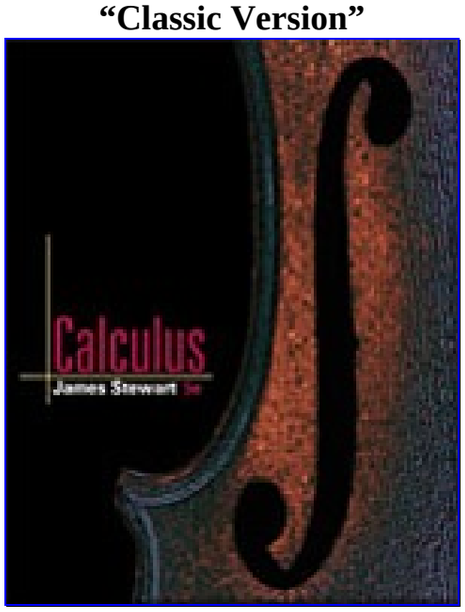
Stewart's Complete Solutions Manual
Calculus 5e

"Classic Version"



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Chapter 1

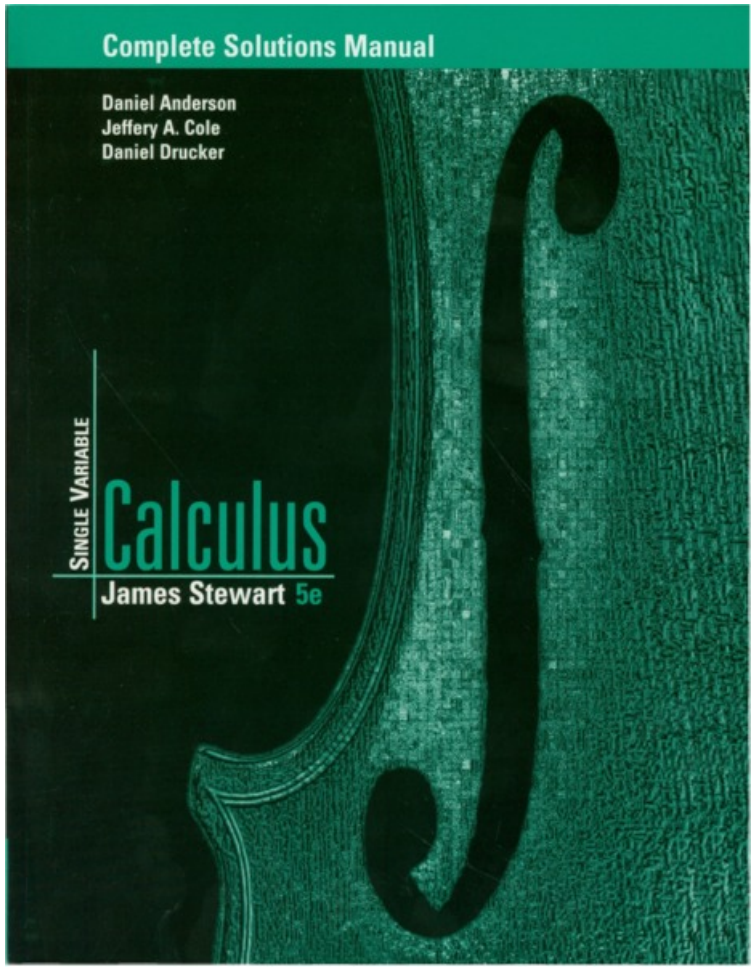
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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1 □ FUNCTIONS AND MODELS

1.1 Four Ways to Represent a Function

In exercises requiring estimations or approximations, your answers may vary slightly from the answers given here.

1. (a) The point $(-1, -2)$ is on the graph of f, so $f(-1) = -2$.

(b) When $x = 2$, y is about 2.8, so $f(2) \approx 2.8$.

(c) $f(x) = 2$ is equivalent to $y = 2$. When $y = 2$, we have $x = -3$ and $x = 1$.

(d) Reasonable estimates for x when $y = 0$ are $x = -2.5$ and $x = 0.3$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-3 \le x \le 3$, or $[-3, 3]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \le y \le 3$, or $[-2, 3]$.

(f) As x increases from -1 to 3, y increases from -2 to 3. Thus, f is increasing on the interval $[-1, 3]$.

2. (a) The point $(-4, -2)$ is on the graph of f, so $f(-4) = -2$. The point $(3, 4)$ is on the graph of g, so $g(3) = 4$.

(b) We are looking for the values of x for which the y-values are equal. The y-values for f and g are equal at the points $(-2, 1)$ and $(2, 2)$, so the desired values of x are -2 and 2.

(c) $f(x) = -1$ is equivalent to $y = -1$. When $y = -1$, we have $x = -3$ and $x = 4$.

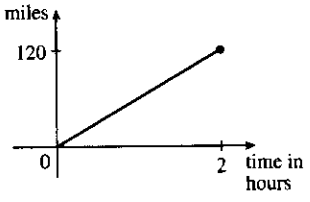
(d) As x increases from 0 to 4, y decreases from 3 to -1. Thus, f is decreasing on the interval $[0, 4]$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-4 \le x \le 4$, or $[-4, 4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \le y \le 3$, or $[-2, 3]$.

(f) The domain of g is $[-4, 3]$ and the range is $[0.5, 4]$.

3. From Figure 1 in the text, the lowest point occurs at about $(t, a) = (12, -85)$. The highest point occurs at about $(17, 115)$. Thus, the range of the vertical ground acceleration is $-85 \le a \le 115$. In Figure 11, the range of the north-south acceleration is approximately $-325 \le a \le 485$. In Figure 12, the range of the east-west acceleration is approximately $-210 \le a \le 200$.

4. *Example 1:* A car is driven at 60 mi/h for 2 hours. The distance d traveled by the car is a function of the time t. The domain of the function is $\{t \mid 0 \le t \le 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0 \le d \le 120\}$, where d is measured in miles.



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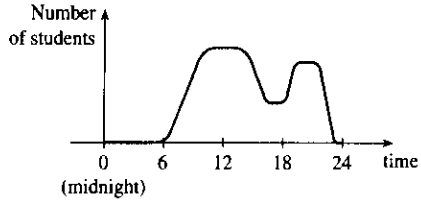
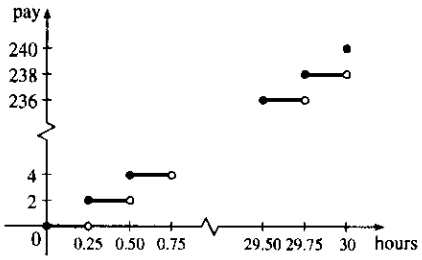
Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0 \le t \le 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0 \le N \le k\}$, where N is an integer and k is the largest number of students on campus at once.



Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h. The domain of the function is $[0, 30]$ and the range of the function is $\{0, 2.00, 4.00, \ldots, 238.00, 240.00\}$.



5. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

6. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2, 2]$ and the range is $[-1, 2]$.

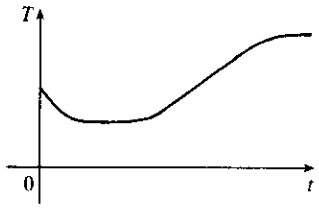
7. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.

8. No, the curve is not the graph of a function since for $x = 0, \pm 1$, and ± 2, there are infinitely many points on the curve.

9. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

10. The salesman travels away from home from 8 to 9 A.M. and is then stationary until 10:00. The salesman travels farther away from 10 until noon. There is no change in his distance from home until 1:00, at which time the distance from home decreases until 3:00. Then the distance starts increasing again, reaching the maximum distance away from home at 5:00. There is no change from 5 until 6, and then the distance decreases rapidly until 7:00 P.M., at which time the salesman reaches home.

11. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.



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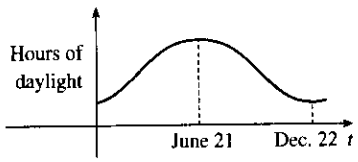
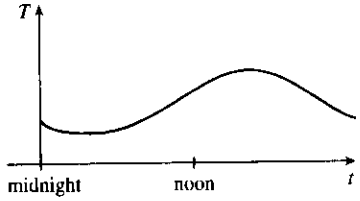
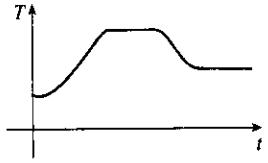
12. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22.



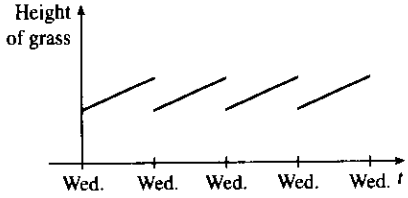
13. Of course, this graph depends strongly on the geographical location!



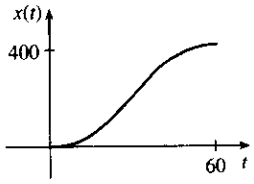
14. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.



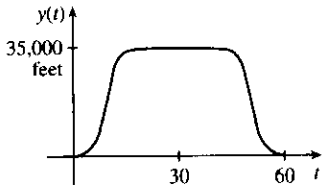
15.



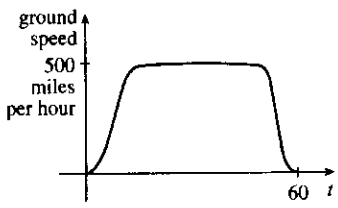
16. (a)



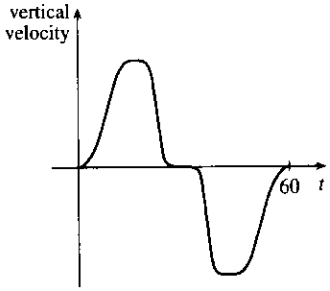
(b)



(c)



(d)



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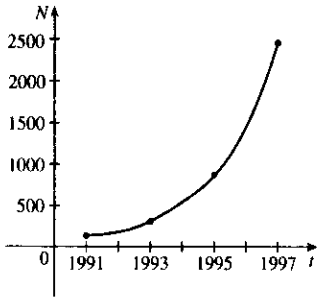
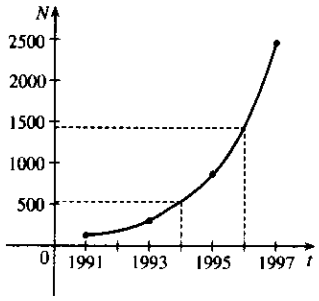
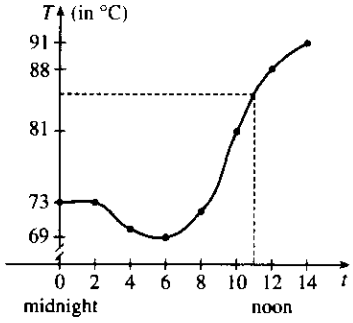
17. (a)



(b) From the graph, we estimate the number of cell-phone subscribers in Malaysia to be about 540 in 1994 and 1450 in 1996.



18. (a)



(b) From the graph in part (a), we estimate the temperature at 11:00 A.M. to be about 84.5°C.

19. $f(x) = 3x^2 - x + 2$.

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12$.

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16$.

$f(a) = 3a^2 - a + 2$.

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2$.

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4$.

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4$.

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2$.

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2$.

$$[f(a)]^2 = \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right)$$
$$= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.$$

$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2$.

20. A spherical balloon with radius $r+1$ has volume $V(r+1) = \frac{4}{3}\pi(r+1)^3 = \frac{4}{3}\pi\left(r^3 + 3r^2 + 3r + 1\right)$. We wish to find the amount of air needed to inflate the balloon from a radius of r to $r+1$. Hence, we need to find the difference $V(r+1) - V(r) = \frac{4}{3}\pi\left(r^3 + 3r^2 + 3r + 1\right) - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(3r^2 + 3r + 1\right)$.

21. $f(x) = x - x^2$, so

$f(2+h) = 2 + h - (2+h)^2 = 2 + h - (4 + 4h + h^2) = 2 + h - 4 - 4h - h^2 = -\left(h^2 + 3h + 2\right)$,

$f(x+h) = x + h - (x+h)^2 = x + h - (x^2 + 2xh + h^2) = x + h - x^2 - 2xh - h^2$, and

$$\frac{f(x+h) - f(x)}{h} = \frac{x + h - x^2 - 2xh - h^2 - x + x^2}{h} = \frac{h - 2xh - h^2}{h} = \frac{h(1 - 2x - h)}{h} = 1 - 2x - h.$$

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22. $f(x) = \dfrac{x}{x+1}$, so $f(2+h) = \dfrac{2+h}{2+h+1} = \dfrac{2+h}{3+h}$, $f(x+h) = \dfrac{x+h}{x+h+1}$, and

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h} = \frac{(x+h)(x+1) - x(x+h+1)}{h(x+h+1)(x+1)} = \frac{1}{(x+h+1)(x+1)}.$$

23. $f(x) = x/(3x-1)$ is defined for all x except when $0 = 3x - 1 \quad \Leftrightarrow \quad x = \frac{1}{3}$, so the domain is $\left\{x \in \mathbb{R} \mid x \neq \frac{1}{3}\right\} = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

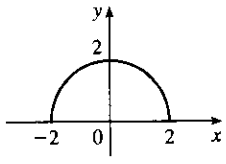
24. $f(x) = (5x+4)/(x^2+3x+2)$ is defined for all x except when $0 = x^2+3x+2 \quad \Leftrightarrow \quad 0 = (x+2)(x+1)$ $\Leftrightarrow \quad x = -2$ or -1, so the domain is $\{x \in \mathbb{R} \mid x \neq -2, -1\} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

25. $f(t) = \sqrt{t} + \sqrt[3]{t}$ is defined when $t \geq 0$. These values of t give real number results for $\sqrt{t}$, whereas any value of t gives a real number result for $\sqrt[3]{t}$. The domain is $[0, \infty)$.

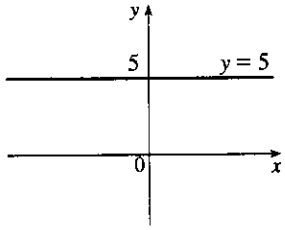
26. $g(u) = \sqrt{u} + \sqrt{4-u}$ is defined when $u \geq 0$ and $4 - u \geq 0 \quad \Leftrightarrow \quad u \leq 4$. Thus, the domain is $0 \leq u \leq 4 = [0, 4]$.

27. $h(x) = 1 / \sqrt[4]{x^2 - 5x}$ is defined when $x^2 - 5x > 0 \quad \Leftrightarrow \quad x(x-5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

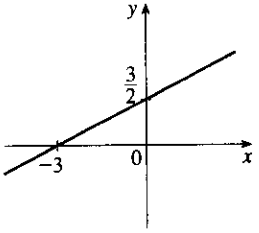
28. $h(x) = \sqrt{4-x^2}$. Now $y = \sqrt{4-x^2} \quad \Rightarrow \quad y^2 = 4 - x^2 \quad \Leftrightarrow \quad x^2 + y^2 = 4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\{x \mid 4 - x^2 \geq 0\} = \{x \mid 4 \geq x^2\} = \{x \mid 2 \geq |x|\} = [-2, 2]$. From the graph, the range is $0 \leq y \leq 2$, or $[0, 2]$.



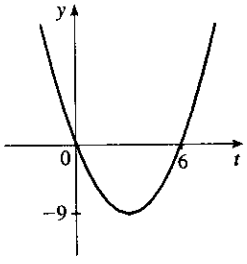
29. $f(x) = 5$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a horizontal line with y-intercept 5.



30. $F(x) = \frac{1}{2}(x+3)$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of F is a line with x-intercept -3 and y-intercept $\frac{3}{2}$.



31. $f(t) = t^2 - 6t$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a parabola opening upward since the coefficient of t^2 is positive. To find the t-intercepts, let $y = 0$ and solve for t. $0 = t^2 - 6t = t(t-6) \quad \Rightarrow \quad t = 0$ and $t = 6$. The t-coordinate of the vertex is halfway between the t-intercepts, that is, at $t = 3$. Since $f(3) = 3^2 - 6 \cdot 3 = -9$, the vertex is $(3, -9)$.



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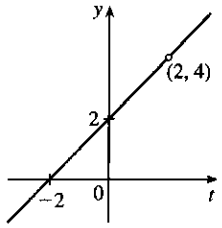
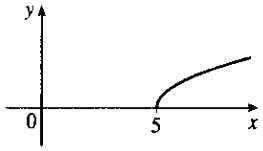
32. $H(t) = \dfrac{4-t^2}{2-t} = \dfrac{(2+t)(2-t)}{2-t}$, so for $t \neq 2$, $H(t) = 2+t$. The domain is $\{t \mid t \neq 2\}$. So the graph of H is the same as the graph of the function $f(t) = t + 2$ (a line) except for the hole at $(2, 4)$.



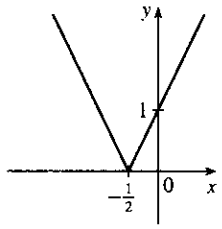
33. $g(x) = \sqrt{x-5}$ is defined when $x - 5 \geq 0$ or $x \geq 5$, so the domain is $[5, \infty)$. Since $y = \sqrt{x-5} \;\Rightarrow\; y^2 = x - 5 \;\Rightarrow\; x = y^2 + 5$, we see that g is the top half of a parabola.



34. $F(x) = |2x+1| = \begin{cases} 2x+1 & \text{if } 2x+1 \geq 0 \\ -(2x+1) & \text{if } 2x+1 < 1 \end{cases}$

$$= \begin{cases} 2x+1 & \text{if } x \geq -\frac{1}{2} \\ -2x-1 & \text{if } x < -\frac{1}{2} \end{cases}$$

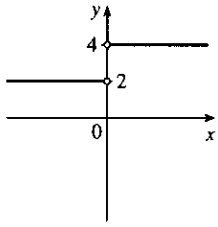
The domain is $\mathbb{R}$, or $(-\infty, \infty)$.



35. $G(x) = \dfrac{3x + |x|}{x}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

$$G(x) = \begin{cases} \dfrac{3x+x}{x} & \text{if } x > 0 \\ \dfrac{3x-x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} \dfrac{4x}{x} & \text{if } x > 0 \\ \dfrac{2x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$

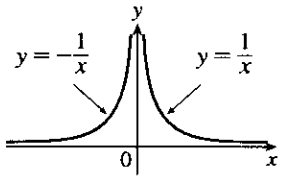
Note that G is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.



36. $g(x) = \dfrac{|x|}{x^2}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

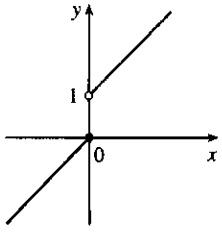
$$g(x) = \begin{cases} \dfrac{x}{x^2} & \text{if } x > 0 \\ \dfrac{-x}{x^2} & \text{if } x < 0 \end{cases} = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x} & \text{if } x < 0 \end{cases}$$

Note that g is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.



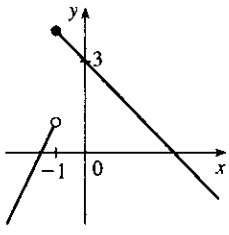
37. $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases}$

Domain is $\mathbb{R}$, or $(-\infty, \infty)$.



38. $f(x) = \begin{cases} 2x+3 & \text{if } x < -1 \\ 3-x & \text{if } x \geq -1 \end{cases}$

Domain is $\mathbb{R}$, or $(-\infty, \infty)$.



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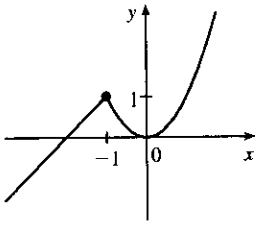
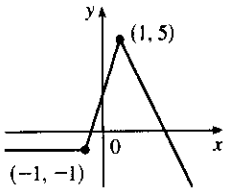
39. $f(x) = \begin{cases} x+2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

Note that for $x = -1$, both $x + 2$ and x^2 are equal to 1. Domain is $\mathbb{R}$.



40. $f(x) = \begin{cases} -1 & \text{if } x \le -1 \\ 3x+2 & \text{if } -1 < x < 1 \\ 7-2x & \text{if } x \ge 1 \end{cases}$

Domain is $\mathbb{R}$.



41. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of this line segment is $\dfrac{-6-1}{4-(-2)} = -\dfrac{7}{6}$, so an equation is $y - 1 = -\frac{7}{6}(x + 2)$. The function is $f(x) = -\frac{7}{6}x - \frac{4}{3}$, $-2 \le x \le 4$.

42. The slope of this line segment is $\dfrac{3-(-2)}{6-(-3)} = \dfrac{5}{9}$, so an equation is $y + 2 = \frac{5}{9}(x + 3)$. The function is $f(x) = \frac{5}{9}x - \frac{1}{3}$, $-3 \le x \le 6$.

43. We need to solve the given equation for y. $x + (y-1)^2 = 0 \iff (y-1)^2 = -x \iff y - 1 = \pm\sqrt{-x} \iff y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$. Note that the domain is $x \le 0$.

44. $(x-1)^2 + y^2 = 1 \iff y = \pm\sqrt{1-(x-1)^2} = \pm\sqrt{2x - x^2}$. The top half is given by the function $f(x) = \sqrt{2x - x^2}$, $0 \le x \le 2$.

45. For $-1 \le x \le 2$, the graph is the line with slope 1 and y-intercept 1, that is, the line $y = x + 1$. For $2 < x \le 4$, the graph is the line with slope $-\frac{3}{2}$ and x-intercept 4 [which corresponds to the point $(4, 0)$], so $y - 0 = -\frac{3}{2}(x - 4) = -\frac{3}{2}x + 6$. So the function is $f(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 2 \\ -\frac{3}{2}x + 6 & \text{if } 2 < x \le 4 \end{cases}$

46. For $x \le 0$, the graph is the line $y = 2$. For $0 < x \le 1$, the graph is the line with slope -2 and y-intercept 2, that is, the line $y = -2x + 2$. For $x > 1$, the graph is the line with slope 1 and x-intercept 1, that is, the line $y = 1\,(x - 1) = x - 1$. So the function is $f(x) = \begin{cases} 2 & \text{if } x \le 0 \\ -2x + 2 & \text{if } 0 < x \le 1 \\ x - 1 & \text{if } 1 < x \end{cases}$.

47. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

48. Let the length and width of the rectangle be L and W. Then the area is $LW = 16$, so that $W = 16/L$. The perimeter is $P = 2L + 2W$, so $P(L) = 2L + 2(16/L) = 2L + 32/L$, and the domain of P is $L > 0$, since lengths must be positive quantities. If we further restrict L to be larger than W, then $L > 4$ would be the domain.

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49. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

50. Let the volume of the cube be V and the length of an edge be L. Then $V = L^3$ so $L = \sqrt[3]{V}$, and the surface area is $S(V) = 6\left(\sqrt[3]{V}\right)^2 = 6V^{2/3}$, with domain $V > 0$.

51. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.

52. The area of the window is $A = xh + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = xh + \dfrac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window. The perimeter is $P = 2h + x + \frac{1}{2}\pi x = 30 \quad \Leftrightarrow \quad 2h = 30 - x - \frac{1}{2}\pi x \quad \Leftrightarrow \quad h = \frac{1}{4}(60 - 2x - \pi x)$. Thus,

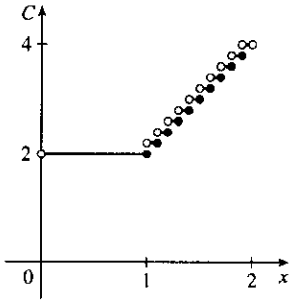
$$A(x) = x\,\frac{60 - 2x - \pi x}{4} + \frac{\pi x^2}{8} = 15x - \tfrac{1}{2}x^2 - \tfrac{\pi}{4}x^2 + \tfrac{\pi}{8}x^2 = 15x - \tfrac{4}{8}x^2 - \tfrac{\pi}{8}x^2 = 15x - x^2\left(\tfrac{\pi+4}{8}\right)$$

Since the lengths x and h must be positive quantities, we have $x > 0$ and $h > 0$. For $h > 0$, we have $2h > 0 \quad \Leftrightarrow$ $30 - x - \frac{1}{2}\pi x > 0 \quad \Leftrightarrow \quad 60 > 2x + \pi x \quad \Leftrightarrow \quad x < \dfrac{60}{2+\pi}$. Hence, the domain of A is $0 < x < \dfrac{60}{2+\pi}$.

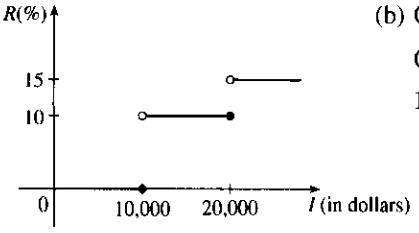
53. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so $V(x) = (20 - 2x)(12 - 2x)\,(x) = 4(10 - x)(6 - x)(x) = 4x\left(60 - 16x + x^2\right) = 4x^3 - 64x^2 + 240x$. The sides L, W, and x must be positive. Thus, $L > 0 \quad \Leftrightarrow \quad 20 - 2x > 0 \quad \Leftrightarrow \quad x < 10$; $W > 0 \quad \Leftrightarrow$ $12 - 2x > 0 \quad \Leftrightarrow \quad x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

54.

$$C(x) = \begin{cases} \$2.00 & \text{if } 0.0 < x \le 1.0 \\ 2.20 & \text{if } 1.0 < x \le 1.1 \\ 2.40 & \text{if } 1.1 < x \le 1.2 \\ 2.60 & \text{if } 1.2 < x \le 1.3 \\ 2.80 & \text{if } 1.3 < x \le 1.4 \\ 3.00 & \text{if } 1.4 < x \le 1.5 \\ 3.20 & \text{if } 1.5 < x \le 1.6 \\ 3.40 & \text{if } 1.6 < x \le 1.7 \\ 3.60 & \text{if } 1.7 < x \le 1.8 \\ 3.80 & \text{if } 1.8 < x \le 1.9 \\ 4.00 & \text{if } 1.9 < x < 2.0 \end{cases}$$



55. (a)



(b) On \$14,000, tax is assessed on \$4000, and $10\%(\$4000) = \$400$.
On \$26,000, tax is assessed on \$16,000, and
$10\%(\$10{,}000) + 15\%(\$6000) = \$1000 + \$900 = \$1900$.

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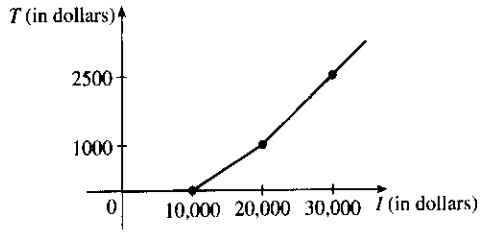
(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from $(10{,}000, 0)$ to $(20{,}000, 1000)$. The tax on \$30,000 is \$2500, so the graph of T for $x > 20{,}000$ is the ray with initial point $(20{,}000, 1000)$ that passes through $(30{,}000, 2500)$.



56. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

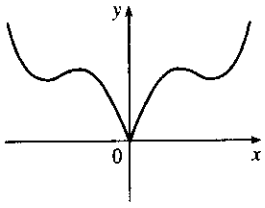
57. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y-axis.

58. f is not an even function since it is not symmetric with respect to the y-axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y-axis.

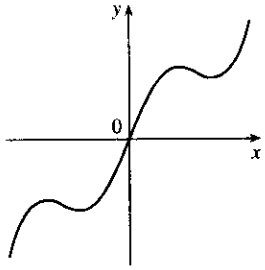
59. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

60. (a) If f is even, we get the rest of the graph by reflecting about the y-axis.



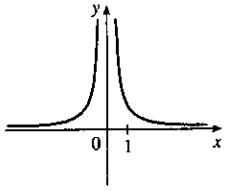
(b) If f is odd, we get the rest of the graph by rotating 180° about the origin.



61. $f(x) = x^{-2}$.

$$f(-x) = (-x)^{-2} = \frac{1}{(-x)^2} = \frac{1}{x^2}$$
$$= x^{-2} = f(x)$$

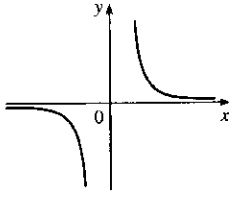
So f is an even function.



62. $f(x) = x^{-3}$.

$$f(-x) = (-x)^{-3} = \frac{1}{(-x)^3} = \frac{1}{-x^3}$$
$$= -\frac{1}{x^3} = -\left(x^{-3}\right) = -f(x)$$

So f is odd.



63. $f(x) = x^2 + x$, so $f(-x) = (-x)^2 + (-x) = x^2 - x$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

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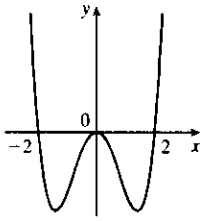
64. $f(x) = x^4 - 4x^2$.

$f(-x) = (-x)^4 - 4(-x)^2$

$= x^4 - 4x^2 = f(x)$

So f is even.

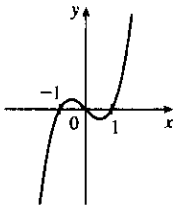


65. $f(x) = x^3 - x$.

$f(-x) = (-x)^3 - (-x) = -x^3 + x$

$= -(x^3 - x) = -f(x)$

So f is odd.



66. $f(x) = 3x^3 + 2x^2 + 1$, so $f(-x) = 3(-x)^3 + 2(-x)^2 + 1 = -3x^3 + 2x^2 + 1$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

1.2 Mathematical Models: A Catalog of Essential Functions

1. (a) $f(x) = \sqrt[5]{x}$ is a root function with $n = 5$.

(b) $g(x) = \sqrt{1 - x^2}$ is an algebraic function because it is a root of a polynomial.

(c) $h(x) = x^9 + x^4$ is a polynomial of degree 9.

(d) $r(x) = \dfrac{x^2 + 1}{x^3 + x}$ is a rational function because it is a ratio of polynomials.

(e) $s(x) = \tan 2x$ is a trigonometric function.

(f) $t(x) = \log_{10} x$ is a logarithmic function.

2. (a) $y = (x - 6)/(x + 6)$ is a rational function because it is a ratio of polynomials.

(b) $y = x + x^2/\sqrt{x - 1}$ is an algebraic function because it involves polynomials and roots of polynomials.

(c) $y = 10^x$ is an exponential function (notice that x is the *exponent*).

(d) $y = x^{10}$ is a power function (notice that x is the *base*).

(e) $y = 2t^6 + t^4 - \pi$ is a polynomial of degree 6.

(f) $y = \cos\theta + \sin\theta$ is a trigonometric function.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $[y = x^5]$ must be f. Since g is flatter than h near the origin, we must have (c) $[y = x^8]$ matched with g and (a) $[y = x^2]$ matched with h.

4. (a) The graph of $y = 3x$ is a line (choice G).

(b) $y = 3^x$ is an exponential function (choice f).

(c) $y = x^3$ is an odd polynomial function or power function (choice F).

(d) $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

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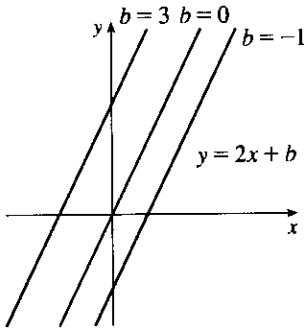
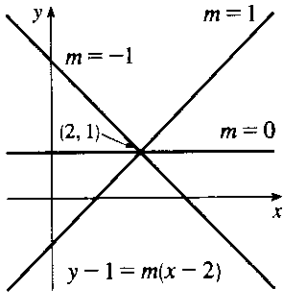
5. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

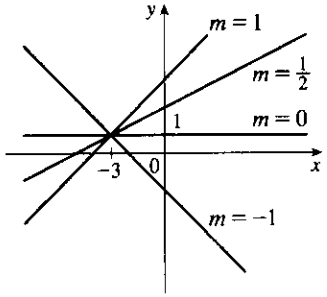


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.

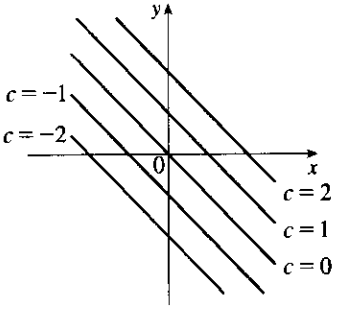


(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

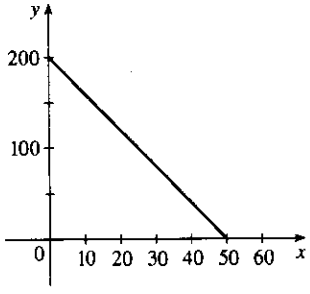
6. All members of the family of linear functions $f(x) = 1 + m(x + 3)$ have graphs that are lines passing through the point $(-3, 1)$.



7. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1. The y-intercept is c.



8. (a)



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4. The y-intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x-intercept of 50 is the smallest rental fee that results in no spaces rented.

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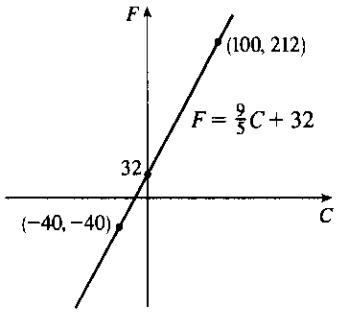
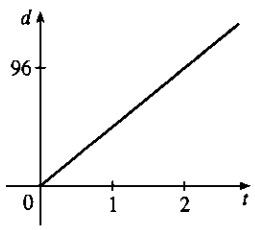
9. (a)



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of 1 °C. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

10. (a) Let d = distance traveled (in miles) and t = time elapsed (in hours). At $t = 0$, $d = 0$ and at $t = 50$ minutes $= 50 \cdot \frac{1}{60} = \frac{5}{6}$ h, $d = 40$. Thus we have two points: $(0, 0)$ and $\left(\frac{5}{6}, 40\right)$, so $m = \dfrac{40 - 0}{\frac{5}{6} - 0} = 48$ and so $d = 48t$.

(b)



(c) The slope is 48 and represents the car's speed in mi/h.

11. (a) Using N in place of x and T in place of y, we find the slope to be $\dfrac{T_2 - T_1}{N_2 - N_1} = \dfrac{80 - 70}{173 - 113} = \dfrac{10}{60} = \dfrac{1}{6}$. So a linear equation is $T - 80 = \frac{1}{6}(N - 173) \Leftrightarrow T - 80 = \frac{1}{6}N - \frac{173}{6} \Leftrightarrow T = \frac{1}{6}N + \frac{307}{6}$ $\left[\frac{307}{6} = 51.1\overline{6}\right]$.

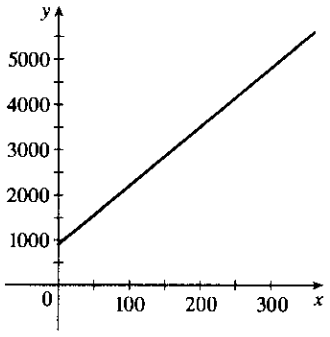
(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1 °F.

(c) When $N = 150$, the temperature is given approximately by $T = \frac{1}{6}(150) + \frac{307}{6} = 76.1\overline{6}\,^\circ\text{F} \approx 76\,^\circ\text{F}$.

12. (a) Let x denote the number of chairs produced in one day and y the associated cost. Using the points $(100, 2200)$ and $(300, 4800)$ we get the slope $\frac{4800-2200}{300-100} = \frac{2600}{200} = 13$. So $y - 2200 = 13(x - 100) \Leftrightarrow y = 13x + 900$.

(b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.

(c) The y-intercept is 900 and it represents the fixed daily costs of operating the factory.



13. (a) We are given $\dfrac{\text{change in pressure}}{10 \text{ feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.

(b) When $P = 100$, then $100 = 0.434d + 15 \Leftrightarrow 0.434d = 85 \Leftrightarrow d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is 100 lb/in^2 at a depth of approximately 196 feet.

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14. (a) Using d in place of x and C in place of y, we find the slope to be

$$\frac{C_2 - C_1}{d_2 - d_1} = \frac{460 - 380}{800 - 480} = \frac{80}{320} = \frac{1}{4}$$

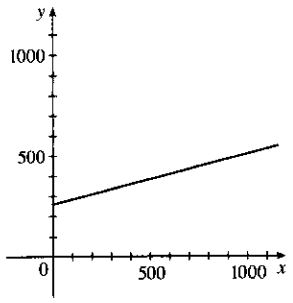
So a linear equation is $C - 460 = \frac{1}{4}(d - 800) \Leftrightarrow$

$C - 460 = \frac{1}{4}d - 200 \Leftrightarrow C = \frac{1}{4}d + 260.$

(b) Letting $d = 1500$ we get $C = \frac{1}{4}(1500) + 260 = 635$.

The cost of driving 1500 miles is \$635.

(c)



The slope of the line represents the cost per mile, \$0.25.

(d) The y-intercept represents the fixed cost, \$260.

(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

15. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

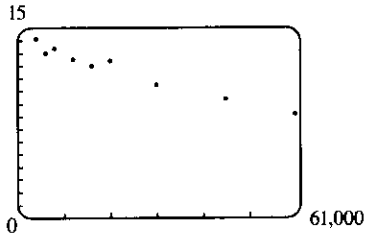
(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

16. (a) The data appear to be increasing exponentially. A model of the form $f(x) = a \cdot b^x$ or $f(x) = a \cdot b^x + c$ seems appropriate.

(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x) = a/x$ seems appropriate.

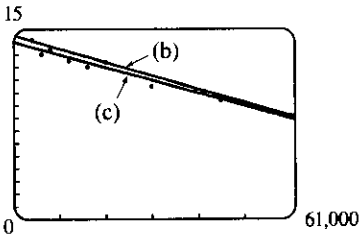
Some values are given to many decimal places. These are the results given by several computer algebra systems—rounding is left to the reader.

17. (a)



A linear model does seem appropriate.

(b) Using the points $(4000, 14.1)$ and $(60{,}000, 8.2)$, we obtain $y - 14.1 = \frac{8.2 - 14.1}{60{,}000 - 4000}(x - 4000)$ or, equivalently, $y \approx -0.000105357x + 14.521429$.



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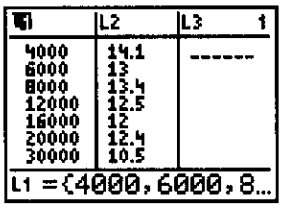
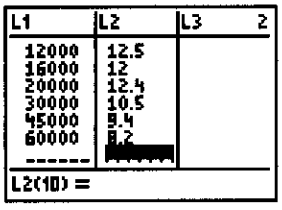
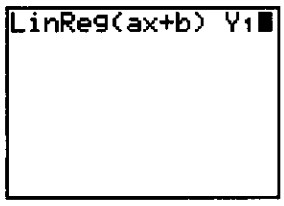
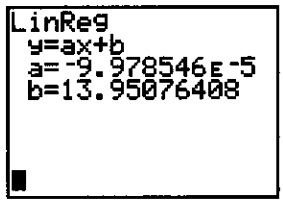
(c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$. The following commands and screens illustrate how to find the least squares regression line on a TI-83 Plus. Enter the data into list one (L1) and list two (L2). Press STAT 1 to enter the editor.

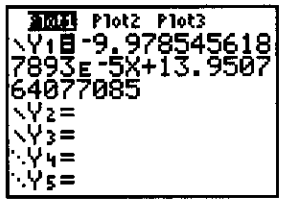




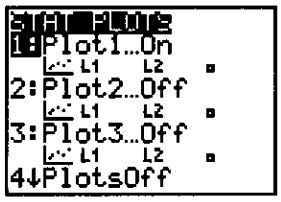
Find the regession line and store it in Y_1. Press 2nd QUIT STAT ▸ 4 VARS ▸ 1 1 ENTER.

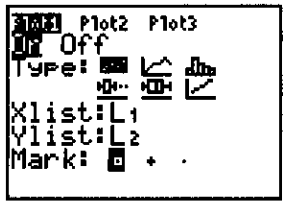




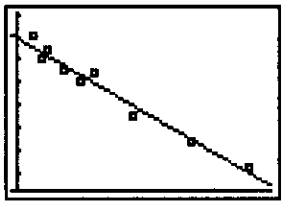


Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing ENTER or by pressing 2nd STAT PLOT 1 ENTER.





Now press ZOOM 9 to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.



(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

(f) When $x = 200{,}000$, y is negative, so the model does not apply.

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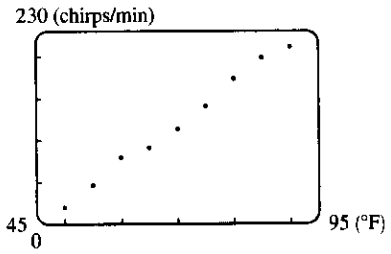
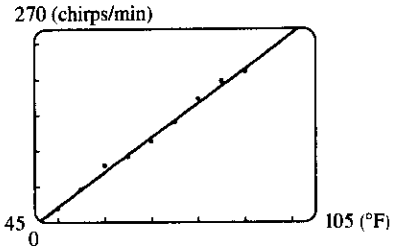
18. (a)



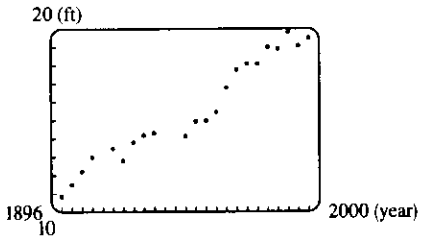
(b)



Using a computing device, we obtain the least squares regression line $y = 4.85\overline{6}x - 220.9\overline{6}$.

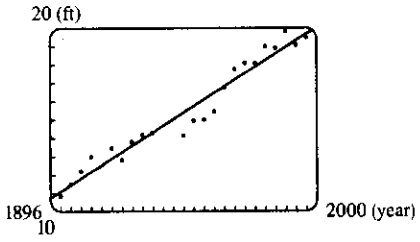
(c) When $x = 100°$ F, $y = 264.7 \approx 265$ chirps/min.

19. (a)



A linear model does seem appropriate.

(b)

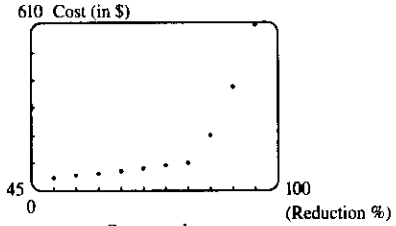


Using a computing device, we obtain the least squares regression line $y = 0.089119747x - 158.2403249$, where x is the year and y is the height in feet.

(c) When $x = 2000$, the model gives $y \approx 20.00$ ft. Note that the actual winning height for the 2000 Olympics is *less than* the winning height for 1996—so much for that prediction.

(d) When $x = 2100$, $y \approx 28.91$ ft. This would be an increase of 9.49 ft from 1996 to 2100. Even though there was an increase of 8.59 ft from 1900 to 1996, it is unlikely that a similar increase will occur over the next 100 years.

20. By looking at the scatter plot of the data, we rule out the linear and logarithmic models.



Scatter plot

We try various models:

Quadratic: $y = 0.49\overline{6}x^2 - 62.28\overline{93}x + 1970.6\overline{39}$

Cubic: $y = 0.0201243201x^3 - 3.88037296x^2 + 247.6754468x - 5163.935198$

Quartic: $y = 0.0002951049x^4 - 0.0654560995x^3 + 5.27525641x^2 - 180.2266511x + 2203.210956$

Exponential: $y = 2.41422994\,(1.054516914)^x$

Power: $y = 0.000022854971x^{3.616078251}$

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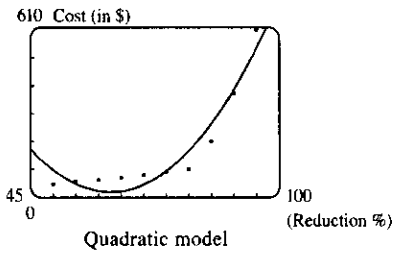
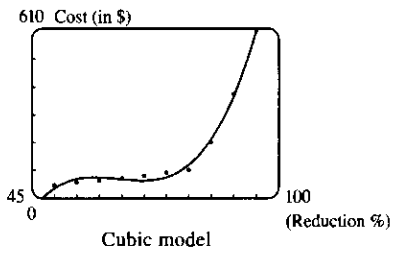
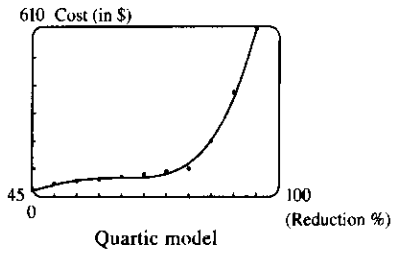


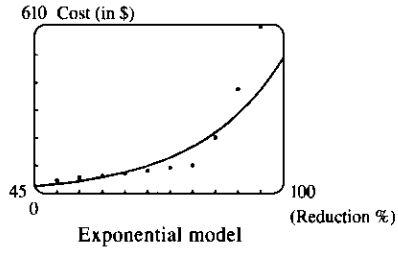
Quadratic model



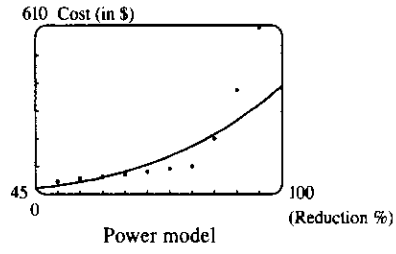
Cubic model



Quartic model



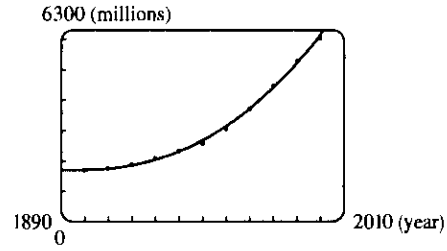
Exponential model



Power model

After examining the graphs of these models, we see that the cubic and quartic models are clearly the best.

21.



Using a computing device, we obtain the cubic function $y = ax^3 + bx^2 + cx + d$ with $a = 0.0012937$, $b = -7.06142$, $c = 12{,}823$, and $d = -7{,}743{,}770$. When $x = 1925$, $y \approx 1914$ (million).

22. (a) $T = 1.000396048d^{1.499661718}$

(b) The power model in part (a) is approximately $T = d^{1.5}$. Squaring both sides gives us $T^2 = d^3$, so the model matches Kepler's Third Law, $T^2 = kd^3$.

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

2. (a) To obtain the graph of $y = 5f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5.

(b) To obtain the graph of $y = f(x - 5)$ from the graph of $y = f(x)$, shift the graph 5 units to the right.

(c) To obtain the graph of $y = -f(x)$ from the graph of $y = f(x)$, reflect the graph about the x-axis.

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(d) To obtain the graph of $y = -5f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5 and reflect it about the x-axis.

(e) To obtain the graph of $y = f(5x)$ from the graph of $y = f(x)$, shrink the graph horizontally by a factor of 5.

(f) To obtain the graph of $y = 5f(x) - 3$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5 and shift it 3 units downward.

3. (a) (graph 3) The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

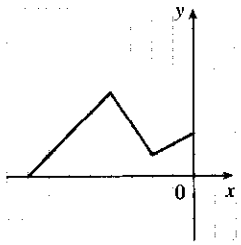
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x + 4)$.

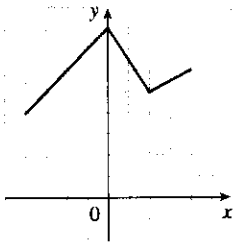
(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

4. (a) To graph $y = f(x + 4)$ we shift the graph of f, 4 units to the left.



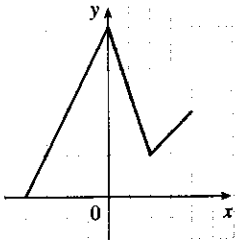
The point $(2, 1)$ on the graph of f corresponds to the point $(2 - 4, 1) = (-2, 1)$.

(b) To graph $y = f(x) + 4$ we shift the graph of f, 4 units upward.



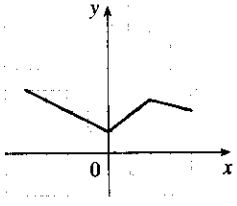
The point $(2, 1)$ on the graph of f corresponds to the point $(2, 1 + 4) = (2, 5)$.

(c) To graph $y = 2f(x)$ we stretch the graph of f vertically by a factor of 2.



The point $(2, 1)$ on the graph of f corresponds to the point $(2, 2 \cdot 1) = (2, 2)$.

(d) To graph $y = -\frac{1}{2}f(x) + 3$, we shrink the graph of f vertically by a factor of 2, then reflect the resulting graph about the x-axis, then shift the resulting graph 3 units upward.



The point $(2, 1)$ on the graph of f corresponds to the point $\left(2, -\frac{1}{2} \cdot 1 + 3\right) = (2, 2.5)$.

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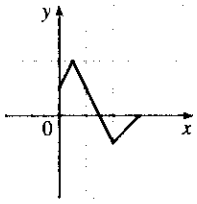
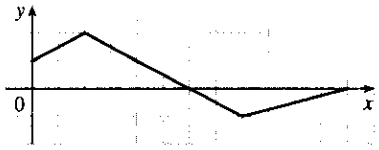
5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



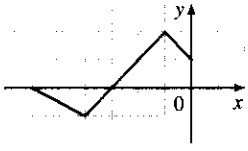
The point $(4, -1)$ on the graph of f corresponds to the point $\left(\frac{1}{2} \cdot 4, -1\right) = (2, -1)$.

(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



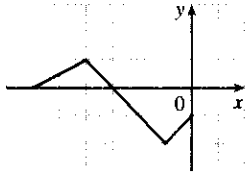
The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

6. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2. Thus, a function describing the graph is

$$\begin{aligned} y &= 2f(x-2) = 2\sqrt{3(x-2) - (x-2)^2} \\ &= 2\sqrt{3x - 6 - (x^2 - 4x + 4)} = 2\sqrt{-x^2 + 7x - 10} \end{aligned}$$

7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x-axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1 \cdot}_{\substack{\text{reflect} \\ \text{about} \\ x\text{-axis}}} \underbrace{f(x+4)}_{\substack{\text{shift} \\ \text{4 units} \\ \text{left}}} \underbrace{-1}_{\substack{\text{shift} \\ \text{1 unit} \\ \text{down}}}$$

This function can be written as

$$\begin{aligned} y &= -f(x+4) - 1 = -\sqrt{3(x+4) - (x+4)^2} - 1 = -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 \\ &= -\sqrt{-x^2 - 5x - 4} - 1 \end{aligned}$$

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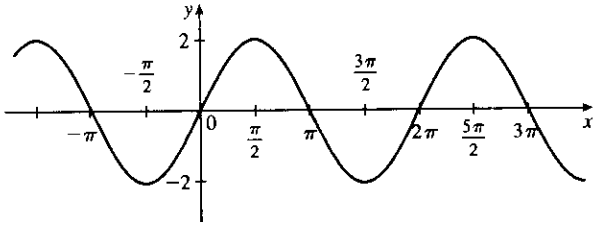
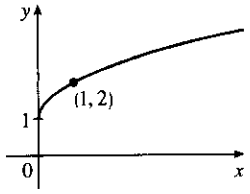
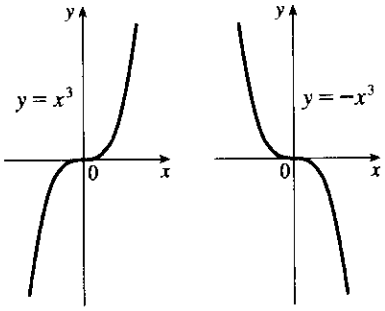
8. (a) The graph of $y = 2\sin x$ can be obtained from the graph of $y = \sin x$ by stretching it vertically by a factor of 2.



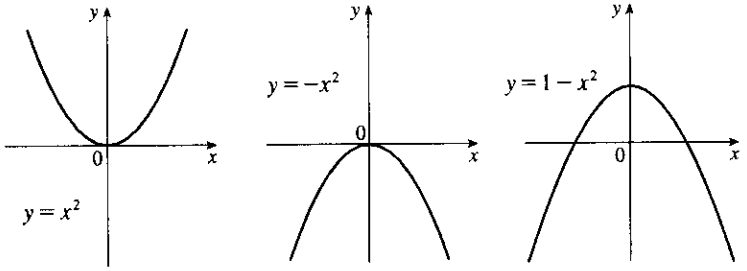
(b) The graph of $y = 1 + \sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting it upward 1 unit.



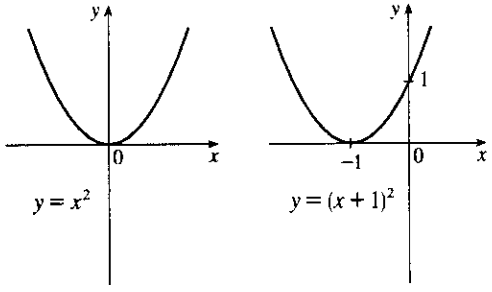
9. $y = -x^3$: Start with the graph of $y = x^3$ and reflect about the x-axis. Note: Reflecting about the y-axis gives the same result since substituting $-x$ for x gives us $y = (-x)^3 = -x^3$.



10. $y = 1 - x^2 = -x^2 + 1$: Start with the graph of $y = x^2$, reflect about the x-axis, and then shift 1 unit upward.



11. $y = (x + 1)^2$: Start with the graph of $y = x^2$ and shift 1 unit to the left.



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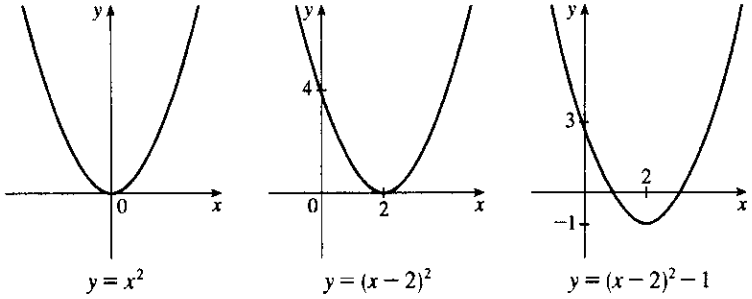
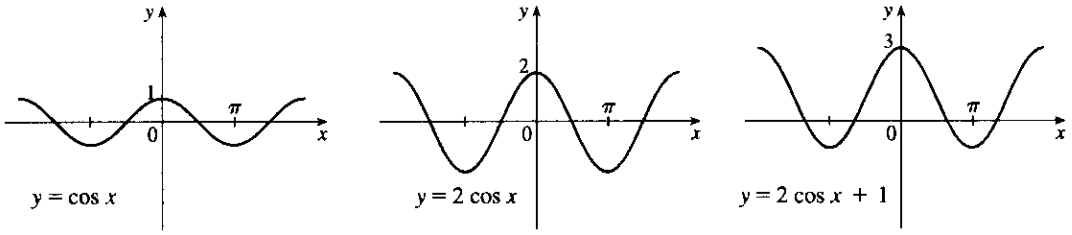
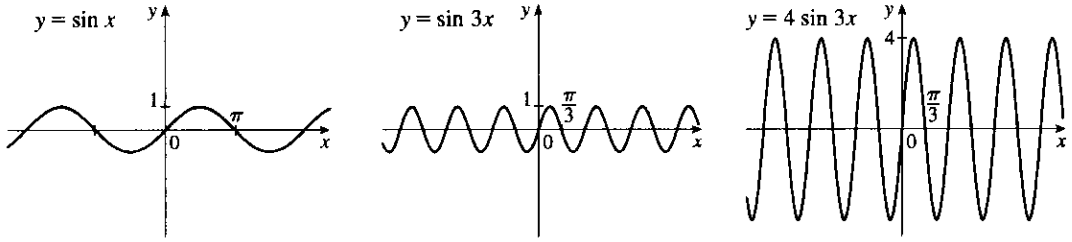
12. $y = x^2 - 4x + 3 = (x^2 - 4x + 4) - 1 = (x - 2)^2 - 1$: Start with the graph of $y = x^2$, shift 2 units to the right, and then shift 1 unit downward.



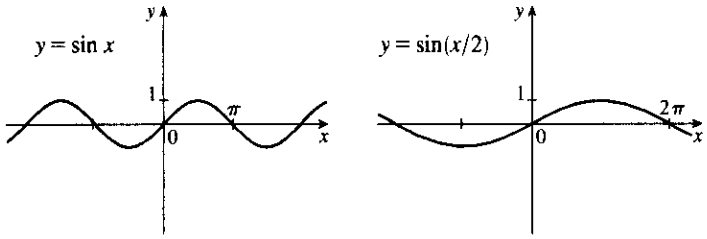
13. $y = 1 + 2\cos x$: Start with the graph of $y = \cos x$, stretch vertically by a factor of 2, and then shift 1 unit upward.



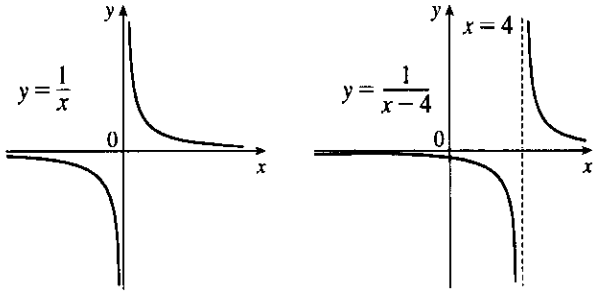
14. $y = 4\sin 3x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 3, and then stretch vertically by a factor of 4.



15. $y = \sin(x/2)$: Start with the graph of $y = \sin x$ and stretch horizontally by a factor of 2.



16. $y = 1/(x - 4)$: Start with the graph of $y = 1/x$ and shift 4 units to the right.



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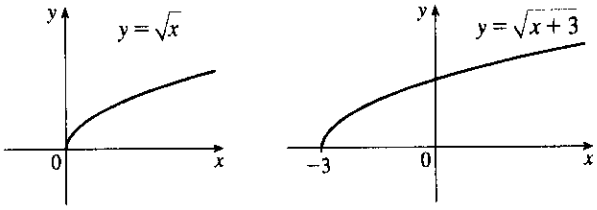
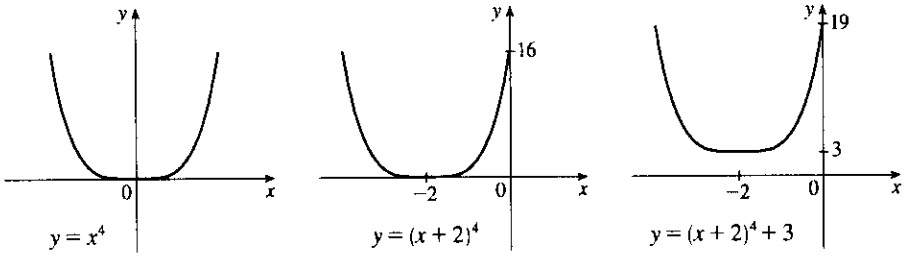
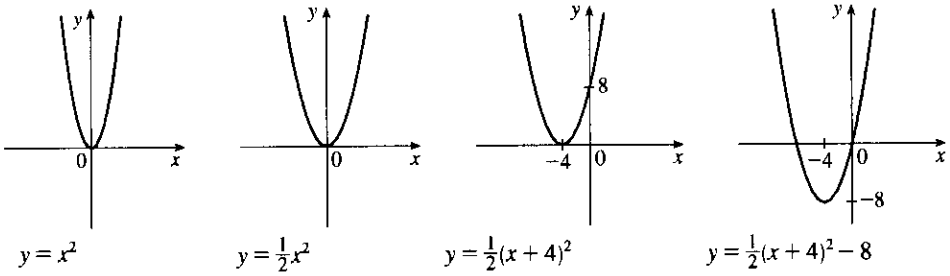
17. $y = \sqrt{x+3}$: Start with the graph of $y = \sqrt{x}$ and shift 3 units to the left.



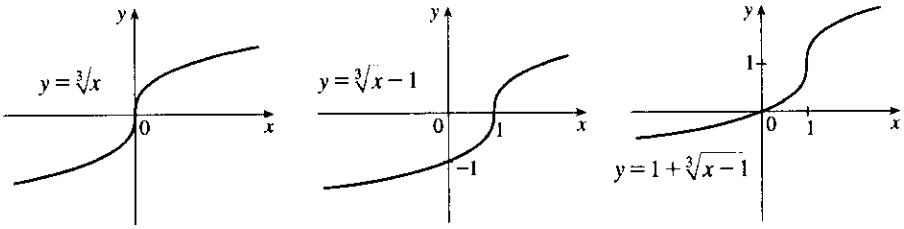
18. $y = (x+2)^4 + 3$: Start with the graph of $y = x^4$, shift 2 units to the left, and then shift 3 units upward.



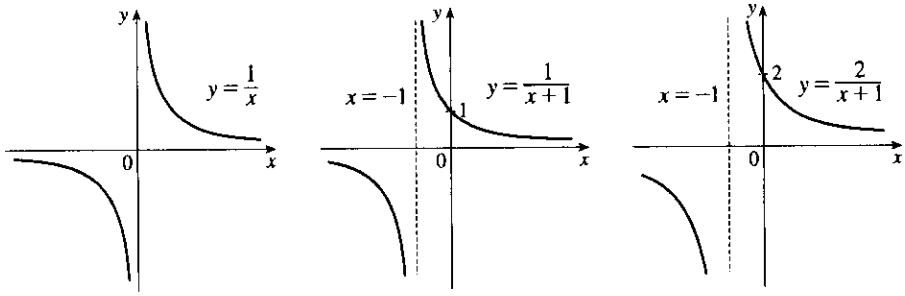
19. $y = \frac{1}{2}(x^2 + 8x) = \frac{1}{2}(x^2 + 8x + 16) - 8 = \frac{1}{2}(x+4)^2 - 8$: Start with the graph of $y = x^2$, compress vertically by a factor of 2, shift 4 units to the left, and then shift 8 units downward.



20. $y = 1 + \sqrt[3]{x-1}$: Start with the graph of $y = \sqrt[3]{x}$, shift 1 unit to the right, and then shift 1 unit upward.



21. $y = 2/(x+1)$: Start with the graph of $y = 1/x$, shift 1 unit to the left, and then stretch vertically by a factor of 2.



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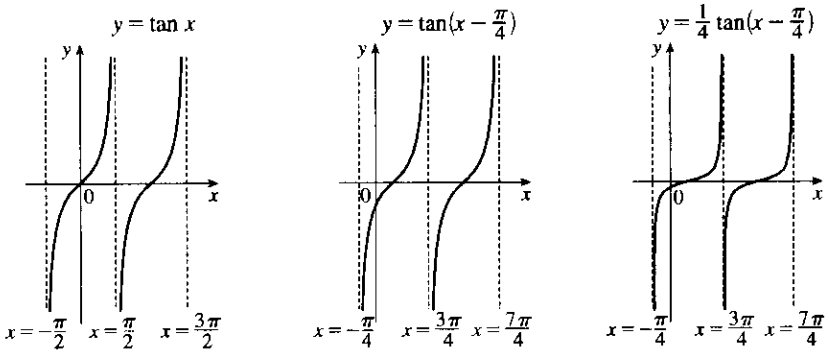
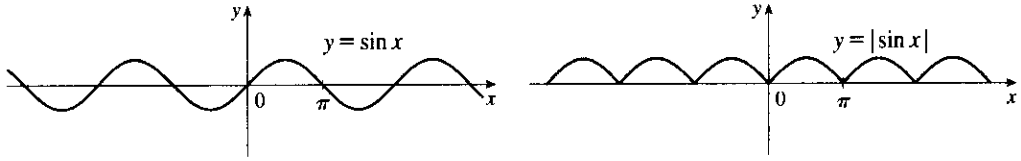
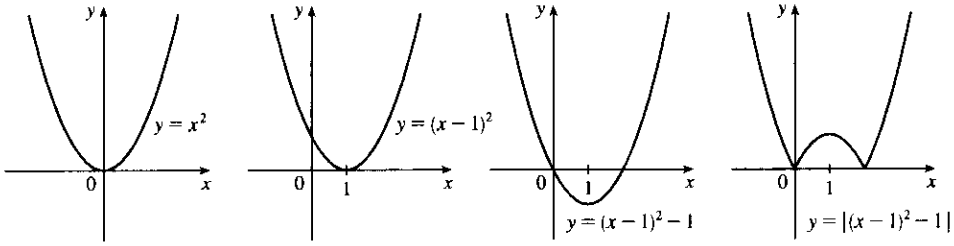
22. $y = \frac{1}{4}\tan(x - \frac{\pi}{4})$: Start with the graph of $y = \tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4.



23. $y = |\sin x|$: Start with the graph of $y = \sin x$ and reflect all the parts of the graph below the x-axis about the x-axis.



24. $y = |x^2 - 2x| = |x^2 - 2x + 1 - 1| = |(x-1)^2 - 1|$: Start with the graph of $y = x^2$, shift 1 unit right, shift 1 unit downward, and reflect the portion of the graph below the x-axis about the x-axis.



25. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 A.M. to 6:18 P.M.) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

26. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0. If we take $t = 0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t) = 4.0 + 0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

27. (a) To obtain $y = f(|x|)$, the portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

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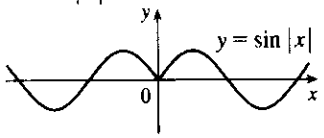
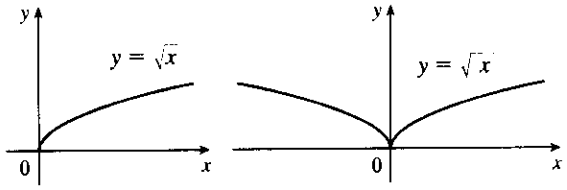
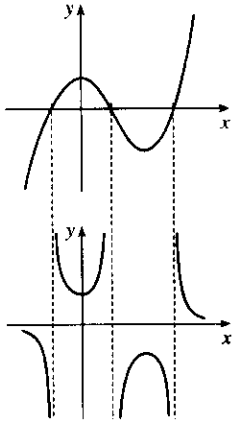
(b) $y = \sin|x|$



(c) $y = \sqrt{|x|}$

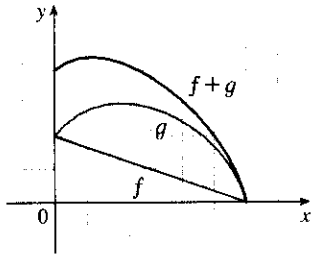


28. The most important features of the given graph are the x-intercepts and the maximum and minimum points. The graph of $y = 1/f(x)$ has vertical asymptotes at the x-values where there are x-intercepts on the graph of $y = f(x)$. The maximum of 1 on the graph of $y = f(x)$ corresponds to a minimum of $1/1 = 1$ on $y = 1/f(x)$. Similarly, the minimum on the graph of $y = f(x)$ corresponds to a maximum on the graph of $y = 1/f(x)$. As the values of y get large (positively or negatively) on the graph of $y = f(x)$, the values of y get close to zero on the graph of $y = 1/f(x)$.



29. Assuming that successive horizontal and vertical gridlines are a unit apart, we can make a table of approximate values as follows.

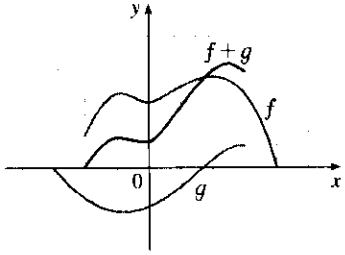
x	0	1	2	3	4	5	6
$f(x)$	2	1.7	1.3	1.0	0.7	0.3	0
$g(x)$	2	2.7	3	2.8	2.4	1.7	0
$f(x)+g(x)$	4	4.4	4.3	3.8	3.1	2.0	0



Connecting the points $(x, f(x)+g(x))$ with a smooth curve gives an approximation to the graph of $f+g$. Extra points can be plotted between those listed above if necessary.

30. First note that the domain of $f+g$ is the intersection of the domains of f and g; that is, $f+g$ is only defined where both f and g are defined. Taking the horizontal and vertical units of length to be the distances between successive vertical and horizontal gridlines, we can make a table of approximate values as follows:

x	-2	-1	0	1	2	2.5	3
$f(x)$	-1	2.2	2.0	2.4	2.7	2.7	2.3
$g(x)$	1	-1.3	-1.2	-0.6	0.3	0.5	0.7
$f(x)+g(x)$	0	0.9	0.8	1.8	3.0	3.2	3.0



Extra values of x (like the value 2.5 in the table above) can be added as needed.

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31. $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$. $D = \mathbb{R}$ for both f and g.

$(f+g)(x) = (x^3 + 2x^2) + (3x^2 - 1) = x^3 + 5x^2 - 1, \quad D = \mathbb{R}.$

$(f-g)(x) = (x^3 + 2x^2) - (3x^2 - 1) = x^3 - x^2 + 1, \quad D = \mathbb{R}.$

$(fg)(x) = (x^3 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2, \quad D = \mathbb{R}.$

$\left(\frac{f}{g}\right)(x) = \frac{x^3 + 2x^2}{3x^2 - 1}, \quad D = \left\{x \mid x \neq \pm\frac{1}{\sqrt{3}}\right\}$ since $3x^2 - 1 \neq 0$.

32. $f(x) = \sqrt{1+x}, \quad D = [-1, \infty)$; $\quad g(x) = \sqrt{1-x}, \quad D = (-\infty, 1]$.

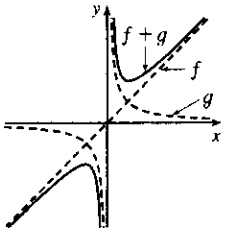
$(f+g)(x) = \sqrt{1+x} + \sqrt{1-x}, \quad D = (-\infty, 1] \cap [-1, \infty) = [-1, 1].$

$(f-g)(x) = \sqrt{1+x} - \sqrt{1-x}, \quad D = [-1, 1].$

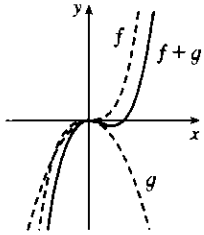
$(fg)(x) = \sqrt{1+x} \cdot \sqrt{1-x} = \sqrt{1-x^2}, \quad D = [-1, 1].$

$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{1+x}}{\sqrt{1-x}}, \quad D = [-1, 1)$. We must exclude $x = 1$ since it would make $\frac{f}{g}$ undefined.

33. $f(x) = x, \quad g(x) = 1/x$



34. $f(x) = x^3, \quad g(x) = -x^2$



35. $f(x) = 2x^2 - x$; $g(x) = 3x + 2$. $D = \mathbb{R}$ for both f and g, and hence for their composites.

$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)^2 - (3x+2) = 2(9x^2 + 12x + 4) - 3x - 2 = 18x^2 + 21x + 6.$

$(g \circ f)(x) = g(f(x)) = g(2x^2 - x) = 3(2x^2 - x) + 2 = 6x^2 - 3x + 2.$

$(f \circ f)(x) = f(f(x)) = f(2x^2 - x) = 2(2x^2 - x)^2 - (2x^2 - x) = 2(4x^4 - 4x^3 + x^2) - 2x^2 + x = 8x^4 - 8x^3 + x.$

$(g \circ g)(x) = g(g(x)) = g(3x+2) = 3(3x+2) + 2 = 9x + 6 + 2 = 9x + 8.$

36. $f(x) = 1 - x^3$, $D = \mathbb{R}$; $g(x) = 1/x$, $D = \{x \mid x \neq 0\}$.

$(f \circ g)(x) = f(g(x)) = f(1/x) = 1 - (1/x)^3 = 1 - 1/x^3$, $D = \{x \mid x \neq 0\}$.

$(g \circ f)(x) = g(f(x)) = g(1 - x^3) = 1/(1 - x^3)$, $D = \{x \mid 1 - x^3 \neq 0\} = \{x \mid x \neq 1\}$.

$(f \circ f)(x) = f(f(x)) = f(1 - x^3) = 1 - (1 - x^3)^3 \quad [= x^9 - 3x^6 + 3x^3]$, $D = \mathbb{R}$.

$(g \circ g)(x) = g(g(x)) = g(1/x) = 1/(1/x) = x$, $D = \{x \mid x \neq 0\}$ because 0 is not in the domain of g.

37. $f(x) = \sin x, \quad D = \mathbb{R}; \quad g(x) = 1 - \sqrt{x}, \quad D = [0, \infty)$.

$(f \circ g)(x) = f(g(x)) = f(1 - \sqrt{x}\,) = \sin(1 - \sqrt{x}\,)$, $D = [0, \infty)$.

$(g \circ f)(x) = g(f(x)) = g(\sin x) = 1 - \sqrt{\sin x}$. For $\sqrt{\sin x}$ to be defined, we must have

$\sin x \geq 0 \quad \Leftrightarrow \quad x \in [0, \pi] \cup [2\pi, 3\pi] \cup [-2\pi, -\pi] \cup [4\pi, 5\pi] \cup [-4\pi, -3\pi] \cup \ldots$, so

$D = \{x \mid x \in [2n\pi, \pi + 2n\pi]$, where n is an integer$\}$.

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$(f \circ f)(x) = f(f(x)) = f(\sin x) = \sin(\sin x)$, $D = \mathbb{R}$.

$(g \circ g)(x) = g(g(x)) = g(1 - \sqrt{x}\,) = 1 - \sqrt{1 - \sqrt{x}}$,
$D = \{x \geq 0 \mid 1 - \sqrt{x} \geq 0\} = \{x \geq 0 \mid 1 \geq \sqrt{x}\} = \{x \geq 0 \mid \sqrt{x} \leq 1\} = [0, 1]$.

38. $f(x) = 1 - 3x$, $D = \mathbb{R}$; $g(x) = 5x^2 + 3x + 2$, $D = \mathbb{R}$.

$$(f \circ g)(x) = f(g(x)) = f(5x^2 + 3x + 2) = 1 - 3(5x^2 + 3x + 2)$$
$$= 1 - 15x^2 - 9x - 6 = -15x^2 - 9x - 5, \quad D = \mathbb{R}.$$

$$(g \circ f)(x) = g(f(x)) = g(1 - 3x) = 5(1 - 3x)^2 + 3(1 - 3x) + 2 = 5(1 - 6x + 9x^2) + 3 - 9x + 2$$
$$= 5 - 30x + 45x^2 - 9x + 5 = 45x^2 - 39x + 10, \quad D = \mathbb{R}.$$

$$(f \circ f)(x) = f(f(x)) = f(1 - 3x) = 1 - 3(1 - 3x) = 1 - 3 + 9x = 9x - 2, \quad D = \mathbb{R}.$$

$$(g \circ g)(x) = g(g(x)) = g(5x^2 + 3x + 2) = 5(5x^2 + 3x + 2)^2 + 3(5x^2 + 3x + 2) + 2$$
$$= 5(25x^4 + 30x^3 + 29x^2 + 12x + 4) + 15x^2 + 9x + 6 + 2$$
$$= 125x^4 + 150x^3 + 145x^2 + 60x + 20 + 15x^2 + 9x + 8$$
$$= 125x^4 + 150x^3 + 160x^2 + 69x + 28, \quad D = \mathbb{R}.$$

39. $f(x) = x + \dfrac{1}{x}$, $D = \{x \mid x \neq 0\}$; $g(x) = \dfrac{x+1}{x+2}$, $D = \{x \mid x \neq -2\}$.

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+1}{x+2}\right) = \frac{x+1}{x+2} + \frac{1}{\frac{x+1}{x+2}} = \frac{x+1}{x+2} + \frac{x+2}{x+1}$$
$$= \frac{(x+1)(x+1) + (x+2)(x+2)}{(x+2)(x+1)} = \frac{(x^2 + 2x + 1) + (x^2 + 4x + 4)}{(x+2)(x+1)} = \frac{2x^2 + 6x + 5}{(x+2)(x+1)}$$

Since $g(x)$ is not defined for $x = -2$ and $f(g(x))$ is not defined for $x = -2$ and $x = -1$, the domain of $(f \circ g)(x)$ is $D = \{x \mid x \neq -2, -1\}$.

$$(g \circ f)(x) = g(f(x)) = g\left(x + \frac{1}{x}\right) = \frac{\left(x + \frac{1}{x}\right) + 1}{\left(x + \frac{1}{x}\right) + 2} = \frac{\frac{x^2 + 1 + x}{x}}{\frac{x^2 + 1 + 2x}{x}} = \frac{x^2 + x + 1}{x^2 + 2x + 1} = \frac{x^2 + x + 1}{(x+1)^2}.$$

Since $f(x)$ is not defined for $x = 0$ and $g(f(x))$ is not defined for $x = -1$, the domain of $(g \circ f)(x)$ is $D = \{x \mid x \neq -1, 0\}$.

$$(f \circ f)(x) = f(f(x)) = f\left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right) + \frac{1}{x + \frac{1}{x}} = x + \frac{1}{x} + \frac{1}{\frac{x^2+1}{x}} = x + \frac{1}{x} + \frac{x}{x^2 + 1}$$
$$= \frac{x(x)(x^2 + 1) + 1(x^2 + 1) + x(x)}{x(x^2 + 1)} = \frac{x^4 + x^2 + x^2 + 1 + x^2}{x(x^2 + 1)}$$
$$= \frac{x^4 + 3x^2 + 1}{x(x^2 + 1)}, \quad D = \{x \mid x \neq 0\}.$$

$$(g \circ g)(x) = g(g(x)) = g\left(\frac{x+1}{x+2}\right) = \frac{\frac{x+1}{x+2} + 1}{\frac{x+1}{x+2} + 2} = \frac{\frac{x+1+1(x+2)}{x+2}}{\frac{x+1+2(x+2)}{x+2}} = \frac{x+1+x+2}{x+1+2x+4} = \frac{2x+3}{3x+5}.$$

Since $g(x)$ is not defined for $x = -2$ and $g(g(x))$ is not defined for $x = -\frac{5}{3}$, the domain of $(g \circ g)(x)$ is $D = \left\{x \mid x \neq -2, -\frac{5}{3}\right\}$.

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40. $f(x) = \sqrt{2x+3}$, $D = \{x \mid x \geq -\frac{3}{2}\}$; $g(x) = x^2 + 1$, $D = \mathbb{R}$.

$(f \circ g)(x) = f(x^2+1) = \sqrt{2(x^2+1)+3} = \sqrt{2x^2+5}$, $D = \mathbb{R}$.

$(g \circ f)(x) = g\big(\sqrt{2x+3}\,\big) = \big(\sqrt{2x+3}\,\big)^2 + 1 = (2x+3)+1 = 2x+4$, $D = \{x \mid x \geq -\frac{3}{2}\}$.

$(f \circ f)(x) = f\big(\sqrt{2x+3}\,\big) = \sqrt{2\big(\sqrt{2x+3}\,\big)+3} = \sqrt{2\sqrt{2x+3}+3}$, $D = \{x \mid x \geq -\frac{3}{2}\}$.

$(g \circ g)(x) = g(x^2+1) = (x^2+1)^2 + 1 = (x^4+2x^2+1)+1 = x^4+2x^2+2$, $D = \mathbb{R}$.

41. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x-1)) = f(2(x-1))$
$= 2(x-1)+1 = 2x-1$

42. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(1-x)) = f((1-x)^2)$
$= 2(1-x)^2 - 1 = 2x^2 - 4x + 1$

43. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+3)) = f((x+3)^2+2)$
$= f(x^2+6x+11) = \sqrt{(x^2+6x+11)-1} = \sqrt{x^2+6x+10}$

44. $(f \circ g \circ h)(x) = f(g(h(x))) = f\big(g\big(\sqrt{x+3}\,\big)\big) = f\big(\cos\sqrt{x+3}\,\big) = \dfrac{2}{\cos\sqrt{x+3}+1}$

45. Let $g(x) = x^2+1$ and $f(x) = x^{10}$. Then $(f \circ g)(x) = f(g(x)) = (x^2+1)^{10} = F(x)$.

46. Let $g(x) = \sqrt{x}$ and $f(x) = \sin x$. Then $(f \circ g)(x) = f(g(x)) = \sin(\sqrt{x}\,) = F(x)$.

47. Let $g(x) = x^2$ and $f(x) = \dfrac{x}{x+4}$. Then $(f \circ g)(x) = f(g(x)) = \dfrac{x^2}{x^2+4} = G(x)$.

48. Let $g(x) = x+3$ and $f(x) = 1/x$. Then $(f \circ g)(x) = f(g(x)) = 1/(x+3) = G(x)$.

49. Let $g(t) = \cos t$ and $f(t) = \sqrt{t}$. Then $(f \circ g)(t) = f(g(t)) = \sqrt{\cos t} = u(t)$.

50. Let $g(t) = \tan t$ and $f(t) = \dfrac{t}{1+t}$. Then $(f \circ g)(t) = f(g(t)) = \dfrac{\tan t}{1+\tan t} = u(t)$.

51. Let $h(x) = x^2$, $g(x) = 3^x$, and $f(x) = 1-x$. Then $(f \circ g \circ h)(x) = 1 - 3^{x^2} = H(x)$.

52. Let $h(x) = \sqrt{x}$, $g(x) = x-1$, and $f(x) = \sqrt[3]{x}$. Then $(f \circ g \circ h)(x) = \sqrt[3]{\sqrt{x}-1} = H(x)$.

53. Let $h(x) = \sqrt{x}$, $g(x) = \sec x$, and $f(x) = x^4$. Then $(f \circ g \circ h)(x) = (\sec\sqrt{x}\,)^4 = \sec^4(\sqrt{x}\,) = H(x)$.

54. (a) $f(g(1)) = f(6) = 5$ (b) $g(f(1)) = g(3) = 2$

(c) $f(f(1)) = f(3) = 4$ (d) $g(g(1)) = g(6) = 3$

(e) $(g \circ f)(3) = g(f(3)) = g(4) = 1$ (f) $(f \circ g)(6) = f(g(6)) = f(3) = 4$

55. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of g. Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of f.

(b) $g(f(0)) = g(0) = 3$

(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$

(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

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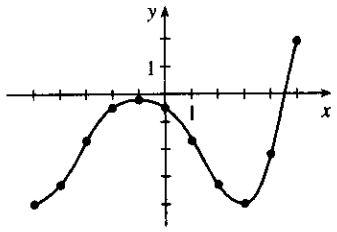
56. To find a particular value of $f(g(x))$, say for $x = 0$, we note from the graph that $g(0) \approx 2.8$ and $f(2.8) \approx -0.5$. Thus, $f(g(0)) \approx f(2.8) \approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
-5	-0.2	-4
-4	1.2	-3.3
-3	2.2	-1.7
-2	2.8	-0.5
-1	3	-0.2

x	$g(x)$	$f(g(x))$
0	2.8	-0.5
1	2.2	-1.7
2	1.2	-3.3
3	-0.2	-4
4	-1.9	-2.2
5	-4.1	1.9



57. (a) Using the relationship *distance* = *rate* · *time* with the radius r as the distance, we have $r(t) = 60t$.

