQueen, R.G., "Phase transitions, Grüneisen parameters and elasticity for shocked iron between 77 GPa and 400 GPa," *J. Geophys. Res.* **91**, 7485–7494 (1986).

Busse, F.H., "Thermal instabilities in rapidly rotating systems," *J. Fluid Mech.* **44**, 441–460 (1970).

Busse, F.H., "Convection driven zonal flows and vortices in the major planets," CHAOS **4**, 123–134 (1994).

Busse, F.H. and Carrigan, C.R., "Convection induced by centrifugal buoyancy," *J. Fluid Mech.* **62**, 579–592 (1974).

Busse, F.H. and Carrigan, C.R., "Laboratory simulation of thermal convection in rotating planets and stars," *Science* **191**, 81–93 (1976).

Cardin, P. and Olson, P., "An experimental approach to thermochemical convection in Earth's core," *Geophpys. Res. Lett.* **19**, 1995–1998 (1992).

Cox, J.P. and Giuli, R.T., *Principles of Stellar Structure*, **1**. New York: Gordon and Breach (1968).

Dziewonski, A.M. and Anderson, D.L., "Preliminary reference Earth model," *Phys. Earth Planet. Inter.* **25**, 297–356 (1981).

Dziewonski, A.M. and Woodhouse, J.H., "Global images of the Earth's interior," *Science* **236**, 37–48 (1987).

Fearn, D.R., Loper, D.E. and Roberts, P.H., "Structure of the Earth's inner core," *Nature* **292**, 232–233 (1981).

Frenkel, Ya.I., *Introduction to the Theory of Metals*, Moscow: Gostechisdat (in Russian) (1958).

Gans, R.F., "Viscosity of the Earth;s core," *J. Geophys. Res.* **77**, 360–366 (1972).

Glatzmaier, G.A. and Olson, P., "Highly supercritical thermal convection in a rotating spherical shell: centrifugal vs. radial gravity," *Geophys. Astrophys. Fluid Dynam.* **70**, 113–136 (1993).

deGroot, S.R. and Mazur, P., *Non-equilibrium Thermodynamics*. Amsterdam: North Holland (1962).

Gubbins, D., "Energetics of the Earth's core," *J. Geophys.* **43**, 453–464 (1977).

Gubbins, D., "The influence of extrinsic pressure changes on the Earth's dynamo," *Phys. Earth Planet. Inter.* **33**, 255–259 (1983).

Gubbins, D., Masters, T.G. and Jacobs, J.A., "Thermal evolution of the Earth's core," *Geophys. J. R. astr. Soc.* **59**, 57–99 (1979).

Hewitt, J.M., McKenzie, D.P. and Weiss, N.O., "Dissipative heating in convective flows," *J. Fluid Mech.*, **68**, 721–738 (1975).

Hollerbach, R., "Magnetohydrodynamic Ekman and Stewartson layers in a rotating spherical shell," *Proc. R. Soc. Lond.*, **A444**, 333–346 (1994).

Hollerbach, R. and Jones, C.A., "Influence of the Earth's outer core on geomagnetic fluctuations and reversals," *Nature* **365**, 541–543 (1993).

Downloaded by [Universite Laval] at 23:08 09 July 2014

Jeffreys, H., "The instability of a comprehensible fluid heated below," *Proc. Camb. Phil. Soc.*, **26**, 170–172 (1930). Reprinted in Saltzman (1962).

Jephcoat, A. and Olson, P., "Is the inner core of the Earth pure iron?" *Nature* **325**, 332–335 (1987).

Keeler, R.N. and Mitchell, A.C., "Electrical conductivity, demagnetization, and the high-pressure phase transition in shock-compressed iron," *Solid State Comm.* **7**, 271–274 (1969).

Kleeorin, N., Rogachevskii, I. and Ruzmaikin, A., "Magnetized flow between two rotating spheres," submitted (1995).

Kubaschewski, O. and Alcock, C.B., *Metallurgical Thermochemistry*, 5^{th} edition, Oxford: Pergamon (1979).

Kumar, S. and Roberts, P.H., "A three-dimensional kinematic dynamo," *Proc. R. Soc. Lond.* **A344**, 235–258 (1975).

Landau, L.D. and Lifshitz, E.M., *Statistical Physics, Part 1. 3^{rd}* Edition. Oxford: Pergamon (1980).

Landau, L.D. and Lifshitz, E.M., *Fluid Mechanics, 2^{nd}* Edition. Oxford: Pergamon (1987).

Loper, D.E., "The gravitationally powered dynamo," *Geophys. J. R. astr. Soc.* **54**, 389–404 (1978).

Loper, D.E. and Roberts, P.H., "On the motion of an iron-alloy core containing a slurry. I. General Theory," *Geophys. and Astrophys. Fluid Dynam.* **9**, 289–321 (1978).

Loper, D.E. and Roberts, P.H., "A study of conditions at the inner core boundary of the Earth," *Phys. Earth Planet. Inter.* **24**, 302–307 (1981).

Loper, D.E. and Roberts. P.H., "Compositional convection and the gravitationally powered dynamo," in: *Stellar and Planetary Dynamos*, (Ed. A.M. Soward), pp. 297–327. New York: Gordon and Breach (1983).

Loper, D.E. and Stacey, F.D., "The dynamical and thermal structure of deep mantle plumes," *Phys. Earth Planet. Inter.*, **33**, 304–317 (1983).

Matassov, G., *The Electrical Conductivity of Iron-Silicon Alloys at High Pressures and the Earth's Core*, Ph.D. thesis, Lawrence Livermore Laboratory. Rpt. UCRL-52322 (1977).

Mathews, P.M., Buffet, B.A., Herring, T.A. and Shapiro, I.I., "Forced nutation of the Earth: influence of inner core dynamics," *J. Geophys. Res.* **96B**, 8243–8257 (1991).

Moffatt, H.K., "Liquid metal MHD and the geodynamo," pp. 403–412 in *Liquid Metal Magnetohydrodynamics*, (Eds. J. Lielpeteris and R. Moreau). Dordrecht: Kluver (1989).

Moffatt, H.K. and Loper, D.E., "Hydromagnetics of the Earth's core, I. The rise of a buoyant, blob," *Geophys. J. Intern.* **117**, 394–402 (1994).

Morelli, A., Dziewonski, A.M. and Woodhouse, J.H., "Anisotropy of the inner core inferred from PKIKP travel times," *Geophys. Res. Lett.*, **13**, 1545–1548 (1986).

Poirier, J.P., "Transport properties of liquid metals and viscosity of the Earth's core," *Geophys. J. Intern.* **92**, 99–105 (1988).

Poirier, J.-P. and Shankland, T.J., "Dislocation melting of iron and the temperature of the inner core, revisited," *Geophys. J. Intern.* **115**, 147–151 (1993).

Pollack, H.N., Hurter, S.J. and Johnson, J.R. "Heat flow from the Earth's interior: analysis of the global data set," *Rev. Geophys.* **31**, 267–280 (1993).

Roberts, P.H., "On the thermal instability of a rotating-fluid sphere containing heat sources," *Phil. Trans. R. Soc. Lond.* **A263**, 93–117 (1968).

Ruzmaikin, A. "On the role of rotation of the internal core relative to the mantle," in: *Solar and Planetary Dynamos*, (Eds. M.R.E. Proctor, P.C. Matthews and A.M. Rucklidge), pp. 265–270. Cambridge UK: University Press (1993).

Saltzman, B., *Selected Papers on the Theory of Thermal Convection with Special Application to the Earth's planetary Atmosphere* New York: Dover (1962).

Shearer, P. and Masters, G., "The density and shear velocity contrast at the inner core boundary," *Geophys. J. Intern.* **102**, 491–496 (1991).

St. Pierre, M.G., "On the local nature of the fluid flow in Earth's outer core". Submitted (1995).

Stacey, F.D., *Physics of the Earth*, Brisbane: Brookfield Press, 3^{rd} Edition (1992).

Stacey, F.D., "Theory of thermal and elastic properties of the lower mantle and core," submitted to *Geophys. Surv.* (1994).

Stacey, F.D. and Loper, D.E., " The thermal boundary layer interpretation of D″ and its role as a plume source," *Phys. Earth Planet. Inter.* **33**, 45–55 (1983).

Stevenson, D.J., "Turbulent thermal convection in the presence of rotation and a magnetic field: a heuristic theory," *Geophys. Astrophys. Fluid Dynam.* **12**, 139–169 (1979).

Verhoogen, J., "Heat balance in the Earth's core:, *Geophys. J.* **4**, 276–281 (1961).

Downloaded by [Universite Laval] at 23:08 09 July 2014

APPENDIX A: NOTATION

Four abbreviations occur frequently in the text:

CMB = core mantle boundary,

ICB = inner core boundary,

FOC = fluid outer core,

SIC = solid inner core.

We use the word 'Nucleus' as an alternative to SIC, the suffix N then being attached to quantities evaluated in the SIC or on its surface. In an effort to make our notation self explanatory, we have adopted a few simple rules: To track the numerous physical variables that arise in our work, we have distinguished each by a combination of a letter and a suffix that have unique meanings. For example, ρ is used everywhere for density while a always refers to the adiabatic reference state. The significance of ρ_a is therefore immediately obvious. The following should also be noted:

1. All small parameters are denoted by ϵ, with an appropriate subscript;
2. Energies per unit mass (with the exception of the gravitational potentials U) are denoted by ε, with an appropriate superscript;
3. All energies per unit volume by u, with an appropriate superscript;
4. All rates of dissipation of volumetric energies are denoted by Q, with an appropriate superscript; all rates of volumetric entropy production are denoted by σ, with an appropriate superscript;
5. When an extensive quantity is integrated over a volume, such as the volume of the core, it is denoted by a calligraphic letter. For example, the integral of a mass density ρ is denoted by $\mathcal{M}$, the integral of a volumetric energy density u is denoted by $\mathcal{E}$, and the integral of the rate of dissipation of such an energy density, Q, is denoted by $\mathcal{Q}$; in Section 7, $\mathcal{A}$ is the rate at which "useful work" is done and $\mathcal{L}$ denotes couple;
6. Material fluxes are denoted by $\mathbf{I}$, with an appropriate superscript; electric current density is denoted by $\mathbf{J}$;
7. Some subscripts and superscripts permanently have specific meanings; these are often omitted when vectors or tensors are written in component form. In particular:
   (a) The suffices $_1$ and $_2$ refer respectively to the CMB and ICB, the volumes they contain being denoted by $\mathcal{V}_1$ (the entire core) and $\mathcal{V}_2$ (the SIC); between them lies $\mathcal{V}_{12}$ (the FOC). Where a quantity carries the suffix 1, it is to be evaluated on the CMB; 2 means it is evaluated on the fluid side of the ICB—if the quantity is discontinuous there, $_N$ means that it is evaluated on the solid of the ICB;
   (b) The subscript a refers to the basic adiabatic state. Note however that properties of the fluid such as α, c_p, $\eta, \ldots$ do not usually carry suffices even when they are evaluated for the basic state. The subscript a is also omitted when it occurs in conjunction with the suffices $_1$ and $_2$ that refer to CMB and ICB. We therefore write R_1 and R_2 in place of R_{1a} and R_{2a}, T_1 and T_2 instead of T_{1a} and T_{2a}, $\mathbf{I}_1^q$ instead of $\mathbf{I}_{1a}^q$, and so forth;

Downloaded by [Universite Laval] at 23:08 09 July 2014

(c) For the FOC, the increment in a variable, through the action of convection, over its value in the basic state carries the subscript c. The fields $\mathbf{V}$ and $\mathbf{B}$ arise only in the convective state and do not carry the subscript c. For the SIC, the suffix c on a variable signifies that it is the deviation from its value in the basic state which is mainly due to the thermal conduction;

(d) The superscript t is used to distinguish transport coefficients associated with turbulent processes from the corresponding molecular coefficients. The turbulent contributions to other fields are denoted by daggers, e.g. $\mathbf{V}^\dagger$, $\mathbf{B}^\dagger$, $C^\dagger, \ldots$ are the fluctuating parts of $\mathbf{V}, \mathbf{B}, C, \ldots$. [For brevity, $\mathbf{V}^\dagger$, $\mathbf{B}^\dagger$, and $C^\dagger$ are replaced by $\mathbf{v}$, $\mathbf{b}\sqrt{(\mu_0\rho_0)}$ and c in Appendix C;

(e) Other superscripts and subscripts that appear are

0	which stands for average over the FOC
B	which stands for magnetic field
e	which stands for effective
J	which stands for Joule
K	which stands for kinetic
M	which stands for mantle
N	which stands for nucleus (SIC)
nd	which stands for non-dimensional
p	which stands for pressure
S	which stands for entropy
T	which stands for temperature
V	which stands for volume or for velocity, depending on context
ν	which stands for viscosity
ξ	which stands for composition

8. Time derivatives of basic quantities are denoted by an overdot, e.g. we write $\dot{\rho}_a$ in place of $\partial\rho_a/\partial t$;
9. There are types of average: $\langle Q\rangle^t$, $\bar{Q}$, $\langle Q\rangle^V$. These are introduced in Subsections 4.2 and 5.1 and in (6.26) and are respectively averages over the *t*urbulent ensemble, over large-scale *c*onvection and over *V*olume. In (6.25a, b) we introduce ρ_0 and T_0 in place of $\langle\rho_a\rangle^V$ and $\langle T_a\rangle^V$;
10. Double square brackets are used to denote the discontinuity of any field at a surface, the location of which is specified. For example, "$[\![\xi_a]\!]$ at $r = R_2$" denotes $\xi_2 - \xi_N$, where $\xi_2 = \xi_a(R_2+)$ and $\xi_N = \xi_a(R_2-)$ are the concentrations of admixture at $r = R_2$ in the FOC and SIC, respectively. We denote this particular jump also by ξ_{2N}.

