PATRICK SUPPES

IS VISUAL SPACE EUCLIDEAN?

Philosophers of past times have claimed that the answer to the question, Is visual space Euclidean?, can be answered by *a priori* or purely philosophical methods. Today such a view is presumably held only in remote philosophical backwaters. It would be generally agreed that one way or another the answer is surely empirical, but the answer might be empirical for indirect reasons. It could be decided by physical arguments that physical space is Euclidean and then by conceptual arguments about perception that necessarily the visual space must be Euclidean. To some extent this must be the view of many laymen who accept that to a high degree of approximation physical space is Euclidean, and therefore automatically hold the view that visual space is Euclidean.

I begin with the question, How do we test the proposition that visual space is Euclidean? The first section is devoted to this problem of methodology. The second section provides a brief overview of the hierarchy of geometries relevant to visual phenomena. The third section reviews a number of answers that have been given to the question of the Euclidean character of visual space. I examine both philosophical and psychological claims. The final section is devoted to central issues raised by the variety of answers that have been given.

I. HOW TO APPROACH THE QUESTION

What would seem to be, in many ways, the most natural mathematical approach to the question has also been the method most used experimentally. It consists of considering a finite set of points. Experimentally, the points are approximated by small point sources of light of low illumination intensity, displayed in a darkened room. The intuitive idea of the setting is to make only a finite number of point-light sources visible and to make these light sources of sufficiently low intensity to exclude illumination of the surroundings. The second step is to ask the person making visual judgments to state whether certain geometrical relations hold

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between the points. For example, are points a and b the same distance from each other as points c and d? (Hereafter in this discussion I shall refer to points but it should be understood that I have in mind the physical realization in terms of point-light sources.) Another kind of question might be, Is the angle formed by points abc congruent or equal in measure to the angle formed by points def?

Another approach to such judgments is not to ask whether given points have a certain relation but rather to permit the individual making the judgments to manipulate some of the points. For example, first fix points a, b and c and then ask him to adjust d so that the distance between c and d is the same as the distance between a and b. Although the formulation I am giving of these questions sounds as if they might be metric in character, they are ordinarily of a qualitative nature – for example, that of congruence of segments, which I formulated as same distance. No metric requirements are imposed upon the individuals making such judgments. For instance, no one would naturally ask subjects in the experiments relevant to our question to set the distance between two points to be approximately 1.3 meters or to determine an angle of, say, 21 degrees.

Once such judgments are obtained, whether on the basis of fixed relations or by adjusting the position of points, the formal or mathematical question to ask is whether the finite relational structure can be embedded in a two- or three-dimensional Euclidean space. The dimensionality depends upon the character of the experiment. In many cases the points will be restricted to a plane and therefore embedding in two dimensions is required; in other cases embedding in three dimensions is appropriate. By a *finite relational structure* I mean a relational structure whose domain is finite. To give a simple example, suppose that A is the finite set of points and the judgments we have asked for are judgments of equidistance of points. Let E be the quaternary relation of equidistance. Then to say that the finite relational structure $\mathfrak{A} = \langle A, E\rangle$ can be embedded in three-dimensional Euclidean space is to say that there exists a function φ defined on A such that φ maps A into the set of triples of real numbers and such that for every a, b, c, and d in A the following relation holds:

$$ab\,E\,cd \quad \text{iff} \quad \sum_{i=1}^{3} (\varphi_i(a) - \varphi_i(b))^2 = \sum_{i=1}^{3} (\varphi_i(c) - \varphi_i(d))^2,$$

where $\varphi_i(a)$ is the ith coordinate of $\varphi(a)$.

Note that the mapping into triples of real numbers is just mapping visual points into a Cartesian representation of three-dimensional Euclidean space.

In principle, it is straightforward to answer the question raised by this embedding procedure. So that, given a set of data from an individual's visual judgments of equidistance between points, we can determine in a definite and constructive mathematical manner whether such embedding is possible.

Immediately, however, a problem arises. This problem can be grasped by considering the analogous physical situation. Suppose we are making observations of the stars and want to test a similar proposition, or some more complex proposition of celestial mechanics. We are faced with the problem recognized early in the history of astronomy, and also in the history of geodetic surveys, that the data are bound not to fit the theoretical model exactly. The classical way of putting this is that errors of measurement arise, and our problem is to determine if the model fits the data within the limits of the error of measurement. In examining data on the advancement of the perihelion of Mercury, which is one of the important tests of Einstein's general theory of relativity, the most tedious and difficult aspect of the data analysis is to determine whether the theory and the observations are in agreement within the estimated error of measurement.

Laplace, for example, used such methods with unparalleled success. He would examine data from some particular aspect of the solar system, for example, irregularities in the motion of Jupiter and Saturn, and would then raise the question of whether these observed irregularities were due to errors of measurement or to the existence of 'constant' causes. When the irregularities were too great to be accounted for by errors of measurement, he then searched for a constant cause to explain the deviations from the simpler model of the phenomena. In the case mentioned, the irregularities in the motion of Jupiter and Saturn, he was able to explain them as being due to the mutual gravitational attraction of the two planets, which had been ignored in the simple theory of their motion. But Laplace's situation is different from the present one in the following important respect. The data he was examining were already rendered in quantitative form and there was no question of having a numerical representation. Our problem is that we start from qualitative judgments and we are faced with the problem of simultaneously assigning a measurement and determining the error of that measurement. Because of the complexity and sublety of the statistical questions concerning errors of measurement in the present setting, for purposes of simplification I shall ignore them, but it is absolutely essential to recognize that they must be dealt with in any detailed analysis of experimental data.

Returning to the formal problem of embedding qualitative relations among a finite set of points into a given space, it is surprising to find that the results of the kinds that are needed in the present context are not really present in the enormous mathematical literature on geometry. There is a very large literature on finite geometries; for example, Dembowski (1968) contains over 1200 references. Moreover, the tradition of considering finite geometries goes back at least to the beginning of this century. Construction of such geometries by Veblen and others was a fruitful source of models for proving independence of axioms, etc. On the other hand, the literature that culminates in Dembowski's magisterial survey consists almost entirely of projective and affine geometries that have a relatively weak structure. From a mathematical standpoint, such structures have been of considerable interest in connection with a variety of problems in abstract algebra. The corresponding theory of finite geometries of a stronger type, for example, finite Euclidean, finite elliptic, or finite hyperbolic geometries, is scarcely developed at all. As a result, the experimental literature does not deal directly with such finite geometries, although they are a natural extension of the weaker finite geometries on the one hand and finite measurement structures on the other.