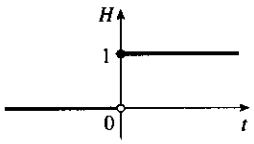
(b) $A = \pi r^2 \Rightarrow (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

58. (a) $d = rt \Rightarrow d(t) = 350t$

(b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s: $d^2 + 1^2 = s^2$. Thus, $s(d) = \sqrt{d^2 + 1}$.

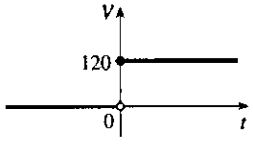
(c) $(s \circ d)(t) = s(d(t)) = s(350t) = \sqrt{(350t)^2 + 1}$

59. (a)



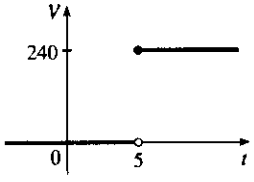
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

(b)



$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases} \quad \text{so } V(t) = 120H(t).$$

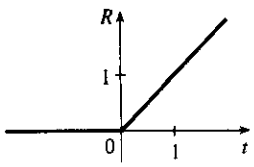
(c)



Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t - 5)$.

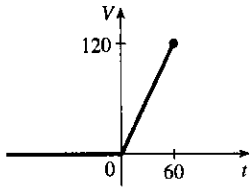
60. (a) $R(t) = tH(t)$

$$= \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$



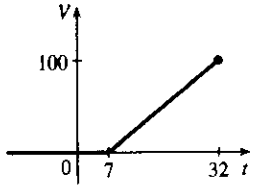
(b) $V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } 0 \leq t \leq 60 \end{cases}$

so $V(t) = 2tH(t), t \leq 60$.



(c) $V(t) = \begin{cases} 0 & \text{if } t < 7 \\ 4(t-7) & \text{if } 7 \leq t \leq 32 \end{cases}$

so $V(t) = 4(t - 7)H(t - 7), t \leq 32$.



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61. (a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 + c = 4x^2 + 4x + (1+c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But $h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$, so we see that $g(x) = x^2 + x - 1$.

62. We need a function g so that $g(f(x)) = g(x+4) = h(x) = 4x - 1 = 4(x+4) - 17$. So we see that the function g must be $g(x) = 4x - 17$.

63. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad \text{[because } g \text{ is even]} \quad = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

64. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

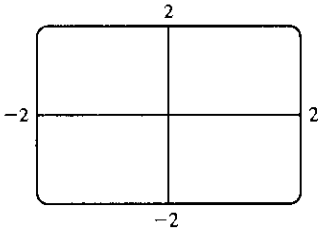
Now suppose f is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so h is odd if both f and g are odd.

Now suppose f is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so h is even if g is odd and f is even.

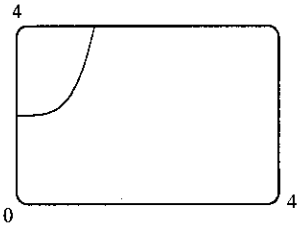
1.4 Graphing Calculators and Computers

1. $f(x) = x^4 + 2$

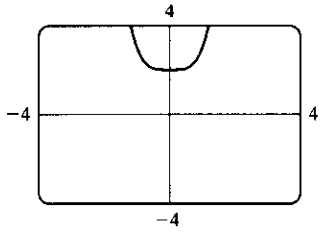
(a) $[-2, 2]$ by $[-2, 2]$



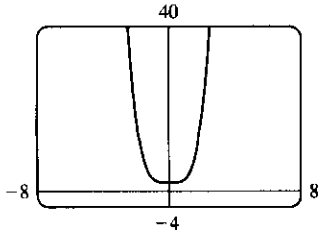
(b) $[0, 4]$ by $[0, 4]$



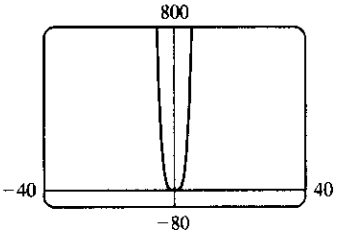
(c) $[-4, 4]$ by $[-4, 4]$



(d) $[-8, 8]$ by $[-4, 40]$



(e) $[-40, 40]$ by $[-80, 800]$



The most appropriate graph is produced in viewing rectangle (d).

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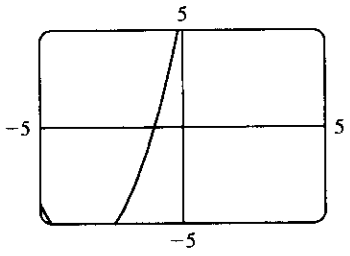
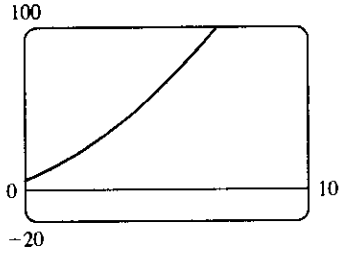
2. $f(x) = x^2 + 7x + 6$

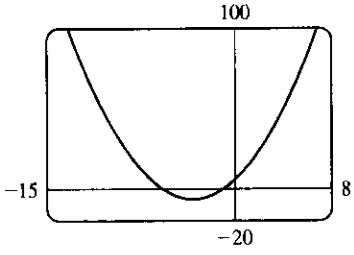
(a) $[-5, 5]$ by $[-5, 5]$



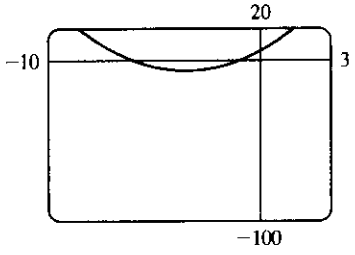
(b) $[0, 10]$ by $[-20, 100]$



(c) $[-15, 8]$ by $[-20, 100]$



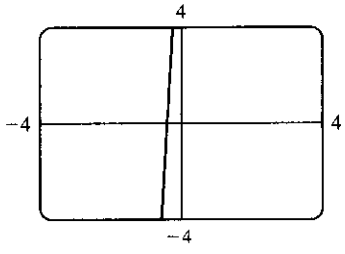
(d) $[-10, 3]$ by $[-100, 20]$



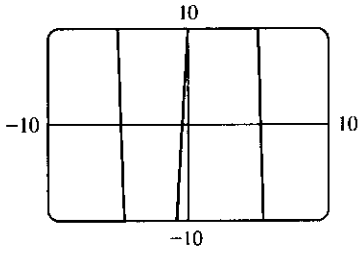
The most appropriate graph is produced in viewing rectangle (c).

3. $f(x) = 10 + 25x - x^3$

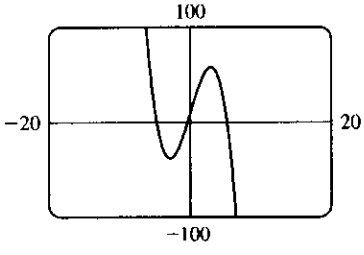
(a) $[-4, 4]$ by $[-4, 4]$



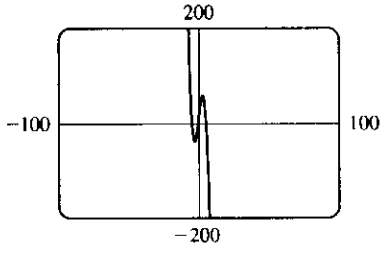
(b) $[-10, 10]$ by $[-10, 10]$



(c) $[-20, 20]$ by $[-100, 100]$



(d) $[-100, 100]$ by $[-200, 200]$



The most appropriate graph is produced in viewing rectangle (c) because the maximum and minimum points are fairly easy to see and estimate.

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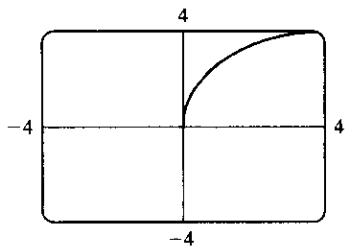
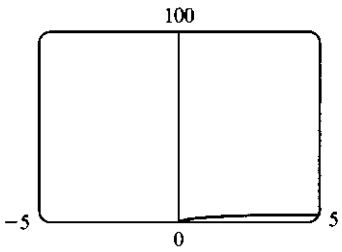
4. $f(x) = \sqrt{8x - x^2}$

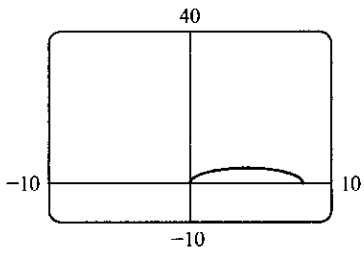
(a) $[-4, 4]$ by $[-4, 4]$



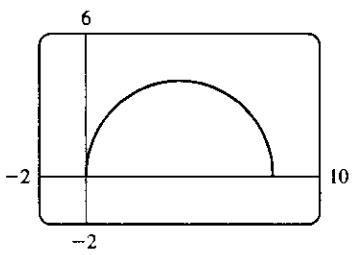
(b) $[-5, 5]$ by $[0, 100]$



(c) $[-10, 10]$ by $[-10, 40]$

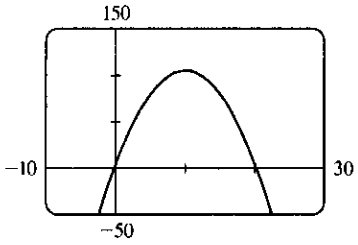


(d) $[-2, 10]$ by $[-2, 6]$

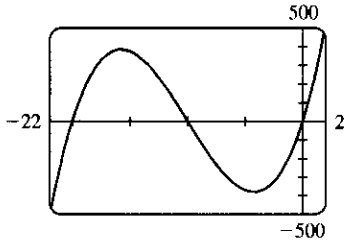


The most appropriate graph is produced in viewing rectangle (d).

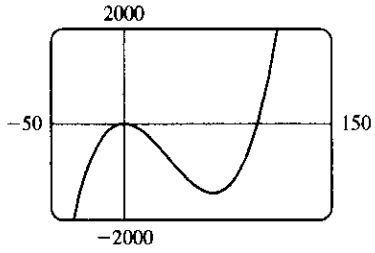
5. Since the graph of $f(x) = 5 + 20x - x^2$ is a parabola opening downward, an appropriate viewing rectangle should include the maximum point.



6. An appropriate viewing rectangle for $f(x) = x^3 + 30x^2 + 200x$ should include the high and low points.



7. $f(x) = 0.01x^3 - x^2 + 5$. Graphing f in a standard viewing rectangle, $[-10, 10]$ by $[-10, 10]$, shows us what appears to be a parabola. But since this is a cubic polynomial, we know that a larger viewing rectangle will reveal a minimum point as well as the maximum point. After some trial and error, we choose the viewing rectangle $[-50, 150]$ by $[-2000, 2000]$.



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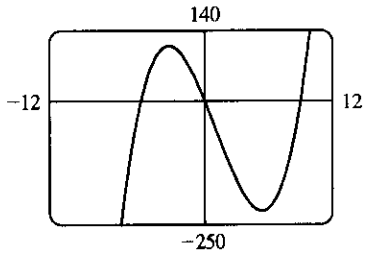
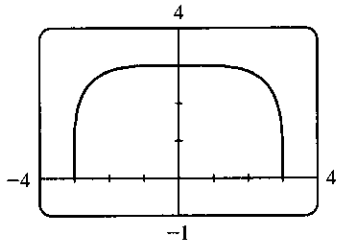
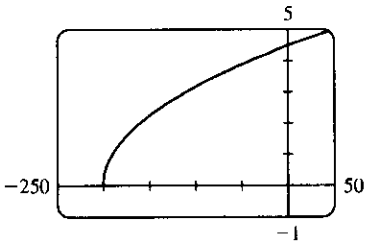
8. $f(x) = x(x+6)(x-9)$



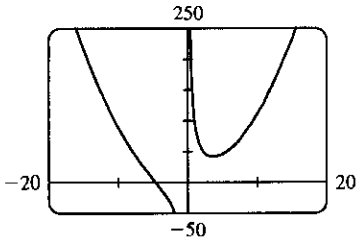
9. $f(x) = \sqrt[4]{81 - x^4}$ is defined when $81 - x^4 \geq 0 \iff x^4 \leq 81 \iff |x| \leq 3$, so the domain of f is $[-3, 3]$. Also $0 \leq \sqrt[4]{81 - x^4} \leq \sqrt[4]{81} = 3$, so the range is $[0, 3]$.



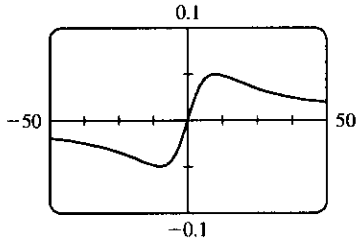
10. $f(x) = \sqrt{0.1x + 20}$ is defined when $0.1x + 20 \geq 0 \iff x \geq -200$, so the domain of f is $[-200, \infty)$.



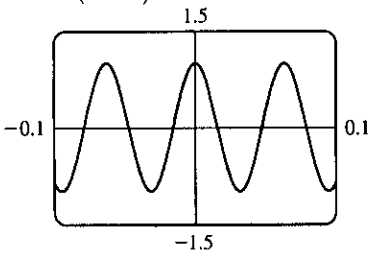
11. The graph of $f(x) = x^2 + (100/x)$ has a vertical asymptote of $x = 0$. As you zoom out, the graph of f looks more and more like that of $y = x^2$.



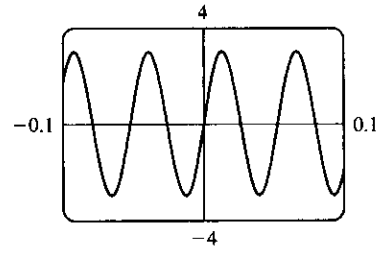
12. The graph of $f(x) = x/(x^2 + 100)$ is symmetric with respect to the origin.



13. $f(x) = \cos(100x)$



14. $f(x) = 3\sin(120x)$



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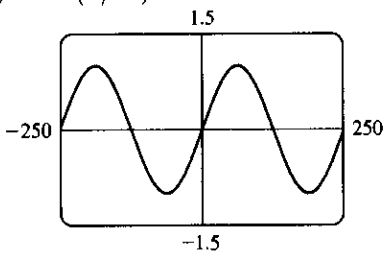
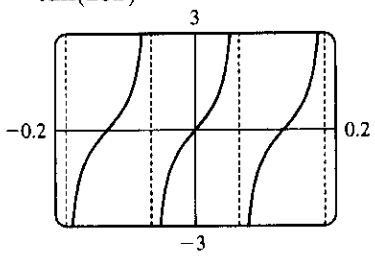
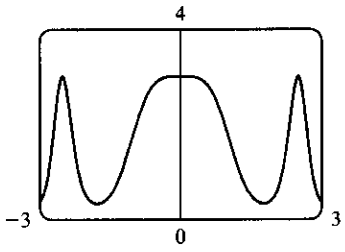
15. $f(x) = \sin(x/40)$

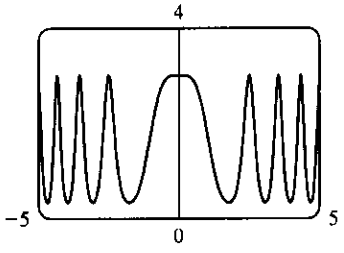


16. $f(x) = \tan(25x)$

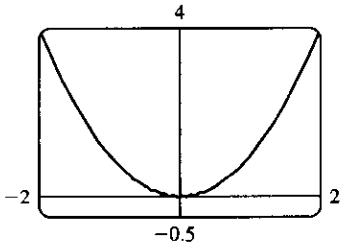


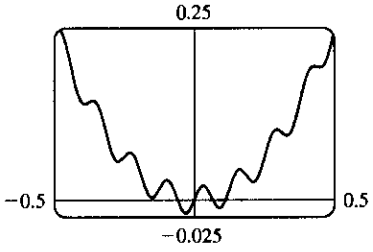
17. $y = 3^{\cos(x^2)}$





18. $y = x^2 + 0.02\sin(50x)$

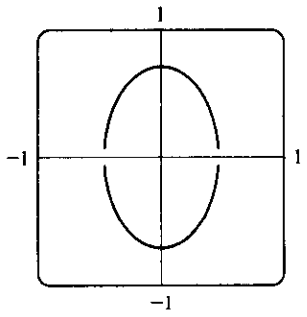




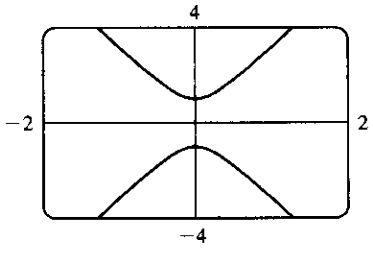
19. We must solve the given equation for y to obtain equations for the upper and lower halves of the ellipse.

$$4x^2 + 2y^2 = 1 \quad \Leftrightarrow \quad 2y^2 = 1 - 4x^2 \quad \Leftrightarrow \quad y^2 = \frac{1 - 4x^2}{2}$$

$$\Leftrightarrow \quad y = \pm\sqrt{\frac{1 - 4x^2}{2}}$$



20. $y^2 - 9x^2 = 1 \quad \Leftrightarrow \quad y^2 = 1 + 9x^2 \quad \Leftrightarrow \quad y = \pm\sqrt{1 + 9x^2}$



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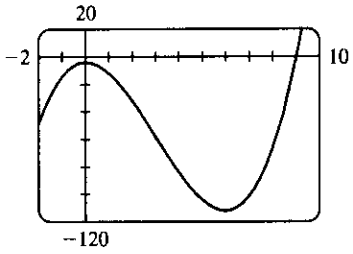
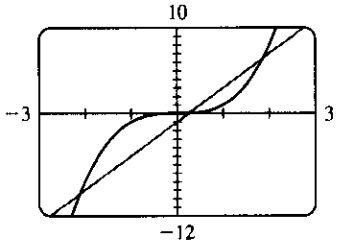
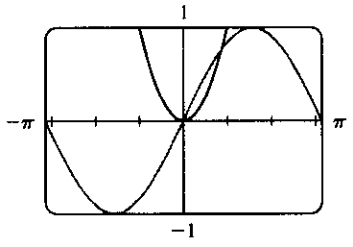
21. From the graph of $f(x) = x^3 - 9x^2 - 4$, we see that there is one solution of the equation $f(x) = 0$ and it is slightly larger than 9. By zooming in or using a root or zero feature, we obtain $x \approx 9.05$.



22. We see that the graphs of $f(x) = x^3$ and $g(x) = 4x - 1$ intersect three times. The x-coordinates of these points (which are the solutions of the equation) are approximately -2.11, 0.25, and 1.86. Alternatively, we could find these values by finding the zeros of $h(x) = x^3 - 4x + 1$.

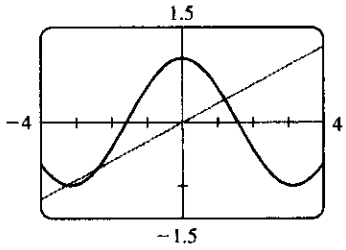


23. We see that the graphs of $f(x) = x^2$ and $g(x) = \sin x$ intersect twice. One solution is $x = 0$. The other solution of $f = g$ is the x-coordinate of the point of intersection in the first quadrant. Using an intersect feature or zooming in, we find this value to be approximately 0.88. Alternatively, we could find that value by finding the positive zero of $h(x) = x^2 - \sin x$.



Note: After producing the graph on a TI-83 Plus, we can find the approximate value 0.88 by using the following keystrokes: 2nd CALC 5 ENTER ENTER 1 ENTER. The "1" is just a guess for 0.88.

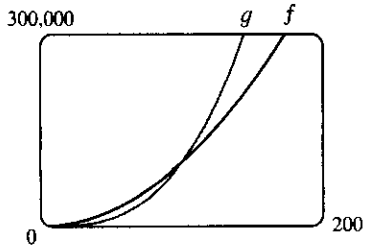
24. (a)



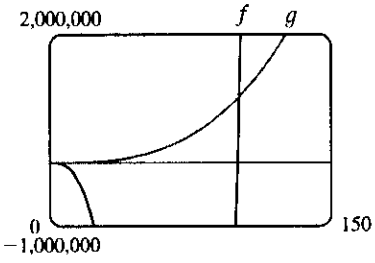
The x-coordinates of the three points of intersection are $x \approx -3.29$, -2.36 and 1.20.

(b) Using trial and error, we find that $m \approx 0.3365$. Note that m could also be negative.

25. $g(x) = x^3/10$ is larger than $f(x) = 10x^2$ whenever $x > 100$.



26. $f(x) = x^4 - 100x^3$ is larger than $g(x) = x^3$ whenever $x > 101$.



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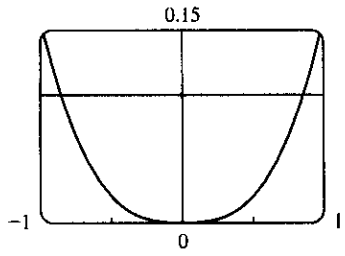
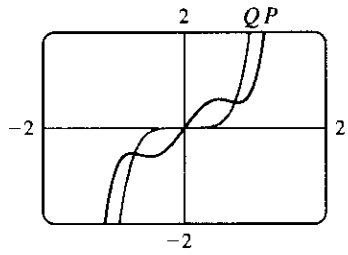
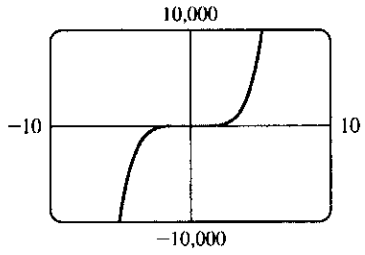
27.



We see from the graphs of $y = |\sin x - x|$ and $y = 0.1$ that there are two solutions to the equation $|\sin x - x| = 0.1$: $x \approx -0.85$ and $x \approx 0.85$. The condition $|\sin x - x| < 0.1$ holds for any x lying between these two values.

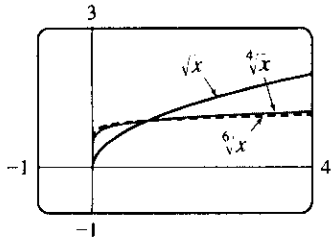
28.



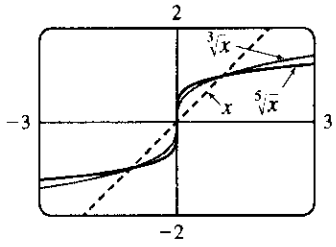


$P(x) = 3x^5 - 5x^3 + 2x$, $Q(x) = 3x^5$. These graphs are significantly different only in the region close to the origin. The larger a viewing rectangle one chooses, the more similar the two graphs look.

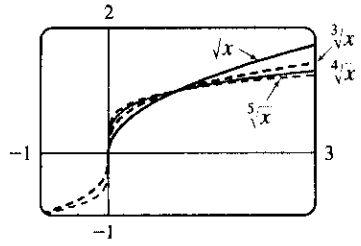
29. (a) The root functions $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$



(b) The root functions $y = x$, $y = \sqrt[3]{x}$ and $y = \sqrt[5]{x}$

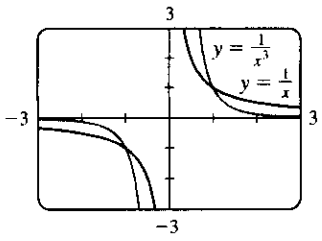


(c) The root functions $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[5]{x}$

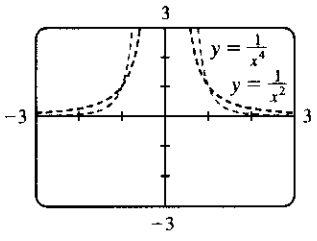


(d)
- For any n, the nth root of 0 is 0 and the nth root of 1 is 1; that is, all nth root functions pass through the points $(0, 0)$ and $(1, 1)$.
- For odd n, the domain of the nth root function is $\mathbb{R}$, while for even n, it is $\{x \in \mathbb{R} \mid x \geq 0\}$.
- Graphs of even root functions look similar to that of $\sqrt{x}$, while those of odd root functions resemble that of $\sqrt[3]{x}$.
- As n increases, the graph of $\sqrt[n]{x}$ becomes steeper near 0 and flatter for $x > 1$.

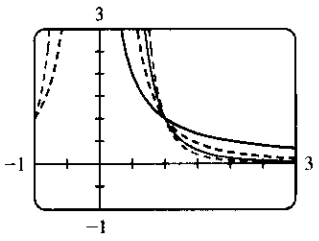
30. (a) The functions $y = 1/x$ and $y = 1/x^3$



(b) The functions $y = 1/x^2$ and $y = 1/x^4$



(c) The functions $y = 1/x$, $y = 1/x^2$, $y = 1/x^3$ and $y = 1/x^4$



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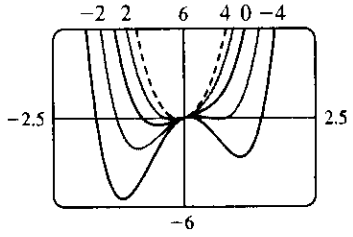
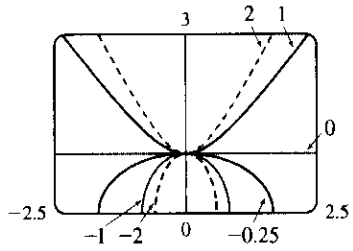
(d)
- The graphs of all functions of the form $y = 1/x^n$ pass through the point $(1, 1)$.
- If n is even, the graph of the function is entirely above the x-axis. The graphs of $1/x^n$ for n even are similar to one another.
- If n is odd, the function is positive for positive x and negative for negative x. The graphs of $1/x^n$ for n odd are similar to one another.
- As n increases, the graphs of $1/x^n$ approach 0 faster as $x \to \infty$.

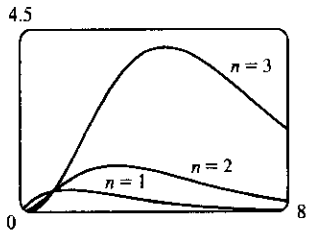
31. $f(x) = x^4 + cx^2 + x$. If $c < 0$, there are three humps: two minimum points and a maximum point. These humps get flatter as c increases, until at $c = 0$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.

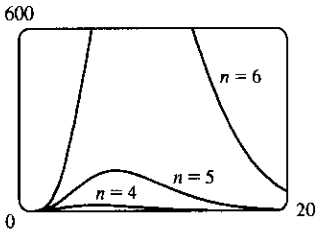


32. $f(x) = \sqrt{1 + cx^2}$. If $c < 0$, the function is only defined on $\left[-1/\sqrt{-c}\,, 1/\sqrt{-c}\,\right]$, and its graph is the top half of an ellipse. If $c = 0$, the graph is the line $y = 1$. If $c > 0$, the graph is the top half of a hyperbola. As c approaches 0, these curves become flatter and approach the line $y = 1$.

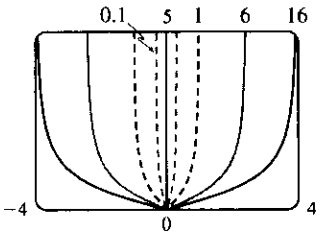


33. $y = x^n 2^{-x}$. As n increases, the maximum of the function moves further from the origin, and gets larger. Note, however, that regardless of n, the function approaches 0 as $x \to \infty$.



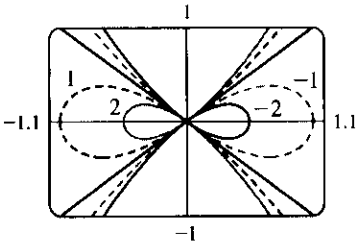


34. $y = \dfrac{|x|}{\sqrt{c - x^2}}$. The "bullet" becomes broader as c increases.



35. $y^2 = cx^3 + x^2$

If $c < 0$, the loop is to the right of the origin, and if c is positive, it is to the left. In both cases, the closer c is to 0, the larger the loop is. (In the limiting case, $c = 0$, the loop is "infinite", that is, it doesn't close.) Also, the larger $|c|$ is, the steeper the slope is on the loopless side of the origin.



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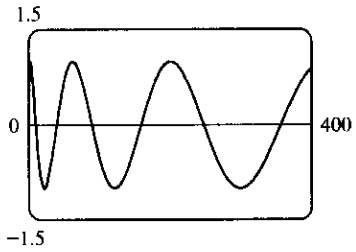
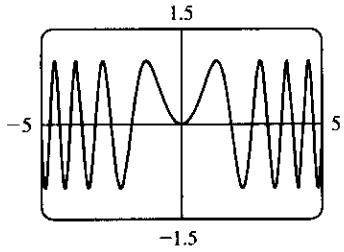
36. (a) $y = \sin(\sqrt{x})$

This function is not periodic; it oscillates less frequently as x increases.



(b) $y = \sin(x^2)$

This function oscillates more frequently as $|x|$ increases. Note also that this function is even, whereas $\sin x$ is odd.



37. The graphing window is 95 pixels wide and we want to start with $x = 0$ and end with $x = 2\pi$. Since there are 94 "gaps" between pixels, the distance between pixels is $\frac{2\pi - 0}{94}$. Thus, the x-values that the calculator actually plots are $x = 0 + \frac{2\pi}{94} \cdot n$, where $n = 0, 1, 2, \ldots, 93, 94$. For $y = \sin 2x$, the actual points plotted by the calculator are $\left(\frac{2\pi}{94} \cdot n, \sin\left(2 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. For $y = \sin 96x$, the points plotted are $\left(\frac{2\pi}{94} \cdot n, \sin\left(96 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. But

$$\begin{aligned}\sin\left(96 \cdot \tfrac{2\pi}{94} \cdot n\right) &= \sin\left(94 \cdot \tfrac{2\pi}{94} \cdot n + 2 \cdot \tfrac{2\pi}{94} \cdot n\right) = \sin\left(2\pi n + 2 \cdot \tfrac{2\pi}{94} \cdot n\right) \\ &= \sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) \quad \text{[by periodicity of sine]}, \quad n = 0, 1, \ldots, 94\end{aligned}$$

So the y-values, and hence the points, plotted for $y = \sin 96x$ are identical to those plotted for $y = \sin 2x$.

Note: Try graphing $y = \sin 94x$. Can you see why all the y-values are zero?

38. As in Exercise 37, we know that the points being plotted for $y = \sin 45x$ are $\left(\frac{2\pi}{94} \cdot n, \sin\left(45 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. But

$$\begin{aligned}\sin\left(45 \cdot \tfrac{2\pi}{94} \cdot n\right) &= \sin\left(47 \cdot \tfrac{2\pi}{94} \cdot n - 2 \cdot \tfrac{2\pi}{94} \cdot n\right) = \sin\left(n\pi - 2 \cdot \tfrac{2\pi}{94} \cdot n\right) \\ &= \sin(n\pi)\cos\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) - \cos(n\pi)\sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) \quad \text{[Subtraction formula for the sine]} \\ &= 0 \cdot \cos\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) - (\pm 1)\sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) = \pm \sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right), \quad n = 0, 1, \ldots, 94\end{aligned}$$

So the y-values, and hence the points, plotted for $y = \sin 45x$ lie on either $y = \sin 2x$ or $y = -\sin 2x$.

1 Review

CONCEPT CHECK

1. (a) A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

(b) If f is a function with domain A, then its **graph** is the set of ordered pairs $\{(x, f(x)) \mid x \in A\}$.

(c) Use the Vertical Line Test on page 17.

2. The four ways to represent a function are: verbally, numerically, visually, and algebraically. An example of each is given below.

Verbally: An assignment of students to chairs in a classroom (a description in words)

Numerically: A tax table that assigns an amount of tax to an income (a table of values)

Visually: A graphical history of the Dow Jones average (a graph)

Algebraically: A relationship between distance, rate, and time: $d = rt$ (an explicit formula)

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3. (a) An **even function** f satisfies $f(-x) = f(x)$ for every number x in its domain. It is symmetric with respect to the y-axis.

(b) An **odd function** g satisfies $g(-x) = -g(x)$ for every number x in its domain. It is symmetric with respect to the origin.

4. A function f is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon.

6. (a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

(b) Power function: $f(x) = x^2$, $f(x) = x^a$

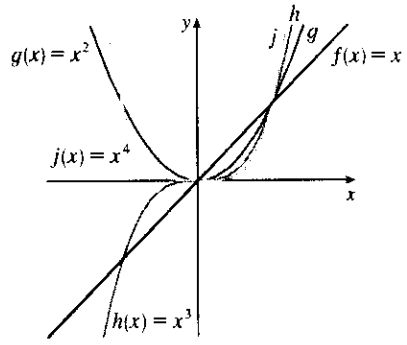
(c) Exponential function: $f(x) = 2^x$, $f(x) = a^x$

(d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

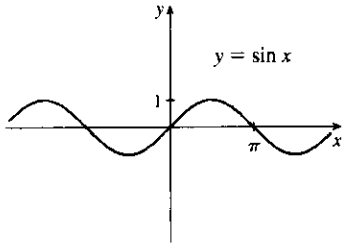
(e) Polynomial of degree 5: $f(x) = x^5 + 2$

(f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials

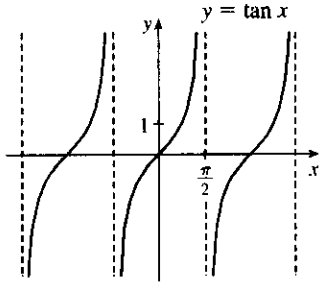
7.



8. (a)



(b)



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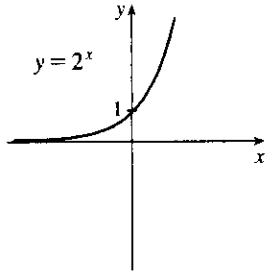

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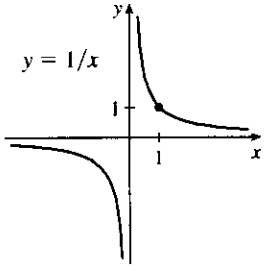
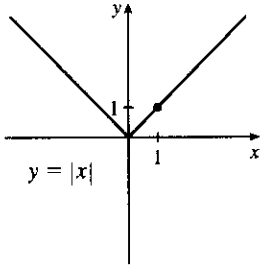
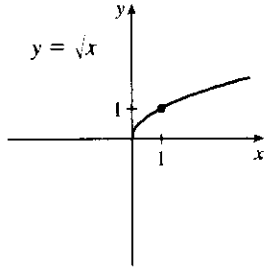


(e)



(f)



9. (a) The domain of $f + g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

(b) The domain of fg is also $A \cap B$.

(c) The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. Given two functions f and g, the **composite** function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. (a) If the graph of f is shifted 2 units upward, its equation becomes $y = f(x) + 2$.

(b) If the graph of f is shifted 2 units downward, its equation becomes $y = f(x) - 2$.

(c) If the graph of f is shifted 2 units to the right, its equation becomes $y = f(x - 2)$.

(d) If the graph of f is shifted 2 units to the left, its equation becomes $y = f(x + 2)$.

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(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 2, its equation becomes $y = 2f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 2, its equation becomes $y = \frac{1}{2}f(x)$.

(i) If the graph of f is stretched horizontally by a factor of 2, its equation becomes $y = f(\frac{1}{2}x)$.

(j) If the graph of f is shrunk horizontally by a factor of 2, its equation becomes $y = f(2x)$.

TRUE-FALSE QUIZ

1. False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s+t) = (-1+1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \neq 0 = f(s+t)$.

2. False. Let $f(x) = x^2$. Then $f(-2) = 4 = f(2)$, but $-2 \neq 2$.

3. False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \neq 3f(x)$.

4. True. If $x_1 < x_2$ and f is a decreasing function, then the y-values get smaller as we move from left to right. Thus, $f(x_1) > f(x_2)$.

5. True. See the Vertical Line Test.

6. False. Let $f(x) = x^2$ and $g(x) = 2x$. Then $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$ and $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$. So $f \circ g \neq g \circ f$.

EXERCISES

1. (a) When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.

(b) $f(x) = 3 \quad \Rightarrow \quad x \approx 2.3, 5.6$

(c) The domain of f is $-6 \leq x \leq 6$, or $[-6, 6]$.

(d) The range of f is $-4 \leq y \leq 4$, or $[-4, 4]$.

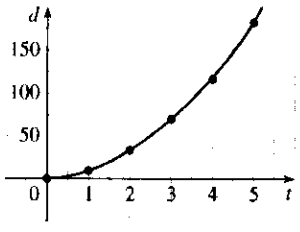
(e) f is increasing on $[-4, 4]$, that is, on $-4 \leq x \leq 4$.

(f) f is odd since its graph is symmetric about the origin.

2. (a) This curve *is not* the graph of a function of x since it *fails* the Vertical Line Test.

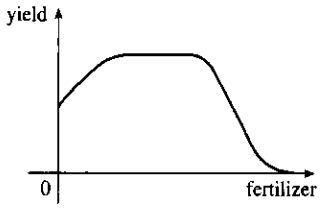
(b) This curve *is* the graph of a function of x since it *passes* the Vertical Line Test. The domain is $[-3, 3]$ and the range is $[-2, 3]$.

3. (a)



(b) From the graph, we see that the distance traveled after 4.5 seconds is slightly less than 150 feet.

4.



There will be some yield with no fertilizer, increasing yields with increasing fertilizer use, a leveling-off of yields at some point, and disaster with too much fertilizer use.

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5. $f(x) = \sqrt{4 - 3x^2}$. Domain: $4 - 3x^2 \geq 0 \;\Rightarrow\; 3x^2 \leq 4 \;\Rightarrow\; x^2 \leq \frac{4}{3} \;\Rightarrow\; |x| \leq \frac{2}{\sqrt{3}}$. Range: $y \geq 0$ and $y \leq \sqrt{4} \;\Rightarrow\; 0 \leq y \leq 2$.

6. $g(x) = \dfrac{1}{x+1}$. Domain: $x + 1 \neq 0 \;\Rightarrow\; x \neq -1$. Range: all reals except 0 ($y = 0$ is the horizontal asymptote for g.)

7. $y = 1 + \sin x$. Domain: $\mathbb{R}$. Range: $-1 \leq \sin x \leq 1 \;\Rightarrow\; 0 \leq 1 + \sin x \leq 2 \;\Rightarrow\; 0 \leq y \leq 2$.

8. $y = \tan 2x$. Domain: $2x \neq \frac{\pi}{2} + \pi n \;\Rightarrow\; x \neq \frac{\pi}{4} + \frac{\pi}{2} n$. Range: the tangent function takes on all real values, so the range is $\mathbb{R}$.

9. (a) To obtain the graph of $y = f(x) + 8$, we shift the graph of $y = f(x)$ up 8 units.

(b) To obtain the graph of $y = f(x + 8)$, we shift the graph of $y = f(x)$ left 8 units.

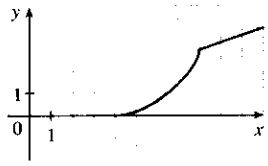
(c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

(d) To obtain the graph of $y = f(x - 2) - 2$, we shift the graph of $y = f(x)$ right 2 units (for the "-2" inside the parentheses), and then shift the resulting graph 2 units downward.

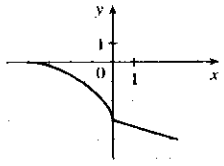
(e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.

(f) To obtain the graph of $y = 3 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 3 units upward.

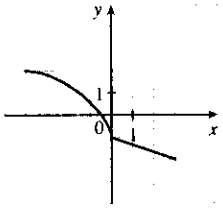
10. (a) To obtain the graph of $y = f(x - 8)$, we shift the graph of $y = f(x)$ right 8 units.



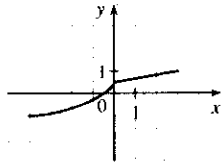
(b) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.



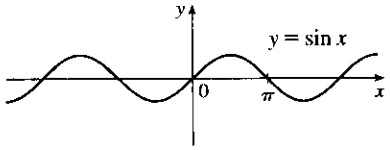
(c) To obtain the graph of $y = 2 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 2 units upward.

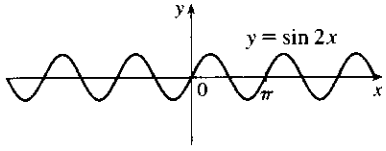


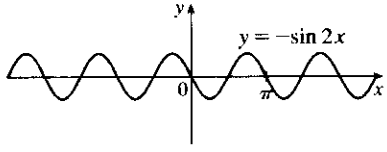
(d) To obtain the graph of $y = \frac{1}{2}f(x) - 1$, we shrink the graph of $y = f(x)$ by a factor of 2, and then shift the resulting graph 1 unit downward.



11. $y = -\sin 2x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 2, and reflect about the x-axis.







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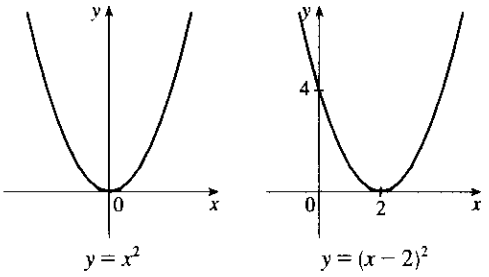
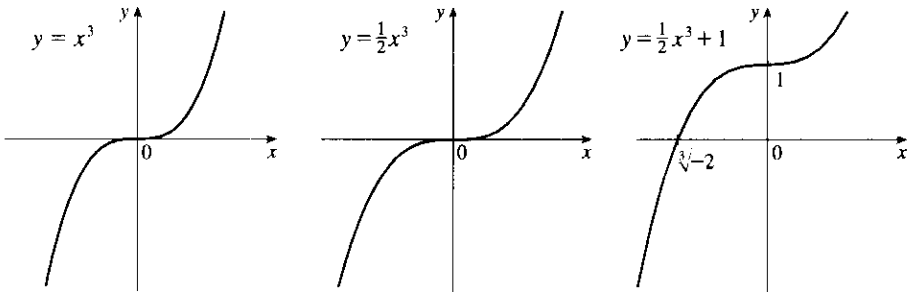
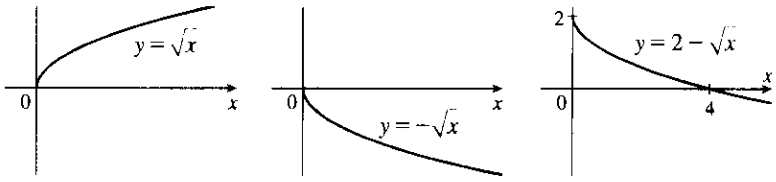
12. $y = (x-2)^2$: Start with the graph of $y = x^2$ and shift 2 units to the right.



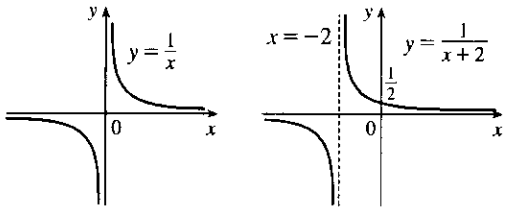
13. $y = 1 + \frac{1}{2}x^3$: Start with the graph of $y = x^3$, compress vertically by a factor of 2, and shift 1 unit upward.



14. $y = 2 - \sqrt{x}$: Start with the graph of $y = \sqrt{x}$, reflect about the x-axis, and shift 2 units upward.



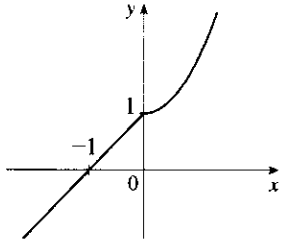
15. $f(x) = \dfrac{1}{x+2}$: Start with the graph of $f(x) = 1/x$ and shift 2 units to the left.



16. $f(x) = \begin{cases} 1+x & \text{if } x < 0 \\ 1+x^2 & \text{if } x \geq 0 \end{cases}$

On $(-\infty, 0)$, graph $y = 1 + x$ (the line with slope 1 and y-intercept 1) with open endpoint $(0, 1)$.

On $[0, \infty)$, graph $y = 1 + x^2$ (the rightmost half of the parabola $y = x^2$ shifted 1 unit upward) with closed endpoint $(0, 1)$.



17. (a) The terms of f are a mixture of odd and even powers of x, so f is neither even nor odd.

(b) The terms of f are all odd powers of x, so f is odd.

(c) $f(-x) = \cos\big((-x)^2\big) = \cos(x^2) = f(x)$, so f is even.

(d) $f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

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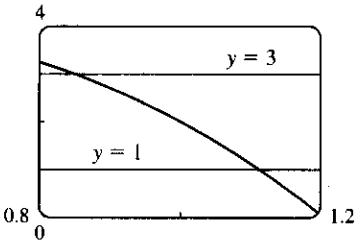
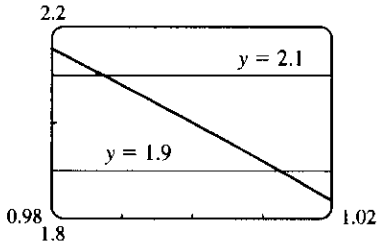
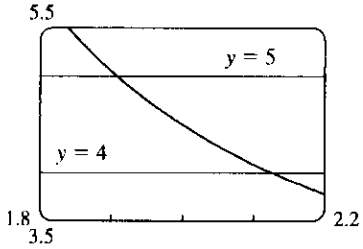
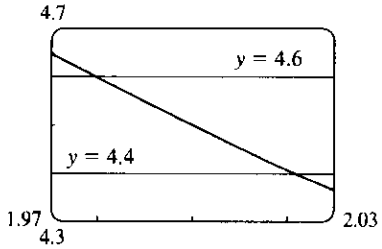
9. For $\varepsilon = 1$, the definition of a limit requires that we find δ such that $|(4 + x - 3x^3) - 2| < 1 \quad \Leftrightarrow$ $1 < 4 + x - 3x^3 < 3$ whenever $0 < |x - 1| < \delta$. If we plot the graphs of $y = 1$, $y = 4 + x - 3x^3$ and $y = 3$ on the same screen, we see that we need $0.86 \le x \le 1.11$. So since $|1 - 0.86| = 0.14$ and $|1 - 1.11| = 0.11$, we choose $\delta = 0.11$ (or any smaller positive number). For $\varepsilon = 0.1$, we must find δ such that $|(4 + x - 3x^3) - 2| < 0.1 \quad \Leftrightarrow \quad 1.9 < 4 + x - 3x^3 < 2.1$ whenever $0 < |x - 1| < \delta$. From the graph, we see that we need $0.988 \le x \le 1.012$. So since $|1 - 0.988| = 0.012$ and $|1 - 1.012| = 0.012$, we choose $\delta = 0.012$ (or any smaller positive number) for the inequality to hold.



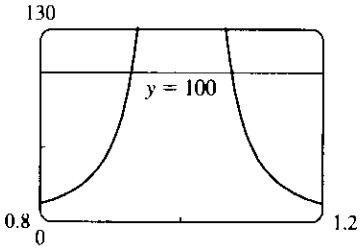


10. For $\varepsilon = 0.5$, we need $1.91 \le x \le 2.125$. So since $|2 - 1.91| = 0.09$ and $|2 - 2.125| = 0.125$, we can take $0 < \delta \le 0.09$. For $\varepsilon = 0.1$, we need $1.980 \le 2.021$. So since $|2 - 1.980| = 0.02$ and $|2 - 2.021| = 0.021$, we can take $\delta = 0.02$ (or any smaller positive number).

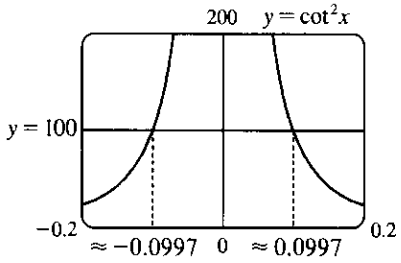




11. From the graph, we see that $\dfrac{x}{(x^2 + 1)(x - 1)^2} > 100$ whenever $0.93 \le x \le 1.07$. So since $|1 - 0.93| = 0.07$ and $|1 - 1.07| = 0.07$, we can take $\delta = 0.07$ (or any smaller positive number).



12. For $M = 100$, we need $-0.0997 < x < 0$ or $0 < x < 0.0997$. Thus, we choose $\delta = 0.0997$ (or any smaller positive number) so that if $0 < |x| < \delta$, then $\cot^2 x > 100$.



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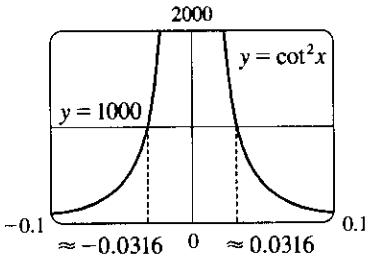
For $M = 1000$, we need $-0.0316 < x < 0$ or $0 < x < 0.0316$. Thus, we choose $\delta = 0.0316$ (or any smaller positive number) so that if $0 < |x| < \delta$, then $\cot^2 x > 1000$.

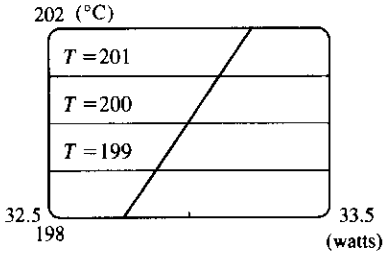


13. (a) $A = \pi r^2$ and $A = 1000\ \text{cm}^2 \Rightarrow \pi r^2 = 1000 \Rightarrow r^2 = \frac{1000}{\pi} \Rightarrow$ $r = \sqrt{\frac{1000}{\pi}}$ $[r > 0] \approx 17.8412$ cm.

(b) $|A - 1000| \le 5 \Rightarrow -5 \le \pi r^2 - 1000 \le 5 \Rightarrow 1000 - 5 \le \pi r^2 \le 1000 + 5 \Rightarrow$ $\sqrt{\frac{995}{\pi}} \le r \le \sqrt{\frac{1005}{\pi}} \Rightarrow 17.7966 \le r \le 17.8858$. $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and $\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

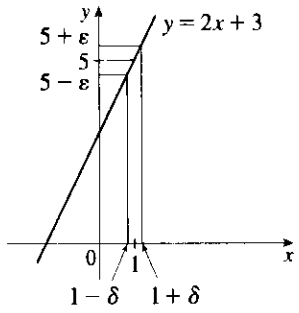
14. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \Rightarrow$ $0.1w^2 + 2.155w + 20 = 200 \Rightarrow$ [by the quadratic formula or from the graph] $w \approx 33.0$ watts $(w > 0)$



(b) From the graph, $199 \le T \le 201 \Rightarrow 32.89 < w < 33.11$.

(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|(2x + 3) - 5| < \varepsilon$. But $|(2x + 3) - 5| < \varepsilon \Leftrightarrow |2x - 2| < \varepsilon \Leftrightarrow 2|x - 1| < \varepsilon \Leftrightarrow |x - 1| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x - 1| < \delta \Rightarrow |(2x + 3) - 5| < \varepsilon$. Thus, $\lim_{x \to 1}(2x + 3) = 5$ by the definition of a limit.



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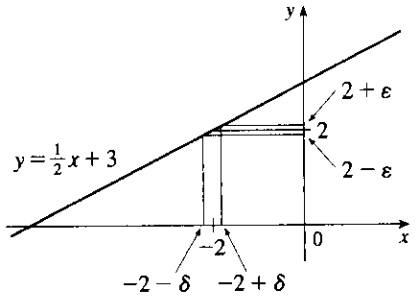
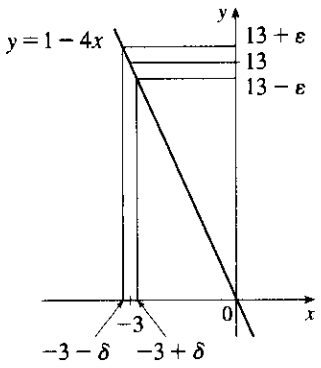
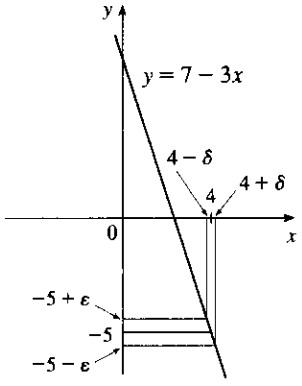
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-2)| < \delta$, then $\left|\left(\frac{1}{2}x+3\right)-2\right| < \varepsilon$. But $\left|\left(\frac{1}{2}x+3\right)-2\right| < \varepsilon \Leftrightarrow \left|\frac{1}{2}x+1\right| < \varepsilon \Leftrightarrow \frac{1}{2}|x+2| < \varepsilon \Leftrightarrow |x-(-2)| < 2\varepsilon$. So if we choose $\delta = 2\varepsilon$, then $0 < |x-(-2)| < \delta \Rightarrow \left|\left(\frac{1}{2}x+3\right)-2\right| < \varepsilon$. Thus, $\lim_{x\to -2}\left(\frac{1}{2}x+3\right) = 2$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-3)| < \delta$, then $|(1-4x)-13| < \varepsilon$. But $|(1-4x)-13| < \varepsilon \Leftrightarrow |-4x-12| < \varepsilon \Leftrightarrow |-4|\,|x+3| < \varepsilon \Leftrightarrow |x-(-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x-(-3)| < \delta \Rightarrow |(1-4x)-13| < \varepsilon$. Thus, $\lim_{x\to -3}(1-4x) = 13$ by the definition of a limit.



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-4| < \delta$, then $|(7-3x)-(-5)| < \varepsilon$. But $|(7-3x)-(-5)| < \varepsilon \Leftrightarrow |-3x+12| < \varepsilon \Leftrightarrow |-3|\,|x-4| < \varepsilon \Leftrightarrow |x-4| < \varepsilon/3$. So if we choose $\delta = \varepsilon/3$, then $0 < |x-4| < \delta \Rightarrow |(7-3x)-(-5)| < \varepsilon$. Thus, $\lim_{x\to 4}(7-3x) = -5$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $\left|\frac{x}{5}-\frac{3}{5}\right| < \varepsilon \Leftrightarrow \frac{1}{5}|x-3| < \varepsilon \Leftrightarrow |x-3| < 5\varepsilon$. So choose $\delta = 5\varepsilon$. Then $0 < |x-3| < \delta \Rightarrow |x-3| < 5\varepsilon \Rightarrow \frac{|x-3|}{5} < \varepsilon \Rightarrow \left|\frac{x}{5}-\frac{3}{5}\right| < \varepsilon$. By the definition of a limit, $\lim_{x\to 3}\frac{x}{5} = \frac{3}{5}$.

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-6| < \delta$, then $\left|\left(\frac{x}{4}+3\right)-\frac{9}{2}\right| < \varepsilon \Leftrightarrow \left|\frac{x}{4}-\frac{3}{2}\right| < \varepsilon \Leftrightarrow \frac{1}{4}|x-6| < \varepsilon \Leftrightarrow |x-6| < 4\varepsilon$. So choose $\delta = 4\varepsilon$. Then $0 < |x-6| < \delta \Rightarrow |x-6| < 4\varepsilon \Rightarrow \frac{|x-6|}{4} < \varepsilon \Rightarrow \left|\frac{x}{4}-\frac{6}{4}\right| < \varepsilon \Rightarrow \left|\left(\frac{x}{4}+3\right)-\frac{9}{2}\right| < \varepsilon$. By the definition of a limit, $\lim_{x\to 6}\left(\frac{x}{4}+3\right) = \frac{9}{2}$.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-5)| < \delta$, then $\left|\left(4-\frac{3}{5}x\right)-7\right| < \varepsilon \Leftrightarrow \left|-\frac{3}{5}x-3\right| < \varepsilon \Leftrightarrow \frac{3}{5}|x+5| < \varepsilon \Leftrightarrow |x-(-5)| < \frac{5}{3}\varepsilon$. So choose $\delta = \frac{5}{3}\varepsilon$. Then $|x-(-5)| < \delta \Rightarrow \left|\left(4-\frac{3}{5}x\right)-7\right| < \varepsilon$. Thus, $\lim_{x\to -5}\left(4-\frac{3}{5}x\right) = 7$ by the definition of a limit.

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22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $\left|\dfrac{x^2+x-12}{x-3} - 7\right| < \varepsilon$. Notice that if $0 < |x-3|$, then $x \neq 3$, so $\dfrac{x^2+x-12}{x-3} = \dfrac{(x+4)(x-3)}{x-3} = x+4$. Thus, when $0 < |x-3|$, we have $\left|\dfrac{x^2+x-12}{x-3} - 7\right| < \varepsilon \quad \Leftrightarrow \quad |(x+4)-7| < \varepsilon \quad \Leftrightarrow \quad |x-3| < \varepsilon$. We take $\delta = \varepsilon$ and see that $0 < |x-3| < \delta \quad \Rightarrow \quad \left|\dfrac{x^2+x-12}{x-3} - 7\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x\to 3} \dfrac{x^2+x-12}{x-3} = 7$.

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|x-a| < \varepsilon$. So $\delta = \varepsilon$ will work.

24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|c-c| < \varepsilon$. But $|c-c| = 0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $\left|x^2-0\right| < \varepsilon \quad \Leftrightarrow \quad x^2 < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $0 < |x-0| < \delta \quad \Rightarrow \quad \left|x^2-0\right| < \varepsilon$. Thus, $\lim\limits_{x\to 0} x^2 = 0$ by the definition of a limit.

26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $\left|x^3-0\right| < \varepsilon \quad \Leftrightarrow \quad |x|^3 < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$. Then $0 < |x-0| < \delta \quad \Rightarrow \quad \left|x^3-0\right| < \delta^3 = \varepsilon$. Thus, $\lim\limits_{x\to 0} x^3 = 0$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $\big||x|-0\big| < \varepsilon$. But $\big||x|\big| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\lim\limits_{x\to 0} |x| = 0$ by the definition of a limit.

28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $9-\delta < x < 9$, then $\left|\sqrt[4]{9-x} - 0\right| < \varepsilon \quad \Leftrightarrow \quad \sqrt[4]{9-x} < \varepsilon \quad \Leftrightarrow \quad 9-x < \varepsilon^4 \quad \Leftrightarrow \quad 9-\varepsilon^4 < x < 9$. So take $\delta = \varepsilon^4$. Then $9-\delta < x < 9 \quad \Rightarrow \quad \left|\sqrt[4]{9-x} - 0\right| < \varepsilon$. Thus, $\lim\limits_{x\to 9^-} \sqrt[4]{9-x} = 0$ by the definition of a limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|\left(x^2-4x+5\right)-1\right| < \varepsilon \quad \Leftrightarrow \quad \left|x^2-4x+4\right| < \varepsilon \quad \Leftrightarrow \quad \left|(x-2)^2\right| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x-2| < \delta \quad \Leftrightarrow \quad |x-2| < \sqrt{\varepsilon} \quad \Leftrightarrow \quad \left|(x-2)^2\right| < \varepsilon$. Thus, $\lim\limits_{x\to 2} \left(x^2-4x+5\right) = 1$ by the definition of a limit.

30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $\left|\left(x^2+x-4\right)-8\right| < \varepsilon \quad \Leftrightarrow \quad \left|x^2+x-12\right| < \varepsilon \quad \Leftrightarrow \quad |(x-3)(x+4)| < \varepsilon$. Notice that if $|x-3| < 1$, then $-1 < x-3 < 1 \quad \Rightarrow \quad 6 < x+4 < 8 \quad \Rightarrow \quad |x+4| < 8$. So take $\delta = \min\{1, \varepsilon/8\}$. Then $0 < |x-3| < \delta \quad \Leftrightarrow \quad |(x-3)(x+4)| \le |8(x-3)| = 8 \cdot |x-3| < 8\delta \le \varepsilon$. Thus, $\lim\limits_{x\to 3} \left(x^2+x-4\right) = 8$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-2)| < \delta$, then $\left|\left(x^2-1\right)-3\right| < \varepsilon$ or upon simplifying we need $\left|x^2-4\right| < \varepsilon$ whenever $0 < |x+2| < \delta$. Notice that if $|x+2| < 1$, then $-1 < x+2 < 1 \quad \Rightarrow \quad -5 < x-2 < -3 \quad \Rightarrow \quad |x-2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x+2| < \delta \quad \Rightarrow \quad |x-2| < 5$ and $|x+2| < \varepsilon/5$, so $\left|\left(x^2-1\right)-3\right| = |(x+2)(x-2)| = |x+2|\,|x-2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x\to -2} \left(x^2-1\right) = 3$.

32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|x^3-8\right| < \varepsilon$. Now $\left|x^3-8\right| = \left|(x-2)\left(x^2+2x+4\right)\right|$. If $|x-2| < 1$, that is, $1 < x < 3$, then $x^2+2x+4 < 3^2+2(3)+4 = 19$ and so $\left|x^3-8\right| = |x-2|\left(x^2+2x+4\right) < 19\,|x-2|$. So if we take $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$, then

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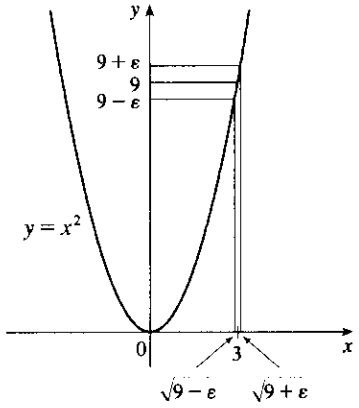
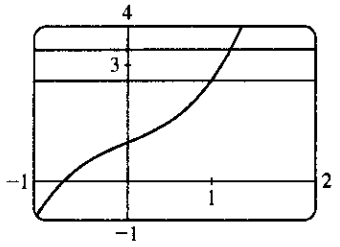
$0 < |x-2| < \delta \;\Rightarrow\; |x^3-8| = |x-2|\left(x^2+2x+4\right) < \frac{\varepsilon}{19}\cdot 19 = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x\to 2} x^3 = 8$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x-3| < \delta$, then $|x-3| < 2 \;\Rightarrow\; -2 < x-3 < 2 \;\Rightarrow\; 4 < x+3 < 8 \;\Rightarrow\; |x+3| < 8$. Also $|x-3| < \frac{\varepsilon}{8}$, so $|x^2-9| = |x+3|\,|x-3| < 8\cdot\frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim\limits_{x\to 3} x^2 = 9$.

34. From the figure, our choices for δ are $\delta_1 = 3-\sqrt{9-\varepsilon}$ and $\delta_2 = \sqrt{9+\varepsilon}-3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1, \delta_2\}$; that is, $\delta = \min\{\delta_1,\delta_2\} = \delta_2 = \sqrt{9+\varepsilon}-3$.



35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1-x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is $x(\varepsilon) = \dfrac{\left(216+108\varepsilon+12\sqrt{336+324\varepsilon+81\varepsilon^2}\right)^{2/3} - 12}{6\left(216+108\varepsilon+12\sqrt{336+324\varepsilon+81\varepsilon^2}\right)^{1/3}}$. Thus, $\delta = x(\varepsilon) - 1$.

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093\,272\,342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. *1. Guessing a value for δ* Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$ whenever $0 < |x-2| < \delta$. But $\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{|2x|} < \varepsilon$. We find a positive constant C such that $\frac{1}{|2x|} < C \;\Rightarrow\; \frac{|x-2|}{|2x|} < C\,|x-2|$ and we can make $C\,|x-2| < \varepsilon$ by taking $|x-2| < \frac{\varepsilon}{C} = \delta$. We restrict x to lie in the interval $|x-2| < 1 \;\Rightarrow\; 1 < x < 3$ so $1 > \frac{1}{x} > \frac{1}{3} \;\Rightarrow\; \frac{1}{6} < \frac{1}{2x} < \frac{1}{2} \;\Rightarrow\; \frac{1}{|2x|} < \frac{1}{2}$. So $C = \frac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\varepsilon\}$.

2. Showing that δ works Given $\varepsilon > 0$ we let $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x-2| < \delta$, then $|x-2| < 1 \;\Rightarrow\; 1 < x < 3 \;\Rightarrow\; \frac{1}{|2x|} < \frac{1}{2}$ (as in part 1). Also $|x-2| < 2\varepsilon$, so $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{|2x|} < \frac{1}{2}\cdot 2\varepsilon = \varepsilon$. This shows that $\lim\limits_{x\to 2}(1/x) = \frac{1}{2}$.

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37. *1. Guessing a value for δ* Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x - a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \dfrac{|x - a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then $\dfrac{|x - a|}{\sqrt{x} + \sqrt{a}} < \dfrac{|x - a|}{C} < \varepsilon$, and we take $|x - a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a. If $|x - a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x - a < \frac{1}{2}a \quad \Rightarrow \quad \frac{1}{2}a < x < \frac{3}{2}a$ $\Rightarrow \quad \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a} + \sqrt{a}$ is a suitable choice for the constant. So $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$. If $0 < |x - a| < \delta$, then $|x - a| < \frac{1}{2}a \quad \Rightarrow \quad \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$ (as in part 1). Also $|x - a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$, so $|\sqrt{x} - \sqrt{a}| = \dfrac{|x - a|}{\sqrt{x} + \sqrt{a}} < \dfrac{\left(\sqrt{a/2} + \sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2} + \sqrt{a}\right)} = \varepsilon$. Therefore, $\lim\limits_{x \to a} \sqrt{x} = \sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim\limits_{t \to 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \quad \Rightarrow \quad |H(t) - L| < \frac{1}{2} \quad \Leftrightarrow$ $L - \frac{1}{2} < H(t) < L + \frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L + \frac{1}{2} \quad \Rightarrow \quad L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L - \frac{1}{2} < 0 \quad \Rightarrow \quad L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim\limits_{t \to 0} H(t)$ does not exist.

39. Suppose that $\lim\limits_{x \to 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \quad \Rightarrow \quad |f(x) - L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L \leq |L| < \frac{1}{2}$. Now take any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim\limits_{x \to 0} f(x)$ does not exist.

40. First suppose that $\lim\limits_{x \to a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x - a| < \delta \quad \Rightarrow$ $|f(x) - L| < \varepsilon$. Then $a - \delta < x < a \quad \Rightarrow \quad 0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Thus, $\lim\limits_{x \to a^-} f(x) = L$. Also $a < x < a + \delta \quad \Rightarrow \quad 0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x \to a^+} f(x) = L$.

Now suppose $\lim\limits_{x \to a^-} f(x) = L = \lim\limits_{x \to a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x \to a^-} f(x) = L$, there exists $\delta_1 > 0$ so that $a - \delta_1 < x < a \quad \Rightarrow \quad |f(x) - L| < \varepsilon$. Since $\lim\limits_{x \to a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a + \delta_2$ $\Rightarrow \quad |f(x) - L| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x - a| < \delta \quad \Rightarrow \quad a - \delta_1 < x < a$ or $a < x < a + \delta_2$ so $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x \to a} f(x) = L$. So we have proved that $\lim\limits_{x \to a} f(x) = L \quad \Leftrightarrow$ $\lim\limits_{x \to a^-} f(x) = L = \lim\limits_{x \to a^+} f(x)$.

41. $\dfrac{1}{(x+3)^4} > 10{,}000 \quad \Leftrightarrow \quad (x+3)^4 < \dfrac{1}{10{,}000} \quad \Leftrightarrow \quad |x+3| < \dfrac{1}{\sqrt[4]{10{,}000}} \quad \Leftrightarrow \quad |x - (-3)| < \dfrac{1}{10}$

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42. Given $M > 0$, we need $\delta > 0$ such that $0 < |x+3| < \delta \Rightarrow 1/(x+3)^4 > M$. Now $\dfrac{1}{(x+3)^4} > M \Leftrightarrow (x+3)^4 < \dfrac{1}{M} \Leftrightarrow |x+3| < \dfrac{1}{\sqrt[4]{M}}$. So take $\delta = \dfrac{1}{\sqrt[4]{M}}$. Then $0 < |x+3| < \delta = \dfrac{1}{\sqrt[4]{M}} \Rightarrow \dfrac{1}{(x+3)^4} > M$, so $\lim\limits_{x \to -3} \dfrac{1}{(x+3)^4} = \infty$.

43. Let $N < 0$ be given. Then, for $x < -1$, we have $\dfrac{5}{(x+1)^3} < N \Leftrightarrow \dfrac{5}{N} < (x+1)^3 \Leftrightarrow \sqrt[3]{\dfrac{5}{N}} < x+1$. Let $\delta = -\sqrt[3]{\dfrac{5}{N}}$. Then $-1-\delta < x < -1 \Rightarrow \sqrt[3]{\dfrac{5}{N}} < x+1 < 0 \Rightarrow \dfrac{5}{(x+1)^3} < N$, so $\lim\limits_{x \to -1^-} \dfrac{5}{(x+1)^3} = -\infty$.

44. (a) Let M be given. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow f(x) > M+1-c$. Since $\lim\limits_{x \to a} g(x) = c$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow |g(x)-c| < 1 \Rightarrow g(x) > c-1$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x-a| < \delta \Rightarrow f(x)+g(x) > (M+1-c)+(c-1) = M$. Thus, $\lim\limits_{x \to a} [f(x)+g(x)] = \infty$.

(b) Let $M > 0$ be given. Since $\lim\limits_{x \to a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow |g(x)-c| < c/2 \Rightarrow g(x) > c/2$. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow f(x) > 2M/c$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x-a| < \delta \Rightarrow f(x)\,g(x) > \dfrac{2M}{c}\,\dfrac{c}{2} = M$, so $\lim\limits_{x \to a} f(x)\,g(x) = \infty$.

(c) Let $N < 0$ be given. Since $\lim\limits_{x \to a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow |g(x)-c| < -c/2 \Rightarrow g(x) < c/2$. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow f(x) > 2N/c$. (Note that $c < 0$ and $N < 0 \Rightarrow 2N/c > 0$.) Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x-a| < \delta \Rightarrow f(x) > 2N/c \Rightarrow f(x)\,g(x) < \dfrac{2N}{c} \cdot \dfrac{c}{2} = N$, so $\lim\limits_{x \to a} f(x)\,g(x) = -\infty$.

2.5 Continuity

1. From Definition 1, $\lim\limits_{x \to 4} f(x) = f(4)$.

2. The graph of f has no hole, jump, or vertical asymptote.

3. (a) The following are the numbers at which f is discontinuous and the type of discontinuity at that number: -4 (removable), -2 (jump), 2 (jump), 4 (infinite).

(b) f is continuous from the left at -2 since $\lim\limits_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim\limits_{x \to 2^+} f(x) = f(2)$ and $\lim\limits_{x \to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

4. g is continuous on $[-4, -2)$, $(-2, 2)$, $[2, 4)$, $(4, 6)$, and $(6, 8)$.

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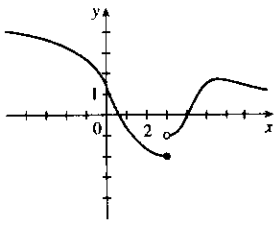
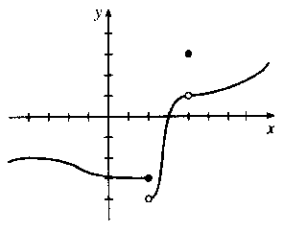
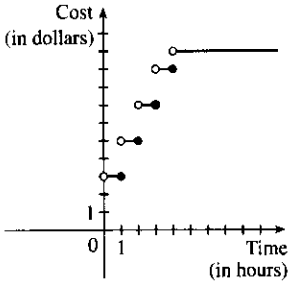
5. The graph of $y = f(x)$ must have a discontinuity at $x = 3$ and must show that $\lim_{x \to 3^-} f(x) = f(3)$.



6.



7. (a)



(b) There are discontinuities at times $t = 1, 2, 3$, and 4. A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

8. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values—at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

9. Since f and g are continuous functions,

$$\begin{aligned}\lim_{x \to 3}[2f(x) - g(x)] &= 2\lim_{x \to 3} f(x) - \lim_{x \to 3} g(x) && \text{[by Limit Laws 2 and 3]}\\ &= 2f(3) - g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x = 3]\\ &= 2 \cdot 5 - g(3) = 10 - g(3)\end{aligned}$$

Since it is given that $\lim_{x \to 3}[2f(x) - g(x)] = 4$, we have $10 - g(3) = 4$, so $g(3) = 6$.

10. $\lim_{x \to 4} f(x) = \lim_{x \to 4}\left(x^2 + \sqrt{7 - x}\right) = \lim_{x \to 4} x^2 + \sqrt{\lim_{x \to 4} 7 - \lim_{x \to 4} x} = 4^2 + \sqrt{7 - 4} = 16 + \sqrt{3} = f(4)$.

By the definition of continuity, f is continuous at $a = 4$.

11. $\lim_{x \to -1} f(x) = \lim_{x \to -1}\left(x + 2x^3\right)^4 = \left(\lim_{x \to -1} x + 2\lim_{x \to -1} x^3\right)^4 = \left[-1 + 2(-1)^3\right]^4 = (-3)^4 = 81 = f(-1)$.

By the definition of continuity, f is continuous at $a = -1$.

12. $\lim_{x \to 4} g(x) = \lim_{x \to 4} \dfrac{x + 1}{2x^2 - 1} = \dfrac{\lim_{x \to 4} x + \lim_{x \to 4} 1}{2\lim_{x \to 4} x^2 - \lim_{x \to 4} 1} = \dfrac{4 + 1}{2(4)^2 - 1} = \dfrac{5}{31} = g(4)$. So g is continuous at 4.

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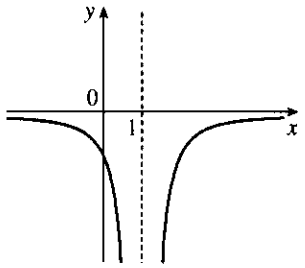
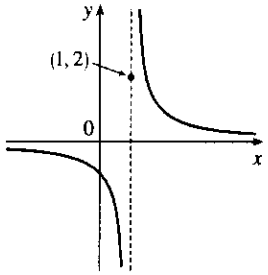
13. For $a > 2$, we have $\lim_{x\to a} f(x) = \lim_{x\to a} \dfrac{2x+3}{x-2} = \dfrac{\lim_{x\to a}(2x+3)}{\lim_{x\to a}(x-2)}$ [Limit Law 5] $= \dfrac{2\lim_{x\to a} x + \lim_{x\to a} 3}{\lim_{x\to a} x - \lim_{x\to a} 2}$ [1, 2, and 3] $= \dfrac{2a+3}{a-2}$ [7 and 8] $= f(a)$. Thus, f is continuous at $x = a$ for every a in $(2, \infty)$; that is, f is continuous on $(2, \infty)$.

14. For $a < 3$, we have $\lim_{x\to a} g(x) = \lim_{x\to a} 2\sqrt{3-x} = 2\lim_{x\to a}\sqrt{3-x}$ [Limit Law 3] $= 2\sqrt{\lim_{x\to a}(3-x)}$ [11] $= 2\sqrt{\lim_{x\to a} 3 - \lim_{x\to a} x}$ [2] $= 2\sqrt{3-a}$ [7 and 8] $= g(a)$, so g is continuous at $x = a$ for every a in $(-\infty, 3)$. Also, $\lim_{x\to 3^-} g(x) = 0 = g(3)$, so g is continuous from the left at 3. Thus, g is continuous on $(-\infty, 3]$.

15. $f(x) = -\dfrac{1}{(x-1)^2}$ is discontinuous at 1 since $f(1)$ is not defined.

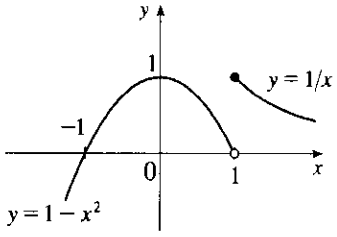


16. $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ is discontinuous at 1 because $\lim_{x\to 1} f(x)$ does not exist.



17. $f(x) = \begin{cases} 1 - x^2 & \text{if } x < 1 \\ 1/x & \text{if } x \geq 1 \end{cases}$

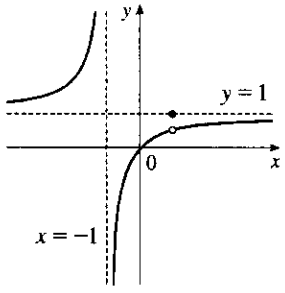
The left-hand limit of f at $a = 1$ is $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(1 - x^2) = 0$. The right-hand limit of f at $a = 1$ is $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(1/x) = 1$. Since these limits are not equal, $\lim_{x\to 1} f(x)$ does not exist and f is discontinuous at 1.



18. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x\to 1} \frac{x(x-1)}{(x+1)(x-1)}$$
$$= \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2},$$

but $f(1) = 1$, so f is discontinous at 1.



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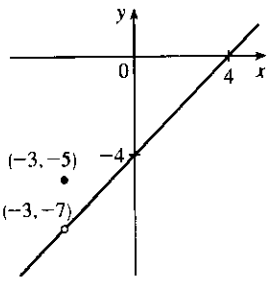
19. $f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x + 3} & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases} = \begin{cases} x - 4 & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases}$

So $\lim\limits_{x \to -3} f(x) = \lim\limits_{x \to -3} (x - 4) = -7$ and $f(-3) = -5$.

Since $\lim\limits_{x \to -3} f(x) \neq f(-3)$, f is discontinuous at -3.

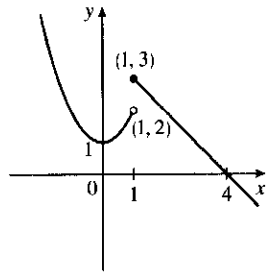


20. $f(x) = \begin{cases} 1 + x^2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (1 + x^2) = 1 + 1^2 = 2$ and

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (4 - x) = 4 - 1 = 3$.

Thus, f is discontinuous at 1 because $\lim\limits_{x \to 1} f(x)$ does not exist.



21. $F(x) = \dfrac{x}{x^2 + 5x + 6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\{x \mid x^2 + 5x + 6 \neq 0\} = \{x \mid (x + 3)(x + 2) \neq 0\} = \{x \mid x \neq -3, -2\}$ or $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

22. By Theorem 7, the root function $\sqrt[3]{x}$ and the polynomial function $1 + x^3$ are continuous on $\mathbb{R}$. By part 4 of Theorem 4, the product $G(x) = \sqrt[3]{x}\,(1 + x^3)$ is continuous on its domain, $\mathbb{R}$.

23. By Theorem 5, the polynomials x^2 and $2x - 1$ are continuous on $(-\infty, \infty)$. By Theorem 7, the root function $\sqrt{x}$ is continuous on $[0, \infty)$. By Theorem 9, the composite function $\sqrt{2x - 1}$ is continuous on its domain, $[\frac{1}{2}, \infty)$. By part 1 of Theorem 4, the sum $R(x) = x^2 + \sqrt{2x - 1}$ is continuous on $[\frac{1}{2}, \infty)$.

24. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x + 1$ are continuous on $\mathbb{R}$. By part 5 of Theorem 4, $h(x) = \dfrac{\sin x}{x + 1}$ is continuous on its domain, $\{x \mid x \neq -1\}$.

25. By Theorem 5, the polynomial $1 - x^2$ is continuous on $(-\infty, \infty)$. By Theorem 7, cos is continuous on its domain, $\mathbb{R}$. By Theorem 9, $\cos(1 - x^2)$ is continuous on its domain, which is $\mathbb{R}$.

26. By Theorem 5, the polynomial $2x$ is continuous on $(-\infty, \infty)$. By Theorem 7, tan is continuous at every number in its domain; that is, $\{x \mid x \neq \frac{\pi}{2} + \pi n\}$. By Theorem 9, $\tan 2x$ is continuous on its domain, which is $\{x \mid 2x \neq \frac{\pi}{2} + \pi n\} = \{x \mid x \neq \frac{\pi}{4} + \frac{\pi}{2} n\}$ (the odd multiples of $\frac{\pi}{4}$).

27. By Theorem 7, the root function $\sqrt{x}$ and the trigonometric function $\sin x$ are continuous on their domains, $[0, \infty)$ and $(-\infty, \infty)$, respectively. Thus, the product $F(x) = \sqrt{x} \sin x$ is continuous on the intersection of those domains, $[0, \infty)$, by part 4 of Theorem 4.

28. The sine and cosine functions are continuous everywhere by Theorem 7, so $F(x) = \sin(\cos(\sin x))$, which is the composite of sine, cosine, and (once again) sine, is continuous everywhere by Theorem 9.

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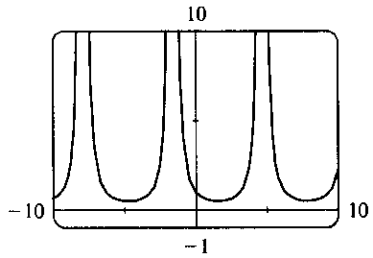
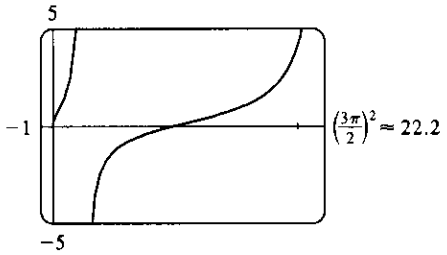
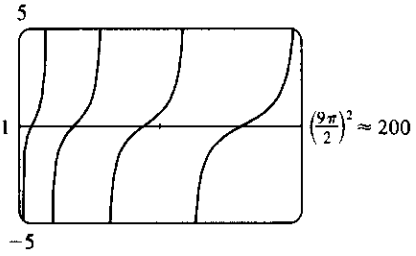
29.



$y = \dfrac{1}{1+\sin x}$ is undefined and hence discontinuous when $1+\sin x = 0 \;\Leftrightarrow\; \sin x = -1 \;\Leftrightarrow\; x = -\frac{\pi}{2} + 2\pi n$, n an integer. The figure shows discontinuities for $n = -1$, 0, and 1; that is, $-\dfrac{5\pi}{2} \approx -7.85$, $-\dfrac{\pi}{2} \approx -1.57$, and $\dfrac{3\pi}{2} \approx 4.71$.

30.





The function $y = f(x) = \tan\sqrt{x}$ is continuous throughout its domain because it is the composite of a trigonometric function and a root function. The square root function has domain $[0, \infty)$ and the tangent function has domain $\left\{x \mid x \neq \frac{\pi}{2} + \pi n\right\}$. So f is discontinuous when $x < 0$ and when $\sqrt{x} = \frac{\pi}{2} + \pi n \;\Rightarrow\; x = \left(\frac{\pi}{2} + \pi n\right)^2$, where n is a nonnegative integer. Note that as x increases, the distance between discontinuities increases.

31. Because we are dealing with root functions, $5 + \sqrt{x}$ is continuous on $[0, \infty)$, $\sqrt{x+5}$ is continuous on $[-5, \infty)$, so the quotient $f(x) = \dfrac{5+\sqrt{x}}{\sqrt{5+x}}$ is continuous on $[0, \infty)$. Since f is continuous at $x = 4$, $\lim\limits_{x\to 4} f(x) = f(4) = \frac{7}{3}$.

32. Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function $f(x) = \sin(x + \sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x\to\pi} f(x) = f(\pi) = \sin(\pi + \sin\pi) = \sin\pi = 0$.

33. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty, 1)$, f is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1, \infty)$, f is continuous on $(1, \infty)$. At $x = 1$, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x^2 = 1$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \sqrt{x} = 1$. Thus, $\lim\limits_{x\to 1} f(x)$ exists and equals 1. Also, $f(1) = \sqrt{1} = 1$. Thus, f is continuous at $x = 1$. We conclude that f is continuous on $(-\infty, \infty)$.

34. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

By Theorem 7, the trigonometric functions are continuous. Since $f(x) = \sin x$ on $(-\infty, \pi/4)$ and $f(x) = \cos x$ on $(\pi/4, \infty)$, f is continuous on $(-\infty, \pi/4) \cup (\pi/4, \infty)$. $\lim\limits_{x\to(\pi/4)^-} f(x) = \lim\limits_{x\to(\pi/4)^-} \sin x = \sin\frac{\pi}{4} = 1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim\limits_{x\to(\pi/4)^+} f(x) = \lim\limits_{x\to(\pi/4)^+} \cos x = 1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim\limits_{x\to(\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty, \infty)$.

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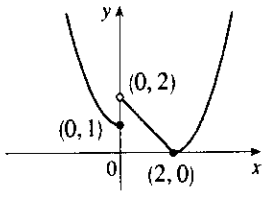
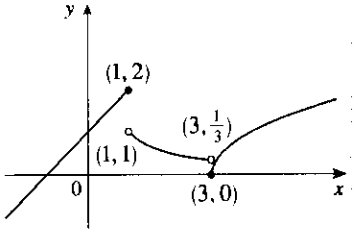
35. $f(x) = \begin{cases} 1+x^2 & \text{if } x \le 0 \\ 2-x & \text{if } 0 < x \le 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$



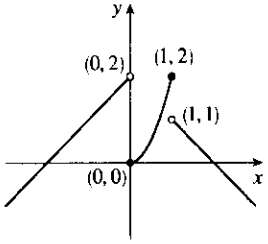
f is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on each of these intervals. Now $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1+x^2) = 1$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2-x) = 2$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the left at 0. Also, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2-x) = 0$, $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x-2)^2 = 0$, and $f(2) = 0$, so f is continuous at 2. The only number at which f is discontinuous is 0.

36. $f(x) = \begin{cases} x+1 & \text{if } x \le 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x-3} & \text{if } x \ge 3 \end{cases}$



f is continuous on $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$, where it is a polynomial, a rational function, and a composite of a root function with a polynomial, respectively. Now $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+1) = 2$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1/x) = 1$, so f is discontinuous at 1. Since $f(1) = 2$, f is continuous from the left at 1. Also, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (1/x) = 1/3$, and $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x-3} = 0 = f(3)$, so f is discontinuous at 3, but it is continuous from the right at 3.

37. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x > 1 \end{cases}$



f is continuous on $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ since on each of these intervals it is a polynomial. Now $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x+2) = 2$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2x^2 = 0$, so f is discontinuous at 0. Since $f(0) = 0$, f is continuous from the right at 0. Also $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x^2 = 2$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2-x) = 1$, so f is discontinuous at 1. Since $f(1) = 2$, f is continuous from the left at 1.

38. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r = R$. $\lim_{r \to R^-} F(r) = \lim_{r \to R^-} \dfrac{GMr}{R^3} = \dfrac{GM}{R^2}$ and $\lim_{r \to R^+} F(r) = \lim_{r \to R^+} \dfrac{GM}{r^2} = \dfrac{GM}{R^2}$, so $\lim_{r \to R} F(r) = \dfrac{GM}{R^2}$. Since $F(R) = \dfrac{GM}{R^2}$, F is continuous at R. Therefore, F is a continuous function of r.

39. f is continuous on $(-\infty, 3)$ and $(3, \infty)$. Now $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (cx+1) = 3c+1$ and $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (cx^2 - 1) = 9c - 1$. So f is continuous $\Leftrightarrow$ $3c+1 = 9c-1$ $\Leftrightarrow$ $6c = 2$ $\Leftrightarrow$ $c = \frac{1}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{1}{3}$.

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40. The functions $x^2 - c^2$ and $cx + 20$, considered on the intervals $(-\infty, 4)$ and $[4, \infty)$ respectively, are continuous for any value of c. So the only possible discontinuity is at $x = 4$. For the function to be continuous at $x = 4$, the left-hand and right-hand limits must be the same. Now $\lim_{x \to 4^-} g(x) = \lim_{x \to 4^-} (x^2 - c^2) = 16 - c^2$ and $\lim_{x \to 4^+} g(x) = \lim_{x \to 4^+} (cx + 20) = 4c + 20 = g(4)$. Thus, $16 - c^2 = 4c + 20 \quad \Leftrightarrow \quad c^2 + 4c + 4 = 0 \quad \Leftrightarrow \quad c = -2$.

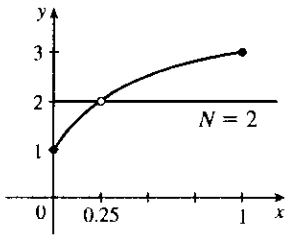
41. (a) $f(x) = \dfrac{x^2 - 2x - 8}{x + 2} = \dfrac{(x - 4)(x + 2)}{x + 2}$ has a removable discontinuity at -2 because $g(x) = x - 4$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq -2$. [The discontinuity is removed by defining $f(-2) = -6$.]

(b) $f(x) = \dfrac{x - 7}{|x - 7|} \quad \Rightarrow \quad \lim_{x \to 7^-} f(x) = -1$ and $\lim_{x \to 7^+} f(x) = 1$. Thus, $\lim_{x \to 7} f(x)$ does not exist, so the discontinuity is not removable. (It is a jump discontinuity.)

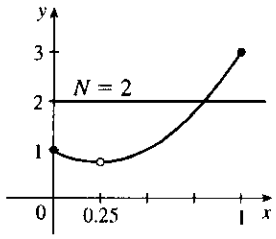
(c) $f(x) = \dfrac{x^3 + 64}{x + 4} = \dfrac{(x + 4)(x^2 - 4x + 16)}{x + 4}$ has a removable discontinuity at -4 because $g(x) = x^2 - 4x + 16$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq -4$. [The discontinuity is removed by defining $f(-4) = 48$.]

(d) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x} = \dfrac{3 - \sqrt{x}}{(3 - \sqrt{x})(3 + \sqrt{x})}$ has a removable discontinuity at 9 because $g(x) = \dfrac{1}{3 + \sqrt{x}}$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq 9$. [The discontinuity is removed by defining $f(9) = \frac{1}{6}$.]

42.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

43. $f(x) = x^3 - x^2 + x$ is continuous on the interval $[2, 3]$, $f(2) = 6$, and $f(3) = 21$. Since $6 < 10 < 21$, there is a number c in $(2, 3)$ such that $f(c) = 10$ by the Intermediate Value Theorem.

44. $f(x) = x^2$ is continuous on the interval $[1, 2]$, $f(1) = 1$, and $f(2) = 4$. Since $1 < 2 < 4$, there is a number c in $(1, 2)$ such that $f(c) = c^2 = 2$ by the Intermediate Value Theorem.

45. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

46. $f(x) = \sqrt[3]{x} + x - 1$ is continuous on the interval $[0, 1]$, $f(0) = -1$, and $f(1) = 1$. Since $-1 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sqrt[3]{x} + x - 1 = 0$, or $\sqrt[3]{x} = 1 - x$, in the interval $(0, 1)$.

47. $f(x) = \cos x - x$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = \cos 1 - 1 \approx -0.46$. Since $-0.46 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x = 0$, or $\cos x = x$, in the interval $(0, 1)$.

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48. $f(x) = \tan x - 2x$ is continuous on the interval $[0, 1.4]$, $f(1) = \tan 1 - 2 \approx -0.44$, and $f(1.4) = \tan 1.4 - 2.8 \approx 3.00$. Since $-0.44 < 0 < 3.00$, there is a number c in $(0, 1.4)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\tan x - 2x = 0$, or $\tan x = 2x$, in the interval $(0, 1.4)$.

49. (a) $f(x) = \sin x - 2 + x$ is continuous on $[0, 2]$, $f(0) = -2$, and $f(2) = \sin 2 \approx 0.91$. Since $-2 < 0 < 0.91$, there is a number c in $(0, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - 2 + x = 0$, or $\sin x = 2 - x$, in the interval $(0, 2)$.

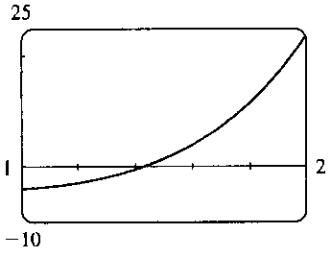
(b) $f(1.10) \approx -0.009 < 0$ and $f(1.11) \approx 0.006 > 0$, so there is a root between 1.10 and 1.11.

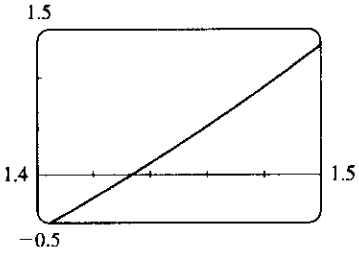
50. (a) $f(x) = x^5 - x^2 + 2x + 3$ is continuous on $[-1, 0]$, $f(-1) = -1 < 0$, and $f(0) = 3 > 0$. Since $-1 < 0 < 3$, there is a number c in $(-1, 0)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^2 + 2x + 3 = 0$ in the interval $(-1, 0)$.

(b) $f(-0.88) \approx -0.062 < 0$ and $f(-0.87) \approx 0.0047 > 0$, so there is a root between -0.88 and -0.87.

51. (a) Let $f(x) = x^5 - x^2 - 4$. Then $f(1) = 1^5 - 1^2 - 4 = -4 < 0$ and $f(2) = 2^5 - 2^2 - 4 = 24 > 0$. So by the Intermediate Value Theorem, there is a number c in $(1, 2)$ such that $f(c) = c^5 - c^2 - 4 = 0$.

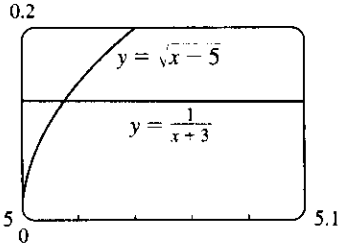
(b) We can see from the graphs that, correct to three decimal places, the root is $x \approx 1.434$.





52. (a) Let $f(x) = \sqrt{x - 5} - \dfrac{1}{x + 3}$. Then $f(5) = -\frac{1}{8} < 0$ and $f(6) = \frac{8}{9} > 0$, and f is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number c in $(5, 6)$ such that $f(c) = 0$. This implies that $\dfrac{1}{c + 3} = \sqrt{c - 5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 5.016$, correct to three decimal places.



53. ($\Rightarrow$) If f is continuous at a, then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h \to 0} f(a + h) = f\left(\lim_{h \to 0} (a + h)\right) = f(a).$$

($\Leftarrow$) Let $\varepsilon > 0$. Since $\lim_{h \to 0} f(a + h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \Rightarrow$ $|f(a + h) - f(a)| < \varepsilon$. So if $0 < |x - a| < \delta$, then $|f(x) - f(a)| = |f(a + (x - a)) - f(a)| < \varepsilon$. Thus, $\lim_{x \to a} f(x) = f(a)$ and so f is continuous at a.

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54. $\lim_{h\to 0}\sin(a+h) = \lim_{h\to 0}(\sin a\cos h + \cos a\sin h) = \lim_{h\to 0}(\sin a\cos h) + \lim_{h\to 0}(\cos a\sin h)$

$$= \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\cos h\right) + \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\sin h\right)$$

$$= (\sin a)(1) + (\cos a)(0) = \sin a$$

55. As in the previous exercise, we must show that $\lim_{h\to 0}\cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\lim_{h\to 0}\cos(a+h) = \lim_{h\to 0}(\cos a\cos h - \sin a\sin h)$$

$$= \lim_{h\to 0}(\cos a\cos h) - \lim_{h\to 0}(\sin a\sin h)$$

$$= \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\cos h\right) - \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\sin h\right)$$

$$= (\cos a)(1) - (\sin a)(0) = \cos a$$

56. (a) Since f is continuous at a, $\lim_{x\to a} f(x) = f(a)$. Thus, using the Constant Multiple Law of Limits, we have $\lim_{x\to a}(cf)(x) = \lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cf(a) = (cf)(a)$. Therefore, cf is continuous at a.

(b) Since f and g are continuous at a, $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. Since $g(a) \neq 0$, we can use the Quotient Law of Limits: $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$. Thus, $\frac{f}{g}$ is continuous at a.

57. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x\to a} f(x) \neq f(a)$.

[In fact, $\lim_{x\to a} f(x)$ does not even exist.]

58. $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$ is continuous at 0. To see why, note that $-|x| \le g(x) \le |x|$, so by the Squeeze Theorem $\lim_{x\to 0} g(x) = 0 = g(0)$. But g is continuous nowhere else. For if $a \neq 0$ and $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a) = 0$ or a, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|g(x) - g(a)| > |a|/2$. Thus, $\lim_{x\to a} g(x) \neq g(a)$.

59. If there is such a number, it satisfies the equation $x^3 + 1 = x \iff x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

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60. (a) $\lim_{x\to 0^+} F(x) = 0$ and $\lim_{x\to 0^-} F(x) = 0$, so $\lim_{x\to 0} F(x) = 0$, which is $F(0)$, and hence F is continuous at $x = a$ if $a = 0$. For $a > 0$, $\lim_{x\to a} F(x) = \lim_{x\to a} x = a = F(a)$. For $a < 0$, $\lim_{x\to a} F(x) = \lim_{x\to a}(-x) = -a = F(a)$. Thus, F is continuous at $x = a$; that is, continuous everywhere.

(b) Assume that f is continuous on the interval I. Then for $a \in I$, $\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right| = |f(a)|$ by Theorem 8. (If a is an endpoint of I, use the appropriate one-sided limit.) So $|f|$ is continuous on I.

(c) No, the converse is false. For example, the function $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at $x = 0$, but $|f(x)| = 1$ is continuous on $\mathbb{R}$.

61. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \quad \Leftrightarrow \quad u(t_0) = d(t_0)$. So at time t_0 after 7:00 A.M., the monk will be at the same place on both days.

2.6 Tangents, Velocities, and Other Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

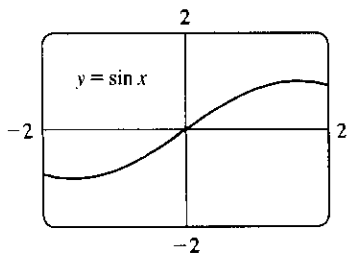
(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim_{x\to 3} \dfrac{f(x) - f(3)}{x - 3}$.

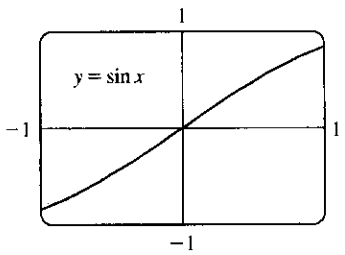
2. (a) Average velocity $= \dfrac{\Delta s}{\Delta t} = \dfrac{f(a+h) - f(a)}{(a+h) - a} = \dfrac{f(a+h) - f(a)}{h}$

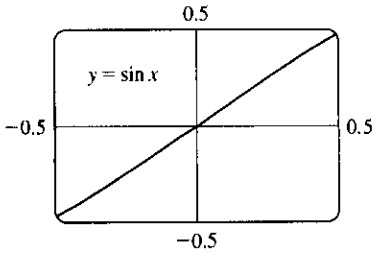
(b) Instantaneous velocity $= \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$

3. The slope at D is the largest positive slope, followed by the positive slope at E. The slope at C is zero. The slope at B is steeper than at A (both are negative). In decreasing order, we have the slopes at: D, E, C, A, and B.

4. The curve looks more like a line as the viewing rectangle gets smaller.







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5. (a) (i) Using Definition 1,

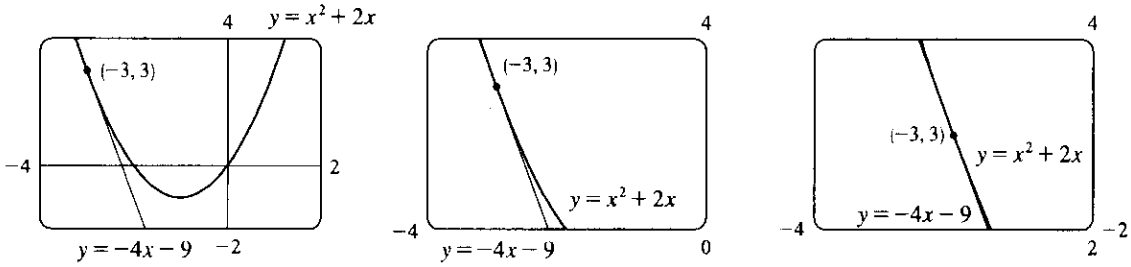
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to -3} \frac{f(x) - f(-3)}{x - (-3)} = \lim_{x \to -3} \frac{(x^2 + 2x) - (3)}{x - (-3)} = \lim_{x \to -3} \frac{(x+3)(x-1)}{x+3}$$
$$= \lim_{x \to -3} (x - 1) = -4$$

(ii) Using Equation 2,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h \to 0} \frac{\left[(-3+h)^2 + 2(-3+h)\right] - (3)}{h}$$
$$= \lim_{h \to 0} \frac{9 - 6h + h^2 - 6 + 2h - 3}{h} = \lim_{h \to 0} \frac{h(h-4)}{h} = \lim_{h \to 0} (h - 4) = -4$$

(b) Using the point-slope form of the equation of a line, an equation of the tangent line is $y - 3 = -4(x + 3)$. Solving for y gives us $y = -4x - 9$, which is the slope-intercept form of the equation of the tangent line.

(c)



6. (a) (i) $m = \lim_{x \to -1} \dfrac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \dfrac{x^3 - (-1)}{x + 1} = \lim_{x \to -1} \dfrac{(x+1)(x^2 - x + 1)}{x + 1}$

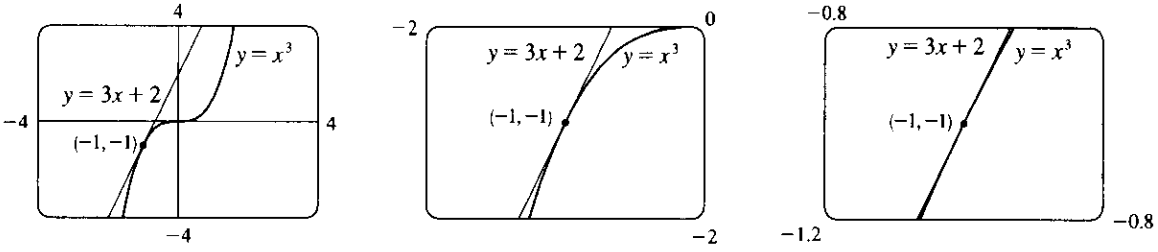
$$= \lim_{x \to -1} (x^2 - x + 1) = 3$$

(ii) $m = \lim_{h \to 0} \dfrac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} \dfrac{(-1+h)^3 - (-1)}{h} = \lim_{h \to 0} \dfrac{h^3 - 3h^2 + 3h - 1 + 1}{h}$

$$= \lim_{h \to 0} (h^2 - 3h + 3) = 3$$

(b) $y - (-1) = 3[x - (-1)] \quad \Leftrightarrow \quad y + 1 = 3x + 3 \quad \Leftrightarrow \quad y = 3x + 2$

(c)



7. Using (2) with $f(x) = 1 + 2x - x^3$ and $P(1, 2)$,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[1 + 2(1+h) - (1+h)^3\right] - 2}{h}$$
$$= \lim_{h \to 0} \frac{1 + 2 + 2h - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h \to 0} \frac{-h^3 - 3h^2 - h}{h}$$
$$= \lim_{h \to 0} \frac{h(-h^2 - 3h - 1)}{h} = \lim_{h \to 0} (-h^2 - 3h - 1) = -1$$

Tangent line: $y - 2 = -1(x - 1) \quad \Leftrightarrow \quad y - 2 = -x + 1 \quad \Leftrightarrow \quad y = -x + 3$

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8. Using (1),

$$m = \lim_{x\to 4} \frac{\sqrt{2x+1}-\sqrt{2(4)+1}}{x-4} = \lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x-4} \cdot \frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3}$$

$$= \lim_{x\to 4} \frac{(2x+1)-3^2}{(x-4)(\sqrt{2x+1}+3)} = \lim_{x\to 4} \frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)}$$

$$= \lim_{x\to 4} \frac{2}{(\sqrt{2x+1}+3)} = \frac{2}{3+3} = \frac{1}{3}.$$

Tangent line: $y-3=\frac{1}{3}(x-4) \quad \Leftrightarrow \quad y-3=\frac{1}{3}x-\frac{4}{3} \quad \Leftrightarrow \quad y=\frac{1}{3}x+\frac{5}{3}$

9. Using (1) with $f(x)=\dfrac{x-1}{x-2}$ and $P(3,2)$,

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to 3} \frac{\dfrac{x-1}{x-2}-2}{x-3} = \lim_{x\to 3} \frac{\dfrac{x-1-2(x-2)}{x-2}}{x-3} = \lim_{x\to 3} \frac{3-x}{(x-2)(x-3)}$$

$$= \lim_{x\to 3} \frac{-1}{x-2} = \frac{-1}{1} = -1.$$

Tangent line: $y-2=-1(x-3) \quad \Leftrightarrow \quad y-2=-x+3 \quad \Leftrightarrow \quad y=-x+5$

10. Using (1), $m = \lim_{x\to 0} \dfrac{\dfrac{2x}{(x+1)^2}-0}{x-0} = \lim_{x\to 0} \dfrac{2x}{x(x+1)^2} = \lim_{x\to 0} \dfrac{2}{(x+1)^2} = \dfrac{2}{1^2} = 2.$

Tangent line: $y-0=2(x-0) \quad \Leftrightarrow \quad y=2x$

11. (a) $m = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim_{x\to a} \dfrac{2/(x+3)-2/(a+3)}{x-a} = \lim_{x\to a} \dfrac{2(a+3)-2(x+3)}{(x-a)(x+3)(a+3)}$

$$= \lim_{x\to a} \frac{2(a-x)}{(x-a)(x+3)(a+3)} = \lim_{x\to a} \frac{-2}{(x+3)(a+3)} = \frac{-2}{(a+3)^2}$$

(b) (i) $a=-1 \quad \Rightarrow \quad m=\dfrac{-2}{(-1+3)^2}=-\dfrac{1}{2}$ (ii) $a=0 \quad \Rightarrow \quad m=\dfrac{-2}{(0+3)^2}=-\dfrac{2}{9}$

(iii) $a=1 \quad \Rightarrow \quad m=\dfrac{-2}{(1+3)^2}=-\dfrac{1}{8}$

12. (a) Using (1),

$$m = \lim_{x\to a} \frac{(1+x+x^2)-(1+a+a^2)}{x-a} = \lim_{x\to a} \frac{x+x^2-a-a^2}{x-a} = \lim_{x\to a} \frac{x-a+(x-a)(x+a)}{x-a}$$

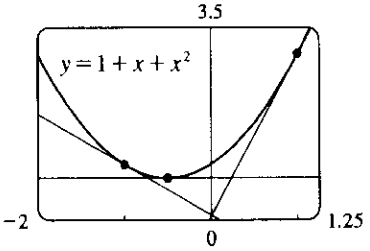
$$= \lim_{x\to a} \frac{(x-a)(1+x+a)}{x-a} = \lim_{x\to a} (1+x+a) = 1+2a$$

(b) (i) $x=-1 \quad \Rightarrow \quad m=1+2(-1)=-1$

(ii) $x=-\frac{1}{2} \quad \Rightarrow \quad m=1+2\left(-\frac{1}{2}\right)=0$

(iii) $x=1 \quad \Rightarrow \quad m=1+2(1)=3$

(c)



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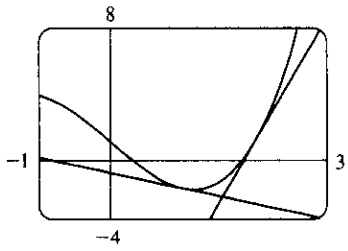
13. (a) Using (1),

$$m = \lim_{x \to a} \frac{(x^3 - 4x + 1) - (a^3 - 4a + 1)}{x - a} = \lim_{x \to a} \frac{(x^3 - a^3) - 4(x - a)}{x - a}$$

$$= \lim_{x \to a} \frac{(x - a)(x^2 + ax + a^2) - 4(x - a)}{x - a} = \lim_{x \to a} (x^2 + ax + a^2 - 4) = 3a^2 - 4$$

(b) At $(1, -2)$: $m = 3(1)^2 - 4 = -1$, so an equation of the tangent line is $y - (-2) = -1(x - 1) \quad \Leftrightarrow \quad y = -x - 1$. At $(2, 1)$: $m = 3(2)^2 - 4 = 8$, so an equation of the tangent line is $y - 1 = 8(x - 2) \quad \Leftrightarrow \quad y = 8x - 15$.

(c)



14. (a) Using (1),

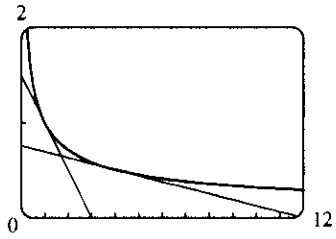
$$m = \lim_{x \to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} = \lim_{x \to a} \frac{\dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}}}{x - a} = \lim_{x \to a} \frac{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})}{\sqrt{ax}(x - a)(\sqrt{a} + \sqrt{x})}$$

$$= \lim_{x \to a} \frac{a - x}{\sqrt{ax}(x - a)(\sqrt{a} + \sqrt{x})} = \lim_{x \to a} \frac{-1}{\sqrt{ax}(\sqrt{a} + \sqrt{x})} = \frac{-1}{\sqrt{a^2}(2\sqrt{a})} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2}$$

(b) At $(1, 1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 1) \quad \Leftrightarrow \quad y = -\frac{1}{2}x + \frac{3}{2}$.

At $\left(4, \frac{1}{2}\right)$: $m = -\frac{1}{16}$, so an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x - 4) \quad \Leftrightarrow \quad y = -\frac{1}{16}x + \frac{3}{4}$.

(c)



15. (a) Since the slope of the tangent at $t = 0$ is 0, the car's initial velocity was 0.

(b) The slope of the tangent is greater at C than at B, so the car was going faster at C.

(c) Near A, the tangent lines are becoming steeper as x increases, so the velocity was increasing, so the car was speeding up. Near B, the tangent lines are becoming less steep, so the car was slowing down. The steepest tangent near C is the one at C, so at C the car had just finished speeding up, and was about to start slowing down.

(d) Between D and E, the slope of the tangent is 0, so the car did not move during that time.

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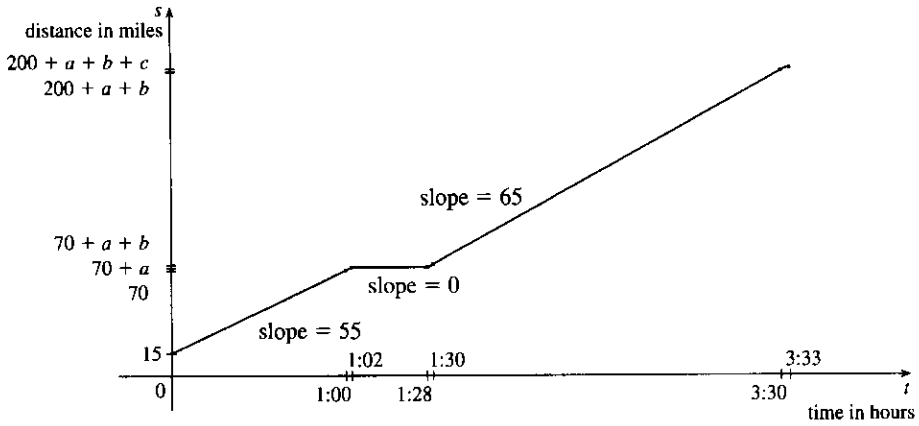
16. Let a denote the distance traveled from 1:00 to 1:02, b from 1:28 to 1:30, and c from 3:30 to 3:33, where all the times are relative to $t = 0$ at the beginning of the trip.



17. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t\to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t\to 2} \frac{(40t - 16t^2) - 16}{t - 2} = \lim_{t\to 2} \frac{-16t^2 + 40t - 16}{t - 2} = \lim_{t\to 2} \frac{-8(2t^2 - 5t + 2)}{t - 2}$$

$$= \lim_{t\to 2} \frac{-8(t - 2)(2t - 1)}{t - 2} = -8 \lim_{t\to 2}(2t - 1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

18. (a) $v(1) = \lim_{h\to 0} \dfrac{H(1 + h) - H(1)}{h}$

$$= \lim_{h\to 0} \frac{(58 + 58h - 0.83 - 1.66h - 0.83h^2) - 57.17}{h} = \lim_{h\to 0}(56.34 - 0.83h) = 56.34 \text{ m/s}$$

(b) $v(a) = \lim_{h\to 0} \dfrac{H(a + h) - H(a)}{h}$

$$= \lim_{h\to 0} \frac{(58a + 58h - 0.83a^2 - 1.66ah - 0.83h^2) - (58a - 0.83a^2)}{h}$$

$$= \lim_{h\to 0} (58 - 1.66a - 0.83h) = 58 - 1.66a \text{ m/s}$$

(c) The arrow strikes the moon when the height is 0, that is, $58t - 0.83t^2 = 0 \quad \Leftrightarrow \quad t(58 - 0.83t) = 0 \quad \Leftrightarrow$ $t = \dfrac{58}{0.83} \approx 69.9$ s (since t can't be 0).

(d) Using the time from part (c), $v\left(\dfrac{58}{0.83}\right) = 58 - 1.66\left(\dfrac{58}{0.83}\right) = -58$ m/s. Thus, the arrow will have a velocity of -58 m/s.

19. $v(a) = \lim_{h\to 0} \dfrac{s(a + h) - s(a)}{h} = \lim_{h\to 0} \dfrac{4(a + h)^3 + 6(a + h) + 2 - (4a^3 + 6a + 2)}{h}$

$$= \lim_{h\to 0} \frac{4a^3 + 12a^2h + 12ah^2 + 4h^3 + 6a + 6h + 2 - 4a^3 - 6a - 2}{h}$$

$$= \lim_{h\to 0} \frac{12a^2h + 12ah^2 + 4h^3 + 6h}{h} = \lim_{h\to 0} (12a^2 + 12ah + 4h^2 + 6) = (12a^2 + 6) \text{ m/s}$$

So $v(1) = 12(1)^2 + 6 = 18$ m/s, $v(2) = 12(2)^2 + 6 = 54$ m/s, and $v(3) = 12(3)^2 + 6 = 114$ m/s.

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20. (a) The average velocity between times t and $t+h$ is

$$\frac{s(t+h)-s(t)}{(t+h)-t} = \frac{(t+h)^2-8(t+h)+18-(t^2-8t+18)}{h}$$

$$= \frac{t^2+2th+h^2-8t-8h+18-t^2+8t-18}{h} = \frac{2th+h^2-8h}{h}$$

$$= (2t+h-8) \text{ m/s}$$

(i) $[3,4]$: $t=3$, $h=4-3=1$, so the average velocity is $2(3)+1-8=-1$ m/s.

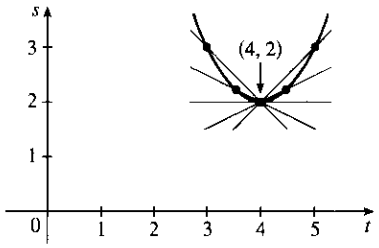
(ii) $[3.5,4]$: $t=3.5$, $h=0.5$, so the average velocity is $2(3.5)+0.5-8=-0.5$ m/s.

(iii) $[4,5]$: $t=4$, $h=1$, so the average velocity is $2(4)+1-8=1$ m/s.

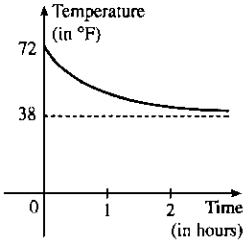
(iv) $[4,4.5]$: $t=4$, $h=0.5$, so the average velocity is $2(4)+0.5-8=0.5$ m/s.

(b) $v(t) = \lim_{h\to 0} \dfrac{s(t+h)-s(t)}{h} = \lim_{h\to 0}(2t+h-8) = 2t-8$, so $v(4)=0$.

(c)

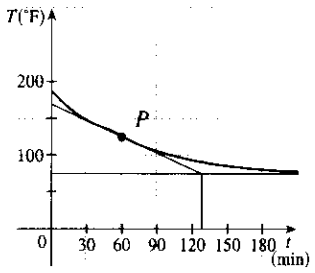


21. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



22. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t=1$ h seems to be about

$$\frac{75-168}{132-0} \approx -0.7 \text{ °F/min}.$$



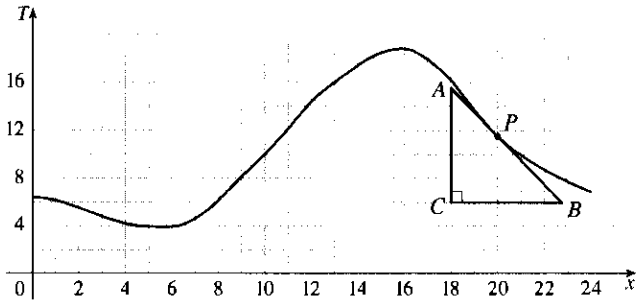
23. (a) (i) $[20,23]$: $\dfrac{7.9-11.5}{23-20} = -1.2$ °C/h

(ii) $[20,22]$: $\dfrac{9.0-11.5}{22-20} = -1.25$ °C/h

(iii) $[20,21]$: $\dfrac{10.2-11.5}{21-20} = -1.3$ °C/h

(b) In the figure, we estimate A to be $(18, 15.5)$ and B as $(23, 6)$. So the slope is

$$\frac{6-15.5}{23-18} = -1.9 \text{ °C/h at 8:00 P.M.}$$



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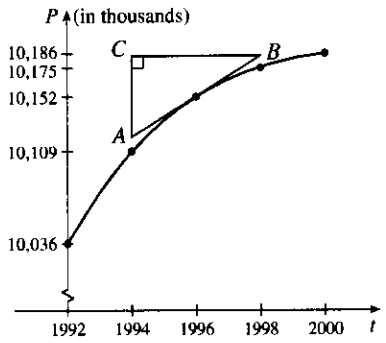
24. (a) (i) $[1992, 1996]$: $\dfrac{P(1996) - P(1992)}{1996 - 1992} = \dfrac{10{,}152 - 10{,}036}{4} = \dfrac{116}{4} = 29$ thousand/year

(ii) $[1994, 1996]$: $\dfrac{P(1996) - P(1994)}{1996 - 1994} = \dfrac{10{,}152 - 10{,}109}{2} = \dfrac{43}{2} = 21.5$ thousand/year

(iii) $[1996, 1998]$: $\dfrac{P(1998) - P(1996)}{1998 - 1996} = \dfrac{10{,}175 - 10{,}152}{2} = \dfrac{23}{2} = 11.5$ thousand/year

(b) Using the values from (ii) and (iii), we have $\dfrac{21.5 + 11.5}{2} = 16.5$ thousand/year.

(c) Estimating A as $(1994, 10{,}125)$ and B as $(1998, 10{,}182)$, the slope at 1996 is $\dfrac{10{,}182 - 10{,}125}{1998 - 1994} = \dfrac{57}{4} = 14.25$ thousand/year.



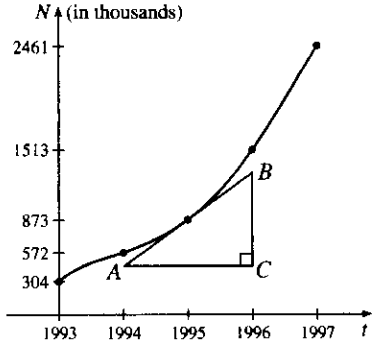
25. (a) (i) $[1995, 1997]$: $\dfrac{N(1997) - N(1995)}{1997 - 1995} = \dfrac{2461 - 873}{2} = \dfrac{1588}{2} = 794$ thousand/year

(ii) $[1995, 1996]$: $\dfrac{N(1996) - N(1995)}{1996 - 1995} = \dfrac{1513 - 873}{1} = 640$ thousand/year

(iii) $[1994, 1995]$: $\dfrac{N(1995) - N(1994)}{1995 - 1994} = \dfrac{873 - 572}{1} = 301$ thousand/year

(b) Using the values from (ii) and (iii), we have $\dfrac{640 + 301}{2} = \dfrac{941}{2} = 470.5$ thousand/year.

(c) Estimating A as $(1994, 420)$ and B as $(1996, 1275)$, the slope at 1995 is $\dfrac{1275 - 420}{1996 - 1994} = \dfrac{855}{2} = 427.5$ thousand/year



26. (a) (i) $[1996, 1998]$: $\dfrac{N(1998) - N(1996)}{1998 - 1996} = \dfrac{1886 - 1015}{2} = \dfrac{871}{2} = 435.5$ locations/year

(ii) $[1997, 1998]$: $\dfrac{N(1998) - N(1997)}{1998 - 1997} = \dfrac{1886 - 1412}{1} = 474$ locations/year

(iii) $[1998, 1999]$: $\dfrac{N(1999) - N(1998)}{1999 - 1998} = \dfrac{2135 - 1886}{1} = 249$ locations/year

(b) Using the values from (ii) and (iii), we have $\dfrac{474 + 249}{2} = \dfrac{723}{2} = 361.5 \approx 362$ locations/year.

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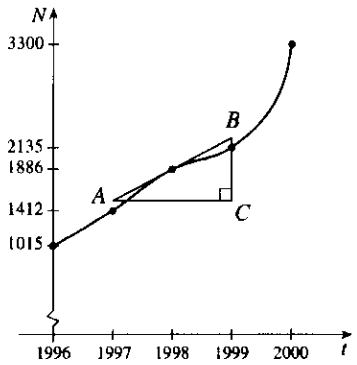
(c) Estimating A as $(1997, 1525)$ and B as $(1999, 2250)$, the slope at 1998 is $\dfrac{2250-1525}{1999-1997} = \dfrac{725}{2} = 362.5$ locations/year.



27. (a) (i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105)-C(100)}{105-100} = \dfrac{6601.25-6500}{5} = \$20.25/\text{unit}.$

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101)-C(100)}{101-100} = \dfrac{6520.05-6500}{1} = \$20.05/\text{unit}.$

(b) $$\frac{C(100+h)-C(100)}{h} = \frac{[5000+10(100+h)+0.05(100+h)^2]-6500}{h} = \frac{20h+0.05h^2}{h}$$
$$= 20+0.05h,\ h \neq 0$$

So the instantaneous rate of change is $\lim\limits_{h\to 0}\dfrac{C(100+h)-C(100)}{h} = \lim\limits_{h\to 0}(20+0.05h) = \$20/\text{unit}.$

28. $$\Delta V = V(t+h)-V(t) = 100{,}000\left(1-\frac{t+h}{60}\right)^2 - 100{,}000\left(1-\frac{t}{60}\right)^2$$
$$= 100{,}000\left[\left(1-\frac{t+h}{30}+\frac{(t+h)^2}{3600}\right)-\left(1-\frac{t}{30}+\frac{t^2}{3600}\right)\right] = 100{,}000\left(-\frac{h}{30}+\frac{2th}{3600}+\frac{h^2}{3600}\right)$$
$$= \frac{100{,}000}{3600}h\,(-120+2t+h) = \frac{250}{9}h\,(-120+2t+h)$$

Dividing ΔV by h and then letting $h \to 0$, we see that the instantaneous rate of change is $\frac{500}{9}\,(t-60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\overline{3}$	100,000
10	$-2777.\overline{7}$	$69,444.\overline{4}$
20	$-2222.\overline{2}$	$44,444.\overline{4}$
30	$-1666.\overline{6}$	25,000
40	$-1111.\overline{1}$	$11,111.\overline{1}$
50	$-555.\overline{5}$	$2,777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

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2 Review

CONCEPT CHECK

1. (a) $\lim_{x \to a} f(x) = L$: See Definition 2.2.1 and Figures 1 and 2 in Section 2.2.

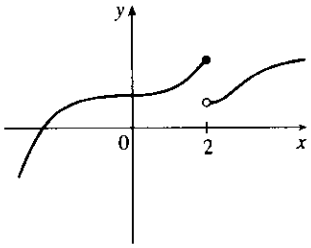
(b) $\lim_{x \to a^+} f(x) = L$: See the paragraph after Definition 2.2.2 and Figure 9(b) in Section 2.2.

(c) $\lim_{x \to a^-} f(x) = L$: See Definition 2.2.2 and Figure 9(a) in Section 2.2.

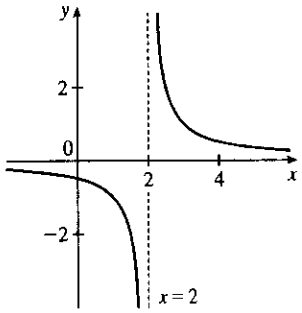
(d) $\lim_{x \to a} f(x) = \infty$: See Definition 2.2.4 and Figure 12 in Section 2.2.

(e) $\lim_{x \to a} f(x) = -\infty$: See Definition 2.2.5 and Figure 13 in Section 2.2.

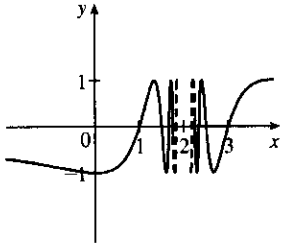
2. In general, the limit of a function fails to exist when the function does not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left- and right-hand limits are not equal.



There is an infinite discontinuity.



There are an infinite number of oscillations.

3. See Definition 2.2.6 and Figure 14 in Section 2.2.

4. (a)–(g) See the statements of Limit Laws 1–6 and 11 in Section 2.3.

5. See Theorem 3 in Section 2.3.

6. (a) A function f is continuous at a number a if $f(x)$ approaches $f(a)$ as x approaches a; that is, $\lim_{x \to a} f(x) = f(a)$.

(b) A function f is continuous on the interval $(-\infty, \infty)$ if f is continuous at every real number a. The graph of such a function has no breaks and every vertical line crosses it.

7. See Theorem 2.5.10.

8. See Definition 2.6.1.

9. See the paragraph containing Formula 3 in Section 2.6.

10. (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$ is $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

(b) The instantaneous rate of change of y with respect to x at $x = x_1$ is $\lim_{x_2 \to x_1} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don't).

2. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

3. True. Limit Law 5 applies.

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4. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as x approaches 5. (The denominator approaches 0 and the numerator doesn't.)

5. False. Consider $\lim_{x\to 5} \dfrac{x(x-5)}{x-5}$ or $\lim_{x\to 5} \dfrac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 3 in Section 2.2, we know that the latter limit exists (and it is equal to 1).

6. False. Consider $\lim_{x\to 6} [f(x)g(x)] = \lim_{x\to 6} \left[(x-6)\dfrac{1}{x-6}\right]$. It exists (its value is 1) but $f(6) = 0$ and $g(6)$ does not exist, so $f(6)g(6) \neq 1$.

7. True. A polynomial is continuous everywhere, so $\lim_{x\to b} p(x)$ exists and is equal to $p(b)$.

8. False. Consider $\lim_{x\to 0} [f(x) - g(x)] = \lim_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

9. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

10. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let $f(x) = \begin{cases} 1 & \text{if } 0 \le x < 3 \\ -1 & \text{if } x = 3 \end{cases}$ There is no number $c \in [0, 3]$ with $f(c) = 0$.

11. True. Use Theorem 2.5.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.

12. True. Use the Intermediate Value Theorem with $a = -1$, $b = 1$, and $N = \pi$, since $3 < \pi < 4$.

13. True, by the definition of a limit with $\varepsilon = 1$.

14. False. For example, let $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

Then $f(x) > 1$ for all x, but $\lim_{x\to 0} f(x) = \lim_{x\to 0} (x^2 + 1) = 1$.

EXERCISES

1. (a) (i) $\lim_{x\to 2^+} f(x) = 3$ (ii) $\lim_{x\to -3^+} f(x) = 0$

(iii) $\lim_{x\to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.) (iv) $\lim_{x\to 4} f(x) = 2$

(v) $\lim_{x\to 0} f(x) = \infty$ (vi) $\lim_{x\to 2^-} f(x) = -\infty$

(b) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.

(c) f is discontinuous at $x = -3$, 0, 2, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

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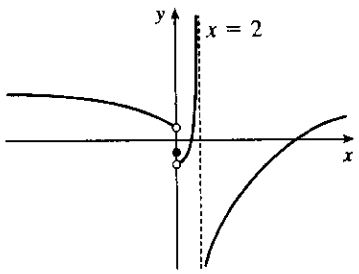
2.



3. $\lim_{x\to 0} \cos(x+\sin x) = \cos\left[\lim_{x\to 0}(x+\sin x)\right]$ [by Theorem 2.5.8] $= \cos 0 = 1$

4. Since rational functions are continuous, $\lim_{x\to 3} \dfrac{x^2-9}{x^2+2x-3} = \dfrac{3^2-9}{3^2+2(3)-3} = \dfrac{0}{12} = 0.$

5. $\lim_{x\to -3} \dfrac{x^2-9}{x^2+2x-3} = \lim_{x\to -3} \dfrac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to -3} \dfrac{x-3}{x-1} = \dfrac{-3-3}{-3-1} = \dfrac{-6}{-4} = \dfrac{3}{2}$

6. $\lim_{x\to 1^+} \dfrac{x^2-9}{x^2+2x-3} = -\infty$ since $x^2+2x-3 \to 0$ as $x \to 1^+$ and $\dfrac{x^2-9}{x^2+2x-3} < 0$ for $1 < x < 3$.

7. $\lim_{h\to 0} \dfrac{(h-1)^3+1}{h} = \lim_{h\to 0} \dfrac{(h^3-3h^2+3h-1)+1}{h} = \lim_{h\to 0} \dfrac{h^3-3h^2+3h}{h} = \lim_{h\to 0} (h^2-3h+3) = 3$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\lim_{h\to 0} \frac{(h-1)^3+1}{h} = \lim_{h\to 0} \frac{(h-1)^3+1^3}{h} = \lim_{h\to 0} \frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h}$$
$$= \lim_{h\to 0} \left[(h-1)^2-h+2\right] = 1-0+2 = 3$$

8. $\lim_{t\to 2} \dfrac{t^2-4}{t^3-8} = \lim_{t\to 2} \dfrac{(t+2)(t-2)}{(t-2)(t^2+2t+4)} = \lim_{t\to 2} \dfrac{t+2}{t^2+2t+4} = \dfrac{2+2}{4+4+4} = \dfrac{4}{12} = \dfrac{1}{3}$

9. $\lim_{r\to 9} \dfrac{\sqrt{r}}{(r-9)^4} = \infty$ since $(r-9)^4 \to 0$ as $r \to 9$ and $\dfrac{\sqrt{r}}{(r-9)^4} > 0$ for $r \neq 9$.

10. $\lim_{v\to 4^+} \dfrac{4-v}{|4-v|} = \lim_{v\to 4^+} \dfrac{4-v}{-(4-v)} = \lim_{v\to 4^+} \dfrac{1}{-1} = -1$

11. $\lim_{s\to 16} \dfrac{4-\sqrt{s}}{s-16} = \lim_{s\to 16} \dfrac{4-\sqrt{s}}{(\sqrt{s}+4)(\sqrt{s}-4)} = \lim_{s\to 16} \dfrac{-1}{\sqrt{s}+4} = \dfrac{-1}{\sqrt{16}+4} = -\dfrac{1}{8}$

12. $\lim_{v\to 2} \dfrac{v^2+2v-8}{v^4-16} = \lim_{v\to 2} \dfrac{(v+4)(v-2)}{(v+2)(v-2)(v^2+4)} = \lim_{v\to 2} \dfrac{v+4}{(v+2)(v^2+4)} = \dfrac{2+4}{(2+2)(2^2+4)} = \dfrac{3}{16}$

13. $$\frac{|x-8|}{x-8} = \begin{cases} \dfrac{x-8}{x-8} & \text{if } x-8>0 \\ \dfrac{-(x-8)}{x-8} & \text{if } x-8<0 \end{cases} = \begin{cases} 1 & \text{if } x>8 \\ -1 & \text{if } x<8 \end{cases}$$

Thus, $\lim_{x\to 8^-} \dfrac{|x-8|}{x-8} = \lim_{x\to 8^-} (-1) = -1.$

14. $\lim_{x\to 9^+} \left(\sqrt{x-9} + [\![x+1]\!]\right) = \lim_{x\to 9^+} \sqrt{x-9} + \lim_{x\to 9^+} [\![x+1]\!] = \sqrt{9-9}+10 = 10$

15. $\lim_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x} \cdot \dfrac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \lim_{x\to 0} \dfrac{1-(1-x^2)}{x\left(1+\sqrt{1-x^2}\right)} = \lim_{x\to 0} \dfrac{x^2}{x\left(1+\sqrt{1-x^2}\right)} = \lim_{x\to 0} \dfrac{x}{1+\sqrt{1-x^2}} = 0$

16. $\lim_{x\to 2} \dfrac{\sqrt{x+2}-\sqrt{2x}}{x(x-2)} \cdot \dfrac{\sqrt{x+2}+\sqrt{2x}}{\sqrt{x+2}+\sqrt{2x}} = \lim_{x\to 2} \dfrac{-(x-2)}{x(x-2)\left(\sqrt{x+2}+\sqrt{2x}\right)} = \lim_{x\to 2} \dfrac{-1}{x\left(\sqrt{x+2}+\sqrt{2x}\right)} = -\dfrac{1}{8}$

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17. Since $2x - 1 \le f(x) \le x^2$ for $0 < x < 3$ and $\lim_{x \to 1} (2x - 1) = 1 = \lim_{x \to 1} x^2$, we have $\lim_{x \to 1} f(x) = 1$ by the Squeeze Theorem.

18. Let $f(x) = -x^2$, $g(x) = x^2 \cos(1/x^2)$ and $h(x) = x^2$. Then since $|\cos(1/x^2)| \le 1$ for $x \neq 0$, we have $f(x) \le g(x) \le h(x)$ for $x \neq 0$, and so $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0 \Rightarrow \lim_{x \to 0} g(x) = 0$ by the Squeeze Theorem.

19. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x - 5| < \delta$, then $|(7x - 27) - 8| < \varepsilon \Leftrightarrow |7x - 35| < \varepsilon \Leftrightarrow |x - 5| < \varepsilon/7$. So take $\delta = \varepsilon/7$. Then $0 < |x - 5| < \delta \Rightarrow |(7x - 27) - 8| < \varepsilon$. Thus, $\lim_{x \to 5} (7x - 27) = 8$ by the definition of a limit.

20. Given $\varepsilon > 0$ we must find $\delta > 0$ so that if $0 < |x - 0| < \delta$, then $|\sqrt[3]{x} - 0| < \varepsilon$. Now $|\sqrt[3]{x} - 0| = |\sqrt[3]{x}| < \varepsilon \Rightarrow |x| = |\sqrt[3]{x}|^3 < \varepsilon^3$. So take $\delta = \varepsilon^3$. Then $0 < |x - 0| = |x| < \varepsilon^3 \Rightarrow |\sqrt[3]{x} - 0| = |\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\varepsilon^3} = \varepsilon$. Therefore, by the definition of a limit, $\lim_{x \to 0} \sqrt[3]{x} = 0$.

21. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x - 2| < \delta$, then $|x^2 - 3x - (-2)| < \varepsilon$. First, note that if $|x - 2| < 1$, then $-1 < x - 2 < 1$, so $0 < x - 1 < 2 \Rightarrow |x - 1| < 2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $0 < |x - 2| < \delta \Rightarrow |x^2 - 3x - (-2)| = |(x - 2)(x - 1)| = |x - 2|\,|x - 1| < (\varepsilon/2)(2) = \varepsilon$.
Thus, $\lim_{x \to 2} (x^2 - 3x) = -2$ by the definition of a limit.

22. Given $M > 0$, we need $\delta > 0$ such that if $0 < x - 4 < \delta$, then $2/\sqrt{x-4} > M$. This is true $\Leftrightarrow \sqrt{x-4} < 2/M \Leftrightarrow x - 4 < 4/M^2$. So if we choose $\delta = 4/M^2$, then $0 < x - 4 < \delta \Rightarrow 2/\sqrt{x-4} > M$. So by the definition of a limit, $\lim_{x \to 4^+} (2/\sqrt{x-4}) = \infty$.

23. (a) $f(x) = \sqrt{-x}$ if $x < 0$, $f(x) = 3 - x$ if $0 \le x < 3$, $f(x) = (x - 3)^2$ if $x > 3$.

(i) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3 - x) = 3$

(ii) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{-x} = 0$

(iii) Because of (i) and (ii), $\lim_{x \to 0} f(x)$ does not exist.

(iv) $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3 - x) = 0$

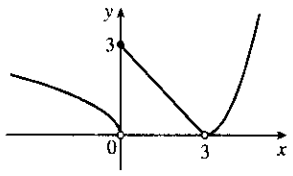
(v) $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 3)^2 = 0$

(vi) Because of (iv) and (v), $\lim_{x \to 3} f(x) = 0$.

(b) f is discontinuous at 0 since $\lim_{x \to 0} f(x)$ does not exist.

f is discontinuous at 3 since $f(3)$ does not exist.

(c)



24. (a) $g(x) = 2x - x^2$ if $0 \le x \le 2$, $g(x) = 2 - x$ if $2 < x \le 3$, $g(x) = x - 4$ if $3 < x < 4$, $g(x) = \pi$ if $x \ge 4$. Therefore,

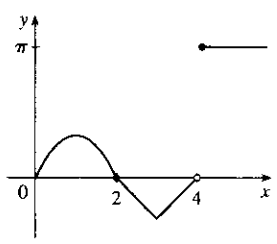
$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (2x - x^2) = 0$ and

$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (2 - x) = 0$. Thus, $\lim_{x \to 2} g(x) = 0 = g(2)$, so g is continuous at 2. $\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} (2 - x) = -1$ and

$\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} (x - 4) = -1$. Thus, $\lim_{x \to 3} g(x) = -1 = g(3)$, so g is continuous at 3.

$\lim_{x \to 4^-} g(x) = \lim_{x \to 4^-} (x - 4) = 0$ and $\lim_{x \to 4^+} g(x) = \lim_{x \to 4^+} \pi = \pi$. Thus, $\lim_{x \to 4} g(x)$ does not exist, so g is discontinuous at 4. But $\lim_{x \to 4^+} g(x) = \pi = g(4)$, so g is continuous from the right at 4.

(b)



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25. x^3 is continuous on $\mathbb{R}$ since it is a polynomial and $\cos x$ is also continuous on $\mathbb{R}$, so the product $x^3 \cos x$ is continuous on $\mathbb{R}$. The root function $\sqrt[4]{x}$ is continuous on its domain, $[0, \infty)$, and so the sum, $h(x) = \sqrt[4]{x} + x^3 \cos x$, is continuous on its domain, $[0, \infty)$.

26. $x^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0, \infty)$, so the composition $\sqrt{x^2 - 9}$ is continuous on $\{x \mid x^2 - 9 \geq 0\} = (-\infty, -3] \cup [3, \infty)$. Note that $x^2 - 2 \neq 0$ on this set and so the quotient function $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$ is continuous on its domain, $(-\infty, -3] \cup [3, \infty)$.

27. $f(x) = 2x^3 + x^2 + 2$ is a polynomial, so it is continuous on $[-2, -1]$ and $f(-2) = -10 < 0 < 1 = f(-1)$. So by the Intermediate Value Theorem there is a number c in $(-2, -1)$ such that $f(c) = 0$, that is, the equation $2x^3 + x^2 + 2 = 0$ has a root in $(-2, -1)$.

28. Let $f(x) = 2 \sin x - 3 + 2x$. Now f is continuous on $[0, 1]$ and $f(0) = -3 < 0$ and $f(1) = 2 \sin 1 - 1 \approx 0.68 > 0$. So by the Intermediate Value Theorem there is a number c in $(0, 1)$ such that $f(c) = 0$, that is, the equation $2 \sin x = 3 - 2x$ has a root in $(0, 1)$.

29. (a) The slope of the tangent line at $(2, 1)$ is

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{9 - 2x^2 - 1}{x - 2} = \lim_{x \to 2} \frac{8 - 2x^2}{x - 2} = \lim_{x \to 2} \frac{-2(x^2 - 4)}{x - 2} = \lim_{x \to 2} \frac{-2(x - 2)(x + 2)}{x - 2}$$
$$= \lim_{x \to 2} [-2(x + 2)] = -2 \cdot 4 = -8$$

(b) An equation of this tangent line is $y - 1 = -8(x - 2)$ or $y = -8x + 17$.

30. For a general point with x-coordinate a, we have

$$m = \lim_{x \to a} \frac{2/(1 - 3x) - 2/(1 - 3a)}{x - a} = \lim_{x \to a} \frac{2(1 - 3a) - 2(1 - 3x)}{(1 - 3a)(1 - 3x)(x - a)}$$
$$= \lim_{x \to a} \frac{6(x - a)}{(1 - 3a)(1 - 3x)(x - a)} = \lim_{x \to a} \frac{6}{(1 - 3a)(1 - 3x)} = \frac{6}{(1 - 3a)^2}$$

For $a = 0$, $m = 6$ and $f(0) = 2$, so an equation of the tangent line is $y - 2 = 6(x - 0)$ or $y = 6x + 2$. For $a = -1$, $m = \frac{3}{8}$ and $f(-1) = \frac{1}{2}$, so an equation of the tangent line is $y - \frac{1}{2} = \frac{3}{8}(x + 1)$ or $y = \frac{3}{8}x + \frac{7}{8}$.

31. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1 + h]$ is

$$v_{\text{ave}} = \frac{s(1 + h) - s(1)}{(1 + h) - 1} = \frac{1 + 2(1 + h) + (1 + h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10 + h}{4}.$$

So for the following intervals the average velocities are:

(i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = (10 + 2)/4 = 3$ m/s

(ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = (10 + 1)/4 = 2.75$ m/s

(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = (10 + 0.5)/4 = 2.625$ m/s

(iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = (10 + 0.1)/4 = 2.525$ m/s

(b) When $t = 1$, the instantaneous velocity is $\lim_{h \to 0} \dfrac{s(1 + h) - s(1)}{h} = \lim_{h \to 0} \dfrac{10 + h}{4} = \dfrac{10}{4} = 2.5$ m/s.

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32. (a) When V increases from 200 in^3 to 250 in^3, we have $\Delta V = 250 - 200 = 50$ in^3, and since $P = 800/V$, $\Delta P = P(250) - P(200) = \dfrac{800}{250} - \dfrac{800}{200} = 3.2 - 4 = -0.8$ lb/in^2. So the average rate of change is $\dfrac{\Delta P}{\Delta V} = \dfrac{-0.8}{50} = -0.016 \dfrac{\text{lb/in}^2}{\text{in}^3}$.

(b) Since $V = 800/P$, the instantaneous rate of change of V with respect to P is

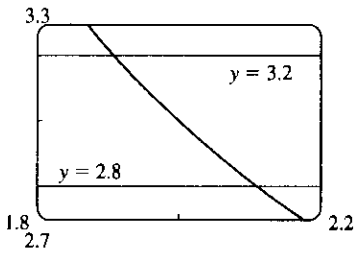
$$\lim_{h\to 0} \frac{\Delta V}{\Delta P} = \lim_{h\to 0} \frac{V(P+h) - V(P)}{h} = \lim_{h\to 0} \frac{800/(P+h) - 800/P}{h}$$

$$= \lim_{h\to 0} \frac{800\,[P - (P+h)]}{h(P+h)P} = \lim_{h\to 0} \frac{-800}{(P+h)P} = -\frac{800}{P^2}$$

which is inversely proportional to the square of P.

33. $\left|\dfrac{x+1}{x-1} - 3\right| < 0.2 \quad \Rightarrow \quad -0.2 < \dfrac{x+1}{x-1} - 3 < 0.2 \quad \Rightarrow$

$2.8 < \dfrac{x+1}{x-1} < 3.2$. Graphing the functions $y = 2.8$, $y = (x+1)/(x-1)$, and $y = 3.2$ on the interval $[1.8, 2.2]$, we see that the inequality holds whenever $1.91 < x < 2.11$ (approximately). So since $|2 - 1.91| = 0.09$ and $|2 - 2.15| = 0.15$, any positive $\delta \le 0.09$ will do.



34. The slope of the tangent to $y = \dfrac{x+1}{x-1}$ is

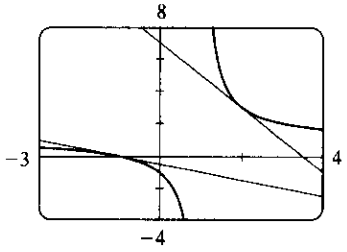
$$\lim_{h\to 0} \frac{\dfrac{(x+h)+1}{(x+h)-1} - \dfrac{x+1}{x-1}}{h} = \lim_{h\to 0} \frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{h(x-1)(x+h-1)}$$

$$= \lim_{h\to 0} \frac{-2h}{h(x-1)(x+h-1)} = -\frac{2}{(x-1)^2}$$

So at $(2,3)$, $m = -\dfrac{2}{(2-1)^2} = -2 \quad \Rightarrow \quad y - 3 = -2(x-2) \quad \Rightarrow$

$y = -2x + 7$. At $(-1,0)$, $m = -\dfrac{2}{(-1-1)^2} = -\dfrac{1}{2} \quad \Rightarrow$

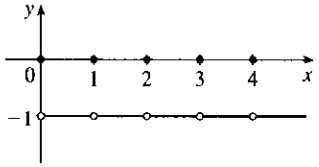
$y = -\frac{1}{2}(x+1) \quad \Rightarrow \quad y = -\frac{1}{2}x - \frac{1}{2}$.



35. $|f(x)| \le g(x) \quad \Leftrightarrow \quad -g(x) \le f(x) \le g(x)$ and $\lim_{x\to a} g(x) = 0 = \lim_{x\to a} -g(x)$. Thus, by the Squeeze Theorem, $\lim_{x\to a} f(x) = 0$.

36. (a) Note that f is an even function since $f(x) = f(-x)$. Now for any integer n, $[\![n]\!] + [\![-n]\!] = n - n = 0$, and for any real number k which is not an integer, $[\![k]\!] + [\![-k]\!] = [\![k]\!] + (-[\![k]\!] - 1) = -1$. So $\lim_{x\to a} f(x)$ exists (and is equal to -1) for all values of a.

(b) f is discontinuous at all integers.



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PROBLEMS PLUS

1. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1} \frac{t^2-1}{t^3-1} = \lim_{t\to 1} \frac{(t-1)(t+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1} \frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x}+1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$.

2. First rationalize the numerator: $\lim_{x\to 0} \dfrac{\sqrt{ax+b}-2}{x} \cdot \dfrac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim_{x\to 0} \dfrac{ax+b-4}{x\left(\sqrt{ax+b}+2\right)}$. Now since the denominator approaches 0 as $x \to 0$, the limit will exist only if the numerator also approaches 0 as $x \to 0$. So we require that $a(0)+b-4=0 \;\Rightarrow\; b=4$. So the equation becomes

$\lim_{x\to 0} \dfrac{a}{\sqrt{ax+4}+2} = 1 \;\Rightarrow\; \dfrac{a}{\sqrt{4}+2} = 1 \;\Rightarrow\; a = 4$. Therefore, $a = b = 4$.

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$.

Therefore, $\lim_{x\to 0} \dfrac{|2x-1|-|2x+1|}{x} = \lim_{x\to 0} \dfrac{-(2x-1)-(2x+1)}{x} = \lim_{x\to 0} \dfrac{-4x}{x} = \lim_{x\to 0} (-4) = -4$.

4. Let R be the midpoint of OP, so the coordinates of R are $\left(\frac{1}{2}x, \frac{1}{2}x^2\right)$ since the coordinates of P are $\left(x, x^2\right)$. Let $Q = (0, a)$. Since the slope $m_{OP} = \dfrac{x^2}{x} = x$, $m_{QR} = -\dfrac{1}{x}$ (negative reciprocal). But

$m_{QR} = \dfrac{\frac{1}{2}x^2 - a}{\frac{1}{2}x - 0} = \dfrac{x^2-2a}{x}$, so we conclude that $-1 = x^2 - 2a \;\Rightarrow\; 2a = x^2+1 \;\Rightarrow\; a = \frac{1}{2}x^2 + \frac{1}{2}$.

As $x \to 0$, $a \to \frac{1}{2}$, and the limiting position of Q is $\left(0, \frac{1}{2}\right)$.

5. Since $[\![x]\!] \le x < [\![x]\!] + 1$, we have $\dfrac{[\![x]\!]}{[\![x]\!]} \le \dfrac{x}{[\![x]\!]} < \dfrac{[\![x]\!]+1}{[\![x]\!]} \;\Rightarrow\; 1 \le \dfrac{x}{[\![x]\!]} < 1 + \dfrac{1}{[\![x]\!]}$ for $x \ge 1$. As $x \to \infty$, $[\![x]\!] \to \infty$, so $\dfrac{1}{[\![x]\!]} \to 0$ and $1 + \dfrac{1}{[\![x]\!]} \to 1$. Thus, $\lim_{x\to\infty} \dfrac{x}{[\![x]\!]} = 1$ by the Squeeze Theorem.

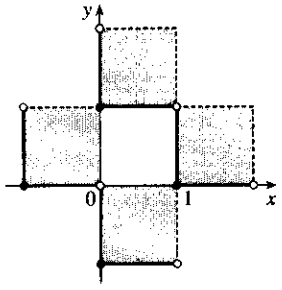
6. (a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$. Since $[\![x]\!]^2$ and $[\![y]\!]^2$ are positive integers or 0, there are only 4 cases:

Case (i): $[\![x]\!] = 1, [\![y]\!] = 0 \;\Rightarrow\; 1 \le x < 2$ and $0 \le y < 1$

Case (ii): $[\![x]\!] = -1, [\![y]\!] = 0 \;\Rightarrow\; -1 \le x < 0$ and $0 \le y < 1$

Case (iii): $[\![x]\!] = 0, [\![y]\!] = 1 \;\Rightarrow\; 0 \le x < 1$ and $1 \le y < 2$

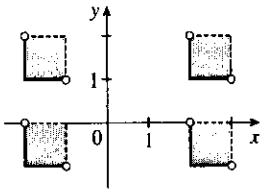
Case (iv): $[\![x]\!] = 0, [\![y]\!] = -1 \;\Rightarrow\; 0 \le x < 1$ and $-1 \le y < 0$



(b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$. The only integral solution of $n^2 - m^2 = 3$ is $n = \pm 2$ and $m = \pm 1$. So the graph is

$\{(x,y) \mid [\![x]\!] = \pm 2,\ [\![y]\!] = \pm 1\} =$

$$\left\{ (x,y) \;\middle|\; \begin{array}{l} 2 \le x \le 3 \text{ or } -2 \le x < 1, \\ 1 \le y < 2 \text{ or } -1 \le y < 0 \end{array} \right\}.$$



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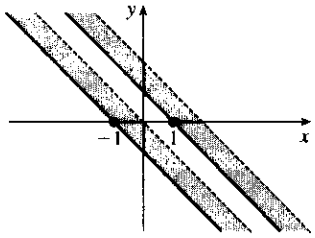
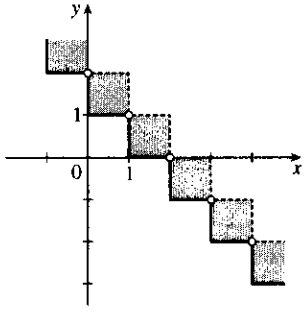
(c) $[\![x+y]\!]^2 = 1 \;\Rightarrow\; [\![x+y]\!] = \pm 1$
$\Rightarrow\; 1 \le x+y < 2$ or
$-1 \le x+y < 0$

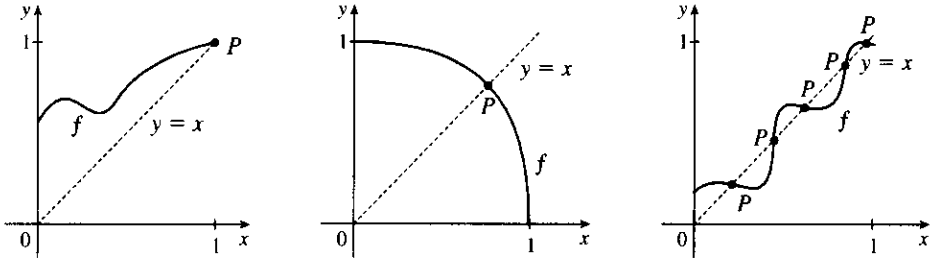


(d) For $n \le x < n+1$, $[\![x]\!] = n$. Then $[\![x]\!] + [\![y]\!] = 1 \;\Rightarrow$ $[\![y]\!] = 1-n \;\Rightarrow\; 1-n \le y < 2-n$. Choosing integer values for n produces the graph.



7. f is continuous on $(-\infty, a)$ and (a, ∞). To make f continuous on $\mathbb{R}$, we must have continuity at a. Thus,
$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \;\Rightarrow\; \lim_{x\to a^+} x^2 = \lim_{x\to a^-} (x+1) \;\Rightarrow\; a^2 = a+1 \;\Rightarrow\; a^2 - a - 1 = 0 \;\Rightarrow$
[by the quadratic formula] $a = \left(1 \pm \sqrt{5}\,\right)/2 \approx 1.618$ or -0.618.

8. (a) Here are a few possibilities:



(b) The "obstacle" is the line $x = y$ (see diagram). Any intersection of the graph of f with the line $y = x$ constitutes a fixed point, and if the graph of the function does not cross the line somewhere in $(0, 1)$, then it must either start at $(0, 0)$ (in which case 0 is a fixed point) or finish at $(1, 1)$ (in which case 1 is a fixed point).

(c) Consider the function $F(x) = f(x) - x$, where f is any continuous function with domain $[0, 1]$ and range in $[0, 1]$. We shall prove that f has a fixed point.
Now if $f(0) = 0$ then we are done: f has a fixed point (the number 0), which is what we are trying to prove. So assume $f(0) \neq 0$. For the same reason we can assume that $f(1) \neq 1$. Then $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there exists some number c in the interval $(0, 1)$ such that $F(c) = f(c) - c = 0$. So $f(c) = c$, and therefore f has a fixed point.

9. $\lim_{x\to a} f(x) = \lim_{x\to a} \left(\frac{1}{2}[f(x) + g(x)] + \frac{1}{2}[f(x) - g(x)]\right)$
$= \frac{1}{2} \lim_{x\to a} [f(x) + g(x)] + \frac{1}{2} \lim_{x\to a} [f(x) - g(x)]$
$= \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{3}{2}$, and

$\lim_{x\to a} g(x) = \lim_{x\to a} \Big([f(x) + g(x)] - f(x) \Big) = \lim_{x\to a} [f(x) + g(x)] - \lim_{x\to a} f(x) = 2 - \frac{3}{2} = \frac{1}{2}$.

So $\lim_{x\to a} [f(x)g(x)] = \left[\lim_{x\to a} f(x)\right] \left[\lim_{x\to a} g(x)\right] = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

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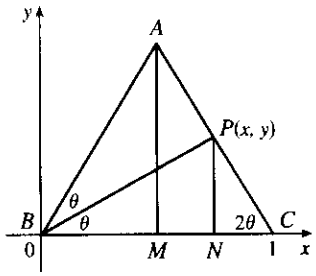
Another solution: Since $\lim_{x\to a}[f(x)+g(x)]$ and $\lim_{x\to a}[f(x)-g(x)]$ exist, we must have

$\lim_{x\to a}[f(x)+g(x)]^2 = \left(\lim_{x\to a}[f(x)+g(x)]\right)^2$ and $\lim_{x\to a}[f(x)-g(x)]^2 = \left(\lim_{x\to a}[f(x)-g(x)]\right)^2$, so

$$\lim_{x\to a}[f(x)\,g(x)] = \lim_{x\to a}\tfrac{1}{4}\left([f(x)+g(x)]^2-[f(x)-g(x)]^2\right) \quad \text{[because all of the } f^2 \text{ and } g^2 \text{ cancel]}$$

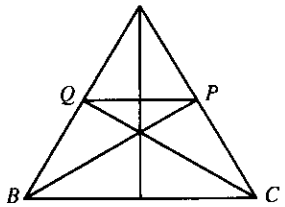
$$= \tfrac{1}{4}\left(\lim_{x\to a}[f(x)+g(x)]^2 - \lim_{x\to a}[f(x)-g(x)]^2\right) = \tfrac{1}{4}(2^2-1^2) = \tfrac{3}{4}.$$

10. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular from P, as shown. We see from $\angle NCP$ that $\tan 2\theta = \dfrac{y}{1-x}$, and from $\angle NBP$ that $\tan\theta = y/x$. Using the double-angle formula for tangents, we get $\dfrac{y}{1-x} = \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2(y/x)}{1-(y/x)^2}$.

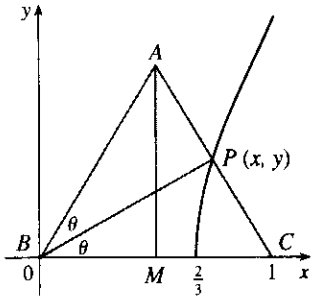


After a bit of simplification, this becomes $\dfrac{1}{1-x} = \dfrac{2x}{x^2-y^2} \quad \Leftrightarrow$ $y^2 = x\,(3x-2)$. As the altitude AM decreases in length, the point P will approach the x-axis, that is, $y \to 0$, so the limiting location of P must be one of the roots of the equation $x(3x-2)=0$. Obviously it is not $x=0$ (the point P can never be to the left of the altitude AM, which it would have to be in order to approach 0) so it must be $3x-2=0$, that is, $x=\frac{2}{3}$.

Solution 2: We add a few lines to the original diagram, as shown. Now note that $\angle BPQ = \angle PBC$ (alternate angles; $QP \parallel BC$ by symmetry) and similarly $\angle CQP = \angle QCB$. So ΔBPQ and ΔCQP are isosceles, and the line segments BQ, QP and PC are all of equal length. As $|AM| \to 0$, P and Q approach points on the base, and the point P is seen to approach a position two-thirds of the way between B and C, as above.



(b) The equation $y^2 = x(3x-2)$ calculated in part (a) is the equation of the curve traced out by P. Now as $|AM| \to \infty$, $2\theta \to \frac{\pi}{2}$, $\theta \to \frac{\pi}{4}$, $x \to 1$, and since $\tan\theta = y/x$, $y \to 1$. Thus, P only traces out the part of the curve with $0 \le y < 1$.



11. (a) Consider $G(x) = T(x+180°) - T(x)$. Fix any number a. If $G(a)=0$, we are done: Temperature at a = Temperature at $a+180°$. If $G(a)>0$, then $G(a+180°) = T(a+360°) - T(a+180°)$ $= T(a) - T(a+180°) = -G(a) < 0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a+180°]$. If $G(a)<0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

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Chapter 3

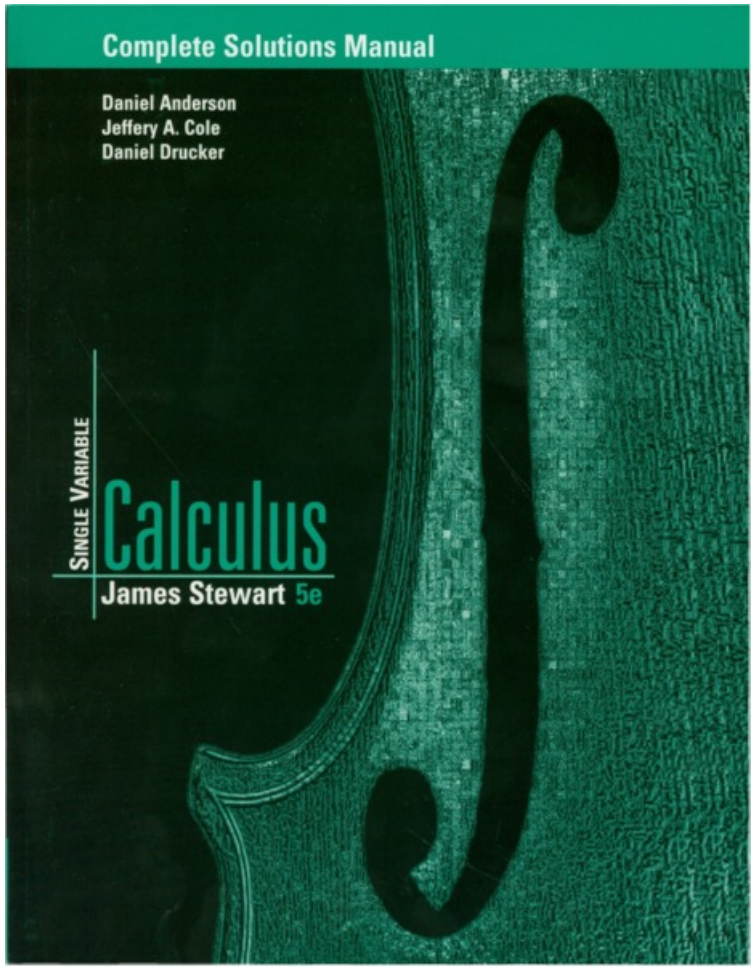
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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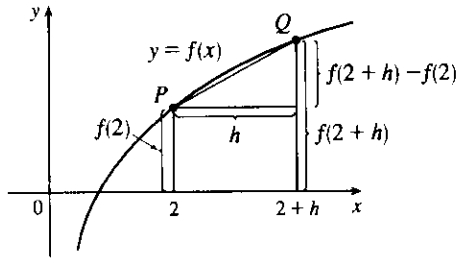
3 □ DERIVATIVES

3.1 Derivatives

1. 

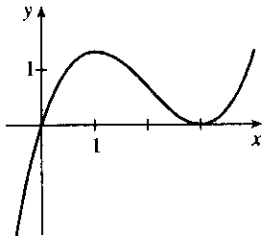
The line from $P(2, f(2))$ to $Q(2+h, f(2+h))$ is the line that has slope $\dfrac{f(2+h)-f(2)}{h}$.

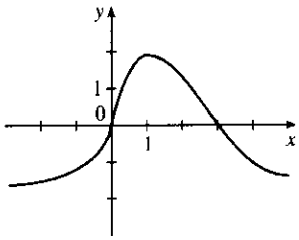
2. As h decreases, the line PQ becomes steeper, so its slope increases. So $0 < \dfrac{f(4)-f(2)}{4-2} < \dfrac{f(3)-f(2)}{3-2} < \lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$. Thus, $0 < \frac{1}{2}\,[f(4)-f(2)] < f(3)-f(2) < f'(2)$.

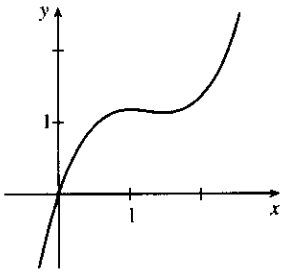
3. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

4. Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

5. We begin by drawing a curve through the origin at a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Lastly, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.





6. 

7. Using Definition 2 with $f(x) = 3x^2 - 5x$ and the point $(2, 2)$, we have

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{[3(2+h)^2 - 5(2+h)] - 2}{h}$$

$$= \lim_{h\to 0} \frac{(12 + 12h + 3h^2 - 10 - 5h) - 2}{h} = \lim_{h\to 0} \frac{3h^2 + 7h}{h} = \lim_{h\to 0} (3h + 7) = 7.$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = 7(x - 2)$ or $y = 7x - 12$.

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8. Using Definition 2 with $g(x) = 1 - x^3$ and the point $(0, 1)$, we have

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{[1-(0+h)^3] - 1}{h} = \lim_{h\to 0} \frac{(1-h^3) - 1}{h} = \lim_{h\to 0} (-h^2) = 0.$$

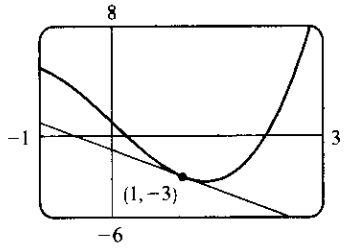
So an equation of the tangent line is $y - 1 = 0(x - 0)$ or $y = 1$.

9. (a) Using Definition 2 with $F(x) = x^3 - 5x + 1$ and the point $(1, -3)$, we have

$$\begin{aligned} F'(1) &= \lim_{h\to 0} \frac{F(1+h) - F(1)}{h} = \lim_{h\to 0} \frac{[(1+h)^3 - 5(1+h) + 1] - (-3)}{h} \\ &= \lim_{h\to 0} \frac{(1 + 3h + 3h^2 + h^3 - 5 - 5h + 1) + 3}{h} = \lim_{h\to 0} \frac{h^3 + 3h^2 - 2h}{h} \\ &= \lim_{h\to 0} \frac{h(h^2 + 3h - 2)}{h} = \lim_{h\to 0} (h^2 + 3h - 2) = -2 \end{aligned}$$

So an equation of the tangent line at $(1, -3)$ is $y - (-3) = -2(x - 1) \quad \Leftrightarrow \quad y = -2x - 1$.

(b)

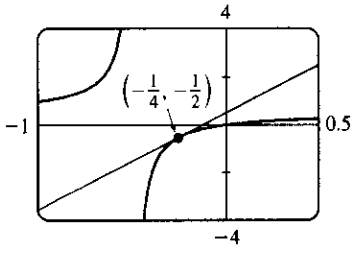


10. (a) $G'(a) = \lim_{h\to 0} \dfrac{G(a+h) - G(a)}{h} = \lim_{h\to 0} \dfrac{\dfrac{a+h}{1+2(a+h)} - \dfrac{a}{1+2a}}{h}$

$$= \lim_{h\to 0} \frac{a + 2a^2 + h + 2ah - a - 2a^2 - 2ah}{h(1+2a+2h)(1+2a)} = \lim_{h\to 0} \frac{1}{(1+2a+2h)(1+2a)} = (1+2a)^{-2}$$

So the slope of the tangent at the point $\left(-\frac{1}{4}, -\frac{1}{2}\right)$ is $m = \left[1 + 2\left(-\frac{1}{4}\right)\right]^{-2} = 4$, and thus an equation is $y + \frac{1}{2} = 4\left(x + \frac{1}{4}\right)$ or $y = 4x + \frac{1}{2}$.