Table A1 Key to Notation

Quantity	Name	Units	Definition
$\mathbf{1}_q$	Unit vector in direction of increasing coordinate q	None	
∂_t	Eulerian time derivative	s^{-1}	$\partial/\partial t$
d_t	Lagrangian time derivative	s^{-1}	$\partial_t + \mathbf{V}\cdot\nabla$
$\mathscr{E}$	Various energies	J	
α	Thermal coefficient of volume expansion	$°K^{-1}$	(D8)
α^S	Entropy coefficient of volume expansion	$kg\,J^{-1}\,°K = s^2m^{-2}\,°K$	(3.7d)

Downloaded by [Universite Laval] at 23:08 09 July 2014

Table A1 (*Continued*)

Quantity	Name	Units	Definition
α_T^ξ	Isothermal compositional expansion coefficient	None	(D9)
α^ξ	Adiabatic compositional expansion coefficient	None	(D17)
γ	Grüneisen parameter	None	(3.7f)
δ_{ij}	Kronecker delta	None	
ϵ	Various small parameters	None	(3.8)–(3.11)
ε	Various energies per unit mass	$m^2s^{-2} = Jkg^{-1}$	
η	Magnetic diffusivity (≥ 0)	m^2s^{-1}	
η_X	Various efficiencies ($X = B, C, D, E, F, G, I$)	None	Section 7
θ	Colatitude	None	
κ^T	Molecular thermal diffusivity (≥ 0)	m^2s^{-1}	(2.38a)
κ^ξ	Molecular compositional diffusivity (≥ 0)	m^2s^{-1}	(D31)
$\overleftrightarrow{\kappa}^t$	Turbulent diffusivity tensor	m^2s^{-1}	
μ_0	Permeability of free space ($4\pi\ 10^{-7}$)	$H\ m^{-1}$	
μ	Chemical potential	m^2s^{-2}	(D4)
μ_T^ξ	Isothermal compositional derivative of chemical potential	m^2s^{-2}	(D10)
μ^ξ	Adiabatic compositional derivative of chemical potential	m^2s^{-2}	(D16)
ν	Kinematic shear viscosity (≥ 0)	m^2s^{-1}	(2.13a)
ν_2	Kinematic second (bulk) viscosity (≥ 0)	m^2s^{-1}	(2.13a)
$\overleftrightarrow{\upsilon}$	Viscosity tensor	m^2s^{-1}	
ξ	Mass fraction of light component of alloy	None	
$\overleftrightarrow{\pi}$	Stress tensor	Nm^{-2}	(2.13)
ρ	Density	kgm^{-3}	
σ_η	Electrical conductivity (≥ 0)	Sm^{-1}	
σ^S	Entropy production per unit volume	$Wm^{-3}\,{}^\circ K^{-1}$	
τ	Various time scales	s	
ϕ	East longitude	None	
Ω	Angular speed of reference frame	s^{-1}	
$\mathbf{\Omega}$	Angular velocity of reference frame	s^{-1}	
A_1, A_2	Area of CMB, area of ICB	m^2	
A_{12}	Area $A_1 + A_2$ of boundaries of FOC	m^2	
$\mathcal{A}$	Rate of working	W	
$\mathbf{B}$	Magnetic field	T	
c_p	Specific heat at constant pressure (≥ 0)	$J kg^{-1}\,{}^\circ K^{-1}$	(D11)
c_v	Specific heat at constant volume	$J kg^{-1}\,{}^\circ K^{-1}$	(D19)
C	Codensity (Fractional density change at constant pressure)	None	(4.12)
$\overleftrightarrow{\mathbf{e}}$	Rate of strain tensor	s^{-1}	(2.13b)
$\overleftrightarrow{\mathbf{D}}$	Dimensionless turbulent diffusivity tensor	None	(8.14e)
$\mathbf{E}$	Electric field	Vm^{-1}	
$\mathbf{F}$	Body force per unit mass	ms^{-2}	
$\mathbf{g}$	Gravitational field ($-\nabla U$)	ms^{-2}	
$\mathbf{g}_a$	Effective gravitational field ($-\nabla U_\epsilon$)	ms^{-2}	
h^ξ	Heat of reaction	$J kg^{-1}$	(D12)
h_L	Latent heat	$J kg^{-1}$	(6.30b)
h_N	Generalized latent heat	$J kg^{-1}$	(6.30a,c)
$\mathbf{I}$	Various fluxes	Various	
$\mathbf{J}$	Electric current density	$A m^{-2}$	
k_N	Newtonian constant of gravitation ($6.673\ 10^{-11}$)	$kg^{-1} m^3 s^{-2}$	
k_p^ξ	Pressure coefficient	None	(D34)

Downloaded by [Universite Laval] at 23:08 09 July 2014

Table A1 (*Continued*)

Quantity	Name	Units	Definition
k_T^ξ	Soret coefficient	None	(D33)
K^T	Thermal conductivity $\rho c_p \kappa^T$ (≥ 0)	$\mathrm{Wm^{-1}\,^\circ K^{-1}}$	(D30)
K_S	Incompressibility ρu_S^2	Pa	
$\mathscr{L}$	Couple	$\mathrm{kg\,m^2 s^{-2}}$	
$\mathscr{M}_1, \mathscr{M}_2$	Mass of entire core, mass of SIC	kg	
$\mathscr{M}_{12}$	Mass of FOC	kg	
p	Pressure	$\mathrm{Nm^{-2}}$	
P	Reduced pressure	$\mathrm{m^2 s^{-2}}$	p_c/ρ_c
Q	Energy dissipation per unit volume	$\mathrm{Wm^{-3}}$	
$\mathscr{Q}$	Energy dissipation	W	
$\mathbf{r}$	Radius vector from geocenter	m	
r	Distance from geocenter	m	$\lvert\mathbf{r}\rvert$
r_{FS}	Rejection coefficient	None	(5.15)
s	Distance from the polar axis	m	
t	Time	s	
t_a	Time on geological scale	s	
t_c	Time on convectional scale	s	
S	Specific entropy	$\mathrm{J\,kg^{-1}\,^\circ K^{-1}}$	
T	Temperature	$^\circ$K	
u_S	(Adiabatic) velocity of sound	$\mathrm{ms^{-1}}$	(D20)
u_T	Isothermal velocity of sound	$\mathrm{ms^{-1}}$	(D21)
U	Potential of gravitational field	$\mathrm{m^2 s^{-2}}$	(B9)
$\mathbf{U}_{1,2}$	Velocities of CMB, ICB	$\mathrm{ms^{-1}}$	
U_e	Effective gravitational potential	$\mathrm{m^2 s^{-2}}$	(2.8b)
$\mathbf{V}$	Fluid velocity	$\mathrm{ms^{-1}}$	
$\mathscr{V}_1, \mathscr{V}_2$	Volume of entire core, volume of SIC	$\mathrm{m^3}$	
$\mathscr{V}_{12}$	Volume of FOC	$\mathrm{m^3}$	
z	Distance northwards from equatorial plane	m	

Downloaded by [Universite Laval] at 23:08 09 July 2014

APPENDIX B: GRAVITATIONAL ENERGY

The theory of Newtonian gravitation bears a close relationship with that of electrostatics; there are also significant differences, of which the opposite sign (attraction of masses rather than repulsion of like charges) is not alone. The basic field equations are

$$\nabla \times \mathbf{g} = \mathbf{0}, \qquad \nabla \cdot \mathbf{g} = -4\pi k_N \rho. \tag{B1,2}$$

As in electrostatic theory, (B1) and (B2) are the pointwise forms of more general integral statements

$$\oint_C \mathbf{g} \cdot \mathbf{dC} = 0, \qquad \oint_A \mathbf{g} \cdot \mathbf{dA} = -4\pi k_N \int_{\mathscr{V}} \rho \, dV, \tag{B3,4}$$

where A is any closed surface containing a volume $\mathscr{V}$, and C is any closed curve. When applied at a surface where ρ changes discontinuously, they imply that

$$[\![\mathbf{n} \times \mathbf{g}]\!] = \mathbf{0}, \qquad [\![\mathbf{n} \cdot \mathbf{g}]\!] = -4\pi k_N \rho_A, \quad \text{on } A, \tag{B5,6}$$

where ρ_A is the concentrated surface mass density and $\mathbf{n}$ is the unit outward normal to A. In most models of gravitational phenomena, ρ_A is zero, and then (B5) and (B6) give

$$[\![\mathbf{g}]\!] = \mathbf{0} \quad \text{on } A \text{ provided } \rho_A = 0. \tag{B6a}$$

When mass is contained only in a bounded volume, $\mathscr{V}$, surrounded by vacuum, $\hat{\mathscr{V}}$, we have

$$\mathbf{g} = \mathrm{O}(r^{-2}), \qquad \text{for} \quad r \to \infty. \tag{B7}$$

Equations (B1) and (B2) allow one to write the gravitational force per unit volume as

$$\rho \mathbf{g} = \nabla \cdot \overleftrightarrow{\pi}^g, \tag{B8}$$

where

$$\pi^g_{ij} = -\frac{1}{4\pi k_N}(g_i g_j - \tfrac{1}{2} g^2 \delta_{ij}), \tag{B8a}$$

is the gravitational stress tensor. With the help of (B8), the gravitational force and couple on a body $\mathscr{V}$ are readily expressed as integrals over its surface A. It is easy to show from (B5), (B6), (B7), (B8) and (B8a) that the self-force and self-couple on $\mathscr{V}$ are zero.