A second basic methodological approach to the geometrical character of visual space is to assume that a standard metric representation already exists and then to examine which kind of space best fits the data. An excellent example of this methodology is to be found in various publications of Foley (1965, 1972). Foley shows experimentally that the size-distance invariance hypothesis, which asserts that the perceived size-distance ratio is equal to the physical size-distance ratio, is grossly incorrect. At the same time he also shows that perceived visual angles are about ten percent greater than physical angles. These studies are conducted on the assumption that a variety of more primitive and elementary axioms are satisfied. In contrast, Luneburg (1948) assumes that the perceived visual angle equals the physical angle, that is, that the transformation between the two is conformal, but what is back of the use of this assumption is a whole variety of assumptions that both physical space and visual space are homogeneous spaces of constant curvature, that is, are Riemannian spaces, and essentially Luneburg does not propose to test in any serious way the many consequences implied by this very rich assumption of having a homogeneous space with constant curvature. In other words, in this second approach there is no serious attempt to provide tests that will show if all of the axioms that hold for a given type of space are satisfied.

A third approach is to go back to the well-known Helmholtz-Lie problem of the nature of space and to replace finiteness by questions of continuity and motion. In a famous lecture of 1854, Riemann (1854/1892) discussed the hypotheses on which the foundations of geometry lie. More than a decade later, Helmholtz (1868) responded in a paper entitled 'Über die Tatsachen, die der Geometrie zugrunde liegen'. The basic argument of Helmholtz's paper was that, although arbitrary Riemannian spaces are conceivable, actual physical space has as an essential feature the free mobility of rigid bodies. From a mathematical standpoint, such motions are characterized in metric geometry as transformations of a space onto itself that preserve distances. Such transformations are called *isometries*. Because of the extensive mathematical development of the topic (for modern review, see Busemann, 1955, Section 48, or Freudenthal, 1965), an excellent body of formal results is available as tools to be used in the investigation of the character of visual space. Under various axiomatizations of the Helmholtz-Lie approach it can be proved that the only spaces satisfying the axioms are the following three kinds of elementary spaces: Euclidean, hyperbolic, and spherical.

From a philosophical standpoint, it is important to recognize that considerations of continuity and motion are probably more fundamental in the analysis of the nature of visual space than the mathematically more elementary properties of finite spaces. Unfortunately, I am not able to report any experimental literature that uses the Helmholtz-Lie approach as a way of investigating the nature of visual space, although it is implicit in some of the results reported below that it would be difficult to interpret the experimental results as satisfying an axiom of free mobility. Let me be clear on this point. Some of the experimental investigations lead to the result that visual space cannot be elementary in the sense just defined, but these investigations do not explicitly use the kind of approach to motion suggested by the rich mathematical developments that have followed in response to the Helmholtz-Lie problem.

A fourth approach that lies outside the main body of the literature to be considered in this paper is the recent approach through picture grammars and the analysis of perceptual scenes. Its growing literature has been in response especially to problems of pattern recognition that center on construction of computer programs and peripheral devices that have rudimentary perceptual capacities. Although this approach has a formal character quite different from the others considered and it has not been used to address directly the question about the Euclidean character of space, it should be mentioned because it does provide an approach that in many respects is very natural psychologically and that is in certain aspects more closely connected to the psychology of perception than most of the classical geometric approaches that have been used thus far in the analysis of visual space. (An elementary introduction and references to the literature are to be found in Suppes and Rottmayer, 1974; an encyclopedic review is given by Fu, 1974.)

A typical picture grammar has the following character. Finite line segments or finite curves of a given length and with a given orientation are concatenated together as basic elements to form geometrical figures of greater complexity. A typical problem in the literature of pattern recognition is to provide such a concatenation (not necessarily one dimensional) so as to construct handwritten characters, or, as a specialized example that has received a fair amount of attention, to recognize handwritten mathematical symbols. These approaches are often labeled picture grammars because they adopt the approach used in mathematical linguistics for writing phrase–structure grammars to generate linguistic utterances. Picture grammars can in fact be characterized as context free, context sensitive, etc., depending upon the exact character of the rules of production. What is missing is the question, Can the set of figures generated by the picture grammars be embedded in Euclidean space or other metric spaces of an elementary character? This question would seem to have some conceptual interest from the standpoint of the theory of perception. It is clearly not of the same importance for the theory of pattern recognition. Picture grammars base perception on a set of primitive concepts that seem much more natural than the more abstract concepts familiar in classical geometry. They would seem to represent an excellent approach for exploration of the character of visual space but I am unable to cite references that test these ideas experimentally.

II. THE HIERARCHY OF GEOMETRIES

Those who have declared that visual space is not Euclidean have usually had a well-defined alternative in mind. The most popular candidates have been claims that visual space is either elliptic or hyperbolic, although some more radical theses are implicit in some of the experimental work.

How the various geometries are to be related hierarchically is not entirely a simple matter, for by different methods of specialization one may be obtained from another. A reasonably natural hierarchy for purposes of talking about visual space is shown in Fig. 1. In the figure, I

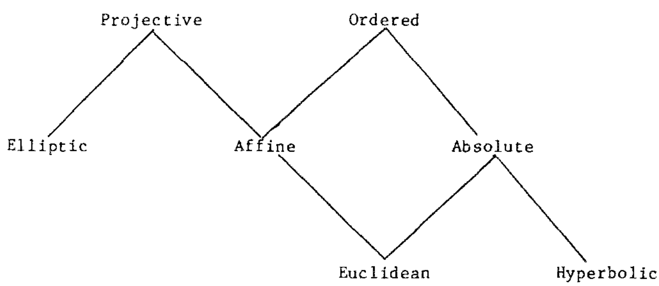

Fig. 1. Hierarchy of geometries.

have also referred to geometries rather than to spaces, although from a certain conceptual standpoint the latter is preferable. I have held to the language of *geometries* in deference to tradition in the literature on visual space. The weakest geometry considered here is either projective geometry on the left-hand side at the top of the figure or ordered geometry at the right. There are various natural primitive concepts for projective geometry. Fundamental in any case is the concept of incidence and, once order is introduced, the concept of separation. In contrast, ordered geometry is based upon the single ternary relation of betweenness holding for three points in the fashion standard for Euclidean geometry, but of course axioms based only upon betweenness are weaker than those required for Euclidean geometry. Without entering into technical details, elliptic geometry of the plane is obtained from projective geometry by defining it as the geometry corresponding to the group of projective collineations that leave an imaginary ellipse invariant in the projective plane. Although elliptic geometry has been important in the consideration of visual space, as we shall see later, the details of elliptic geometry are complicated and subtle, and as far as I know have not actually been adequately studied in detail in relation to any serious body of experimental data.