(b)



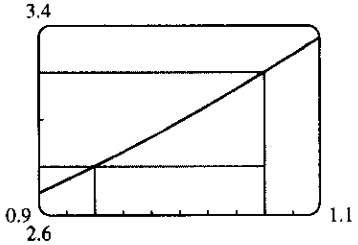
11. (a) $f'(1) = \lim_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \dfrac{3^{1+h} - 3^1}{h}$.

So let $F(h) = \dfrac{3^{1+h} - 3}{h}$. We calculate:

h	$F(h)$	h	$F(h)$
0.1	3.484	−0.1	3.121
0.01	3.314	−0.01	3.278
0.001	3.298	−0.001	3.294
0.0001	3.296	−0.0001	3.296

We estimate that $f'(1) \approx 3.296$.

(b)



From the graph, we estimate that the slope of the tangent is about

$$\frac{3.2 - 2.8}{1.06 - 0.94} = \frac{0.4}{0.12} \approx 3.3.$$

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12. (a) $g'\left(\frac{\pi}{4}\right) = \lim_{h\to 0} \dfrac{g\left(\frac{\pi}{4}+h\right) - g\left(\frac{\pi}{4}\right)}{h}$

$$= \lim_{h\to 0} \frac{\tan\left(\frac{\pi}{4}+h\right) - \tan\left(\frac{\pi}{4}\right)}{h}.$$

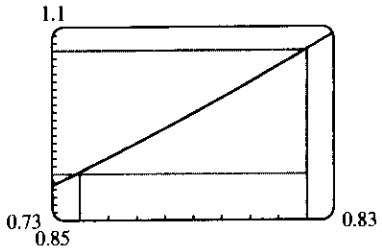
So let $G(h) = \dfrac{\tan\left(\frac{\pi}{4}+h\right) - 1}{h}$.

We calculate:

h	$G(h)$	h	$G(h)$
0.1	2.2305	−0.1	1.8237
0.01	2.0203	−0.01	1.9803
0.001	2.0020	−0.001	1.9980
0.0001	2.0002	−0.0001	1.9998

We estimate that $g'\left(\frac{\pi}{4}\right) = 2$.

(b)



From the graph, we estimate that the slope of the tangent is about $\dfrac{1.07-0.91}{0.82-0.74} = \dfrac{0.16}{0.08} = 2$.

13. Use Definition 2 with $f(x) = 3 - 2x + 4x^2$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\left[3-2(a+h)+4(a+h)^2\right] - \left(3-2a+4a^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(3-2a-2h+4a^2+8ah+4h^2\right) - \left(3-2a+4a^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{-2h+8ah+4h^2}{h} = \lim_{h\to 0} \frac{h(-2+8a+4h)}{h} = \lim_{h\to 0}(-2+8a+4h) = -2+8a$$

14. $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \dfrac{\left[(a+h)^4 - 5(a+h)\right] - \left(a^4-5a\right)}{h}$

$$= \lim_{h\to 0} \frac{\left(a^4+4a^3h+6a^2h^2+4ah^3+h^4-5a-5h\right) - \left(a^4-5a\right)}{h}$$

$$= \lim_{h\to 0} \frac{4a^3h+6a^2h^2+4ah^3+h^4-5h}{h} = \lim_{h\to 0} \frac{h\left(4a^3+6a^2h+4ah^2+h^3-5\right)}{h}$$

$$= \lim_{h\to 0} \left(4a^3+6a^2h+4ah^2+h^3-5\right) = 4a^3-5$$

15. Use Definition 2 with $f(t) = (2t+1)/(t+3)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{2(a+h)+1}{(a+h)+3} - \dfrac{2a+1}{a+3}}{h}$$

$$= \lim_{h\to 0} \frac{(2a+2h+1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h\to 0} \frac{\left(2a^2+6a+2ah+6h+a+3\right) - \left(2a^2+2ah+6a+a+h+3\right)}{h(a+h+3)(a+3)}$$

$$= \lim_{h\to 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h\to 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2}$$

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16. $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \dfrac{\dfrac{(a+h)^2+1}{(a+h)-2} - \dfrac{a^2+1}{a-2}}{h}$

$$= \lim_{h\to 0} \frac{(a^2+2ah+h^2+1)(a-2)-(a^2+1)(a+h-2)}{h(a+h-2)(a-2)}$$

$$= \lim_{h\to 0} \frac{(a^3-2a^2+2a^2h-4ah+ah^2-2h^2+a-2)-(a^3+a^2h-2a^2+a+h-2)}{h(a+h-2)(a-2)}$$

$$= \lim_{h\to 0} \frac{a^2h-4ah+ah^2-2h^2-h}{h(a+h-2)(a-2)} = \lim_{h\to 0} \frac{h\,(a^2-4a+ah-2h-1)}{h(a+h-2)(a-2)}$$

$$= \lim_{h\to 0} \frac{a^2-4a+ah-2h-1}{(a+h-2)(a-2)} = \frac{a^2-4a-1}{(a-2)^2}$$

17. Use Definition 2 with $f(x) = 1/\sqrt{x+2}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{(a+h)+2}} - \dfrac{1}{\sqrt{a+2}}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{\sqrt{a+2}-\sqrt{a+h+2}}{\sqrt{a+h+2}\sqrt{a+2}}}{h} = \lim_{h\to 0} \left[\frac{\sqrt{a+2}-\sqrt{a+h+2}}{h\sqrt{a+h+2}\sqrt{a+2}} \cdot \frac{\sqrt{a+2}+\sqrt{a+h+2}}{\sqrt{a+2}+\sqrt{a+h+2}}\right]$$

$$= \lim_{h\to 0} \frac{(a+2)-(a+h+2)}{h\sqrt{a+h+2}\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$$

$$= \lim_{h\to 0} \frac{-h}{h\sqrt{a+h+2}\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{a+h+2}\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$$

$$= \frac{-1}{\left(\sqrt{a+2}\right)^2\left(2\sqrt{a+2}\right)} = -\frac{1}{2\,(a+2)^{3/2}}$$

18. $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \dfrac{\sqrt{3\,(a+h)+1}-\sqrt{3a+1}}{h}$

$$= \lim_{h\to 0} \frac{\left(\sqrt{3a+3h+1}-\sqrt{3a+1}\right)\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)}{h\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)}$$

$$= \lim_{h\to 0} \frac{(3a+3h+1)-(3a+1)}{h\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)} = \lim_{h\to 0} \frac{3h}{h\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3a+3h+1}+\sqrt{3a+1}} = \frac{3}{2\sqrt{3a+1}}$$

Note that the answers to Exercises 19–24 are not unique.

19. By Definition 2, $\lim_{h\to 0} \dfrac{(1+h)^{10}-1}{h} = f'(1)$, where $f(x) = x^{10}$ and $a = 1$.

Or: By Definition 2, $\lim_{h\to 0} \dfrac{(1+h)^{10}-1}{h} = f'(0)$, where $f(x) = (1+x)^{10}$ and $a = 0$.

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20. By Definition 2, $\lim_{h\to 0} \dfrac{\sqrt[4]{16+h}-2}{h} = f'(16)$, where $f(x) = \sqrt[4]{x}$ and $a = 16$.

Or: By Definition 2, $\lim_{h\to 0} \dfrac{\sqrt[4]{16+h}-2}{h} = f'(0)$, where $f(x) = \sqrt[4]{16+x}$ and $a = 0$.

21. By Equation 3, $\lim_{x\to 5} \dfrac{2^x - 32}{x-5} = f'(5)$, where $f(x) = 2^x$ and $a = 5$.

22. By Equation 3, $\lim_{x\to \pi/4} \dfrac{\tan x - 1}{x - \pi/4} = f'(\pi/4)$, where $f(x) = \tan x$ and $a = \pi/4$.

23. By Definition 2, $\lim_{h\to 0} \dfrac{\cos(\pi+h)+1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By Definition 2, $\lim_{h\to 0} \dfrac{\cos(\pi+h)+1}{h} = f'(0)$, where $f(x) = \cos(\pi + x)$ and $a = 0$.

24. By Equation 3, $\lim_{t\to 1} \dfrac{t^4+t-2}{t-1} = f'(1)$, where $f(t) = t^4 + t$ and $a = 1$.

25.
$$v(2) = f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\left[(2+h)^2 - 6(2+h) - 5\right] - \left[2^2 - 6(2) - 5\right]}{h}$$
$$= \lim_{h\to 0} \frac{(4 + 4h + h^2 - 12 - 6h - 5) - (-13)}{h} = \lim_{h\to 0} \frac{h^2 - 2h}{h} = \lim_{h\to 0} (h-2) = -2 \text{ m/s}$$

26.
$$v(2) = f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0} \frac{\left[2(2+h)^3 - (2+h) + 1\right] - \left[2(2)^3 - 2 + 1\right]}{h}$$
$$= \lim_{h\to 0} \frac{\left(2h^3 + 12h^2 + 24h + 16 - 2 - h + 1\right) - 15}{h}$$
$$= \lim_{h\to 0} \frac{2h^3 + 12h^2 + 23h}{h} = \lim_{h\to 0} \left(2h^2 + 12h + 23\right) = 23 \text{ m/s}$$

27. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

28. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

29. (a) $f'(v)$ is the rate at which the fuel consumption is changing with respect to the speed. Its units are (gal/h)/(mi/h).

(b) The fuel consumption is decreasing by 0.05 (gal/h)/(mi/h) as the car's speed reaches 20 mi/h. So if you increase your speed to 21 mi/h, you could expect to decrease your fuel consumption by about 0.05 (gal/h)/(mi/h).

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30. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).

(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

31. $T'(10)$ is the rate at which the temperature is changing at 10:00 A.M. To estimate the value of $T'(10)$, we will average the difference quotients obtained using the times $t = 8$ and $t = 12$. Let

$A = \dfrac{T(8) - T(10)}{8 - 10} = \dfrac{72 - 81}{-2} = 4.5$ and $B = \dfrac{T(12) - T(10)}{12 - 10} = \dfrac{88 - 81}{2} = 3.5$. Then

$$T'(10) = \lim_{t \to 10} \frac{T(t) - T(10)}{t - 10} \approx \frac{A + B}{2} = \frac{4.5 + 3.5}{2} = 4^\circ\text{F/h}.$$

32. For 1910: We will average the difference quotients obtained using the years 1900 and 1920.

Let $A = \dfrac{E(1900) - E(1910)}{1900 - 1910} = \dfrac{48.3 - 51.1}{-10} = 0.28$ and

$B = \dfrac{E(1920) - E(1910)}{1920 - 1910} = \dfrac{55.2 - 51.1}{10} = 0.41$. Then

$E'(1910) = \lim\limits_{t \to 1910} \dfrac{E(t) - E(1910)}{t - 1910} \approx \dfrac{A + B}{2} = 0.345$. This means that life expectancy at birth was increasing at about 0.345 year/year in 1910.

For 1950: Using data for 1940 and 1960 in a similar fashion, we obtain $E'(1950) \approx [0.31 + 0.10]/2 = 0.205$. So life expectancy at birth was increasing at about 0.205 year/year in 1950.

33. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\text{mg/L})/^\circ\text{C}$.

(b) For $T = 16^\circ\text{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \dfrac{6 - 14}{32 - 0} = -\dfrac{8}{32} = -0.25$ $(\text{mg/L})/^\circ\text{C}$. This means that as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of 0.25 $(\text{mg/L})/^\circ\text{C}$.

34. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are $(\text{cm/s})/^\circ\text{C}$.

(b) For $T = 15\ ^\circ\text{C}$, it appears the tangent line to the curve goes through the points $(10, 25)$ and $(20, 32)$. So $S'(15) \approx \dfrac{32 - 25}{20 - 10} = 0.7$ $(\text{cm/s})/^\circ\text{C}$. This tells us that at $T = 15\ ^\circ\text{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of 0.7 $(\text{cm/s})/^\circ\text{C}$. In a similar fashion for $T = 25\ ^\circ\text{C}$, we can use the points $(20, 35)$ and $(25, 25)$ to obtain $S'(25) \approx \dfrac{25 - 35}{25 - 20} = -2$ $(\text{cm/s})/^\circ\text{C}$. As it gets warmer than $20\ ^\circ\text{C}$, the maximum sustainable speed decreases rapidly.

35. Since $f(x) = x \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h \sin(1/h) - 0}{h} = \lim\limits_{h \to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 4 in Section 2.2.)

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36. Since $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{h^2 \sin(1/h) - 0}{h} = \lim_{h \to 0} h \sin(1/h)$. Since $-1 \leq \sin \dfrac{1}{h} \leq 1$, we have

$-|h| \leq |h| \sin \dfrac{1}{h} \leq |h| \quad \Rightarrow \quad -|h| \leq h \sin \dfrac{1}{h} \leq |h|$. Because $\lim_{h \to 0} (-|h|) = 0$ and $\lim_{h \to 0} |h| = 0$, we know that

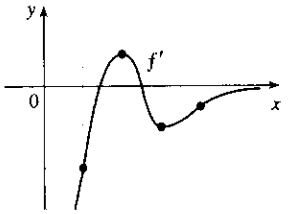
$\lim_{h \to 0} \left(h \sin \dfrac{1}{h} \right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

3.2 The Derivative as a Function

1. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f, it appears that

(a) $f'(1) \approx -2$ (b) $f'(2) \approx 0.8$

(c) $f'(3) \approx -1$ (d) $f'(4) \approx -0.5$

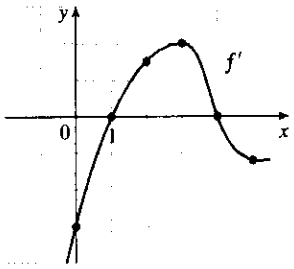


2. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f, it appears that

(a) $f'(0) \approx -3$ (b) $f'(1) \approx 0$

(c) $f'(2) \approx 1.5$ (d) $f'(3) \approx 2$

(e) $f'(4) \approx 0$ (f) $f'(5) \approx -1.2$



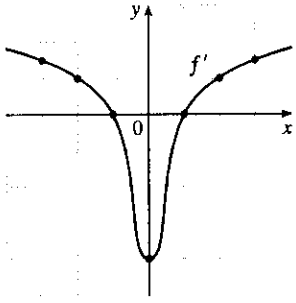
3. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

(a) $f'(-3) \approx 1.5$ (b) $f'(-2) \approx 1$

(c) $f'(-1) \approx 0$ (d) $f'(0) \approx -4$

(e) $f'(1) \approx 0$ (f) $f'(2) \approx 1$

(g) $f'(3) \approx 1.5$



4. (a)′ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

(b)′ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)′ = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)′ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

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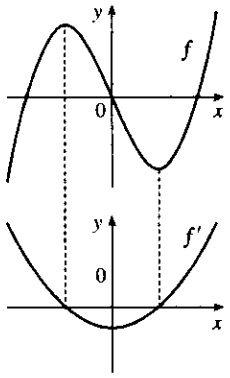
Hints for Exercises 5–13: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f — there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

5.



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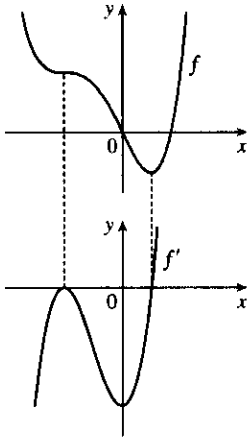

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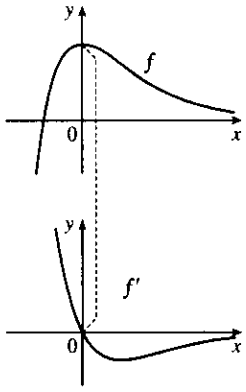

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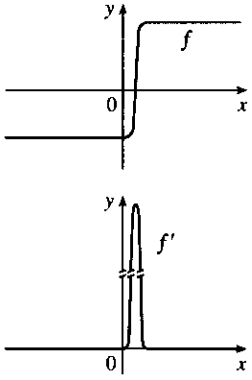

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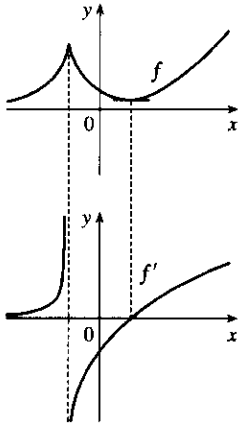

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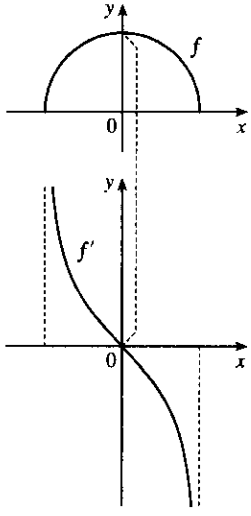
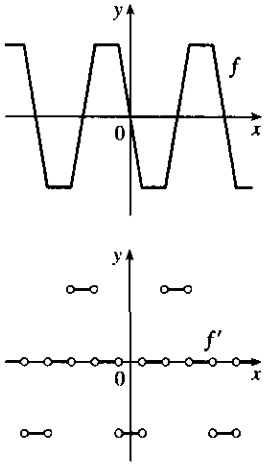


11.



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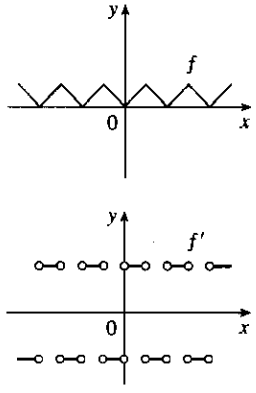
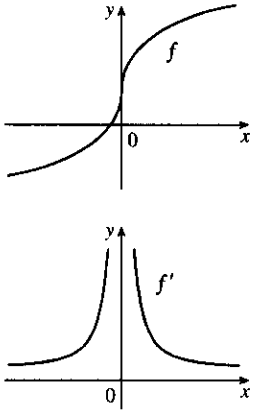


13.



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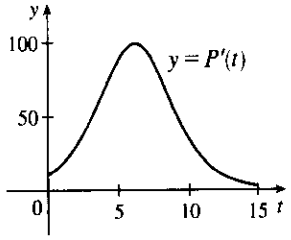
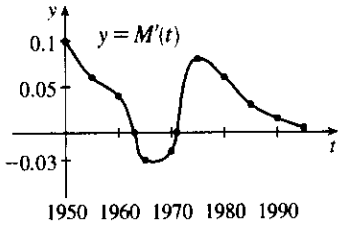
14. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y-values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y-values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.

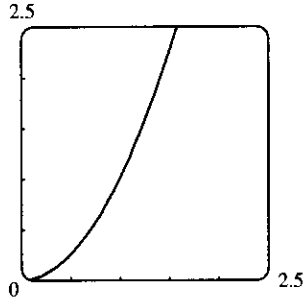


16. See Figure 1 in Section 3.5.

17. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.

(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.



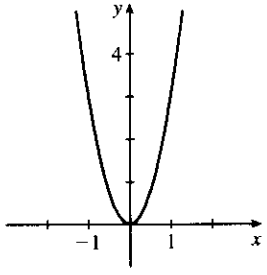
(d) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{\left(x^2 + 2hx + h^2\right) - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x + h) = 2x$$

18. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

(b) By symmetry, $f'(-x) = f'(x)$. So $f'\left(-\frac{1}{2}\right) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

(d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$. Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

(c)



(e) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h} = \lim_{h\to 0} \left(3x^2 + 3xh + h^2\right) = 3x^2$$

19. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{37 - 37}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

Domain of f = domain of $f' = \mathbb{R}$.

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20. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[12+7(x+h)]-(12+7x)}{h}$

$= \lim_{h\to 0} \dfrac{12+7x+7h-12-7x}{h} = \lim_{h\to 0} \dfrac{7h}{h} = \lim_{h\to 0} 7 = 7$

Domain of f = domain of $f' = \mathbb{R}$.

21. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[1-3(x+h)^2]-(1-3x^2)}{h}$

$= \lim_{h\to 0} \dfrac{[1-3(x^2+2xh+h^2)]-(1-3x^2)}{h} = \lim_{h\to 0} \dfrac{1-3x^2-6xh-3h^2-1+3x^2}{h}$

$= \lim_{h\to 0} \dfrac{-6xh-3h^2}{h} = \lim_{h\to 0} \dfrac{h(-6x-3h)}{h} = \lim_{h\to 0}(-6x-3h) = -6x$

Domain of f = domain of $f' = \mathbb{R}$.

22. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[5(x+h)^2+3(x+h)-2]-(5x^2+3x-2)}{h}$

$= \lim_{h\to 0} \dfrac{5x^2+10xh+5h^2+3x+3h-2-5x^2-3x+2}{h} = \lim_{h\to 0} \dfrac{10xh+5h^2+3h}{h}$

$= \lim_{h\to 0} \dfrac{h(10x+5h+3)}{h} = \lim_{h\to 0}(10x+5h+3) = 10x+3$

Domain of f = domain of $f' = \mathbb{R}$.

23. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[(x+h)^3-3(x+h)+5]-(x^3-3x+5)}{h}$

$= \lim_{h\to 0} \dfrac{(x^3+3x^2h+3xh^2+h^3-3x-3h+5)-(x^3-3x+5)}{h}$

$= \lim_{h\to 0} \dfrac{3x^2h+3xh^2+h^3-3h}{h} = \lim_{h\to 0} \dfrac{h(3x^2+3xh+h^2-3)}{h}$

$= \lim_{h\to 0} (3x^2+3xh+h^2-3) = 3x^2-3$

Domain of f = domain of $f' = \mathbb{R}$.

24. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h+\sqrt{x+h})-(x+\sqrt{x})}{h}$

$= \lim_{h\to 0} \left(\dfrac{h}{h} + \dfrac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \right) = \lim_{h\to 0} \left[1 + \dfrac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \right]$

$= \lim_{h\to 0} \left(1 + \dfrac{1}{\sqrt{x+h}+\sqrt{x}} \right) = 1 + \dfrac{1}{\sqrt{x}+\sqrt{x}} = 1 + \dfrac{1}{2\sqrt{x}}$

Domain of $f = [0,\infty)$, domain of $f' = (0,\infty)$.

25. $g'(x) = \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h} \left[\dfrac{\sqrt{1+2(x+h)}+\sqrt{1+2x}}{\sqrt{1+2(x+h)}+\sqrt{1+2x}} \right]$

$= \lim_{h\to 0} \dfrac{(1+2x+2h)-(1+2x)}{h\left[\sqrt{1+2(x+h)}+\sqrt{1+2x}\right]} = \lim_{h\to 0} \dfrac{2}{\sqrt{1+2x+2h}+\sqrt{1+2x}} = \dfrac{2}{2\sqrt{1+2x}} = \dfrac{1}{\sqrt{1+2x}}$

Domain of $g = \left[-\frac{1}{2},\infty\right)$, domain of $g' = \left(-\frac{1}{2},\infty\right)$.

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26. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{3+(x+h)}{1-3(x+h)} - \dfrac{3+x}{1-3x}}{h}$

$= \lim_{h\to 0} \dfrac{(3+x+h)(1-3x)-(3+x)(1-3x-3h)}{h(1-3x-3h)(1-3x)}$

$= \lim_{h\to 0} \dfrac{(3-9x+x-3x^2+h-3hx)-(3-9x-9h+x-3x^2-3hx)}{h(1-3x-3h)(1-3x)}$

$= \lim_{h\to 0} \dfrac{10h}{h(1-3x-3h)(1-3x)} = \lim_{h\to 0} \dfrac{10}{(1-3x-3h)(1-3x)} = \dfrac{10}{(1-3x)^2}$

Domain of f = domain of $f' = (-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$.

27. $G'(t) = \lim_{h\to 0} \dfrac{G(t+h)-G(t)}{h} = \lim_{h\to 0} \dfrac{\dfrac{4(t+h)}{(t+h)+1} - \dfrac{4t}{t+1}}{h} = \lim_{h\to 0} \dfrac{4(t+h)(t+1)-4t(t+h+1)}{(t+h+1)(t+1)}$

$= \lim_{h\to 0} \dfrac{(4t^2+4ht+4t+4h)-(4t^2+4ht+4t)}{h(t+h+1)(t+1)}$

$= \lim_{h\to 0} \dfrac{4h}{h(t+h+1)(t+1)} = \lim_{h\to 0} \dfrac{4}{(t+h+1)(t+1)} = \dfrac{4}{(t+1)^2}$

Domain of G = domain of $G' = (-\infty, -1) \cup (-1, \infty)$.

28. $g'(x) = \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = \lim_{h\to 0} \dfrac{x^2-(x+h)^2}{h(x+h)^2x^2}$

$= \lim_{h\to 0} \dfrac{x^2-(x^2+2xh+h^2)}{h} = \lim_{h\to 0} \dfrac{-2xh-h^2}{h(x+h)^2x^2} = \lim_{h\to 0} \dfrac{-2x-h}{(x+h)^2x^2} = \dfrac{-2x}{x^4}$

$= -2x^{-3}$

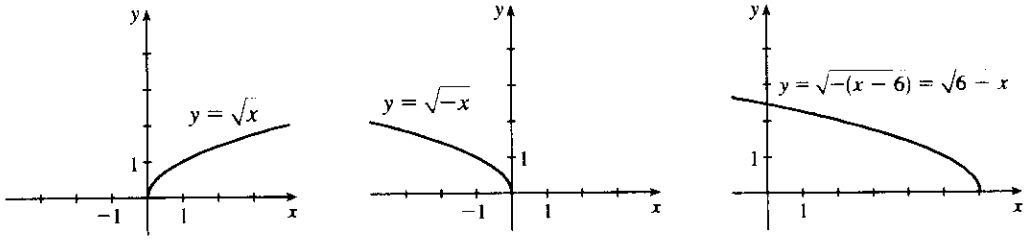
Domain of g = domain of $g' = \{x \mid x \neq 0\}$.

29. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^4-x^4}{h} = \lim_{h\to 0} \dfrac{(x^4+4x^3h+6x^2h^2+4xh^3+h^4)-x^4}{h}$

$= \lim_{h\to 0} \dfrac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0} (4x^3+6x^2h+4xh^2+h^3) = 4x^3$

Domain of f = domain of $f' = \mathbb{R}$.

30. (a)



(b) Note that the third graph in part (a) has small negative values for its slope, f'; but as $x \to 6^-$, $f' \to -\infty$. See the graph in part (d).

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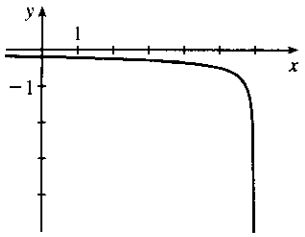
(c) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{6-(x+h)}-\sqrt{6-x}}{h}\left[\dfrac{\sqrt{6-(x+h)}+\sqrt{6-x}}{\sqrt{6-(x+h)}+\sqrt{6-x}}\right]$

$= \lim_{h\to 0} \dfrac{[6-(x+h)]-(6-x)}{h\left[\sqrt{6-(x+h)}+\sqrt{6-x}\right]} = \lim_{h\to 0} \dfrac{-h}{h\left(\sqrt{6-x-h}+\sqrt{6-x}\right)}$

$= \lim_{h\to 0} \dfrac{-1}{\sqrt{6-x-h}+\sqrt{6-x}} = \dfrac{-1}{2\sqrt{6-x}}$

Domain of $f = (-\infty, 6]$, domain of $f' = (-\infty, 6)$.

(d)

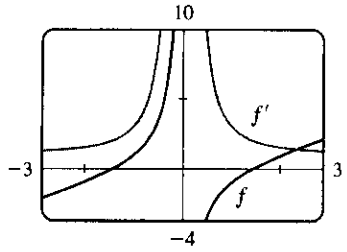


31. (a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\left[x+h-\left(\dfrac{2}{x+h}\right)\right]-\left[x-\left(\dfrac{2}{x}\right)\right]}{h}$

$= \lim_{h\to 0} \left[\dfrac{h-\dfrac{2}{(x+h)}+\dfrac{2}{x}}{h}\right] = \lim_{h\to 0}\left[1+\dfrac{-2x+2(x+h)}{hx(x+h)}\right] = \lim_{h\to 0}\left[1+\dfrac{2h}{hx(x+h)}\right]$

$= \lim_{h\to 0}\left[1+\dfrac{2}{x(x+h)}\right] = 1+\dfrac{2}{x^2}$

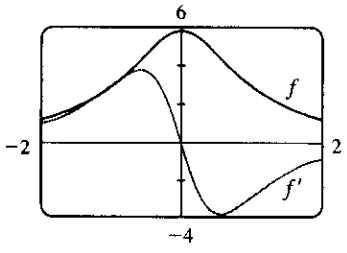
(b) Notice that when f has steep tangent lines, $f'(x)$ is very large. When f is flatter, $f'(x)$ is smaller.



32. (a) $f'(t) = \lim_{h\to 0} \dfrac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \dfrac{\dfrac{6}{1+(t+h)^2}-\dfrac{6}{1+t^2}}{h} = \lim_{h\to 0} \dfrac{6+6t^2-6-6(t+h)^2}{h\left[1+(t+h)^2\right](1+t^2)}$

$= \lim_{h\to 0} \dfrac{-12th-6h^2}{h\left[1+(t+h)^2\right](1+t^2)} = \lim_{h\to 0} \dfrac{-12t-6h}{\left[1+(t+h)^2\right](1+t^2)} = \dfrac{-12t}{(1+t^2)^2}$

(b) Notice that f has a horizontal tangent when $t = 0$. This corresponds to $f'(0) = 0$. f' is positive when f is increasing and negative when f is decreasing.



33. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

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(b) To find $U'(t)$, we use $\lim_{h\to 0} \dfrac{U(t+h)-U(t)}{h} \approx \dfrac{U(t+h)-U(t)}{h}$ for small values of h.

For 1991: $U'(1991) = \dfrac{U(1992)-U(1991)}{1992-1991} = \dfrac{7.5-6.8}{1} = 0.70$

For 1992: We estimate $U'(1992)$ by using $h=-1$ and $h=1$, and then average the two results to obtain a final estimate.

$h=-1 \quad \Rightarrow \quad U'(1992) \approx \dfrac{U(1991)-U(1992)}{1991-1992} = \dfrac{6.8-7.5}{-1} = 0.70;$

$h=1 \quad \Rightarrow \quad U'(1992) \approx \dfrac{U(1993)-U(1992)}{1993-1992} = \dfrac{6.9-7.5}{1} = -0.60.$

So we estimate that $U'(1992) \approx \frac{1}{2}[0.70+(-0.60)] = 0.05$.

t	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
$U'(t)$	0.70	0.05	−0.70	−0.65	−0.35	−0.35	−0.45	−0.35	−0.25	−0.20

34. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h\to 0} \dfrac{P(t+h)-P(t)}{h} \approx \dfrac{P(t+h)-P(t)}{h}$ for small values of h.

For 1950: $P'(1950) = \dfrac{P(1960)-P(1950)}{1960-1950} = \dfrac{35.7-31.1}{10} = 0.46$

For 1960: We estimate $P'(1960)$ by using $h=-10$ and $h=10$, and then average the two results to obtain a final estimate.

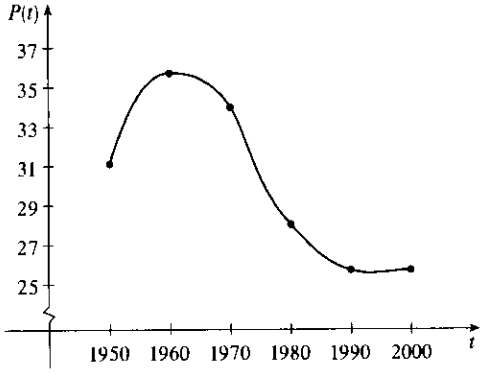
$h=-10 \quad \Rightarrow \quad P'(1960) \approx \dfrac{P(1950)-P(1960)}{1950-1960} = \dfrac{31.1-35.7}{-10} = 0.46$

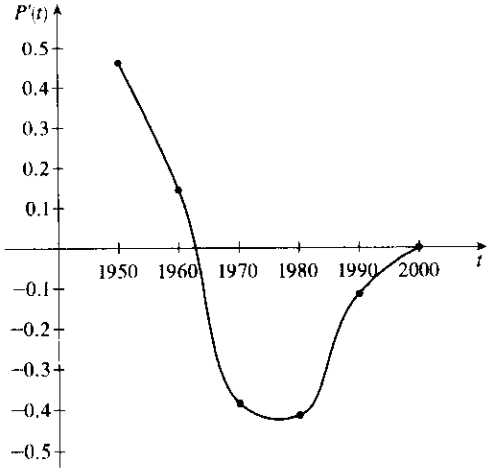
$h=10 \quad \Rightarrow \quad P'(1960) \approx \dfrac{P(1970)-P(1960)}{1970-1960} = \dfrac{34.0-35.7}{10} = -0.17$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46+(-0.17)] = 0.145$.

t	1950	1960	1970	1980	1990	2000
$P'(t)$	0.460	0.145	−0.385	−0.415	−0.115	0.000

(c)





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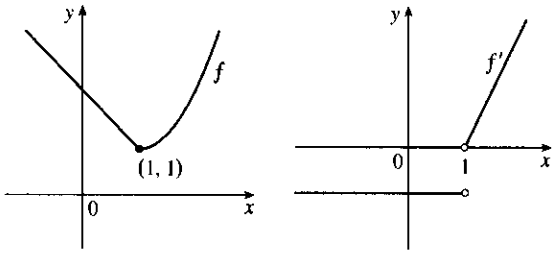
79. $f(x) = 2 - x$ if $x \le 1$ and $f(x) = x^2 - 2x + 2$ if $x > 1$. Now we compute the right- and left-hand derivatives defined in Exercise 3.2.44:

$$f'_-(1) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{2-(1+h)-1}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1 \text{ and}$$

$$f'_+(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h)^2-2(1+h)+2-1}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0.$$

Thus, $f'(1)$ does not exist since $f'_-(1) \ne f'_+(1)$, so f is not differentiable at 1. But $f'(x) = -1$ for $x < 1$ and $f'(x) = 2x - 2$ if $x > 1$.



80. $g(x) = \begin{cases} -1-2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$

$$\lim_{h\to 0^-} \frac{g(-1+h)-g(-1)}{h} = \lim_{h\to 0^-} \frac{[-1-2(-1+h)]-1}{h} = \lim_{h\to 0^-} \frac{-2h}{h} = \lim_{h\to 0^-} (-2) = -2 \text{ and}$$

$$\lim_{h\to 0^+} \frac{g(-1+h)-g(-1)}{h} = \lim_{h\to 0^+} \frac{(-1+h)^2-1}{h} = \lim_{h\to 0^+} \frac{-2h+h^2}{h} = \lim_{h\to 0^+} (-2+h) = -2,$$

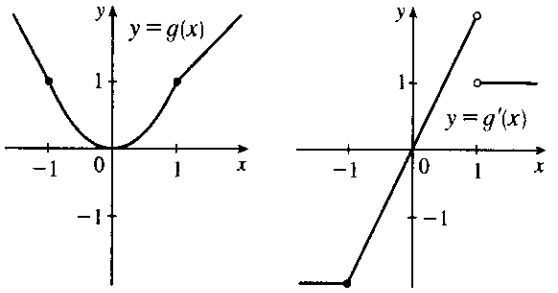
so g is differentiable at -1 and $g'(-1) = -2$.

$$\lim_{h\to 0^-} \frac{g(1)-g(1)}{h} = \lim_{h\to 0^-} \frac{(1)^2-1}{h} = \lim_{h\to 0^-} \frac{2h+h^2}{h} = \lim_{h\to 0^-} (2+h) = 2 \text{ and}$$

$$\lim_{h\to 0^+} \frac{g(1)-g(1)}{h} = \lim_{h\to 0^+} \frac{(1)-1}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1, \text{ so } g'(1) \text{ does not exist.}$$

Thus, g is differentiable except when $x = 1$, and

$$g'(x) = \begin{cases} -2 & \text{if } x < -1 \\ 2x & \text{if } -1 \le x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$



81. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \iff |x| < 3 \iff -3 < x < 3$. So

$$f(x) = \begin{cases} x^2-9 & \text{if } x \le -3 \\ -x^2+9 & \text{if } -3 < x < 3 \\ x^2-9 & \text{if } x \ge 3 \end{cases} \Rightarrow$$

$$f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

To show that $f'(3)$ does not exist we investigate $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 3.2.44.

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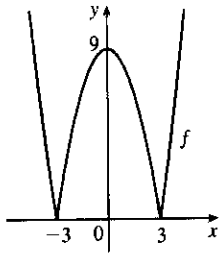
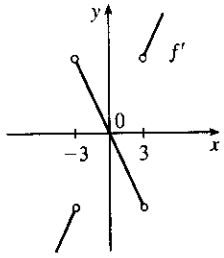
$$f'_-(3) = \lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \frac{[-(3+h)^2+9]-0}{h} = \lim_{h\to 0^-} (-6-h) = -6 \quad \text{and}$$

$$f'_+(3) = \lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+} \frac{[(3+h)^2-9]-0}{h} = \lim_{h\to 0^+} \frac{6h+h^2}{h} = \lim_{h\to 0^+} (6+h) = 6.$$

Since the left and right limits are different, $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

(b)





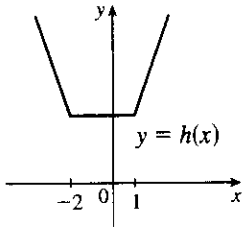
82. If $x \geq 1$, then $h(x) = |x-1| + |x+2| = x - 1 + x + 2 = 2x + 1$.

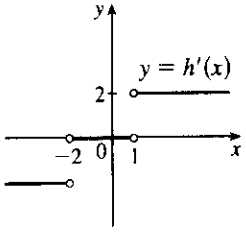
If $-2 < x < 1$, then $h(x) = -(x-1) + x + 2 = 3$.

If $x \leq -2$, then $h(x) = -(x-1) - (x+2) = -2x - 1$. Therefore,

$$h(x) = \begin{cases} -2x-1 & \text{if } x \leq -2 \\ 3 & \text{if } -2 < x < 1 \\ 2x+1 & \text{if } x \geq 1 \end{cases} \quad \Rightarrow \quad h'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$$

To see that $h'(1) = \lim_{x\to 1} \dfrac{h(x)-h(1)}{x-1}$ does not exist, observe that $\lim_{x\to 1^-} \dfrac{h(x)-h(1)}{x-1} = \lim_{x\to 1^-} \dfrac{3-3}{3-1} = 0$ but $\lim_{x\to 1^+} \dfrac{h(x)-h(1)}{x-1} = \lim_{x\to 1^+} \dfrac{2x-2}{x-1} = 2$. Similarly, $h'(-2)$ does not exist.





83. $y = f(x) = ax^2 \quad \Rightarrow \quad f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \quad \Leftrightarrow \quad y = -2x + b$, is seen to be -2, so we must have $4a = -2 \quad \Leftrightarrow \quad a = -\frac{1}{2}$. So when $x = 2$, the point in question has y-coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \quad \Leftrightarrow \quad b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

84. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.

85. $F = f/g \quad \Rightarrow \quad f = Fg \quad \Rightarrow \quad f' = F'g + Fg' \quad \Rightarrow \quad F' = \dfrac{f' - Fg'}{g} = \dfrac{f' - (f/g)g'}{g} = \dfrac{f'g - fg'}{g^2}$

86. (a) $xy = c \quad \Rightarrow \quad y = \dfrac{c}{x}$. Let $P = \left(a, \dfrac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\dfrac{c}{a^2}$. Its equation is $y - \dfrac{c}{a} = -\dfrac{c}{a^2}(x - a)$ or $y = -\dfrac{c}{a^2}x + \dfrac{2c}{a}$, so its y-intercept is $\dfrac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \dfrac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \dfrac{c}{a}\right) = P$.

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(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

87. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative,

$f'(1) = \lim_{x\to 1} \dfrac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \dfrac{x^{1000}-1}{x-1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim_{x\to 1} \dfrac{x^{1000}-1}{x-1} = 1000$.

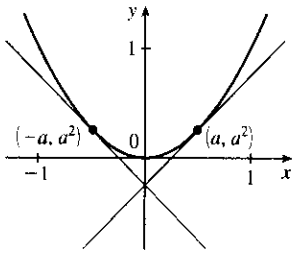
Solution 2: Note that $\left(x^{1000}-1\right) = (x-1)\left(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1\right)$. So

$$\lim_{x\to 1} \frac{x^{1000}-1}{x-1} = \lim_{x\to 1} \frac{(x-1)\left(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1\right)}{x-1}$$

$$= \lim_{x\to 1} \left(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1\right) = \underbrace{1+1+1+\cdots+1+1+1}_{1000 \text{ ones}}$$

$= 1000$, as above.

88.



In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y = x^2$ is symmetric about the y-axis. Say the points of tangency are $\left(a, a^2\right)$ and $\left(-a, a^2\right)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $dy/dx = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x+a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation $y - a^2 = 2a(x-a)$, or $y = 2ax - a^2$. So the two lines intersect at $\left(0, -a^2\right)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a) = -1 \quad \Leftrightarrow \quad a^2 = \frac{1}{4} \quad \Leftrightarrow \quad a = \frac{1}{2}$. So the lines intersect at $\left(0, -\frac{1}{4}\right)$.

3.4 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^2 - 10t + 12 \quad \Rightarrow \quad v(t) = f'(t) = 2t - 10$

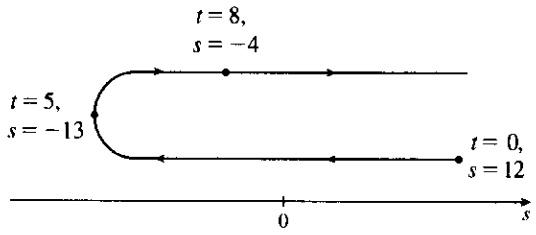
(b) $v(3) = 2(3) - 10 = -4$ ft/s

(c) The particle is at rest when $v(t) = 0 \quad \Leftrightarrow \quad 2t - 10 = 0 \quad \Leftrightarrow \quad t = 5$ s.

(d) The particle is moving in the positive direction when $v(t) > 0 \quad \Leftrightarrow \quad 2t - 10 > 0 \quad \Leftrightarrow \quad 2t > 10 \quad \Leftrightarrow \quad t > 5$.

(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 5]$ and $[5, 8]$ separately. $|f(5) - f(0)| = |-13 - 12| = 25$ ft and $|f(8) - f(5)| = |-4 - (-13)| = 9$ ft. The total distance traveled during the first 8 s is $25 + 9 = 34$ ft.

(f)



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2. (a) $s = f(t) = t^3 - 9t^2 + 15t + 10 \quad \Rightarrow \quad v(t) = f'(t) = 3t^2 - 18t + 15 = 3(t-1)(t-5)$

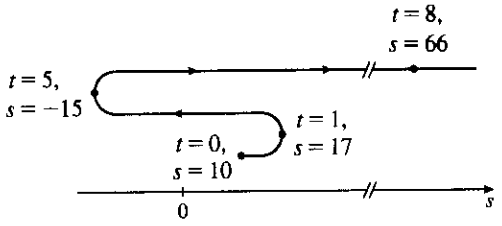
(b) $v(3) = 3(2)(-2) = -12$ ft/s

(c) $v(t) = 0 \quad \Leftrightarrow \quad t = 1$ s or 5 s

(d) $v(t) > 0 \quad \Leftrightarrow \quad 0 \le t < 1$ or $t > 5$

(e) $|f(1) - f(0)| = |17 - 10| = 7$,
$|f(5) - f(1)| = |-15 - 17| = 32$, and
$|f(8) - f(5)| = |66 - (-15)| = 81$.
Total distance $= 7 + 32 + 81 = 120$ ft.

(f)



3. (a) $s = f(t) = t^3 - 12t^2 + 36t \quad \Rightarrow \quad v(t) = f'(t) = 3t^2 - 24t + 36$

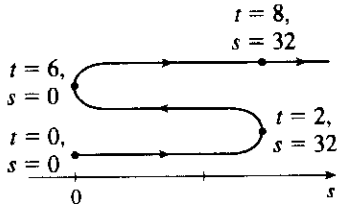
(b) $v(3) = 27 - 72 + 36 = -9$ ft/s

(c) The particle is at rest when $v(t) = 0$. $3t^2 - 24t + 36 = 0 \quad \Rightarrow \quad 3(t-2)(t-6) = 0 \quad \Rightarrow \quad t = 2$ s or 6 s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $3(t-2)(t-6) > 0 \quad \Leftrightarrow \quad 0 \le t < 2$ or $t > 6$.

(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 2]$, $[2, 6]$, and $[6, 8]$ separately.
$|f(2) - f(0)| = |32 - 0| = 32$.
$|f(6) - f(2)| = |0 - 32| = 32$.
$|f(8) - f(6)| = |32 - 0| = 32$.
The total distance is $32 + 32 + 32 = 96$ ft.

(f)



4. (a) $s = f(t) = t^4 - 4t + 1 \quad \Rightarrow \quad v(t) = f'(t) = 4t^3 - 4$

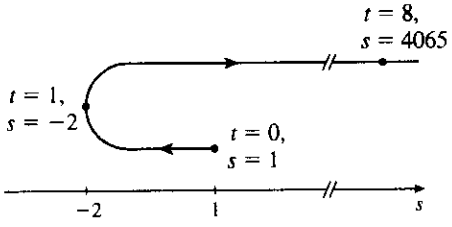
(b) $v(3) = 4(3)^3 - 4 = 104$ ft/s

(c) It is at rest when $v(t) = 4(t^3 - 1) = 4(t-1)(t^2 + t + 1) = 0 \quad \Leftrightarrow \quad t = 1$ s.

(d) It moves in the positive direction when $4(t^3 - 1) > 0 \quad \Leftrightarrow \quad t > 1$.

(e) Distance in positive direction $= |f(8) - f(1)| = |4065 - (-2)| = 4067$ ft
Distance in negative direction $= |f(1) - f(0)| = |-2 - 1| = 3$ ft
Total distance traveled $= 4067 + 3 = 4070$ ft

(f)



5. (a) $s = \dfrac{t}{t^2 + 1} \quad \Rightarrow \quad v(t) = s'(t) = \dfrac{(t^2 + 1)(1) - t(2t)}{(t^2 + 1)^2} = \dfrac{1 - t^2}{(t^2 + 1)^2}$

(b) $v(3) = \dfrac{1 - (3)^2}{(3^2 + 1)^2} = \dfrac{1 - 9}{10^2} = \dfrac{-8}{100} = -\dfrac{2}{25}$ ft/s

(c) It is at rest when $v = 0 \quad \Leftrightarrow \quad 1 - t^2 = 0 \quad \Leftrightarrow \quad t = 1$ s $\quad [t \ne -1$ since $t \ge 0]$.

(d) It moves in the positive direction when $v > 0 \quad \Leftrightarrow \quad 1 - t^2 > 0 \quad \Leftrightarrow \quad t^2 < 1 \quad \Leftrightarrow \quad 0 \le t < 1$.

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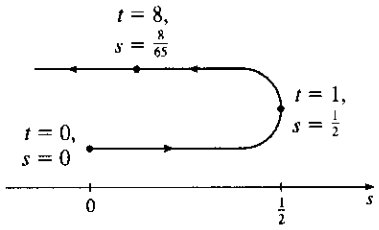
(e) Distance in positive direction $= |s(1) - s(0)| = \left|\frac{1}{2} - 0\right| = \frac{1}{2}$ ft

Distance in negative direction $= |s(8) - s(1)| = \left|\frac{8}{65} - \frac{1}{2}\right| = \frac{49}{130}$ ft

Total distance traveled $= \frac{1}{2} + \frac{49}{130} = \frac{57}{65}$ ft

(f)



6. (a) $s = \sqrt{t}\,(3t^2 - 35t + 90) = 3t^{5/2} - 35t^{3/2} + 90t^{1/2} \;\Rightarrow$

$$v(t) = s'(t) = \tfrac{15}{2}t^{3/2} - \tfrac{105}{2}t^{1/2} + 45t^{-1/2} = \tfrac{15}{2}t^{-1/2}(t^2 - 7t + 6) = \frac{15}{2\sqrt{t}}(t-1)(t-6)$$

(b) $v(3) = \frac{15}{2\sqrt{3}}(2)(-3) = -15\sqrt{3}$ ft/s

(c) It is at rest when $v = 0 \;\Leftrightarrow\; t = 1$ s or 6 s.

(d) It moves in the positive direction when $v > 0 \;\Leftrightarrow\; (t-1)(t-6) > 0 \;\Leftrightarrow\; 0 \le t < 1$ or $t > 6$.

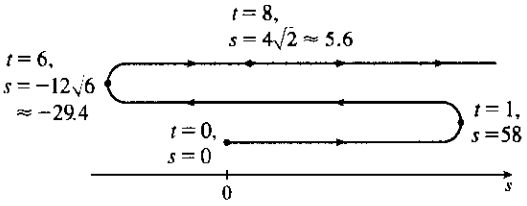
(e) Distance in positive direction $= |s(1) - s(0)| + |s(8) - s(6)| = |58 - 0| + \left|4\sqrt{2} - \left(-12\sqrt{6}\right)\right|$

$$= 58 + 4\sqrt{2} + 12\sqrt{6} \approx 93.05 \text{ ft}$$

Distance in negative direction $= |s(6) - s(1)| = \left|-12\sqrt{6} - 58\right| = 58 + 12\sqrt{6} \approx 87.39$ ft

Total distance traveled $= 58 + 4\sqrt{2} + 12\sqrt{6} + 58 + 12\sqrt{6} = 116 + 4\sqrt{2} + 24\sqrt{6} \approx 180.44$ ft

(f)



7. $s(t) = t^3 - 4.5t^2 - 7t \;\Rightarrow\; v(t) = s'(t) = 3t^2 - 9t - 7 = 5 \;\Leftrightarrow\; 3t^2 - 9t - 12 = 0 \;\Leftrightarrow$ $3(t-4)(t+1) = 0 \;\Leftrightarrow\; t = 4$ or -1. Since $t \ge 0$, the particle reaches a velocity of 5 m/s at $t = 4$ s.

8. (a) $s = 5t + 3t^2 \;\Rightarrow\; v(t) = \dfrac{ds}{dt} = 5 + 6t$, so $v(2) = 5 + 6(2) = 17$ m/s.

(b) $v(t) = 35 \;\Rightarrow\; 5 + 6t = 35 \;\Rightarrow\; 6t = 30 \;\Rightarrow\; t = 5$ s.

9. (a) $h = 10t - 0.83t^2 \;\Rightarrow\; v(t) = \dfrac{dh}{dt} = 10 - 1.66t$, so $v(3) = 10 - 1.66(3) = 5.02$ m/s.

(b) $h = 25 \;\Rightarrow\; 10t - 0.83t^2 = 25 \;\Rightarrow\; 0.83t^2 - 10t + 25 = 0 \;\Rightarrow\; t = \dfrac{10 \pm \sqrt{17}}{1.66} \approx 3.54$ or 8.51.

The value $t_1 = \left(10 - \sqrt{17}\right)/1.66$ corresponds to the time it takes for the stone to rise 25 m and $t_2 = \left(10 + \sqrt{17}\right)/1.66$ corresponds to the time when the stone is 25 m high on the way down. Thus, $v(t_1) = 10 - 1.66\left[\left(10 - \sqrt{17}\right)/1.66\right] = \sqrt{17} \approx 4.12$ m/s.

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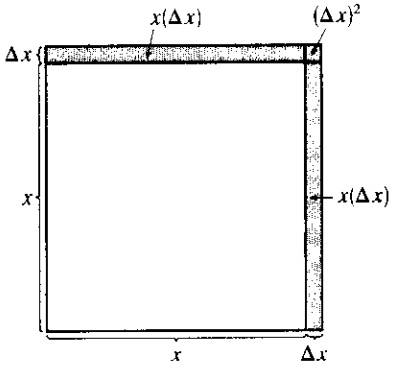
10. (a) At maximum height the velocity of the ball is 0 ft/s. $v(t) = s'(t) = 80 - 32t = 0 \quad \Leftrightarrow \quad 32t = 80 \quad \Leftrightarrow \quad t = \frac{5}{2}$. So the maximum height is $s\left(\frac{5}{2}\right) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = 200 - 100 = 100$ ft.

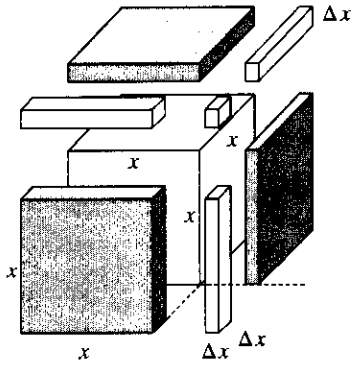
(b) $s(t) = 80t - 16t^2 = 96 \quad \Leftrightarrow \quad 16t^2 - 80t + 96 = 0 \quad \Leftrightarrow \quad 16\left(t^2 - 5t + 6\right) = 0 \quad \Leftrightarrow \quad 16(t-3)(t-2) = 0.$ So the ball has a height of 96 ft on the way up at $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 80 - 32(2) = 16$ ft/s and $v(3) = 80 - 32(3) = -16$ ft/s, respectively.

11. (a) $A(x) = x^2 \quad \Rightarrow \quad A'(x) = 2x$. $A'(15) = 30$ mm^2/mm is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.



(b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx. From the figure, $\Delta A = 2x(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.

12. (a) $V(x) = x^3 \quad \Rightarrow \quad \dfrac{dV}{dx} = 3x^2$. $\left.\dfrac{dV}{dx}\right|_{x=3} = 3(3)^2 = 27$ mm^3/mm is the rate at which the volume is increasing as x increases past 3 mm.



(b) The surface area is $S(x) = 6x^2$, so $V'(x) = 3x^2 = \frac{1}{2}\left(6x^2\right) = \frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx. From the figure, $\Delta V = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$. If Δx is small, then $\Delta V \approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x \approx 3x^2$.

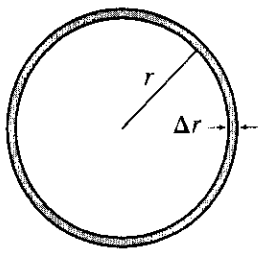
13. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

(i) $\dfrac{A(3) - A(2)}{3 - 2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$ (ii) $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

(iii) $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) $A(r) = \pi r^2 \quad \Rightarrow \quad A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr. Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr, so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically,



$\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$.

So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.

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14. After t seconds the radius is $r = 60t$, so the area is $A(t) = \pi(60t)^2 = 3600\pi t^2 \Rightarrow A'(t) = 7200\pi t \Rightarrow$

(a) $A'(1) = 7200\pi \text{ cm}^2/\text{s}$ (b) $A'(3) = 21{,}600\pi \text{ cm}^2/\text{s}$ (c) $A'(5) = 36{,}000\pi \text{ cm}^2/\text{s}$

As time goes by, the area grows at an increasing rate. In fact, the rate of change is linear with respect to time.

15. $S(r) = 4\pi r^2 \Rightarrow S'(r) = 8\pi r \Rightarrow$

(a) $S'(1) = 8\pi \text{ ft}^2/\text{ft}$ (b) $S'(2) = 16\pi \text{ ft}^2/\text{ft}$ (c) $S'(3) = 24\pi \text{ ft}^2/\text{ft}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

16. (a) Using $V(r) = \frac{4}{3}\pi r^3$, we find that the average rate of change is:

(i) $\dfrac{V(8) - V(5)}{8 - 5} = \dfrac{\frac{4}{3}\pi(512) - \frac{4}{3}\pi(125)}{3} = 172\pi \ \mu\text{m}^3/\mu\text{m}$

(ii) $\dfrac{V(6) - V(5)}{6 - 5} = \dfrac{\frac{4}{3}\pi(216) - \frac{4}{3}\pi(125)}{1} = 121.\overline{3}\pi \ \mu\text{m}^3/\mu\text{m}$

(iii) $\dfrac{V(5.1) - V(5)}{5.1 - 5} = \dfrac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = 102.01\overline{3}\pi \ \mu\text{m}^3/\mu\text{m}$

(b) $V'(r) = 4\pi r^2$, so $V'(5) = 100\pi \ \mu\text{m}^3/\mu\text{m}$.

(c) $V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = 4\pi r^2 = S(r)$. By analogy with Exercise 13(c), we can say that the change in the volume of the spherical shell, ΔV, is approximately equal to its thickness, Δr, times the surface area of the inner sphere. Thus, $\Delta V \approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r \approx 4\pi r^2$.

17. The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6 \text{ kg/m}$ (b) $\rho(2) = 12 \text{ kg/m}$ (c) $\rho(3) = 18 \text{ kg/m}$

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

18. $V(t) = 5000\left(1 - \frac{1}{40}t\right)^2 = 5000\left(1 - \frac{1}{20}t + \frac{1}{1600}t^2\right) \Rightarrow V'(t) = 5000\left(-\frac{1}{20} + \frac{1}{800}t\right) = -250\left(1 - \frac{1}{40}t\right)$

(a) $V'(5) = -250\left(1 - \frac{5}{40}\right) = -218.75 \text{ gal/min}$ (b) $V'(10) = -250\left(1 - \frac{10}{40}\right) = -187.5 \text{ gal/min}$

(c) $V'(20) = -250\left(1 - \frac{20}{40}\right) = -125 \text{ gal/min}$ (d) $V'(40) = -250\left(1 - \frac{40}{40}\right) = 0 \text{ gal/min}$

The water is flowing out the fastest at the beginning — when $t = 0$, $V'(t) = -250$ gal/min. The water is flowing out the slowest at the end — when $t = 40$, $V'(t) = 0$. As the tank empties, the water flows out more slowly.

19. The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75 \text{ A}$ (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5 \text{ A}$

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3}$ s.

20. (a) $F = \dfrac{GmM}{r^2} = (GmM)r^{-2} \Rightarrow \dfrac{dF}{dr} = -2(GmM)r^{-3} = -\dfrac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Given $F'(20{,}000) = -2$, find $F'(10{,}000)$. $-2 = -\dfrac{2GmM}{20{,}000^3} \Rightarrow GmM = 20{,}000^3$.

$$F'(10{,}000) = -\frac{2(20{,}000^3)}{10{,}000^3} = -2 \cdot 2^3 = -16 \text{ N/km}$$

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21. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P.

$$PV = C \quad \Rightarrow \quad V = \frac{C}{P} \quad \Rightarrow \quad \frac{dV}{dP} = -\frac{C}{P^2}.$$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases. Thus, the volume is decreasing more rapidly at the beginning.

$$\text{(c) } \beta = -\frac{1}{V}\frac{dV}{dP} = -\frac{1}{V}\left(-\frac{C}{P^2}\right) = \frac{C}{(PV)P} = \frac{C}{CP} = \frac{1}{P}$$

22. (a) (i) $\dfrac{C(6)-C(2)}{6-2} = \dfrac{0.0295-0.0570}{4}$

$= -0.006875$ (moles/L)/min

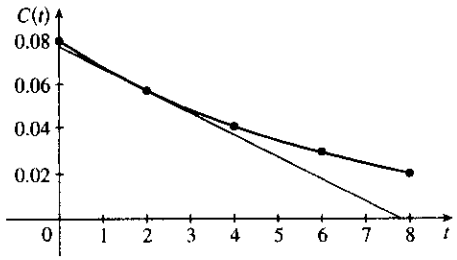
(ii) $\dfrac{C(4)-C(2)}{4-2} = \dfrac{0.0408-0.0570}{2}$

$= -0.008$ (moles/L)/min

(iii) $\dfrac{C(2)-C(0)}{2-0} = \dfrac{0.0570-0.0800}{2}$

$= -0.0115$ (moles/L)/min

(b) Slope $= \dfrac{\Delta C}{\Delta t} \approx -\dfrac{0.077}{7.8} \approx -0.01$ (moles/L)/min



23. (a) **1920:** $m_1 = \dfrac{1860-1750}{1920-1910} = \frac{110}{10} = 11$, $m_2 = \dfrac{2070-1860}{1930-1920} = \frac{210}{10} = 21$,

$(m_1+m_2)/2 = (11+21)/2 = 16$ million/year

1980: $m_1 = \dfrac{4450-3710}{1980-1970} = \frac{740}{10} = 74$, $m_2 = \dfrac{5280-4450}{1990-1980} = \frac{830}{10} = 83$,

$(m_1+m_2)/2 = (74+83)/2 = 78.5$ million/year

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx 0.0012937063$, $b \approx -7.061421911$, $c \approx 12{,}822.97902$, and $d \approx -7{,}743{,}770.396$.

(c) $P(t) = at^3 + bt^2 + ct + d \quad \Rightarrow \quad P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d) $P'(1920) = 3(0.0012937063)(1920)^2 + 2(-7.061421911)(1920) + 12{,}822.97902$

≈ 14.48 million/year [smaller than the answer in part (a), but close to it]

$P'(1980) \approx 75.29$ million/year (smaller, but close)

(e) $P'(1985) \approx 81.62$ million/year, so the rate of growth in 1985 was about 81.62 million/year.

24. (a) $A(t) = at^4 + bt^3 + ct^2 + dt + e$, where $a = -5.8275058275396 \times 10^{-6}$, $b = 0.0460458430461$, $c = -136.43277039706$, $d = 179{,}661.02676871$, and $e = -88{,}717{,}597.060767$.

(b) $A(t) = at^4 + bt^3 + ct^2 + dt + e \quad \Rightarrow \quad A'(t) = 4at^3 + 3bt^2 + 2ct + d$

(c) $A'(1990) \approx 0.0833$ years of age per year

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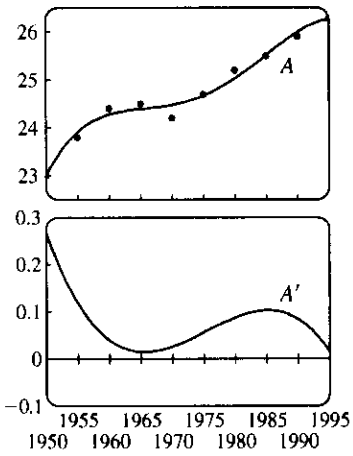
(d)



25. (a) $[\mathrm{C}] = \dfrac{a^2kt}{akt+1} \quad \Rightarrow$

$$\text{rate of reaction} = \frac{d[\mathrm{C}]}{dt} = \frac{(akt+1)(a^2k) - (a^2kt)(ak)}{(akt+1)^2} = \frac{a^2k(akt+1-akt)}{(akt+1)^2} = \frac{a^2k}{(akt+1)^2}$$

(b) If $x = [\mathrm{C}]$, then $a - x = a - \dfrac{a^2kt}{akt+1} = \dfrac{a^2kt + a - a^2kt}{akt+1} = \dfrac{a}{akt+1}$.

So $k(a-x)^2 = k\left(\dfrac{a}{akt+1}\right)^2 = \dfrac{a^2k}{(akt+1)^2} = \dfrac{d[\mathrm{C}]}{dt}$ [from part (a)] $= \dfrac{dx}{dt}$.

26. $\dfrac{1}{p} = \dfrac{1}{f} - \dfrac{1}{q} \quad \Leftrightarrow \quad \dfrac{1}{p} = \dfrac{q-f}{fq} \quad \Leftrightarrow \quad p = \dfrac{fq}{q-f}$. So $\dfrac{dp}{dq} = \dfrac{f(q-f) - fq}{(q-f)^2} = -\dfrac{f^2}{(q-f)^2}$.

27. (a) Using $v = \dfrac{P}{4\eta l}\left(R^2 - r^2\right)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r:

$$v(r) = \frac{3000}{4(0.027)3}\left(0.01^2 - r^2\right).\ v(0) = 0.\overline{925}\ \text{cm/s},\ v(0.005) = 0.69\overline{4}\ \text{cm/s},\ v(0.01) = 0.$$

(b) $v(r) = \dfrac{P}{4\eta l}\left(R^2 - r^2\right) \quad \Rightarrow \quad v'(r) = \dfrac{P}{4\eta l}(-2r) = -\dfrac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$$v'(r) = -\frac{3000r}{2(0.027)3}.\ v'(0) = 0,\ v'(0.005) = -92.\overline{592}\ (\text{cm/s})/\text{cm}, \text{ and } v'(0.01) = -185.\overline{185}\ (\text{cm/s})/\text{cm}.$$

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

28. (a) (i) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-1} \quad \Rightarrow \quad \dfrac{df}{dL} = -\left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-2} = -\dfrac{1}{2L^2}\sqrt{\dfrac{T}{\rho}}$

(ii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{1/2} \quad \Rightarrow \quad \dfrac{df}{dT} = \dfrac{1}{2}\left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{-1/2} = \dfrac{1}{4L\sqrt{T\rho}}$

(iii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-1/2} \quad \Rightarrow \quad \dfrac{df}{d\rho} = -\dfrac{1}{2}\left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-3/2} = -\dfrac{\sqrt{T}}{4L\rho^{3/2}}$

(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\dfrac{df}{dL} < 0$ and L is decreasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(ii) $\dfrac{df}{dT} > 0$ and T is increasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

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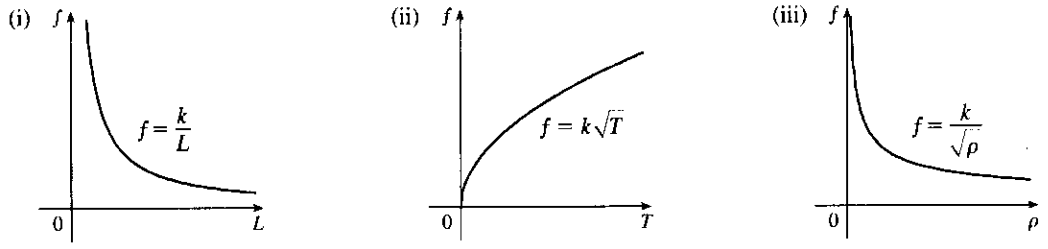
(iii) $\dfrac{df}{d\rho} < 0$ and ρ is increasing $\Rightarrow$ f is decreasing $\Rightarrow$ lower note



29. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \Rightarrow C'(x) = 3 + 0.02x + 0.0006x^2$

(b) $C'(100) = 3 + 0.02(100) + 0.0006(10{,}000) = 3 + 2 + 6 = \$11$/pair. $C'(100)$ is the rate at which the cost is increasing as the 100th pair of jeans is produced. It predicts the cost of the 101st pair.

(c) The cost of manufacturing the 101st pair of jeans is

$$\begin{aligned} C(101) - C(100) &= (2000 + 303 + 102.01 + 206.0602) - (2000 + 300 + 100 + 200) \\ &= 11.0702 \approx \$11.07 \end{aligned}$$

30. (a) $C(x) = 84 + 0.16x - 0.0006x^2 + 0.000003x^3 \Rightarrow C'(x) = 0.16 - 0.0012x + 0.000009x^2 \Rightarrow$ $C'(100) = 0.13$. This is the rate at which the cost is increasing as the 100th item is produced.

(b) $C(101) - C(100) = 97.13030299 - 97 \approx \$0.13$.

31. (a) $A(x) = \dfrac{p(x)}{x} \Rightarrow A'(x) = \dfrac{xp'(x) - p(x) \cdot 1}{x^2} = \dfrac{xp'(x) - p(x)}{x^2}$. $A'(x) > 0 \Rightarrow A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

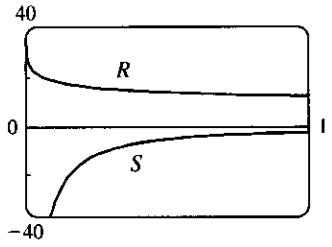
(b) $p'(x)$ is greater than the average productivity $\Rightarrow p'(x) > A(x) \Rightarrow p'(x) > \dfrac{p(x)}{x} \Rightarrow$

$xp'(x) > p(x) \Rightarrow xp'(x) - p(x) > 0 \Rightarrow \dfrac{xp'(x) - p(x)}{x^2} > 0 \Rightarrow A'(x) > 0.$

32. (a) $S = \dfrac{dR}{dx} = \dfrac{(1 + 4x^{0.4})(9.6x^{-0.6}) - (40 + 24x^{0.4})(1.6x^{-0.6})}{(1 + 4x^{0.4})^2}$

$$= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} = -\frac{54.4x^{-0.6}}{(1 + 4x^{0.4})^2}$$

(b)



At low levels of brightness, R is quite large [$R(0) = 40$] and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

33. $PV = nRT \Rightarrow T = \dfrac{PV}{nR} = \dfrac{PV}{(10)(0.0821)} = \dfrac{1}{0.821}(PV)$. Using the Product Rule, we have

$$\frac{dT}{dt} = \frac{1}{0.821}\left[P(t)V'(t) + V(t)P'(t)\right] = \frac{1}{0.821}\left[(8)(-0.15) + (10)(0.10)\right] \approx -0.2436 \text{ K/min}.$$

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34. (a) If $dP/dt = 0$, the population is stable (it is constant).

(b) $\dfrac{dP}{dt} = 0 \;\Rightarrow\; \beta P = r_0\left(1 - \dfrac{P}{P_c}\right)P \;\Rightarrow\; \dfrac{\beta}{r_0} = 1 - \dfrac{P}{P_c} \;\Rightarrow\; \dfrac{P}{P_c} = 1 - \dfrac{\beta}{r_0} \;\Rightarrow\; P = P_c\left(1 - \dfrac{\beta}{r_0}\right)$.

If $P_c = 10{,}000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10{,}000\left(1 - \frac{4}{5}\right) = 2000$.

(c) If $\beta = 0.05$, then $P = 10{,}000\left(1 - \frac{5}{5}\right) = 0$. There is no stable population.

35. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is,

$$\frac{dC}{dt} = 0 \text{ and } \frac{dW}{dt} = 0.$$

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\dfrac{dC}{dt} = aC - bCW$ and $\dfrac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ **(1)** and $-0.05W + 0.0001CW = 0$ **(2)**. Adding 10 times **(2)** to **(1)** eliminates the CW-terms and gives us $0.05C - 0.5W = 0 \;\Rightarrow\; C = 10W$. Substituting $C = 10W$ into **(1)** results in $0.05(10W) - 0.001(10W)W = 0 \;\Leftrightarrow\; 0.5W - 0.01W^2 = 0 \;\Leftrightarrow\; 50W - W^2 = 0 \;\Leftrightarrow\; W(50 - W) = 0 \;\Leftrightarrow\; W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs (C, W) that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

3.5 Derivatives of Trigonometric Functions

1. $f(x) = x - 3\sin x \;\Rightarrow\; f'(x) = 1 - 3\cos x$

2. $f(x) = x\sin x \;\Rightarrow\; f'(x) = x\cdot\cos x + (\sin x)\cdot 1 = x\cos x + \sin x$

3. $y = \sin x + 10\tan x \;\Rightarrow\; y' = \cos x + 10\sec^2 x$

4. $y = 2\csc x + 5\cos x \;\Rightarrow\; y' = -2\csc x\cot x - 5\sin x$

5. $g(t) = t^3\cos t \;\Rightarrow\; g'(t) = t^3(-\sin t) + (\cos t)\cdot 3t^2 = 3t^2\cos t - t^3\sin t$ or $t^2(3\cos t - t\sin t)$

6. $g(t) = 4\sec t + \tan t \;\Rightarrow\; g'(t) = 4\sec t\tan t + \sec^2 t$

7. $h(\theta) = \theta\csc\theta - \cot\theta \;\Rightarrow\; h'(\theta) = \theta(-\csc\theta\cot\theta) + (\csc\theta)\cdot 1 - \left(-\csc^2\theta\right) = \csc\theta - \theta\csc\theta\cot\theta + \csc^2\theta$

8. $y = u(a\cos u + b\cot u) \;\Rightarrow$

$y' = u(-a\sin u - b\csc^2 u) + (a\cos u + b\cot u)\cdot 1 = a\cos u + b\cot u - au\sin u - bu\csc^2 u$

9. $y = \dfrac{x}{\cos x} \;\Rightarrow\; y' = \dfrac{(\cos x)(1) - (x)(-\sin x)}{(\cos x)^2} = \dfrac{\cos x + x\sin x}{\cos^2 x}$

10. $y = \dfrac{1 + \sin x}{x + \cos x} \;\Rightarrow$

$$y' = \frac{(x + \cos x)(\cos x) - (1 + \sin x)(1 - \sin x)}{(x + \cos x)^2} = \frac{x\cos x + \cos^2 x - (1 - \sin^2 x)}{(x + \cos x)^2}$$

$$= \frac{x\cos x + \cos^2 x - (\cos^2 x)}{(x + \cos x)^2} = \frac{x\cos x}{(x + \cos x)^2}$$

11. $f(\theta) = \dfrac{\sec\theta}{1 + \sec\theta} \;\Rightarrow$

$$f'(\theta) = \frac{(1 + \sec\theta)(\sec\theta\tan\theta) - (\sec\theta)(\sec\theta\tan\theta)}{(1 + \sec\theta)^2} = \frac{(\sec\theta\tan\theta)[(1 + \sec\theta) - \sec\theta]}{(1 + \sec\theta)^2} = \frac{\sec\theta\tan\theta}{(1 + \sec\theta)^2}$$

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12. $y = \dfrac{\tan x - 1}{\sec x} \Rightarrow$

$$\frac{dy}{dx} = \frac{\sec x \sec^2 x - (\tan x - 1)\sec x \tan x}{\sec^2 x} = \frac{\sec x\left(\sec^2 x - \tan^2 x + \tan x\right)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

Another method: Simplify y first: $y = \sin x - \cos x \Rightarrow y' = \cos x + \sin x$.

13. $y = \dfrac{\sin x}{x^2} \Rightarrow y' = \dfrac{x^2 \cos x - (\sin x)(2x)}{(x^2)^2} = \dfrac{x(x\cos x - 2\sin x)}{x^4} = \dfrac{x\cos x - 2\sin x}{x^3}$

14. $y = \csc\theta\,(\theta + \cot\theta) \Rightarrow$

$$y' = \csc\theta\left(1 - \csc^2\theta\right) + (\theta + \cot\theta)(-\csc\theta\cot\theta) = \csc\theta\left(1 - \csc^2\theta - \theta\cot\theta - \cot^2\theta\right)$$
$$= \csc\theta\left(-\cot^2\theta - \theta\cot\theta - \cot^2\theta\right) \qquad \left[1 + \cot^2\theta = \csc^2\theta\right]$$
$$= \csc\theta\left(-\theta\cot\theta - 2\cot^2\theta\right) = -\csc\theta\cot\theta\,(\theta + 2\cot\theta)$$

15. $y = \sec\theta\,\tan\theta \Rightarrow y' = \sec\theta\left(\sec^2\theta\right) + \tan\theta\,(\sec\theta\tan\theta) = \sec\theta\left(\sec^2\theta + \tan^2\theta\right)$

Using the identity $1 + \tan^2\theta = \sec^2\theta$, we can write alternative forms of the answer as

$$\sec\theta\left(1 + 2\tan^2\theta\right) \qquad \text{or} \qquad \sec\theta\left(2\sec^2\theta - 1\right)$$

16. Recall that if $y = fgh$, then $y' = f'gh + fg'h + fgh'$. $y = x\sin x\cos x \Rightarrow$

$$\frac{dy}{dx} = \sin x\cos x + x\cos x\cos x + x\sin x\,(-\sin x) = \sin x\cos x + x\cos^2 x - x\sin^2 x$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x} = -\csc x\cot x$

18. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x} = \sec x\tan x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

20. $f(x) = \cos x \Rightarrow$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$
$$= \lim_{h\to 0}\left(\cos x\,\frac{\cos h - 1}{h} - \sin x\,\frac{\sin h}{h}\right) = \cos x\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h}$$
$$= (\cos x)(0) - (\sin x)(1) = -\sin x$$

21. $y = \tan x \Rightarrow y' = \sec^2 x \Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{4}, 1\right)$ is $\sec^2\frac{\pi}{4} = \left(\sqrt{2}\right)^2 = 2$ and an equation of the tangent line is $y - 1 = 2\left(x - \frac{\pi}{4}\right)$ or $y = 2x + 1 - \frac{\pi}{2}$.

22. $y = (1 + x)\cos x \Rightarrow y' = (1 + x)(-\sin x) + \cos x \cdot 1$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

23. $y = x + \cos x \Rightarrow y' = 1 - \sin x$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

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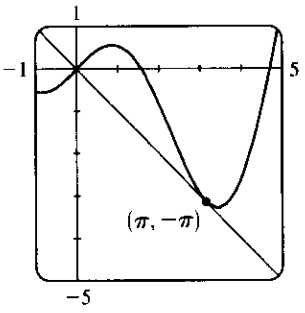
24. $y = \dfrac{1}{\sin x + \cos x} \Rightarrow y' = -\dfrac{\cos x - \sin x}{(\sin x + \cos x)^2}$ [Reciprocal Rule]. At $(0, 1)$, $y' = -\dfrac{1-0}{(0+1)^2} = -1$, and an equation of the tangent line is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

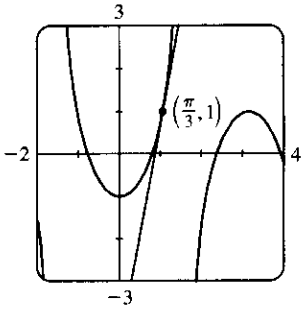
25. (a) $y = x\cos x \Rightarrow y' = x(-\sin x) + \cos x(1) = \cos x - x\sin x$.
So the slope of the tangent at the point $(\pi, -\pi)$ is
$\cos\pi - \pi\sin\pi = -1 - \pi(0) = -1$, and an equation is
$y + \pi = -(x - \pi)$ or $y = -x$.

(b)



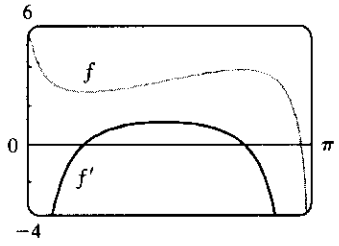
26. (a) $y = \sec x - 2\cos x \Rightarrow y' = \sec x\tan x + 2\sin x \Rightarrow$
the slope of the tangent line at $\left(\frac{\pi}{3}, 1\right)$ is
$\sec\frac{\pi}{3}\tan\frac{\pi}{3} + 2\sin\frac{\pi}{3} = 2\cdot\sqrt{3} + 2\cdot\frac{\sqrt{3}}{2} = 3\sqrt{3}$ and an equation is
$y - 1 = 3\sqrt{3}\left(x - \frac{\pi}{3}\right)$ or $y = 3\sqrt{3}\,x + 1 - \pi\sqrt{3}$.

(b)



27. (a) $f(x) = 2x + \cot x \Rightarrow f'(x) = 2 - \csc^2 x$

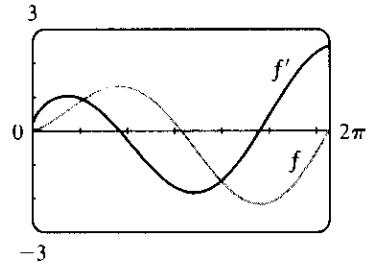
(b)



Notice that $f'(x) = 0$ when f has a horizontal tangent. f' is positive when f is increasing and f' is negative when f is decreasing. Also, $f'(x)$ is large negative when the graph of f is steep.

28. (a) $f(x) = \sqrt{x}\sin x \Rightarrow f'(x) = \sqrt{x}\cos x + (\sin x)\left(\frac{1}{2}x^{-1/2}\right) = \sqrt{x}\cos x + \dfrac{\sin x}{2\sqrt{x}}$

(b)



Notice that $f'(x) = 0$ when f has a horizontal tangent. f' is positive when f is increasing and f' is negative when f is decreasing.

29. $f(x) = x + 2\sin x$ has a horizontal tangent when $f'(x) = 0 \Leftrightarrow 1 + 2\cos x = 0 \Leftrightarrow \cos x = -\frac{1}{2} \Leftrightarrow$ $x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n+1)\pi \pm \frac{\pi}{3}$, n an integer.

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30. $y = \dfrac{\cos x}{2 + \sin x} \quad \Rightarrow$

$y' = \dfrac{(2+\sin x)(-\sin x) - \cos x \cos x}{(2+\sin x)^2} = \dfrac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2} = \dfrac{-2\sin x - 1}{(2+\sin x)^2} = 0$ when

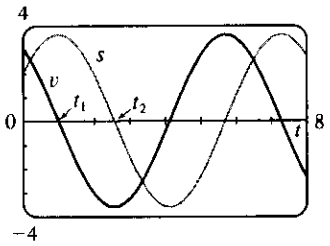
$-2\sin x - 1 = 0 \quad \Leftrightarrow \quad \sin x = -\frac{1}{2} \quad \Leftrightarrow \quad x = \frac{11\pi}{6} + 2\pi n$ or $x = \frac{7\pi}{6} + 2\pi n$, n an integer. So $y = \frac{1}{\sqrt{3}}$ or $y = -\frac{1}{\sqrt{3}}$ and the points on the curve with horizontal tangents are: $\left(\frac{11\pi}{6} + 2\pi n, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{7\pi}{6} + 2\pi n, -\frac{1}{\sqrt{3}}\right)$, n an integer.

31. (a) $x(t) = 8\sin t \quad \Rightarrow \quad v(t) = x'(t) = 8\cos t$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8\sin\frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$ and velocity $v\left(\frac{2\pi}{3}\right) = 8\cos\frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

32. (a) $s(t) = 2\cos t + 3\sin t \quad \Rightarrow \quad v(t) = -2\sin t + 3\cos t$

(b)

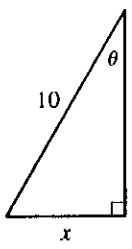


(c) $s = 0 \quad \Rightarrow \quad t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \quad \Rightarrow \quad t_1 \approx 0.98$, $s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$; that is, when $t = t_2 + n\pi$, n a positive integer.

33.



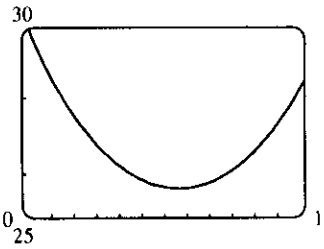
From the diagram we can see that $\sin\theta = x/10 \quad \Leftrightarrow \quad x = 10\sin\theta$. We want to find the rate of change of x with respect to θ; that is, $dx/d\theta$. Taking the derivative of the above expression, $dx/d\theta = 10(\cos\theta)$. So when $\theta = \frac{\pi}{3}$,

$$dx/d\theta = 10\cos\tfrac{\pi}{3} = 10\left(\tfrac{1}{2}\right) = 5 \text{ ft/rad}$$

34. (a) $F = \dfrac{\mu W}{\mu\sin\theta + \cos\theta} \quad \Rightarrow \quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}$

(b) $\dfrac{dF}{d\theta} = 0 \quad \Rightarrow \quad \mu W(\sin\theta - \mu\cos\theta) = 0 \quad \Rightarrow \quad \sin\theta = \mu\cos\theta \quad \Rightarrow \quad \tan\theta = \mu \quad \Rightarrow \quad \theta = \tan^{-1}\mu$

(c)



From the graph of $F = \dfrac{0.6(50)}{0.6\sin\theta + \cos\theta}$ for $0 \le \theta \le 1$, we see that $\dfrac{dF}{d\theta} = 0 \quad \Rightarrow \quad \theta \approx 0.54$. Checking this with part (b) and $\mu = 0.6$, we calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the value from the graph is consistent with the value in part (b).