According to (B1) and (B2), we have

$$\mathbf{g} = -\nabla U, \qquad \nabla^2 U = 4\pi k_N \rho, \tag{B9, 10}$$

and (B5) and (B6) are satisfied if we apply

$$[\![U]\!] = 0, \qquad [\![\mathbf{n} \cdot \nabla U]\!] = 4\pi k_N \rho_A, \quad \text{on } A, \tag{B11, 12}$$

while (B6a) becomes

$$[\![U]\!] = 0, \qquad [\![\mathbf{n} \cdot \nabla U]\!] = 0, \quad \text{on } A \text{ provided } \rho_A = 0. \tag{B11a, 12a}$$

Condition (B7) reduces to

$$U = \mathrm{O}(r^{-1}), \qquad \text{for} \quad r \to \infty. \tag{B13}$$

The gravitational energy of a mass distribution is defined to be the energy required to assemble it from masses brought "from infinity". It is (for $\rho_A = 0$)

$$\mathscr{E}^g = \tfrac{1}{2} \int \rho U \, dV. \tag{B14}$$

This is in fact negative since energy is extracted during the process of assembly. By using (B10), (B11a) and (B12a), we find that (B14) can be written as

$$\mathscr{E}^g = \frac{1}{8\pi k_N} \int_{\mathscr{V}} U \nabla^2 U \, dV = \frac{1}{8\pi k_N} \oint_{A_\infty} U d\mathbf{A} \cdot \nabla U - \frac{1}{8\pi k_N} \int_{\mathscr{V}_\infty} (\nabla U)^2 \, dV, \tag{B15}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

where the surface integral is taken at infinity and vanishes by (B13); $\mathscr{V}_\infty$ is all space. We then obtain an alternative to (B14), namely

$$\mathscr{E}^g = \int_{\mathscr{V}_\infty} u^g \, dV, \quad \text{where} \quad u^g = -\frac{g^2}{8\pi k_N}. \tag{B16, 16a}$$

This provides a definition of the energy density for gravitation that is, apart from sign, exactly analogous to the energy density that arises in electrostatics. It is interesting to note that even a second alternative expression for $\mathscr{E}^g$ exists. By (B7), (B8), (B8a) and an application of the divergence theorem, we have

$$0 = \oint_{A_\infty} x_i \pi^g_{ij} dA_j = \int_{\mathscr{V}_\infty} \pi^g_{jj} dV + \int_{\mathscr{V}_\infty} x_i \rho g_i \, dV = \int_{\mathscr{V}_\infty} \frac{g^2}{8\pi k_N} dV + \int_{\mathscr{V}_\infty} \rho \mathbf{g} \cdot \mathbf{r} \, dV, \tag{B17}$$

where the volume integrals are taken over all space so that the surface integral is at infinity. By (B16, B16a) we now see that

$$\mathscr{E}^g = \int_{\mathscr{V}_\infty} \rho \mathbf{g} \cdot \mathbf{r} \, dV. \tag{B18}$$

We may use (B16a) to derive a pointwise expression of gravitational energy conservation. By (B9), (B10) and (2.2), we have

$$\begin{aligned} \partial_t u^g &= -\frac{1}{4\pi k_N} g_i \partial_t g_i = -\frac{1}{4\pi k_N} \nabla_i U \nabla_i (\partial_t U) = -\frac{1}{4\pi k_N} \nabla_i (U \partial_t \nabla_i U) + \frac{1}{4\pi k_N} U \partial_t \nabla^2 U \\ &= \frac{1}{4\pi k_N} \mathbf{\nabla} \cdot (U \partial_t \mathbf{g}) + U \partial_t \rho = \mathbf{\nabla} \cdot \left(\frac{1}{4\pi k_N} U \partial_t \mathbf{g} - U \rho \mathbf{V} \right) + \rho \mathbf{V} \cdot \mathbf{\nabla} U, \end{aligned} \tag{B19}$$

which may be written as

$$\partial_t u^g + \mathbf{\nabla} \cdot \mathbf{I}^g = -\rho \mathbf{V} \cdot \mathbf{g}, \tag{B20}$$

where

$$\mathbf{I}^g = U \left(\rho \mathbf{V} - \frac{1}{4\pi k_N} \partial_t \mathbf{g} \right). \tag{B20a}$$

Despite the possible discontinuity in $\rho U \mathbf{n} \cdot \mathbf{V}$ at a surface A of discontinuity in ρ, we have

$$[\![\mathbf{n} \cdot \mathbf{I}^g]\!] = 0, \quad \text{on } A. \tag{B21}$$

To see this, we take the motional derivative of (B11) and (B6) with respect to the velocity, $\mathbf{n} \cdot \mathbf{V}_A$, of A along its normal and obtain

$$[\![\partial_t U]\!] = -\mathbf{n} \cdot \mathbf{V}_A [\![\mathbf{n} \cdot \mathbf{\nabla} U]\!], \quad \text{on } A, \tag{B22}$$

$$[\![\mathbf{n} \cdot \partial_t \mathbf{g}]\!] = -\mathbf{n} \cdot \mathbf{V}_A [\![\mathbf{n} \cdot \mathbf{\nabla g}]\!] - 4\pi k_N \dot{\rho}_A, \quad \text{on } A. \tag{B23}$$

On applying (B12) and (B2) and using $\dot{\rho}_A = -[\![\rho \mathbf{n} \cdot (\mathbf{V} - \mathbf{V}_A)]\!]$, we reduce these to

$$[\![\partial_t U]\!] = -4\pi k_N \rho_A \mathbf{n} \cdot \mathbf{V}_A, \qquad [\![\mathbf{n} \cdot \partial_t \mathbf{g}]\!] = 4\pi k_N [\![\rho \mathbf{n} \cdot \mathbf{V}]\!], \quad \text{on } A. \tag{B22a, 23a}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

Taking the scalar product of (B20a) with $\mathbf{n}$ and again applying (B11), we obtain (B21) from (B23a). (In fact, $\mathbf{I}^g = I^g \mathbf{1}_r$ vanishes identically for a spherically symmetric system.) It is also worth noticing that, according to (B7) and (B13),

$$\mathbf{I}^g = \mathrm{O}(r^{-3}), \qquad \text{for} \quad r \to \infty. \tag{B24}$$

By a transformation similar to (B19), it is possible to show that (B20) also holds when (B16a) and (B20a) are replaced by

$$u^g = \tfrac{1}{2}\rho U, \qquad \mathbf{I}^g = \rho U \mathbf{V} + \frac{1}{8\pi k_N}(\mathbf{g}\partial_t U - U\partial_t \mathbf{g}). \tag{B25, 26}$$

Though more symmetrical, these are perhaps not quite convenient as the other forms, since $\mathbf{n}\cdot\mathbf{I}^g$ would in general be discontinuous on A.

Many of these results have been derived on the assumption that the gravitational field is self-generated; see (B7) and (B13). If an externally generated field,

$$\mathbf{g}^{\text{ext}} = -\nabla U^{\text{ext}}, \tag{B27}$$

is present in addition, it is necessary to add $u^{g\,\text{ext}}$ to the right-hand sides of (B16a) and (B25) and to add $\mathbf{I}^{g\,\text{ext}}$ to the right-hand sides of (B20a) and (B26), where

$$u^{g\,\text{ext}} = \rho U^{\text{ext}}, \qquad \mathbf{I}^{g\,\text{ext}} = \rho U^{\text{ext}}\mathbf{V}. \tag{B28, 29}$$

Equation (7.33) provided an estimate of the power, $\mathscr{A}^\xi$, released by gravitational settling. This estimate presumed, consistently with the basis of the Boussinesq model of Section 8, that the density of the FOC is almost constant. We conclude this Section by attempting, again through the use of (7.33), to derive a more accurate estimate of $\mathscr{A}^\xi$. It is possible to represent the density of the FOC with an inaccuracy of at most $\sim 0.5\%$ by a simple parabolic law which, replacing r by the non-dimensional r/R_1, is

$$\rho_a = \rho_1 + \rho_d(1 - r^2), \qquad \text{where} \qquad \rho_d = \frac{\rho_2 - \rho_1}{1 - r_2^2}, \qquad r_2 = \frac{R_2}{R_1}, \tag{B30, 30a, 30b}$$

where ρ_1 and ρ_2 are given by PREM and are listed in Table 1, so that $\rho_d \approx 2580.81\,\mathrm{kg\,m^{-3}}$. The resulting mass, $\mathscr{M}_{12} = 1.8367 \times 10^{24}\,\mathrm{kg}$, of the FOC agrees well with the value $1.8411 \times 10^{24}\,\mathrm{kg}$ given by PREM. We adjust the mass of the SIC to give the g_2 obtained from PREM and listed in Table 1. The value, $g_1 = 10.66\,\mathrm{m\,s^{-2}}$ implied by (B30) then agrees well with the PREM value listed in Table 1. Since $\partial U/\partial r = g = k_N \mathscr{M}(r)/r^2$, where $\mathscr{M}(r)$ is the mass contained in the sphere of radius r centered on O, we find that

$$U - U_2 = g_2 R_1 \frac{(r - r_2)}{r} r_2 + \frac{2\pi k_N(\rho_1 + \rho_d) R_1^2}{3} \frac{(r - r_2)^2}{r}(r + 2r_2)$$
$$- \frac{\pi k_N \rho_d R_1^2}{15} \frac{(r - r_2)^2}{r}(r^3 + 2r_2 r^2 + 3r_2^2 r + 4r_2^3). \tag{B31}$$

From (7.33) we have

$$\mathscr{A}^\xi = -\dot{\xi}_a \int_{\mathscr{V}_{12}} (\mu_a - \mu_2)\rho_a \, dV = -\frac{\dot{\mathscr{V}}_2 \Delta_\xi \rho}{\mathscr{M}_{12}} \int_{\mathscr{V}_{12}} (U_2 - U_a)\rho_a \, dV. \tag{B32}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

Downloaded by [Universite Laval] at 23:08 09 July 2014

We have here appealed to (8.35), which holds for nonuniform ρ_a provided that we assume, as we shall, that α^ξ is constant in the FOC; we have also used (6.23b) and (D48c). The final integral in (B32) is easily evaluated with the help of (B30) and (B31). Writing also $\dot{\mathscr{V}} = \mathscr{V}/t_2$, we find that

$$\mathscr{A}^\xi = 0.38\frac{\Delta\rho}{\Delta\rho_0}\frac{t_{20}}{t_2}10^{12}\,\mathrm{W}. \tag{B32a}$$

We have used the full $\Delta\rho_0 = 0.6\times10^3\,\mathrm{kg\,m^{-3}}$ in this evaluation instead of $\Delta_\xi\rho \approx 0.5\times10^3\,\mathrm{kg\,m^{-3}}$. The coefficient in (B32a) is about 10% greater than that appearing in (8.37a), where the approximations $\rho = \rho_0$ and $g = g_1 r$ were used.

It is clear from the derivation of (7.32) that, in general, $\mathscr{A}^\xi \neq \dot{\mathscr{E}}^g$. Changes in ρ (and therefore in $\mathscr{E}^g$) arise from variations in p, S and ξ. All three are properly accounted for in Sections 6–8, but $\mathscr{A}^\xi$ involves only the ξ–created ρ–variations. It is also apparent that, if α^ξ is not constant, even variations in ξ will cause the volume of the core and therefore R_1 to change, with a concomitant modification to the distribution of ρ in the mantle that will make a nonzero contribution to $\dot{\mathscr{E}}^g$. When α^ξ is constant however, the mantle and SIC do not contribute to $\dot{\mathscr{E}}^g$, and (B32) has a simple interpretation. In time δt, a fraction $\alpha^\xi\xi_{2N}\delta t$ of the mass in a unit volume situated at a distance of r from the geocenter is effectively carried to the ICB, releasing gravitational energy of $\rho_a\alpha^\xi\xi_{2N}\delta t\,[U_a(r) - U_2]$. Integrating this over the FOC, we obtain (B32) as the total gravitational energy release. In this case therefore $\mathscr{A}^\xi = \dot{\mathscr{E}}^g$.

We may use the constant–α^ξ model to estimate $\partial_t p_2$, a quantity arising in Section 6 (see footnote 8). According to this model R_1 and p_1 are independent of t and $\partial_t\rho_a$ is independent of r in the FOC. We have

$$\partial_t\rho_a = -\frac{3r_2\dot{r}_2}{1-r_2^3}\Delta\rho, \qquad \partial_t g = -\frac{4\pi k_N R_1}{3r^2}(1-r^3)\,\partial_t\rho_a. \tag{B33, 34}$$

Differentiating the equation, $\partial_r p = -\rho g$, of hydrostatic equilibrium with respect to t and integrating over r, we find that

$$\partial_t p = \left[U_1 - U - \frac{4\pi k_N R_1^2}{3}\int_r^1 \rho_a(1-r^3)\frac{dr}{r^2}\right]\partial_t\rho_a, \tag{B35}$$

where we have used (B9) and $\partial_t p_1 = 0$. Combining (B33) and (B35), we obtain

$$\partial_t p_2 = f g_2 \dot{R}_2 \Delta\rho, \tag{B36}$$

where, for the model defined by (B30), we have by (B31)

$$f = \frac{3r_2}{1+r_2+r_2^2}\left\{\frac{4\pi k_N R_1}{3g_2}(1-r_2^2)\left[\rho_1(1+r_2) + \frac{\rho_d}{20}(5-2r_2-21r_2^2-12r_2^3) - \frac{g_1 r_2^2}{g_2}\right\}, \tag{B36a}$$

i.e. $f \approx 0.69$.