Turning now to the right-hand side of Figure 1, affine geometry is obtained from ordered geometry by adding Euclid's axiom that, given a line and a point external to the line, there is at most one line (i) through the point, (ii) in the plane formed by the point and the line, and (iii) that does not meet the line. Going in the other direction from ordered geometry in Figure 1, we obtain absolute geometry by adding the concept of congruence of segments, which is just the notion of equidistance mentioned earlier. We add Euclid's axiom to absolute geometry to obtain Euclidean geometry, and we add the negation of Euclid's axiom to absolute geometry to obtain hyperbolic geometry. These are the only two extensions of absolute geometry. Given the fundamental character of absolute geometry in relation to the claims often made that visual space is either Euclidean or hyperbolic, it is somewhat surprising that there has been no more detailed investigation experimentally of whether the axioms of absolute geometry hold for visual space.

There is another way of organizing the hierarchy of geometries in terms of metric spaces. Recall that a *metric space* is a pair $\langle A, d\rangle$ such that A is a nonempty set, d is a real-valued function defined on the Cartesian product $A \times A$, and for all a, b, c in A,

Axiom 1. $d(a, a) = 0$ and if $a \neq b, d(a, b) > 0$;

Axiom 2. $d(a, b) = d(b, a)$;

Axiom 3. $d(a, b) + d(b, c) \geq d(a, c)$.

The elements of the set A are called *points*. The first axiom asserts that distances are positive, except for the distance between identical points, which is zero. The second axiom asserts that distance is symmetric; that is, it is a function only of the unordered pair of points, not a function of their order. The third axiom is the triangle inequality. Most of the metric spaces important for the theory of perception have the property that any two points can be joined by a segment. Such spaces are called metric spaces with additive segments. These spaces are naturally divided into two broad subclasses, affine metrics and coordinate-free metrics. By further specialization of each of these subclasses we are led naturally to the Euclidean, hyperbolic, and spherical spaces, as well as to generalizations of the Euclidean metric in terms of what are called Minkowski metrics. An important subclass of the coordinate-free metrics is the Riemannian metrics. It may be shown that the only spaces that are Riemannian and affine metric are either Euclidean or hyperbolic. We shall not use these concepts in detail, but it is important to mention that this alternative hierarchy of metric spaces is as natural to use as the more classical hierarchy exhibited in Figure 1.

All of the concepts I have introduced in this brief survey of the hierarchy of geometries are familiar in the mathematical literature of geometry.

III. EXPERIMENTAL AND PHILOSOPHICAL ANSWERS

My main purpose in this section is to provide a survey of the answers that have been given. A summary is provided in Table 1.

The natural place to begin is with Euclid's *Optics*, the oldest extant treatise on mathematical optics. It is important to emphasize that Euclid's *Optics* is really a theory of vision and not a treatise on physical optics. A large number of the propositions are concerned with vision from the standpoint of perspective in monocular vision. Indeed, Euclid's *Optics* could be characterized as a treatise on perspective within Euclidean geometry. The tone of Euclid's treatise can be seen from quoting the initial part, which consists of seven 'definitions'.

1. Let it be assumed that lines drawn directly from the eye pass through a space of great extent;
2. and that the form of the space included within our vision is a cone, with its apex in the eye and its base at the limits of our vision;
3. and that those things upon which the vision falls are seen, and that those things upon which the vision does not fall are not seen;

TABLE I
Is visual space Euclidean?

Name	Claim	Answer
Euclid (300 B.C.)	Theory of perspective	Yes
Reid (1764), Daniels (1972), Angell (1974)	Geometry of visibles spherical	No
Blumenfeld (1913)	Parallel alleys not equal to equidistance alleys	No
Luneburg (1947, 1948, 1950)	Visual space is hyperbolic	No
Blank (1953, 1957, 1958a, 1958b, 1961)	Essentially same as Luneburg	No
Hardy *et al.* (1953)	Essentially same as Luneburg	No
Zajaczkowska (1956)	Positive results on experimental test of Luneburg theory	No
Schelling (1956)	Hyperbolic relative to given fixation point	No
Gogel (1956a, 1956b, 1963, 1964a, 1964b, 1965)	Equidistance tendency evidence for contextual geometry	No
Foley (1964, 1965, 1966, 1969, 1972)	Visual space is nonhomogeneous	No but
Indow (1967, 1968, 1974a, 1974b, 1975)	MDS methods yield good Euclidean fit	Not sure
Indow *et al.* (1962a, 1962b, 1963)	Close to Indow	Not sure
Nishikawa (1967)	Close to Indow	Not sure
Matsushima and Noguchi (1967)	Close to Indow	Not sure
Grünbaum (1963)	Questions Luneburg theory	Yes
Strawson (1966)	Phenomenal geometry is Euclidean	Yes

4. and that those things seen within a larger angle appear larger, and those seen within a smaller angle appear smaller, and those seen within equal angles appear to be of the same size;
5. and that those things seen within the higher visual range appear higher, while those within the lower range appear lower;
6. and, similarly, that those seen within the visual range on the right appear on the right, while those within that on the left appear on the left;
7. but that things seen within several angles appear to be more clear.

(The translation is taken from that given by Burton in 1945.)
The development of Euclid's *Optics* is mathematical in character, but it is not axiomatic in the same way that the *Elements* are. For example, later

Euclid proves two propositions, 'to know how great is a given elevation when the sun is shining' and 'to know how great is a given elevation when the sun is not shining'. As would be expected, there is no serious introduction of the concept of the sun or of shining but they are treated in an informal, commonsense, physical way with the essential thing for the proof being rays from the sun falling upon the end of a line. Visual space is of course treated by Euclid as Euclidean in character.