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35. $\lim_{x\to 0}\dfrac{\sin 3x}{x} = \lim_{x\to 0}\dfrac{3\sin 3x}{3x}$ [multiply numerator and denominator by 3]

$= 3\lim_{3x\to 0}\dfrac{\sin 3x}{3x}$ [as $x\to 0$, $3x\to 0$]

$= 3\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}$ [let $\theta = 3x$]

$= 3(1)$ [Equation 2]

$= 3$

36. $\lim_{x\to 0}\dfrac{\sin 4x}{\sin 6x} = \lim_{x\to 0}\left(\dfrac{\sin 4x}{x}\cdot\dfrac{x}{\sin 6x}\right) = \lim_{x\to 0}\dfrac{4\sin 4x}{4x}\cdot\lim_{x\to 0}\dfrac{6x}{6\sin 6x}$

$= 4\lim_{x\to 0}\dfrac{\sin 4x}{4x}\cdot\dfrac{1}{6}\lim_{x\to 0}\dfrac{6x}{\sin 6x} = 4(1)\cdot\dfrac{1}{6}(1) = \dfrac{2}{3}$

37. $\lim_{t\to 0}\dfrac{\tan 6t}{\sin 2t} = \lim_{t\to 0}\left(\dfrac{\sin 6t}{t}\cdot\dfrac{1}{\cos 6t}\cdot\dfrac{t}{\sin 2t}\right) = \lim_{t\to 0}\dfrac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\dfrac{1}{\cos 6t}\cdot\lim_{t\to 0}\dfrac{2t}{2\sin 2t}$

$= 6\lim_{t\to 0}\dfrac{\sin 6t}{6t}\cdot\lim_{t\to 0}\dfrac{1}{\cos 6t}\cdot\dfrac{1}{2}\lim_{t\to 0}\dfrac{2t}{\sin 2t} = 6(1)\cdot\dfrac{1}{1}\cdot\dfrac{1}{2}(1) = 3$

38. $\lim_{\theta\to 0}\dfrac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0}\dfrac{\dfrac{\cos\theta-1}{\theta}}{\dfrac{\sin\theta}{\theta}} = \dfrac{\lim_{\theta\to 0}\dfrac{\cos\theta-1}{\theta}}{\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}} = \dfrac{0}{1} = 0$

39. $\lim_{\theta\to 0}\dfrac{\sin(\cos\theta)}{\sec\theta} = \dfrac{\sin\left(\lim_{\theta\to 0}\cos\theta\right)}{\lim_{\theta\to 0}\sec\theta} = \dfrac{\sin 1}{1} = \sin 1$

40. $\lim_{t\to 0}\dfrac{\sin^2 3t}{t^2} = \lim_{t\to 0}\left(\dfrac{\sin 3t}{t}\cdot\dfrac{\sin 3t}{t}\right) = \lim_{t\to 0}\dfrac{\sin 3t}{t}\cdot\lim_{t\to 0}\dfrac{\sin 3t}{t}$

$= \left(\lim_{t\to 0}\dfrac{\sin 3t}{t}\right)^2 = \left(3\lim_{t\to 0}\dfrac{\sin 3t}{3t}\right)^2 = (3\cdot 1)^2 = 9$

41. $\lim_{x\to 0}\dfrac{\cot 2x}{\csc x} = \lim_{x\to 0}\dfrac{\cos 2x\sin x}{\sin 2x} = \lim_{x\to 0}\cos 2x\left[\dfrac{(\sin x)/x}{(\sin 2x)/x}\right] = \lim_{x\to 0}\cos 2x\left[\dfrac{\lim_{x\to 0}[(\sin x)/x]}{2\lim_{x\to 0}[(\sin 2x)/2x]}\right]$

$= 1\cdot\dfrac{1}{2\cdot 1} = \dfrac{1}{2}$

42. $\lim_{x\to\pi/4}\dfrac{\sin x - \cos x}{\cos 2x} = \lim_{x\to\pi/4}\dfrac{\sin x - \cos x}{\cos^2 x - \sin^2 x} = \lim_{x\to\pi/4}\dfrac{\sin x - \cos x}{(\cos x + \sin x)(\cos x - \sin x)}$

$= \lim_{x\to\pi/4}\dfrac{-1}{\cos x + \sin x} = \dfrac{-1}{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}} = \dfrac{-1}{\sqrt{2}}$

43. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta} = \lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1+\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}} = \frac{\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}}{1+\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}\lim_{\theta\to 0}\dfrac{1}{\cos\theta}} = \frac{1}{1+1\cdot 1} = \frac{1}{2}$$

44. $\lim_{x\to 1}\dfrac{\sin(x-1)}{x^2+x-2} = \lim_{x\to 1}\dfrac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x\to 1}\dfrac{1}{x+2}\lim_{x\to 1}\dfrac{\sin(x-1)}{x-1} = \frac{1}{3}\cdot 1 = \frac{1}{3}$

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45. (a) $\dfrac{d}{dx}\tan x = \dfrac{d}{dx}\dfrac{\sin x}{\cos x} \;\Rightarrow\; \sec^2 x = \dfrac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \dfrac{1}{\cos^2 x}$.

(b) $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\dfrac{1}{\cos x} \;\Rightarrow\; \sec x \tan x = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x \tan x = \dfrac{\sin x}{\cos^2 x}$.

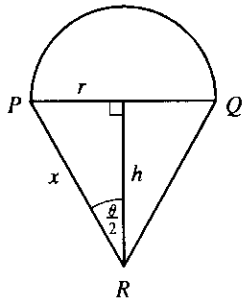
(c) $\dfrac{d}{dx}(\sin x + \cos x) = \dfrac{d}{dx}\dfrac{1+\cot x}{\csc x} \;\Rightarrow$

$$\cos x - \sin x = \frac{\csc x\left(-\csc^2 x\right) - (1+\cot x)(-\csc x \cot x)}{\csc^2 x} = \frac{\csc x\left[-\csc^2 x + (1+\cot x)\cot x\right]}{\csc^2 x}$$

$$= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1 + \cot x}{\csc x}$$

So $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$.

46. Let $|PR| = x$. Then we get the following formulas for r and h in terms of θ and x:



$\sin\dfrac{\theta}{2} = \dfrac{r}{x} \;\Rightarrow\; r = x\sin\dfrac{\theta}{2}$ and $\cos\dfrac{\theta}{2} = \dfrac{h}{x} \;\Rightarrow\; h = x\cos\dfrac{\theta}{2}$.

Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \tfrac{1}{2}\pi \lim_{\theta\to 0^+}\frac{r}{h} = \tfrac{1}{2}\pi \lim_{\theta\to 0^+}\frac{x\sin(\theta/2)}{x\cos(\theta/2)}$$

$$= \tfrac{1}{2}\pi \lim_{\theta\to 0^+}\tan(\theta/2) = 0.$$

47. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle.

By drawing the bisector of the angle θ, we can see that $\sin\dfrac{\theta}{2} = \dfrac{d/2}{r} \;\Rightarrow\; d = 2r\sin\dfrac{\theta}{2}$.

So $\lim\limits_{\theta\to 0^+}\dfrac{s}{d} = \lim\limits_{\theta\to 0^+}\dfrac{r\theta}{2r\sin(\theta/2)} = \lim\limits_{\theta\to 0^+}\dfrac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim\limits_{\theta\to 0}\dfrac{\theta/2}{\sin(\theta/2)} = 1$. [This is just the reciprocal of the limit $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta \to 0$, $\frac{\theta}{2} \to 0$ also.]

3.6 The Chain Rule

1. Let $u = g(x) = 4x$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)(4) = 4\cos 4x$.

2. Let $u = g(x) = 4 + 3x$ and $y = f(u) = \sqrt{u} = u^{1/2}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \frac{1}{2}u^{-1/2}(3) = \dfrac{3}{2\sqrt{u}} = \dfrac{3}{2\sqrt{4+3x}}$.

3. Let $u = g(x) = 1 - x^2$ and $y = f(u) = u^{10}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(10u^9\right)(-2x) = -20x\left(1-x^2\right)^9$.

4. Let $u = g(x) = \sin x$ and $y = f(u) = \tan u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(\sec^2 u\right)(\cos x) = \sec^2(\sin x)\cdot\cos x$, or equivalently, $[\sec(\sin x)]^2\cos x$.

5. Let $u = g(x) = \sin x$ and $y = f(u) = \sqrt{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \frac{1}{2}u^{-1/2}\cos x = \dfrac{\cos x}{2\sqrt{u}} = \dfrac{\cos x}{2\sqrt{\sin x}}$.

6. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)\left(\frac{1}{2}x^{-1/2}\right) = \dfrac{\cos u}{2\sqrt{x}} = \dfrac{\cos\sqrt{x}}{2\sqrt{x}}$.

7. $F(x) = \left(x^3 + 4x\right)^7 \;\Rightarrow\; F'(x) = 7\left(x^3+4x\right)^6\left(3x^2+4\right)$ [or $7x^6\left(x^2+4\right)^6\left(3x^2+4\right)$]

8. $F(x) = \left(x^2 - x + 1\right)^3 \;\Rightarrow\; F'(x) = 3\left(x^2 - x + 1\right)^2(2x-1)$

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9. $F(x) = \sqrt[4]{1+2x+x^3} = (1+2x+x^3)^{1/4} \Rightarrow$

$$F'(x) = \tfrac{1}{4}(1+2x+x^3)^{-3/4} \cdot \frac{d}{dx}(1+2x+x^3) = \frac{1}{4(1+2x+x^3)^{3/4}} \cdot (2+3x^2)$$

$$= \frac{2+3x^2}{4(1+2x+x^3)^{3/4}} = \frac{2+3x^2}{4\sqrt[4]{(1+2x+x^3)^3}}$$

10. $f(x) = (1+x^4)^{2/3} \Rightarrow f'(x) = \frac{2}{3}(1+x^4)^{-1/3}(4x^3) = \dfrac{8x^3}{3\sqrt[3]{1+x^4}}$

11. $g(t) = \dfrac{1}{(t^4+1)^3} = (t^4+1)^{-3} \Rightarrow g'(t) = -3(t^4+1)^{-4}(4t^3) = -12t^3(t^4+1)^{-4} = \dfrac{-12t^3}{(t^4+1)^4}$

12. $f(t) = \sqrt[3]{1+\tan t} = (1+\tan t)^{1/3} \Rightarrow f'(t) = \frac{1}{3}(1+\tan t)^{-2/3}\sec^2 t = \dfrac{\sec^2 t}{3\sqrt[3]{(1+\tan t)^2}}$

13. $y = \cos(a^3+x^3) \Rightarrow y' = -\sin(a^3+x^3)\cdot 3x^2$ [a^3 is just a constant] $= -3x^2\sin(a^3+x^3)$

14. $y = a^3 + \cos^3 x \Rightarrow y' = 3(\cos x)^2(-\sin x)$ [a^3 is just a constant] $= -3\sin x\cos^2 x$

15. $y = \cot(x/2) \Rightarrow y' = -\csc^2(x/2)\cdot\frac{1}{2} = -\frac{1}{2}\csc^2(x/2)$

16. $y = 4\sec 5x \Rightarrow y' = 4\sec 5x\tan 5x(5) = 20\sec 5x\tan 5x$

17. $g(x) = (1+4x)^5(3+x-x^2)^8 \Rightarrow$

$$g'(x) = (1+4x)^5\cdot 8(3+x-x^2)^7(1-2x) + (3+x-x^2)^8\cdot 5(1+4x)^4\cdot 4$$
$$= 4(1+4x)^4(3+x-x^2)^7\left[2(1+4x)(1-2x) + 5(3+x-x^2)\right]$$
$$= 4(1+4x)^4(3+x-x^2)^7\left[(2+4x-16x^2) + (15+5x-5x^2)\right]$$
$$= 4(1+4x)^4(3+x-x^2)^7(17+9x-21x^2)$$

18. $h(t) = (t^4-1)^3(t^3+1)^4 \Rightarrow$

$$h'(t) = (t^4-1)^3\cdot 4(t^3+1)^3(3t^2) + (t^3+1)^4\cdot 3(t^4-1)^2(4t^3)$$
$$= 12t^2(t^4-1)^2(t^3+1)^3\left[(t^4-1) + t(t^3+1)\right] = 12t^2(t^4-1)^2(t^3+1)^3(2t^4+t-1)$$

19. $y = (2x-5)^4(8x^2-5)^{-3} \Rightarrow$

$$y' = 4(2x-5)^3(2)(8x^2-5)^{-3} + (2x-5)^4(-3)(8x^2-5)^{-4}(16x)$$
$$= 8(2x-5)^3(8x^2-5)^{-3} - 48x(2x-5)^4(8x^2-5)^{-4}$$

[This simplifies to $8(2x-5)^3(8x^2-5)^{-4}(-4x^2+30x-5)$.]

20. $y = (x^2+1)(x^2+2)^{1/3} \Rightarrow$

$$y' = 2x(x^2+2)^{1/3} + (x^2+1)\left(\tfrac{1}{3}\right)(x^2+2)^{-2/3}(2x) = 2x(x^2+2)^{1/3}\left[1 + \frac{x^2+1}{3(x^2+2)}\right]$$

21. $y = x^3\cos nx \Rightarrow y' = x^3(-\sin nx)(n) + \cos nx\,(3x^2) = x^2(3\cos nx - nx\sin nx)$

22. $y = x\sin\sqrt{x} \Rightarrow y' = x\cos\sqrt{x}\cdot\frac{1}{2}x^{-1/2} + \sin\sqrt{x}\cdot 1 = \frac{1}{2}\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}$

23. $y = \sin(x\cos x) \Rightarrow y' = \cos(x\cos x)\cdot[x(-\sin x) + \cos x\cdot 1] = (\cos x - x\sin x)\cos(x\cos x)$

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24. $f(x) = \dfrac{x}{\sqrt{7-3x}} \Rightarrow$

$$f'(x) = \frac{\sqrt{7-3x}(1) - x \cdot \frac{1}{2}(7-3x)^{-1/2} \cdot (-3)}{(\sqrt{7-3x})^2} = \frac{\sqrt{7-3x} + \dfrac{3x}{2\sqrt{7-3x}}}{(7-3x)^1}$$

$$= \frac{2(7-3x)+3x}{2(7-3x)^{3/2}} = \frac{14-3x}{2(7-3x)^{3/2}}$$

25. $F(z) = \sqrt{\dfrac{z-1}{z+1}} = \left(\dfrac{z-1}{z+1}\right)^{1/2} \Rightarrow$

$$F'(z) = \frac{1}{2}\left(\frac{z-1}{z+1}\right)^{-1/2} \cdot \frac{d}{dz}\left(\frac{z-1}{z+1}\right) = \frac{1}{2}\left(\frac{z+1}{z-1}\right)^{1/2} \cdot \frac{(z+1)(1)-(z-1)(1)}{(z+1)^2}$$

$$= \frac{1}{2}\frac{(z+1)^{1/2}}{(z-1)^{1/2}} \cdot \frac{z+1-z+1}{(z+1)^2} = \frac{1}{2}\frac{(z+1)^{1/2}}{(z-1)^{1/2}} \cdot \frac{2}{(z+1)^2} = \frac{1}{(z-1)^{1/2}(z+1)^{3/2}}$$

26. $G(y) = \dfrac{(y-1)^4}{(y^2+2y)^5} \Rightarrow$

$$G'(y) = \frac{(y^2+2y)^5 \cdot 4(y-1)^3 \cdot 1 - (y-1)^4 \cdot 5(y^2+2y)^4(2y+2)}{\left[(y^2+2y)^5\right]^2}$$

$$= \frac{2(y^2+2y)^4(y-1)^3[2(y^2+2y) - 5(y-1)(y+1)]}{(y^2+2y)^{10}}$$

$$= \frac{2(y-1)^3[(2y^2+4y)+(-5y^2+5)]}{(y^2+2y)^6} = \frac{2(y-1)^3(-3y^2+4y+5)}{(y^2+2y)^6}$$

27. $y = \dfrac{r}{\sqrt{r^2+1}} \Rightarrow$

$$y' = \frac{\sqrt{r^2+1}\,(1) - r \cdot \frac{1}{2}(r^2+1)^{-1/2}(2r)}{\left(\sqrt{r^2+1}\right)^2} = \frac{\sqrt{r^2+1} - \dfrac{r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2} = \frac{\dfrac{\sqrt{r^2+1}\sqrt{r^2+1} - r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2}$$

$$= \frac{(r^2+1)-r^2}{\left(\sqrt{r^2+1}\right)^3} = \frac{1}{(r^2+1)^{3/2}} \text{ or } (r^2+1)^{-3/2}$$

Another solution: Write y as a product and make use of the Product Rule. $y = r(r^2+1)^{-1/2} \Rightarrow$

$$y' = r \cdot -\tfrac{1}{2}(r^2+1)^{-3/2}(2r) + (r^2+1)^{-1/2} \cdot 1$$

$$= (r^2+1)^{-3/2}\left[-r^2 + (r^2+1)^1\right] = (r^2+1)^{-3/2}(1) = (r^2+1)^{-3/2}$$

The step that students usually have trouble with is factoring out $(r^2+1)^{-3/2}$. But this is no different than factoring out x^2 from x^2+x^5; that is, we are just factoring out a factor with the *smallest* exponent that appears on it. In this case, $-\frac{3}{2}$ is smaller than $-\frac{1}{2}$.

28. $y = \dfrac{\cos \pi x}{\sin \pi x + \cos \pi x} \Rightarrow$

$$y' = \frac{(\sin \pi x + \cos \pi x)(-\pi \sin \pi x) - (\cos \pi x)(\pi \cos \pi x - \pi \sin \pi x)}{(\sin \pi x + \cos \pi x)^2}$$

$$= \frac{-\pi \sin^2 \pi x - \pi \sin \pi x \cos \pi x - \pi \cos^2 \pi x + \pi \sin \pi x \cos \pi x}{(\sin \pi x + \cos \pi x)^2}$$

$$= \frac{-\pi(\sin^2 \pi x + \cos^2 \pi x)}{(\sin \pi x + \cos \pi x)^2} = \frac{-\pi}{(\sin \pi x + \cos \pi x)^2} \quad \text{or} \quad \frac{-\pi}{1 + 2 \sin \pi x \cos \pi x}$$

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29. $y = \tan(\cos x) \quad \Rightarrow \quad y' = \sec^2(\cos x) \cdot (-\sin x) = -\sin x \sec^2(\cos x)$

30. $y = \dfrac{\sin^2 x}{\cos x} \quad \Rightarrow$

$$y' = \frac{\cos x\,(2\sin x \cos x) - \sin^2 x\,(-\sin x)}{\cos^2 x} = \frac{\sin x\,(2\cos^2 x + \sin^2 x)}{\cos^2 x} = \frac{\sin x\,(1 + \cos^2 x)}{\cos^2 x}$$
$$= \sin x\,(1 + \sec^2 x)$$

Another method: $y = \tan x \sin x \quad \Rightarrow \quad y' = \sec^2 x \sin x + \tan x \cos x = \sec^2 x \sin x + \sin x$

31. $y = \sin\sqrt{1+x^2} \quad \Rightarrow \quad y' = \cos\sqrt{1+x^2} \cdot \tfrac{1}{2}(1+x^2)^{-1/2} \cdot 2x = \left(x \cos\sqrt{1+x^2}\,\right)/\sqrt{1+x^2}$

32. $y = \tan^2(3\theta) = (\tan 3\theta)^2 \quad \Rightarrow \quad y' = 2(\tan 3\theta) \cdot \dfrac{d}{d\theta}(\tan 3\theta) = 2\tan 3\theta \cdot \sec^2 3\theta \cdot 3 = 6\tan 3\theta \sec^2 3\theta$

33. $y = (1 + \cos^2 x)^6 \quad \Rightarrow \quad y' = 6(1+\cos^2 x)^5 \cdot 2\cos x\,(-\sin x) = -12\,\cos x\,\sin x\,(1+\cos^2 x)^5$

34. $y = x \sin\dfrac{1}{x} \quad \Rightarrow \quad y' = \sin\dfrac{1}{x} + x\cos\dfrac{1}{x}\left(-\dfrac{1}{x^2}\right) = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}$

35. $y = \sec^2 x + \tan^2 x = (\sec x)^2 + (\tan x)^2 \quad \Rightarrow$
$y' = 2(\sec x)(\sec x \tan x) + 2(\tan x)(\sec^2 x) = 2\sec^2 x \tan x + 2\sec^2 x \tan x = 4\sec^2 x \tan x$

36. $y = \cot(x^2) + \cot^2 x = \cot(x^2) + (\cot x)^2 \quad \Rightarrow$
$y' = -\csc^2(x^2) \cdot 2x + 2(\cot x)^1(-\csc^2 x) = -2x\csc^2(x^2) - 2\cot x \csc^2 x$

37. $y = \cot^2(\sin\theta) = [\cot(\sin\theta)]^2 \quad \Rightarrow$
$y' = 2\,[\cot(\sin\theta)] \cdot \dfrac{d}{d\theta}\,[\cot(\sin\theta)] = 2\cot(\sin\theta) \cdot \left[-\csc^2(\sin\theta) \cdot \cos\theta\right] = -2\cos\theta\,\cot(\sin\theta)\,\csc^2(\sin\theta)$

38. $y = \sin(\sin(\sin x)) \quad \Rightarrow \quad y' = \cos(\sin(\sin x))\,\dfrac{d}{dx}(\sin(\sin x)) = \cos(\sin(\sin x))\,\cos(\sin x)\,\cos x$

39. $y = \sqrt{x+\sqrt{x}} \quad \Rightarrow \quad y' = \tfrac{1}{2}\left(x+\sqrt{x}\,\right)^{-1/2}\left(1 + \tfrac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

40. $y = \sqrt{x+\sqrt{x+\sqrt{x}}} \quad \Rightarrow \quad y' = \tfrac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left[1 + \tfrac{1}{2}\left(x+\sqrt{x}\,\right)^{-1/2}\left(1 + \tfrac{1}{2}x^{-1/2}\right)\right]$

41. $y = \sin\left(\tan\sqrt{\sin x}\,\right) \quad \Rightarrow$

$$y' = \cos\left(\tan\sqrt{\sin x}\,\right) \cdot \frac{d}{dx}\left(\tan\sqrt{\sin x}\,\right) = \cos\left(\tan\sqrt{\sin x}\,\right)\sec^2\sqrt{\sin x} \cdot \frac{d}{dx}(\sin x)^{1/2}$$
$$= \cos\left(\tan\sqrt{\sin x}\,\right)\sec^2\sqrt{\sin x} \cdot \tfrac{1}{2}(\sin x)^{-1/2} \cdot \cos x$$
$$= \cos\left(\tan\sqrt{\sin x}\,\right)\left(\sec^2\sqrt{\sin x}\,\right)\left(\frac{1}{2\sqrt{\sin x}}\right)(\cos x)$$

42. $y = \sqrt{\cos\left(\sin^2 x\right)} \quad \Rightarrow \quad y' = \tfrac{1}{2}\left(\cos\left(\sin^2 x\right)\right)^{-1/2}\left[-\sin\left(\sin^2 x\right)\right](2\sin x \cos x) = -\dfrac{\sin\left(\sin^2 x\right)\sin x \cos x}{\sqrt{\cos\left(\sin^2 x\right)}}$

43. $y = (1+2x)^{10} \quad \Rightarrow \quad y' = 10(1+2x)^9 \cdot 2 = 20(1+2x)^9$. At $(0,1)$, $y' = 20(1+0)^9 = 20$, and an equation of the tangent line is $y - 1 = 20(x-0)$, or $y = 20x + 1$.

44. $y = \sin x + \sin^2 x \quad \Rightarrow \quad y' = \cos x + 2\sin x \cos x$. At $(0,0)$, $y' = 1$, and an equation of the tangent line is $y - 0 = 1(x-0)$, or $y = x$.

45. $y = \sin(\sin x) \quad \Rightarrow \quad y' = \cos(\sin x) \cdot \cos x$. At $(\pi, 0)$, $y' = \cos(\sin\pi) \cdot \cos\pi = \cos(0) \cdot (-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x - \pi)$, or $y = -x + \pi$.

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46. $y = \sqrt{5+x^2} \;\Rightarrow\; y' = \frac{1}{2}(5+x^2)^{-1/2}(2x) = x/\sqrt{5+x^2}$. At $(2,3)$, $y' = \frac{2}{3}$, and an equation of the tangent line is $y - 3 = \frac{2}{3}(x-2)$, or $y = \frac{2}{3}x + \frac{5}{3}$.

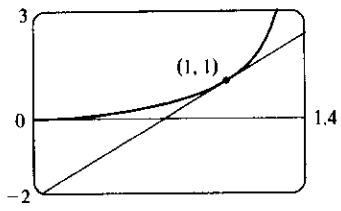
47. (a) $y = f(x) = \tan\left(\frac{\pi}{4}x^2\right) \;\Rightarrow\; f'(x) = \sec^2\left(\frac{\pi}{4}x^2\right)\left(2 \cdot \frac{\pi}{4}x\right)$.

The slope of the tangent at $(1,1)$ is thus

$f'(1) = \sec^2\frac{\pi}{4}\left(\frac{\pi}{2}\right) = 2 \cdot \frac{\pi}{2} = \pi$, and its equation is

$y - 1 = \pi(x-1)$ or $y = \pi x - \pi + 1$.

(b)



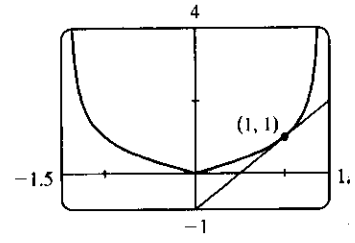
48. (a) For $x > 0$, $|x| = x$, and $y = f(x) = \dfrac{x}{\sqrt{2-x^2}} \;\Rightarrow$

$$f'(x) = \frac{\sqrt{2-x^2}\,(1) - x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2} \cdot \frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}}$$

$$= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}}$$

So at $(1,1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y - 1 = 2(x-1)$ or $y = 2x - 1$.

(b)

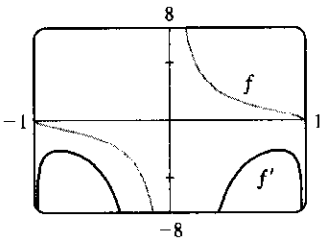


49. (a) $f(x) = \dfrac{\sqrt{1-x^2}}{x} \;\Rightarrow$

$$f'(x) = \frac{x \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x) - \sqrt{1-x^2}\,(1)}{x^2} \cdot \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

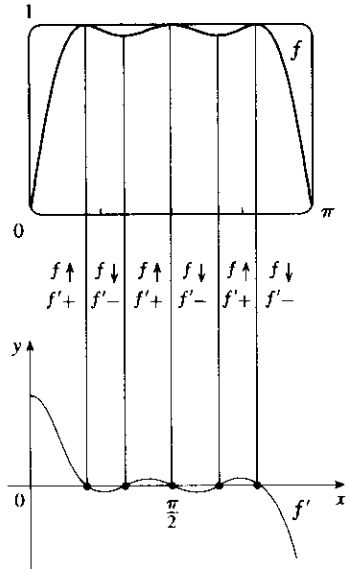
$$= \frac{-x^2 - (1-x^2)}{x^2\sqrt{1-x^2}} = \frac{-1}{x^2\sqrt{1-x^2}}$$

(b)



Notice that all tangents to the graph of f have negative slopes and $f'(x) < 0$ always.

50. (a)



From the graph of f, we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f'. From the symmetry of the graph of f, we must have the graph of f' as high at $x = 0$ as it is low at $x = \pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

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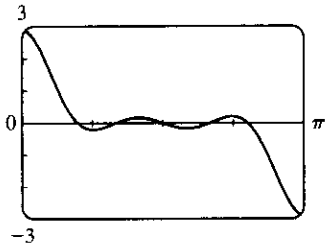
(b) $f(x) = \sin(x + \sin 2x) \Rightarrow$

$$f'(x) = \cos(x + \sin 2x) \cdot \frac{d}{dx}(x + \sin 2x)$$
$$= \cos(x + \sin 2x)(1 + 2\cos 2x)$$



51. For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2\sin x + \sin^2 x \Rightarrow$ $f'(x) = 2\cos x + 2\sin x \cos x = 0 \Leftrightarrow 2\cos x\,(1 + \sin x) = 0 \Leftrightarrow \cos x = 0$ or $\sin x = -1$, so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right) = 3$ and $f\left(\frac{3\pi}{2}\right) = -1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2} + 2n\pi, 3\right)$ and $\left(\frac{3\pi}{2} + 2n\pi, -1\right)$, where n is any integer.

52. $f(x) = \sin 2x - 2\sin x \Rightarrow f'(x) = 2\cos 2x - 2\cos x = 4\cos^2 x - 2\cos x - 2$, and $4\cos^2 x - 2\cos x - 2 = 0 \Leftrightarrow (\cos x - 1)(4\cos x + 2) = 0 \Leftrightarrow \cos x = 1$ or $\cos x = -\frac{1}{2}$. So $x = 2n\pi$ or $(2n+1)\pi \pm \frac{\pi}{3}$, n any integer.

53. $F(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(3) = f'(g(3)) \cdot g'(3) = f'(6) \cdot g'(3) = 7 \cdot 4 = 28$. Notice that we did not use $f'(3) = 2$.

54. $w = u \circ v \Rightarrow w(x) = u(v(x)) \Rightarrow w'(x) = u'(v(x)) \cdot v'(x)$, so $w'(0) = u'(v(0)) \cdot v'(0) = u'(2) \cdot v'(0) = 4 \cdot 5 = 20$. The other pieces of information, $u(0) = 1$, $u'(0) = 3$, and $v'(2) = 6$, were not needed.

55. (a) $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \Rightarrow H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

56. (a) $F(x) = f(f(x)) \Rightarrow F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

(b) $G(x) = g(g(x)) \Rightarrow G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.

57. (a) $u(x) = f(g(x)) \Rightarrow u'(x) = f'(g(x))g'(x)$. So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2, 4)$ to $(6, 3)$, so its slope is $\frac{3-4}{6-2} = -\frac{1}{4}$. To find $g'(1)$, note that g is linear from $(0, 6)$ to $(2, 0)$, so its slope is $\frac{0-6}{2-0} = -3$. Thus, $f'(3)g'(1) = \left(-\frac{1}{4}\right)(-3) = \frac{3}{4}$.

(b) $v(x) = g(f(x)) \Rightarrow v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.

(c) $w(x) = g(g(x)) \Rightarrow w'(x) = g'(g(x))g'(x)$. So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2, 0)$ to $(5, 2)$, so its slope is $\frac{2-0}{5-2} = \frac{2}{3}$. Thus, $g'(3)g'(1) = \left(\frac{2}{3}\right)(-3) = -2$.

58. (a) $h(x) = f(f(x)) \Rightarrow h'(x) = f'(f(x))f'(x)$.
So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

(b) $g(x) = f(x^2) \Rightarrow g'(x) = f'(x^2) \cdot \frac{d}{dx}(x^2) = f'(x^2)(2x)$.
So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(1.5) = 6$.

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5. $f(x) = x^5 + 6x^2 - 7x \Rightarrow f'(x) = 5x^4 + 12x - 7 \Rightarrow f''(x) = 20x^3 + 12$

6. $f(t) = t^8 - 7t^6 + 2t^4 \Rightarrow f'(t) = 8t^7 - 42t^5 + 8t^3 \Rightarrow f''(t) = 56t^6 - 210t^4 + 24t^2$

7. $y = \cos 2\theta \Rightarrow y' = -2\sin 2\theta \Rightarrow y'' = -4\cos 2\theta$

8. $y = \theta \sin\theta \Rightarrow y' = \theta\cos\theta + \sin\theta \Rightarrow y'' = \theta(-\sin\theta) + \cos\theta \cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$

9. $F(t) = (1-7t)^6 \Rightarrow F'(t) = 6(1-7t)^5(-7) = -42(1-7t)^5 \Rightarrow$
$F''(t) = -42 \cdot 5(1-7t)^4(-7) = 1470(1-7t)^4$

10. $g(x) = \dfrac{2x+1}{x-1} \Rightarrow g'(x) = \dfrac{(x-1)(2) - (2x+1)(1)}{(x-1)^2} = \dfrac{2x-2-2x-1}{(x-1)^2} = \dfrac{-3}{(x-1)^2}$ or $-3(x-1)^{-2}$

$\Rightarrow g''(x) = -3(-2)(x-1)^{-3} = 6(x-1)^{-3}$ or $\dfrac{6}{(x-1)^3}$

11. $h(u) = \dfrac{1-4u}{1+3u} \Rightarrow h'(u) = \dfrac{(1+3u)(-4) - (1-4u)(3)}{(1+3u)^2} = \dfrac{-4-12u-3+12u}{(1+3u)^2} = \dfrac{-7}{(1+3u)^2}$ or

$-7(1+3u)^{-2} \Rightarrow h''(u) = -7(-2)(1+3u)^{-3}(3) = 42(1+3u)^{-3}$ or $\dfrac{42}{(1+3u)^3}$

12. $H(s) = a\sqrt{s} + \dfrac{b}{\sqrt{s}} = as^{1/2} + bs^{-1/2} \Rightarrow$

$H'(s) = a \cdot \frac{1}{2}s^{-1/2} + b\left(-\frac{1}{2}s^{-3/2}\right) = \frac{1}{2}as^{-1/2} - \frac{1}{2}bs^{-3/2} \Rightarrow$

$H''(s) = \frac{1}{2}a\left(-\frac{1}{2}s^{-3/2}\right) - \frac{1}{2}b\left(-\frac{3}{2}s^{-5/2}\right) = -\frac{1}{4}as^{-3/2} + \frac{3}{4}bs^{-5/2}$

13. $h(x) = \sqrt{x^2+1} \Rightarrow h'(x) = \dfrac{1}{2}(x^2+1)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+1}} \Rightarrow$

$$h''(x) = \frac{\sqrt{x^2+1}\cdot 1 - x\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right]}{\left(\sqrt{x^2+1}\right)^2} = \frac{(x^2+1)^{-1/2}\left[(x^2+1) - x^2\right]}{(x^2+1)^1} = \frac{1}{(x^2+1)^{3/2}}$$

14. $y = x^n \Rightarrow y' = nx^{n-1} \Rightarrow y'' = n(n-1)x^{n-2}$

15. $y = (x^3+1)^{2/3} \Rightarrow y' = \frac{2}{3}(x^3+1)^{-1/3}(3x^2) = 2x^2(x^3+1)^{-1/3} \Rightarrow$
$y'' = 2x^2\left(-\frac{1}{3}\right)(x^3+1)^{-4/3}(3x^2) + (x^3+1)^{-1/3}(4x) = 4x(x^3+1)^{-1/3} - 2x^4(x^3+1)^{-4/3}$

16. $y = \dfrac{4x}{\sqrt{x+1}} \Rightarrow$

$$y' = \frac{\sqrt{x+1}\cdot 4 - 4x\cdot\frac{1}{2}(x+1)^{-1/2}}{\left(\sqrt{x+1}\right)^2} = \frac{4\sqrt{x+1} - 2x/\sqrt{x+1}}{x+1} = \frac{4(x+1)-2x}{(x+1)^{3/2}} = \frac{2x+4}{(x+1)^{3/2}} \Rightarrow$$

$$y'' = \frac{(x+1)^{3/2}\cdot 2 - (2x+4)\cdot\frac{3}{2}(x+1)^{1/2}}{\left[(x+1)^{3/2}\right]^2} = \frac{(x+1)^{1/2}[2(x+1) - 3(x+2)]}{(x+1)^3}$$

$$= \frac{2x+2-3x-6}{(x+1)^{5/2}} = \frac{-x-4}{(x+1)^{5/2}}$$

17. $H(t) = \tan 3t \Rightarrow H'(t) = 3\sec^2 3t \Rightarrow$

$H''(t) = 2\cdot 3\sec 3t\,\dfrac{d}{dt}(\sec 3t) = 6\sec 3t\,(3\sec 3t\tan 3t) = 18\sec^2 3t\tan 3t$

18. $g(s) = s^2\cos s \Rightarrow g'(s) = 2s\cos s - s^2\sin s \Rightarrow$
$g''(s) = 2\cos s - 2s\sin s - 2s\sin s - s^2\cos s = (2-s^2)\cos s - 4s\sin s$

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19. $g(\theta) = \theta \csc\theta \;\Rightarrow\; g'(\theta) = -\theta \csc\theta \cot\theta + \csc\theta \;\Rightarrow$

$$g''(\theta) = (-1)\csc\theta\cot\theta + (-\theta)(-\csc\theta\cot\theta)\cot\theta + (-\theta\csc\theta)(-\csc^2\theta) - \csc\theta\cot\theta$$
$$= -\csc\theta\cot\theta + \theta\csc\theta\cot^2\theta + \theta\csc^3\theta - \csc\theta\cot\theta$$
$$= \csc\theta(\theta\csc^2\theta + \theta\cot^2\theta - 2\cot\theta)$$

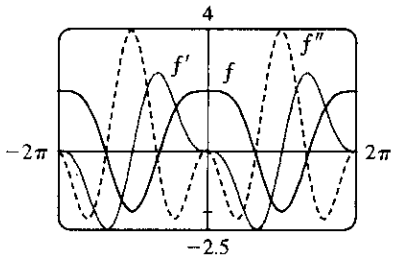
20. $h(x) = \dfrac{x+3}{x^2+2x} \;\Rightarrow$

$$h'(x) = \frac{(x^2+2x)(1) - (x+3)(2x+2)}{(x^2+2x)^2} = \frac{(x^2+2x) - (2x^2+8x+6)}{(x^2+2x)^2} = -\frac{x^2+6x+6}{(x^2+2x)^2} \;\Rightarrow$$

$$h''(x) = -\frac{(x^2+2x)^2(2x+6) - (x^2+6x+6)2(x^2+2x)(2x+2)}{\left[(x^2+2x)^2\right]^2}$$
$$= -\frac{2(x^2+2x)\left[(x^2+2x)(x+3) - (x^2+6x+6)(2x+2)\right]}{(x^2+2x)^4}$$
$$= -\frac{2\left[(x^3+5x^2+6x) - (2x^3+14x^2+24x+12)\right]}{(x^2+2x)^3} = \frac{2(x^3+9x^2+18x+12)}{(x^2+2x)^3}$$

21. (a) $f(x) = 2\cos x + \sin^2 x \;\Rightarrow\; f'(x) = 2(-\sin x) + 2\sin x\,(\cos x) = \sin 2x - 2\sin x \;\Rightarrow$
$f''(x) = 2\cos 2x - 2\cos x = 2(\cos 2x - \cos x)$

(b)

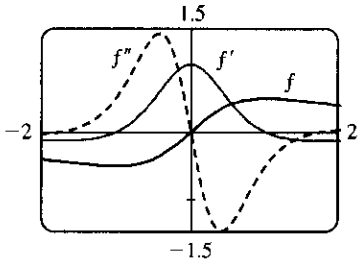


We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

22. (a) $f(x) = \dfrac{x}{x^2+1} \;\Rightarrow\; f'(x) = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \;\Rightarrow$

$$f''(x) = \frac{(x^2+1)^2(-2x) - (1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2x(2x^2-2-x^2-1)}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

23. $y = \sqrt{2x+3} = (2x+3)^{1/2} \;\Rightarrow\; y' = \frac{1}{2}(2x+3)^{-1/2}\cdot 2 = (2x+3)^{-1/2} \;\Rightarrow$
$y'' = -\frac{1}{2}(2x+3)^{-3/2}\cdot 2 = -(2x+3)^{-3/2} \;\Rightarrow\; y''' = \frac{3}{2}(2x+3)^{-5/2}\cdot 2 = 3(2x+3)^{-5/2}$

24. $y = \dfrac{x}{2x-1} \;\Rightarrow\; y' = \dfrac{(2x-1)(1) - x(2)}{(2x-1)^2} = \dfrac{-1}{(2x-1)^2}$ or $-1(2x-1)^{-2} \;\Rightarrow$
$y'' = -1(-2)(2x-1)^{-3}(2) = 4(2x-1)^{-3} \;\Rightarrow$
$y''' = 4(-3)(2x-1)^{-4}(2) = -24(2x-1)^{-4}$ or $-24/(2x-1)^4$

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25. $f(t) = t\cos t \;\Rightarrow\; f'(t) = t(-\sin t) + \cos t \cdot 1 \;\Rightarrow\; f''(t) = t(-\cos t) - \sin t \cdot 1 - \sin t \;\Rightarrow$
$f'''(t) = t\sin t - \cos t \cdot 1 - \cos t - \cos t = t\sin t - 3\cos t$, so $f'''(0) = 0 - 3 = -3$.

26. $g(x) = \sqrt{5-2x} \;\Rightarrow\; g'(x) = \frac{1}{2}(5-2x)^{-1/2}(-2) = -(5-2x)^{-1/2} \;\Rightarrow$
$g''(x) = \frac{1}{2}(5-2x)^{-3/2}(-2) = -(5-2x)^{-3/2} \;\Rightarrow\; g'''(x) = \frac{3}{2}(5-2x)^{-5/2}(-2) = -3(5-2x)^{-5/2}$, so
$g'''(2) = -3(1)^{-5/2} = -3$.

27. $f(\theta) = \cot\theta \;\Rightarrow\; f'(\theta) = -\csc^2\theta \;\Rightarrow\; f''(\theta) = -2\csc\theta\,(-\csc\theta\cot\theta) = 2\csc^2\theta\cot\theta \;\Rightarrow$
$f'''(\theta) = 2\left(-2\csc^2\theta\cot\theta\right)\cot\theta + 2\csc^2\theta\left(-\csc^2\theta\right) = -2\csc^2\theta\left(2\cot^2\theta + \csc^2\theta\right) \;\Rightarrow$
$f'''\left(\frac{\pi}{6}\right) = -2(2)^2\left[2\left(\sqrt{3}\right)^2 + (2)^2\right] = -80$

28. $g(x) = \sec x \;\Rightarrow\; g'(x) = \sec x\tan x \;\Rightarrow$
$g''(x) = \sec x\sec^2 x + \tan x\,(\sec x\tan x) = \sec^3 x + \sec x\tan^2 x = \sec^3 x + \sec x\left(\sec^2 x - 1\right)$
$= 2\sec^3 x - \sec x \;\Rightarrow$
$g'''(x) = 6\sec^2 x\,(\sec x\tan x) - \sec x\tan x = \sec x\tan x\left(6\sec^2 x - 1\right) \;\Rightarrow$
$g'''\left(\frac{\pi}{4}\right) = \sqrt{2}\,(1)(6\cdot 2 - 1) = 11\sqrt{2}$

29. $9x^2 + y^2 = 9 \;\Rightarrow\; 18x + 2yy' = 0 \;\Rightarrow\; 2yy' = -18x \;\Rightarrow\; y' = -9x/y \;\Rightarrow$
$y'' = -9\left(\frac{y\cdot 1 - x\cdot y'}{y^2}\right) = -9\left(\frac{y - x(-9x/y)}{y^2}\right) = -9\cdot\frac{y^2+9x^2}{y^3} = -9\cdot\frac{9}{y^3}$ [since x and y must satisfy the original equation, $9x^2 + y^2 = 9$]. Thus, $y'' = -81/y^3$.

30. $\sqrt{x} + \sqrt{y} = 1 \;\Rightarrow\; \frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0 \;\Rightarrow\; y' = -\sqrt{y}/\sqrt{x} \;\Rightarrow$
$y'' = -\dfrac{\sqrt{x}\left[1/(2\sqrt{y})\right]y' - \sqrt{y}\left[1/(2\sqrt{x})\right]}{x} = -\dfrac{\sqrt{x}\,(1/\sqrt{y})\,(-\sqrt{y}/\sqrt{x}) - \sqrt{y}\,(1/\sqrt{x})}{2x} = \dfrac{1 + \sqrt{y}/\sqrt{x}}{2x}$
$= \dfrac{\sqrt{x}+\sqrt{y}}{2x\sqrt{x}} = \dfrac{1}{2x\sqrt{x}}$ since x and y must satisfy the original equation, $\sqrt{x} + \sqrt{y} = 1$.

31. $x^3 + y^3 = 1 \;\Rightarrow\; 3x^2 + 3y^2y' = 0 \;\Rightarrow\; y' = -\frac{x^2}{y^2} \;\Rightarrow$
$y'' = -\dfrac{y^2(2x) - x^2\cdot 2yy'}{(y^2)^2} = -\dfrac{2xy^2 - 2x^2y\left(-x^2/y^2\right)}{y^4} = -\dfrac{2xy^4 + 2x^4y}{y^6} = -\dfrac{2xy(y^3+x^3)}{y^6} = -\dfrac{2x}{y^5}$,
since x and y must satisfy the original equation, $x^3 + y^3 = 1$.

32. $x^4 + y^4 = a^4 \;\Rightarrow\; 4x^3 + 4y^3y' = 0 \;\Rightarrow\; 4y^3y' = -4x^3 \;\Rightarrow\; y' = -x^3/y^3 \;\Rightarrow$
$y'' = -\left(\dfrac{y^3\cdot 3x^2 - x^3\cdot 3y^2y'}{(y^3)^2}\right) = -3x^2y^2\cdot\dfrac{y - x(-x^3/y^3)}{y^6} = -3x^2\cdot\dfrac{y^4+x^4}{y^4y^3} = -3x^2\cdot\dfrac{a^4}{y^7} = \dfrac{-3a^4x^2}{y^7}$

33. $f(x) = x^n \;\Rightarrow\; f'(x) = nx^{n-1} \;\Rightarrow\; f''(x) = n(n-1)x^{n-2} \;\Rightarrow\; \cdots \;\Rightarrow$
$f^{(n)}(x) = n(n-1)(n-2)\cdots 2\cdot 1x^{n-n} = n!$

34. $f(x) = \dfrac{1}{5x-1} = (5x-1)^{-1} \;\Rightarrow\; f'(x) = -1(5x-1)^{-2}\cdot 5 \;\Rightarrow\; f''(x) = (-1)(-2)(5x-1)^{-3}\cdot 5^2 \;\Rightarrow$
$f'''(x) = (-1)(-2)(-3)(5x-1)^{-4}\cdot 5^3 \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = (-1)^n n!\,5^n(5x-1)^{-(n+1)}$

35. $f(x) = (1+x)^{-1} \;\Rightarrow\; f'(x) = -1(1+x)^{-2}$, $f''(x) = 1\cdot 2(1+x)^{-3}$, $f^{(3)}(x) = -1\cdot 2\cdot 3(1+x)^{-4}$,
$f^{(4)}(x) = 1\cdot 2\cdot 3\cdot 4(1+x)^{-5}$, $\ldots$, $f^{(n)}(x) = (-1)^n n!(1+x)^{-(n+1)}$

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36. $f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} \Rightarrow f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} \Rightarrow$

$f^{(4)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2} = -\frac{1\cdot 3\cdot 5}{2^4}x^{-7/2} \Rightarrow$

$f^{(5)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)x^{-9/2} = \frac{1\cdot 3\cdot 5\cdot 7}{2^5}x^{-9/2} \Rightarrow \cdots \Rightarrow$

$f^{(n)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-n+1\right)x^{-(2n-1)/2} = (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n}x^{-(2n-1)/2}$

37. $f(x) = 1/(3x^3) = \frac{1}{3}x^{-3} \Rightarrow f'(x) = \frac{1}{3}(-3)x^{-4} \Rightarrow f''(x) = \frac{1}{3}(-3)(-4)x^{-5} \Rightarrow$

$f'''(x) = \frac{1}{3}(-3)(-4)(-5)x^{-6} \Rightarrow \cdots \Rightarrow$

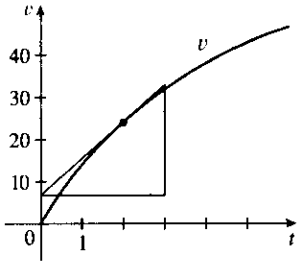
$f^{(n)}(x) = \frac{1}{3}(-3)(-4)\cdots[-(n+2)]\,x^{-(n+3)} = \frac{(-1)^n\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n+2)}{3x^{n+3}}\cdot\frac{2}{2} = \frac{(-1)^n\,(n+2)!}{6x^{n+3}}$

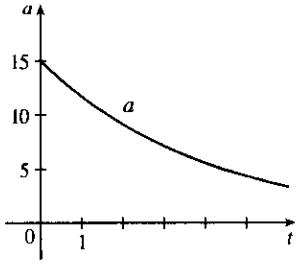
38. $D\sin x = \cos x \Rightarrow D^2\sin x = -\sin x \Rightarrow D^3\sin x = -\cos x \Rightarrow D^4\sin x = \sin x$. The derivatives of $\sin x$ occur in a cycle of four. Since $74 = 4(18) + 2$, we have $D^{74}\sin x = D^2\sin x = -\sin x$.

39. Let $f(x) = \cos x$. Then $Df(2x) = 2f'(2x)$, $D^2f(2x) = 2^2f''(2x)$, $D^3f(2x) = 2^3f'''(2x)$, $\ldots$, $D^{(n)}f(2x) = 2^nf^{(n)}(2x)$. Since the derivatives of $\cos x$ occur in a cycle of four, and since $103 = 4(25) + 3$, we have $f^{(103)}(x) = f^{(3)}(x) = \sin x$ and $D^{103}\cos 2x = 2^{103}f^{(103)}(2x) = 2^{103}\sin 2x$.

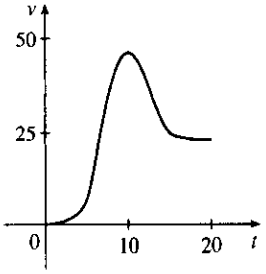
40. Let $f(x) = x\sin x$ and $h(x) = \sin x$, so $f(x) = xh(x)$. Then
$f'(x) = h(x) + xh'(x)$, $f''(x) = h'(x) + h'(x) + xh''(x) = 2h'(x) + xh''(x)$,
$f'''(x) = 2h''(x) + h''(x) + xh'''(x) = 3h''(x) + xh'''(x)$, $\ldots$, $f^{(n)}(x) = nh^{(n-1)}(x) + xh^{(n)}(x)$. Since $34 = 4(8) + 2$, we have $h^{(34)}(x) = h^{(2)}(x) = D^2\sin x = -\sin x$ and $h^{(35)}(x) = -\cos x$. Thus, $D^{(35)}x\sin x = 35h^{(34)}(x) + xh^{(35)}(x) = -35\sin x - x\cos x$.

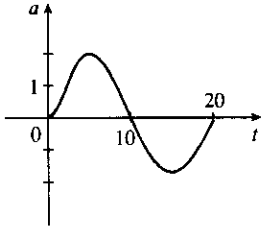
41. By measuring the slope of the graph of $s = f(t)$ at $t = 0, 1, 2, 3, 4$, and 5, and using the method of Example 1 in Section 3.2, we plot the graph of the velocity function $v = f'(t)$ in the first figure. The acceleration when $t = 2$ s is $a = f''(2)$, the slope of the tangent line to the graph of f' when $t = 2$. We estimate the slope of this tangent line to be $a(2) = f''(2) = v'(2) \approx \frac{27}{3} = 9$ ft/s^2. Similar measurements enable us to graph the acceleration function in the second figure.





42. (a) Since we estimate the velocity to be a maximum at $t = 10$, the acceleration is 0 at $t = 10$.





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(b) Drawing a tangent line at $t = 10$ on the graph of a, a appears to decrease by $10\ \text{ft/s}^2$ over a period of 20 s. So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5\ \ (\text{ft/s}^2)/\text{s}$ or ft/s^3.

43. (a) $s = 2t^3 - 15t^2 + 36t + 2 \ \Rightarrow\ v(t) = s'(t) = 6t^2 - 30t + 36 \ \Rightarrow\ a(t) = v'(t) = 12t - 30$

(b) $a(1) = 12 \cdot 1 - 30 = -18\ \text{m/s}^2$

(c) $v(t) = 6(t^2 - 5t + 6) = 6(t-2)(t-3) = 0$ when $t = 2$ or 3 and $a(2) = 24 - 30 = -6\ \text{m/s}^2$, $a(3) = 36 - 30 = 6\ \text{m/s}^2$.

44. (a) $s = 2t^3 - 3t^2 - 12t \ \Rightarrow\ v(t) = s'(t) = 6t^2 - 6t - 12 \ \Rightarrow\ a(t) = v'(t) = 12t - 6$

(b) $a(1) = 12 \cdot 1 - 6 = 6\ \text{m/s}^2$

(c) $v(t) = 6(t^2 - t - 2) = 6(t+1)(t-2) = 0$ when $t = -1$ or 2. Since $t \geq 0$, $t \neq -1$ and $a(2) = 24 - 6 = 18\ \text{m/s}^2$.

45. (a) $s = \sin\left(\frac{\pi}{6}t\right) + \cos\left(\frac{\pi}{6}t\right), 0 \leq t \leq 2$. $v(t) = s'(t) = \cos\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} - \sin\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} = \frac{\pi}{6}\left[\cos\left(\frac{\pi}{6}t\right) - \sin\left(\frac{\pi}{6}t\right)\right]$

$\Rightarrow\ a(t) = v'(t) = \frac{\pi}{6}\left[-\sin\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} - \cos\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6}\right] = -\dfrac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6}t\right) + \cos\left(\frac{\pi}{6}t\right)\right]$

(b) $a(1) = -\dfrac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6} \cdot 1\right) + \cos\left(\frac{\pi}{6} \cdot 1\right)\right] = -\dfrac{\pi^2}{36}\left[\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\right] = -\dfrac{\pi^2}{72}\left(1 + \sqrt{3}\right) \approx -0.3745\ \text{m/s}^2$

(c) $v(t) = 0$ for $0 \leq t \leq 2 \ \Rightarrow\ \cos\left(\frac{\pi}{6}t\right) = \sin\left(\frac{\pi}{6}t\right) \ \Rightarrow\ 1 = \dfrac{\sin\left(\frac{\pi}{6}t\right)}{\cos\left(\frac{\pi}{6}t\right)} \ \Rightarrow$

$\tan\left(\frac{\pi}{6}t\right) = 1 \ \Rightarrow\ \frac{\pi}{6}t = \tan^{-1} 1 \ \Rightarrow\ t = \frac{6}{\pi} \cdot \frac{\pi}{4} = \frac{3}{2} = 1.5$ s. Thus,

$a\left(\frac{3}{2}\right) = -\dfrac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6} \cdot \frac{3}{2}\right) + \cos\left(\frac{\pi}{6} \cdot \frac{3}{2}\right)\right] = -\dfrac{\pi^2}{36}\left[\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\right] = -\dfrac{\pi^2}{36}\sqrt{2} \approx -0.3877\ \text{m/s}^2$.

46. (a) $s = 2t^3 - 7t^2 + 4t + 1 \ \Rightarrow\ v(t) = s'(t) = 6t^2 - 14t + 4 \ \Rightarrow\ a(t) = v'(t) = 12t - 14$

(b) $a(1) = 12 - 14 = -2\ \text{m/s}^2$

(c) $v(t) = 2\left(3t^2 - 7t + 2\right) = 2(3t-1)(t-2) = 0$ when $t = \frac{1}{3}$ or 2 and $a\left(\frac{1}{3}\right) = 12\left(\frac{1}{3}\right) - 14 = -10\ \text{m/s}^2$, $a(2) = 12(2) - 14 = 10\ \text{m/s}^2$.

47. (a) $s(t) = t^4 - 4t^3 + 2 \ \Rightarrow\ v(t) = s'(t) = 4t^3 - 12t^2 \ \Rightarrow\ a(t) = v'(t) = 12t^2 - 24t = 12t(t-2) = 0$ when $t = 0$ or 2.

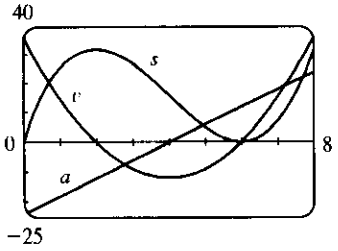
(b) $s(0) = 2$ m, $v(0) = 0$ m/s, $s(2) = -14$ m, $v(2) = -16$ m/s

48. (a) $s(t) = 2t^3 - 9t^2 \ \Rightarrow\ v(t) = s'(t) = 6t^2 - 18t \ \Rightarrow\ a(t) = v'(t) = 12t - 18 = 0$ when $t = 1.5$.

(b) $s(1.5) = -13.5$ m, $v(1.5) = -13.5$ m/s

49. (a) $s = f(t) = t^3 - 12t^2 + 36t, t \geq 0 \ \Rightarrow\ v(t) = f'(t) = 3t^2 - 24t + 36$.
$a(t) = v'(t) = 6t - 24$. $a(3) = 6(3) - 24 = -6\ (\text{m/s})/\text{s}$ or m/s^2.

(b)



(c) The particle is speeding up when v and a have the same sign. This occurs when $2 < t < 4$ and when $t > 6$. It is slowing down when v and a have opposite signs; that is, when $0 \leq t < 2$ and when $4 < t < 6$.

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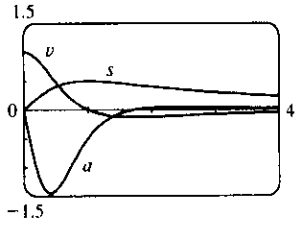
50. (a) $x(t) = \dfrac{t}{1+t^2} \;\Rightarrow\; v(t) = x'(t) = \dfrac{(1+t^2)(1) - t(2t)}{(1+t^2)^2} = \dfrac{1-t^2}{(1+t^2)^2}$.

$a(t) = v'(t) = \dfrac{2t(t^2-3)}{(1+t^2)^3}$. $a(t) = 0 \;\Rightarrow\; 2t(t^2-3) = 0 \;\Rightarrow\; t = 0$ or $\sqrt{3}$

(b)



(c) v and a have the same sign and the particle is speeding up when $1 < t < \sqrt{3}$. The particle is slowing down and v and a have opposite signs when $0 < t < 1$ and when $t > \sqrt{3}$.

51. (a) $y(t) = A\sin\omega t \;\Rightarrow\; v(t) = y'(t) = A\omega\cos\omega t \;\Rightarrow\; a(t) = v'(t) = -A\omega^2\sin\omega t$

(b) $a(t) = -A\omega^2\sin\omega t = -\omega^2(A\sin\omega t) = -\omega^2 y(t)$, so $a(t)$ is proportional to $y(t)$.

(c) speed $= |v(t)| = A\omega\,|\cos\omega t|$ is a maximum when $\cos\omega t = \pm 1$. But when $\cos\omega t = \pm 1$, we have $\sin\omega t = 0$, and $a(t) = -A\omega^2\sin\omega t = -A\omega^2(0) = 0$.

52. By the Chain Rule, $a(t) = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \dfrac{dv}{ds}\,v(t) = v(t)\,\dfrac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

53. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \;\Rightarrow\; 2a = 2 \;\Rightarrow\; a = 1$.
$P'(2) = 3 \;\Rightarrow\; 2(1)(2) + b = 3 \;\Rightarrow\; 4 + b = 3 \;\Rightarrow\; b = -1$.
$P(2) = 5 \;\Rightarrow\; 1(2)^2 + (-1)(2) + c = 5 \;\Rightarrow\; 2 + c = 5 \;\Rightarrow\; c = 3$. So $P(x) = x^2 - x + 3$.

54. Let $Q(x) = ax^3 + bx^2 + cx + d$. Then $Q'(x) = 3ax^2 + 2bx + c$, $Q''(x) = 6ax + 2b$ and $Q'''(x) = 6a$. Thus, $Q(1) = a + b + c + d = 1$, $Q'(1) = 3a + 2b + c = 3$, $Q''(1) = 6a + 2b = 6$ and $Q'''(1) = 6a = 12$. Solving these four equations in four unknowns a, b, c and d we get $a = 2$, $b = -3$, $c = 3$ and $d = -1$, so $Q(x) = 2x^3 - 3x^2 + 3x - 1$.

55. $y = A\sin x + B\cos x \;\Rightarrow\; y' = A\cos x - B\sin x \;\Rightarrow\; y'' = -A\sin x - B\cos x$. Substituting into $y'' + y' - 2y = \sin x$ gives us $(-3A - B)\sin x + (A - 3B)\cos x = 1\sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$. Solving for A and B, we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

56. $y = Ax^2 + Bx + C \;\Rightarrow\; y' = 2Ax + B \;\Rightarrow\; y'' = 2A$. We substitute these expressions into the equation $y'' + y' - 2y = x^2$ to get

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$
$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$
$$(-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = (1)x^2 + (0)x + (0)$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \;\Rightarrow\; A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \;\Rightarrow\; A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \;\Rightarrow\; -1 - \frac{1}{2} - 2C = 0 \;\Rightarrow\; C = -\frac{3}{4}$.

57. $f(x) = xg(x^2) \;\Rightarrow\; f'(x) = x \cdot g'(x^2) \cdot 2x + g(x^2) \cdot 1 = g(x^2) + 2x^2 g'(x^2) \;\Rightarrow$
$f''(x) = g'(x^2) \cdot 2x + 2x^2 \cdot g''(x^2) \cdot 2x + g'(x^2) \cdot 4x = 6xg'(x^2) + 4x^3 g''(x^2)$

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58. $f(x) = \dfrac{g(x)}{x} \Rightarrow f'(x) = \dfrac{xg'(x) - g(x)}{x^2} \Rightarrow$

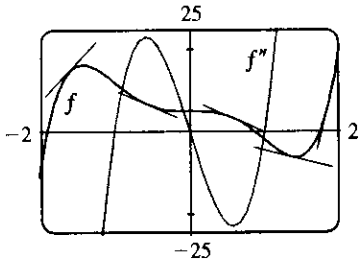
$$f''(x) = \frac{x^2[g'(x) + xg''(x) - g'(x)] - 2x\,[xg'(x) - g(x)]}{x^4} = \frac{x^2g''(x) - 2xg'(x) + 2g(x)}{x^3}$$

59. $f(x) = g(\sqrt{x}) \Rightarrow f'(x) = g'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} = \dfrac{g'(\sqrt{x})}{2\sqrt{x}} \Rightarrow$

$$f''(x) = \frac{2\sqrt{x} \cdot g''(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} - g'(\sqrt{x}) \cdot 2 \cdot \frac{1}{2}x^{-1/2}}{(2\sqrt{x})^2} = \frac{x^{-1/2}\,[\sqrt{x}\,g''(\sqrt{x}) - g'(\sqrt{x})]}{4x}$$

$$= \frac{\sqrt{x}\,g''(\sqrt{x}) - g'(\sqrt{x})}{4x\sqrt{x}}$$

60.



$f(x) = 3x^5 - 10x^3 + 5 \Rightarrow f'(x) = 15x^4 - 30x^2 \Rightarrow$
$f''(x) = 60x^3 - 60x = 60x(x^2 - 1) = 60x(x+1)(x-1)$
So $f''(x) > 0$ when $-1 < x < 0$ or $x > 1$, and on these intervals the graph of f lies above its tangent lines; and $f''(x) < 0$ when $x < -1$ or $0 < x < 1$, and on these intervals the graph of f lies below its tangent lines.

61. (a) $f(x) = \dfrac{1}{x^2 + x} \Rightarrow f'(x) = \dfrac{-(2x+1)}{(x^2+x)^2} \Rightarrow$

$$f''(x) = \frac{(x^2+x)^2(-2) + (2x+1)(2)(x^2+x)(2x+1)}{(x^2+x)^4} = \frac{2(3x^2+3x+1)}{(x^2+x)^3} \Rightarrow$$

$$f'''(x) = \frac{(x^2+x)^3(2)(6x+3) - 2(3x^2+3x+1)(3)(x^2+x)^2(2x+1)}{(x^2+x)^6}$$

$$= \frac{-6(4x^3+6x^2+4x+1)}{(x^2+x)^4} \Rightarrow$$

$$f^{(4)}(x) = \frac{(x^2+x)^4(-6)(12x^2+12x+4) + 6(4x^3+6x^2+4x+1)(4)(x^2+x)^3(2x+1)}{(x^2+x)^8}$$

$$= \frac{24(5x^4+10x^3+10x^2+5x+1)}{(x^2+x)^5}$$

$f^{(5)}(x) = ?$

(b) $f(x) = \dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1} \Rightarrow f'(x) = -x^{-2} + (x+1)^{-2} \Rightarrow f''(x) = 2x^{-3} - 2(x+1)^{-3} \Rightarrow$

$f'''(x) = (-3)(2)x^{-4} + (3)(2)(x+1)^{-4} \Rightarrow \cdots \Rightarrow f^{(n)}(x) = (-1)^n n!\left[x^{-(n+1)} - (x+1)^{-(n+1)}\right]$

62. (a) For $f(x) = \dfrac{7x+17}{2x^2-7x-4}$, a CAS gives us $f'''(x) = \dfrac{-6(56x^4+544x^3-2184x^2+6184x-6139)}{(2x^2-7x-4)^4}$

(b) Using a CAS we get $f(x) = \dfrac{7x+17}{2x^2-7x-4} = \dfrac{-3}{2x+1} + \dfrac{5}{x-4}$. Now we differentiate three times to obtain

$$f'''(x) = \frac{144}{(2x+1)^4} - \frac{30}{(x-4)^4}.$$

63. We will show that for each positive integer n, the nth derivative $f^{(n)}$ exists and equals one of f, f', f'', f''', $\ldots$, $f^{(p-1)}$. Since $f^{(p)} = f$, the first p derivatives of f are f', f'', f''', $\ldots$, $f^{(p-1)}$, and f. In particular, our statement is true for $n = 1$. Suppose that k is an integer, $k \geq 1$, for which f is k-times differentiable with $f^{(k)}$ in the set

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$S = \{f, f', f'', \ldots, f^{(p-1)}\}$. Since f is p-times differentiable, every member of S [including $f^{(k)}$] is differentiable, so $f^{(k+1)}$ exists and equals the derivative of some member of S. Thus, $f^{(k+1)}$ is in the set $\{f', f'', f''', \ldots, f^{(p)}\}$, which equals S since $f^{(p)} = f$. We have shown that the statement is true for $n = 1$ and that its truth for $n = k$ implies its truth for $n = k + 1$. By mathematical induction, the statement is true for all positive integers n.

64. (a) Use the Product Rule repeatedly: $F = fg \Rightarrow F' = f'g + fg' \Rightarrow$
$F'' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''$.

(b) $F''' = f'''g + f''g' + 2(f''g' + f'g'') + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg''' \Rightarrow$
$F^{(4)} = f^{(4)}g + f'''g' + 3(f'''g' + f''g'') + 3(f''g'' + f'g''') + f'g''' + fg^{(4)}$
$= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)}$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g'' + \cdots + \binom{n}{k}f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)}$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

65. The Chain Rule says that $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \quad \text{[Product Rule]}$$

$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

66. From Exercise 65, $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{dy}{du}\dfrac{d^2u}{dx^2} \Rightarrow$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2\right] + \frac{d}{dx}\left[\frac{dy}{du}\frac{d^2u}{dx^2}\right]$$

$$= \left[\frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\right]\left(\frac{du}{dx}\right)^2 + \left[\frac{d}{dx}\left(\frac{du}{dx}\right)^2\right]\frac{d^2y}{du^2} + \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{d^2u}{dx^2} + \left[\frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\right]\frac{dy}{du}$$

$$= \left[\frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\right]\left(\frac{du}{dx}\right)^2 + 2\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\left(\frac{d^2u}{dx^2}\right) + \frac{d^3u}{dx^3}\frac{dy}{du}$$

$$= \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 3\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d^3u}{dx^3}$$

APPLIED PROJECT Where Should a Pilot Start Descent?

1. Condition (i) will hold if and only if all of the following four conditions hold:

(α) $P(0) = 0$

(β) $P'(0) = 0$ (for a smooth landing)

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(γ) $P'(\ell) = 0$ (since the plane is cruising horizontally when it begins its descent)

(δ) $P(\ell) = h$.

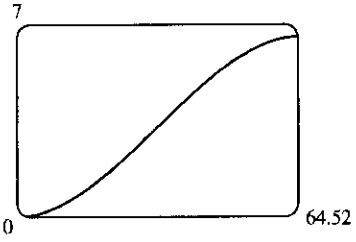
First of all, condition α implies that $P(0) = d = 0$, so $P(x) = ax^3 + bx^2 + cx \Rightarrow P'(x) = 3ax^2 + 2bx + c$. But $P'(0) = c = 0$ by condition β. So $P'(\ell) = 3a\ell^2 + 2b\ell = \ell(3a\ell + 2b)$. Now by condition γ, $3a\ell + 2b = 0 \Rightarrow a = -\dfrac{2b}{3\ell}$. Therefore, $P(x) = -\dfrac{2b}{3\ell}x^3 + bx^2$. Setting $P(\ell) = h$ for condition δ, we get $P(\ell) = -\dfrac{2b}{3\ell}\ell^3 + b\ell^2 = h \Rightarrow -\dfrac{2}{3}b\ell^2 + b\ell^2 = h \Rightarrow \dfrac{1}{3}b\ell^2 = h \Rightarrow b = \dfrac{3h}{\ell^2} \Rightarrow a = -\dfrac{2h}{\ell^3}$. So $y = P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$.

2. By condition (ii), $\dfrac{dx}{dt} = -v$ for all t, so $x(t) = \ell - vt$. Condition (iii) states that $\left|\dfrac{d^2y}{dt^2}\right| \le k$. By the Chain Rule, we have $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = -\dfrac{2h}{\ell^3}(3x^2)\dfrac{dx}{dt} + \dfrac{3h}{\ell^2}(2x)\dfrac{dx}{dt} = \dfrac{6hx^2v}{\ell^3} - \dfrac{6hxv}{\ell^2}$ (for $x \le \ell$) $\Rightarrow$

$\dfrac{d^2y}{dt^2} = \dfrac{6hv}{\ell^3}(2x)\dfrac{dx}{dt} - \dfrac{6hv}{\ell^2}\dfrac{dx}{dt} = -\dfrac{12hv^2}{\ell^3}x + \dfrac{6hv^2}{\ell^2}$. In particular, when $t = 0$, $x = \ell$ and so $\left.\dfrac{d^2y}{dt^2}\right|_{t=0} = -\dfrac{12hv^2}{\ell^3}\ell + \dfrac{6hv^2}{\ell^2} = -\dfrac{6hv^2}{\ell^2}$. Thus, $\left|\left.\dfrac{d^2y}{dt^2}\right|_{t=0}\right| = \dfrac{6hv^2}{\ell^2} \le k$. (This condition also follows from taking $x = 0$.)

3. We substitute $k = 860$ mi/h^2, $h = 35{,}000\text{ ft} \times \dfrac{1\text{ mi}}{5280\text{ ft}}$, and $v = 300$ mi/h into the result of part (b): $\dfrac{6\left(35{,}000 \cdot \frac{1}{5280}\right)(300)^2}{\ell^2} \le 860 \Rightarrow \ell \ge 300\sqrt{6 \cdot \dfrac{35{,}000}{5280 \cdot 860}} \approx 64.5$ miles.

4. Substituting the values of h and ℓ in Problem 3 into $P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$ gives us $P(x) = ax^3 + bx^2$, where $a \approx 4.937 \times 10^{-5}$ and $b \approx 4.78 \times 10^{-3}$.



APPLIED PROJECT Building a Better Roller Coaster

1. (a) $f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$.

The origin is at P	: $f(0) = 0$	$\Rightarrow$	$c = 0$
The slope of the ascent is 0.8:	$f'(0) = 0.8$	$\Rightarrow$	$b = 0.8$
The slope of the drop is -1.6:	$f'(100) = -1.6$	$\Rightarrow$	$200a + b = -1.6$

(b) $b = 0.8$, so $200a + b = -1.6 \Rightarrow 200a + 0.8 = -1.6 \Rightarrow 200a = -2.4 \Rightarrow a = -\dfrac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

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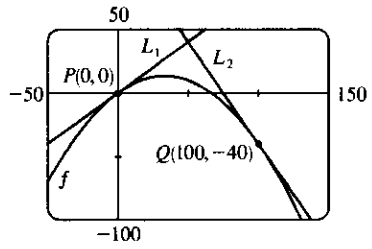
(c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$. The horizontal distance between P and Q is 100, so the y-coordinate at Q is $f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the point $(100, -40)$ and has slope -1.6, it has equation $y + 40 = -1.6(x - 100)$ or $y = -1.6x + 120$.



(d) The difference in elevation between $P(0, 0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L_1'(x) = 0.8$	$L_1''(x) = 0$
$[0, 10)$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L_2'(x) = -1.6$	$L_2''(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition — these are found after solving the system in part (b).

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L_1'(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L_1''(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729{,}000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24{,}300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1{,}000{,}000p + 10{,}000q + 100r + s = -40$
	$h'(100) = L_2'(100)$	$-\frac{8}{5}$	$30{,}000p + 200q + r = -1.6$
	$h''(100) = L_2''(100)$	0	$600p + 2q = 0$

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37. $y = f(x) = \sec x \;\Rightarrow\; f'(x) = \sec x \tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is $f(0) + f'(0)(x - 0) = 1 + 0(x) = 1$. Since 0.08 is close to 0, approximating $\sec 0.08$ with 1 is reasonable.

38. If $y = x^6$, $y' = 6x^5$ and the tangent line approximation at $(1, 1)$ has slope 6. If the change in x is 0.01, the change in y on the tangent line is 0.06, and approximating $(1.01)^6$ with 1.06 is reasonable.

39. (a) If x is the edge length, then $V = x^3 \;\Rightarrow\; dV = 3x^2\,dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3. The relative error is calculated by dividing the change in V, ΔV, by V. We approximate ΔV with dV.

$$\text{Relative error} = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{3x^2\,dx}{x^3} = 3\frac{dx}{x} = 3\left(\frac{0.1}{30}\right) = 0.01.$$

Percentage error = relative error $\times\, 100\% = 0.01 \times 100\% = 1\%$.

(b) $S = 6x^2 \;\Rightarrow\; dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2.

$$\text{Relative error} = \frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{12x\,dx}{6x^2} = 2\frac{dx}{x} = 2\left(\frac{0.1}{30}\right) = 0.00\overline{6}.$$

Percentage error = relative error $\times\, 100\% = 0.00\overline{6} \times 100\% = 0.\overline{6}\%$.

40. (a) $A = \pi r^2 \;\Rightarrow\; dA = 2\pi r\,dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30$ cm^2.

(b) $$\text{Relative error} = \frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r\,dr}{\pi r^2} = \frac{2\,dr}{r} = \frac{2(0.2)}{24} = \frac{0.2}{12} = \frac{1}{60} = 0.01\overline{6}.$$

Percentage error = relative error $\times\, 100\% = 0.01\overline{6} \times 100\% = 1.\overline{6}\%$.

41. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so $r = C/(2\pi) \;\Rightarrow\; S = 4\pi(C/2\pi)^2 = C^2/\pi \;\Rightarrow\; dS = (2/\pi)C\,dC$. When $C = 84$ and $dC = 0.5$, $dS = \frac{2}{\pi}(84)(0.5) = \frac{84}{\pi}$, so the maximum error is about $\frac{84}{\pi} \approx 27$ cm^2. Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012$

(b) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{C}{2\pi}\right)^3 = \frac{C^3}{6\pi^2} \;\Rightarrow\; dV = \frac{1}{2\pi^2}C^2\,dC$. When $C = 84$ and $dC = 0.5$, $dV = \frac{1}{2\pi^2}(84)^2(0.5) = \frac{1764}{\pi^2}$, so the maximum error is about $\frac{1764}{\pi^2} \approx 179$ cm^3. The relative error is approximately $\dfrac{dV}{V} = \dfrac{1764/\pi^2}{(84)^3/(6\pi^2)} = \dfrac{1}{56} \approx 0.018$.

42. For a hemispherical dome, $V = \frac{2}{3}\pi r^3 \;\Rightarrow\; dV = 2\pi r^2\,dr$. When $r = \frac{1}{2}(50) = 25$ m and $dr = 0.05$ cm $= 0.0005$ m, $dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8} \approx 2$ m^3.

43. (a) $V = \pi r^2 h \;\Rightarrow\; \Delta V \approx dV = 2\pi r h\,dr = 2\pi r h\,\Delta r$

(b) The error is

$$\begin{aligned}\Delta V - dV &= [\pi(r + \Delta r)^2 h - \pi r^2 h] - 2\pi r h\,\Delta r = \pi r^2 h + 2\pi r h\,\Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi r h\,\Delta r \\ &= \pi(\Delta r)^2 h\end{aligned}$$

44. $F = kR^4 \;\Rightarrow\; dF = 4kR^3\,dR \;\Rightarrow\; \dfrac{dF}{F} = \dfrac{4kR^3\,dR}{kR^4} = 4\left(\dfrac{dR}{R}\right)$. Thus, the relative change in F is about 4 times the relative change in R. So a 5% increase in the radius corresponds to a 20% increase in blood flow.

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45. (a) $dc = \dfrac{dc}{dx}\,dx = 0\,dx = 0$

(b) $d(cu) = \dfrac{d}{dx}(cu)\,dx = c\,\dfrac{du}{dx}\,dx = c\,du$

(c) $d(u+v) = \dfrac{d}{dx}(u+v)\,dx = \left(\dfrac{du}{dx} + \dfrac{dv}{dx}\right)dx = \dfrac{du}{dx}\,dx + \dfrac{dv}{dx}\,dx = du + dv$

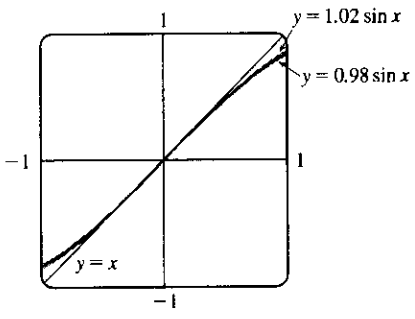
(d) $d(uv) = \dfrac{d}{dx}(uv)\,dx = \left(u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}\right)dx = u\,\dfrac{dv}{dx}\,dx + v\,\dfrac{du}{dx}\,dx = u\,dv + v\,du$

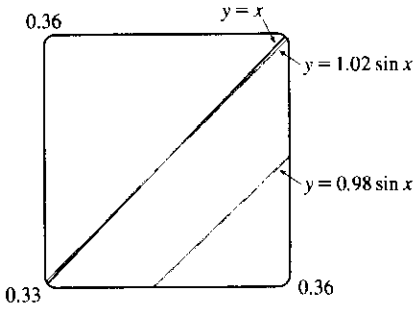
(e) $d\left(\dfrac{u}{v}\right) = \dfrac{d}{dx}\left(\dfrac{u}{v}\right)dx = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}\,dx = \dfrac{v\,\dfrac{du}{dx}\,dx - u\,\dfrac{dv}{dx}\,dx}{v^2} = \dfrac{v\,du - u\,dv}{v^2}$

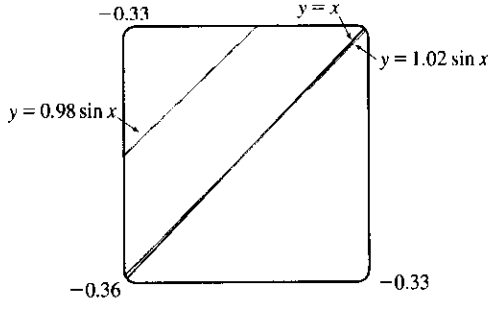
(f) $d\,(x^n) = \dfrac{d}{dx}\,(x^n)\,dx = nx^{n-1}\,dx$

46. (a) $f(x) = \sin x \;\Rightarrow\; f'(x) = \cos x$, so $f(0) = 0$ and $f'(0) = 1$.
Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 0 + 1(x - 0) = x$.

(b)







We want to know the values of x for which $y = x$ approximates $y = \sin x$ with less than a 2% difference; that is, the values of x for which

$$\left|\frac{x - \sin x}{\sin x}\right| < 0.02 \quad\Leftrightarrow\quad -0.02 < \frac{x - \sin x}{\sin x} < 0.02 \quad\Leftrightarrow$$

$$\begin{cases} -0.02\sin x < x - \sin x < 0.02\sin x & \text{if } \sin x > 0 \\ -0.02\sin x > x - \sin x > 0.02\sin x & \text{if } \sin x < 0 \end{cases} \quad\Leftrightarrow\quad \begin{cases} 0.98\sin x < x < 1.02\sin x & \text{if } \sin x > 0 \\ 1.02\sin x < x < 0.98\sin x & \text{if } \sin x < 0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $x = 0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y = x$ intersects $y = 1.02\sin x$ at $x \approx 0.344$. By symmetry, they also intersect at $x \approx -0.344$ (see the third figure.). Converting 0.344 radians to degrees, we get $0.344\left(\frac{180^\circ}{\pi}\right) \approx 19.7^\circ \approx 20^\circ$, which verifies the statement.

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47. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x - 1) = 5 + 2(x - 1) = 2x + 3$. $f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

48. (a) $g'(x) = \sqrt{x^2 + 5} \Rightarrow g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95 - 2) = -4 + 3(-0.05) = -4.15$. $g(2.05) \approx g(2) + g'(2)(2.05 - 2) = -4 + 3(0.05) = -3.85$.

(b) The formula $g'(x) = \sqrt{x^2 + 5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g. Hence, the estimates in part (a) are too small.

LABORATORY PROJECT Taylor Polynomials

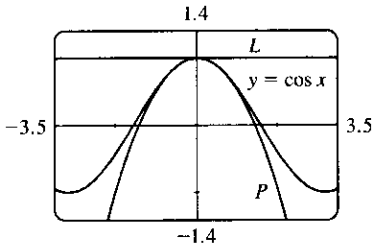
1. We first write the functions described in conditions (i), (ii), and (iii):

$$\begin{aligned} P(x) &= A + Bx + Cx^2 & f(x) &= \cos x \\ P'(x) &= B + 2Cx & f'(x) &= -\sin x \\ P''(x) &= 2C & f''(x) &= -\cos x \end{aligned}$$

So, taking $a = 0$, our three conditions become

$$\begin{aligned} P(0) = f(0): &\quad A = \cos 0 = 1 \\ P'(0) = f'(0): &\quad B = -\sin 0 = 0 \\ P''(0) = f''(0): &\quad 2C = -\cos 0 = -1 \Rightarrow C = -\tfrac{1}{2} \end{aligned}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.