APPENDIX C: LOCAL TURBULENCE

Turbulence plays a crucial role in the MHD of Earth's core, but no theory has yet been developed to describe it fully. The discussions that have so far been published might be better described as intelligent speculations than as deductive theories. As we have mentioned in Section 4, proposals have fallen into two main categories: developments of the classical ideas of large-scale turbulence in which energy is injected at a macroscale and cascades to dissipation at a microscale, and new ideas concerning local turbulence in which energy enters at the microscale level, at which it is also dissipated (see Braginsky 1964b; Braginsky & Meytlis, 1990). Both proposals fully recognize the importance of Coriolis and Lorentz forces, and both therefore visualize turbulence that is far from isotropic. Despite its small length scale, turbulence of the second kind enormously enhances the diffusion of heat and composition. We speculated in Section 4 that this diffusion is so large that it would quench instabilities on the macroscale that might otherwise have been expected to provide the source of classical turbulence of the first kind. We therefore concentrate in this appendix on turbulence of Braginsky–Meytlis type. We summarize their proposal, which they call "local turbulence", and derive a new relationship that has considerable bearing on the arguments of Sections 4 and 6.

Local turbulence is caused by a simple local instability: that of a heavy fluid overlying a lighter one. Because of the strong influence of Coriolis and Lorentz forces, this instability has a fundamentally different character from the usual buoyancy-driven instability. We will exhibit the instability through the simplest possible example; namely the growth of a disturbance in a plane layer in which $\mathbf{\Omega}$ and $\mathbf{g}$ are constant and parallel: $\mathbf{\Omega} = \Omega \mathbf{1}_z$, $\mathbf{g} = -g\mathbf{1}_z$, and in which the main flow and magnetic field are constant and horizontal: $\mathbf{V} = V\mathbf{1}_y$, $\mathbf{B} = B\mathbf{1}_y$. The cylindrical coordinates (s, ϕ, z) for the core therefore correspond to (x, y, z), in that order. We suppose that this equilibrium state is slightly perturbed, so that $\mathbf{V}$, $\mathbf{B}$, C and P become $\mathbf{V}+\mathbf{v}$, $\mathbf{B}+\mathbf{b}$, $C+c$ and $P+p$, where $\mathbf{v}$, $\mathbf{b}$, c and p are infinitesimal quantities whose squares and products can be neglected, i.e. we appeal to linear stability theory and we seek to find the growth rate, γ_α, of the resulting motion. In the notation of Section 4, $\mathbf{v}$, $\mathbf{b}$, c and p correspond to $\mathbf{V}^\dagger$, $\mathbf{B}^\dagger(\mu_0\rho_0)^{-1/2}$, $C^\dagger$ and $P^\dagger$. Since we have no small letter counterparts for S and ξ (s being used—see above—for cylindrical radius), we continue to use $S^\dagger$ and $\xi^\dagger$ for the departures of S and ξ from their equilibrium values. The effect of V is merely to Doppler shift Im γ_α, and we shall for simplicity transform to the frame in which $V=0$.

The linearized equations (6.1)–(6.4), (6.6) and (6.7) give

$$\mathbf{0} = -\nabla\tilde{p} - 2\mathbf{\Omega}\times\mathbf{v} + c\mathbf{g} + \mathbf{B}\cdot\nabla\mathbf{b}, \tag{C1}$$

$$\mathbf{0} = \mathbf{B}\cdot\nabla\mathbf{v} + \eta\nabla^2\mathbf{b}, \tag{C2}$$

$$\partial_t c = -\mathbf{v}\cdot\nabla C, \tag{C3}$$

$$\nabla\cdot\mathbf{v} = 0, \qquad \nabla\cdot\mathbf{b} = 0, \tag{C4, 5}$$

where $\tilde{p} = p + \mathbf{B}\cdot\mathbf{b}$. We have assumed here that the microscale magnetic Reynolds number is negligibly small so that $d_t\mathbf{b}$, which would otherwise have replace 0 on the

Downloaded by [Universite Laval] at 23:08 09 July 2014

left-hand side of (C2), has been discarded. We have divided **B** and **b** by $\sqrt{(\mu_0 \rho_0)}$, so that these fields now have dimensions of velocity; $1\,\mathrm{cm\,s^{-1}}$ is equivalent to 11.7 G. The Lorentz force has been written as the divergence of the magnetic stress tensor, and the term $\nabla(\mathbf{B}\cdot\mathbf{b})$ has been absorbed into the pressure gradient. We have neglected the molecular fluxes $\mathbf{I}^{\xi}$ and $\mathbf{I}^{S}$ appearing in (6.3) and (6.4), and ξ and S therefore obey equations of identical form. This marks it possible to combine them together into a single equation governing the perturbed codensity,

$$c = -\alpha^S S_c^{\dagger} - \alpha^{\xi} \xi_c^{\dagger}, \tag{C6}$$

instead of two equations governing $\xi_c^{\dagger}$ and $S_c^{\dagger}$ separately; see (6.17). The linearized form of this single equation is (C3).

Solutions to (C1)–(C5) can be sought as a superposition of elementary perturbations that, near the point $\mathbf{x} = \mathbf{x}_0$, have the form

$$(\mathbf{v}, \mathbf{b}, c, p) = (\mathbf{v}_k, \mathbf{b}_k, c_k, p_k)\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_0)}, \tag{C7}$$

where $\mathbf{k}$ is a constant wave vector; the suffix k will usually be omitted in what follows and ∇ will often be replace by $i\mathbf{k}$. Substituting (C7) into (C2) we obtain a linear relation between $\mathbf{b}$ and $\mathbf{v}$:

$$\mathbf{b} = \frac{1}{\eta k^2}(\mathbf{B}\cdot\nabla)\,\mathbf{v}. \tag{C8}$$

The Lorentz force can then be expressed as an anisotropic frictional force:

$$\mathbf{f}^b = \frac{1}{\eta k^2}(\mathbf{B}\cdot\nabla)^2\mathbf{v} = -\gamma_*\mathbf{v}, \tag{C9}$$

where

$$\gamma_* = \gamma_B \frac{k_y^2}{k^2}, \qquad \gamma_B = \frac{B^2}{\eta}. \tag{C9a, b}$$

In conditions prevailing in Earth's core, the 'coefficient of magnetic friction', γ_B, is very large. For example, if $B \sim 10^2$ G $\approx 8.5\,\mathrm{cm\,s^{-1}}$, then $\gamma_B \sim 4\times 10^{-3}\,\mathrm{s^{-1}}$, which is much greater than even the frequency Ω of the Earth's rotation, which itself is so large that Coriolis forces dominate large scale motions in the core.

Operating on (C1) by $\nabla\times$ in order to eliminate the pressure gradient, we may express $\mathbf{v}$ in terms of c:

$$\mathbf{v} = \frac{g_z\gamma_*}{\gamma_*^2 + \Omega_*^2}\frac{k_\perp^2}{k^2}\left[\mathbf{1}_z c + \frac{1}{k_\perp^2}\left(\nabla_\perp - \frac{2}{\gamma_*}\boldsymbol{\Omega}\times\nabla\right)\nabla_z c\right], \tag{C10}$$

where

$$k_\perp^2 = k_x^2 + k_y^2, \qquad \nabla_\perp = \mathbf{1}_x\nabla_x + \mathbf{1}_y\nabla_y = \nabla - \mathbf{1}_z\nabla_z, \qquad \Omega_* = 2\Omega k_z/k. \tag{C10a, b, c}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

Applying (C3) we now see that the perturbation grows exponentially as $\exp(\gamma_\alpha t)$, where

$$\gamma_\alpha = \frac{\omega_\alpha^2 \gamma_*}{\gamma_*^2 + \Omega_*^2} \frac{k_\perp^2}{k^2}, \qquad \omega_\alpha^2 = -g_z \nabla_z C. \tag{C11, 12}$$

The assumption $\nabla C = \mathbf{1}_z \nabla_z C$ is used here. The Archimedean frequency, ω_α, is the Brunt frequency, but for unstable rather than stable situations.

The growth rate γ_α is real and depends on the squares of k_x, k_y and k_z. Hence, from the elementary solutions (C7), growing cells can be constructed with sinusoidal coordinate dependence, i.e. standing rather than progressing cells. The magnitude of ω_α in the core can be estimated as gC_0/L, where $C_0 \sim 10^{-8}$. This gives $\omega_\alpha \sim 3 \times 10^{-7}\,\mathrm{s}^{-1}$ (or $2\pi/\omega_\alpha \sim 1$ yr) which is much smaller than both Ω and γ_*. If all components of $\mathbf{k}$ were of the same order of magnitude, γ_α would be of order ω_α^2/γ_B or (in the case of small k_y) of order ω_α^2/Ω, i.e. the characteristic time over which such cells would grow would be of the order of a thousand years. In other words, the magnetic frictional force and the Coriolis force both strongly suppress the growth of instabilities that are more or less isotropic. It is well known however that Coriolis forces have a much weaker influence on perturbations that are elongated in the direction of $\mathbf{\Omega}$, and it is immediately seen from (C10c) that, if $k_z \ll k$, then the $\mathbf{\Omega}_*$, that enters (C11) is much smaller than 2Ω. In an analogous way, perturbations that are elongated in the direction of $\mathbf{B}$ (i.e. those for which $k_y \ll k$) experience less magnetic friction; for these $\gamma_* \ll \gamma_B$. It follows that plate-like cells, that are elongated in the directions both of $\mathbf{\Omega}$ and of $\mathbf{B}$, are less suppressed by Coriolis and magnetic forces and will grow fastest. This conclusion was reached by Braginsky (1964b), who argued that small-scale turbulence in the core consists of a collection of such plate-like cells. This simple idea, of elongation of turbulent cells in directions parallel to both $\mathbf{\Omega}$ and $\mathbf{B}$, provides us with a plausible qualitative picture of anisotropic local turbulence.

A quantitative theory of turbulence based on these ideas is still lacking, but a 'heuristic theory' was developed by Braginsky & Meytlis (1990). They argued that, for the dominating cells, $k_y \sim k_z \ll k_x$ so $k_\perp \sim k_x \simeq k$. They concluded that the relative dimensions of the cells in the three coordinate directions is given by $\gamma_* \sim \Omega_*$, which according to (C11) maximizes the growth rate γ_*, and which implies that

$$\frac{k_y}{k_x} \sim \epsilon_*, \qquad \frac{k_z}{k_x} \sim \epsilon_*, \tag{C13a, b}$$

where

$$\epsilon_* \sim \frac{2\Omega}{\gamma_B}, \qquad \text{that is} \qquad \epsilon_* \sim \frac{B_*^2}{B^2}. \tag{C14a, b}$$

Here $B_* = \sqrt{(2\Omega\eta)}$, which corresponds to approximately 20G, is the natural scale for measuring dynamo created field strength. Considered as a function of position in the core, ϵ_* varies strongly; $\epsilon_* \sim 1/25$ might be taken as representative of the FOC as

Downloaded by [Universite Laval] at 23:08 09 July 2014

a whole. For the modes (C13a, b), (C11) may now be rewritten as

$$\gamma_\alpha \sim \gamma_{\alpha\max} \sim \omega_\alpha^2/4\Omega\epsilon_*. \tag{C11a}$$

Equation (C8) may be written, in order of magnitude, as $b \sim (k_y/k_x^2)(B/\eta)v$ and, taking into account (C13a) and $k_x = k_\perp$, we obtain

$$b/v \sim 2\Omega/Bk_\perp. \tag{C8a}$$

On the basis of qualitative arguments, Braginsky & Meytlis (1990) concluded that a typical turbulent velocity, v_z, is of order γ_α/k_z.