The restriction to monocular vision is one that we shall meet repeatedly in this survey. However, it should be noted that Euclid proves several propositions involving more than one eye; for example, 'If the distance between the eyes is greater than the diameter of the sphere, more than the hemispheres will be seen'. Euclid is not restricted to some simple geometric optics but is indeed concerned with the theory of vision, as is evident from the proposition that 'if an arc of a circle is placed on the same plane as the eye, the arc appears to be a straight line'. This kind of proposition is a precursor of later theories – for example, that of Thomas Reid – which emphasize the non-Euclidean character of visual space.

I skip rapidly through the period after Euclid to the eighteenth century, not because there are not matters of interest in this long intervening period but because there do not seem to be salient changes of opinion about the character of visual space, or at least if there are they are not known to me. I looked, for example, at the recent translation by David C. Lindberg (1970) of the thirteenth-century treatise *Perspectiva Communis* of John Pecham and found nothing to report in the present context, although the treatise itself and Lindberg's comments on it are full of interesting matter of great importance concerning other questions in optics, as, for example, theories about the causes of light.

Newton's *Opticks* (1704/1931) is in marked contrast to Euclid's. The initial definitions do not make any mention of the eye until Axiom VIII, and then in very restrained fashion. Almost without exception, the propositions of Newton's optics are concerned with geometrical and especially physical properties of light. Only really in several of the Queries at the end are there any conjectures about the mechanisms of the eye, and these conjectures do not bear on the topic at hand.

Five years after the publication of the first edition of Newton's *Opticks*, Berkeley's *An Essay Towards a New Theory of Vision* (1709/1901) appeared in 1709. Berkeley does not really have much of interest to say about the geometry of visual space, except in a negative way. He makes the point that distance cannot be seen directly and, in fact, seems to categorize the perception of distance as a matter of tactile rather than visual sensation because the muscular convergence of the eyes is tactile in character. He emphatically makes the point that we are not able geometrically to observe or compute the optical angle generated by a remote point as a vertex with sides pointing toward the centers of the two eyes. Here is what he says about the perception of optical angles. "Since therefore those angles and lines are not themselves perceived by sight, it follows, . . . that the mind does not by them judge the distance of objects" (# 13). What he says about distance he also says about magnitude not being directly perceived visually. In this passage (paragraph 53), he is especially negative about trying to use the geometry of the visual world as a basis for visual perception.

It is clear from these and other passages that for Berkeley visual space is not Euclidean because there is no proper perception of distance or magnitude; at least, visual space is not a three-dimensional Euclidean space. What he seems to say is sufficiently ambiguous as to whether one should argue that it is at least a two-dimensional Euclidean space. My own inclination is to judge that his views on this are more negative than positive. Perhaps a sound negative argument can be made up from his insistence on there being a minimum visible. As he puts it, "It is certain sensible extension is not infinitely divisible. There is a minimum tangible, and a minimum visible, beyond which sense cannot perceive. This everyone's experience will inform him" (# 54).

In fact, toward the end of the essay, Berkeley makes it clear that even two-dimensional geometry is not a proper part of visual space or, as we might say, the visual field. As he says in the final paragraph of the essay, "By this time, I suppose, it is clear that neither abstract nor visible extension makes the object of geometry."

Of much greater interest here is Thomas Reid's *Inquiry into the Human Mind*, first published in 1764 (1764/1967). Chapter 6 deals with seeing, and Section 9 is the celebrated one entitled 'Of the geometry of visibles'. It is sometimes said that this section is a proper precursor of non-Euclidean geometry, but if so, it must be regarded as an implicit precursor because the geometry explicitly discussed by Reid as the geometry of visibles is wholly formulated in terms of spherical geometry, which had of course been recognized as a proper part of geometry since ancient times. The viewpoint of Reid's development is clearly set forth at the beginning of the section: "Supposing the eye placed in the centre of a sphere, every great circle of the sphere will have the same appearance to the eye as if it was a straight line; for the curvature of the circle being turned directly toward the eye, is not perceived by it. And, for the same reason, any line which is drawn in the plane of a great circle of the sphere, whether it be in reality straight or curve, will appear to the eye." It is important to note that Reid's geometry of visibles is a geometry of monocular vision. He mentions in other places binocular vision, but the detailed geometrical development is restricted to the geometry of a single eye. The important contrast between Berkeley and Reid is that Reid develops in some detail the geometry in a straightforward, informal, mathematical fashion. No such comparable development occurs in Berkeley.

Daniels (1972) has argued vigorously that Reid's geometry of visibles is not simply a use of spherical geometry but is an introduction by Reid of a double elliptic space. A similar argument is made by Angell (1974). I am sympathetic with these arguments, but it seems to me that they go too far and for a fairly straightforward reason not discussed by either Daniels or Angell. Let us recall how elliptic geometry was created by Felix Klein at the end of the nineteenth century. He recognized that a natural geometry very similar to Euclidean geometry or hyperbolic geometry could be obtained from spherical geometry by identifying antipodal points as a single point. The difficulty with spherical geometry as a geometry having a development closely parallel to that of Euclidean geometry is that two great circles, which correspond to lines, have two points, not one point, of intersection. However, by identifying the two antipodal points as a single point, a fair number of standard Euclidean postulates remain valid. It is quite clear that no such identification of antipodal points was made by Reid, for he says quite clearly in the fifth of his propositions, 'Any two right lines being produced will meet in two points, and mutually bisect each other'. This property of meeting in two points rather than one is what keeps his geometry of visibles from being a proper elliptic geometry and forces us to continue to think of it in terms of the spherical model used directly by Reid himself.

In spite of the extensive empirical and theoretical work of Helmholtz on vision, he does not have a great deal to say that directly bears on this question, and I move along to experiments and relevant psychological theory in the twentieth century. The first stopping point is Blumenfeld (1913).