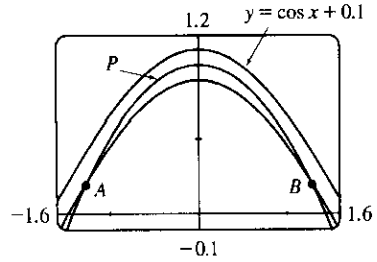


The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $\left|\cos x - \left(1 - \frac{1}{2}x^2\right)\right| < 0.1 \Leftrightarrow$

$-0.1 < \cos x - \left(1 - \frac{1}{2}x^2\right) < 0.1 \Leftrightarrow 0.1 > \left(1 - \frac{1}{2}x^2\right) - \cos x > -0.1 \Leftrightarrow$

$\cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \Leftrightarrow \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1$.



From the figure we see that this is true between A and B. Zooming in or using an intersect feature, we find that the x-coordinates of B and A are about ± 1.26. Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

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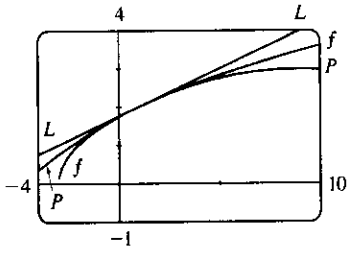
3. If $P(x) = A + B(x-a) + C(x-a)^2$, then $P'(x) = B + 2C(x-a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$\begin{aligned} P(a) = f(a): &\quad A = f(a) \\ P'(a) = f'(a): &\quad B = f'(a) \\ P''(a) = f''(a): &\quad 2C = f''(a) \quad \Rightarrow \quad C = \tfrac{1}{2}f''(a) \end{aligned}$$

Thus, $P(x) = A + B(x-a) + C(x-a)^2$ can be written in the form $P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

4. From Example 2 in Section 3.10, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and $f'(x) = \frac{1}{2}(x+3)^{-1/2}$. So $f''(x) = -\frac{1}{4}(x+3)^{-3/2} \Rightarrow f''(1) = -\frac{1}{32}$. From Problem 3, the quadratic approximation $P(x)$ is

$$\begin{aligned} \sqrt{x+3} &\approx f(1) + f'(1)(x-1) + \tfrac{1}{2}f''(1)(x-1)^2 \\ &= 2 + \tfrac{1}{4}(x-1) - \tfrac{1}{64}(x-1)^2 \end{aligned}$$



The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373...	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118...	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015...	2.04937500

5. $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$. If we put $x = a$ in this equation, then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain $T_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1}$. Substituting $x = a$ gives $T_n'(a) = c_1$. Differentiating again, we have $T_n''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a^2) + \cdots + (n-1)nc_n(x-a)^{n-2}$ and so $T_n''(a) = 2c_2$. Continuing in this manner, we get $T_n'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3}$ and $T_n'''(a) = 2 \cdot 3c_3$.

By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain $T_n^{(4)}(a) = 2 \cdot 3 \cdot 4c_4$ and in general, for any integer k between 1 and n,

$$T_n^{(k)}(a) = 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot kc_k = k!\,c_k \quad \Rightarrow \quad c_k = \frac{T_n^{(k)}(a)}{k!}$$

Because we want T_n and f to have the same derivatives at a, we require that $c_k = \dfrac{f^{(k)}(a)}{k!}$ for $k = 1, 2, \ldots, n$.

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6. $T_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

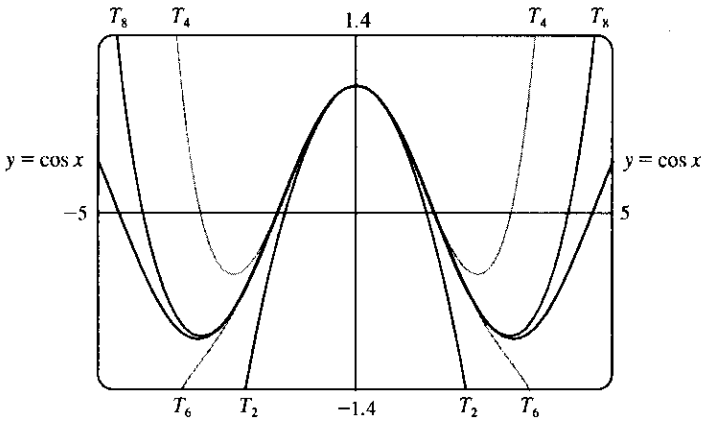
$$\begin{aligned} f(x) &= \cos x & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \end{aligned}$$

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$. From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

$$\begin{aligned} T_8(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8 \\ &= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 + \frac{-1}{6!}x^6 + 0 \cdot x^7 + \frac{1}{8!}x^8 \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \end{aligned}$$

and the desired approximation is $\cos x \approx 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!}$. The Taylor polynomials T_2, T_4, and T_6 consist of the initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \dfrac{x^2}{2!}$, $T_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$, and $T_6(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$.

We graph T_2, T_4, T_6, T_8, and f:



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

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3 Review

CONCEPT CHECK

1. See Definition 3.1.2 and the subsequent discussions on the interpretations of the derivative as the slope of a tangent and as a rate of change.

2. (a) A function f is differentiable at a number a if its derivative f' exists at $x = a$; that is, if $f'(a)$ exists.

(b) See Theorem 3.2.4. This theorem also tells us that if f is *not* continuous at a, then f is *not* differentiable at a.

3. (a) The Power Rule: If n is any real number, then $\dfrac{d}{dx}(x^n) = nx^{n-1}$. The derivative of a variable base raised to a constant power is the power times the base raised to the power minus one.

(b) The Constant Multiple Rule: If c is a constant and f is a differentiable function, then $\dfrac{d}{dx}[cf(x)] = c\dfrac{d}{dx}f(x)$. The derivative of a constant times a function is the constant times the derivative of the function.

(c) The Sum Rule: If f and g are both differentiable, then $\dfrac{d}{dx}[f(x) + g(x)] = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$. The derivative of a sum of functions is the sum of the derivatives.

(d) The Difference Rule: If f and g are both differentiable, then $\dfrac{d}{dx}[f(x) - g(x)] = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$. The derivative of a difference of functions is the difference of the derivatives.

(e) The Product Rule: If f and g are both differentiable, then $\dfrac{d}{dx}[f(x)g(x)] = f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x)$. The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

(f) The Quotient Rule: If f and g are both differentiable, then $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\dfrac{d}{dx}f(x) - f(x)\dfrac{d}{dx}g(x)}{[g(x)]^2}$. The derivative of a quotient of functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

(g) The Chain Rule: If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x))g'(x)$. The derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

4. (a) $y = x^n \Rightarrow y' = nx^{n-1}$ (b) $y = \sin x \Rightarrow y' = \cos x$

(c) $y = \cos x \Rightarrow y' = -\sin x$ (d) $y = \tan x \Rightarrow y' = \sec^2 x$

(e) $y = \csc x \Rightarrow y' = -\csc x \cot x$ (f) $y = \sec x \Rightarrow y' = \sec x \tan x$

(g) $y = \cot x \Rightarrow y' = -\csc^2 x$

5. Implicit differentiation consists of differentiating both sides of an equation involving x and y with respect to x, and then solving the resulting equation for y'.

6. The second derivative of a function f is the rate of change of the first derivative f'. The third derivative is the derivative (rate of change) of the second derivative. If f is the position function of an object, f' is its velocity function, f'' is its acceleration function, and f''' is its jerk function.

7. (a) The linearization L of f at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$.

(b) If $y = f(x)$, then the differential dy is given by $dy = f'(x)\,dx$.

(c) See Figure 6 in Section 3.10.

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TRUE-FALSE QUIZ

1. False. See the warning after Theorem 3.2.4.

2. True. This is the Sum Rule.

3. False. See the warning before the Product Rule.

4. True. This is the Chain Rule.

5. True by the Chain Rule.

6. False. $\dfrac{d}{dx}f(\sqrt{x}\,) = \dfrac{f'(\sqrt{x}\,)}{2\sqrt{x}}$ by the Chain Rule.

7. False. $f(x) = |x^2 + x| = x^2 + x$ for $x \geq 0$ or $x \leq -1$ and $|x^2 + x| = -(x^2 + x)$ for $-1 < x < 0$. So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \geq -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.

8. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim\limits_{x \to r} f(x) = f(r)$.

9. True. $g(x) = x^5 \Rightarrow g'(x) = 5x^4 \Rightarrow g'(2) = 5(2)^4 = 80$, and by the definition of the derivative, $\lim\limits_{x \to 2} \dfrac{g(x) - g(2)}{x - 2} = g'(2) = 80$.

10. False. $\dfrac{d^2y}{dx^2}$ is the second derivative while $\left(\dfrac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\dfrac{d^2y}{dx^2} = 0$, but $\left(\dfrac{dy}{dx}\right)^2 = 1$.

11. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x + 2)$. [The given equation, $y - 4 = 2x(x + 2)$, is not even linear!]

12. True. $\dfrac{d}{dx}\left(\tan^2 x\right) = 2 \tan x \sec^2 x$, and $\dfrac{d}{dx}\left(\sec^2 x\right) = 2 \sec x\,(\sec x \tan x) = 2 \tan x \sec^2 x$.

EXERCISES

1. Estimating the slopes of the tangent lines at $x = 2$, 3, and 5, we obtain approximate values 0.4, 2, and 0.1. The slope of the tangent line at $x = 7$ is negative, so $f'(7) < 0$. Arranging the numbers in increasing order, we have: $f'(7) < 0 < f'(5) < f'(2) < 1 < f'(3)$.

2. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.

3. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

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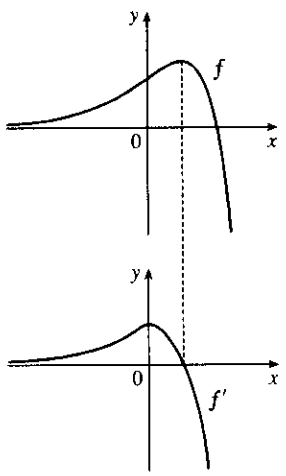
For Exercises 4–6, see the hints before Exercise 5 in Section 3.2.

4.



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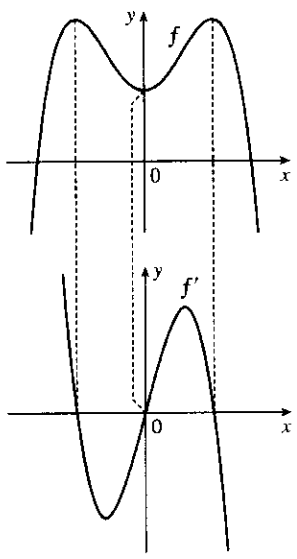

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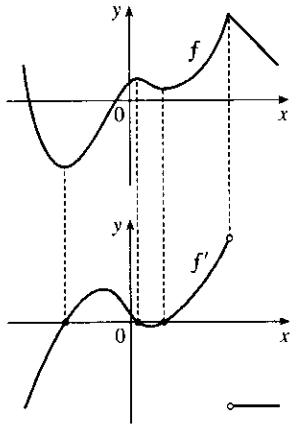

7. The graph of a has tangent lines with positive slope for $x < 0$ and negative slope for $x > 0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a. The graph of c has horizontal tangent lines to the left and right of the x-axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c. Therefore, a is the graph of f, c is the graph of f', and b is the graph of f''.

8. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$, $F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

(c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

9. $B'(1990)$ is the rate at which the total value of U.S. banknotes in circulation is changing in billions of dollars per year. To estimate the value of $B'(1990)$, we will average the difference quotients obtained using the times $t = 1985$ and $t = 1995$. Let $A = \dfrac{B(1985) - B(1990)}{1985 - 1990} = \dfrac{182.0 - 268.2}{-5} = 17.24$ and

$C = \dfrac{B(1995) - B(1990)}{1995 - 1990} = \dfrac{401.5 - 268.2}{5} = 26.66$. Then

$$B'(1990) = \lim_{t \to 1990} \frac{B(t) - B(1990)}{t - 1990} \approx \frac{A + C}{2} = \frac{17.24 + 26.66}{2} = 21.95 \text{ billions of dollars/year.}$$

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10. $f(x) = \dfrac{4-x}{3+x} \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{4-(x+h)}{3+(x+h)} - \dfrac{4-x}{3+x}}{h}$$
$$= \lim_{h\to 0} \frac{(4-x-h)(3+x)-(4-x)(3+x+h)}{h(3+x+h)(3+x)} = \lim_{h\to 0} \frac{-7h}{h(3+x+h)(3+x)}$$
$$= \lim_{h\to 0} \frac{-7}{(3+x+h)(3+x)} = -\frac{7}{(3+x)^2}$$

11. $f(x) = x^3 + 5x + 4 \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3+5(x+h)+4-(x^3+5x+4)}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3+5h}{h} = \lim_{h\to 0}(3x^2+3xh+h^2+5) = 3x^2+5$$

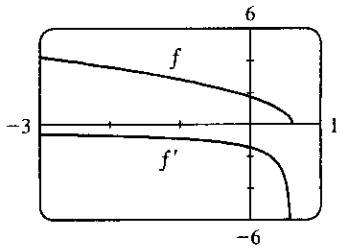
12. (a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{3-5(x+h)}-\sqrt{3-5x}}{h} \cdot \dfrac{\sqrt{3-5(x+h)}+\sqrt{3-5x}}{\sqrt{3-5(x+h)}+\sqrt{3-5x}}$

$$= \lim_{h\to 0} \frac{[3-5(x+h)]-(3-5x)}{h\left(\sqrt{3-5(x+h)}+\sqrt{3-5x}\right)} = \lim_{h\to 0} \frac{-5}{\sqrt{3-5(x+h)}+\sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

(b) Domain of f: (the radicand must be nonnegative) $3-5x \geq 0 \Rightarrow 5x \leq 3 \Rightarrow x \in (-\infty, \frac{3}{5}]$

Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero; $x \in (-\infty, \frac{3}{5})$

(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.



13. $y = (x^4 - 3x^2 + 5)^3 \Rightarrow$

$$y' = 3(x^4-3x^2+5)^2 \frac{d}{dx}(x^4-3x^2+5) = 3(x^4-3x^2+5)^2(4x^3-6x) = 6x(x^4-3x^2+5)^2(2x^2-3)$$

14. $y = \cos(\tan x) \Rightarrow y' = -\sin(\tan x)\dfrac{d}{dx}(\tan x) = -\sin(\tan x)(\sec^2 x)$

15. $y = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}} = x^{1/2} + x^{-4/3} \Rightarrow y' = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3} = \dfrac{1}{2\sqrt{x}} - \dfrac{4}{3\sqrt[3]{x^7}}$

16. $y = \dfrac{3x-2}{\sqrt{2x+1}} \Rightarrow$

$$y' = \frac{\sqrt{2x+1}\,(3)-(3x-2)\frac{1}{2}(2x+1)^{-1/2}(2)}{\left(\sqrt{2x+1}\right)^2} \cdot \frac{(2x+1)^{1/2}}{(2x+1)^{1/2}} = \frac{3(2x+1)-(3x-2)}{(2x+1)^{3/2}} = \frac{3x+5}{(2x+1)^{3/2}}$$

17. $y = 2x\sqrt{x^2+1} \Rightarrow$

$$y' = 2x \cdot \tfrac{1}{2}(x^2+1)^{-1/2}(2x) + \sqrt{x^2+1}\,(2) = \frac{2x^2}{\sqrt{x^2+1}} + 2\sqrt{x^2+1} = \frac{2x^2+2(x^2+1)}{\sqrt{x^2+1}} = \frac{2(2x^2+1)}{\sqrt{x^2+1}}$$

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18. $y = (x + 1/x^2)^{\sqrt{7}} \quad \Rightarrow \quad y' = \sqrt{7}\,(x + 1/x^2)^{\sqrt{7}-1}(1 - 2/x^3)$

19. $y = \dfrac{t}{1-t^2} \quad \Rightarrow \quad y' = \dfrac{(1-t^2)(1) - t(-2t)}{(1-t^2)^2} = \dfrac{1 - t^2 + 2t^2}{(1-t^2)^2} = \dfrac{t^2+1}{(1-t^2)^2}$

20. $y = \sin(\cos x) \quad \Rightarrow \quad y' = \cos(\cos x)(-\sin x) = -\sin x \cos(\cos x)$

21. $y = \tan\sqrt{1-x} \quad \Rightarrow \quad y' = (\sec^2\sqrt{1-x})\left(\dfrac{1}{2\sqrt{1-x}}\right)(-1) = -\dfrac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$

22. Using the Reciprocal Rule, $g(x) = \dfrac{1}{f(x)} \quad \Rightarrow \quad g'(x) = -\dfrac{f'(x)}{[f(x)]^2}$, we have $y = \dfrac{1}{\sin(x - \sin x)} \quad \Rightarrow$

$$y' = -\frac{\cos(x - \sin x)(1 - \cos x)}{\sin^2(x - \sin x)}.$$

23. $\dfrac{d}{dx}(xy^4 + x^2y) = \dfrac{d}{dx}(x + 3y) \quad \Rightarrow \quad x \cdot 4y^3y' + y^4 \cdot 1 + x^2 \cdot y' + y \cdot 2x = 1 + 3y' \quad \Rightarrow$

$$y'(4xy^3 + x^2 - 3) = 1 - y^4 - 2xy \quad \Rightarrow \quad y' = \frac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$$

24. $y = \sec(1 + x^2) \quad \Rightarrow \quad y' = 2x\sec(1 + x^2)\tan(1 + x^2)$

25. $y = \dfrac{\sec 2\theta}{1 + \tan 2\theta} \quad \Rightarrow$

$$y' = \frac{(1 + \tan 2\theta)(\sec 2\theta \tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1 + \tan 2\theta)^2} = \frac{2\sec 2\theta\left[(1 + \tan 2\theta)\tan 2\theta - \sec^2 2\theta\right]}{(1 + \tan 2\theta)^2}$$

$$= \frac{2\sec 2\theta\,(\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1 + \tan 2\theta)^2} = \frac{2\sec 2\theta\,(\tan 2\theta - 1)}{(1 + \tan 2\theta)^2} \qquad [1 + \tan^2 x = \sec^2 x]$$

26. $\dfrac{d}{dx}(x^2\cos y + \sin 2y) = \dfrac{d}{dx}(xy) \quad \Rightarrow \quad x^2(-\sin y \cdot y') + (\cos y)(2x) + \cos 2y \cdot 2y' = x \cdot y' + y \cdot 1 \quad \Rightarrow$

$$y'(-x^2\sin y + 2\cos 2y - x) = y - 2x\cos y \quad \Rightarrow \quad y' = \frac{y - 2x\cos y}{2\cos 2y - x^2\sin y - x}$$

27. $y = (1 - x^{-1})^{-1} \quad \Rightarrow$

$$y' = -1(1 - x^{-1})^{-2}[-(-1x^{-2})] = -(1 - 1/x)^{-2}x^{-2} = -((x-1)/x)^{-2}x^{-2} = -(x-1)^{-2}$$

28. $y = (x + \sqrt{x}\,)^{-1/3} \quad \Rightarrow \quad y' = -\frac{1}{3}(x + \sqrt{x}\,)^{-4/3}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

29. $\sin(xy) = x^2 - y \quad \Rightarrow \quad \cos(xy)(xy' + y \cdot 1) = 2x - y' \quad \Rightarrow \quad x\cos(xy)y' + y' = 2x - y\cos(xy) \quad \Rightarrow$

$$y'[x\cos(xy) + 1] = 2x - y\cos(xy) \quad \Rightarrow \quad y' = \frac{2x - y\cos(xy)}{x\cos(xy) + 1}$$

30. $y = \sqrt{\sin\sqrt{x}} \quad \Rightarrow \quad y' = \frac{1}{2}(\sin\sqrt{x}\,)^{-1/2}(\cos\sqrt{x})\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$

31. $y = \cot(3x^2 + 5) \quad \Rightarrow \quad y' = -\csc^2(3x^2 + 5)(6x) = -6x\csc^2(3x^2 + 5)$

32. $y = \dfrac{(x + \lambda)^4}{x^4 + \lambda^4} \quad \Rightarrow \quad y' = \dfrac{(x^4 + \lambda^4)(4)(x + \lambda)^3 - (x + \lambda)^4(4x^3)}{(x^4 + \lambda^4)^2} = \dfrac{4(x + \lambda)^3(\lambda^4 - \lambda x^3)}{(x^4 + \lambda^4)^2}$

33. $y = \sin(\tan\sqrt{1 + x^3}\,) \quad \Rightarrow \quad y' = \cos(\tan\sqrt{1 + x^3}\,)(\sec^2\sqrt{1 + x^3}\,)\left[3x^2/(2\sqrt{1 + x^3}\,)\right]$

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34. $y = (\sin mx)/x \quad \Rightarrow \quad y' = (mx\cos mx - \sin mx)/x^2$

35. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \quad \Rightarrow \quad y' = 2[\tan(\sin\theta)]\cdot\sec^2(\sin\theta)\cdot\cos\theta$

36. $x\tan y = y - 1 \quad \Rightarrow \quad \tan y + (x\sec^2 y)\,y' = y' \quad \Rightarrow \quad y' = \dfrac{\tan y}{1 - x\sec^2 y}$

37. $y = (x\tan x)^{1/5} \quad \Rightarrow \quad y' = \frac{1}{5}(x\tan x)^{-4/5}(\tan x + x\sec^2 x)$

38. $y = \dfrac{(x-1)(x-4)}{(x-2)(x-3)} = \dfrac{x^2-5x+4}{x^2-5x+6} \quad \Rightarrow$

$$y' = \frac{(x^2-5x+6)(2x-5) - (x^2-5x+4)(2x-5)}{(x^2-5x+6)^2} = \frac{2(2x-5)}{(x-2)^2(x-3)^2}$$

39. $f(t) = \sqrt{4t+1} \quad \Rightarrow \quad f'(t) = \frac{1}{2}(4t+1)^{-1/2}\cdot 4 = 2(4t+1)^{-1/2} \quad \Rightarrow$

$f''(t) = 2(-\frac{1}{2})(4t+1)^{-3/2}\cdot 4 = -4/(4t+1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\frac{4}{27}$.

40. $g(\theta) = \theta\sin\theta \quad \Rightarrow \quad g'(\theta) = \theta\cos\theta + \sin\theta\cdot 1 \quad \Rightarrow \quad g''(\theta) = \theta(-\sin\theta) + \cos\theta\cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$,
so $g''(\pi/6) = 2\cos(\pi/6) - (\pi/6)\sin(\pi/6) = 2\left(\sqrt{3}/2\right) - (\pi/6)(1/2) = \sqrt{3} - \pi/12$.

41. $x^6 + y^6 = 1 \quad \Rightarrow \quad 6x^5 + 6y^5y' = 0 \quad \Rightarrow \quad y' = -x^5/y^5 \quad \Rightarrow$

$$y'' = -\frac{y^5(5x^4) - x^5(5y^4y')}{(y^5)^2} = -\frac{5x^4y^4\left[y - x(-x^5/y^5)\right]}{y^{10}} = -\frac{5x^4\left[(y^6+x^6)/y^5\right]}{y^6} = -\frac{5x^4}{y^{11}}$$

42. $f(x) = (2-x)^{-1} \quad \Rightarrow \quad f'(x) = (2-x)^{-2} \quad \Rightarrow \quad f''(x) = 2(2-x)^{-3} \quad \Rightarrow \quad f'''(x) = 2\cdot 3(2-x)^{-4} \quad \Rightarrow$

$f^{(4)}(x) = 2\cdot 3\cdot 4(2-x)^{-5}$. In general, $f^{(n)}(x) = 2\cdot 3\cdot 4\cdot\cdots\cdot n(2-x)^{-(n+1)} = \dfrac{n!}{(2-x)^{(n+1)}}$.

43. $\lim\limits_{x\to 0}\dfrac{\sec x}{1-\sin x} = \dfrac{\sec 0}{1-\sin 0} = \dfrac{1}{1-0} = 1$

44. $\lim\limits_{t\to 0}\dfrac{t^3}{\tan^3 2t} = \lim\limits_{t\to 0}\dfrac{t^3\cos^3 2t}{\sin^3 2t} = \lim\limits_{t\to 0}\cos^3 2t\cdot\dfrac{1}{8\dfrac{\sin^3 2t}{(2t)^3}} = \lim\limits_{t\to 0}\dfrac{\cos^3 2t}{8\left(\lim\limits_{t\to 0}\dfrac{\sin 2t}{2t}\right)^3} = \dfrac{1}{8\cdot 1^3} = \dfrac{1}{8}$

45. $y = 4\sin^2 x \quad \Rightarrow \quad y' = 4\cdot 2\sin x\cos x$. At $\left(\frac{\pi}{6}, 1\right)$, $y' = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line is $y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$, or $y = 2\sqrt{3}\,x + 1 - \pi\sqrt{3}/3$.

46. $y = \dfrac{x^2-1}{x^2+1} \quad \Rightarrow \quad y' = \dfrac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$. At $(0,-1)$, $y' = 0$, so an equation of the tangent line is $y + 1 = 0(x-0)$, or $y = -1$.

47. $y = \sqrt{1+4\sin x} \quad \Rightarrow \quad y' = \frac{1}{2}(1+4\sin x)^{-1/2}\cdot 4\cos x = \dfrac{2\cos x}{\sqrt{1+4\sin x}}$. At $(0,1)$, $y' = \dfrac{2}{\sqrt{1}} = 2$, so an equation of the tangent line is $y - 1 = 2(x-0)$, or $y = 2x + 1$.

48. $x^2 + 4xy + y^2 = 13 \quad \Rightarrow \quad 2x + 4(xy' + y\cdot 1) + 2yy' = 0 \quad \Rightarrow \quad x + 2xy' + 2y + yy' = 0 \quad \Rightarrow$

$2xy' + yy' = -x - 2y \quad \Rightarrow \quad y'(2x+y) = -x - 2y \quad \Rightarrow \quad y' = \dfrac{-x-2y}{2x+y}$. At $(2,1)$, $y' = \dfrac{-2-2}{4+1} = -\dfrac{4}{5}$, so an equation of the tangent line is $y - 1 = -\frac{4}{5}(x-2)$, or $y = -\frac{4}{5}x + \frac{13}{5}$.

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49. (a) $f(x) = x\sqrt{5-x} \;\Rightarrow$

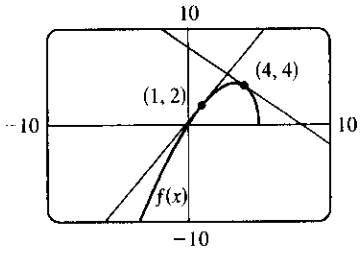
$$f'(x) = x\left[\tfrac{1}{2}(5-x)^{-1/2}(-1)\right] + \sqrt{5-x} = \frac{-x}{2\sqrt{5-x}} + \sqrt{5-x}\cdot\frac{2\sqrt{5-x}}{2\sqrt{5-x}}$$

$$= \frac{-x}{2\sqrt{5-x}} + \frac{2(5-x)}{2\sqrt{5-x}} = \frac{-x+10-2x}{2\sqrt{5-x}} = \frac{10-3x}{2\sqrt{5-x}}$$

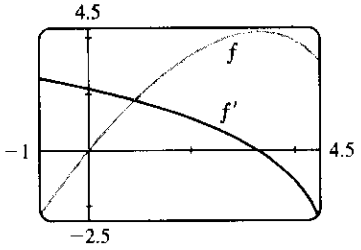
(b) At $(1,2)$: $f'(1) = \frac{7}{4}$. So an equation of the tangent line is $y - 2 = \frac{7}{4}(x-1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

At $(4,4)$: $f'(4) = -\frac{2}{2} = -1$. So an equation of the tangent line is $y - 4 = -1(x-4)$ or $y = -x + 8$.

(c)



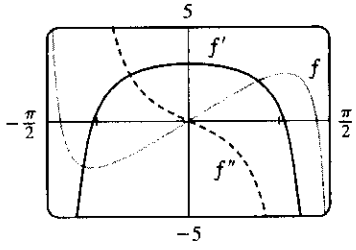
(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

50. (a) $f(x) = 4x - \tan x \;\Rightarrow\; f'(x) = 4 - \sec^2 x \;\Rightarrow\; f''(x) = -2\sec x(\sec x \tan x) = -2\sec^2 x \tan x$.

(b)



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f''.

51. $y = \sin x + \cos x \;\Rightarrow\; y' = \cos x - \sin x = 0 \;\Leftrightarrow\; \cos x = \sin x$ and $0 \le x \le 2\pi \;\Leftrightarrow\; x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

52. $x^2 + 2y^2 = 1 \;\Rightarrow\; 2x + 4yy' = 0 \;\Rightarrow\; y' = -x/(2y) = 1 \;\Leftrightarrow\; x = -2y$. Since the points lie on the ellipse, we have $(-2y)^2 + 2y^2 = 1 \;\Rightarrow\; 6y^2 = 1 \;\Rightarrow\; y = \pm\frac{1}{\sqrt{6}}$. The points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

53. $f(x) = (x-a)(x-b)(x-c) \;\Rightarrow\; f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$. So

$$\frac{f'(x)}{f(x)} = \frac{(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)}{(x-a)(x-b)(x-c)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}.$$

54. (a) $\cos 2x = \cos^2 x - \sin^2 x \;\Rightarrow\; -2\sin 2x = -2\cos x \sin x - 2\sin x \cos x \;\Leftrightarrow\; \sin 2x = 2\sin x \cos x$

(b) $\sin(x+a) = \sin x \cos a + \cos x \sin a \;\Rightarrow\; \cos(x+a) = \cos x \cos a - \sin x \sin a$.

55. (a) $h(x) = f(x)g(x) \;\Rightarrow\; h'(x) = f(x)g'(x) + g(x)f'(x) \;\Rightarrow$

$h'(2) = f(2)g'(2) + g(2)f'(2) = (3)(4) + (5)(-2) = 12 - 10 = 2$

(b) $F(x) = f(g(x)) \;\Rightarrow\; F'(x) = f'(g(x))g'(x) \;\Rightarrow\; F'(2) = f'(g(2))g'(2) = f'(5)(4) = 11 \cdot 4 = 44$

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56. (a) $P(x) = f(x)g(x) \Rightarrow P'(x) = f(x)g'(x) + g(x)f'(x) \Rightarrow$

$$P'(2) = f(2)g'(2) + g(2)f'(2) = (1)\left(\tfrac{6-0}{3-0}\right) + (4)\left(\tfrac{0-3}{3-0}\right) = (1)(2) + (4)(-1) = 2 - 4 = -2$$

(b) $Q(x) = \dfrac{f(x)}{g(x)} \Rightarrow Q'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \Rightarrow$

$$Q'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(4)(-1) - (1)(2)}{4^2} = \frac{-6}{16} = -\frac{3}{8}$$

(c) $C(x) = f(g(x)) \Rightarrow C'(x) = f'(g(x))g'(x) \Rightarrow$

$$C'(2) = f'(g(2))g'(2) = f'(4)g'(2) = \left(\tfrac{6-0}{5-3}\right)(2) = (3)(2) = 6$$

57. $f(x) = x^2 g(x) \Rightarrow f'(x) = x^2 g'(x) + g(x)(2x) = x\,[xg'(x) + 2g(x)]$

58. $f(x) = g(x^2) \Rightarrow f'(x) = g'(x^2)(2x) = 2xg'(x^2)$

59. $f(x) = [g(x)]^2 \Rightarrow f'(x) = 2[g(x)]^1 \cdot g'(x) = 2g(x)g'(x)$

60. $f(x) = x^a g(x^b) \Rightarrow f'(x) = ax^{a-1}g(x^b) + x^a g'(x^b)(bx^{b-1}) = ax^{a-1}g(x^b) + bx^{a+b-1}g'(x^b)$

61. $f(x) = g(g(x)) \Rightarrow f'(x) = g'(g(x))g'(x)$

62. $f(x) = \sin(g(x)) \Rightarrow f'(x) = \cos(g(x)) \cdot g'(x)$

63. $f(x) = g(\sin x) \Rightarrow f'(x) = g'(\sin x) \cdot \cos x$

64. $f(x) = g(\tan\sqrt{x}) \Rightarrow$

$$f'(x) = g'(\tan\sqrt{x}) \cdot \frac{d}{dx}(\tan\sqrt{x}) = g'(\tan\sqrt{x}) \cdot \sec^2\sqrt{x} \cdot \frac{d}{dx}(\sqrt{x}) = \frac{g'(\tan\sqrt{x})\sec^2\sqrt{x}}{2\sqrt{x}}$$

65. $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)} \Rightarrow$

$$h'(x) = \frac{[f(x)+g(x)]\,[f(x)g'(x) + g(x)f'(x)] - f(x)g(x)\,[f'(x) + g'(x)]}{[f(x)+g(x)]^2}$$

$$= \frac{[f(x)]^2 g'(x) + f(x)g(x)f'(x) + f(x)g(x)g'(x) + [g(x)]^2 f'(x) - f(x)g(x)f'(x) - f(x)g(x)g'(x)}{[f(x)+g(x)]^2}$$

$$= \frac{f'(x)[g(x)]^2 + g'(x)[f(x)]^2}{[f(x)+g(x)]^2}$$

66. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}} \Rightarrow h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{2\sqrt{f(x)/g(x)}\,[g(x)]^2} = \dfrac{f'(x)g(x) - f(x)g'(x)}{2[g(x)]^{3/2}\sqrt{f(x)}}$

67. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \Rightarrow$

$$h'(x) = f'(g(\sin 4x)) \cdot \frac{d}{dx}(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \frac{d}{dx}(\sin 4x)$$

$$= f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4)$$

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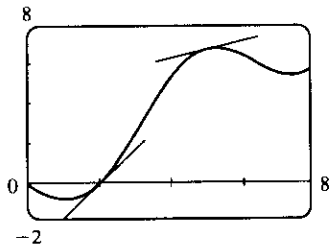
68. (a)



(b) The average rate of change is larger on $[2, 3]$.

(c) The instantaneous rate of change (the slope of the tangent) is larger at $x = 2$.

(d) $f(x) = x - 2\sin x \;\Rightarrow\; f'(x) = 1 - 2\cos x$, so $f'(2) = 1 - 2\cos 2 \approx 1.8323$ and $f'(5) = 1 - 2\cos 5 \approx 0.4327$. So $f'(2) > f'(5)$, as predicted in part (c).

69. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

70. (a) $x = \sqrt{b^2 + c^2t^2} \;\Rightarrow\; v(t) = x' = \left[1/\left(2\sqrt{b^2 + c^2t^2}\right)\right] 2c^2t = c^2t/\sqrt{b^2 + c^2t^2} \;\Rightarrow$

$$a(t) = v'(t) = \frac{c^2\sqrt{b^2 + c^2t^2} - c^2t\left(c^2t/\sqrt{b^2 + c^2t^2}\right)}{b^2 + c^2t^2} = \frac{b^2c^2}{(b^2 + c^2t^2)^{3/2}}$$

(b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

71. (a) $y = t^3 - 12t + 3 \;\Rightarrow\; v(t) = y' = 3t^2 - 12 \;\Rightarrow\; a(t) = v'(t) = 6t$

(b) $v(t) = 3(t^2 - 4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3) - y(2) = -6 - (-13) = 7$,
Distance downward $= y(0) - y(2) = 3 - (-13) = 16$. Total distance $= 7 + 16 = 23$.

72. (a) $V = \frac{1}{3}\pi r^2h \;\Rightarrow\; dV/dh = \frac{1}{3}\pi r^2$ [r constant]

(b) $V = \frac{1}{3}\pi r^2h \;\Rightarrow\; dV/dr = \frac{2}{3}\pi rh$ [h constant]

73. The linear density ρ is the rate of change of mass m with respect to length x. $m = x(1 + \sqrt{x}) = x + x^{3/2} \;\Rightarrow$ $\rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2}\sqrt{4} = 4$ kg/m.

74. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \;\Rightarrow\; C'(x) = 2 - 0.04x + 0.00021x^2$

(b) $C'(100) = 2 - 4 + 2.1 = \$0.10$/unit. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

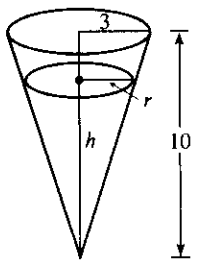
(c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \$0.10107$, slightly larger than $C'(100)$.

75. If $x =$ edge length, then $V = x^3 \;\Rightarrow\; dV/dt = 3x^2\,dx/dt = 10 \;\Rightarrow\; dx/dt = 10/(3x^2)$ and $S = 6x^2 \;\Rightarrow$ $dS/dt = (12x)\,dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}$ cm^2/min.

76. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2h$ and, from similar triangles, $\dfrac{r}{h} = \dfrac{3}{10} \;\Rightarrow\; V = \dfrac{\pi}{3}\left(\dfrac{3h}{10}\right)^2 h = \dfrac{3\pi}{100}h^3$, so

$$2 = \frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt} \;\Rightarrow\; \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi(5)^2} = \frac{8}{9\pi} \text{ cm/s}$$

when $h = 5$.



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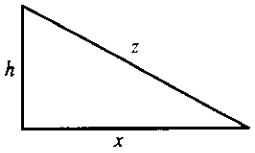
77. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \Rightarrow$

$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 2h\,\dfrac{dh}{dt} \Rightarrow \dfrac{dz}{dt} = \dfrac{1}{z}(15x + 5h)$. When $t = 3$,

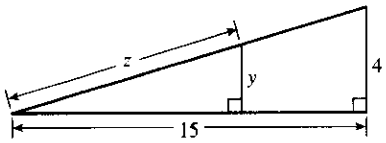
$h = 45 + 3(5) = 60$ and $x = 15(3) = 45 \Rightarrow z = \sqrt{45^2 + 60^2} = 75$, so

$\dfrac{dz}{dt} = \frac{1}{75}\,[15(45) + 5(60)] = 13$ ft/s.



78. We are given $dz/dt = 30$ ft/s. By similar triangles, $\dfrac{y}{z} = \dfrac{4}{\sqrt{241}}$

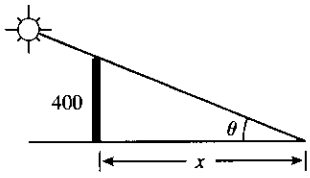
$\Rightarrow \quad y = \dfrac{4}{\sqrt{241}}z$, so $\dfrac{dy}{dt} = \dfrac{4}{\sqrt{241}}\,\dfrac{dz}{dt} = \dfrac{120}{\sqrt{241}} \approx 7.7$ ft/s.



79. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \Rightarrow$

$x = 400\cot\theta \Rightarrow \dfrac{dx}{dt} = -400\csc^2\theta\,\dfrac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$,

$\dfrac{dx}{dt} = -400(2)^2(-0.25) = 400$ ft/h.

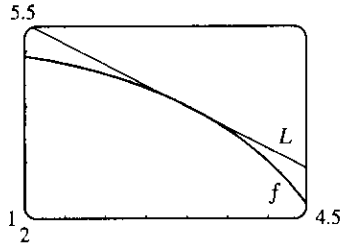


80. (a) $f(x) = \sqrt{25 - x^2} \Rightarrow$

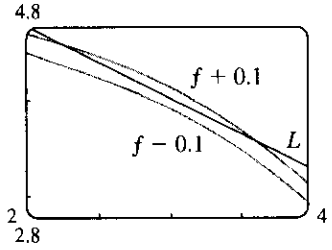
$f'(x) = \dfrac{-2x}{2\sqrt{25 - x^2}} = -x(25 - x^2)^{-1/2}$. So the linear approximation to $f(x)$ near 3 is

$f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3)$.

(b)

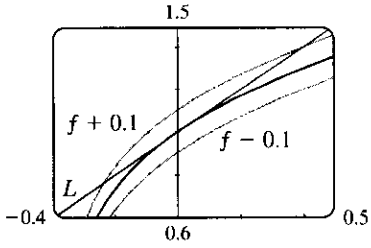


(c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$ and $4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that these both hold for $2.24 < x < 3.66$.



81. (a) $f(x) = \sqrt[3]{1 + 3x} = (1 + 3x)^{1/3} \Rightarrow f'(x) = (1 + 3x)^{-2/3}$, so the linearization of f at $a = 0$ is $L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1 + 3x} \approx 1 + x \Rightarrow$ $\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)} \approx 1 + (0.01) = 1.01$.

(b) The linear approximation is $\sqrt[3]{1 + 3x} \approx 1 + x$, so for the required accuracy we want $\sqrt[3]{1 + 3x} - 0.1 < 1 + x < \sqrt[3]{1 + 3x} + 0.1$. From the graph, it appears that this is true when $-0.23 < x < 0.40$.



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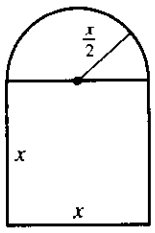
82. $y = x^3 - 2x^2 + 1 \;\Rightarrow\; dy = (3x^2 - 4x)\,dx$. When $x = 2$ and $dx = 0.2$, $dy = [3(2)^2 - 4(2)](0.2) = 0.8$.

83. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2 \;\Rightarrow\; dA = \left(2 + \frac{\pi}{4}\right)x\,dx$.

When $x = 60$ and $dx = 0.1$, $dA = \left(2 + \frac{\pi}{4}\right)60(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is approximately $12 + \frac{3\pi}{2} \approx 16.7\ \text{cm}^2$.



84. $\lim_{x\to 1} \dfrac{x^{17} - 1}{x - 1} = \left[\dfrac{d}{dx}\,x^{17}\right]_{x=1} = 17(1)^{16} = 17$

85. $\lim_{h\to 0} \dfrac{\sqrt[4]{16+h} - 2}{h} = \left[\dfrac{d}{dx}\,\sqrt[4]{x}\right]_{x=16} = \left.\frac{1}{4}x^{-3/4}\right|_{x=16} = \dfrac{1}{4\left(\sqrt[4]{16}\right)^3} = \dfrac{1}{32}$

86. $\lim_{\theta\to\pi/3} \dfrac{\cos\theta - 0.5}{\theta - \pi/3} = \left[\dfrac{d}{d\theta}\cos\theta\right]_{\theta=\pi/3} = -\sin\dfrac{\pi}{3} = -\dfrac{\sqrt{3}}{2}$

87.
$$\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} = \lim_{x\to 0} \frac{\left(\sqrt{1+\tan x} - \sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)}$$

$$= \lim_{x\to 0} \frac{(1+\tan x) - (1+\sin x)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)} = \lim_{x\to 0} \frac{\sin x\,(1/\cos x - 1)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)} \cdot \frac{\cos x}{\cos x}$$

$$= \lim_{x\to 0} \frac{\sin x\,(1-\cos x)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)\cos x} \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0} \frac{\sin x \cdot \sin^2 x}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$

$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^3 \lim_{x\to 0} \frac{1}{\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$

$$= 1^3 \cdot \frac{1}{\left(\sqrt{1}+\sqrt{1}\right)\cdot 1\cdot(1+1)} = \frac{1}{4}$$

88. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain $f(g(x)) = x \;\Rightarrow\; f'(g(x))g'(x) = 1 \;\Rightarrow\; g'(x) = \dfrac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression gives $g'(x) = \dfrac{1}{1 + [f(g(x))]^2}$. But the first given equation states that $f(g(x)) = x$, so $g'(x) = \dfrac{1}{1+x^2}$.

89. $\dfrac{d}{dx}\,[f(2x)] = x^2 \;\Rightarrow\; f'(2x)\cdot 2 = x^2 \;\Rightarrow\; f'(2x) = \frac{1}{2}x^2$. Let $t = 2x$. Then $f'(t) = \frac{1}{2}\left(\frac{1}{2}t\right)^2 = \frac{1}{8}t^2$, so $f'(x) = \frac{1}{8}x^2$.

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90. Let (b, c) be on the curve, that is, $b^{2/3} + c^{2/3} = a^{2/3}$. Now $x^{2/3} + y^{2/3} = a^{2/3} \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\left(\frac{y}{x}\right)^{1/3}$, so at (b, c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is $y - c = -(c/b)^{1/3}(x - b)$ or $y = -(c/b)^{1/3}x + (c + b^{2/3}c^{1/3})$. Setting $y = 0$, we find that the x-intercept is $b^{1/3}c^{2/3} + b = b^{1/3}(c^{2/3} + b^{2/3})$ and setting $x = 0$ we find that the y-intercept is $c + b^{2/3}c^{1/3} = c^{1/3}(c^{2/3} + b^{2/3})$. So the length of the tangent line between these two points is

$$\begin{aligned}\sqrt{\left[b^{1/3}(c^{2/3} + b^{2/3})\right]^2 + \left[c^{1/3}(c^{2/3} + b^{2/3})\right]^2} &= \sqrt{b^{2/3}(a^{2/3})^2 + c^{2/3}(a^{2/3})^2} \\ &= \sqrt{(b^{2/3} + c^{2/3})a^{4/3}} = \sqrt{a^{2/3}a^{4/3}} \\ &= \sqrt{a^2} = a = \text{constant}\end{aligned}$$

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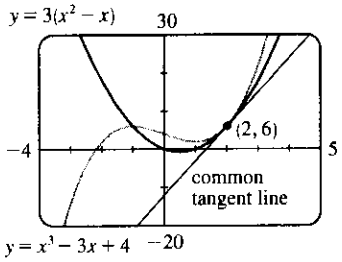
PROBLEMS PLUS

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3}$ $\Rightarrow$ $a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

2. $y = x^3 - 3x + 4$ $\Rightarrow$ $y' = 3x^2 - 3$, and $y = 3(x^2 - x)$ $\Rightarrow$ $y' = 6x - 3$. The slopes of the tangents of the two curves are equal when $3x^2 - 3 = 6x - 3$; that is, when $x = 0$ or 2. At $x = 0$, both tangents have slope -3, but the curves do not intersect. At $x = 2$, both tangents have slope 9 and the curves intersect at $(2, 6)$. So there is a common tangent line at $(2, 6)$, $y = 9x - 12$.



3. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0 + 0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2$ $\Rightarrow$ $f(0) = 2f(0)$. Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

(b) $$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{\left[f(0) + f(h) + 0^2 h + 0h^2\right] - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{f(h)}{h} = \lim_{x \to 0} \frac{f(x)}{x} = 1$$

(c) $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[f(x) + f(h) + x^2 h + xh^2\right] - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{f(h) + x^2 h + xh^2}{h} = \lim_{h \to 0} \left[\frac{f(h)}{h} + x^2 + xh\right] = 1 + x^2$$

4. We find the equation of the parabola by substituting the point $(-100, 100)$, at which the car is situated, into the general equation $y = ax^2$: $100 = a(-100)^2$ $\Rightarrow$ $a = \frac{1}{100}$. Now we find the equation of a tangent to the parabola at the point (x_0, y_0). We can show that $y' = a(2x) = \frac{1}{100}(2x) = \frac{1}{50}x$, so an equation of the tangent is $y - y_0 = \frac{1}{50}x_0(x - x_0)$. Since the point (x_0, y_0) is on the parabola, we must have $y_0 = \frac{1}{100}x_0^2$, so our equation of the tangent can be simplified to $y = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(x - x_0)$. We want the statue to be located on the tangent line, so we substitute its coordinates $(100, 50)$ into this equation: $50 = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(100 - x_0)$ $\Rightarrow$ $x_0^2 - 200x_0 + 5000 = 0$ $\Rightarrow$ $x_0 = \frac{1}{2}\left[200 \pm \sqrt{200^2 - 4\,(5000)}\right]$ $\Rightarrow$ $x_0 = 100 \pm 50\sqrt{2}$. But $x_0 < 100$, so the car's headlights illuminate the statue when it is located at the point $\left(100 - 50\sqrt{2}, 150 - 100\sqrt{2}\right) \approx (29.3, 8.6)$, that is, about 29.3 m east and 8.6 m north of the origin.

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5. We use mathematical induction. Let S_n be the statement that $\dfrac{d^n}{dx^n}\left(\sin^4 x+\cos^4 x\right)=4^{n-1}\cos(4x+n\pi/2)$.

S_1 is true because

$$\begin{aligned}\frac{d}{dx}\left(\sin^4 x+\cos^4 x\right)&=4\sin^3 x\cos x-4\cos^3 x\sin x=4\sin x\cos x\left(\sin^2 x-\cos^2 x\right)\\&=-4\sin x\cos x\cos 2x=-2\sin 2x\cos 2x=-\sin 4x=\sin(-4x)\\&=\cos\left(\tfrac{\pi}{2}-(-4x)\right)=\cos\left(\tfrac{\pi}{2}+4x\right)=4^{n-1}\cos\left(4x+n\tfrac{\pi}{2}\right)\ \text{when } n=1\end{aligned}$$

Now assume S_k is true, that is, $\dfrac{d^k}{dx^k}\left(\sin^4 x+\cos^4 x\right)=4^{k-1}\cos\left(4x+k\tfrac{\pi}{2}\right)$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\left(\sin^4 x+\cos^4 x\right)&=\frac{d}{dx}\left[\frac{d^k}{dx^k}\left(\sin^4 x+\cos^4 x\right)\right]=\frac{d}{dx}\left[4^{k-1}\cos\left(4x+k\tfrac{\pi}{2}\right)\right]\\&=-4^{k-1}\sin\left(4x+k\tfrac{\pi}{2}\right)\cdot\frac{d}{dx}\left(4x+k\tfrac{\pi}{2}\right)=-4^k\sin\left(4x+k\tfrac{\pi}{2}\right)\\&=4^k\sin\left(-4x-k\tfrac{\pi}{2}\right)=4^k\cos\left(\tfrac{\pi}{2}-\left(-4x-k\tfrac{\pi}{2}\right)\right)\\&=4^k\cos\left(4x+(k+1)\tfrac{\pi}{2}\right)\end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\dfrac{d^n}{dx^n}\left(\sin^4 x+\cos^4 x\right)=4^{n-1}\cos\left(4x+n\tfrac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another proof: First write

$\sin^4 x+\cos^4 x=\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x=1-\frac{1}{2}\sin^2 2x=1-\frac{1}{4}(1-\cos 4x)=\frac{3}{4}+\frac{1}{4}\cos 4x$.

Then we have $\dfrac{d^n}{dx^n}\left(\sin^4 x+\cos^4 x\right)=\dfrac{d^n}{dx^n}\left(\frac{3}{4}+\frac{1}{4}\cos 4x\right)=\frac{1}{4}\cdot 4^n\cos\left(4x+n\tfrac{\pi}{2}\right)=4^{n-1}\cos\left(4x+n\tfrac{\pi}{2}\right)$.

6. If we divide $1-x$ into x^n by long division, we find that $f(x)=\dfrac{x^n}{1-x}=-x^{n-1}-x^{n-2}-\cdots-x-1+\dfrac{1}{1-x}$.

This can also be seen by multiplying the last expression by $1-x$ and canceling terms on the right-hand side. So we let $g(x)=1+x+x^2+\cdots+x^{n-1}$, so that $f(x)=\dfrac{1}{1-x}-g(x)\quad\Rightarrow\quad f^{(n)}(x)=\left(\dfrac{1}{1-x}\right)^{(n)}-g^{(n)}(x)$. But g is a polynomial of degree $(n-1)$, so its nth derivative is 0, and therefore $f^{(n)}(x)=\left(\dfrac{1}{1-x}\right)^{(n)}$. Now

$\dfrac{d}{dx}(1-x)^{-1}=(-1)(1-x)^{-2}(-1)=(1-x)^{-2}$, $\dfrac{d^2}{dx^2}(1-x)^{-1}=(-2)(1-x)^{-3}(-1)=2(1-x)^{-3}$,

$\dfrac{d^3}{dx^3}(1-x)^{-1}=(-3)\cdot 2(1-x)^{-4}(-1)=3\cdot 2(1-x)^{-4}$, $\dfrac{d^4}{dx^4}(1-x)^{-1}=4\cdot 3\cdot 2(1-x)^{-5}$, and so on. So after n differentiations, we will have $f^{(n)}(x)=\left(\dfrac{1}{1-x}\right)^{(n)}=\dfrac{n!}{(1-x)^{n+1}}$.

7. We must find a value x_0 such that the normal lines to the parabola $y=x^2$ at $x=\pm x_0$ intersect at a point one unit from the points $(\pm x_0, x_0^2)$. The normals to $y=x^2$ at $x=\pm x_0$ have slopes $-\dfrac{1}{\pm 2x_0}$ and pass through $(\pm x_0, x_0^2)$ respectively, so the normals have the equations $y-x_0^2=-\dfrac{1}{2x_0}(x-x_0)$ and $y-x_0^2=\dfrac{1}{2x_0}(x+x_0)$. The

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common y-intercept is $x_0^2 + \frac{1}{2}$. We want to find the value of x_0 for which the distance from $\left(0, x_0^2 + \frac{1}{2}\right)$ to $\left(x_0, x_0^2\right)$ equals 1. The square of the distance is $(x_0 - 0)^2 + \left[x_0^2 - \left(x_0^2 + \frac{1}{2}\right)\right]^2 = x_0^2 + \frac{1}{4} = 1 \quad \Leftrightarrow \quad x_0 = \pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $\left(0, \frac{5}{4}\right)$.

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$. Solving with the equation of the parabola, $y = x^2$, we get $x^2 + \left(x^2 - a\right)^2 = 1 \quad \Leftrightarrow \quad x^2 + x^4 - 2ax^2 + a^2 = 1$ $\Leftrightarrow \quad x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4\left(a^2 - 1\right) = 0 \quad \Leftrightarrow$ $1 - 4a + 4a^2 - 4a^2 + 4 = 0 \quad \Leftrightarrow \quad 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $\left(0, \frac{5}{4}\right)$.

8. $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} = \lim\limits_{x \to a} \left[\dfrac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} \cdot \dfrac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}\right] = \lim\limits_{x \to a} \left[\dfrac{f(x) - f(a)}{x - a} \cdot \left(\sqrt{x} + \sqrt{a}\right)\right]$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} \left(\sqrt{x} + \sqrt{a}\right) = f'(a) \cdot \left(\sqrt{a} + \sqrt{a}\right) = 2\sqrt{a}\, f'(a)$$

9. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $= 360 \cdot (2\pi \text{ rad})/(60 \text{ s}) = 12\pi$ rad/s.] Then the position of A as a function of time is $A = (40\cos\theta, 40\sin\theta) = (40\cos 12\pi t, 40\sin 12\pi t)$, so $\sin\alpha = \dfrac{y}{1.2 \text{ m}} = \dfrac{40\sin\theta}{120} = \dfrac{\sin\theta}{3} = \frac{1}{3}\sin 12\pi t$.

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha \cdot \dfrac{d\alpha}{dt} = \frac{1}{3} \cdot 12\pi \cdot \cos 12\pi t = 4\pi\cos\theta$.

When $\theta = \frac{\pi}{3}$, we have $\sin\alpha = \frac{1}{3}\sin\theta = \frac{\sqrt{3}}{6}$, so $\cos\alpha = \sqrt{1 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{11}{12}}$ and

$$\frac{d\alpha}{dt} = \frac{4\pi\cos\frac{\pi}{3}}{\cos\alpha} = \frac{2\pi}{\sqrt{11/12}} = \frac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56 \text{ rad/s}.$$

(b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2\,|OA|\,|OP|\cos\theta \quad \Rightarrow$

$120^2 = 40^2 + |OP|^2 - 2 \cdot 40\,|OP|\cos\theta \quad \Rightarrow \quad |OP|^2 - (80\cos\theta)\,|OP| - 12{,}800 = 0 \quad \Rightarrow$

$$|OP| = \tfrac{1}{2}\left(80\cos\theta \pm \sqrt{6400\cos^2\theta + 51{,}200}\right) = 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8}$$

$$= 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right) \text{ cm} \quad (\text{since } |OP| > 0)$$

As a check, note that $|OP| = 160$ cm when $\theta = 0$ and $|OP| = 80\sqrt{2}$ cm when $\theta = \frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x = 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = 40\left(-\sin\theta - \frac{2\cos\theta\sin\theta}{2\sqrt{8 + \cos^2\theta}}\right) \cdot 12\pi = -480\pi\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8 + \cos^2\theta}}\right) \text{ cm/s}.$$

In particular, $dx/dt = 0$ cm/s when $\theta = 0$ and $dx/dt = -480\pi$ cm/s when $\theta = \frac{\pi}{2}$.

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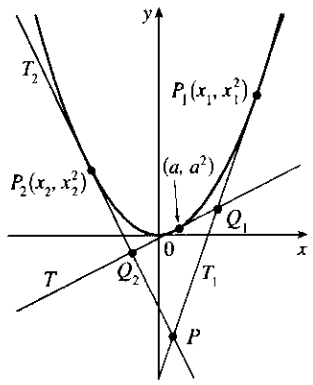
10. The equation of T_1 is $y - x_1^2 = 2x_1(x - x_1) = 2x_1x - 2x_1^2$ or $y = 2x_1x - x_1^2$. The equation of T_2 is $y = 2x_2x - x_2^2$. Solving for the point of intersection, we get $2x(x_1 - x_2) = x_1^2 - x_2^2 \quad \Rightarrow \quad x = \frac{1}{2}(x_1 + x_2)$. Therefore, the coordinates of P are $\left(\frac{1}{2}(x_1 + x_2), x_1x_2\right)$. So if the point of contact of T is (a, a^2), then Q_1 is $\left(\frac{1}{2}(a + x_1), ax_1\right)$ and Q_2 is $\left(\frac{1}{2}(a + x_2), ax_2\right)$.



Therefore, $|PQ_1|^2 = \frac{1}{4}(a - x_2)^2 + x_1^2(a - x_2)^2 = (a - x_2)^2\left(\frac{1}{4} + x_1^2\right)$ and $|PP_1|^2 = \frac{1}{4}(x_1 - x_2)^2 + x_1^2(x_1 - x_2)^2 = (x_1 - x_2)^2\left(\frac{1}{4} + x_1^2\right)$. So $\dfrac{|PQ_1|^2}{|PP_1|^2} = \dfrac{(a - x_2)^2}{(x_1 - x_2)^2}$, and similarly $\dfrac{|PQ_2|^2}{|PP_2|^2} = \dfrac{(x_1 - a)^2}{(x_1 - x_2)^2}$. Finally, $\dfrac{|PQ_1|}{|PP_1|} + \dfrac{|PQ_2|}{|PP_2|} = \dfrac{a - x_2}{x_1 - x_2} + \dfrac{x_1 - a}{x_1 - x_2} = 1$.

11. It seems from the figure that as P approaches the point $(0, 2)$ from the right, $x_T \to \infty$ and $y_T \to 2^+$. As P approaches the point $(3, 0)$ from the left, it appears that $x_T \to 3^+$ and $y_T \to \infty$. So we guess that $x_T \in (3, \infty)$ and $y_T \in (2, \infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N \in (0, 3)$, and $y_N \in (-\infty, 0)$ or $(-2, 0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \quad \Rightarrow \quad \dfrac{2x}{9} + \dfrac{2y}{4}y' = 0$, so $y' = -\dfrac{4}{9}\dfrac{x}{y}$. So at the point (x_0, y_0) on the ellipse, an equation of the tangent line is $y - y_0 = -\dfrac{4}{9}\dfrac{x_0}{y_0}(x - x_0)$ or $4x_0x + 9y_0y = 4x_0^2 + 9y_0^2$. This can be written as $\dfrac{x_0x}{9} + \dfrac{y_0y}{4} = \dfrac{x_0^2}{9} + \dfrac{y_0^2}{4} = 1$, because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is $\dfrac{x_0x}{9} + \dfrac{y_0y}{4} = 1$.

Therefore, the x-intercept x_T for the tangent line is given by $\dfrac{x_0x_T}{9} = 1 \quad \Leftrightarrow \quad x_T = \dfrac{9}{x_0}$, and the y-intercept y_T is given by $\dfrac{y_0y_T}{4} = 1 \quad \Leftrightarrow \quad y_T = \dfrac{4}{y_0}$.

So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$. At the point (x_0, y_0) on the ellipse, the slope of the normal line is $-\dfrac{1}{y'(x_0, y_0)} = \dfrac{9}{4}\dfrac{y_0}{x_0}$, and its equation is $y - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x - x_0)$. So the x-intercept x_N for the normal line is given by $0 - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x_N - x_0) \quad \Rightarrow \quad x_N = -\dfrac{4x_0}{9} + x_0 = \dfrac{5x_0}{9}$, and the y-intercept y_N is given by $y_N - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(0 - x_0) \quad \Rightarrow \quad y_N = -\dfrac{9y_0}{4} + y_0 = -\dfrac{5y_0}{4}$.

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $\left(0, \frac{5}{3}\right)$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $\left(-\frac{5}{2}, 0\right)$.

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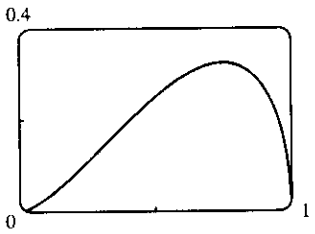
and $f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^4 - 3\left(\frac{1}{4}\right)^3 + 3\left(\frac{1}{4}\right)^2 - \frac{1}{4} = -\frac{27}{256}$ (minimum). At the right endpoint we have $f(2) = 2^4 - 3 \cdot 2^3 + 3 \cdot 2^2 - 2 = 2$ (maximum).

61. (a)

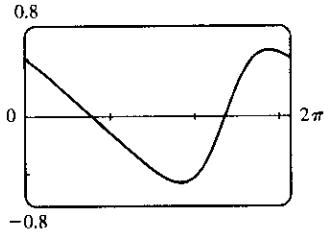


From the graph, it appears that the absolute maximum value is about $f(0.75) = 0.32$, and the absolute minimum value is $f(0) = f(1) = 0$; that is, at both endpoints.

(b) $f(x) = x\sqrt{x - x^2} \quad \Rightarrow \quad f'(x) = x \cdot \dfrac{1 - 2x}{2\sqrt{x - x^2}} + \sqrt{x - x^2} = \dfrac{(x - 2x^2) + (2x - 2x^2)}{2\sqrt{x - x^2}} = \dfrac{3x - 4x^2}{2\sqrt{x - x^2}}$.

So $f'(x) = 0 \quad \Rightarrow \quad 3x - 4x^2 = 0 \quad \Rightarrow \quad x(3 - 4x) = 0 \quad \Rightarrow \quad x = 0$ or $\frac{3}{4}$. $f(0) = f(1) = 0$ [minimum], and $f\left(\frac{3}{4}\right) = \frac{3}{4}\sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3\sqrt{3}}{16}$ [maximum].

62. (a)



From the graph, it appears that the absolute maximum value is about $f(5.76) = 0.58$, and the absolute minimum value is about $f(3.67) = -0.58$.

(b) $f(x) = \dfrac{\cos x}{2 + \sin x} \quad \Rightarrow \quad f'(x) = \dfrac{(2 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(2 + \sin x)^2} = \dfrac{-1 - 2\sin x}{(2 + \sin x)^2}$.

So $f'(x) = 0 \quad \Rightarrow \quad \sin x = -\frac{1}{2} \quad \Rightarrow \quad x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. Now $f\left(\frac{7\pi}{6}\right) = \frac{-\sqrt{3}/2}{3/2} = -\frac{1}{\sqrt{3}}$ [minimum], and $f\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}/2}{3/2} = \frac{1}{\sqrt{3}}$ [maximum].

63. The density is defined as $\rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{1000}{V(T)}$ (in g/cm^3). But a critical point of ρ will also be a critical point of V [since $\dfrac{d\rho}{dT} = -1000V^{-2}\dfrac{dV}{dT}$ and V is never 0], and V is easier to differentiate than ρ.

$V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \quad \Rightarrow$

$V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2$. Setting this equal to 0 and using the quadratic formula to find T, we get $T = \dfrac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot 0.0002037 \cdot 0.06426}}{2(-0.0002037)} \approx 3.9665°\text{C}$ or $79.5318°\text{C}$. Since we are only interested in the region $0°\text{C} \le T \le 30°\text{C}$, we check the density ρ at the endpoints and at $3.9665°\text{C}$: $\rho(0) \approx \dfrac{1000}{999.87} \approx 1.00013$; $\rho(30) \approx \dfrac{1000}{1003.7628} \approx 0.99625$; $\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at about $3.9665°\text{C}$.

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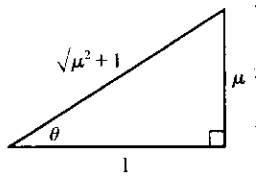
64. $F = \dfrac{\mu W}{\mu \sin\theta + \cos\theta} \quad \Rightarrow \quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{-\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}.$

So $\dfrac{dF}{d\theta} = 0 \quad \Rightarrow \quad \mu\cos\theta - \sin\theta = 0 \quad \Rightarrow \quad \mu = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$. Substituting $\tan\theta$ for μ in F gives us

$$F = \frac{(\tan\theta)W}{(\tan\theta)\sin\theta + \cos\theta} = \frac{W\tan\theta}{\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta} = \frac{W\tan\theta\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{W\sin\theta}{1} = W\sin\theta.$$

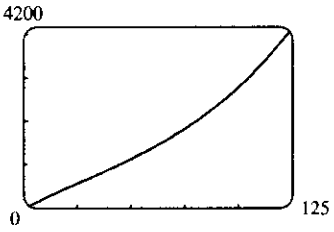
If $\tan\theta = \mu$, then $\sin\theta = \dfrac{\mu}{\sqrt{\mu^2+1}}$ (see the figure), so $F = \dfrac{\mu}{\sqrt{\mu^2+1}}W$. We compare this with the value of F at the endpoints: $F(0) = \mu W$ and $F\left(\frac{\pi}{2}\right) = W$.



Now because $\dfrac{\mu}{\sqrt{\mu^2+1}} \le 1$ and $\dfrac{\mu}{\sqrt{\mu^2+1}} \le \mu$, we have that $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is less than or equal to each of $F(0)$ and $F\left(\frac{\pi}{2}\right)$. Hence, $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is the absolute minimum value of $F(\theta)$, and it occurs when $\tan\theta = \mu$.

65. We apply the Closed Interval Method to the continuous function $I(t) = 0.00009045t^5 + 0.001438t^4 - 0.06561t^3 + 0.4598t^2 - 0.6270t + 99.33$ on $[0, 10]$. Its derivative is $I'(t) = 0.00045225t^4 + 0.005752t^3 - 0.19683t^2 + 0.9196t - 0.6270$. Since I' exists for all t, the only critical numbers of I occur when $I'(t) = 0$. We use a root-finder on a computer algebra system (or a graphing device) to find that $I'(t) = 0$ when $t \approx -29.7186$, 0.8231, 5.1309, or 11.0459, but only the second and third roots lie in the interval $[0, 10]$. The values of I at these critical numbers are $I(0.8231) \approx 99.09$ and $I(5.1309) \approx 100.67$. The values of I at the endpoints of the interval are $I(0) = 99.33$ and $I(10) \approx 96.86$. Comparing these four numbers, we see that food was most expensive at $t \approx 5.1309$ (corresponding roughly to August, 1989) and cheapest at $t = 10$ (midyear 1994).

66. (a)



The equation of the graph in the figure is

$$v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872.$$

(b) $a(t) = v'(t) = 0.00438t^2 - 0.23106t + 24.98169 \quad \Rightarrow \quad a'(t) = 0.00876t - 0.23106$. $a'(t) = 0 \quad \Rightarrow$ $t_1 = \frac{0.23106}{0.00876} \approx 26.4$. $a(0) \approx 24.98$, $a(t_1) \approx 21.93$, and $a(125) \approx 64.54$. The maximum acceleration is about 64.5 ft/s^2 and the minimum acceleration is about 21.93 ft/s^2.

67. (a) $v(r) = k(r_0 - r)r^2 = kr_0r^2 - kr^3 \quad \Rightarrow \quad v'(r) = 2kr_0r - 3kr^2$. $v'(r) = 0 \quad \Rightarrow \quad kr(2r_0 - 3r) = 0$ $\Rightarrow \quad r = 0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0, we get $v\left(\frac{1}{2}r_0\right) = \frac{1}{8}kr_0^3$, $v\left(\frac{2}{3}r_0\right) = \frac{4}{27}kr_0^3$, and $v(r_0) = 0$. Since $\frac{4}{27} > \frac{1}{8}$, v attains its maximum value at $r = \frac{2}{3}r_0$. This supports the statement in the text.

(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

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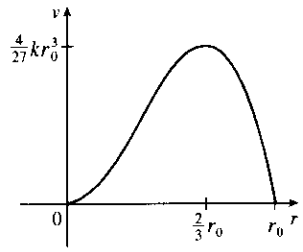
(c)



68. $g(x) = 2 + (x-5)^3 \Rightarrow g'(x) = 3(x-5)^2 \Rightarrow g'(5) = 0$, so 5 is a critical number. But $g(5) = 2$ and g takes on values > 2 and values < 2 in any open interval containing 5, so g does not have a local maximum or minimum at 5.

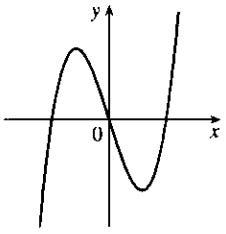
69. $f(x) = x^{101} + x^{51} + x + 1 \Rightarrow f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1$ for all x, so $f'(x) = 0$ has no solution. Thus, $f(x)$ has no critical number, so $f(x)$ can have no local maximum or minimum.

70. Suppose that f has a minimum value at c, so $f(x) \geq f(c)$ for all x near c. Then $g(x) = -f(x) \leq -f(c) = g(c)$ for all x near c, so $g(x)$ has a maximum value at c.

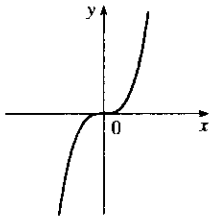
71. If f has a local minimum at c, then $g(x) = -f(x)$ has a local maximum at c, so $g'(c) = 0$ by the case of Fermat's Theorem proved in the text. Thus, $f'(c) = -g'(c) = 0$.

72. (a) $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. So $f'(x) = 3ax^2 + 2bx + c$ is a quadratic and hence has either 2, 1, or 0 real roots, so $f(x)$ has either 2, 1 or 0 critical numbers.

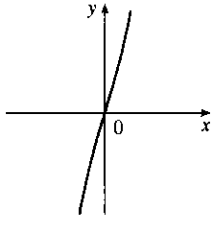
Case (i) (2 critical numbers):
$f(x) = x^3 - 3x \Rightarrow$
$f'(x) = 3x^2 - 3$, so $x = -1, 1$
are critical numbers.



Case (ii) (1 critical number):
$f(x) = x^3 \Rightarrow$
$f'(x) = 3x^2$, so $x = 0$
is the only critical number.



Case (iii) (no critical number):
$f(x) = x^3 + 3x \Rightarrow$
$f'(x) = 3x^2 + 3$,
so there are no real roots.



(b) Since there are at most two critical numbers, it can have at most two local extreme values and by (i) this can occur. By (iii) it can have no local extreme value. However, if there is only one critical number, then there is no local extreme value.

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APPLIED PROJECT The Calculus of Rainbows

1. From Snell's Law, we have $\sin\alpha = \frac{4}{3}\sin\beta \;\Rightarrow\; \frac{d}{d\alpha}(\sin\alpha) = \frac{4}{3}\frac{d}{d\alpha}(\sin\beta) \;\Rightarrow\; \cos\alpha = \frac{4}{3}\cos\beta\,\frac{d\beta}{d\alpha} \;\Rightarrow$ $\frac{d\beta}{d\alpha} = \frac{3\cos\alpha}{4\cos\beta}$. Now $D(\alpha) = \pi + 2\alpha - 4\beta \;\Rightarrow\; D'(\alpha) = 2 - 4\frac{d\beta}{d\alpha} = 2 - 3\frac{\cos\alpha}{\cos\beta}$. So $D'(\alpha) = 0 \;\Leftrightarrow$ $2\cos\beta = 3\cos\alpha$. Thus, $4\cos^2\beta = 9\cos^2\alpha \;\Rightarrow\; 4 - 4\sin^2\beta = 9 - 9\sin^2\alpha$. Since $3\sin\alpha = 4\sin\beta$, $\sin\beta = \frac{3}{4}\sin\alpha \;\Rightarrow\; 4 - 4\left(\frac{3}{4}\sin\alpha\right)^2 = 9 - 9\sin^2\alpha \;\Rightarrow\; \left(9 - \frac{9}{4}\right)\sin^2\alpha = 9 - 4 = 5 \;\Rightarrow\; \sin^2\alpha = \frac{20}{27}$ $\Rightarrow\; \sin\alpha = \sqrt{\frac{20}{27}}$. So $\alpha \approx 1.037$ radians or $59.4°$. We show that this α does give the minimum on $\left[0, \frac{\pi}{2}\right]$: When $\alpha = 0$, $\sin\alpha = \frac{4}{3}\sin\beta \;\Rightarrow\; \beta = 0$, or $D(0) = \pi \approx 3.14$. When $\alpha = \frac{\pi}{2}$, $1 = \sin\frac{\pi}{2} = \frac{4}{3}\sin\beta \;\Rightarrow\; \sin\beta = \frac{3}{4}$ $\Rightarrow\; \beta \approx 0.85$. So $D\left(\frac{\pi}{2}\right) \approx \pi + \pi - 4(0.85) \approx 2.88$. For $\alpha \approx 1.037$, $\sin\beta = \frac{3}{4}\sin\alpha = \frac{3}{4}\sqrt{\frac{20}{27}}$, so $\beta \approx 0.702$ $\Rightarrow\; D(\alpha) \approx \pi + 2(1.036) - 4(0.702) \approx 2.41$. So the minimum occurs when $\alpha \approx 1.04$ radians or $59.4°$.

2. We repeat Problem 1 with k in place of $\frac{4}{3}$. So $\sin\alpha = k\sin\beta \;\Rightarrow\; \frac{d\beta}{d\alpha} = \frac{1}{k}\frac{\cos\alpha}{\cos\beta}$. Then

$D'(\alpha) = 2 - 4\frac{d\beta}{d\alpha} = 2 - \frac{4}{k}\frac{\cos\alpha}{\cos\beta}$ and $D'(\alpha) = 0 \;\Leftrightarrow\; k\cos\beta = 2\cos\alpha$. So $k^2\cos^2\beta = 4\cos^2\alpha \;\Rightarrow$

$k^2 - k^2\sin^2\beta = 4 - 4\sin^2\alpha \;\Rightarrow\; k^2 - \sin^2\alpha = 4 - 4\sin^2\alpha \;\Rightarrow\; 3\sin^2\alpha = 4 - k^2 \;\Rightarrow$

$\sin\alpha = \sqrt{\frac{4 - k^2}{3}}$. So for $k \approx 1.3318$ (red light) the minimum occurs when $\sin\alpha_1 = \sqrt{\frac{4 - (1.3318)^2}{3}}$ or $\alpha_1 \approx 1.038$ radians, so the rainbow angle is about $\pi - D(\alpha_1) \approx 42.3°$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx 1.026$ radians, and so the rainbow angle is about $\pi - D(\alpha_2) \approx 40.6°$.

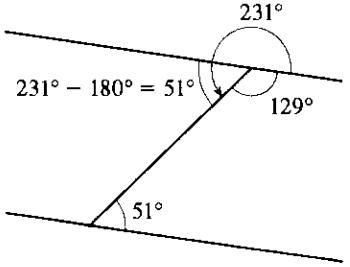
3. At each reflection or refraction, the light is bent in a counterclockwise direction: the bend at A is $\alpha - \beta$, the bend at B is $\pi - 2\beta$, the bend at C is again $\pi - 2\beta$, and the bend at D is $\alpha - \beta$. So the total bend is

$D(\alpha) = 2(\alpha - \beta) + 2(\pi - 2\beta) = 2\alpha - 6\beta + 2\pi$, as required. Now $\sin\alpha = k\sin\beta \;\Rightarrow\; \frac{d\beta}{d\alpha} = \frac{1}{k}\frac{\cos\alpha}{\cos\beta}$. So

$D'(\alpha) = 2 - 6\frac{d\beta}{d\alpha} = 2 - \frac{6}{k}\frac{\cos\alpha}{\cos\beta}$ and $D'(\alpha) = 0 \;\Leftrightarrow\; k\cos\beta = 3\cos\alpha$. So $k^2\cos^2\beta = 9\cos^2\alpha \;\Rightarrow$

$k^2 - k^2\sin^2\beta = 9 - 9\sin^2\alpha \;\Rightarrow\; k^2 - \sin^2\alpha = 9 - 9\sin^2\alpha \;\Rightarrow\; \sin^2\alpha = \frac{9 - k^2}{8} \;\Rightarrow$

$\sin\alpha = \sqrt{\frac{9 - k^2}{8}}$. If $k = \frac{4}{3}$, then the minimum occurs when $\sin\alpha_1 = \sqrt{\frac{9 - (4/3)^2}{8}}$ or $\alpha_1 \approx 1.254$ radians. Thus, the minimum *counterclockwise* rotation is $D(\alpha_1) \approx 231°$, which is equivalent to a *clockwise* rotation of $360° - 231° = 129°$ (see the figure). So the rainbow angle for the secondary rainbow is about $180° - 129° = 51°$, as required. In general, the rainbow angle for the secondary rainbow is $\pi - [2\pi - D(\alpha)] = D(\alpha) - \pi$.



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4. In the primary rainbow, the rainbow angle gets smaller as k gets larger, as we found in Problem 2, so the colors appear from top to bottom in order of increasing k. But in the secondary rainbow, the rainbow angle gets larger as k gets larger. To see this, we find the minimum deviations for red light and for violet light in the secondary rainbow.

For $k \approx 1.3318$ (red light) the minimum occurs when $\sin\alpha_1 = \sqrt{\dfrac{9-1.3318^2}{8}}$ or $\alpha_1 \approx 1.254$ radians, and so the rainbow angle is $D(\alpha_1) - \pi \approx 50.6°$. For $k \approx 1.3435$ (violet light) the minimum occurs when $\sin\alpha_2 = \sqrt{\dfrac{9-1.3435^2}{8}}$ or $\alpha_2 \approx 1.248$ radians, and so the rainbow angle is $D(\alpha_2) - \pi \approx 53.6°$. Consequently, the rainbow angle is larger for colors with higher indices of refraction, and the colors appear from bottom to top in order of increasing k, the reverse of their order in the primary rainbow.

Note that our calculations above also explain why the secondary rainbow is more spread-out than the primary rainbow: in the primary rainbow, the difference between rainbow angles for red and violet light is about $1.7°$, whereas in the secondary rainbow it is about $3°$.

4.2 The Mean Value Theorem

1. $f(x) = x^2 - 4x + 1$, $[0, 4]$. Since f is a polynomial, it is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Also, $f(0) = 1 = f(4)$. $f'(c) = 0 \quad \Leftrightarrow \quad 2c - 4 = 0 \quad \Leftrightarrow \quad c = 2$, which is in the open interval $(0, 4)$, so $c = 2$ satisfies the conclusion of Rolle's Theorem.

2. $f(x) = x^3 - 3x^2 + 2x + 5$, $[0, 2]$. f is continuous on $[0, 2]$ and differentiable on $(0, 2)$. Also, $f(0) = 5 = f(2)$. $f'(c) = 0 \quad \Leftrightarrow \quad 3c^2 - 6c + 2 = 0 \quad \Leftrightarrow \quad c = \dfrac{6 \pm \sqrt{36-24}}{6} = 1 \pm \frac{1}{3}\sqrt{3}$, both in $(0, 2)$.

3. $f(x) = \sin 2\pi x$, $[-1, 1]$. f, being the composite of the sine function and the polynomial $2\pi x$, is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Also, $f(-1) = 0 = f(1)$. $f'(c) = 0 \quad \Leftrightarrow \quad 2\pi\cos 2\pi c = 0 \quad \Leftrightarrow \quad \cos 2\pi c = 0 \quad \Leftrightarrow \quad 2\pi c = \pm\frac{\pi}{2} + 2\pi n \quad \Leftrightarrow \quad c = \pm\frac{1}{4} + n$. If $n = 0$ or ± 1, then $c = \pm\frac{1}{4}, \pm\frac{3}{4}$ is in $(-1, 1)$.

4. $f(x) = x\sqrt{x+6}$, $[-6, 0]$. f is continuous on its domain, $[-6, \infty)$, and differentiable on $(-6, \infty)$, so it is continuous on $[-6, 0]$ and differentiable on $(-6, 0)$. Also, $f(-6) = 0 = f(0)$. $f'(c) = 0 \quad \Leftrightarrow \quad \dfrac{3c+12}{2\sqrt{c+6}} = 0$ $\Leftrightarrow \quad c = -4$, which is in $(-6, 0)$.

5. $f(x) = 1 - x^{2/3}$. $f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0 = f(1)$. $f'(x) = -\frac{2}{3}x^{-1/3}$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1, 1)$.

6. $f(x) = (x-1)^{-2}$. $f(0) = (0-1)^{-2} = 1 = (2-1)^{-2} = f(2)$. $f'(x) = -2(x-1)^{-3} \quad \Rightarrow \quad f'(x)$ is never 0. This does not contradict Rolle's Theorem since $f'(1)$ does not exist.

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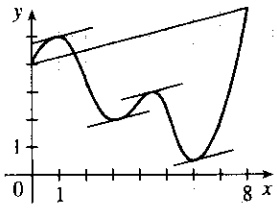
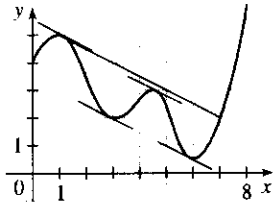
7. $\dfrac{f(8)-f(0)}{8-0}=\dfrac{6-4}{8}=\dfrac{1}{4}$. The values of c which satisfy $f'(c)=\frac{1}{4}$ seem to be about $c=0.8$, 3.2, 4.4, and 6.1.

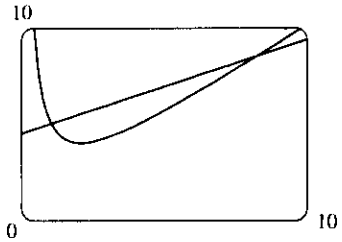


8. $\dfrac{f(7)-f(1)}{7-1}=\dfrac{2-5}{6}=-\dfrac{1}{2}$. The values of c which satisfy $f'(c)=-\frac{1}{2}$ seem to be about $c=1.1$, 2.8, 4.6, and 5.8.