It will be seen that nothing in the foregoing discussion determines the absolute dimensions of the cells. This would not have been the case had we included thermal and/or viscous diffusion, but the molecular transport coefficients for these processes are so small that they are surely irrelevant even on the dimensions of the turbulent cells. It seems much more likely that the dimensions of the cells are determined by the nonlinear processes that have been omitted from this linear stability analysis. The inclusion of these processes adds considerable complexity, and so far their effects have not been quantitatively analyzed. In applying their qualitative analysis to core turbulence, Braginsky & Meytlis (1990) visualized a statistical ensemble of cells of all shapes and sizes but predominantly those having dimensions $\ell_x \sim \pi/k_x \equiv \ell_\perp$, and $\ell(\sim \ell_y \sim \pi/k_y \sim \ell_z \sim \pi/k_z)$, with characteristic velocities $v_\perp \sim v_x$ and $v \sim v_y \sim v_z$; correspondingly $b \sim b_y \sim b_z$. On the basis of heuristic arguments, Braginsky & Meytlis (1990) concluded that, if the viscosity is as small as that typically quoted for the core (see Appendix E), approximate equipartition will establish itself in local core turbulence:

$$b \sim v. \tag{C15}$$

Applying this to (C8a), we obtain $k_\perp \sim 2\Omega/B$ or $\ell_\perp \sim 2\,\mathrm{km}$; (C13a, b) give $\ell \sim \ell_\perp/\epsilon_* = 50\,\mathrm{km}$. Using $v \sim \gamma_\alpha/k_z$ (see above) and (C11a), we can estimate v. Based on the arguments adumbrated above, Braginsky & Meytlis (1990) estimated that

$$\kappa_{zz}^t \sim \ell v \sim 1\,\mathrm{m^2\,s^{-1}} \sim \eta, \qquad \kappa_{xx}^t \sim \ell_\perp v_\perp \sim \epsilon_*^2 \kappa_{zz}^t \sim 10^{-3}\,\mathrm{m^2\,s^{-1}}. \tag{C15a, b}$$

Clearly κ_{xx}^t is much smaller than κ_{zz}^t but it nevertheless greatly exceeds the molecular diffusivities $\kappa^T \sim 10^{-5}\,\mathrm{m^2\,s^{-1}}$ and $\kappa^\xi \sim 10^{-8}\,\mathrm{m^2\,s^{-1}}$. The magnetic Reynolds number, $\mathcal{R}_\ell \sim v\ell_\perp/\eta \sim \epsilon_*$, of this turbulent motion is approximately $\mathcal{R}_\ell \sim \ell_\perp/\ell \ll 1$. This justifies the neglect of $d_t\mathbf{b}$ in (C2) because $d_t b/\eta\nabla^2 b \sim \gamma_\alpha/\eta k_\perp^2 \sim k_z v/\eta k_\perp^2 \sim \mathcal{R}_\ell \ell_\perp/\ell \sim \mathcal{R}_\ell^2 \ll 1$.

It may be recalled that expression (C11) rested on the assumption that $\nabla C = \mathbf{1}_z\nabla_z C$, so that terms perpendicular to $\mathbf{1}_z$ disappear when we calculate $\mathbf{v}\cdot\nabla C$. For plate-like perturbations and for $\nabla C = \mathbf{1}_z\nabla_z C + \mathbf{1}_x\nabla_x C$ this term is of order $v_x/v_z \sim k_z/k_x$, i.e. small, and the result (C11) is still approximately correct; only the component $\nabla_z C$ of ∇C is influential in exciting the instability. The model on which the present discussion is based is therefore not quite as narrow as might at first sight appear.

Downloaded by [Universite Laval] at 23:08 09 July 2014

Assuming that turbulence in the core is of this type, it is possible to relate the entropy source, σ^S, directly to the ohmic dissipation, Q^j, of the microscale currents. It should be stressed here that, although $|\mathbf{b}| \ll |\mathbf{B}|$, it is not true that $|\nabla \times \mathbf{b}| \sim |\mathbf{b}|/\ell_\perp$ is small compared with $|\nabla \times \mathbf{B}| \sim |\mathbf{B}|/L$. Thus Q^j is not a negligibly small part of the total ohmic heating, $Q^J + Q^j$. To establish the stated relationship, we first note that the neglect of $\partial_t \mathbf{b}$ in (C2) is equivalent to assuming that the microscale electric field, $\mathbf{e}$, is dominated by its potential part, which we write here as $-\nabla[\sqrt{(\mu_0 \rho_0)}\,\varphi]$, i.e. (C2) is a consequence of Ohms law in the form

$$\eta \mathbf{j} = -\nabla\varphi + \mathbf{v} \times \mathbf{B}, \tag{C16}$$

where in present units $\mathbf{j} = \nabla \times \mathbf{b}$. It follows from this expressions that

$$\eta j^2 = \mathbf{B}\cdot(\mathbf{j} \times \mathbf{v}) - \nabla\cdot(\varphi\mathbf{j}). \tag{C17}$$

The equation of motion (C1) and (C4) give

$$c\mathbf{v}\cdot\mathbf{g} = \mathbf{B}\cdot(\mathbf{j} \times \mathbf{v}) + \nabla\cdot(p\mathbf{v}). \tag{C18}$$

The ohmic dissipation due to the microscale currents $\mathbf{j}$ is

$$Q^j = \rho_0 \eta \langle j^2 \rangle^t. \tag{C19}$$

By (C17) and (C18), we have

$$Q^j = \rho_0 \mathbf{g}\cdot\langle c\mathbf{v}\rangle^t - \langle \rho_0 \nabla\cdot(p\mathbf{v} + \varphi\mathbf{j})\rangle^t. \tag{C19a}$$

The final term vanishes when we interpret the average over the turbulent ensemble as a local average over space. We therefore have

$$Q^j = \rho_0 \mathbf{g}\cdot\langle c\mathbf{v}\rangle^t = \mathbf{g}\cdot\mathbf{I}^C, \tag{C20}$$

where

$$\mathbf{I}^C = -\alpha^S \rho_0 \langle S_c^\dagger \mathbf{v}\rangle^t - \alpha^\xi \rho_0 \langle \xi_c^\dagger \mathbf{v}\rangle^t = -\alpha^S \mathbf{I}^{S_t} - \alpha^\xi \mathbf{I}^{\xi_t}. \tag{C20a}$$

We have here used (C6), (4.26) and (4.30); see also (6.18a). It follows from (C20) and (4.37) that

$$\sigma^t = Q^j/T_a, \qquad \text{or} \qquad \sigma^t = \mathbf{g}\cdot\mathbf{I}^C/T_a. \tag{C21, 21a}$$

It should be no surprise that σ^t is given by the Joule dissipation of the electric currents associated with the turbulence. The right-hand side of (C20) is the rate of working of the gravitational force on the rising mass flux, $\mathbf{I}^C$. All this energy is dissipated locally into heat, in this case through Joule heating. If other dissipation mechanisms acted, e.g. the viscous regeneration of heat, Q^v, by the microscale motions, they would also have to be included, along with Q^j, in (C21). According to

Downloaded by [Universite Laval] at 23:08 09 July 2014

(C19) and (C15),

$$Q^v \equiv \rho_0 \nu \langle (\nabla \times \mathbf{v})^2 \rangle^t \sim (\nu/\eta)(\langle v^2 \rangle^t / \langle b^2 \rangle^t) Q^j \sim (\nu/\eta) Q^j, \tag{C22}$$

so that $Q^v \ll Q^j$ for the linear plate-like cells under consideration. If however strong nonlinearities developed in the small-scale turbulence and as a result local viscous dissipation became of the same order as the rate of working, $-\mathbf{g}\cdot\mathbf{I}^C$, of the buoyancy force, the latter would have to make good both ohmic and viscous losses, i.e. (C21) would be replaced by $\sigma^t = (Q^j + Q^v)/T_a$ but (C21a) would still be valid.

The estimation of turbulent momentum transport is not straightforward. Adopting the Reynolds analogy for

$$\pi_{ij}^{Vt} = -\langle \rho_0 v_i v_j \rangle^t, \tag{C23}$$

one might write in a first approximation

$$\pi_{ij}^{Vt} = \rho_0 \nu_{ijkl}^{Vt} \nabla_k \langle V_l \rangle^t, \tag{C24}$$

where, by (C23), $\nu_{jikl}^{Vt} = \nu_{ijkl}^{Vt}$ and, if π_{ij}^{Vt} is to vanish identically when $\langle \mathbf{V} \rangle^t$ is solid body rotation, $\nu_{ijlk}^{Vt} = \nu_{ijkl}^{Vt}$. Arguments of the type used in elementary treatments of the kinetic theory of gases suggest that

$$\nu_{xyxy}^{Vt} \sim \ell_x v_y \sim \ell_\perp v, \tag{C25}$$

with similar results for other off-diagonal elements of the "viscosity tensor" ν_{ijkl}^{Vt}. Fluctuating turbulent magnetic fields, of strength $b \sim v$, contribute terms of the same order of magnitude to (C24). The estimates (C24) and (C15a) imply that $\nu_{xyxy}^{Vt} \sim \kappa_{zz}^t \ell_x / \ell_z \sim 3 \times 10^{-2}\,\mathrm{m^2 s^{-1}}$. Although this exceeds the molecular viscosity of the core, the transfer of mean momentum by Reynolds stresses on length scales of order L is negligible in the main body of the core. Nevertheless, because it acts on large-scale motions, it is this turbulent viscosity that should be employed in the description of internal shear layers that may exist within the core, such as the shear layer surrounding the tangent cylinder that has recently been studied by Ruzmaikin (1993), Hollerbach (1994) and by Kleorin *et al.* (1995). It is therefore this turbulent velocity that should be used in computing the viscous dissipation, Q^{Vt}, in that shear layer, which might conceivably be a significant part of the total viscous dissipation Q^v in the core.

APPENDIX D: THERMODYNAMICS

This appendix has three objectives: (1) to provide a summary of thermodynamic relations needed in the main body of the paper, (2) to provide a synopsis of the derivation of (2.36)–(2.40), and (3) to set up a simple model of the core as a binary alloy.

Downloaded by [Universite Laval] at 23:08 09 July 2014

The starting point for the first objective is the internal energy per unit mass $\epsilon^I(\rho, S, \xi)$ for which

$$d\epsilon^I = \frac{p}{\rho^2}d\rho + TdS + \mu d\xi, \tag{D1}$$

and from which it therefore follows that

$$p = \rho^2\left(\frac{\partial \epsilon^I}{\partial \rho}\right)_{S,\xi}, \quad T = \left(\frac{\partial \epsilon^I}{\partial S}\right)_{\rho,\xi}, \quad \mu = \left(\frac{\partial \epsilon^I}{\partial \xi}\right)_{\rho,S}. \tag{D2, 3, 4}$$

In terms of p, T and ξ, the differentials of ρ, S and μ are

$$\rho^{-1}d\rho = (\rho u_T^2)^{-1}dp - \alpha dT - \alpha_T^\xi d\xi, \tag{D5}$$

$$dS = -(\alpha/\rho)dp + (c_p/T)dT + (h^\xi/T)d\xi, \tag{D6}$$

$$d\mu = (\alpha_T^\xi/\rho)dp - (h^\xi/T)dT + \mu_T^\xi d\xi, \tag{D7}$$

from which we have

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p,\xi} = -\rho\left(\frac{\partial S}{\partial p}\right)_{T,\xi}, \quad \alpha_T^\xi = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \xi}\right)_{p,T} = \rho\left(\frac{\partial \mu}{\partial p}\right)_{T,\xi}, \tag{D8, 9}$$

$$\mu_T^\xi = \left(\frac{\partial \mu}{\partial \xi}\right)_{p,T}, \quad c_p = T\left(\frac{\partial S}{\partial T}\right)_{p,\xi}, \quad h^\xi = T\left(\frac{\partial S}{\partial \xi}\right)_{p,T} = -T\left(\frac{\partial \mu}{\partial T}\right)_{p,\xi}. \tag{D10, 11, 12}$$

It is sometimes convenient however to use p, S and ξ as independent variables instead of ρ, S and ξ. In this case the enthalpy, $\varepsilon^H = \varepsilon^I + p/\rho$, plays the role of $\varepsilon^I(\rho, S, \xi)$ and (D5)–(D7) are replaced by

$$\rho^{-1}d\rho = (\rho u_S^2)^{-1}dp - \alpha^S dS - \alpha^\xi d\xi, \tag{D13}$$

$$dT = (\alpha^S/\rho)dp + (T/c_p)dS - (h^\xi/c_p)d\xi, \tag{D14}$$

$$d\mu = (\alpha^\xi/\rho)dp + (h^\xi/c_p)dS + \mu^\xi d\xi, \tag{D15}$$

from which it follows that

$$\mu^\xi = \left(\frac{\partial \mu}{\partial \xi}\right)_{p,S}, \quad \alpha^\xi = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \xi}\right)_{p,S} = \rho\left(\frac{\partial \mu}{\partial p}\right)_{S,\xi}, \tag{D16, 17}$$

$$\alpha^S = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial S}\right)_{p,\xi} = \rho\left(\frac{\partial T}{\partial p}\right)_{S,\xi} = \frac{\alpha T}{c_p}. \tag{D18}$$

Note that all extensive quantities (ε^I, S, etc.) are per unit mass (not per mole) and that correspondingly ξ is the mass fraction (not the molar fraction) of the light constituent.