Blumenfeld was among the first to perform a specific experiment to show that, in one sense, phenomenological visual judgments do not satisfy all Euclidean properties. Blumenfeld performed experiments with so-called parallel and equidistance alleys. In a darkened room the subject sits at a table, looking straight ahead, and he is asked to adjust two rows of point sources of light placed on either side of the normal plane, i.e., the vertical plane that bisects the horizontal segment joining the centers of the two eyes. The two furthest lights are fixed and are placed symmetrically and equidistant from the normal plane. The subject is then asked to arrange the other lights so that they form a parallel alley extending toward him from the fixed lights. His task is to arrange the lights so that he perceives them as being straight and parallel to each other in his visual space. This is the task for construction of a parallel alley. The second task is to construct a distance alley. In this case, all the lights except the two fixed lights are turned off and a pair of lights is presented, which are adjusted as being at the same physical distance apart as the fixed lights – the kind of equidistance judgments discussed earlier. That pair of lights is then turned off and another pair of lights closer to him is presented for adjustment, and so forth. The physical configurations do not coincide, but in Euclidean geometry straight lines are parallel if and only if they are equidistant from each other along any mutual perpendiculars. The discrepancies observed in Blumenfeld's experiment are taken to be evidence that visual space is not Euclidean. In both the parallel-alley and equidistance-alley judgments the lines diverge as you move away from the subject, but the angle of divergence tends to be greater in the case of parallel than in the case of equidistance alleys. The divergence of the alleys as one moves away from the subject has been taken by Luneburg to support his hypothesis that visual space is hyperbolic.

In fact, Luneburg, in several publications in the late forties, has been by far the strongest supporter of the view that visual space is hyperbolic. He, in conjunction with his collaborators, has set forth a detailed mathematical theory of binocular vision and at the same time has generated a series of experimental investigations to test the basic tenets of the theory.

In many respects, Luneburg's article (1947) remains the best detailed mathematical treatment of the theory of binocular vision. Without extensive discussion, Luneburg restricts himself to Riemannian geometries of constant curvature in order to preserve rigid motions, that is, free mobility of rigid bodies. Luneburg develops in a coordinate system natural for binocular vision the theory of Riemannian spaces of constant curvature in a quite satisfactory form, although an explicit axiomatic treatment is missing. On the other hand, he nowhere examines with any care or explicitness the more general and primitive assumptions that lead to assuming that visual space is a Riemannian space of constant curvature. After these general developments he turns to the detailed arguments for the view that the appropriate space of constant curvature for visual space is hyperbolic. It is not possible to enter into the details of Luneburg's argument here, but he bases it on three main lines of considerations, all of which have had a great deal of attention in the theory of vision: first, the data arising from the frontal-plane horopter where curves which appear as straight are physically curved (data on these phenomena go back to the time before Helmholtz); second, the kind of alley phenomena concerning judgments of parallelness mentioned earlier; and, third, accounting for judgments of distorted rooms in which appropriate perspective lines are drawn and which consequently appear as rectangular or regular (here, Luneburg draws on some classic and spectacular demonstrations by A. Ames, Jr.). One of the difficulties of this field is that the kind of detailed mathematical and quantitative arguments presented by Luneburg in connection with these three typical kinds of problems are not satisfactorily analyzed in the later literature. Rather, new data of a different sort are presented to show that different phenomena argue against Luneburg's hypothesis that visual space is hyperbolic.

Luneburg died in 1949, but a number of his former students and collaborators have continued his work and provided additional experimental support as well as additional mathematically based arguments in favor of his views. I refer especially to Blank (1953, 1957, 1958a, 1958b, 1961) and Hardy, Rand, Rittler, Blank, and Boeder (1953), although this is by no means an exhaustive list. Another positive experimental test was provided by Zajaczkowska (1956).

Schelling (1956) basically agrees with Luneburg but makes an important point of modification, namely, the metrics of negative curvature – that is, of the hyperbolic spaces that Luneburg argues for – are essentially momentary metrics. At a given instant the eye has a certain fixation point, and relative to this fixation point Luneburg's theory is, according to Schelling, probably approximately correct, but the applicability of the theory is severely restricted because the eyes are normally moving about continuously and the points of fixation are continually changing. This fundamental fact of change must be taken account of in any fully adequate theory.

Gogel (1956a, 1956b, 1963, 1964a, 1964b, 1965) has studied what is called the equidistance tendency, or what in the context of this paper we might term the Berkeley tendency. Remember that Berkeley held that distance was not a visual idea at all but derived from the tactile sense. Without entering into a precise analysis of Berkeley's views, Gogel has provided an important body of evidence that when other cues are missing there is a strong tendency to view objects as being at the same distance from the observer. These careful and meticulous studies of Gogel are important for establishing not only the equidistance tendency but also its sensitivity to individual variation, on the one hand, and to the presence of additional visual cues, on the other. The equidistance tendency is certainly present as a central effect, but any detailed theory of visual space has a bewildering complexity of contextual and individual differences to account for, and it seems to me that Gogel's experiments are essentially decisive on this point. In the papers referred to, Gogel does not really give a sharp answer to the question about the character of visual space, but I have listed him in Table I because it seems to me that the impact of his studies is to argue strongly for skepticism about fixing the geometry of visual space very far up in the standard hierarchy and, rather, to insist on the point that the full geometry is strongly contextual in character and therefore quite deviant from the classical hierarchy.

A number of interesting experimental studies of the geometry of visual space have been conducted by John Foley. In Foley (1964) an experiment using finite configurations of small point sources of light was conducted to test the Desarguesian property of visual space. (Of course, the property was tested on the assumption that a number of other axioms were valid for visual space.) The results confirmed the Desarguesian property for most observers but not for all. In Foley (1966), perceived equidistance was studied as a function of viewing distance. Like most of Foley's experiments, this was conducted in the horizontal eye-level plane. The locus of perceived equidistance was determined at distances of 1.2, 2.2, 3.2, and 4.2 meters from the observer. As in other Foley experiments, the stimuli were small, point-like light sources viewed in complete darkness. The observer's head was held fixed but his eyes were permitted to move freely. There were five lights, one in the normal plane, which was fixed, and two variable lights on each side of the normal plane at angles of 12 degrees and 24 degrees with respect to the normal plane. The locus of perceived equidistance was found to be concave toward the observer at all distances. Perhaps most importantly, the locus was found to vary with viewing distance, which indicates that the visual space does not depend on the spatial distribution of retinal stimulation alone. Again, there is here a direct argument for a contextual geometry and results are not consistent with Luneburg's theory. The equidistance judgments were of the following sort: A subject was instructed to set each of the lights, except the fixed light, in the normal plane to be at the same distance from himself as the fixed light. Thus, it should appear to him that the lights lie on a circle, with himself as observer at the center. The important point is that for none of the ten subjects in the experiment did the judgments of the locus for equidistance lie on the Vieth-Mueller horopter or circle mentioned earlier as one of the supporting arguments for Luneburg's theory. Also important for the fundamental geometry of visual space is the fact that the loci determined by the observers were not symmetric about the normal plane.