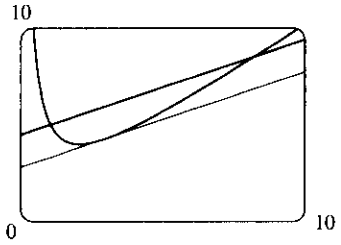


9. (a), (b) The equation of the secant line is

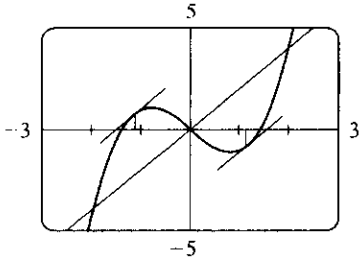
$$y-5=\frac{8.5-5}{8-1}(x-1) \quad \Leftrightarrow \quad y=\tfrac{1}{2}x+\tfrac{9}{2}.$$



(c) $f(x)=x+4/x \quad \Rightarrow \quad f'(x)=1-4/x^2$.
So $f'(c)=\frac{1}{2} \quad \Rightarrow \quad c^2=8 \quad \Rightarrow \quad c=2\sqrt{2}$, and $f(c)=2\sqrt{2}+\frac{4}{2\sqrt{2}}=3\sqrt{2}$. Thus, an equation of the tangent line is $y-3\sqrt{2}=\frac{1}{2}\left(x-2\sqrt{2}\right) \quad \Leftrightarrow$ $y=\frac{1}{2}x+2\sqrt{2}$.



10. (a)



It seems that the tangent lines are parallel to the secant at $x\approx\pm1.2$.

(b) The slope of the secant line is 2, and its equation is $y=2x$. $f(x)=x^3-2x \quad \Rightarrow \quad f'(x)=3x^2-2$, so we solve $f'(c)=2 \quad \Rightarrow \quad 3c^2=4 \quad \Rightarrow$ $c=\pm\frac{2\sqrt{3}}{3}\approx1.155$. Our estimates were off by about 0.045 in each case.

11. $f(x)=3x^2+2x+5$, $[-1,1]$. f is continuous on $[-1,1]$ and differentiable on $(-1,1)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c)=\dfrac{f(b)-f(a)}{b-a} \quad \Leftrightarrow \quad 6c+2=\dfrac{f(1)-f(-1)}{1-(-1)}=\dfrac{10-6}{2}=2 \quad \Leftrightarrow$ $6c=0 \quad \Leftrightarrow \quad c=0$, which is in $(-1,1)$.

12. $f(x)=x^3+x-1$, $[0,2]$. f is continuous on $[0,2]$ and differentiable on $(0,2)$. $f'(c)=\dfrac{f(2)-f(0)}{2-0} \quad \Leftrightarrow$ $3c^2+1=\dfrac{9-(-1)}{2} \quad \Leftrightarrow \quad 3c^2=5-1 \quad \Leftrightarrow \quad c^2=\frac{4}{3} \quad \Leftrightarrow \quad c=\pm\frac{2}{\sqrt{3}}$, but only $\frac{2}{\sqrt{3}}$ is in $(0,2)$.

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13. $f(x) = \sqrt[3]{x}$, $[0, 1]$. f is continuous on $\mathbb{R}$ and differentiable on $(-\infty, 0) \cup (0, \infty)$, so f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad \dfrac{1}{3c^{2/3}} = \dfrac{f(1) - f(0)}{1 - 0} \quad \Leftrightarrow \quad \dfrac{1}{3c^{2/3}} = \dfrac{1 - 0}{1} \quad \Leftrightarrow$

$3c^{2/3} = 1 \quad \Leftrightarrow \quad c^{2/3} = \frac{1}{3} \quad \Leftrightarrow \quad c^2 = \left(\frac{1}{3}\right)^3 = \frac{1}{27} \quad \Leftrightarrow \quad c = \pm\sqrt{\frac{1}{27}} = \pm\frac{\sqrt{3}}{9}$, but only $\frac{\sqrt{3}}{9}$ is in $(0, 1)$.

14. $f(x) = \dfrac{x}{x + 2}$, $[1, 4]$. f is continuous on $[1, 4]$ and differentiable on $(1, 4)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow$

$\dfrac{2}{(c + 2)^2} = \dfrac{\frac{2}{3} - \frac{1}{3}}{4 - 1} \quad \Leftrightarrow \quad (c + 2)^2 = 18 \quad \Leftrightarrow \quad c = -2 \pm 3\sqrt{2}$. $-2 + 3\sqrt{2} \approx 2.24$ is in $(1, 4)$.

15. $f(x) = |x - 1|$. $f(3) - f(0) = |3 - 1| - |0 - 1| = 1$. Since $f'(c) = -1$ if $c < 1$ and $f'(c) = 1$ if $c > 1$, $f'(c)(3 - 0) = \pm 3$ and so is never equal to 1. This does not contradict the Mean Value Theorem since $f'(1)$ does not exist.

16. $f(x) = \dfrac{x + 1}{x - 1}$. $f(2) - f(0) = 3 - (-1) = 4$. $f'(x) = \dfrac{1(x - 1) - 1(x + 1)}{(x - 1)^2} = \dfrac{-2}{(x - 1)^2}$. Since $f'(x) < 0$ for all x (except $x = 1$), $f'(c)(2 - 0)$ is always < 0 and hence cannot equal 4. This does not contradict the Mean Value Theorem since f is not continuous at $x = 1$.

17. Let $f(x) = 1 + 2x + x^3 + 4x^5$. Then $f(-1) = -6 < 0$ and $f(0) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem says that there is a number c between -1 and 0 such that $f(c) = 0$. Thus, the given equation has a real root. Suppose the equation has distinct real roots a and b with $a < b$. Then $f(a) = f(b) = 0$. Since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. By Rolle's Theorem, there is a number r in (a, b) such that $f'(r) = 0$. But $f'(x) = 2 + 3x^2 + 20x^4 \geq 2$ for all x, so $f'(x)$ can never be 0. This contradiction shows that the equation can't have two distinct real roots. Hence, it has exactly one real root.

18. Let $f(x) = 2x - 1 - \sin x$. Then $f(0) = -1 < 0$ and $f(\pi/2) = \pi - 2 > 0$. f is the sum of the polynomial $2x - 1$ and the scalar multiple $(-1) \cdot \sin x$ of the trigonometric function $\sin x$, so f is continuous (and differentiable) for all x. By the Intermediate Value Theorem, there is a number c in $(0, \pi/2)$ such that $f(c) = 0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 2 - \cos r > 0$ since $\cos r \leq 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one real root.

19. Let $f(x) = x^3 - 15x + c$ for x in $[-2, 2]$. If f has two real roots a and b in $[-2, 2]$, with $a < b$, then $f(a) = f(b) = 0$. Since the polynomial f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. Now $f'(r) = 3r^2 - 15$. Since r is in (a, b), which is contained in $[-2, 2]$, we have $|r| < 2$, so $r^2 < 4$. It follows that $3r^2 - 15 < 3 \cdot 4 - 15 = -3 < 0$. This contradicts $f'(r) = 0$, so the given equation can't have two real roots in $[-2, 2]$. Hence, it has at most one real root in $[-2, 2]$.

20. $f(x) = x^4 + 4x + c$. Suppose that $f(x) = 0$ has three distinct real roots a, b, d where $a < b < d$. Then $f(a) = f(b) = f(d) = 0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a < c_1 < b$ and $b < c_2 < d$ and $0 = f'(c_1) = f'(c_2)$, so $f'(x) = 0$ must have at least two real solutions. However $0 = f'(x) = 4x^3 + 4 = 4\left(x^3 + 1\right) = 4(x + 1)\left(x^2 - x + 1\right)$ has as its only real solution $x = -1$. Thus, $f(x)$ can have at most two real roots.

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21. (a) Suppose that a cubic polynomial $P(x)$ has roots $a_1 < a_2 < a_3 < a_4$, so $P(a_1) = P(a_2) = P(a_3) = P(a_4)$. By Rolle's Theorem there are numbers c_1, c_2, c_3 with $a_1 < c_1 < a_2$, $a_2 < c_2 < a_3$ and $a_3 < c_3 < a_4$ and $P'(c_1) = P'(c_2) = P'(c_3) = 0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real roots, which is impossible.

(b) We prove by induction that a polynomial of degree n has at most n real roots. This is certainly true for $n = 1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n + 1$. Suppose that $P(x)$ has more than $n + 1$ real roots, say $a_1 < a_2 < a_3 < \cdots < a_{n+1} < a_{n+2}$. Then $P(a_1) = P(a_2) = \cdots = P(a_{n+2}) = 0$. By Rolle's Theorem there are real numbers $c_1, \ldots, c_{n+1}$ with $a_1 < c_1 < a_2, \ldots, a_{n+1} < c_{n+1} < a_{n+2}$ and $P'(c_1) = \cdots = P'(c_{n+1}) = 0$. Thus, the nth degree polynomial $P'(x)$ has at least $n + 1$ roots. This contradiction shows that $P(x)$ has at most $n + 1$ real roots.

22. (a) Suppose that $f(a) = f(b) = 0$ where $a < b$. By Rolle's Theorem applied to f on $[a, b]$ there is a number c such that $a < c < b$ and $f'(c) = 0$.

(b) Suppose that $f(a) = f(b) = f(c) = 0$ where $a < b < c$. By Rolle's Theorem applied to $f(x)$ on $[a, b]$ and $[b, c]$ there are numbers $a < d < b$ and $b < e < c$ with $f'(d) = 0$ and $f'(e) = 0$. By Rolle's Theorem applied to $f'(x)$ on $[d, e]$ there is a number g with $d < g < e$ such that $f''(g) = 0$.

(c) Suppose that f is n times differentiable on $\mathbb{R}$ and has $n + 1$ distinct real roots. Then $f^{(n)}$ has at least one real root.

23. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4 - 1)$ for some $c \in (1, 4)$. But for every $c \in (1, 4)$ we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4 - 1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.

24. If $3 \leq f'(x) \leq 5$ for all x, then by the Mean Value Theorem, $f(8) - f(2) = f'(c) \cdot (8 - 2)$ for some c in $[2, 8]$. (f is differentiable for all x, so, in particular, f is differentiable on $(2, 8)$ and continuous on $[2, 8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8) - f(2) = 6f'(c)$ and $3 \leq f'(c) \leq 5$, it follows that $6 \cdot 3 \leq 6f'(c) \leq 6 \cdot 5 \quad \Rightarrow \quad 18 \leq f(8) - f(2) \leq 30$.

25. Suppose that such a function f exists. By the Mean Value Theorem there is a number $0 < c < 2$ with $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{5}{2}$. But this is impossible since $f'(x) \leq 2 < \frac{5}{2}$ for all x, so no such function can exist.

26. Let $h = f - g$. Then since f and g are continuous on $[a, b]$ and differentiable on (a, b), so is h, and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a < c < b$ such that $h(b) = h(b) - h(a) = h'(c)(b - a)$. Since $h'(c) < 0$, $h'(c)(b - a) < 0$, so $f(b) - g(b) = h(b) < 0$ and hence $f(b) < g(b)$.

27. We use Exercise 26 with $f(x) = \sqrt{1 + x}$, $g(x) = 1 + \frac{1}{2}x$, and $a = 0$. Notice that $f(0) = 1 = g(0)$ and $f'(x) = \dfrac{1}{2\sqrt{1 + x}} < \dfrac{1}{2} = g'(x)$ for $x > 0$. So by Exercise 26, $f(b) < g(b) \quad \Rightarrow \quad \sqrt{1 + b} < 1 + \frac{1}{2}b$ for $b > 0$.

Another method: Apply the Mean Value Theorem directly to either $f(x) = 1 + \frac{1}{2}x - \sqrt{1 + x}$ or $g(x) = \sqrt{1 + x}$ on $[0, b]$.

28. f satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b, b]$: $\dfrac{f(b) - f(-b)}{b - (-b)} = f'(c)$ for some $c \in (-b, b)$. But since f is odd, $f(-b) = -f(b)$. Substituting this into the above equation, we get $\dfrac{f(b) + f(b)}{2b} = f'(c) \quad \Rightarrow \quad \dfrac{f(b)}{b} = f'(c)$.

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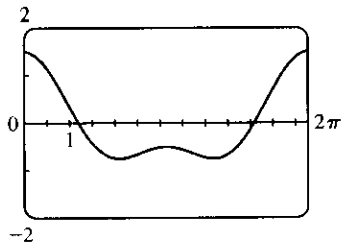
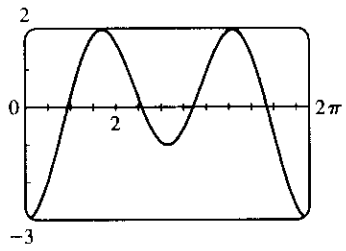
43. $f(x) = \cos x + \frac{1}{2}\cos 2x \Rightarrow f'(x) = -\sin x - \sin 2x \Rightarrow f''(x) = -\cos x - 2\cos 2x$

(a)



From the graph of f, it seems that f is CD on $(0, 1)$, CU on $(1, 2.5)$, CD on $(2.5, 3.7)$, CU on $(3.7, 5.3)$, and CD on $(5.3, 2\pi)$. The points of inflection appear to be at $(1, 0.4)$, $(2.5, -0.6)$, $(3.7, -0.6)$, and $(5.3, 0.4)$.

(b)



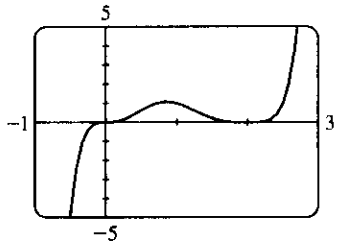
From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(0, 0.94)$, CU on $(0.94, 2.57)$, CD on $(2.57, 3.71)$, CU on $(3.71, 5.35)$, and CD on $(5.35, 2\pi)$. Refined estimates of the inflection points are $(0.94, 0.44)$, $(2.57, -0.63)$, $(3.71, -0.63)$, and $(5.35, 0.44)$.

44. $f(x) = x^3(x-2)^4 \Rightarrow$

$$f'(x) = x^3 \cdot 4(x-2)^3 + (x-2)^4 \cdot 3x^2 = x^2(x-2)^3[4x + 3(x-2)] = x^2(x-2)^3(7x-6) \Rightarrow$$

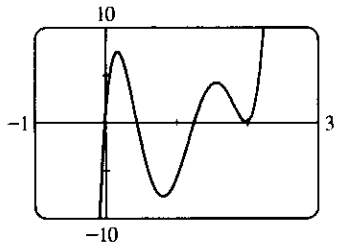
$$\begin{aligned} f''(x) &= (2x)(x-2)^3(7x-6) + x^2 \cdot 3(x-2)^2(7x-6) + x^2(x-2)^3(7) \\ &= x(x-2)^2[2(x-2)(7x-6) + 3x(7x-6) + 7x(x-2)] \\ &= x(x-2)^2[42x^2 - 72x + 24] = 6x(x-2)^2(7x^2 - 12x + 4) \end{aligned}$$

(a)



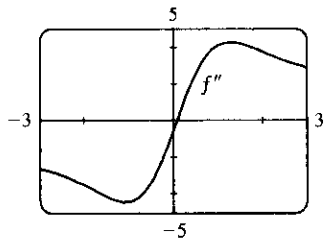
From the graph of f, it seems that f is CD on $(-\infty, 0)$, CU on $(0, 0.5)$, CD on $(0.5, 1.3)$, and CU on $(1.3, \infty)$. The points of inflection appear to be at $(0, 0)$, $(0.5, 0.5)$, and $(1.3, 0.6)$.

(b)



From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(-\infty, 0)$, CU on $(0, 0.45)$, CD on $(0.45, 1.26)$, and CU on $(1.26, \infty)$. Refined estimates of the inflection points are $(0, 0)$, $(0.45, 0.53)$, and $(1.26, 0.60)$.

45. In Maple, we define f and then use the command `plot(diff(diff(f,x),x),x=-3..3);`. In Mathematica, we define f and then use `Plot[Dt[Dt[f,x],x],{x,-3,3}]`. We see that $f'' > 0$ for $x > 0.1$ and $f'' < 0$ for $x < 0.1$. So f is concave up on $(0.1, \infty)$ and concave down on $(-\infty, 0.1)$.



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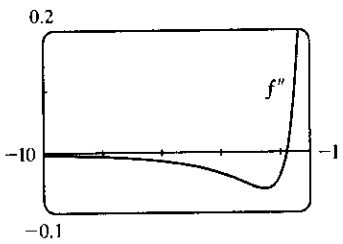
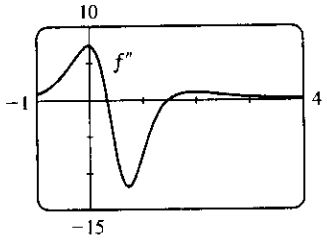
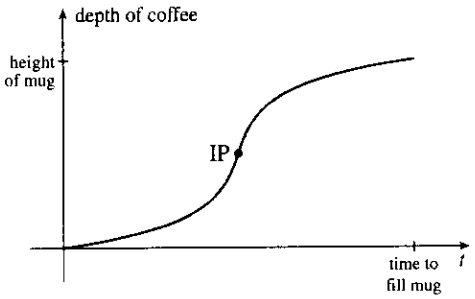
46. It appears that f'' is positive (and thus f is concave up) on $(-1.8, 0.3)$ and $(1.5, \infty)$ and negative (so f is concave down) on $(-\infty, -1.8)$ and $(0.3, 1.5)$.





47. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

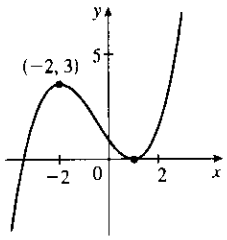
48. At first the depth increases slowly because the base of the mug is wide. But as the mug narrows, the coffee rises more quickly. Thus, the depth d increases at an increasing rate and its graph is concave upward. The rate of increase of d has a maximum where the mug is narrowest; that is, when the mug is half full. It is there that the inflection point (IP) occurs. Then the rate of increase of d starts to decrease as the mug widens and the graph becomes concave down.



49. $f(x) = \tan x - x \quad \Rightarrow \quad f'(x) = \sec^2 x - 1 > 0$ for $0 < x < \frac{\pi}{2}$ since $\sec^2 x > 1$ for $0 < x < \frac{\pi}{2}$. So f is increasing on $\left(0, \frac{\pi}{2}\right)$. Thus, $f(x) > f(0) = 0$ for $0 < x < \frac{\pi}{2} \quad \Rightarrow \quad \tan x - x > 0 \quad \Rightarrow \quad \tan x > x$ for $0 < x < \frac{\pi}{2}$.

50. Let $f(x) = 2\sqrt{x} - 3 + 1/x$. Then $f'(x) = 1/\sqrt{x} - 1/x^2 > 0$ for $x > 1$ since for $x > 1$, $x^2 > x > \sqrt{x}$. Hence, f is increasing, so for $x > 1$, $f(x) > f(1) = 0$ or $2\sqrt{x} - 3 + 1/x > 0$ for $x > 1$. Hence, $2\sqrt{x} > 3 - 1/x$ for $x > 1$.

51. $f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$. We are given that $f(1) = 0$ and $f(-2) = 3$, so $f(1) = a + b + c + d = 0$ and $f(-2) = -8a + 4b - 2c + d = 3$. Also $f'(1) = 3a + 2b + c = 0$ and $f'(-2) = 12a - 4b + c = 0$ by Fermat's Theorem. Solving these four equations, we get $a = \frac{2}{9}, b = \frac{1}{3}, c = -\frac{4}{3}, d = \frac{7}{9}$, so the function is $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$.



52. $f(x) = x^3 + ax^2 + bx + 2 \quad \Rightarrow \quad f'(x) = 3x^2 + 2ax + b$. If $x = -3$ is an extremum, then $f'(-3) = 27 - 6a + b = 0 \quad \Leftrightarrow \quad b = 6a - 27$. If $x = -1$ is an extremum, then $f'(-1) = 3 - 2a + b = 0$ $\Leftrightarrow \quad b = 2a - 3$. So $b = 2a - 3$ and $b = 6a - 27 \quad \Rightarrow \quad b = 9, a = 6$. Then $f'(x) = 3x^2 + 12x + 9 = 3(x + 1)(x + 3)$ and the First Derivative Test shows that f has a local maximum when $x = -3$ and a local minimum when $x = -1$.

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53. Suppose that f is differentiable on an interval I and $f'(x) > 0$ for all x in I except $x = c$. To show that f is increasing on I, let x_1, x_2 be two numbers in I with $x_1 < x_2$.

Case 1 $x_1 < x_2 < c$. Let J be the interval $\{x \in I \mid x < c\}$. By applying the Increasing/Decreasing Test to f on J, we see that f is increasing on J, so $f(x_1) < f(x_2)$.

Case 2 $c < x_1 < x_2$. Apply the Increasing/Decreasing Test to f on $K = \{x \in I \mid x > c\}$.

Case 3 $x_1 < x_2 = c$. Apply the proof of the Increasing/Decreasing Test, using the Mean Value Theorem (MVT) on the interval $[x_1, x_2]$ and noting that the MVT does not require f to be differentiable at the endpoints of $[x_1, x_2]$.

Case 4 $c = x_1 < x_2$. Same proof as in Case 3.

Case 5 $x_1 < c < x_2$. By Cases 3 and 4, f is increasing on $[x_1, c]$ and on $[c, x_2]$, so $f(x_1) < f(c) < f(x_2)$.

In all cases, we have shown that $f(x_1) < f(x_2)$. Since x_1, x_2 were any numbers in I with $x_1 < x_2$, we have shown that f is increasing on I.

54. (a) We will make use of the converse of the Concavity Test (along with the stated assumptions); that is, if f is concave upward on I, then $f'' > 0$ on I. If f and g are CU on I, then $f'' > 0$ and $g'' > 0$ on I, so $(f+g)'' = f'' + g'' > 0$ on $I \;\Rightarrow\; f+g$ is CU on I.

(b) Since f is positive and CU on I, $f > 0$ and $f'' > 0$ on I. So $g(x) = [f(x)]^2 \;\Rightarrow\; g' = 2ff' \;\Rightarrow$ $g'' = 2f'f' + 2ff'' = 2(f')^2 + 2ff'' > 0 \;\Rightarrow\; g$ is CU on I.

55. (a) Since f and g are positive, increasing, and CU on I with f'' and g'' never equal to 0, we have $f > 0$, $f' \geq 0$, $f'' > 0$, $g > 0$, $g' \geq 0$, $g'' > 0$ on I. Then $(fg)' = f'g + fg' \;\Rightarrow$ $(fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \;\Rightarrow\; fg$ is CU on I.

(b) In part (a), if f and g are both decreasing instead of increasing, then $f' \leq 0$ and $g' \leq 0$ on I, so we still have $2f'g' \geq 0$ on I. Thus, $(fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \;\Rightarrow\; fg$ is CU on I as in part (a).

(c) Suppose f is increasing and g is decreasing [with f and g positive and CU]. Then $f' \geq 0$ and $g' \leq 0$ on I, so $2f'g' \leq 0$ on I and the argument in parts (a) and (b) fails.

Example 1. $I = (0, \infty)$, $f(x) = x^3$, $g(x) = 1/x$. Then $(fg)(x) = x^2$, so $(fg)'(x) = 2x$ and $(fg)''(x) = 2 > 0$ on I. Thus, fg is CU on I.

Example 2. $I = (0, \infty)$, $f(x) = 4x\sqrt{x}$, $g(x) = 1/x$. Then $(fg)(x) = 4\sqrt{x}$, so $(fg)'(x) = 2/\sqrt{x}$ and $(fg)''(x) = -1/\sqrt{x^3} < 0$ on I. Thus, fg is CD on I.

Example 3. $I = (0, \infty)$, $f(x) = x^2$, $g(x) = 1/x$. Thus, $(fg)(x) = x$, so fg is linear on I.

56. Since f and g are CU on $(-\infty, \infty)$, $f'' > 0$ and $g'' > 0$ on $(-\infty, \infty)$.

$h(x) = f(g(x)) \;\Rightarrow\; h'(x) = f'(g(x))g'(x) \;\Rightarrow$

$h''(x) = f''(g(x))g'(x)g'(x) + f'(g(x))g''(x) = f''(g(x))[g'(x)]^2 + f'(g(x))g''(x) > 0$ if $f' > 0$.

So h is CU if f is increasing.

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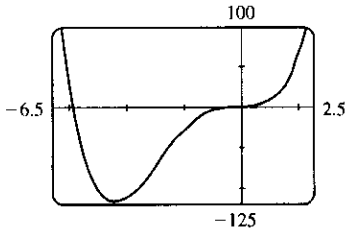
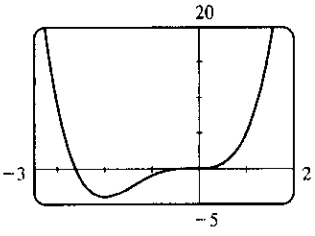
57. Let the cubic function be $f(x) = ax^3 + bx^2 + cx + d \;\Rightarrow\; f'(x) = 3ax^2 + 2bx + c \;\Rightarrow\; f''(x) = 6ax + 2b$. So f is CU when $6ax + 2b > 0 \;\Leftrightarrow\; x > -b/(3a)$, CD when $x < -b/(3a)$, and so the only point of inflection occurs when $x = -b/(3a)$. If the graph has three x-intercepts x_1, x_2 and x_3, then the expression for $f(x)$ must factor as $f(x) = a(x - x_1)(x - x_2)(x - x_3)$. Multiplying these factors together gives us $f(x) = a\big[x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3\big]$. Equating the coefficients of the x^2-terms for the two forms of f gives us $b = -a(x_1 + x_2 + x_3)$. Hence, the x-coordinate of the point of inflection is $-\dfrac{b}{3a} = -\dfrac{-a(x_1 + x_2 + x_3)}{3a} = \dfrac{x_1 + x_2 + x_3}{3}$.

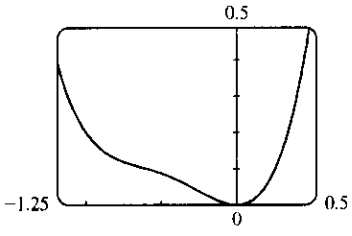
58. $P(x) = x^4 + cx^3 + x^2 \;\Rightarrow\; P'(x) = 4x^3 + 3cx^2 + 2x \;\Rightarrow\; P''(x) = 12x^2 + 6cx + 2$. The graph of $P''(x)$ is a parabola. If $P''(x)$ has two roots, then it changes sign twice and so has two inflection points. This happens when the discriminant of $P''(x)$ is positive, that is, $(6c)^2 - 4 \cdot 12 \cdot 2 > 0 \;\Leftrightarrow\; 36c^2 - 96 > 0 \;\Leftrightarrow\; |c| > \frac{2\sqrt{6}}{3} \approx 1.63$. If $36c^2 - 96 = 0 \;\Leftrightarrow\; c = \pm\frac{2\sqrt{6}}{3}$, $P''(x)$ is 0 at one point, but there is still no inflection point since $P''(x)$ never changes sign, and if $36c^2 - 96 < 0 \;\Leftrightarrow\; |c| < \frac{2\sqrt{6}}{3}$, then $P''(x)$ never changes sign, and so there is no inflection point.



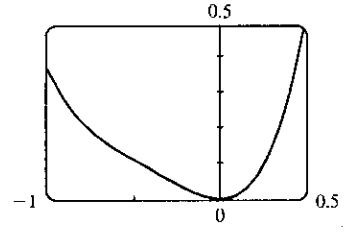
$c = 6$



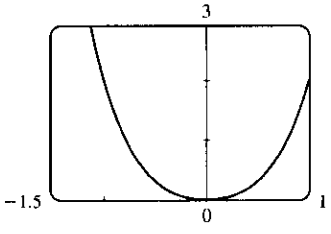
$c = 3$



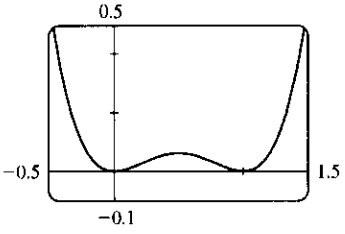
$c = 1.8$



$c = \frac{2\sqrt{6}}{3}$



$c = 0$



$c = -2$

For large positive c, the graph of f has two inflection points and a large dip to the left of the y-axis. As c decreases, the graph of f becomes flatter for $x < 0$, and eventually the dip rises above the x-axis, and then disappears entirely, along with the inflection points. As c continues to decrease, the dip and the inflection points reappear, to the right of the origin.

59. By hypothesis $g = f'$ is differentiable on an open interval containing c. Since $(c, f(c))$ is a point of inflection, the concavity changes at $x = c$, so $f''(x)$ changes signs at $x = c$. Hence, by the First Derivative Test, f' has a local extremum at $x = c$. Thus, by Fermat's Theorem $f''(c) = 0$.

60. $f(x) = x^4 \;\Rightarrow\; f'(x) = 4x^3 \;\Rightarrow\; f''(x) = 12x^2 \;\Rightarrow\; f''(0) = 0$. For $x < 0$, $f''(x) > 0$, so f is CU on $(-\infty, 0)$; for $x > 0$, $f''(x) > 0$, so f is also CU on $(0, \infty)$. Since f does not change concavity at 0, $(0, 0)$ is not an inflection point.

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61. Using the fact that $|x| = \sqrt{x^2}$, we have that $g(x) = x\sqrt{x^2} \Rightarrow g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2|x| \Rightarrow$ $g''(x) = 2x\left(x^2\right)^{-1/2} = \dfrac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$, so $(0, 0)$ is an inflection point. But $g''(0)$ does not exist.

62. There must exist some interval containing c on which f''' is positive, since $f'''(c)$ is positive and f''' is continuous. On this interval, f'' is increasing (since f''' is positive), so $f'' = (f')'$ changes from negative to positive at c. So by the First Derivative Test, f' has a local minimum at $x = c$ and thus cannot change sign there, so f has no maximum or minimum at c. But since f'' changes from negative to positive at c, f has a point of inflection at c (it changes from concave down to concave up).

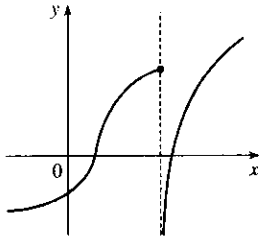
4.4 Limits at Infinity; Horizontal Asymptotes

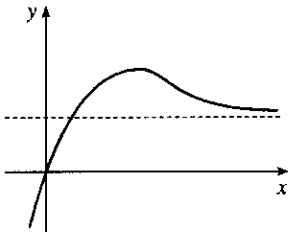
1. (a) As x becomes large, the values of $f(x)$ approach 5.

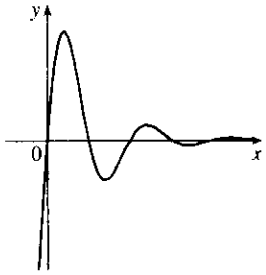
(b) As x becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

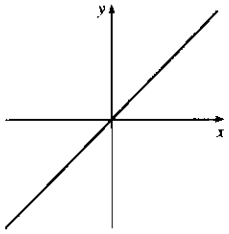
The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.



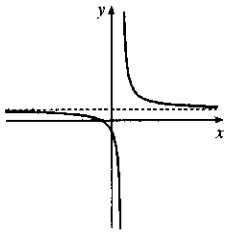




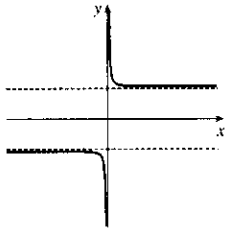
(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote



One horizontal asymptote



Two horizontal asymptotes

3. (a) $\lim_{x \to 2} f(x) = \infty$

(b) $\lim_{x \to -1^-} f(x) = \infty$

(c) $\lim_{x \to -1^+} f(x) = -\infty$

(d) $\lim_{x \to \infty} f(x) = 1$

(e) $\lim_{x \to -\infty} f(x) = 2$

(f) Vertical: $x = -1$, $x = 2$; Horizontal: $y = 1$, $y = 2$

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4. (a) $\lim_{x\to\infty} g(x) = 2$

(b) $\lim_{x\to-\infty} g(x) = -2$

(c) $\lim_{x\to 3} g(x) = \infty$

(d) $\lim_{x\to 0} g(x) = -\infty$

(e) $\lim_{x\to -2^+} g(x) = -\infty$

(f) Vertical: $x = -2$, $x = 0$, $x = 3$; Horizontal: $y = -2$, $y = 2$

5. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$.

It appears that $\lim_{x\to\infty} (x^2/2^x) = 0$.

6. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10{,}000]$ by $[0, 0.2]$, we estimate that $\lim_{x\to\infty} f(x) = 0.14$ (to two decimal places.)

(b)

x	$f(x)$
10,000	0.135308
100,000	0.135333
1,000,000	0.135335

From the table, we estimate that $\lim_{x\to\infty} f(x) = 0.1353$ (to four decimal places.)

7.
$$\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8} = \lim_{x\to\infty} \frac{(3x^2 - x + 4)/x^2}{(2x^2 + 5x - 8)/x^2}$$ [divide both the numerator and denominator by x^2 (the highest power of x that appears in the denominator)]

$$= \frac{\lim_{x\to\infty} (3 - 1/x + 4/x^2)}{\lim_{x\to\infty} (2 + 5/x - 8/x^2)}$$ [Limit Law 5]

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} (1/x) + \lim_{x\to\infty} (4/x^2)}{\lim_{x\to\infty} 2 + \lim_{x\to\infty} (5/x) - \lim_{x\to\infty} (8/x^2)}$$ [Limit Laws 1 and 2]

$$= \frac{3 - \lim_{x\to\infty} (1/x) + 4 \lim_{x\to\infty} (1/x^2)}{2 + 5 \lim_{x\to\infty} (1/x) - 8 \lim_{x\to\infty} (1/x^2)}$$ [Limit Laws 7 and 3]

$$= \frac{3 - 0 + 4(0)}{2 + 5(0) - 8(0)}$$ [Theorem 4]

$$= \frac{3}{2}$$

8.
$$\lim_{x\to\infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}} = \sqrt{\lim_{x\to\infty} \frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$$ [Limit Law 11]

$$= \sqrt{\lim_{x\to\infty} \frac{12 - 5/x^2 + 2/x^3}{1/x^3 + 4/x + 3}}$$ [divide by x^3]

$$= \sqrt{\frac{\lim_{x\to\infty} (12 - 5/x^2 + 2/x^3)}{\lim_{x\to\infty} (1/x^3 + 4/x + 3)}}$$ [Limit Law 5]

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$$= \sqrt{\frac{\lim\limits_{x\to\infty} 12 - \lim\limits_{x\to\infty} (5/x^2) + \lim\limits_{x\to\infty} (2/x^3)}{\lim\limits_{x\to\infty} (1/x^3) + \lim\limits_{x\to\infty} (4/x) + \lim\limits_{x\to\infty} 3}} \quad \text{[Limit Laws 1 and 2]}$$

$$= \sqrt{\frac{12 - 5\lim\limits_{x\to\infty} (1/x^2) + 2\lim\limits_{x\to\infty} (1/x^3)}{\lim\limits_{x\to\infty} (1/x^3) + 4\lim\limits_{x\to\infty} (1/x) + 3}} \quad \text{[Limit Laws 7 and 3]}$$

$$= \sqrt{\frac{12 - 5(0) + 2(0)}{0 + 4(0) + 3}} \quad \text{[Theorem 4]}$$

$$= \sqrt{\frac{12}{3}} = \sqrt{4} = 2$$

9. $\lim\limits_{x\to\infty} \dfrac{1}{2x+3} = \lim\limits_{x\to\infty} \dfrac{1/x}{(2x+3)/x} = \dfrac{\lim\limits_{x\to\infty} (1/x)}{\lim\limits_{x\to\infty} (2+3/x)} = \dfrac{\lim\limits_{x\to\infty} (1/x)}{\lim\limits_{x\to\infty} 2 + 3\lim\limits_{x\to\infty} (1/x)} = \dfrac{0}{2+3(0)} = \dfrac{0}{2} = 0$

10. $\lim\limits_{x\to\infty} \dfrac{3x+5}{x-4} = \lim\limits_{x\to\infty} \dfrac{(3x+5)/x}{(x-4)/x} = \lim\limits_{x\to\infty} \dfrac{3+5/x}{1-4/x} = \dfrac{\lim\limits_{x\to\infty} 3 + 5\lim\limits_{x\to\infty} \dfrac{1}{x}}{\lim\limits_{x\to\infty} 1 - 4\lim\limits_{x\to\infty} \dfrac{1}{x}} = \dfrac{3+5(0)}{1-4(0)} = 3$

11. $\lim\limits_{x\to-\infty} \dfrac{1-x-x^2}{2x^2-7} = \lim\limits_{x\to-\infty} \dfrac{(1-x-x^2)/x^2}{(2x^2-7)/x^2} = \dfrac{\lim\limits_{x\to-\infty} (1/x^2 - 1/x - 1)}{\lim\limits_{x\to-\infty} (2 - 7/x^2)}$

$$= \frac{\lim\limits_{x\to-\infty} (1/x^2) - \lim\limits_{x\to-\infty} (1/x) - \lim\limits_{x\to-\infty} 1}{\lim\limits_{x\to-\infty} 2 - 7\lim\limits_{x\to-\infty} (1/x^2)} = \frac{0-0-1}{2-7(0)} = -\frac{1}{2}$$

12. $\lim\limits_{y\to\infty} \dfrac{2-3y^2}{5y^2+4y} = \lim\limits_{y\to\infty} \dfrac{(2-3y^2)/y^2}{(5y^2+4y)/y^2} = \dfrac{\lim\limits_{y\to\infty} (2/y^2 - 3)}{\lim\limits_{y\to\infty} (5+4/y)} = \dfrac{2\lim\limits_{y\to\infty} (1/y^2) - \lim\limits_{y\to\infty} 3}{\lim\limits_{y\to\infty} 5 + 4\lim\limits_{y\to\infty} (1/y)} = \dfrac{2(0)-3}{5+4(0)} = -\dfrac{3}{5}$

13. Divide both the numerator and denominator by x^3 (the highest power of x that occurs in the denominator).

$$\lim_{x\to\infty} \frac{x^3+5x}{2x^3-x^2+4} = \lim_{x\to\infty} \frac{\dfrac{x^3+5x}{x^3}}{\dfrac{2x^3-x^2+4}{x^3}} = \lim_{x\to\infty} \frac{1+\dfrac{5}{x^2}}{2-\dfrac{1}{x}+\dfrac{4}{x^3}} = \frac{\lim\limits_{x\to\infty} \left(1+\dfrac{5}{x^2}\right)}{\lim\limits_{x\to\infty} \left(2-\dfrac{1}{x}+\dfrac{4}{x^3}\right)}$$

$$= \frac{\lim\limits_{x\to\infty} 1 + 5\lim\limits_{x\to\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \dfrac{1}{x} + 4\lim\limits_{x\to\infty} \dfrac{1}{x^3}} = \frac{1+5(0)}{2-0+4(0)} = \frac{1}{2}$$

14. $\lim\limits_{t\to-\infty} \dfrac{t^2+2}{t^3+t^2-1} = \lim\limits_{t\to-\infty} \dfrac{(t^2+2)/t^3}{(t^3+t^2-1)/t^3} = \lim\limits_{t\to-\infty} \dfrac{1/t+2/t^3}{1+1/t-1/t^3} = \dfrac{0+0}{1+0-0} = 0$

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15. First, multiply the factors in the denominator. Then divide both the numerator and denominator by u^4.

$$\lim_{u\to\infty}\frac{4u^4+5}{(u^2-2)(2u^2-1)}=\lim_{u\to\infty}\frac{4u^4+5}{2u^4-5u^2+2}=\lim_{u\to\infty}\frac{\dfrac{4u^4+5}{u^4}}{\dfrac{2u^4-5u^2+2}{u^4}}=\lim_{u\to\infty}\frac{4+\dfrac{5}{u^4}}{2-\dfrac{5}{u^2}+\dfrac{2}{u^4}}$$

$$=\frac{\lim\limits_{u\to\infty}\left(4+\dfrac{5}{u^4}\right)}{\lim\limits_{u\to\infty}\left(2-\dfrac{5}{u^2}+\dfrac{2}{u^4}\right)}=\frac{\lim\limits_{u\to\infty}4+5\lim\limits_{u\to\infty}\dfrac{1}{u^4}}{\lim\limits_{u\to\infty}2-5\lim\limits_{u\to\infty}\dfrac{1}{u^2}+2\lim\limits_{u\to\infty}\dfrac{1}{u^4}}=\frac{4+5(0)}{2-5(0)+2(0)}$$

$$=\frac{4}{2}=2$$

16. $\displaystyle\lim_{x\to\infty}\frac{x+2}{\sqrt{9x^2+1}}=\lim_{x\to\infty}\frac{(x+2)/x}{\sqrt{9x^2+1}/\sqrt{x^2}}=\lim_{x\to\infty}\frac{1+2/x}{\sqrt{9+1/x^2}}=\frac{1+0}{\sqrt{9+0}}=\frac{1}{3}$

17. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{9x^6-x}}{x^3+1}=\lim_{x\to\infty}\frac{\sqrt{9x^6-x}\,/x^3}{(x^3+1)/x^3}=\frac{\lim\limits_{x\to\infty}\sqrt{(9x^6-x)/x^6}}{\lim\limits_{x\to\infty}(1+1/x^3)}$ [since $x^3=\sqrt{x^6}$ for $x>0$]

$$=\frac{\lim\limits_{x\to\infty}\sqrt{9-1/x^5}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}(1/x^3)}=\frac{\sqrt{\lim\limits_{x\to\infty}9-\lim\limits_{x\to\infty}(1/x^5)}}{1+0}$$

$$=\sqrt{9-0}=3$$

18. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{9x^6-x}}{x^3+1}=\lim_{x\to-\infty}\frac{\sqrt{9x^6-x}\,/x^3}{(x^3+1)/x^3}=\frac{\lim\limits_{x\to-\infty}-\sqrt{(9x^6-x)/x^6}}{\lim\limits_{x\to-\infty}(1+1/x^3)}$ [since $x^3=-\sqrt{x^6}$ for $x<0$]

$$=\frac{\lim\limits_{x\to-\infty}-\sqrt{9-1/x^5}}{\lim\limits_{x\to-\infty}1+\lim\limits_{x\to-\infty}(1/x^3)}=\frac{-\sqrt{\lim\limits_{x\to-\infty}9-\lim\limits_{x\to-\infty}(1/x^5)}}{1+0}$$

$$=-\sqrt{9-0}=-3$$

19. $\displaystyle\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+4}}=\lim_{x\to-\infty}\frac{x/\sqrt{x^2}}{\sqrt{x^2+4}/\sqrt{x^2}}=\lim_{x\to-\infty}\frac{-1}{\sqrt{1+4/x^2}}$ [since $-x=|x|=\sqrt{x^2}$ for $x<0$] $=-1$

20. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^4+6x^2}-x^2\right)=\lim_{x\to\infty}\frac{\sqrt{x^4+6x^2}-x^2}{1}\cdot\frac{\sqrt{x^4+6x^2}+x^2}{\sqrt{x^4+6x^2}+x^2}=\lim_{x\to\infty}\frac{\left(x^4+6x^2\right)-x^4}{\sqrt{x^4+6x^2}+x^2}$

$$=\lim_{x\to\infty}\frac{6x^2}{\sqrt{x^4+6x^2}+x^2}=\lim_{x\to\infty}\frac{6x^2/x^2}{\left(\sqrt{x^4+6x^2}+x^2\right)/\sqrt{x^4}}$$

$$=\lim_{x\to\infty}\frac{6}{\sqrt{1+6/x^2}+1}=\frac{6}{1+1}=3$$

21. $\displaystyle\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}$

$$=\lim_{x\to\infty}\frac{\left(9x^2+x\right)-9x^2}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}\cdot\frac{1/x}{1/x}$$

$$=\lim_{x\to\infty}\frac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x}=\lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

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22. $\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)=\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)\left[\frac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}}\right]=\lim_{x\to-\infty}\frac{x^2-\left(x^2+2x\right)}{x-\sqrt{x^2+2x}}$

$$=\lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2+2x}}=\lim_{x\to-\infty}\frac{-2}{1+\sqrt{1+2/x}}=\frac{-2}{1+\sqrt{1+2\,(0)}}=-1$$

Note: In dividing numerator and denominator by x, we used the fact that for $x<0$, $x=-\sqrt{x^2}$.

23. $\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$

$$=\lim_{x\to\infty}\frac{(x^2+ax)-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\lim_{x\to\infty}\frac{[(a-b)x]/x}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)/\sqrt{x^2}}$$

$$=\lim_{x\to\infty}\frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}}=\frac{a-b}{\sqrt{1+0}+\sqrt{1+0}}=\frac{a-b}{2}$$

24. $\lim_{x\to\infty}\cos x$ does not exist because as x increases $\cos x$ does not approach any one value, but oscillates between 1 and -1.

25. $\sqrt{x}$ is large when x is large, so $\lim_{x\to\infty}\sqrt{x}=\infty$.

26. $\sqrt[3]{x}$ is large negative when x is large negative, so $\lim_{x\to-\infty}\sqrt[3]{x}=-\infty$.

27. $\lim_{x\to\infty}\left(x-\sqrt{x}\right)=\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x}-1\right)=\infty$ since $\sqrt{x}\to\infty$ and $\sqrt{x}-1\to\infty$ as $x\to\infty$.

28. $\lim_{x\to\infty}\frac{x^3-2x+3}{5-2x^2}=\lim_{x\to\infty}\frac{(x^3-2x+3)/x^2}{(5-2x^2)/x^2}$ [divide by the highest power of x in the denominator]

$=\lim_{x\to\infty}\frac{x-2/x+3/x^2}{5/x^2-2}=-\infty$ because $x-2/x+3/x^2\to\infty$ and $5/x^2-2\to-2$ as $x\to\infty$.

29. $\lim_{x\to-\infty}\left(x^4+x^5\right)=\lim_{x\to-\infty}x^5\left(\frac{1}{x}+1\right)$ [factor out the largest power of x] $=-\infty$ because $x^5\to-\infty$ and $1/x+1\to1$ as $x\to-\infty$.

30. $\lim_{x\to\infty}\left(x^2-x^4\right)=\lim_{x\to\infty}x^2\left(1-x^2\right)=-\infty$ since $x^2\to\infty$ and $1-x^2\to-\infty$.

31. $\lim_{x\to\infty}\frac{x+x^3+x^5}{1-x^2+x^4}=\lim_{x\to\infty}\frac{(x+x^3+x^5)/x^4}{(1-x^2+x^4)/x^4}$ [divide by the highest power of x in the denominator]

$$=\lim_{x\to\infty}\frac{1/x^3+1/x+x}{1/x^4-1/x^2+1}=\infty$$

because $(1/x^3+1/x+x)\to\infty$ and $(1/x^4-1/x^2+1)\to1$ as $x\to\infty$.

32. If $t=\frac{1}{x}$, then $\lim_{x\to\infty}x\sin\frac{1}{x}=\lim_{t\to0^+}\frac{1}{t}\sin t=\lim_{t\to0^+}\frac{\sin t}{t}=1$.

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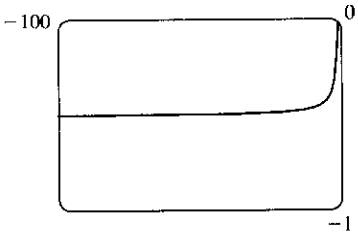
33. (a)



From the graph of $f(x) = \sqrt{x^2 + x + 1} + x$, we estimate the value of $\lim_{x \to -\infty} f(x)$ to be -0.5.

(b)

x	$f(x)$
$-10{,}000$	-0.4999625
$-100{,}000$	-0.4999962
$-1{,}000{,}000$	-0.4999996

From the table, we estimate the limit to be -0.5.

(c) $$\lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right) = \lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right) \left[\frac{\sqrt{x^2 + x + 1} - x}{\sqrt{x^2 + x + 1} - x}\right] = \lim_{x \to -\infty} \frac{(x^2 + x + 1) - x^2}{\sqrt{x^2 + x + 1} - x}$$

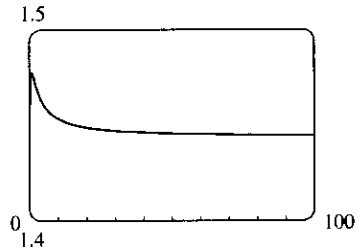
$$= \lim_{x \to -\infty} \frac{(x + 1)(1/x)}{\left(\sqrt{x^2 + x + 1} - x\right)(1/x)} = \lim_{x \to -\infty} \frac{1 + (1/x)}{-\sqrt{1 + (1/x) + (1/x^2)} - 1}$$

$$= \frac{1 + 0}{-\sqrt{1 + 0 + 0} - 1} = -\frac{1}{2}$$

Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x, with $x < 0$, we get

$$\frac{1}{x}\sqrt{x^2 + x + 1} = -\frac{1}{\sqrt{x^2}}\sqrt{x^2 + x + 1} = -\sqrt{1 + (1/x) + (1/x^2)}.$$

34. (a)



From the graph of $f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$, we estimate (to one decimal place) the value of $\lim_{x \to \infty} f(x)$ to be 1.4.

(b)

x	$f(x)$
10,000	1.44339
100,000	1.44338
1,000,000	1.44338

From the table, we estimate (to four decimal places) the limit to be 1.4434.

(c) $$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\left(\sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}\right)\left(\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}\right)}{\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}}$$

$$= \lim_{x \to \infty} \frac{\left(3x^2 + 8x + 6\right) - \left(3x^2 + 3x + 1\right)}{\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}}$$

$$= \lim_{x \to \infty} \frac{(5x + 5)(1/x)}{\left(\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}\right)(1/x)}$$

$$= \lim_{x \to \infty} \frac{5 + 5/x}{\sqrt{3 + 8/x + 6/x^2} + \sqrt{3 + 3/x + 1/x^2}} = \frac{5}{\sqrt{3} + \sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376$$

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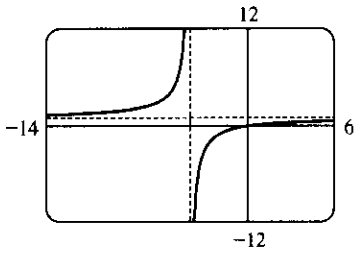
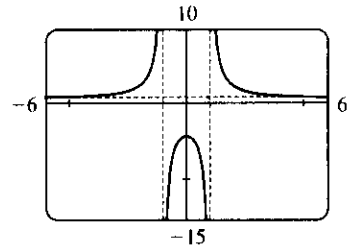
35. $\lim_{x\to\pm\infty}\dfrac{x}{x+4}=\lim_{x\to\pm\infty}\dfrac{1}{1+4/x}=\dfrac{1}{1+0}=1$, so $y=1$ is a horizontal asymptote. $\lim_{x\to-4^-}\dfrac{x}{x+4}=\infty$ and $\lim_{x\to-4^+}\dfrac{x}{x+4}=-\infty$, so $x=-4$ is a vertical asymptote. The graph confirms these calculations.

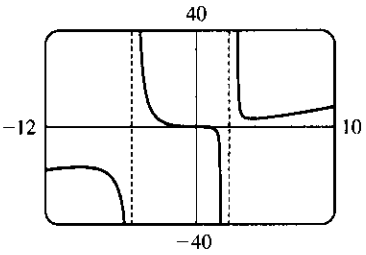


36. Since $x^2-1\to 0$ as $x\to\pm1$ and $y<0$ for $-1<x<1$ and $y>0$ for $x<-1$ and $x>1$, we have $\lim_{x\to1^-}\dfrac{x^2+4}{x^2-1}=-\infty$, $\lim_{x\to1^+}\dfrac{x^2+4}{x^2-1}=\infty$, $\lim_{x\to-1^-}\dfrac{x^2+4}{x^2-1}=\infty$, and $\lim_{x\to-1^+}\dfrac{x^2+4}{x^2-1}=-\infty$, so $x=1$ and $x=-1$ are vertical asymptotes. Also $\lim_{x\to\pm\infty}\dfrac{x^2+4}{x^2-1}=\lim_{x\to\pm\infty}\dfrac{1+4/x^2}{1-1/x^2}=\dfrac{1+0}{1-0}=1$, so $y=1$ is a horizontal asymptote. The graph confirms these calculations.

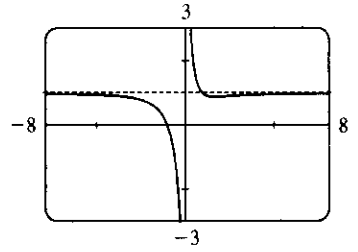


37. $\lim_{x\to\pm\infty}\dfrac{x^3}{x^2+3x-10}=\lim_{x\to\pm\infty}\dfrac{x}{1+(3/x)-(10/x^2)}=\pm\infty$, so there is no horizontal asymptote.

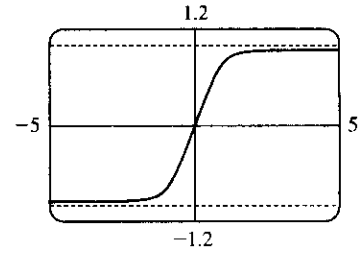
$\lim_{x\to2^+}\dfrac{x^3}{x^2+3x-10}=\lim_{x\to2^+}\dfrac{x^3}{(x+5)(x-2)}=\infty$, since $\dfrac{x^3}{(x+5)(x-2)}>0$ for $x>2$. Similarly, $\lim_{x\to2^-}\dfrac{x^3}{x^2+3x-10}=-\infty$, $\lim_{x\to-5^-}\dfrac{x^3}{x^2+3x-10}=-\infty$, and $\lim_{x\to-5^+}\dfrac{x^3}{x^2+3x-10}=\infty$, so $x=2$ and $x=-5$ are vertical asymptotes. The graph confirms these calculations.



38. $\lim_{x\to\pm\infty}\dfrac{x^3+1}{x^3+x}=\lim_{x\to\pm\infty}\dfrac{1+1/x^3}{1+1/x^2}=1$, so $y=1$ is a horizontal asymptote. Since $y=\dfrac{x^3+1}{x^3+x}=\dfrac{x^3+1}{x(x^2+1)}>0$ for $x>0$ and $y<0$ for $-1<x<0$, $\lim_{x\to0^+}\dfrac{x^3+1}{x^3+x}=\infty$ and $\lim_{x\to0^-}\dfrac{x^3+1}{x^3+x}=-\infty$, so $x=0$ is a vertical asymptote.



39. $\lim_{x\to\infty}\dfrac{x}{\sqrt[4]{x^4+1}}\cdot\dfrac{1/x}{1/\sqrt[4]{x^4}}=\lim_{x\to\infty}\dfrac{1}{\sqrt[4]{1+\dfrac{1}{x^4}}}=\dfrac{1}{\sqrt[4]{1+0}}=1$ and $\lim_{x\to-\infty}\dfrac{x}{\sqrt[4]{x^4+1}}\cdot\dfrac{1/x}{-1/\sqrt[4]{x^4}}=\lim_{x\to-\infty}\dfrac{1}{-\sqrt[4]{1+\dfrac{1}{x^4}}}=\dfrac{1}{-\sqrt[4]{1+0}}=-1$, so $y=\pm1$ are horizontal asymptotes. There is no vertical asymptote.



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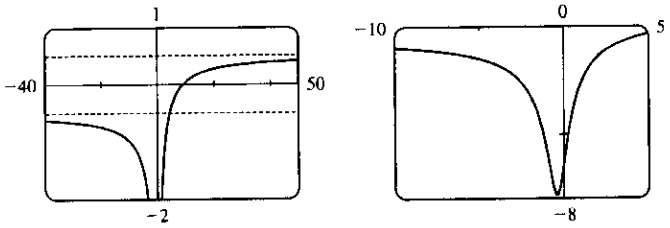
40. $\lim_{x\to\infty} \dfrac{x-9}{\sqrt{4x^2+3x+2}} = \lim_{x\to\infty} \dfrac{1-9/x}{\sqrt{4+(3/x)+(2/x^2)}} = \dfrac{1-0}{\sqrt{4+0+0}} = \dfrac{1}{2}$.

Using the fact that $\sqrt{x^2} = |x| = -x$ for $x < 0$, we divide the numerator by $-x$ and the denominator by $\sqrt{x^2}$.

Thus, $\lim_{x\to-\infty} \dfrac{x-9}{\sqrt{4x^2+3x+2}} = \lim_{x\to-\infty} \dfrac{-1+9/x}{\sqrt{4+(3/x)+(2/x^2)}} = \dfrac{-1+0}{\sqrt{4+0+0}} = -\dfrac{1}{2}$.

The horizontal asymptotes are $y = \pm\frac{1}{2}$. The polynomial $4x^2+3x+2$ is positive for all x, so the denominator never approaches zero, and thus there is no vertical asymptote.



41. Let's look for a rational function.

(1) $\lim_{x\to\pm\infty} f(x) = 0 \quad\Rightarrow\quad$ degree of numerator < degree of denominator

(2) $\lim_{x\to0} f(x) = -\infty \quad\Rightarrow\quad$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim_{x\to3} f(x) = \infty$ and $\lim_{x\to3^+} f(x) = -\infty \quad\Rightarrow\quad$ vertical asymptote at $x = 3$; there is a factor of $(x-3)$ in the denominator.

(4) $f(2) = 0 \quad\Rightarrow\quad 2$ is an x-intercept; there is at least one factor of $(x-2)$ in the numerator.

Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us $f(x) = \dfrac{2-x}{x^2(x-3)}$ as one possibility.

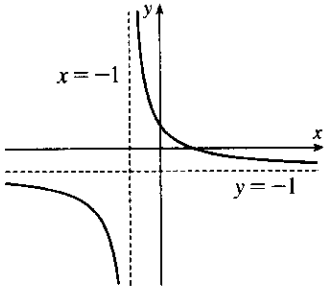
42. Since the function has vertical asymptotes $x = 1$ and $x = 3$, the denominator of the rational function we are looking for must have factors $(x-1)$ and $(x-3)$. Because the horizontal asymptote is $y = 1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility is $f(x) = \dfrac{x^2}{(x-1)(x-3)}$.

43. $y = \dfrac{1-x}{1+x}$ has domain $(-\infty,-1)\cup(-1,\infty)$.

$\lim_{x\to\pm\infty} \dfrac{1-x}{1+x} = \lim_{x\to\pm\infty} \dfrac{1/x-1}{1/x+1} = \dfrac{0-1}{0+1} = -1$, so $y = -1$ is a HA.

The line $x = -1$ is a VA.

$y' = \dfrac{(1+x)(-1)-(1-x)(1)}{(1+x)^2} = \dfrac{-2}{(1+x)^2} < 0$ for $x \neq 1$. Thus, $(-\infty,-1)$ and $(-1,\infty)$ are intervals of decrease.



$y'' = -2\cdot\dfrac{-2(1+x)}{[(1+x)^2]^2} = \dfrac{4}{(1+x)^3} < 0$ for $x < -1$ and $y'' > 0$ for $x > -1$, so the curve is CD on $(-\infty,-1)$ and CU on $(-1,\infty)$. Since $x = -1$ is not in the domain, there is no IP.

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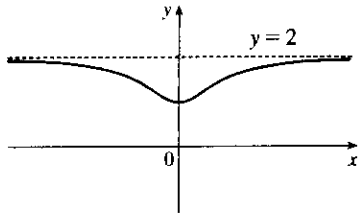
44. $y = \dfrac{1+2x^2}{1+x^2}$ has domain $\mathbb{R}$.



$\lim\limits_{x\to\pm\infty} \dfrac{1+2x^2}{1+x^2} = \lim\limits_{x\to\pm\infty} \dfrac{1/x^2+2}{1/x^2+1} = \dfrac{0+2}{0+1} = 2$, so $y = 2$ is a HA.

There is no VA. $y' = \dfrac{(1+x^2)(4x) - (1+2x^2)(2x)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2} > 0$

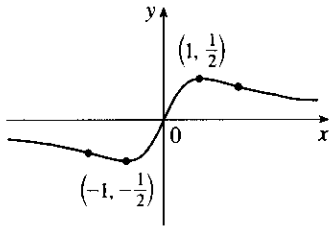
$\Leftrightarrow \quad x > 0$,

and $y' < 0 \quad \Leftrightarrow \quad x < 0$. Thus, y is increasing on $(0,\infty)$ and y is decreasing on $(-\infty, 0)$. There is a local (and absolute) minimum at $(0,1)$. $y'' = \dfrac{(1+x^2)^2(2) - (2x)\cdot 2(1+x^2)(2x)}{\left[(1+x^2)^2\right]^2} = \dfrac{2-6x^2}{(1+x^2)^3} = 0 \quad \Leftrightarrow \quad x = \pm\frac{1}{\sqrt{3}}$.

$y'' > 0 \quad \Leftrightarrow \quad -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, so the curve is CU on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and CD on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

There are IP at $\left(\pm\frac{1}{\sqrt{3}}, \frac{5}{4}\right)$.

45. $\lim\limits_{x\to\pm\infty} \dfrac{x}{x^2+1} = \lim\limits_{x\to\pm\infty} \dfrac{1/x}{1+1/x^2} = \dfrac{0}{1+0} = 0$, so $y = 0$ is a horizontal asymptote.

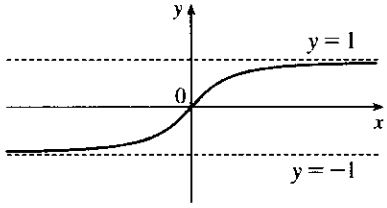


$y' = \dfrac{x^2+1-x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = 0$ when $x = \pm 1$ and $y' > 0 \quad \Leftrightarrow$

$x^2 < 1 \quad \Leftrightarrow \quad -1 < x < 1$, so y is increasing on $(-1,1)$ and decreasing on $(-\infty,-1)$ and $(1,\infty)$.

$y'' = \dfrac{(1+x^2)^2(-2x) - (1-x^2)2(x^2+1)2x}{(1+x^2)^4} = \dfrac{2x(x^2-3)}{(1+x^2)^3} > 0 \quad \Leftrightarrow \quad x > \sqrt{3}$ or $-\sqrt{3} < x < 0$, so y is CU on $(\sqrt{3},\infty)$ and $(-\sqrt{3}, 0)$ and CD on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

46. $y = \dfrac{x}{\sqrt{x^2+1}} = \dfrac{x/|x|}{\sqrt{1+1/x^2}}$ has domain $\mathbb{R}$. As $x \to \pm\infty$, $y \to \pm 1$, so $y = \pm 1$ are HA. There is no VA. $y = x(x^2+1)^{-1/2} \quad \Rightarrow$



$$y' = x\left(-\tfrac{1}{2}\right)(x^2+1)^{-3/2}(2x) + (x^2+1)^{-1/2}(1)$$

$$= (x^2+1)^{-3/2}\left[-x^2 + (x^2+1)\right]$$

$$= (x^2+1)^{-3/2} > 0 \text{ for all } x$$

Thus, y is increasing for all x. $y'' = \left(-\frac{3}{2}\right)(x^2+1)^{-5/2}(2x) = \dfrac{-3x}{(x^2+1)^{5/2}} > 0$ for $x < 0$. So the curve is CU on $(-\infty, 0)$ and CD on $(0,\infty)$. There is an inflection point at $(0,0)$.

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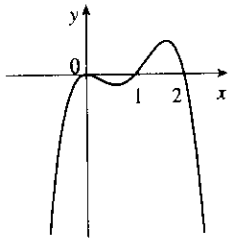
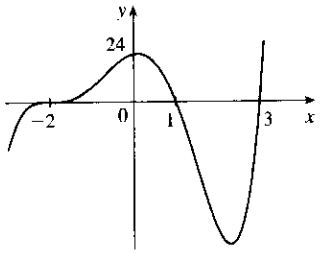
47. $y = f(x) = x^2(x-2)(1-x)$. The y-intercept is $f(0) = 0$, and the x-intercepts occur when $y = 0 \Rightarrow x = 0$, 1, and 2. Notice that, since x^2 is always positive, the graph does not cross the x-axis at 0, but does cross the x-axis at 1 and 2.

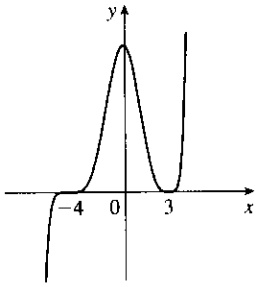
$\lim_{x\to\infty} x^2(x-2)(1-x) = -\infty$, since the first two factors are large positive and the third large negative when x is large positive. $\lim_{x\to-\infty} x^2(x-2)(1-x) = -\infty$ because the first and third factors are large positive and the second large negative as $x \to -\infty$.



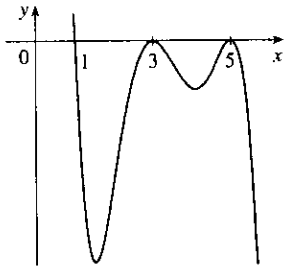
48. $y = (2+x)^3(1-x)(3-x)$. As $x \to \infty$, the first factor is large positive, and the second and third factors are large negative. Therefore, $\lim_{x\to\infty} f(x) = \infty$. As $x \to -\infty$, the first factor is large negative, and the second and third factors are large positive. Therefore, $\lim_{x\to-\infty} f(x) = -\infty$. Now the y-intercept is $f(0) = (2)^3(1)(3) = 24$ and the x-intercepts are the solutions to $f(x) = 0 \Rightarrow x = -2$, 1 and 3, and the graph crosses the x-axis at all of these points.



49. $y = f(x) = (x+4)^5(x-3)^4$. The y-intercept is $f(0) = 4^5(-3)^4 = 82{,}944$. The x-intercepts occur when $y = 0 \Rightarrow x = -4$, 3. Notice that the graph does not cross the x-axis at 3 because $(x-3)^4$ is always positive, but does cross the x-axis at -4. $\lim_{x\to\infty} (x+4)^5(x-3)^4 = \infty$ since both factors are large positive when x is large positive. $\lim_{x\to-\infty} (x+4)^5(x-3)^4 = -\infty$ since the first factor is large negative and the second factor is large positive when x is large negative.



50. $y = (1-x)(x-3)^2(x-5)^2$. As $x \to \infty$, the first factor approaches $-\infty$ while the second and third factors approach ∞. Therefore, $\lim_{x\to\infty} (x) = -\infty$. As $x \to -\infty$, the factors all approach ∞. Therefore, $\lim_{x\to-\infty} (x) = \infty$. Now the y-intercept is $f(0) = (1)(-3)^2(-5)^2 = 225$ and the x-intercepts are the solutions to $f(x) = 0 \Rightarrow x = 1$, 3, and 5. Notice that $f(x)$ does not change sign at $x = 3$ or $x = 5$ because the factors $(x-3)^2$ and $(x-5)^2$ are always positive, so the graph does not cross the x-axis at $x = 3$ or $x = 5$, but does cross the x-axis at $x = 1$.



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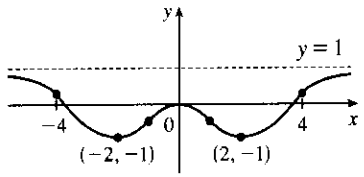
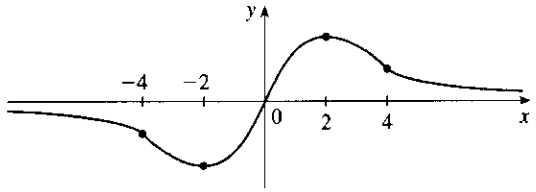
51. First we plot the points which are known to be on the graph: $(2, -1)$ and $(0, 0)$. We can also draw a short line segment of slope 0 at $x = 2$, since we are given that $f'(2) = 0$. Now we know that $f'(x) < 0$ (that is, the function is decreasing) on $(0, 2)$, and that $f''(x) < 0$ on $(0, 1)$ and $f''(x) > 0$ on $(1, 2)$. So we must join the points $(0, 0)$ and $(2, -1)$ in such a way that the curve is concave down on $(0, 1)$ and concave up on $(1, 2)$. The curve must be concave up and increasing on $(2, 4)$ and concave down and increasing toward $y = 1$ on $(4, \infty)$. Now we just need to reflect the curve in the y-axis, since we are given that f is an even function [the condition that $f(-x) = f(x)$ for all x].

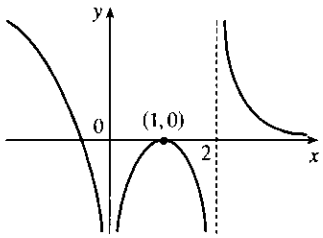


52. The diagram shows one possible function satisfying all of the given conditions.

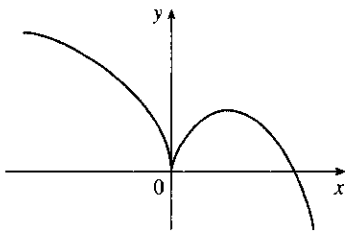


53. We are given that $f(1) = f'(1) = 0$. So we can draw a short horizontal line at the point $(1, 0)$ to represent this situation. We are given that $x = 0$ and $x = 2$ are vertical asymptotes, with $\lim_{x \to 0} f(x) = -\infty$, $\lim_{x \to 2^+} f(x) = \infty$ and $\lim_{x \to 2^-} f(x) = -\infty$, so we can draw the parts of the curve which approach these asymptotes.

On the interval $(-\infty, 0)$, the graph is concave down, and $f(x) \to \infty$ as $x \to -\infty$. Between the asymptotes the graph is concave down. On the interval $(2, \infty)$ the graph is concave up, and $f(x) \to 0$ as $x \to \infty$, so $y = 0$ is a horizontal asymptote. The diagram shows one possible function satisfying all of the given conditions.

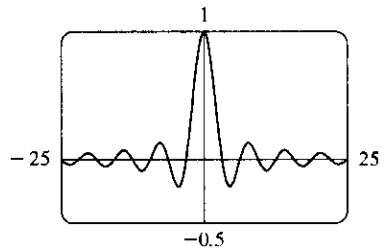


54. The diagram shows one possible function satisfying all of the given conditions.



55. (a) Since $-1 \le \sin x \le 1$ for all x, $-\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$ for $x > 0$. As $x \to \infty$, $-1/x \to 0$ and $1/x \to 0$, so by the Squeeze Theorem, $(\sin x)/x \to 0$. Thus, $\lim_{x \to \infty} \dfrac{\sin x}{x} = 0$.

(b) From part (a), the horizontal asymptote is $y = 0$. The function $y = (\sin x)/x$ crosses the horizontal asymptote whenever $\sin x = 0$; that is, at $x = \pi n$ for every integer n. Thus, the graph crosses the asymptote *an infinite number of times.*



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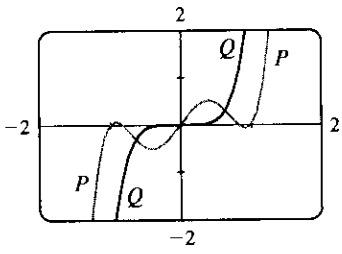
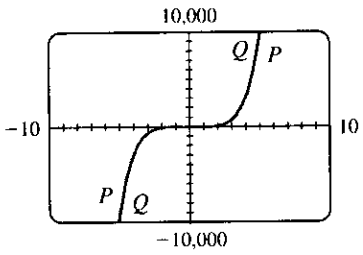
56. (a) In both viewing rectangles, $\lim_{x\to\infty} P(x) = \lim_{x\to\infty} Q(x) = \infty$ and $\lim_{x\to-\infty} P(x) = \lim_{x\to-\infty} Q(x) = -\infty$. In the larger viewing rectangle, P and Q become less distinguishable.





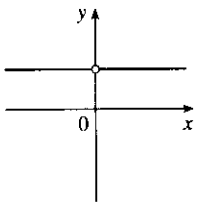
(b) $\lim_{x\to\infty} \dfrac{P(x)}{Q(x)} = \lim_{x\to\infty} \dfrac{3x^5 - 5x^3 + 2x}{3x^5} = \lim_{x\to\infty} \left(1 - \dfrac{5}{3}\cdot\dfrac{1}{x^2} + \dfrac{2}{3}\cdot\dfrac{1}{x^4}\right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \Rightarrow$ P and Q have the same end behavior.

57. Divide the numerator and the denominator by the highest power of x in $Q(x)$.

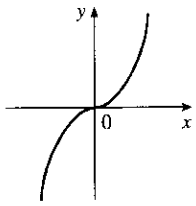
(a) If $\deg P < \deg Q$, then the numerator $\to 0$ but the denominator doesn't. So $\lim_{x\to\infty} [P(x)/Q(x)] = 0$.

(b) If $\deg P > \deg Q$, then the numerator $\to \pm\infty$ but the denominator doesn't, so $\lim_{x\to\infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

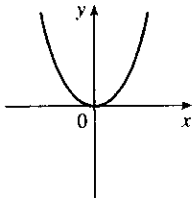
58.



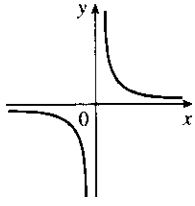
(i) $n = 0$



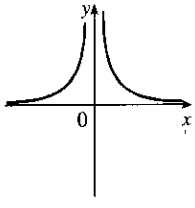
(ii) $n > 0$ (n odd)



(iii) $n > 0$ (n even)



(iv) $n < 0$ (n odd)



(v) $n < 0$ (n even)

From these sketches we see that

(a) $\lim_{x\to0^+} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ \infty & \text{if } n < 0 \end{cases}$

(b) $\lim_{x\to0^-} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ -\infty & \text{if } n < 0,\ n \text{ odd} \\ \infty & \text{if } n < 0,\ n \text{ even} \end{cases}$

(c) $\lim_{x\to\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \\ 0 & \text{if } n < 0 \end{cases}$

(d) $\lim_{x\to-\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ -\infty & \text{if } n > 0,\ n \text{ odd} \\ \infty & \text{if } n > 0,\ n \text{ even} \\ 0 & \text{if } n < 0 \end{cases}$

59. $\lim_{x\to\infty} \dfrac{4x-1}{x} = \lim_{x\to\infty} \left(4 - \dfrac{1}{x}\right) = 4$, and $\lim_{x\to\infty} \dfrac{4x^2+3x}{x^2} = \lim_{x\to\infty} \left(4 + \dfrac{3}{x}\right) = 4$. Therefore, by the Squeeze Theorem, $\lim_{x\to\infty} f(x) = 4$.

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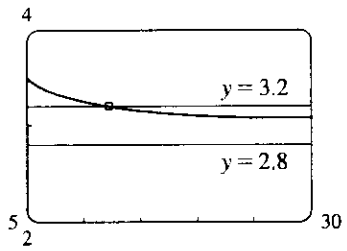
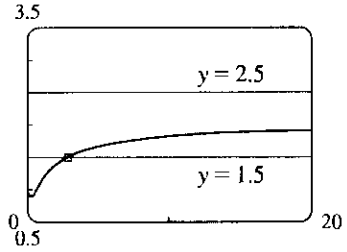
60. (a) After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000 + 25t)$ liters of water and $25t \cdot 30 = 750t$ grams of salt. Therefore, the salt concentration at time t will be $C(t) = \dfrac{750t}{5000 + 25t} = \dfrac{30t}{200 + t}$ $\dfrac{\text{g}}{\text{L}}$.

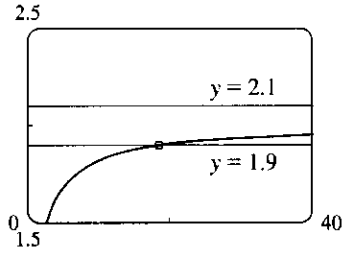
(b) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty} \dfrac{30t}{200 + t} = \lim\limits_{t\to\infty} \dfrac{30t/t}{200/t + t/t} = \dfrac{30}{0 + 1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

61. $\left|\dfrac{6x^2 + 5x - 3}{2x^2 - 1} - 3\right| < 0.2 \quad \Leftrightarrow \quad 2.8 < \dfrac{6x^2 + 5x - 3}{2x^2 - 1} < 3.2$. So we graph the three parts of this inequality on the same screen, and find that the curve $y = \dfrac{6x^2 + 5x - 3}{2x^2 - 1}$ seems to lie between the lines $y = 2.8$ and $y = 3.2$ whenever $x > 12.8$. So we can choose $N = 13$ (or any larger number) so that the inequality holds whenever $x \geq N$.



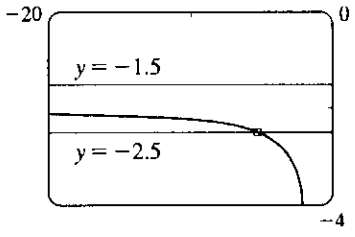
62. For $\varepsilon = 0.5$, we must find N such that whenever $x \geq N$, we have $\left|\dfrac{\sqrt{4x^2 + 1}}{x + 1} - 2\right| < 0.5 \quad \Leftrightarrow \quad 1.5 < \dfrac{\sqrt{4x^2 + 1}}{x + 1} < 2.5$. We graph the three parts of this inequality on the same screen, and find that it holds whenever $x \geq 3$. So we choose $N = 3$ (or any larger number). For $\varepsilon = 0.1$, we must have $1.9 < \dfrac{\sqrt{4x^2 + 1}}{x + 1} < 2.1$, and the graphs show that this holds whenever $x \geq 19$. So we choose $N = 19$ (or any larger number).

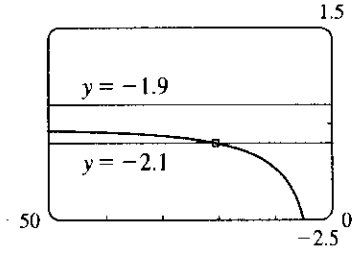




63. For $\varepsilon = 0.5$, we need to find N such that $\left|\dfrac{\sqrt{4x^2 + 1}}{x + 1} - (-2)\right| < 0.5$ $\Leftrightarrow \quad -2.5 < \dfrac{\sqrt{4x^2 + 1}}{x + 1} < -1.5$ whenever $x \leq N$. We graph the three parts of this inequality on the same screen, and see that the inequality holds for $x \leq -6$. So we choose $N = -6$ (or any smaller number).

For $\varepsilon = 0.1$, we need $-2.1 < \dfrac{\sqrt{4x^2 + 1}}{x + 1} < -1.9$ whenever $x \leq N$. From the graph, it seems that this inequality holds for $x \leq -22$. So we choose $N = -22$ (or any smaller number).





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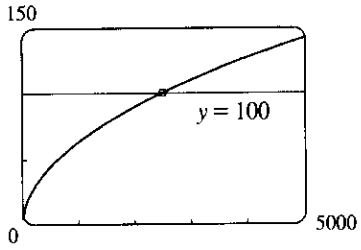
64. We need N such that $\dfrac{2x+1}{\sqrt{x+1}} > 100$ whenever $x \geq N$. From the graph, we see that this inequality holds for $x \geq 2500$. So we choose $N = 2500$ (or any larger number).



65. (a) $1/x^2 < 0.0001 \quad \Leftrightarrow \quad x^2 > 1/0.0001 = 10{,}000 \quad \Leftrightarrow \quad x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \quad \Leftrightarrow \quad x^2 > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\sqrt{\varepsilon}} \quad \Rightarrow \quad \left|\dfrac{1}{x^2} - 0\right| = \dfrac{1}{x^2} < \varepsilon$, so $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$.

66. (a) $1/\sqrt{x} < 0.0001 \quad \Leftrightarrow \quad \sqrt{x} > 1/0.0001 = 10^4 \quad \Leftrightarrow \quad x > 10^8$

(b) If $\varepsilon > 0$ is given, then $1/\sqrt{x} < \varepsilon \quad \Leftrightarrow \quad \sqrt{x} > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\varepsilon^2$. Let $N = 1/\varepsilon^2$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\varepsilon^2} \quad \Rightarrow \quad \left|\dfrac{1}{\sqrt{x}} - 0\right| = \dfrac{1}{\sqrt{x}} < \varepsilon$, so $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x}} = 0$.

67. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \quad \Leftrightarrow \quad x < -1/\varepsilon$.

Take $N = -1/\varepsilon$. Then $x < N \quad \Rightarrow \quad x < -1/\varepsilon \quad \Rightarrow \quad |(1/x) - 0| = -1/x < \varepsilon$, so $\lim\limits_{x\to-\infty} (1/x) = 0$.

68. Given $M > 0$, we need $N > 0$ such that $x > N \quad \Rightarrow \quad x^3 > M$. Now $x^3 > M \quad \Leftrightarrow \quad x > \sqrt[3]{M}$, so take $N = \sqrt[3]{M}$. Then $x > N = \sqrt[3]{M} \quad \Rightarrow \quad x^3 > M$, so $\lim\limits_{x\to\infty} x^3 = \infty$.

69. Suppose that $\lim\limits_{x\to\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \quad \Leftrightarrow \quad 0 < 1/x < 1/N \quad \Leftrightarrow \quad 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that $\lim\limits_{t\to0^+} f(1/t) = L = \lim\limits_{x\to\infty} f(x)$.

Now suppose that $\lim\limits_{x\to-\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \quad \Leftrightarrow \quad 1/N < 1/x < 0 \quad \Leftrightarrow \quad 1/N < t < 0$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\lim\limits_{t\to0^-} f(1/t) = L = \lim\limits_{x\to-\infty} f(x)$.

70. Definition Let f be a function defined on some interval $(-\infty, a)$. Then $\lim\limits_{x\to-\infty} f(x) = -\infty$ means that for every negative number M there is a corresponding negative number N such that $f(x) < M$ whenever $x < N$. Now we use the definition to prove that $\lim\limits_{x\to-\infty} (1 + x^3) = -\infty$. Given a negative number M, we need a negative number N such that $x < N \quad \Rightarrow \quad 1 + x^3 < M$. Now $1 + x^3 < M \quad \Leftrightarrow \quad x^3 < M - 1 \quad \Leftrightarrow \quad x < \sqrt[3]{M-1}$. Thus, we take $N = \sqrt[3]{M-1}$ and find that $x < N \quad \Rightarrow \quad 1 + x^3 < M$. This proves that $\lim\limits_{x\to-\infty} (1 + x^3) = -\infty$.

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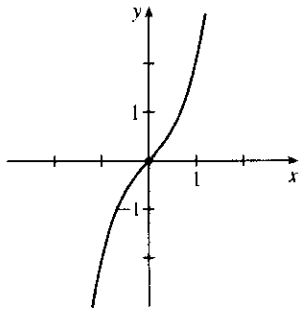
4.5 Summary of Curve Sketching

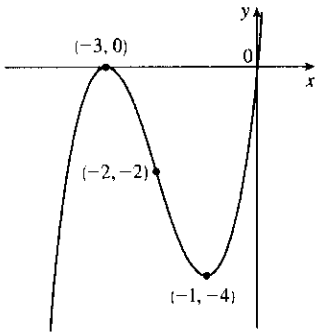
1. $y = f(x) = x^3 + x = x(x^2 + 1)$ **A.** f is a polynomial, so $D = \mathbb{R}$. **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** f is a polynomial, so there is no asymptote. **E.** $f'(x) = 3x^2 + 1 > 0$, so f is increasing on $(-\infty, \infty)$. **F.** There is no critical number and hence, no local maximum or minimum value. **G.** $f''(x) = 6x > 0$ on $(0, \infty)$ and $f''(x) < 0$ on $(-\infty, 0)$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. Since the concavity changes at $x = 0$, there is an inflection point at $(0, 0)$.