Downloaded by [Universite Laval] at 23:08 09 July 2014

A comparison of (D5)–(D7) with (D13)–(D15) suggests that, in an analogy with the names 'thermal coefficient of volume expansion' for α, 'isothermal compositional expansion coefficient' for α_T^ξ, and 'isothermal compositional derivative of chemical potential' for μ_T^ξ, we might name α^S the 'entropy coefficient of volume expansion', α^ξ the 'adiabatic compositional expansion coefficient', and μ^ξ the 'adiabatic compositional derivative of chemical potential'. Central roles are played by α^ξ and α^S in the theory developed in this paper.

Three other useful quantities that arise are

$$c_v = T\left(\frac{\partial S}{\partial T}\right)_{\rho,\xi}, \quad u_S^2 = \left(\frac{\partial p}{\partial \rho}\right)_{S,\xi}, \quad u_T^2 = \left(\frac{\partial p}{\partial \rho}\right)_{T,\xi}. \qquad \text{(D19, 20, 21)}$$

It follows in the usual way that

$$\left(\frac{u_S}{u_T}\right)^2 = \frac{\alpha^\xi}{\alpha_T^\xi} = \frac{c_p}{c_v}, \quad c_p - c_v = \alpha^2 T u_T^2, \quad \alpha_T^\xi - \alpha^\xi = \frac{\alpha^S h^\xi}{T}. \qquad \text{(D22, 23, 24)}$$

All these results are readily generalized if the core is modeled by a multi-constituent alloy. The starting point that replaces (D1) is

$$d\varepsilon^I = \frac{p}{\rho^2}d\rho + T dS + \sum_{k=1}^{K} \mu_k d\xi_k, \qquad \text{(D1a)}$$

the summation being over all K constituents of the alloy. By the definition of ξ_k as a mass fraction, it follows that

$$\sum_{k=1}^{K} \xi_k = 1,$$

so that (D1a) may be rewritten as

$$d\varepsilon^I = \frac{p}{\rho^2}d\rho + T dS + \sum_{k=1}^{K-1} (\mu_k - \mu_K) d\xi_k. \qquad \text{(D1b)}$$

Evidently our binary alloy, of a light constituent (**L**) and a heavy constituent (**H**), concerns $K = 2$. In this case (D1b) coincides with (D1) when we define $\mu = \mu_L - \mu_H$, $\xi = \xi_L$ and $\xi_H = 1 - \xi_L$. For other K, results (D2)–(D24) may be generalized in an obvious way.

We now turn to our second objective: a summary of the facts that we need from diffusion theory for the case $K = 2$. For a more complete discussion, see chapter VI of Landau & Lifshitz (1987). They derive linear relations for the fluxes of light constituent and entropy which, in a notation that differs from theirs, are

$$\mathbf{I}^\xi = -\alpha' \nabla \mu - \beta' \nabla T, \qquad \text{(D25)}$$

$$\mathbf{I}^S = -\delta' \nabla \mu - \gamma' \nabla T. \qquad \text{(D26)}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

Onsager's reciprocity principle implies that

$$\delta' = \beta'; \tag{D27}$$

see for example Landau & Lifshitz (1980) or deGroot & Mazur (1962). It is convenient to eliminate $\nabla\mu$ from (D26) and (2.34) by writing (D25) as

$$\nabla\mu = -\frac{1}{\alpha'}\mathbf{I}^{\xi} - \frac{\beta'}{\alpha'}\nabla T. \tag{D25a}$$

It follows that

$$\mathbf{I}^S = \frac{\beta'}{\alpha'}\mathbf{I}^{\xi} - \left(\gamma' - \frac{\beta'^2}{\alpha'}\right)\nabla T = -\frac{K^T}{T}\nabla T + \frac{\beta'}{\alpha'}\mathbf{I}^{\xi}. \tag{D28}$$

The rate of entropy production, σ^S, is given by [cf. Landau and Lifshitz (1980)]

$$T\sigma^S = \frac{K^T}{T}(\nabla T)^2 + \frac{1}{\alpha'}(\mathbf{I}^{\xi})^2 + Q^v + Q^J + Q^R, \tag{D29}$$

where

$$K^T = T\left(\gamma' - \frac{\beta'^2}{\alpha'}\right). \tag{D30}$$

It is clear from (D29) that positivity of σ^S requires that α' and the thermal conductivity K^T should be non-negative.

It is convenient to introduce a different notation that eliminates α', β' and γ':

$$\kappa^{\xi} = \frac{\alpha'\mu_T^{\xi}}{\rho}, \qquad \mu' = \frac{T\beta'}{\alpha'}, \qquad k_T^{\xi} = \frac{\mu' - h^{\xi}}{\mu_T^{\xi}}, \qquad k_p^{\xi} = \frac{\alpha_T^{\xi} p}{\rho\mu_T^{\xi}}. \tag{D31, 32, 33, 34}$$

The three original transport coefficients (α', β' and γ') are apparently replaced by five, namely $K^T, \kappa^{\xi}, k_T^{\xi}, k_p^{\xi}$ and μ', but $\mu' - \mu_T^{\xi}k_T^{\xi}$ is in fact a thermodynamic property of the fluid (h^{ξ}), as is k_p^{ξ}. In this new notation, we may write (D25) and (D28) as

$$\mathbf{I}^{\xi} = -\rho\kappa^{\xi}\left(\nabla\xi + \frac{k_T^{\xi}}{T}\nabla T + \frac{k_p^{\xi}}{p}\nabla p\right), \tag{D35}$$

$$T\mathbf{I}^S = \mathbf{I}^T + \mu'\mathbf{I}^{\xi}, \qquad \mathbf{I}^T = -K^T\nabla T. \tag{D36, 37}$$

Equations (2.36)–(2.40) now follow from (2.33) and (2.34) in a straightforward way.

The ICB is a surface on which the solid and liquid core are in phase equilibrium and on which therefore

$$[\![\mu]\!] = 0, \quad [\![\varepsilon^G - \mu\xi]\!] = 0 \quad \text{on the ICB;} \tag{D38, 39}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

see for example Loper & Roberts (1978). In Section 6, we introduced two different latent heats (6.30a, b):

$$h_L = \varepsilon_2^H - \varepsilon_N^H, \quad h_N = h_L - \mu_2 \xi_{2N}. \tag{D40, 41}$$

Since T is continuous and $\varepsilon^H = \varepsilon^G + TS$, it follows from (D38)–(D41) that (D40) is equivalent to

$$h_N = T_2(S_2 - S_N). \tag{D42}$$

Finally, in setting up a simple model of the core, we should recognize at the outset that information about the physical chemistry of the core is largely non-existent. Even its composition is uncertain. It is generally agreed only that its principal constituent is iron (Fe) and that light admixtures are also present. Which element dominates those admixtures is not known, the competing merits of oxygen (O), sulfur (S), and silicon (Si) being vigorously but inconclusively argued. In view of these uncertainties, it seems justified to take a simple view of the core, in which the specific volume, ρ^{-1}, simply depends on the total amounts of heavy (H) and light (L) constituent in the core, the volume that each occupies depending only on p and T:

$$\frac{1}{\rho} = \frac{\xi}{\rho_{eL}} + \frac{1-\xi}{\rho_H}, \tag{D43}$$

where ρ_H is the density of heavy fluid (iron) and ρ_{eL}, is the effective density of the light constituent, which (because of volume changes that occur when it is dissolved in iron) differs from its actual density, ρ_{Ad}, in the absence of iron. Equation (D43) can be rewritten as

$$\frac{\rho_H}{\rho} = 1 + \xi\delta_{HL}, \quad \text{where} \quad \delta_{HL} = \frac{\rho_H}{\rho_{eL}} - 1. \tag{D44, 45}$$

By (D17) and (D44) we have[12]

$$\alpha^\xi = \rho\left(\frac{\partial \rho^{-1}}{\partial \xi}\right)_{p,S} = \frac{\rho}{\rho_H}\delta_{HL}. \tag{D46}$$

We shall assume for simplicity that ρ_H/ρ_{eL} and hence δ_{HL} are independent of p, T and ξ and are the same for solid and fluid phases. It then follows that, since ξ and S are constant in the FOC, so are α^ξ, ρ/ρ_H and the mean molecular weight of the core, $\bar{A}$, which is related to those of the heavy and light elements by

$$\frac{1}{\bar{A}} = \frac{\xi_a}{A_L} + \frac{1-\xi_a}{A_H}. \tag{D47}$$

An expression for the density jump $\Delta\rho = \rho_N - \rho_2$ at the ICB $r = R_2$ can be obtained by applying (D44) both to the FOC (where the density of iron is ρ_H and the admixture

[12] We ignore here the difference between α^ξ and α^ξ_T. Taking from Appendix E $\alpha_2^S = 6.28 \times 10^{-5}\,\text{kg J}^{-1}\,°\text{K}$, $h^\xi = -0.5 \times 10^7\,\text{J kg}^{-1}$, $T_2 = 5{,}300\,°\text{K}$ and $\alpha^\xi \sim 0.6$, we see from (D24) that $\alpha^\xi - \alpha^\xi_T = -h^\xi\alpha^S/T \sim 6 \times 10^{-2} \sim 0.1 \times \alpha^\xi$ at $r = r_2$.

Downloaded by [Universite Laval] at 23:08 09 July 2014

concentration is $\xi_2 = \xi_a$) and to the SIC (where they are ρ_H^S and ξ_N). After some algebra, we obtain

$$\Delta\rho = \Delta_s\rho + \Delta_\xi\rho, \tag{D48}$$

where $\Delta_s\rho$ is the density jump through solidification and $\Delta_\xi\rho$ is the density jump arising from the difference in ξ between liquidus and solidus:

$$\Delta_s\rho = \rho_N(1 - \rho_H/\rho_H^S), \quad \Delta_\xi\rho = \rho_N(\rho_H/\rho_H^S)\alpha^\xi\xi_{2N}. \tag{D48a, b}$$

Here $1 - \rho_H/\rho_H^S \sim 10^{-2}$ is very small, and we shall therefore replace (D48b) by

$$\Delta_\xi\rho = \rho_N\alpha^\xi\xi_{2N}. \tag{D48c}$$

We now recall the *rejection factor*, so called because it quantifies the amount of light constituent rejected by the solid when the fluid freezes onto the SIC:

$$r_{FS} \equiv \frac{\xi_2 - \xi_N}{\xi_2}. \tag{D49}$$

Dividing (D48c) by $\rho_N\alpha^\xi\xi_2$ making use of (D44), (D46) and (D48), we find that

$$r_{FS} = \frac{(\Delta\rho - \Delta_s\rho)/\rho_N}{1 - \rho/\rho_H}. \tag{D50}$$

[It is not necessary to specify where the denominator in (D50) is to be evaluated, since ρ/ρ_H is the same everywhere in the FOC in our model.] The right-hand side of (D50) does not depend on any special property of the alloy, such as δ_{HL}, ξ_a, or even its composition! It is a potentially useful method of discriminating between the rival claims of Ad = O, S and Si. If the phase diagrams of FeAd for these three elements could be measured at megabar pressures, it would be discovered which of the resulting values of r_{FS} agreed best with the value deduced from (D50); see Appendix E. That element might then be considered to be the most likely light constituent predominating in the core.

To make the model defined by (D44) definite, we must estimate two of the three unknowns δ_{HL}, ξ_a and ρ/ρ_H, the third then following from (D44). We know ρ from Earth models such as PREM and ρ_H from shock wave experiments on iron; we can therefore find ρ/ρ_H; see Appendix E. The final datum is, in the case of Si, obtained by estimating δ_{HL} from laboratory experiments at normal pressures; in the case of S, it is the ξ_a implied by a conjecture by Boness and Brown (1990); in the case of O, we adopt a value for δ_{HL} derived by Loper (1978).