Foley's (1972) study shows experimentally that, on the one hand, the size-distance invariance hypothesis is incorrect, and that in fact the ratio of perceived frontal extent to perceived egocentric distance greatly exceeds the physical ratio, while, on the other hand, perceived visual angles are quite close to physical ones. These results, together with other standard assumptions, are inconsistent with the Luneburg theory that visual space is hyperbolic. Foley describes the third experiment in this paper in the following way:

How can it be that in the primary visual space reports of perceived size-distance ratio are not related to reports of perceived visual angle in a Euclidean way? One possibility is that the two kinds of judgments are in part the product of different and independent perceptual processes. . . . The results are consistent with the hypothesis that the two kinds of judgments are the product of independent processes. They also show that no one geometrical model can be appropriate to all stimulus situations, and they suggest that the geometry may approach Euclidean geometry with the introduction of cues to distance.

Again, there is in Foley's detailed analysis a strong case for a contextual geometry. A number of other detailed experimental studies of Foley that have not been referenced here build a case for the same general contextual view, which I discuss in more detail below.

A number of detailed investigations on the geometry of visual space have been conducted by Tarow Indow (1967, 1968, 1974a, 1974b, 1975) and other Japanese investigators closely associated with him (Indow *et al.*, 1962a, 1962b, 1963; Matsushima and Noguchi, 1967; Nishikawa, 1967). They have found, for example, that multidimensional scaling methods (MDS), which have been intensively developed in psychology over the past decade and a half, in many cases yield extremely good fits to Euclidean space. Indow has experimentally tested the Luneburg theory based upon the kind of alley experiments that go back to Blumenfeld (1913). As might be expected, he duplicates the result that the equidistance alleys always lie outside the parallel alleys, which under the other assumptions that are standard implies that the curvature of the space is negative and therefore it must be hyperbolic. But Indow (1974) properly challenges the simplicity of the Luneburg assumptions, especially the constancy of curvature. It is in this context that he has also tried the alternative approach of determining how well multidimensional scaling will work to fit a Euclidean metric. As he emphasizes also, the Luneburg approach is fundamentally based upon differential geometry as a method of characterizing Riemannian spaces with constant curvature, but for visual judgments it is probably more appropriate to depend upon the judgments in the large and therefore upon a different conceptual basis for visual geometry. Throughout his writings, Indow recognizes the complexity and difficulty of reaching for any simple answer to give the proper characterization of visual space. The wealth of detail in his articles and those of his collaborators is commended to the reader who wants to pursue these matters in greater depth.

In his important book on the philosophy of space and time, Grünbaum (1963) rejects the Luneburg theory and affirms that, in order to yield the right kinds of perceptual judgments, visual space must be Euclidean. His argument is rather brief and I shall not examine it in any detail. It would be my own view that he has not given proper weight to the detailed experimental studies or to the details of the various theoretical proposals that have been made.

I close this survey by returning to a philosophical response to the question, that of Strawson (1966) in his book on Kant's *Critique of Pure Reason*. From the standpoint of the large psychological literature I have surveyed, it is astounding to find Strawson asserting as a necessary proposition that phenomenal geometry is Euclidean. The following quotation states the matter bluntly:

> With certain reservations and qualifications, to be considered later, it seems that Euclidean geometry may also be interpreted as a body of unfalsifiable propositions about phenomenal straight lines, triangles, circles, etc.; as a body of a priori propositions about spatial appearances of these kinds and hence, of course, as a theory whose application is restricted to such appearances. (p. 286)

The astounding feature of Strawson's view is the absence of any consideration that phenomenal geometry could be other than Euclidean and that it surely must be a matter, one way or another, of empirical investigation to determine what is the case. The qualifications he gives later do not bear on this matter but pertain rather to questions of idealization and of the nature of constructions, etc. The absence of any attempt to deal in any fashion whatsoever with the large theoretical and experimental literature on the nature of visual space is hard to understand.

IV. SOME REMARKS ON THE ISSUES

In this final section, I center my remarks around three clusters of issues. The first is concerned with the contextual character of visual geometry, the second with problems of distance perception and motion, and the third with the problem of characterizing the nature of the objects of visual space.

A. *Contextual Geometry*

A wide variety of experiments and ordinary experience as well testify to the highly contextual character of visual space. The presence or absence of 'extraneous' points can sharply affect perceptual judgments. The whole range of visual illusions, which I have not discussed here, provides a broad body of evidence for the surprising strength of these contextual effects.

As far as I can tell, no one has tried seriously to take account of these contextual effects from the standpoint of the axiomatic foundations of visual geometry. In a way it is not surprising, for the implications for the axiomatic foundations are, from the ordinary standpoint, horrendous. Let us take a simple example to illustrate the point.

In ordinary Euclidean geometry, three points form an isosceles triangle just when two sides of the triangle are of the same length. Suppose now that Euclidean geometry had the much more complicated aspect that whether a triangle were isosceles or not depended not simply on the configuration of the three points but also on whether there was a distinguished point lying just outside the triangle alongside one of the equal sides. This asymmetry may well make the visual triangle no longer isosceles. This is but one simple instance of a combinatorial nightmare of contextual effects that can easily be imagined and, without much imagination or experimental skill, verified as being real effects.

What are we to say about such effects? It seems to me the most important thing is to recognize that perceptual geometry is not really the same as classical geometry at all, but in terms of the kinds of judgments we are making it is much closer to physics. Consider, for example, the corresponding situation with bodies that attract each other by gravitation. The introduction of a third body makes all the difference to the motions of the two original bodies and it would be considered bizarre for the situation to be otherwise. This also applies to electromagnetic forces, mechanical forces of impact, etc. Contextual effects are the order of the day in physics, and the relevant physical theories are built to take account of such effects.

Note that physical theories depend upon distinguished objects located in particular places in space and time. Space-time itself is a continuum of undistinguished points, and it is characteristic of the axiomatic foundations of classical geometry that there are no distinguished points in the space. But it is just a feature of perception that we are always dealing with distinguished points which are analogous to physical objects, not geometrical points. Given this viewpoint, we are as free to say that we have contextual effects in visual geometry as we are to make a similar claim in general relativity due to the presence of large masses in a given region.

Interestingly enough, there is some evidence that as we increase the visual cues, that is, we fill up the visual field with an increasingly complex context of visual imagery, the visual space becomes more and more Euclidean. It is possible that we have here the exact opposite of the situation that exists in general relativity. In the case of perception it may be that spaces consisting of a very small number of visible points may be easily made to deviate from any standard geometry.

The geometric viewpoint can be brought into close alignment with the physical one, when the embedding of finite sets of points in some standard geometry is taken as the appropriate analysis of the nature of visual space. This approach was mentioned earlier and is implicit in some of the experimental literature discussed. It has not sufficiently been brought to the surface, and the full range of qualitative axioms that must be satisfied for the embedding of a finite collection of points in a unique way in a given space, whether Euclidean, hyperbolic, elliptic, or what not, needs more explicit and detailed attention.

It also seems satisfactory to avoid the problems of contextual effects in initial study of this kind by deliberately introducing symmetries and also certain special additional assumptions such as quite special relations of a fixed kind to the observer. The many different experimental studies and the kind of mathematical analysis that has arisen out of the Luneburg tradition suggest that a good many positive and almost definitive results could be achieved under special restrictive assumptions. It seems to me that making these results as definitive as possible, admitting at the same time their specialized character and accepting the fact that the general situation is contextual in character, is an appropriate research strategy. It also seems to me likely that for these special situations one can give a definitely negative answer to the question, Is visual space Euclidean?, and respond that, to high approximations, in many special situations it is hyperbolic and possibly in certain others elliptic in character. This restricted answer is certainly negative. A general answer at the present time does not seem available as to how to characterize the geometry in a fully satisfactory way that takes account of the contextual effects that are characteristic of visual illusions, equidistance tendencies, etc.

B. *Distance Perception and Motion*

As indicated earlier in the brief discussion of the Helmholtz-Lie problem, most of the work surveyed in the preceding section has not taken sufficient account of problems of motion. There is an excellent survey article of Foley (in press) on distance perception which indicates that eye motion during the initial stage of focusing on an object is especially critical in obtaining information about perceptual distance. In spite of the views of Berkeley, philosophical traditions in perception have tended to ignore the complicated problems of motion of the eyes or head as an integral part of visual perception, but the most elementary considerations are sufficient to demonstrate their fundamental importance. It was a fundamental insight of Luneburg to recognize that it is important to characterize invariance properties of motions of the eyes and head that compensate each other. The deeper aspects of scanning as determining the character of the visual field have not really been studied in a thoroughly mathematical and quantitative fashion, and there is little doubt in my mind that this is the area most important for future developments in the theory of visual perception. We should, I would assume, end up with a kinematics of visual perception replacing the geometry of visual perception. For example, Lamb (1919) proves that under Donders' law, which asserts that the position of the eyeball is completely determined by the primary position and the visual axis aligned to the fixation point, it is not possible for every physically straight line segment to be seen as straight. This kinematical theorem of Lamb's, which is set forth in detail in Roberts and Suppes (1967), provides a strong kinematical argument against the Euclidean character of visual space. I cite it here simply as an example of the kind of results that one should expect to obtain in a more thoroughly developed kinematics of visual perception.

C. *Objects of Visual Space*

Throughout the analysis given in this paper the exact characterization of what are to be considered as the objects of visual space has not been settled in any precise or definitive way. This ambiguity has been deliberate because the wide range of literature to which I have referred does not have any settled account of what are to be regarded as the objects of visual space. The range of views is extreme – from Berkeley, who scarcely even wants to admit a geometry of pure visual space, to those who hold that visual space is simply a standard Euclidean space and there is little real distinction between visual objects and physical objects. In building up the subject axiomatically and systematically, clearly some commitments are needed, and yet it seems that one can have an intelligible discussion of the range of literature considered here without having to fix upon a precise characterization, because there is broad agreement on the look of things in the field of vision. From the standpoint of the geometry of visual space, we can even permit such wide disagreement as to whether the objects are two dimensional or three dimensional in order to discuss the character of the geometry. Thomas Reid would lean strongly toward the two-dimensional character of visual space. Foley would hold that visual space is three dimensional; note, however, that most of his experiments have been restricted to two dimensions. At the very least, under several different natural characterizations of the objects of visual space it is apparent that strong claims can be made that visual space is not Euclidean, and this is a conclusion of some philosophical interest.

Stanford University

REFERENCES

Angell, R. B.: 'The Geometry of Visibles', *Noûs* **8** (1974), 87–117.

Berkeley, G.: 'An Essay towards a New Theory of Vision', in A. C. Fraser (ed.), *Berkeley's Complete Works* (Vol. 1), Oxford University Press, Oxford, 1901. (Originally published, 1709).

Blank, A. A.: 'The Luneburg Theory of Binocular Visual Space', *Journal of the Optical Society of America* **43** (1953), 717–727.

Blank, A. A.: 'The Geometry of Vision', *British Journal of Physiological Optics* **14** (1957), 154–169.

Blank, A. A.: 'Analysis of Experiments in Binocular Space Perception', *Journal of the Optical Society of America* **48** (1958a), 911–925.

Blank, A. A.: 'Axiomatics of Binocular Vision: The Foundations of Metric Geometry in Relation to Space Perception', *Journal of the Optical Society of America* **48** (1958b), 328–334.

Blank, A. A.: 'Curvature of Binocular Visual Space: An Experiment', *Journal of the Optical Society of America* **51** (1961), 335–339.

Blumenfeld, W.: 'Untersuchungen über die scheinbare Grösse in Sehraume', *Zeitschrift für Psychologie und Physiologie der Sinnesorgane* **65** (1913), 241–404.

Busemann, H.: *The Geometry of Geodesics*, Academic Press, New York, 1955.

Daniels, N.: 'Thomas Reid's Discovery of a Non-Euclidean Geometry', *Philosophy of Science* **39** (1972), 219–234.

Dembowski, P.: *Finite Geometries*, Springer-Verlag, New York, 1968.