APPENDIX E: NUMERICAL VALUES

Many parameters play parts, of varying degrees of importance, in determining how the core behaves. These parameters can be conveniently thought of as falling into four categories: geometrical, thermodynamic, physico-chemical and electromagnetic.

Downloaded by [Universite Laval] at 23:08 09 July 2014

Those accessible to seismology are the best determined. They have already been listed in Table 1 in Section 3.

The temperature of the core is known with much less precision. It is constrained by the condition $T_2 = T_m(p_2, \xi_a)$, which expresses the fact that the ICB is at the melting temperature corresponding to pressure p_2 and composition ξ_a, but the functional relationship $T_m = T_m(p, \xi)$ is unknown. This is hardly surprising while uncertainty persists about which light element predominates amongst the light admixtures of the iron alloy, the competing merits of S, Si and O being variously, but inconclusively, argued. There is even current uncertainty about what T_m is for pure iron at core pressures, different experimental techniques leading to significantly different T_m. The present situation is described, and the existing contradictions are analyzed, by Anderson (1994, 1995). Direct static measurements of the melting point of iron have been performed by Boehler (1993) up to pressures of 2 Mbar. He also obtained, by extrapolation to $p_2 = 3.3$ Mbar, an ICB temperature of $T_2 \equiv T_m(p_2) = 4850°$K. In contrast, shock wave measurements of T_m at pressures near 3.3 Mbar have been made by many authors and have generally shown a markedly greater value of T_m. According to Anderson (1994), the reason for the difference may be the existence of a phase transition to a new, high pressure, fcc form of iron, with a triple point near 2 Mbar, and which deflects the melting curve, $T = T_m(p)$, upwards. Until uncertainty is removed, Anderson (1994) recommends that the value $T_m(p_2) = 6000°$K be adopted for pure iron. The theoretical calculations of Poirier & Shankland (1993) of the melting point of fcc iron at 3.3 Mbar gave 6060°K, and they further suggested a rather large depression of the melting point, namely 500–1000°K, through the presence of the alloying elements. Anderson (private communication) also recommended to us that we should suppose that this depression is of order 700°K, so implying that $T_2 = 5300°$K. In contrast, Boehler (1993), who measured T_m for an Fe–O alloy, found that the depression of the melting point due to oxygen is very small, at least up to pressures of 2 Mbar, implying that $T_2 = 4850°$K.

Estimates of γ, α, c_p and other thermodynamic properties of Earth's core have been made by many authors. Convenient tabulations have been provided by Stacey (1992), who gave for example $\gamma_1 \equiv \gamma(R_1) = 1.27$ and $\gamma_2 \equiv \gamma(R_2) = 1.44$. In later work (Stacey, 1994), he modified several of his estimates, and in particular took, as we shall, $\gamma_1 = 1.27$ and $\gamma_2 = 1.35$. This smallness of the reduction ($\sim 6\%$) in γ_2 underscores the robustness of this parameter in studies of Earth's interior. As (3.7e,f) show, the Grüneisen constant is in fact the only thermodynamic parameter needed when estimating the adiabatic temperature gradient, $-\nabla_r T_a(r)$. Given the temperature $T_2 = 5300°$K of the CMB, T_a follows throughout the core by integration in r. We performed this integration in a simple (and perhaps simplistic) way. By (3.7e, f) we have

$$-\frac{1}{T_a}\frac{dT_a}{dr} = \frac{\gamma g}{u_S^2} = \frac{2\varphi}{R_2}, \qquad \text{say.} \tag{E1}$$

Since g is roughly proportional to r, we represented the right-hand side of (E1) by the following simple interpolant:

Downloaded by [Universite Laval] at 23:08 09 July 2014

$$\varphi(x) \equiv \frac{\gamma g R_2}{2u_S^2} = x[a_0 + a_1 x^2], \qquad x = \frac{r}{R_2}, \tag{E2a, b}$$

where $a_0 = 0.029619$ and $a_1 = 0.002207$ are constants chosen so that (E2a) agrees with the φ_1 and φ_2 implied by the values of γ_1 and γ_2 quoted above. From (E1) and (E2a, b) it follows that

$$T = T_2 \exp\{-(x^2 - 1)[a_0 + \tfrac{1}{2}a_1(x^2 + 1)]\}. \tag{E3}$$

This gives $T_1 = 4000°K$. The adiabatic gradient at the CMB is then $0.89°K\,km^{-1}$; at the ICB it is $0.276°K\,km^{-1}$. The 'average' temperature (6.25b) of the FOC implied by this model is $T_0 = 4590°K$.

The shock Hugoniot for pure iron, as determined experimentally by Brown and McQueen (1986), intersects the pT–curve implied by (E3) and PREM, at a radius of approximately 2780 km within the FOC, where $p = 204$ GPa, T = 4520°K and the core density is $\rho = 10.84\,10^3\ kg\,m^{-3}$. The shock data for this p and T gives the corresponding ρ_H as $11.98\,10^3\,kg\,m^{-3}$, so that

$$\rho_H/\rho = 1.1057, \tag{E4}$$

which is the same throughout the core according to model (D44).

We already noted in Appendix D that the density discontinuity, $\nabla\rho$ at the ICB is composed of two parts: $\Delta_s\rho$ due to contraction on solidification, and $\Delta_\xi\rho$ due to the difference in liquidus and solidus compositions, the former of which is comparatively small. Taking $\Delta_s\rho = 0.1 10^3\,kg\,m^{-3}$ so that $\Delta_\xi\rho = \Delta\rho - \Delta_s\rho \approx 0.5 \times 10^3\,kg\,m^{-3}$, we find from (D50) that the rejection coefficient, ρ_{FS}, is approximately 0.41. Because the difference between the atomic radii of O and Fe is probably large under core conditions, we would expect that O would have a low solubility in solid Fe and that the rejection factor, r_{FS}, and the density discontinuity, $\Delta\rho$, would be rather larger if O were the principal alloying constituent. Correspondingly, the small difference between the atomic radii of Fe, S, and Si (see below) suggests that, r_{FS} and $\Delta\rho$ should be small (though not zero) for Ad = S or Si. Our estimate of $r_{FS} \sim 41\%$ leads us to favor Ad = S and Si over Ad = O. We now consider how δ_{HL} and ξ_a can be estimated for Ad = O, S and Si.

Boness and Brown (1990) noted that the (Wigner–Seitz) atomic radii of Fe and S are close to one another when p is between 100 and 350 Gpa and that these elements can therefore readily form solid–solution alloys. They used this fact to model FeS mixtures under core conditions. They found, from very detailed quantum mechanical calculations, that the dependence of ρ on p for Fe_3S is (for core temperatures) rather close to that of the PREM model of Dziewonski and Anderson (1981). While conceding that it is somewhat speculative to do so, we adopt the Fe_3S model of Boness and Brown (1990) as the basis for our estimates for Ad = S, i.e. we take $\xi = A_S/(A_S + 3A_{Fe}) = 0.16$ which by (D44) and (E4) implies that $\delta_{HL} = 0.66$.

Matassov (1977) measured the density, $\rho(\xi)$, FeSi at atmospheric pressure over a range of ξ expected to cover core compositions. Although he found that the specific

Downloaded by [Universite Laval] at 23:08 09 July 2014

Downloaded by [Universite Laval] at 23:08 09 July 2014

volume depends nonlinearly on ξ, we found that his data is reasonably well fitted by (D44) with $\delta_{HL} = 0.68$ and $\rho_H = 7.87 \times 10^3\,\mathrm{kg\,m^{-3}}$, the density of iron at NTP. This value of δ_{HL} was self-consistently determined so that the implied ρ falls onto the relevant segment of the $\xi\rho$-plot shown in Figure 4.10 of Matassov (1977) which it did at $\xi = 15.75\%$. Also $\rho^0_{eL} = \rho^0_H/(1+\delta_{HL}) = 4.68 \times 10^3\,\mathrm{kg\,m^{-3}}$. This is significantly greater than the density of solid crystalline silicon, which is $\rho^0_{Si} = 2.42 \times 10^3\,\mathrm{kg\,m^{-3}}$. This underscores the importance of allowing for the effects of chemical interactions when modeling alloys.

For Ad = O, Loper (1978) gave $\rho^0_{eL} = 4.4 \times 10^3\,\mathrm{kg\,m^{-3}}$ and $\rho^0_H = 8.57 \times 10^3\,\mathrm{kg\,m^{-3}}$. The former was obtained on the basis of laboratory data; the latter resulted from correcting the density $\rho_H = 7.87 \times 10^3\,\mathrm{kg\,m^{-3}}$ quoted above to allow for close packing at high pressure. It follows from (D45) that $\delta_{HL} = 0.95$, and then (D44) and (E4) give $\xi_a = 14.1\%$.

Our values for δ_{HL} for Ad = S, Si and O led via (D46), D47) and (D49) to the values for α^ξ, $\bar{A}$ and $\xi_2 - \xi_N$ shown in Table E1. Our estimates of the mean molecular weight Ad = Si and for Ad = S are close to the value, $\bar{A} = 48.1$, to which Stacey (1994) was led from other considerations, while that for O is somewhat smaller. Despite being hampered by a serious lack of information about the effects of high pressure on the phase diagrams of FeAd where Ad = S, Si or O, we are favorably impressed by the consistency the Ad = Si and Ad = S models present. While conceding that S is, according to the arguments given here, an equally plausible candidate for Ad, we shall nevertheless concentrate on an FeSi core below.

Table E1 Compositional Parameters

Ad	A	δ_{HL}	α^ξ	$\xi_a(\%)$	$\xi_a - \xi_N(\%)$	$\bar{A}$
S	32.07	0.66	0.60	16.0	6.6	49.9
Si	28.09	0.68	0.61	15.8	6.5	48.3
O	16.00	0.95	0.84	14.1	5.8	41.3

Stacey (1994) recently revised his earlier estimates (Stacey 1992) of the specific heats. He argued that the electronic contribution to these had previously been under-estimated, and he proposed that $c_v \approx 4.5\,R$ per mole, where $R = 8314\,\mathrm{J\,mole^{-1}\,{}^\circ K^{-1}}$ is the gas constant. For iron ($A_{Fe} = 55.85$), we therefore have $c_v^{Fe} = 670\,\mathrm{J\,kg^{-1}\,{}^\circ K^{-1}}$, and more generally for a material of mean atomic weight $\bar{A}$, we have

$$c_v = c_v^{Fe}\frac{A^{Fe}}{\bar{A}} = 670\frac{A^{Fe}}{\bar{A}}\,\mathrm{J\,kg^{-1}\,{}^\circ K}. \tag{E5}$$

Since

$$c_p = c_v(1 + \gamma\alpha T), \tag{E6}$$

we have, after using (3.7f) to eliminate α from (E6),

$$c_p = \frac{c_v^{Fe}}{(\bar{A}/A_{Fe}) - \gamma^2 c_v^{Fe} T/u_S^2}, \tag{E7}$$

so that, substituting values from Table 1 and adopting the values of γ and T at the CMB and ICB given earlier, we obtain

$$c_{p1} = \frac{670\,\text{J kg}^{-1}{}^\circ\text{K}^{-1}}{(\bar{A}/A_{\text{Fe}}) - 0.0751}, \qquad c_{p2} = \frac{670\,\text{J kg}^{-1}{}^\circ\text{K}^{-1}}{(\bar{A}/A_{\text{Fe}}) - 0.0534}. \tag{E7a, b}$$

Choosing $\bar{A} = 48.3$ (see above), we find the values of c_{p1} and c_{p2} shown in Table E2. These may be compared with the recent values of Stacey (1994): $c_{p1} = 845\,\text{J kg}^{-1}{}^\circ\text{K}^{-1}$, $c_{p2} = 826\,\text{J kg}^{-1}{}^\circ\text{K}^{-1}$. If we took S instead of Si, we would have $c_{p1} = 819\,\text{J kg}^{-1}{}^\circ\text{K}^{-1}$ and $c_{p2} = 798\,\text{J kg}^{-1}{}^\circ\text{K}^{-1}$, whereas O would give $c_{p1} = 1006\,\text{J kg}^{-1}{}^\circ\text{K}^{-1}$ and $c_{p2} = 974\,\text{J kg}^{-1}{}^\circ\text{K}^{-1}$. Since γ is so robust, any change in c_p implies, according to (3.7f), a corresponding revision in the coefficient of volume expansion, α. We obtain (for $\bar{A} = 48.3$) the values shown in Table E2, which are close to those of Stacey (1994): $\alpha_1 = 1.68 \times 10^{-5}{}^\circ\text{K}^{-1}$, $\alpha_2 = 1.00 \times 10^{-5}{}^\circ\text{K}^{-1}$; Anderson (1994) gives $\alpha_1 = 1.62 \times 10^{-5}{}^\circ\text{K}^{-1}$, $\alpha_2 = 0.83 \times 10^{-5}{}^\circ\text{K}^{-1}$.

Table E2 Values of some Thermodynamic Parameters of the Core

T_1	$= 4000^\circ$K	Temperature of the CMB,
T_2	$= 5300^\circ$K	Temperature of the ICB,
T_0	$= 4590^\circ$K	Average temperature of the FOC,
ΔT_{12}	$\equiv T_2 - T_1 = 1300^\circ$K	Temperature contrast across the FOC,
γ_1	$= 1.35$	Grüneisen constant at the CMB,
γ_2	$= 1.27$	Grüneisen constant at the ICB,
α_1	$= 1.76 \times 10^{-5}{}^\circ\text{K}^{-1}$	Thermal coefficient of volume expansion at the CMB,
α_2	$= 0.98 \times 10^{-5}{}^\circ\text{K}^{-1}$	Thermal coefficient of volume expansion at the ICB,
c_{p1}	$= 848\,\text{J kg}^{-1}{}^\circ\text{K}^{-1}$	Specific heat at constant pressure at the CMB,
c_{p2}	$= 826\,\text{J kg}^{-1}{}^\circ\text{K}^{-1}$	Specific heat at constant pressure at the ICB,
α_1^S	$= 8.30 \times 10^{-5}\,\text{kg J}^{-1}{}^\circ\text{K}$,	Entropy coefficient of volume expansion at CMB,
α_2^S	$= 6.28 \times 10^{-5}\,\text{kg J}^{-1}{}^\circ\text{K}$,	Entropy coefficient of volume expansion at ICB,

Equations (D8)–(D12) and (D16)–(D24) contains a number of further physico-chemical parameters whose values depend on pressure and on the specific admixture involved. They are unknown, and can be estimated only very roughly. The value $h^\xi \sim -10^7\,\text{J kg}^{-1}$ was given by Gubbins *et al.* (1979) for FeS. According to Kubaschewski & Alcock (1979), $h^\xi \sim -3 \times 10^6\,\text{J kg}^{-1}$ for Si at NTP, while for O it exceeds $10^7\,\text{J kg}^{-1}$. We have adopted $h^\xi \sim -5 \times 10^6\,\text{J kg}^{-1}$. We should emphasize that this value is very uncertain, and its dependence on pressure is unknown. The situation is scarcely better for h_N and h_L. We assume that $h_N \sim h_L$. We follow Gubbins *et al.* (1979) by taking $h_L = 10^6\,\text{J kg}^{-1}$. This value falls into the interval of uncertainty, $0.8 \times 10^6\,\text{J kg}^{-1} < h_L < 1.5 \times 10^6\,\text{J kg}^{-1}$, given by Anderson & Young (1988) for pure iron.

The kinetic coefficient in which we are most interested is the magnetic diffusivity, η. Its value is important for dynamo theory. The electrical resistivity of metals arises from the scattering of the conduction electrons by thermal oscillations of the ions (phonons) and impurities (admixtures). The first of these processes introduces a linear increase of η with T, although increasing p tends to offset this. These dependencies are considerably complicated by phase transitions and by structural and chemical changes.

Measurements of electrical conductivity at high pressure were made by Keeler & Mitchell (1986) for Fe and by Matassov (1977) for FeSi. Matassov also includes the

Downloaded by [Universite Laval] at 23:08 09 July 2014

results Keeler and Mitchell for pure iron. The measurements were made by shock wave techniques, so that an increase in p is accompanied by the increase in T along the Hugoniot. Matassov (1977) found that an increase in the Si content decreases the conductivity. He also found that, for a geophysically relevant concentration of Si, namely 25% atomic or $\xi = 14.4\%$ (which is rather close to our estimate; see Table E1), the change in conductivity produced by an increase in pressure from 521 kbar to 2518 kbar and by an increase in temperature from 672°K to 2518°K nearly compensated each other. For $p = 1422$ kbar, which is scarcely more than p_1, the conductivity was found to be $8.7 \times 10^5\,\mathrm{Sm}^{-1}$. To adapt this to core conditions, we should make a reduction because of the greater temperature, $T_1 \approx 4000$°K, of the CMB, and also decrease it further because the experimental specimens were solid, not fluid like the core; melting always reduces the conductivity. While these corrections are somewhat uncertain, a conductivity of $4 \times 10^5\,\mathrm{S\,m}^{-1}$ seems quite plausible, corresponding to $\eta = 2\,\mathrm{m}^2\mathrm{s}^{-1}$. Because the increases in pressure and temperature with depth in the core tend to change the conductivity in opposing ways, we anticipate that η does not vary strongly across the FOC. It may be anticipated however that the conductivity of the SIC will be greater than that of the FOC for three reasons: solids conduct better than fluids, p increases with depth in the SIC but the increase in T is slight, and the SIC contains smaller amounts of impurities (the admixtures) that reduce its conductivity. According to the data of Keeler and Mitchell, as cited by Matassov (1977), the electrical conductivity of pure Fe is, at the same p and T, approximately twice that of FeSi at 14.4%. Having no other experimental data available, we shall suppose that this result its typical, and shall assume that $\eta = 2\,\mathrm{m}^2\mathrm{s}^{-1}$ in the FOC but that $\eta_N = 1.5\,\mathrm{m}^2\mathrm{s}^{-1}$ in the SIC.

The thermal conductivity of the core can be estimated through its electrical conductivity by using the Wiedemann-Franz law, $K^T = k_L T/\eta$, where $k_L = 0.02\,\mathrm{W\,m\,s}^{-1}{}^\circ\mathrm{K}^{-2}$ is the Lorentz constant (appropriately modified because η has been used, instead of the more usual electrical conductivity, in the expression for K^T). Taking $\eta = 2\,\mathrm{m}^2\mathrm{s}^{-1}$ and $T_1 = 4 \times 10^{3}$°K, we find that $K_1^T = 40\,\mathrm{W\,m}^{-1}{}^\circ\mathrm{K}^{-1}$ and $\kappa_1^T = 5.7 \times 10^{-6}\,\mathrm{m}^2\mathrm{s}^{-1}$. This led to the estimate of $\mathcal{Q}_1^T = 5.4 \times 10^{12}$ W for the heat flux out of the core down the adiabat. This is about 8 times less than the heat flux through the surface of Earth, namely 42×10^{12} W, according to Pollack *et al.* (1993).

The measured viscosity of liquid iron at atmospheric pressure is $\nu \sim 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$. Increasing T decreases ν, while increases in p tend to increase ν because larger p makes the relative displacement of atoms more difficult. The estimate of core viscosity most often cited, namely $\nu \sim 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$ (Gans, 1972), stems from a statement made in the modern theory of fluids: fluid viscosity does not change along the melting curve, $T = T_m(p)$. And the whole fluid core is near (though above) the melting temperature. Poirier (1988) gives $\nu_1 = 3 \times 10^{-7}\,\mathrm{m}^2\,\mathrm{s}^{-1}$ and $\nu_2 = 6 \times 10^{-7}\,\mathrm{m}^2\,\mathrm{s}^{-1}$. These values, which we adopt here, should nevertheless be used with caution. No experimental measurements of ν have yet been made at core pressures, and the theory of fluids, from which the constancy of ν on the melting curve was inferred, contains a number of strongly simplfying assumptions. It is not impossible that, at the megabar pressures prevailing in the FOC, a large increase in ν with depth occurs.

The viscosity of a fluid is related to its compositional diffusivity by a relation of the form

$$\nu\kappa^{\xi} \sim k_B T/6\pi\rho a, \tag{E8}$$

Downloaded by [Universite Laval] at 23:08 09 July 2014

where $k_B = 1.38 \times 10^{-23}\,\mathrm{J\,{}^\circ K^{-1}}$ is Boltzmann's constant and a is of the order of a few inter-atomic distances (see Frenkel, 1958). If ν is the same as at atmospheric pressure, then according to (E8) laboratory measurements of κ^ξ are applicable to the core. On this basis and using laboratory measurements of the diffusion of S and Si, Loper & Roberts (1983) made the estimate $\kappa^\xi \sim 3 \times 10^{-9}\,\mathrm{m^2s^{-1}}$. Poirier (1988) gave $\kappa^\xi \sim 6 \times 10^{-9}\,\mathrm{m^2s^{-1}}$. This diffusivity might be even smaller if the kinematic viscosity is greater than $10^{-6}\,\mathrm{m^2s^{-1}}$. We will suppose that κ^ξ lies between $10^{-9}\,\mathrm{m^2s^{-1}}$ and $10^{-8}\,\mathrm{m^2s^{-1}}$, but both of these values are so small that molecular diffusion of composition may be safely ignored except over very small distances. For example, taking $\kappa^\xi \sim 3 \times 10^{-9}\,\mathrm{m^2s^{-1}}$, we find that the characteristic diffusion distance, $(\kappa^\xi t)^{1/2}$ over a time-interval of $t = 10^3$ yr is only about 10 m. This weak molecular diffusion is completely insignificant, compared with turbulent mixing.

We conclude this Appendix with the argument that let us to adopt 0.05 as an estimate of Δ_2. Referring to (6.37d) and using values given in Tables 1 and E2, we see that $g_2\gamma_2 R_2/u_S^2 \approx 0.064$. According to Lindemann's law,

$$T_m^{-1}\,\partial T_m/\partial p = 2(\gamma - \tfrac{1}{3})K_T^{-1}, \tag{E9}$$

where K_T is the isothermal incompressibility which may, with an error of less than 10%, be taken to be the adiabatic incompressibility, $K_S = \rho u_S^2$. This, by (3.7e, f), is also $\gamma T(\partial p/\partial T)_{S,\xi}$. It follows that

$$\frac{dT_m/dp_2}{dT_a/dp_2} - 1 = 1 - \frac{2}{3\gamma}, \tag{E10}$$

but we should recognize that, depending as it does on the difference between two gradients, estimates made on the basis of (E10) are unlikely to be robust. Taking $\gamma_2 = 1.27$ (see Table E2), we find from (6.37d) that $\Delta_{ma} \sim 0.03$. The estimation of $\Delta_{m\xi}$ from (6.41) is even more uncertain. The pre-factor, $3\mathscr{V}_2\rho_N/\mathscr{V}_{12}\rho_0$, is 0.159. Taking $\xi_{2N} = 0.06$ and $h^\xi = -5 \times 10^6\,\mathrm{J\,kg^{-1}}$, we find that the first term in square brackets, $\xi_2 h^\xi/c_p T_a$, is approximately -0.067. The second term depends on the unknown physical chemistry of the core, and is very hard to estimate reliably. Taking $(\partial T_m/\partial \xi_a)_p \sim -\Delta T_m/\xi_a$, where $\Delta T_m \sim 700^\circ$K is the depression of the melting temperature through the alloying elements, we find that the second term is about -0.05. In total $\Delta_{m\xi} \sim 0.02$ and, by (6.40), $\Delta_2 \sim 0.05$.

Note added in Proof:

As stated below (2.8), we have not considered the effects of variable rotation and have ignored the Poincaré force, $-\rho\dot{\mathbf{\Omega}} \times \mathbf{r}$, that strictly should be present on the right-hand side of (2.1). We wish to draw attention here to a recently published review (Malkus, 1994) that discusses the dynamical implications of a varying $\mathbf{\Omega}$, especially for the luni–solar precession, and that considers the consequences for core energetics.

Malkus, W.V.R., "Energy sources for planetary dynamos," in: *Lectures on Solar and Planetary Dynamos*, (Eds. M.R.E. Proctor and A.D. Gilbert). Cambridge UK: University Press, pp. 161–179 (1994).

Downloaded by [Universite Laval] at 23:08 09 July 2014