Euclid: *Optics* (H. E. Burton, trans.), *Journal of the Optical Society of America* **35** (1945), 357–372.
Foley, J. M.: 'Desarguesian Property in Visual Space', *Journal of the Optical Society of America* **54** (1964), 684–692.
Foley, J. M.: 'Visual Space: A Scale of Perceived Relative Direction', *Proceedings of the 73rd Annual Convention of the American Psychological Association* **1** (1965), 49–50.
Foley, J. M.: 'Locus of Perceived Equidistance as a Function of Viewing Distance', *Journal of the Optical Society of America* **56** (1966), 822–827.
Foley, J. M.: 'Distance in Stereoscopic Vision: The Three-Point Problem', *Vision Research* **9** (1969), 1505–1521.
Foley, J. M.: 'The Size-Distance Relation and Intrinsic Geometry of Visual Space: Implications for Processing', *Vision Research* **12** (1972), 323–332.
Foley, J. M.: 'Distance Perception', in R. Held, H. Leibowitz, and H.-L. Teuber (eds.), *Handbook of Physiology* (Vol. 8: Perception), in press.
Freudenthal, H.: 'Lie Groups in the Foundations of Geometry', *Advances in Mathematics* **1** (1965), 145–190.
Fu, K. S.: *Syntactic Methods in Pattern Recognition*, Academic Press, New York, 1974.
Gogel, W. C.: 'Relative Visual Direction as a Factor in Relative Distance Perceptions', *Psychological Monographs* **70** (1956a), (11, Whole No. 418).
Gogel, W. C.: 'The Tendency to See Objects as Equidistant and Its Inverse Relation to Lateral Separation', *Psychological Monographs* **70** (1956b), (4, Whole No. 411).
Gogel, W. C.: 'The Visual Perception of Size and Distance', *Vision Research* **3** (1963), 101–120.
Gogel, W. C.: 'The Perception of Depth from Binocular Disparity', *Journal of Experimental Psychology* **67** (1964a), 379–386.
Gogel, W. C.: 'Size Cue to Visually Perceived Distance', *Psychological Bulletin* **62** (1964b), 217–235.
Gogel, W. C.: 'Equidistance Tendency and Its Consequences', *Psychological Bulletin* **64** (1965), 153–163.
Grünbaum, A.: *Philosophical Problems of Space and Time*, Knopf, New York, 1963.
Hardy, L. H., Rand, G., Rittler, M. C., Blank, A. A., and Bòeder, P.: *The Geometry of Binocular Space Perception*, Schiller, Elizabeth, N. J., 1953.
Helmholtz, H.: Über die Tatsachen, die der Geometrie zugrunde liegen', *Wissenschaftliche Abhandlungen* **2** (1868), 618–637.
Indow, T.: Two Interpretations of Binocular Visual Space: Hyperbolic and Euclidean', *Annals of the Japan Association for Philosophy of Science* **3** (1967), 51–64.
Indow, T.: 'Multidimensional Mapping of Visual Space with Real and Simulated Stars', *Perception and Psychophysics* **3** (1B) (1968), 45–53.
Indow, T.: 'Applications of Multidimensional Scaling in Perception', in E. C. Carterette and M. P. Friedman (eds.), *Handbook of Perception* (Vol. 2: Psychophysical Judgment and Measurement), Academic Press, New York, 1974a.
Indow, T.: 'On Geometry of Frameless Binocular Perceptual Space', *Psychologia* **17** (1974b), 50–63.
Indow, T.: 'An Application of MDS to Study of Binocular Visual Space', in *Theory, Methods and Applications of Multidimensional Scaling and Related Techniques*, U.S.-Japan Seminar (sponsored by the National Science Foundation and Japan Society for the Promotion of Science), University of California, San Diego, 1975.

Indow, I., Inoue, E., and Matsushima, K.: 'An Experimental Study of the Luneburg Theory of Binocular Space Perception (1): The 3- and 4-point Experiments', *Japanese Psychological Research* **4** (1) (1962a), 6–16.

Indow, I., Inoue, E., and Matsushima, K.: 'An Experimental Study of the Luneburg Theory of Binocular Space Perception (2): The Alley Experiments', *Japanese Psychological Research* **4** (1) (1962b), 17–24.

Indow, I., Inoue, E., and Matsushima, K.: 'An Experimental Study of the Luneburg Theory of Binocular Space Perception (3): The Experiments in a Spacious Field', *Japanese Psychological Research* **5** (1) (1963), 10–27.

Lamb, H.: 'The Kinematics of the Eye', *Philosophical Magazine* **38** (1919), 685–695.

Lindberg, D. C.: *John Pecham and the Science of Optics:* Perspectiva Communis, University of Wisconsin Press, Madison, 1970.

Luneberg, R. K. *Mathematical Analysis of Binocular Vision*, Princeton University Press, Princeton, N.J., 1947.

Luneberg, R. K.: 'Metric Methods in Binocular Visual Perception', in *Studies and Essays* (Courant Anniversary Volume), Interscience, New York, 1948.

Luneberg, R. K.: 'The Metric of Binocular Visual Space', *Journal of the Optical Society of America* **40** (1950), 627–642.

Matsushima, K. and Noguchi, H.: Multidimensional Representation of Binocular Visual Space', *Japanese Psychological Research* **9** (1967), 85–94.

Newton, I.: *Opticks* (Reprinted from the 4th ed.), Bell, London, 1931. (Originally published, 1704).

Nishikawa, Y.: 'Euclidean Interpretation of Binocular Visual Space', *Japanese Psychological Research* **9** (1967), 191–198.

Reid, T.: 'Inquiry into the Human Mind', in *Philosophical Works of Thomas Reid* (Vol. 1), George Olms, Verlag's Buchhandlung, Hildesheim, Germany, 1967. (Originally published, 1764).

Riemann, B.: 'Über die Hypothesen, welche der Geometrie zugrunde liegen', reprinted in *Mathematische Werke*, Leipzig, 1892, pp. 272–287.

Roberts, F. S. and Suppes, P.: 'Some Problems in the Geometry of Visual Perception', *Synthese* **17** (1967), 173–201.

Schelling, H.: 'Concept of Distance in Affine Geometry and Its Applications in Theories of Vision', *Journal of the Optical Society of America* **46** (1956), 309–315.

Strawson, P. F.: *The Bounds of Sense: An Essay on Kent's* Critique of Pure Reason, Methuen, London, 1966.

Suppes, P. and Rottmayer, W.: 'Automata', in E. C. Carterette and M. P. Friedman (eds.), *Handbook of Perception* (Vol. 1: Historical and Philosophical Roots of Perception), Academic Press, New York, 1974.

Zajaczkowska, A. 'Experimental Test of Luneburg's Theory: Horopter and Alley Experiments', *Journal of the Optical Society of America* **46** (1956), 514–527.