H.



2. $y = f(x) = x^3 + 6x^2 + 9x = x(x + 3)^2$ **A.** $D = \mathbb{R}$ **B.** x-intercepts are -3 and 0, y-intercept $= 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 3x^2 + 12x + 9 = 3(x + 1)(x + 3) < 0 \Leftrightarrow -3 < x < -1$, so f is decreasing on $(-3, -1)$ and increasing on $(-\infty, -3)$ and $(-1, \infty)$. **F.** Local maximum value $f(-3) = 0$, local minimum value $f(-1) = -4$ **G.** $f''(x) = 6x + 12 = 6(x + 2) > 0 \Leftrightarrow x > -2$, so f is CU on $(-2, \infty)$ and CD on $(-\infty, -2)$. IP at $(-2, -2)$

H.

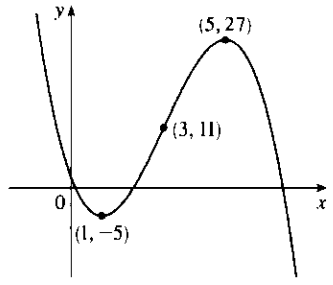


3. $y = f(x) = 2 - 15x + 9x^2 - x^3 = -(x - 2)(x^2 - 7x + 1)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$; x-intercepts: $f(x) = 0 \Rightarrow x = 2$ or (by the quadratic formula) $x = \frac{7 \pm \sqrt{45}}{2} \approx 0.15, 6.85$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = -15 + 18x - 3x^2 = -3(x^2 - 6x + 5)$

$= -3(x - 1)(x - 5) > 0 \Leftrightarrow 1 < x < 5$

so f is increasing on $(1, 5)$ and decreasing on $(-\infty, 1)$ and $(5, \infty)$. **F.** Local maximum value $f(5) = 27$, local minimum value $f(1) = -5$ **G.** $f''(x) = 18 - 6x = -6(x - 3) > 0 \Leftrightarrow x < 3$, so f is CU on $(-\infty, 3)$ and CD on $(3, \infty)$. IP at $(3, 11)$

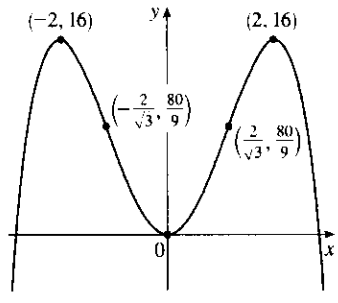
H.



4. $y = f(x) = 8x^2 - x^4 = x^2(8 - x^2)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 0, \pm 2\sqrt{2}$ $(\approx \pm 2.83)$ **C.** $f(-x) = f(x)$, so f is even and symmetric about the y-axis. **D.** No asymptote **E.** $f'(x) = 16x - 4x^3 = 4x(4 - x^2) = 4x(2 + x)(2 - x) > 0 \Leftrightarrow$ $x < -2$ or $0 < x < 2$, so f is increasing on $(-\infty, -2)$ and $(0, 2)$ and decreasing on $(-2, 0)$ and $(2, \infty)$. **F.** Local maximum value $f(\pm 2) = 16$, local minimum value $f(0) = 0$ **G.** $f''(x) = 16 - 12x^2 = 4(4 - 3x^2) = 0 \Leftrightarrow x = \pm \frac{2}{\sqrt{3}}$.

$f''(x) > 0 \Leftrightarrow -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$, so f is CU on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and CD on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$. IP at $\left(\pm\frac{2}{\sqrt{3}}, \frac{80}{9}\right)$

H.



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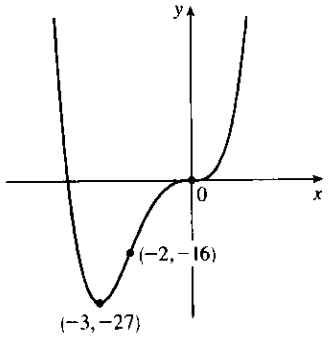
5. $y = f(x) = x^4 + 4x^3 = x^3(x+4)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = -4, 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3) > 0 \Leftrightarrow x > -3$, so f is increasing on $(-3, \infty)$ and decreasing on $(-\infty, -3)$. **F.** Local minimum value $f(-3) = -27$, no local maximum **G.** $f''(x) = 12x^2 + 24x = 12x(x+2) < 0 \Leftrightarrow -2 < x < 0$, so f is CD on $(-2, 0)$ and CU on $(-\infty, -2)$ and $(0, \infty)$. IP at $(0, 0)$ and $(-2, -16)$

H.



6. $y = f(x) = x(x+2)^3$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = -2, 0$ **C.** No symmetry **D.** No asymptote

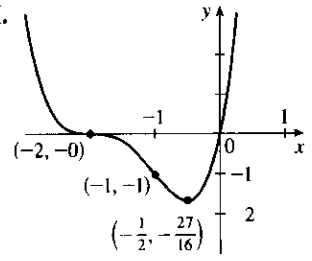
E. $f'(x) = 3x(x+2)^2 + (x+2)^3 = (x+2)^2\,[3x + (x+2)] = (x+2)^2(4x+2)$. $f'(x) > 0 \Leftrightarrow x > -\frac{1}{2}$, and $f'(x) < 0 \Leftrightarrow x < -2$ or $-2 < x < -\frac{1}{2}$, so f is increasing on $\left(-\frac{1}{2}, \infty\right)$ and decreasing on $(-\infty, -2)$ and $\left(-2, -\frac{1}{2}\right)$. [Hence f is decreasing on $\left(-\infty, -\frac{1}{2}\right)$ by the analogue of Exercise 4.3.53 for decreasing functions.]

F. Local minimum value $f\left(-\frac{1}{2}\right) = -\frac{27}{16}$, no local maximum

G.
$$\begin{aligned} f''(x) &= (x+2)^2(4) + (4x+2)(2)(x+2) \\ &= 2(x+2)[(x+2)(2) + 4x + 2] \\ &= 2(x+2)(6x+6) = 12(x+1)(x+2) \end{aligned}$$

$f''(x) < 0 \Leftrightarrow -2 < x < -1$, so f is CD on $(-2, -1)$ and CU on $(-\infty, -2)$ and $(-1, \infty)$. IP at $(-2, 0)$ and $(-1, -1)$

H.



7. $y = f(x) = 2x^5 - 5x^2 + 1$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = 10x^4 - 10x = 10x(x^3 - 1) = 10x(x-1)(x^2 + x + 1)$, so $f'(x) < 0 \Leftrightarrow 0 < x < 1$ and $f'(x) > 0 \Leftrightarrow x < 0$ or $x > 1$. Thus, f is increasing on $(-\infty, 0)$ and $(1, \infty)$ and decreasing on $(0, 1)$.

F. Local maximum value $f(0) = 1$, local minimum value $f(1) = -2$

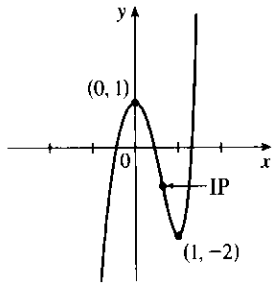
G. $f''(x) = 40x^3 - 10 = 10(4x^3 - 1)$ so $f''(x) = 0 \Leftrightarrow x = 1/\sqrt[3]{4}$.

$f''(x) > 0 \Leftrightarrow x > 1/\sqrt[3]{4}$ and $f''(x) < 0 \Leftrightarrow x < 1/\sqrt[3]{4}$,

so f is CD on $\left(-\infty, 1/\sqrt[3]{4}\right)$ and CU on $\left(1/\sqrt[3]{4}, \infty\right)$.

IP at $\left(\dfrac{1}{\sqrt[3]{4}}, 1 - \dfrac{9}{2\left(\sqrt[3]{4}\right)^2}\right) \approx (0.630, -0.786)$

H.



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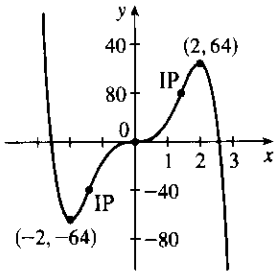
8. $y = f(x) = 20x^3 - 3x^5$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow -3x^3\left(x^2 - \frac{20}{3}\right) = 0 \Leftrightarrow x = 0$ or $\pm\sqrt{20/3} \approx \pm 2.582$ **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** No asymptote

E. $f'(x) = 60x^2 - 15x^4 = -15x^2(x^2 - 4) = -15x^2(x+2)(x-2)$, so $f'(x) > 0 \Leftrightarrow -2 < x < 0$ or $0 < x < 2$ and $f'(x) < 0 \Leftrightarrow x < -2$ or $x > 2$. Thus, f is increasing on $(-2, 0)$ and $(0, 2)$ [hence on $(-2, 2)$ by Exercise 4.3.53] and f is decreasing on $(-\infty, -2)$ and $(2, \infty)$.

F. Local minimum value $f(-2) = -64$, local maximum value $f(2) = 64$ **G.** $f''(x) = 120x - 60x^3 = -60x(x^2 - 2)$. $f''(x) > 0 \Leftrightarrow x < -\sqrt{2}$ or $0 < x < \sqrt{2}$; $f''(x) < 0 \Leftrightarrow -\sqrt{2} < x < 0$ or $x > \sqrt{2}$. Thus, f is CU on $\left(-\infty, -\sqrt{2}\right)$ and $\left(0, \sqrt{2}\right)$, and f is CD on $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, \infty\right)$. IP at $\left(-\sqrt{2}, -28\sqrt{2}\right) \approx (-1.414, -39.598)$, $(0, 0)$, and $\left(\sqrt{2}, 28\sqrt{2}\right)$

H.



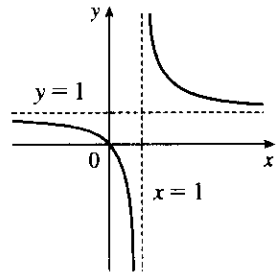
9. $y = f(x) = x/(x-1)$ **A.** $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry **D.** $\lim_{x \to \pm\infty} \dfrac{x}{x-1} = 1$, so $y = 1$ is a HA.

$\lim_{x \to 1^-} \dfrac{x}{x-1} = -\infty$, $\lim_{x \to 1^+} \dfrac{x}{x-1} = \infty$, so $x = 1$ is a VA.

E. $f'(x) = \dfrac{(x-1) - x}{(x-1)^2} = \dfrac{-1}{(x-1)^2} < 0$ for $x \neq 1$, so f is decreasing on $(-\infty, 1)$ and $(1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{2}{(x-1)^3} > 0 \Leftrightarrow x > 1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, 1)$. No IP

H.



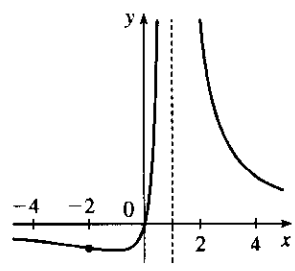
10. $y = x/(x-1)^2$ **A.** $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry **D.** $\lim_{x \to \pm\infty} \dfrac{x}{(x-1)^2} = 0$, so $y = 0$ is a HA. $\lim_{x \to 1} \dfrac{x}{(x-1)^2} = \infty$, so $x = 1$ is a VA.

E. $f'(x) = \dfrac{(x-1)^2(1) - x(2)(x-1)}{(x-1)^4} = \dfrac{-x-1}{(x-1)^3}$. This is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$, so $f(x)$ is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.

F. Local minimum value $f(-1) = -\frac{1}{4}$, no local maximum.

G. $f''(x) = \dfrac{(x-1)^3(-1) + (x+1)(3)(x-1)^2}{(x-1)^6} = \dfrac{2(x+2)}{(x-1)^4}$. This is negative on $(-\infty, -2)$, and positive on $(-2, 1)$ and $(1, \infty)$. So f is CD on $(-\infty, -2)$ and CU on $(-2, 1)$ and $(1, \infty)$. IP at $\left(-2, -\frac{2}{9}\right)$

H.



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11. $y = f(x) = 1/(x^2 - 9)$ **A.** $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

B. y-intercept $= f(0) = -\frac{1}{9}$, no x-intercept **C.** $f(-x) = f(x) \Rightarrow f$ is even; the curve is symmetric about the y-axis. **D.** $\lim_{x \to \pm\infty} \dfrac{1}{x^2 - 9} = 0$, so $y = 0$ is a HA. $\lim_{x \to 3^-} \dfrac{1}{x^2 - 9} = -\infty$, $\lim_{x \to 3^+} \dfrac{1}{x^2 - 9} = \infty$,

$\lim_{x \to -3^-} \dfrac{1}{x^2 - 9} = \infty$, $\lim_{x \to -3^+} \dfrac{1}{x^2 - 9} = -\infty$, so $x = 3$ and $x = -3$ are VA.

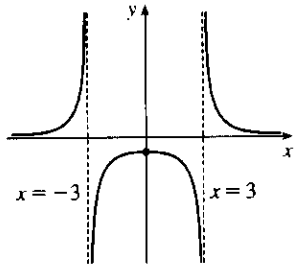
E. $f'(x) = -\dfrac{2x}{(x^2 - 9)^2} > 0 \Leftrightarrow x < 0$ $(x \neq -3)$ so f is increasing on $(-\infty, -3)$ and $(-3, 0)$ and decreasing on $(0, 3)$ and $(3, \infty)$.

F. Local maximum value $f(0) = -\frac{1}{9}$.

G. $y'' = \dfrac{-2(x^2 - 9)^2 + (2x)2(x^2 - 9)(2x)}{(x^2 - 9)^4} = \dfrac{6(x^2 + 3)}{(x^2 - 9)^3} > 0 \Leftrightarrow$

$x^2 > 9 \Leftrightarrow x > 3$ or $x < -3$, so f is CU on $(-\infty, -3)$ and $(3, \infty)$ and CD on $(-3, 3)$. No IP

H.



12. $y = f(x) = x/(x^2 - 9)$ **A.** $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$. **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin.

D. $\lim_{x \to \pm\infty} \dfrac{x}{x^2 - 9} = 0$, so $y = 0$ is a HA. $\lim_{x \to 3^+} \dfrac{x}{x^2 - 9} = \infty$, $\lim_{x \to 3^-} \dfrac{x}{x^2 - 9} = -\infty$,

$\lim_{x \to -3^+} \dfrac{x}{x^2 - 9} = \infty$, $\lim_{x \to -3^-} \dfrac{x}{x^2 - 9} = -\infty$, so $x = 3$ and $x = -3$ are VA.

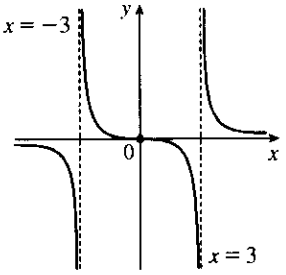
E. $f'(x) = \dfrac{(x^2 - 9) - x(2x)}{(x^2 - 9)^2} = -\dfrac{x^2 + 9}{(x^2 - 9)^2} < 0$ $(x \neq \pm 3)$ so f is decreasing on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$. **F.** No extreme values

G. $f''(x) = -\dfrac{2x(x^2 - 9)^2 - (x^2 + 9) \cdot 2(x^2 - 9)(2x)}{(x^2 - 9)^4}$

$= \dfrac{2x(x^2 + 27)}{(x^2 - 9)^3} > 0$ when $-3 < x < 0$ or $x > 3$,

so f is CU on $(-3, 0)$ and $(3, \infty)$; CD on $(-\infty, -3)$ and $(0, 3)$. IP at $(0, 0)$

H.



13. $y = f(x) = x/(x^2 + 9)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$

C. $f(-x) = -f(x)$, so f is odd and the curve is symmetric about the origin. **D.** $\lim_{x \to \pm\infty} \left[x/(x^2 + 9)\right] = 0$, so $y = 0$ is a HA; no VA **E.** $f'(x) = \dfrac{(x^2 + 9)(1) - x(2x)}{(x^2 + 9)^2} = \dfrac{9 - x^2}{(x^2 + 9)^2} = \dfrac{(3 + x)(3 - x)}{(x^2 + 9)^2} > 0 \Leftrightarrow$

$-3 < x < 3$, so f is increasing on $(-3, 3)$ and decreasing on $(-\infty, -3)$ and $(3, \infty)$.

F. Local minimum value $f(-3) = -\frac{1}{6}$, local maximum value $f(3) = \frac{1}{6}$

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G. $f''(x) = \dfrac{(x^2+9)^2(-2x) - (9-x^2)\cdot 2(x^2+9)(2x)}{\left[(x^2+9)^2\right]^2}$

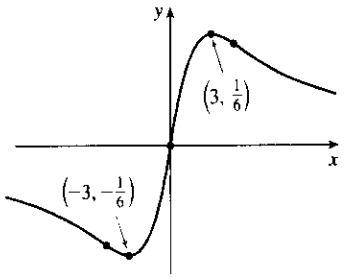
$= \dfrac{(2x)(x^2+9)\left[-(x^2+9) - 2(9-x^2)\right]}{(x^2+9)^4}$

$= \dfrac{2x(x^2-27)}{(x^2+9)^3} = 0 \quad \Leftrightarrow \quad x = 0, \pm\sqrt{27} = \pm 3\sqrt{3}$

$f''(x) > 0 \quad \Leftrightarrow \quad -3\sqrt{3} < x < 0$ or $x > 3\sqrt{3}$, so f is CU on $(-3\sqrt{3}, 0)$ and $(3\sqrt{3}, \infty)$, and CD on $(-\infty, -3\sqrt{3})$ and $(0, 3\sqrt{3})$.

There are three inflection points: $(0,0)$ and $(\pm 3\sqrt{3}, \pm\frac{1}{12}\sqrt{3})$.

H.



14. $y = f(x) = x^2/(x^2+9)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \quad \Leftrightarrow \quad x = 0$

C. $f(-x) = f(x)$, so f is even and symmetric about the y-axis. **D.** $\lim_{x\to\pm\infty}\left[x^2/(x^2+9)\right] = 1$, so $y = 1$ is a HA; no VA **E.** $f'(x) = \dfrac{(x^2+9)(2x) - x^2(2x)}{(x^2+9)^2} = \dfrac{18x}{(x^2+9)^2} > 0 \quad \Leftrightarrow \quad x > 0$, so f is increasing on $(0,\infty)$ and decreasing on $(-\infty, 0)$. **F.** Local minimum value $f(0) = 0$; no local maximum

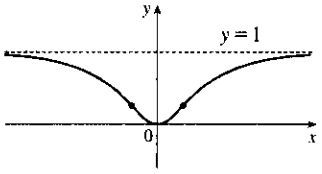
G. $f''(x) = \dfrac{(x^2+9)^2(18) - 18x\cdot 2(x^2+9)\cdot 2x}{\left[(x^2+9)^2\right]^2} = \dfrac{18(x^2+9)\left[(x^2+9) - 4x^2\right]}{(x^2+9)^4} = \dfrac{18(9-3x^2)}{(x^2+9)^3}$

$= \dfrac{-54(x+\sqrt{3})(x-\sqrt{3})}{(x^2+9)^3} > 0 \quad \Leftrightarrow \quad -\sqrt{3} < x < \sqrt{3}$

so f is CU on $(-\sqrt{3}, \sqrt{3})$ and CD on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$.

There are two inflection points: $(\pm\sqrt{3}, \frac{1}{4})$.

H.



15. $y = f(x) = \dfrac{x-1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No y-intercept; x-intercept: $f(x) = 0 \quad \Leftrightarrow \quad x = 1$ **C.** No symmetry **D.** $\lim_{x\to\pm\infty}\dfrac{x-1}{x^2} = 0$, so $y = 0$ is a HA. $\lim_{x\to 0}\dfrac{x-1}{x^2} = -\infty$, so $x = 0$ is a VA.

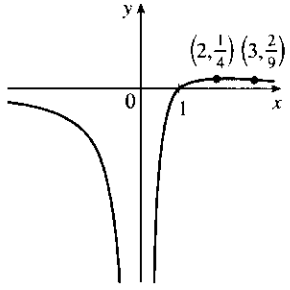
E. $f'(x) = \dfrac{x^2\cdot 1 - (x-1)\cdot 2x}{(x^2)^2} = \dfrac{-x^2+2x}{x^4} = \dfrac{-(x-2)}{x^3}$, so $f'(x) > 0 \quad \Leftrightarrow \quad 0 < x < 2$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < 0$ or $x > 2$. Thus, f is increasing on $(0,2)$ and decreasing on $(-\infty, 0)$ and $(2,\infty)$.

F. No local minimum, local maximum value $f(2) = \frac{1}{4}$.

G. $f''(x) = \dfrac{x^3\cdot(-1) - [-(x-2)]\cdot 3x^2}{(x^3)^2} = \dfrac{2x^3 - 6x^2}{x^6} = \dfrac{2(x-3)}{x^4}$.

$f''(x)$ is negative on $(-\infty, 0)$ and $(0,3)$ and positive on $(3,\infty)$, so f is CD on $(-\infty, 0)$ and $(0,3)$ and CU on $(3,\infty)$. IP at $(3, \frac{2}{9})$

H.



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16. $y = f(x) = \dfrac{x^2 - 2}{x^4}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No y-intercept; x-intercepts: $f(x) = 0$ $\Leftrightarrow$ $x = \pm\sqrt{2}$ **C.** $f(-x) = f(x)$, so f is even; the curve is symmetric about the y-axis.

D. $\lim\limits_{x \to \pm\infty} \dfrac{x^2 - 2}{x^4} = 0$, so $y = 0$ is a HA. $\lim\limits_{x \to 0} \dfrac{x^2 - 2}{x^4} = -\infty$, so $x = 0$ is a VA.

E. $f'(x) = \dfrac{x^4 \cdot 2x - (x^2 - 2)(4x^3)}{(x^4)^2} = \dfrac{-2x^5 + 8x^3}{x^8} = \dfrac{-2(x^2 - 4)}{x^5} = \dfrac{-2(x + 2)(x - 2)}{x^5}$.

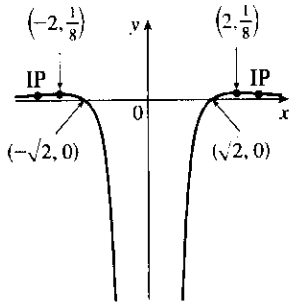
$f'(x)$ is negative on $(-2, 0)$ and $(2, \infty)$ and positive on $(-\infty, -2)$ and $(0, 2)$, so f is decreasing on $(-2, 0)$ and $(2, \infty)$ and increasing on $(-\infty, -2)$ and $(0, 2)$. **F.** Local maximum value $f(\pm 2) = \frac{1}{8}$, no local minimum.

G. $f''(x) = \dfrac{x^5 \cdot (-4x) + 2(x^2 - 4) \cdot 5x^4}{(x^5)^2} = \dfrac{2x^4\left[-2x^2 + 5(x^2 - 4)\right]}{x^{10}} = \dfrac{2(3x^2 - 20)}{x^6}$

$f''(x)$ is positive on $\left(-\infty, -\sqrt{\frac{20}{3}}\right)$ and $\left(\sqrt{\frac{20}{3}}, \infty\right)$ and negative on $\left(-\sqrt{\frac{20}{3}}, 0\right)$ and $\left(0, \sqrt{\frac{20}{3}}\right)$, so f is CU on $\left(-\infty, -\sqrt{\frac{20}{3}}\right)$ and $\left(\sqrt{\frac{20}{3}}, \infty\right)$ and CD on $\left(-\sqrt{\frac{20}{3}}, 0\right)$ and $\left(0, \sqrt{\frac{20}{3}}\right)$.

IP at $\left(\pm\sqrt{\frac{20}{3}}, \frac{21}{200}\right) \approx (\pm 2.5820, 0.105)$

H.

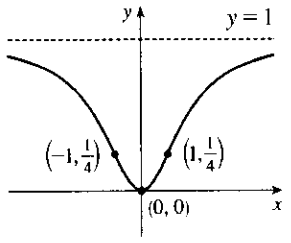


17. $y = f(x) = \dfrac{x^2}{x^2 + 3} = \dfrac{(x^2 + 3) - 3}{x^2 + 3} = 1 - \dfrac{3}{x^2 + 3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = 0$ **C.** $f(-x) = f(x)$, so f is even; the graph is symmetric about the y-axis.

D. $\lim\limits_{x \to \pm\infty} \dfrac{x^2}{x^2 + 3} = 1$, so $y = 1$ is a HA. No VA. **E.** Using the Reciprocal Rule, $f'(x) = -3 \cdot \dfrac{-2x}{(x^2 + 3)^2} = \dfrac{6x}{(x^2 + 3)^2}$. $f'(x) > 0 \Leftrightarrow x > 0$ and $f'(x) < 0 \Leftrightarrow x < 0$, so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. **F.** Local minimum value $f(0) = 0$, no local maximum.

G.
$$f''(x) = \frac{(x^2 + 3)^2 \cdot 6 - 6x \cdot 2(x^2 + 3) \cdot 2x}{\left[(x^2 + 3)^2\right]^2} = \frac{6(x^2 + 3)\left[(x^2 + 3) - 4x^2\right]}{(x^2 + 3)^4} = \frac{6(3 - 3x^2)}{(x^2 + 3)^3} = \frac{-18(x + 1)(x - 1)}{(x^2 + 3)^3}$$

H.



$f''(x)$ is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$ and CU on $(-1, 1)$. IP at $\left(\pm 1, \frac{1}{4}\right)$

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18. $y = f(x) = \dfrac{x^3 - 1}{x^3 + 1}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** x-intercept $= 1$, y-intercept $= f(0) = -1$ **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} \dfrac{x^3 - 1}{x^3 + 1} = \lim\limits_{x \to \pm\infty} \dfrac{1 - 1/x^3}{1 + 1/x^3} = 1$, so $y = 1$ is a HA.

$\lim\limits_{x \to -1^-} \dfrac{x^3 - 1}{x^3 + 1} = \infty$ and $\lim\limits_{x \to -1^+} \dfrac{x^3 - 1}{x^3 + 1} = -\infty$, so $x = -1$ is a VA.

E. $f'(x) = \dfrac{(x^3 + 1)(3x^2) - (x^3 - 1)(3x^2)}{(x^3 + 1)^2} = \dfrac{6x^2}{(x^3 + 1)^2} > 0$ $(x \neq -1)$ so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

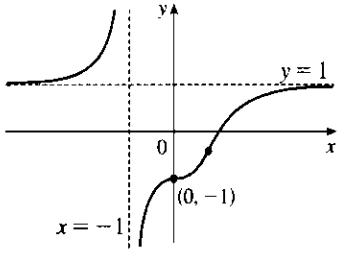
G. $y'' = \dfrac{12x(x^3 + 1)^2 - 6x^2 \cdot 2(x^3 + 1) \cdot 3x^2}{(x^3 + 1)^4}$

$= \dfrac{12x(1 - 2x^3)}{(x^3 + 1)^3} > 0 \quad \Leftrightarrow \quad x < -1$ or $0 < x < \dfrac{1}{\sqrt[3]{2}}$,

so f is CU on $(-\infty, -1)$ and $\left(0, \frac{1}{\sqrt[3]{2}}\right)$ and CD on $(-1, 0)$ and $\left(\frac{1}{\sqrt[3]{2}}, \infty\right)$.

IP at $(0, -1)$, $\left(\frac{1}{\sqrt[3]{2}}, -\frac{1}{3}\right)$

H.



19. $y = f(x) = x\sqrt{5 - x}$ **A.** The domain is $\{x \mid 5 - x \geq 0\} = (-\infty, 5]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = 0, 5$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = x \cdot \frac{1}{2}(5 - x)^{-1/2}(-1) + (5 - x)^{1/2} \cdot 1 = \frac{1}{2}(5 - x)^{-1/2}[-x + 2(5 - x)] = \dfrac{10 - 3x}{2\sqrt{5 - x}} > 0 \quad \Leftrightarrow$

$x < \frac{10}{3}$, so f is increasing on $(-\infty, \frac{10}{3})$ and decreasing on $(\frac{10}{3}, 5)$.

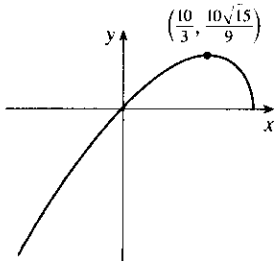
F. Local maximum value $f(\frac{10}{3}) = \frac{10}{9}\sqrt{15} \approx 4.3$; no local minimum

G. $f''(x) = \dfrac{2(5 - x)^{1/2}(-3) - (10 - 3x) \cdot 2(\frac{1}{2})(5 - x)^{-1/2}(-1)}{\left(2\sqrt{5 - x}\right)^2}$

$= \dfrac{(5 - x)^{-1/2}[-6(5 - x) + (10 - 3x)]}{4(5 - x)} = \dfrac{3x - 20}{4(5 - x)^{3/2}}$

$f''(x) < 0$ for $x < 5$, so f is CD on $(-\infty, 5)$. No IP

H.



20. $y = f(x) = 2\sqrt{x} - x$ **A.** $D = [0, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Rightarrow$ $2\sqrt{x} = x \quad \Rightarrow \quad 4x = x^2 \quad \Rightarrow \quad 4x - x^2 = 0 \quad \Rightarrow \quad x(4 - x) = 0 \quad \Rightarrow \quad x = 0, 4$ **C.** No symmetry

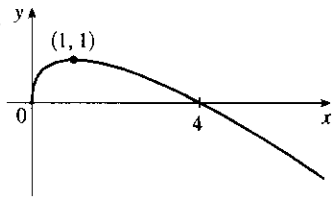
D. No asymptote **E.** $f'(x) = \dfrac{1}{\sqrt{x}} - 1 = \dfrac{1}{\sqrt{x}}(1 - \sqrt{x})$. This is positive for $x < 1$ and negative for $x > 1$, so f is increasing on $(0, 1)$ and decreasing on $(1, \infty)$.

F. Local maximum value $f(1) = 1$, no local minimum.

G. $f''(x) = \left(x^{-1/2} - 1\right)' = -\frac{1}{2}x^{-3/2} = \dfrac{-1}{2x^{3/2}} < 0$ for $x > 0$,

so f is CD on $(0, \infty)$. No IP

H.



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21. $y = f(x) = \sqrt{x^2+1} - x$ **A.** $D = \mathbb{R}$ **B.** No x-intercept, y-intercept $= 1$ **C.** No symmetry

D. $\lim_{x\to-\infty} \left(\sqrt{x^2+1} - x\right) = \infty$ and

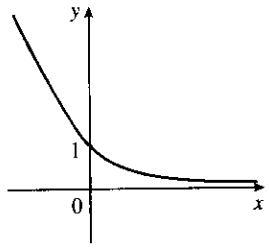
$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0,$$

so $y = 0$ is a HA. **E.** $f'(x) = \dfrac{x}{\sqrt{x^2+1}} - 1 = \dfrac{x - \sqrt{x^2+1}}{\sqrt{x^2+1}} \Rightarrow$

$f'(x) < 0$, so f is decreasing on $\mathbb{R}$. **F.** No extreme values

G. $f''(x) = \dfrac{1}{(x^2+1)^{3/2}} > 0$, so f is CU on $\mathbb{R}$. No IP

H.



22. $y = f(x) = \sqrt{x/(x-5)}$ **A.** $D = \{x \mid x/(x-5) \geq 0\} = (-\infty, 0] \cup (5, \infty)$. **B.** Intercepts are 0.

C. No symmetry **D.** $\lim_{x\to\pm\infty} \sqrt{\dfrac{x}{x-5}} = \lim_{x\to\pm\infty} \sqrt{\dfrac{1}{1-5/x}} = 1$, so $y = 1$ is a HA. $\lim_{x\to5^+} \sqrt{\dfrac{x}{x-5}} = \infty$,

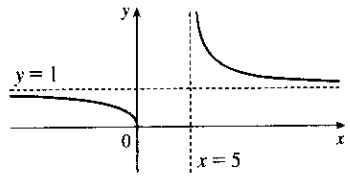
so $x = 5$ is a VA. **E.** $f'(x) = \dfrac{1}{2}\left(\dfrac{x}{x-5}\right)^{-1/2} \dfrac{(-5)}{(x-5)^2} = -\dfrac{5}{2}\left[x(x-5)^3\right]^{-1/2} < 0$, so f is decreasing on

$(-\infty, 0)$ and $(5, \infty)$. **F.** No extreme values

G. $f''(x) = \frac{5}{4}\left[x\,(x-5)^3\right]^{-3/2}(x-5)^2(4x-5) > 0$ for $x > 5$, and

$f''(x) < 0$ for $x < 0$, so f is CU on $(5, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



23. $y = f(x) = x/\sqrt{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 0$

C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \dfrac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to\infty} \dfrac{x/x}{\sqrt{x^2+1}/\sqrt{x^2}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{1+1/x^2}}$

$$= \frac{1}{\sqrt{1+0}} = 1$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}\Big/\left(-\sqrt{x^2}\right)}$$

$$= \lim_{x\to-\infty} \frac{1}{-\sqrt{1+1/x^2}} = \frac{1}{-\sqrt{1+0}} = -1$$

so $y = \pm1$ are HA. No VA.

E. $f'(x) = \dfrac{\sqrt{x^2+1} - x \cdot \dfrac{2x}{2\sqrt{x^2+1}}}{\left[(x^2+1)^{1/2}\right]^2} = \dfrac{x^2+1-x^2}{(x^2+1)^{3/2}} = \dfrac{1}{(x^2+1)^{3/2}} > 0$ for all x, so f is increasing on $\mathbb{R}$.

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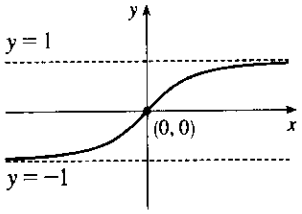
F. No extreme values

G. $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2} \cdot 2x = \dfrac{-3x}{(x^2+1)^{5/2}}$, so $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$. Thus, f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $(0, 0)$

H.

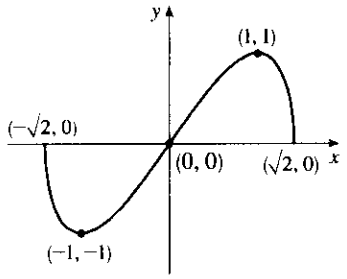


24. $y = f(x) = x\sqrt{2-x^2}$ **A.** $D = [-\sqrt{2}, \sqrt{2}\,]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 0, \pm\sqrt{2}$. **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote

E. $f'(x) = x \cdot \dfrac{-x}{\sqrt{2-x^2}} + \sqrt{2-x^2} = \dfrac{-x^2+2-x^2}{\sqrt{2-x^2}} = \dfrac{2(1+x)(1-x)}{\sqrt{2-x^2}}$. $f'(x)$ is negative for $-\sqrt{2} < x < -1$ and $1 < x < \sqrt{2}$, and positive for $-1 < x < 1$, so f is decreasing on $(-\sqrt{2}, -1)$ and $(1, \sqrt{2}\,)$ and increasing on $(-1, 1)$. **F.** Local minimum value $f(-1) = -1$, local maximum value $f(1) = 1$.

G.
$$f''(x) = \frac{\sqrt{2-x^2}(-4x) - (2-2x^2)\dfrac{-x}{\sqrt{2-x^2}}}{\left[(2-x^2)^{1/2}\right]^2}$$
$$= \frac{(2-x^2)(-4x) + (2-2x^2)x}{(2-x^2)^{3/2}}$$
$$= \frac{2x^3 - 6x}{(2-x^2)^{3/2}} = \frac{2x(x^2-3)}{(2-x^2)^{3/2}}$$

H.



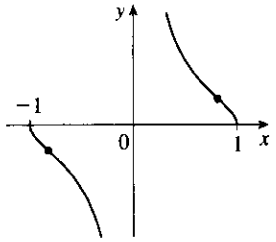
Since $x^2 - 3 < 0$ for x in $[-\sqrt{2}, \sqrt{2}\,]$, $f''(x) > 0$ for $-\sqrt{2} < x < 0$ and $f''(x) < 0$ for $0 < x < \sqrt{2}$. Thus, f is CU on $(-\sqrt{2}, 0)$ and CD on $(0, \sqrt{2}\,)$. The only IP is $(0, 0)$.

25. $y = f(x) = \sqrt{1-x^2}/x$ **A.** $D = \left\{ x \;\middle|\; |x| \le 1, x \ne 0 \right\} = [-1, 0) \cup (0, 1]$ **B.** x-intercepts ± 1, no y-intercept

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\lim\limits_{x \to 0^+} \dfrac{\sqrt{1-x^2}}{x} = \infty$, $\lim\limits_{x \to 0^-} \dfrac{\sqrt{1-x^2}}{x} = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = \dfrac{\left(-x^2/\sqrt{1-x^2}\right) - \sqrt{1-x^2}}{x^2} = -\dfrac{1}{x^2\sqrt{1-x^2}} < 0$, so f is decreasing on $(-1, 0)$ and $(0, 1)$. **F.** No extreme values

G. $f''(x) = \dfrac{2-3x^2}{x^3(1-x^2)^{3/2}} > 0 \;\Leftrightarrow\; -1 < x < -\sqrt{\frac{2}{3}}$ or $0 < x < \sqrt{\frac{2}{3}}$, so f is CU on $\left(-1, -\sqrt{\frac{2}{3}}\right)$ and $\left(0, \sqrt{\frac{2}{3}}\right)$ and CD on $\left(-\sqrt{\frac{2}{3}}, 0\right)$ and $\left(\sqrt{\frac{2}{3}}, 1\right)$. IP at $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{2}}\right)$

H.



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26. $y = f(x) = x/\sqrt{x^2-1}$ **A.** $D = (-\infty,-1) \cup (1,\infty)$ **B.** No intercepts **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2-1}} = 1$ and $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2-1}} = -1$, so $y = \pm 1$ are HA.

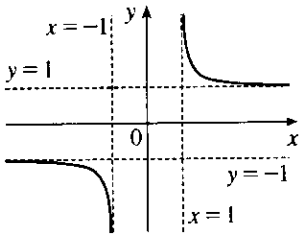
$\lim_{x\to1^+} f(x) = +\infty$ and $\lim_{x\to-1^-} f(x) = -\infty$, so $x = \pm 1$ are VA.

E. $f'(x) = \dfrac{\sqrt{x^2-1} - x\cdot\dfrac{x}{\sqrt{x^2-1}}}{\left[(x^2-1)^{1/2}\right]^2} = \dfrac{x^2-1-x^2}{(x^2-1)^{3/2}} = \dfrac{-1}{(x^2-1)^{3/2}} < 0$, so f is decreasing on $(-\infty,-1)$ and $(1,\infty)$. **F.** No extreme values

G. $f''(x) = (-1)\left(-\frac{3}{2}\right)(x^2-1)^{-5/2}\cdot 2x = \dfrac{3x}{(x^2-1)^{5/2}}$.

$f''(x) < 0$ on $(-\infty,-1)$ and $f''(x) > 0$ on $(1,\infty)$, so f is CD on $(-\infty,-1)$ and CU on $(1,\infty)$. No IP

H.

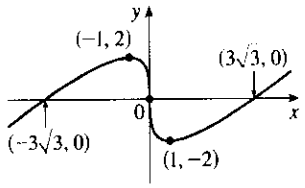


27. $y = f(x) = x - 3x^{1/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 3x^{1/3} \Rightarrow x^3 = 27x \Rightarrow x^3 - 27x = 0 \Rightarrow x(x^2-27) = 0 \Rightarrow x = 0, \pm 3\sqrt{3}$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x) = 1 - x^{-2/3} = 1 - \dfrac{1}{x^{2/3}} = \dfrac{x^{2/3}-1}{x^{2/3}}$.

$f'(x) > 0$ when $|x| > 1$ and $f'(x) < 0$ when $0 < |x| < 1$, so f is increasing on $(-\infty,-1)$ and $(1,\infty)$, and decreasing on $(-1,0)$ and $(0,1)$ [hence decreasing on $(-1,1)$ since f is continuous on $(-1,1)$]. **F.** Local maximum value $f(-1) = 2$, local minimum value $f(1) = -2$ **G.** $f''(x) = \frac{2}{3}x^{-5/3} < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$, so f is CD on $(-\infty,0)$ and CU on $(0,\infty)$.

IP at $(0,0)$

H.



28. $y = f(x) = x^{5/3} - 5x^{2/3} = x^{2/3}(x-5)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts 0, 5; y-intercept 0 **C.** No symmetry

D. $\lim_{x\to\pm\infty} x^{2/3}(x-5) = \pm\infty$, so there is no asymptote

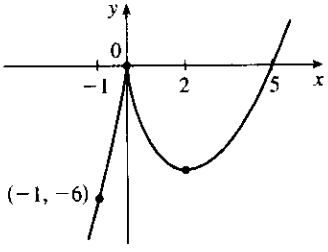
E. $f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2) > 0 \Leftrightarrow x < 0$ or $x > 2$, so f is increasing on $(-\infty,0)$, $(2,\infty)$ and decreasing on $(0,2)$.

F. Local maximum value $f(0) = 0$, local minimum value $f(2) = -3\sqrt[3]{4}$

G. $f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1) > 0 \Leftrightarrow$ $x > -1$, so f is CU on $(-1,0)$ and $(0,\infty)$, CD on $(-\infty,-1)$.

IP at $(-1,-6)$

H.



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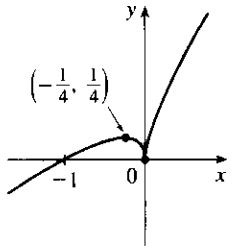
29. $y = f(x) = x + \sqrt{|x|}$ **A.** $D = \mathbb{R}$ **B.** x-intercepts $0, -1$; y-intercept 0 **C.** No symmetry

D. $\lim_{x\to\infty}\left(x+\sqrt{|x|}\right) = \infty$, $\lim_{x\to-\infty}\left(x+\sqrt{|x|}\right) = -\infty$. No asymptote **E.** For $x > 0$, $f(x) = x + \sqrt{x} \Rightarrow$ $f'(x) = 1 + \dfrac{1}{2\sqrt{x}} > 0$, so f increases on $(0,\infty)$. For $x < 0$, $f(x) = x + \sqrt{-x} \Rightarrow f'(x) = 1 - \dfrac{1}{2\sqrt{-x}} > 0$ $\Leftrightarrow$ $2\sqrt{-x} > 1 \Leftrightarrow -x > \frac{1}{4} \Leftrightarrow x < -\frac{1}{4}$, so f increases on $\left(-\infty, -\frac{1}{4}\right)$ and decreases on $\left(-\frac{1}{4}, 0\right)$.

F. Local maximum value $f\left(-\frac{1}{4}\right) = \frac{1}{4}$, local minimum value $f(0) = 0$

G. For $x > 0$, $f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(x) < 0$, so f is CD on $(0,\infty)$. For $x < 0$, $f''(x) = -\frac{1}{4}(-x)^{-3/2} \Rightarrow f''(x) < 0$, so f is CD on $(-\infty, 0)$. No IP

H.



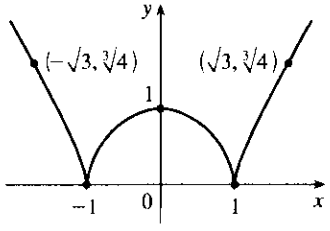
30. $y = f(x) = \sqrt[3]{(x^2-1)^2} = (x^2-1)^{2/3}$ **A.** $D = \mathbb{R}$ **B.** x-intercepts ± 1, y-intercept 1 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim_{x\to\pm\infty}(x^2-1)^{2/3} = \infty$, no asymptote

E. $f'(x) = \frac{4}{3}x(x^2-1)^{-1/3} \Rightarrow f'(x) > 0 \Leftrightarrow x > 1$ or $-1 < x < 0$, $f'(x) < 0 \Leftrightarrow x < -1$ or $0 < x < 1$. So f is increasing on $(-1,0)$, $(1,\infty)$ and decreasing on $(-\infty,-1)$, $(0,1)$. **F.** Local minimum values $f(-1) = f(1) = 0$, local maximum value $f(0) = 1$

G. $f''(x) = \frac{4}{3}(x^2-1)^{-1/3} + \frac{4}{3}x\left(-\frac{1}{3}\right)(x^2-1)^{-4/3}(2x)$
$= \frac{4}{9}(x^2-3)(x^2-1)^{-4/3} > 0 \Leftrightarrow |x| > \sqrt{3}$

so f is CU on $\left(-\infty, -\sqrt{3}\right)$, $\left(\sqrt{3},\infty\right)$ and CD on $\left(-\sqrt{3}, -1\right)$, $(-1,1)$,$\left(1,\sqrt{3}\right)$. IPs at $\left(\pm\sqrt{3}, \sqrt[3]{4}\right)$

H.



31. $y = f(x) = 3\sin x - \sin^3 x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow$ $\sin x\,(3 - \sin^2 x) = 0 \Rightarrow \sin x = 0$ [since $\sin^2 x \le 1 < 3$] $\Rightarrow x = n\pi$, n an integer.

C. $f(-x) = -f(x)$, so f is odd; the graph (shown for $-2\pi \le x \le 2\pi$) is symmetric about the origin and periodic with period 2π. **D.** No asymptote **E.** $f'(x) = 3\cos x - 3\sin^2 x\cos x = 3\cos x\,(1 - \sin^2 x) = 3\cos^3 x$. $f'(x) > 0 \Leftrightarrow \cos x > 0 \Leftrightarrow x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$ for each integer n, and $f'(x) < 0 \Leftrightarrow$ $\cos x < 0 \Leftrightarrow x \in \left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$ for each integer n. Thus, f is increasing on $\left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$ for each integer n, and f is decreasing on $\left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$ for each integer n.

F. f has local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 2$ and local minimum values $f\left(2n\pi + \frac{3\pi}{2}\right) = -2$.

G. $f''(x) = -9\sin x\cos^2 x = -9\sin x\,(1 - \sin^2 x) = -9\sin x\,(1 - \sin x)(1 + \sin x)$. $f''(x) < 0 \Leftrightarrow$

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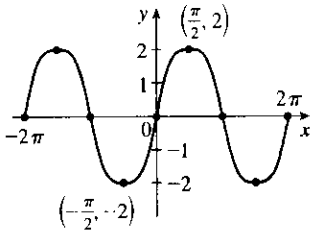
$\sin x > 0$ and $\sin x \neq \pm 1 \quad \Leftrightarrow \quad x \in \left(2n\pi, 2n\pi + \frac{\pi}{2}\right) \cup \left(2n\pi + \frac{\pi}{2}, 2n\pi + \pi\right)$ for some integer n.

$f''(x) > 0 \quad \Leftrightarrow \quad \sin x < 0$ and $\sin x \neq \pm 1 \quad \Leftrightarrow \quad x \in \left((2n-1)\pi, (2n-1)\pi + \frac{\pi}{2}\right) \cup \left((2n-1)\pi + \frac{\pi}{2}, 2n\pi\right)$ for some integer n. Thus, f is CD on the intervals $\left(2n\pi, \left(2n + \frac{1}{2}\right)\pi\right)$ and $\left(\left(2n + \frac{1}{2}\right)\pi, (2n+1)\,\pi\right)$ [hence CD on the intervals $(2n\pi, (2n+1)\,\pi)$] for each integer n, and f is CU on the intervals $\left((2n-1)\pi, \left(2n - \frac{1}{2}\right)\pi\right)$ and $\left(\left(2n - \frac{1}{2}\right)\pi, 2n\pi\right)$ [hence CU on the intervals $((2n-1)\pi, 2n\pi)$] for each integer n. f has inflection points at $(n\pi, 0)$ for each integer n.

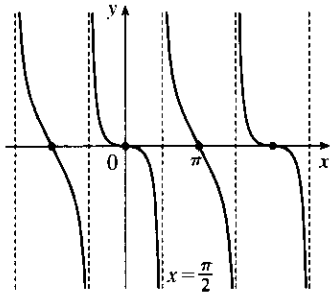
H.



32. $y = f(x) = \sin x - \tan x$ **A.** $D = \left\{x \mid x \neq (2n+1)\frac{\pi}{2}\right\}$ **B.** $y = 0 \quad \Leftrightarrow \quad \sin x = \tan x = \dfrac{\sin x}{\cos x} \quad \Leftrightarrow$ $\sin x = 0$ or $\cos x = 1 \quad \Leftrightarrow \quad x = n\pi$ (x-intercepts), y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. Also periodic with period 2π **D.** $\lim\limits_{x \to (\pi/2)^-} (\sin x - \tan x) = -\infty$ and $\lim\limits_{x \to (\pi/2)^+} (\sin x - \tan x) = \infty$, so $x = n\pi + \frac{\pi}{2}$ are VA.

E. $f'(x) = \cos x - \sec^2 x \leq 0$, so f decreases on each interval in its domain, that is, on $\left((2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right)$. **F.** No extreme values **G.** $f''(x) = -\sin x - 2\sec^2 x \tan x = -\sin x\left(1 + 2\sec^3 x\right)$. Note that $1 + 2\sec^3 x \neq 0$ since $\sec^3 x \neq -\frac{1}{2}$. $f''(x) > 0$ for $-\frac{\pi}{2} < x < 0$ and $\frac{3\pi}{2} < x < 2\pi$, so f is CU on $\left(\left(n - \frac{1}{2}\right)\pi, n\pi\right)$ and CD on $\left(n\pi, \left(n + \frac{1}{2}\right)\pi\right)$. f has IPs at $(n\pi, 0)$. Note also that $f'(0) = 0$, but $f'(\pi) = -2$.

H.

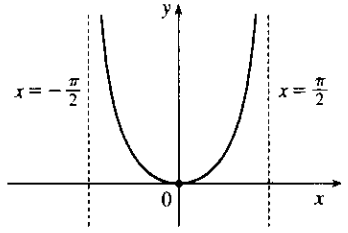


33. $y = f(x) = x \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** Intercepts are 0 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x \to (\pi/2)^-} x \tan x = \infty$ and $\lim\limits_{x \to -(\pi/2)^+} x \tan x = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = \tan x + x\sec^2 x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$, so f increases on $\left(0, \frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2}, 0\right)$.

F. Absolute and local minimum value $f(0) = 0$.

G. $y'' = 2\sec^2 x + 2x \tan x \sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP

H.



34. $y = f(x) = 2x - \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(0) = 0$ $\Leftrightarrow \quad 2x = \tan x \quad \Leftrightarrow \quad x = 0$ or $x \approx \pm 1.17$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim\limits_{x \to (-\pi/2)^+} (2x - \tan x) = \infty$ and $\lim\limits_{x \to (\pi/2)^-} (2x - \tan x) = -\infty$, so $x = \pm\frac{\pi}{2}$ are VA. No HA.

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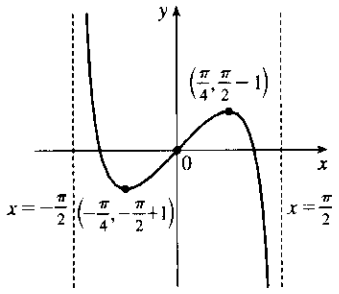
E. $f'(x) = 2 - \sec^2 x < 0 \quad \Leftrightarrow \quad |\sec x| > \sqrt{2}$ and $f'(x) > 0 \quad \Leftrightarrow \quad |\sec x| < \sqrt{2}$, so f is decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$, increasing on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, and decreasing again on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ **F.** Local maximum value $f\left(\frac{\pi}{4}\right) = \frac{\pi}{2} - 1$, local minimum value $f\left(-\frac{\pi}{4}\right) = -\frac{\pi}{2} + 1$

H.

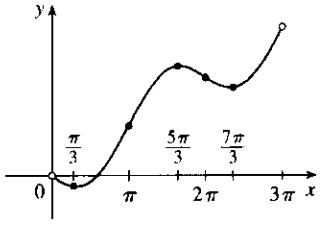


G. $f''(x) = -2\sec x \cdot \sec x \tan x = -2\tan x \sec^2 x$
$= -2\tan x(\tan^2 x + 1)$

so $f''(x) > 0 \quad \Leftrightarrow \quad \tan x < 0 \quad \Leftrightarrow \quad -\frac{\pi}{2} < x < 0$, and $f''(x) < 0 \quad \Leftrightarrow \quad \tan x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$. Thus, f is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. f has an IP at $(0, 0)$.

35. $y = f(x) = \frac{1}{2}x - \sin x, 0 < x < 3\pi$ **A.** $D = (0, 3\pi)$ **B.** No y-intercept. The x-intercept, approximately 1.9, can be found using Newton's Method. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \frac{1}{2} - \cos x > 0 \quad \Leftrightarrow \quad \cos x < \frac{1}{2} \quad \Leftrightarrow \quad \frac{\pi}{3} < x < \frac{5\pi}{3}$ or $\frac{7\pi}{3} < x < 3\pi$, so f is increasing on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3}, 3\pi\right)$ and decreasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$. **F.** Local minimum value $f\left(\frac{\pi}{3}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$, local maximum value $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2}$, local minimum value $f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{6} - \frac{\sqrt{3}}{2}$ **G.** $f''(x) = \sin x > 0 \quad \Leftrightarrow \quad 0 < x < \pi$ or $2\pi < x < 3\pi$, so f is CU on $(0, \pi)$ and $(2\pi, 3\pi)$ and CD on $(\pi, 2\pi)$. IPs at $\left(\pi, \frac{\pi}{2}\right)$ and $(2\pi, \pi)$.

H.



36. $y = f(x) = \cos^2 x - 2\sin x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$ **C.** No symmetry, but f has period 2π. **D.** No asymptote **E.** $y' = 2\cos x(-\sin x) - 2\cos x = -2\cos x(\sin x + 1)$. $y' = 0 \quad \Leftrightarrow \quad \cos x = 0$ or $\sin x = -1 \quad \Leftrightarrow \quad x = (2n+1)\frac{\pi}{2}$. $y' > 0$ when $\cos x < 0$ since $\sin x + 1 \geq 0$ for all x. So $y' > 0$ and f is increasing on $\left((4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2}\right)$; $y' < 0$ and f is decreasing on $\left((4n-1)\frac{\pi}{2}, (4n+1)\frac{\pi}{2}\right)$. **F.** Local maximum values $f\left((4n+3)\frac{\pi}{2}\right) = 2$, local minimum values $f\left((4n+1)\frac{\pi}{2}\right) = -2$

G. $y' = -2\cos x(\sin x + 1) = -\sin 2x - 2\cos x \quad \Rightarrow$

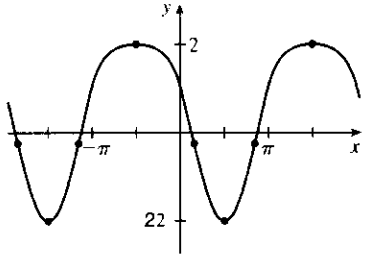
$y'' = -2\cos 2x + 2\sin x = -2\left(1 - 2\sin^2 x\right) + 2\sin x$
$= 4\sin^2 x + 2\sin x - 2 = 2(2\sin x - 1)(\sin x + 1)$

$y'' = 0 \quad \Leftrightarrow \quad \sin x = \frac{1}{2}$ or $-1 \quad \Rightarrow \quad x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$, or $\frac{3\pi}{2} + 2n\pi$. $y'' > 0$ and f is CU on $\left(\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi\right)$;

$y'' \leq 0$ and f is CD on $\left(\frac{5\pi}{6} + 2n\pi, \frac{\pi}{6} + 2(n+1)\pi\right)$.

IPs at $\left(\frac{\pi}{6} + 2n\pi, -\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6} + 2n\pi, -\frac{1}{4}\right)$.

H.



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37. $y = f(x) = \sin 2x - 2\sin x$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 0$. $y = 0 \Leftrightarrow$ $2\sin x = \sin 2x = 2\sin x \cos x \Leftrightarrow \sin x = 0$ or $\cos x = 1 \Leftrightarrow x = n\pi$ (x-intercepts)

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$.

Note: f is periodic with period 2π, so we determine E–G for $-\pi \le x \le \pi$. **D.** No asymptotes

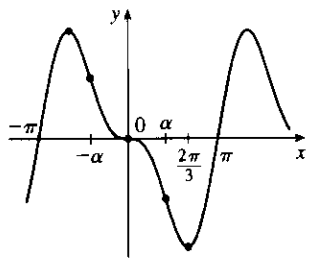
E. $f'(x) = 2\cos 2x - 2\cos x = 2\left(2\cos^2 x - 1 - \cos x\right) = 2(2\cos x + 1)(\cos x - 1) > 0 \Leftrightarrow \cos x < -\frac{1}{2}$ $\Leftrightarrow -\pi < x < -\frac{2\pi}{3}$ or $\frac{2\pi}{3} < x < \pi$, so f is increasing on $\left(-\pi, -\frac{2\pi}{3}\right)$, $\left(\frac{2\pi}{3}, \pi\right)$ and decreasing on $\left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$.

F. Local maximum value $f\left(-\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{2}$, local minimum value $f\left(\frac{2\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$

G. $f''(x) = -4\sin 2x + 2\sin x = 2\sin x\,(1 - 4\cos x) = 0$ when $x = 0, \pm\pi$ or $\cos x = \frac{1}{4}$. If $\alpha = \cos^{-1}\frac{1}{4}$, then f is CU on $(-\alpha, 0)$ and (α, π) and CD on $(-\pi, -\alpha)$ and $(0, \alpha)$.

IPs at $(0, 0)$, $(\pm\pi, 0)$, $\left(\alpha, -\frac{3\sqrt{15}}{8}\right)$, $\left(-\alpha, \frac{3\sqrt{15}}{8}\right)$.

H.

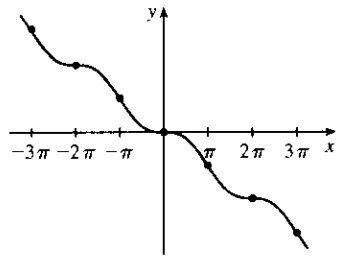


38. $f(x) = \sin x - x$ **A.** $D = \mathbb{R}$ **B.** x-intercept $= 0 = y$-intercept

C. $f(-x) = \sin(-x) - (-x) = -(\sin x - x) = -f(x)$, so f is odd.

D. No asymptote **E.** $f'(x) = \cos x - 1 \le 0$ for all x, so f is decreasing on $(-\infty, \infty)$. **F.** No extreme values **G.** $f''(x) = -\sin x \Rightarrow$ $f''(x) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow (2n-1)\pi < x < 2n\pi$, so f is CU on $((2n-1)\pi, 2n\pi)$ and CD on $(2n\pi, (2n+1)\pi)$, n an integer. Points of inflection occur when $x = n\pi$.

H.



39. $y = f(x) = \dfrac{\sin x}{1 + \cos x}$ $\left[\overset{\text{when } \cos x \neq -1}{=} \dfrac{\sin x}{1 + \cos x} \cdot \dfrac{1 - \cos x}{1 - \cos x} = \dfrac{\sin x\,(1 - \cos x)}{\sin^2 x} = \dfrac{1 - \cos x}{\sin x} = \csc x - \cot x\right]$

A. The domain of f is the set of all real numbers except odd integer multiples of π. **B.** y-intercept: $f(0) = 0$; x-intercepts: $x = n\pi$, n an even integer. **C.** $f(-x) = -f(x)$, so f is an odd function; the graph is symmetric about the origin and has period 2π. **D.** When n is an odd integer, $\lim_{x \to (n\pi)^-} f(x) = \infty$ and $\lim_{x \to (n\pi)^+} f(x) = -\infty$, so $x = n\pi$ is a VA for each odd integer n. No HA.

E. $f'(x) = \dfrac{(1 + \cos x) \cdot \cos x - \sin x(-\sin x)}{(1 + \cos x)^2} = \dfrac{1 + \cos x}{(1 + \cos x)^2} = \dfrac{1}{1 + \cos x}$. $f'(x) > 0$ for all x except odd multiples of π, so f is increasing on $((2k-1)\pi, (2k+1)\pi)$ for each integer k. **F.** No extreme values

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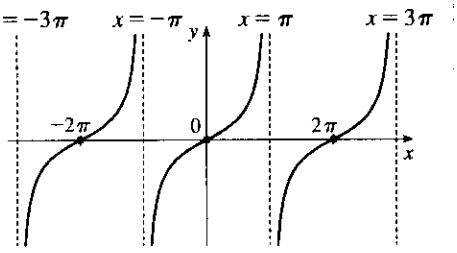
G. $f''(x) = \dfrac{\sin x}{(1+\cos x)^2} > 0 \;\Rightarrow\; \sin x > 0 \;\Rightarrow$

$x \in (2k\pi, (2k+1)\pi)$ and $f''(x) < 0$ on $((2k-1)\pi, 2k\pi)$ for each integer k. f is CU on $(2k\pi, (2k+1)\pi)$ and CD on $((2k-1)\pi, 2k\pi)$ for each integer k. f has IPs at $(2k\pi, 0)$ for each integer k.

H.



40. $y = f(x) = \cos x/(2+\sin x)$ **A.** $D = \mathbb{R}$ *Note:* f is periodic with period 2π, so we determine B–G on $[0, 2\pi]$.

B. x-intercepts $\frac{\pi}{2}, \frac{3\pi}{2}$, y-intercept $= f(0) = \frac{1}{2}$ **C.** No symmetry other than periodicity **D.** No asymptote

E. $f'(x) = \dfrac{(2+\sin x)(-\sin x) - \cos x\,(\cos x)}{(2+\sin x)^2} = -\dfrac{2\sin x + 1}{(2+\sin x)^2}$. $f'(x) > 0 \;\Leftrightarrow\; 2\sin x + 1 < 0 \;\Leftrightarrow$

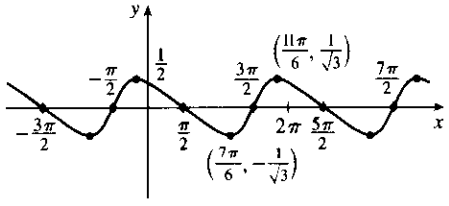
$\sin x < -\frac{1}{2} \;\Leftrightarrow\; \frac{7\pi}{6} < x < \frac{11\pi}{6}$, so f is increasing on $\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$ and decreasing on $\left(0, \frac{7\pi}{6}\right), \left(\frac{11\pi}{6}, 2\pi\right)$.

F. Local minimum value $f\left(\frac{7\pi}{6}\right) = -\frac{1}{\sqrt{3}}$, local maximum value $f\left(\frac{11\pi}{6}\right) = \frac{1}{\sqrt{3}}$

G. $f''(x) = -\dfrac{(2+\sin x)^2(2\cos x) - (2\sin x + 1)2(2+\sin x)\cos x}{(2+\sin x)^4} = -\dfrac{2\cos x\,(1-\sin x)}{(2+\sin x)^3} > 0 \;\Leftrightarrow$

$\cos x < 0 \;\Leftrightarrow\; \frac{\pi}{2} < x < \frac{3\pi}{2}$, so f is CU on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$. IP at $\left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$

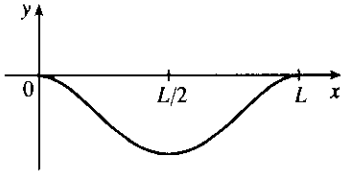
H.



41. $y = -\dfrac{W}{24EI}x^4 + \dfrac{WL}{12EI}x^3 - \dfrac{WL^2}{24EI}x^2 = -\dfrac{W}{24EI}x^2(x^2 - 2Lx + L^2)$

$= \dfrac{-W}{24EI}x^2(x-L)^2 = cx^2(x-L)^2$

where $c = -\dfrac{W}{24EI}$ is a negative constant and $0 \le x \le L$. We sketch



$f(x) = cx^2(x-L)^2$ for $c = -1$. $f(0) = f(L) = 0$.

$f'(x) = cx^2[2(x-L)] + (x-L)^2(2cx) = 2cx(x-L)\,[x + (x-L)] = 2cx(x-L)(2x-L)$. So for $0 < x < L$, $f'(x) > 0 \;\Leftrightarrow\; x(x-L)(2x-L) < 0$ (since $c < 0$) $\;\Leftrightarrow\; L/2 < x < L$ and $f'(x) < 0 \;\Leftrightarrow$ $0 < x < L/2$. So f is increasing on $(L/2, L)$ and decreasing on $(0, L/2)$, and there is a local and absolute minimum at $(L/2, f(L/2)) = (L/2, cL^4/16)$.

$f'(x) = 2c\,[x(x-L)(2x-L)] \;\Rightarrow$

$f''(x) = 2c\,[1(x-L)(2x-L) + x(1)(2x-L) + x(x-L)(2)] = 2c(6x^2 - 6Lx + L^2) = 0 \;\Leftrightarrow$

$x = \dfrac{6L \pm \sqrt{12L^2}}{12} = \frac{1}{2}L \pm \frac{\sqrt{3}}{6}L$, and these are the x-coordinates of the two inflection points.

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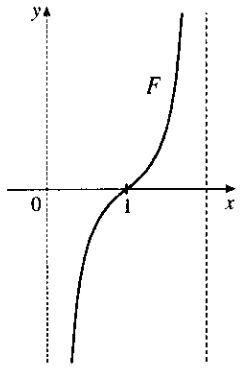
42. $F(x) = -\dfrac{k}{x^2} + \dfrac{k}{(x-2)^2}$, where $k > 0$ and $0 < x < 2$. For $0 < x < 2$,



$x - 2 < 0$, so $F'(x) = \dfrac{2k}{x^3} - \dfrac{2k}{(x-2)^3} > 0$ and F is increasing.

$\lim\limits_{x \to 0^+} F(x) = -\infty$ and $\lim\limits_{x \to 2^-} F(x) = \infty$, so $x = 0$ and $x = 2$ are vertical asymptotes. Notice that when the middle particle is at $x = 1$, the net force acting on it is 0. When $x > 1$, the net force is positive, meaning that it acts to the right. And if the particle approaches $x = 2$, the force on it rapidly becomes very large. When $x < 1$, the net force is negative, so it acts to the left. If the particle approaches 0, the force becomes very large to the left.

43. $y = \dfrac{x^2+1}{x+1}$. Long division gives us:

$$\begin{array}{r} x - 1 \\ x+1 \overline{)\,x^2 \quad +1} \\ \underline{x^2 + x} \\ -x+1 \\ \underline{-x-1} \\ 2 \end{array}$$

Thus, $y = f(x) = \dfrac{x^2+1}{x+1} = x - 1 + \dfrac{2}{x+1}$ and $f(x) - (x-1) = \dfrac{2}{x+1} = \dfrac{\frac{2}{x}}{1+\frac{1}{x}}$ [for $x \neq 0$] $\to 0$ as $x \to \pm\infty$. So the line $y = x - 1$ is a slant asymptote (SA).

44. $y = \dfrac{2x^3+x^2+x+3}{x^2+2x}$. Long division gives us:

$$\begin{array}{r} 2x - 3 \\ x^2+2x \overline{)\,2x^3 + \ x^2 + \ x+3} \\ \underline{2x^3 + 4x^2} \\ -3x^2 + \ x \\ \underline{-3x^2 - 6x} \\ 7x+3 \end{array}$$

Thus, $y = f(x) = \dfrac{2x^3+x^2+x+3}{x^2+2x} = 2x - 3 + \dfrac{7x+3}{x^2+2x}$ and $f(x) - (2x-3) = \dfrac{7x+3}{x^2+2x} = \dfrac{\frac{7}{x} + \frac{3}{x^2}}{1+\frac{2}{x}}$ [for $x \neq 0$] $\to 0$ as $x \to \pm\infty$. So the line $y = 2x - 3$ is a SA.

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45. $y = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3}$. Long division gives us:

$$\begin{array}{r} 2x - 2 \\ 2x^2 + x - 3 \overline{)\,4x^3 - 2x^2 \quad + 5} \\ \underline{4x^3 + 2x^2 - 6x} \\ -4x^2 + 6x + 5 \\ \underline{-4x^2 - 2x + 6} \\ 8x - 1 \end{array}$$

Thus, $y = f(x) = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3} = 2x - 2 + \dfrac{8x - 1}{2x^2 + x - 3}$ and

$f(x) - (2x - 2) = \dfrac{8x - 1}{2x^2 + x - 3} = \dfrac{\frac{8}{x} - \frac{1}{x^2}}{2 + \frac{1}{x} - \frac{3}{x^2}}$ [for $x \neq 0$] $\to 0$ as $x \to \pm\infty$. So the line $y = 2x - 2$ is a SA.

46. $y = \dfrac{5x^4 + x^2 + x}{x^3 - x^2 + 2}$. Long division gives us:

$$\begin{array}{r} 5x + 5 \\ x^3 - x^2 + 2 \overline{)\,5x^4 \quad + x^2 + x} \\ \underline{5x^4 - 5x^3 \quad + 10x} \\ 5x^3 + x^2 - 9x \\ \underline{5x^3 - 5x^2 \quad + 10} \\ 6x^2 - 9x - 10 \end{array}$$

Thus, $y = f(x) = \dfrac{5x^4 + x^2 + x}{x^3 - x^2 + 2} = 5x + 5 + \dfrac{6x^2 - 9x - 10}{x^3 - x^2 + 2}$ and

$f(x) - (5x + 5) = \dfrac{6x^2 - 9x - 10}{x^3 - x^2 + 2} = \dfrac{\frac{6}{x} - \frac{9}{x^2} - \frac{10}{x^3}}{1 - \frac{1}{x} + \frac{2}{x^3}}$ [for $x \neq 0$] $\to 0$ as $x \to \pm\infty$. So the line $y = 5x + 5$ is a SA.

47. $y = f(x) = \dfrac{-2x^2 + 5x - 1}{2x - 1} = -x + 2 + \dfrac{1}{2x - 1}$ **A.** $D = \{x \in \mathbb{R} \mid x \neq \frac{1}{2}\} = (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

B. y-intercept: $f(0) = 1$; x-intercepts: $f(x) = 0 \;\Rightarrow\; -2x^2 + 5x - 1 = 0 \;\Rightarrow\; x = \dfrac{-5 \pm \sqrt{17}}{-4} \;\Rightarrow$ $x \approx 0.22, 2.28$. **C.** No symmetry

D. $\lim\limits_{x \to (1/2)^-} f(x) = -\infty$ and $\lim\limits_{x \to (1/2)^+} f(x) = \infty$, so $x = \frac{1}{2}$ is a VA.

$\lim\limits_{x \to \pm\infty} [f(x) - (-x + 2)] = \lim\limits_{x \to \pm\infty} \dfrac{1}{2x - 1} = 0$, so the line $y = -x + 2$ is a SA.

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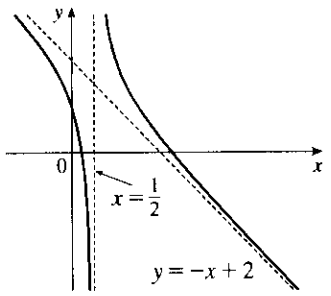
E. $f'(x) = -1 - \dfrac{2}{(2x-1)^2} < 0$ for $x \neq \frac{1}{2}$, so f is decreasing on $\left(-\infty, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \infty\right)$. **F.** No extreme values

G. $f'(x) = -1 - 2(2x-1)^{-2} \Rightarrow$

$f''(x) = -2(-2)(2x-1)^{-3}(2) = \dfrac{8}{(2x-1)^3}$, so $f''(x) > 0$ when $x > \frac{1}{2}$ and $f''(x) < 0$ when $x < \frac{1}{2}$. Thus, f is CU on $\left(\frac{1}{2}, \infty\right)$ and CD on $\left(-\infty, \frac{1}{2}\right)$. No IP

H.



48. $y = f(x) = \dfrac{x^2+12}{x-2} = x + 2 + \dfrac{16}{x-2}$ **A.** $D = \{x \in \mathbb{R} \mid x \neq 2\} = (-\infty, 2) \cup (2, \infty)$

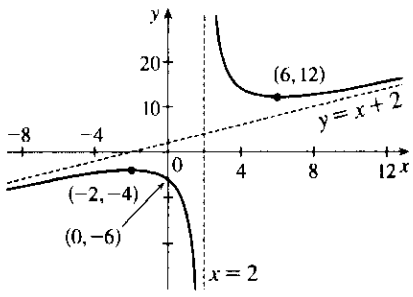
B. y-intercept: $f(0) = -6$; no x-intercepts. **C.** No symmetry **D.** $\lim\limits_{x \to 2^-} f(x) = -\infty$ and $\lim\limits_{x \to 2^+} f(x) = \infty$, so $x = 2$ is a VA. $\lim\limits_{x \to \pm\infty} [f(x) - (x+2)] = \lim\limits_{x \to \pm\infty} \dfrac{16}{x-2} = 0$, so the line $y = x + 2$ is a slant asymptote.

E. $f'(x) = 1 - \dfrac{16}{(x-2)^2} = \dfrac{x^2-4x-12}{(x-2)^2} = \dfrac{(x-6)(x+2)}{(x-2)^2}$, so $f'(x) > 0$ when $x < -2$ or $x > 6$ and $f'(x) < 0$ when $-2 < x < 2$ or $2 < x < 6$. Thus, f is increasing on $(-\infty, -2)$ and $(6, \infty)$ and decreasing on $(-2, 2)$ and $(2, 6)$.

F. Local maximum value $f(-2) = -4$, local minimum value $f(6) = 12$

G. $f''(x) = 16(-2)(x-2)^{-3} = \dfrac{32}{(x-2)^3}$, so $f''(x) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$. f is CU on $(2, \infty)$ and CD on $(-\infty, 2)$. No IP

H.



49. $y = f(x) = (x^2+4)/x = x + 4/x$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept

C. $f(-x) = -f(x) \Rightarrow$ symmetry about the origin **D.** $\lim\limits_{x \to \infty} (x + 4/x) = \infty$ but $f(x) - x = 4/x \to 0$ as $x \to \pm\infty$, so $y = x$ is a slant asymptote. $\lim\limits_{x \to 0^+} (x + 4/x) = \infty$ and $\lim\limits_{x \to 0^-} (x + 4/x) = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = 1 - 4/x^2 > 0$ $\Leftrightarrow$ $x^2 > 4$ $\Leftrightarrow$ $x > 2$ or $x < -2$, so f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and decreasing on $(-2, 0)$ and $(0, 2)$.

F. Local maximum value $f(-2) = -4$, local minimum value $f(2) = 4$

G. $f''(x) = 8/x^3 > 0 \Leftrightarrow x > 0$ so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

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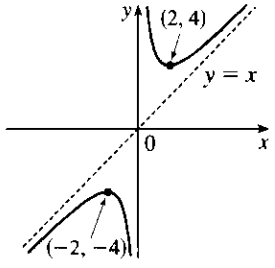

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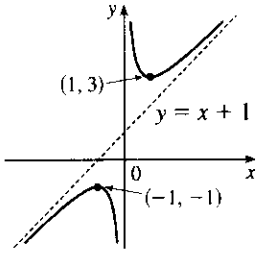
50. $y = f(x) = \dfrac{x^2+x+1}{x} = x + 1 + \dfrac{1}{x}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept (x-intercepts would occur when $x^2 + x + 1 = 0$ but this equation has no real roots since $b^2 - 4ac = -3 < 0$.) **C.** No symmetry **D.** $\lim\limits_{x\to\infty} (x + 1 + 1/x) = \pm\infty$, so no HA. But $(x + 1 + 1/x) - (x + 1) = 1/x \to 0$ as $x \to \pm\infty$, so $y = x + 1$ is a slant asymptote. Also $\lim\limits_{x\to 0^+} (x + 1 + 1/x) = \infty$, $\lim\limits_{x\to 0^-} (x + 1 + 1/x) = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = 1 - 1/x^2 > 0$ when $x^2 > 1 \Leftrightarrow x > 1$ or $x < -1$; $f'(x) < 0 \Leftrightarrow -1 < x < 1$. So f is increasing on $(-\infty, -1)$, $(1, \infty)$ and decreasing on $(-1, 0)$, $(0, 1)$. **F.** $f(1) = 3$ is a local minimum, $f(-1) = -1$ is a local maximum. **G.** $f''(x) = 2/x^3 > 0 \Leftrightarrow x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP **H.**



51. $y = f(x) = \dfrac{2x^3+x^2+1}{x^2+1} = 2x + 1 + \dfrac{-2x}{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0$ $\Rightarrow 0 = 2x^3 + x^2 + 1 = (x + 1)(2x^2 - x + 1) \Rightarrow x = -1$ **C.** No symmetry **D.** No VA

$\lim\limits_{x\to\pm\infty} [f(x) - (2x + 1)] = \lim\limits_{x\to\pm\infty} \dfrac{-2x}{x^2+1} = \lim\limits_{x\to\pm\infty} \dfrac{-2/x}{1+1/x^2} = 0$, so the line $y = 2x + 1$ is a slant asymptote.

E. $f'(x) = 2 + \dfrac{(x^2+1)(-2) - (-2x)(2x)}{(x^2+1)^2} = \dfrac{2(x^4+2x^2+1) - 2x^2 - 2 + 4x^2}{(x^2+1)^2}$

$= \dfrac{2x^4+6x^2}{(x^2+1)^2} = \dfrac{2x^2(x^2+3)}{(x^2+1)^2}$

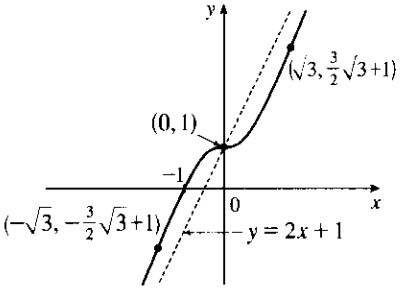
so $f'(x) > 0$ if $x \neq 0$. Thus, f is increasing on $(-\infty, 0)$ and $(0, \infty)$. Since f is continuous at 0, f is increasing on $\mathbb{R}$. **F.** No extreme values

G. $f''(x) = \dfrac{(x^2+1)^2 \cdot (8x^3+12x) - (2x^4+6x^2) \cdot 2(x^2+1)(2x)}{[(x^2+1)^2]^2}$

$= \dfrac{4x(x^2+1)[(x^2+1)(2x^2+3) - 2x^4 - 6x^2]}{(x^2+1)^4}$

$= \dfrac{4x(-x^2+3)}{(x^2+1)^3}$

so $f''(x) > 0$ for $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$, and $f''(x) < 0$ for $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$. f is CU on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$, and CD on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$. There are three IPs: $(0, 1)$, $(-\sqrt{3}, -\frac{3}{2}\sqrt{3} + 1) \approx (-1.73, -1.60)$, and $(\sqrt{3}, \frac{3}{2}\sqrt{3} + 1) \approx (1.73, 3.60)$. **H.**



52. $y = f(x) = \dfrac{(x+1)^3}{(x-1)^2} = \dfrac{x^3+3x^2+3x+1}{x^2-2x+1} = x + 5 + \dfrac{12x-4}{(x-1)^2}$

A. $D = \{x \in \mathbb{R} \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 1$;

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x-intercept: $f(x) = 0 \quad \Rightarrow \quad x = -1$ **C.** No symmetry **D.** $\lim_{x \to 1} f(x) = \infty$, so $x = 1$ is a VA.

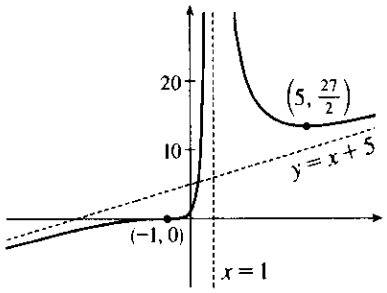
$$\lim_{x \to \pm\infty} [f(x) - (x+5)] = \lim_{x \to \pm\infty} \frac{12x - 4}{x^2 - 2x + 1} = \lim_{x \to \pm\infty} \frac{\frac{12}{x} - \frac{4}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} = 0, \text{ so the line } y = x + 5 \text{ is a SA.}$$

E. $f'(x) = \dfrac{(x-1)^2 \cdot 3(x+1)^2 - (x+1)^3 \cdot 2(x-1)}{[(x-1)^2]^2}$

$$= \frac{(x-1)(x+1)^2[3(x-1) - 2(x+1)]}{(x-1)^4} = \frac{(x+1)^2(x-5)}{(x-1)^3}$$

so $f'(x) > 0$ when $x < -1$, $-1 < x < 1$, or $x > 5$, and $f'(x) < 0$ when $1 < x < 5$. f is increasing on $(-\infty, 1)$ and $(5, \infty)$ and decreasing on $(1, 5)$.

H.



F. Local minimum value $f(5) = \frac{216}{16} = \frac{27}{2}$, no local maximum

G. $f''(x) = \dfrac{(x-1)^3[(x-1)^2 + (x-5) \cdot 2(x+1)] - (x+1)^2(x-5) \cdot 3(x-1)^2}{[(x-1)^3]^2}$

$$= \frac{(x-1)^2(x+1)\{(x-1)[(x+1) + 2(x-5)] - 3(x+1)(x-5)\}}{(x-1)^6}$$

$$= \frac{(x+1)\{(x-1)[3x-9] - 3(x^2 - 4x - 5)\}}{(x-1)^4} = \frac{(x+1)(24)}{(x-1)^4}$$

so $f''(x) > 0$ if $-1 < x < 1$ or $x > 1$, and $f''(x) < 0$ if $x < -1$. Thus, f is CU on $(-1, 1)$ and $(1, \infty)$ and CD on $(-\infty, -1)$. IP at $(-1, 0)$

53. $y = f(x) = \sqrt{4x^2 + 9} \quad \Rightarrow \quad f'(x) = \dfrac{4x}{\sqrt{4x^2 + 9}} \quad \Rightarrow$

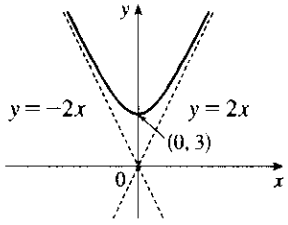
$$f''(x) = \frac{\sqrt{4x^2+9} \cdot 4 - 4x \cdot 4x/\sqrt{4x^2+9}}{4x^2 + 9} = \frac{4(4x^2+9) - 16x^2}{(4x^2+9)^{3/2}} = \frac{36}{(4x^2+9)^{3/2}}.$$

f is defined on $(-\infty, \infty)$. $f(-x) = f(x)$, so f is even, which means its graph is symmetric about the y-axis. The y-intercept is $f(0) = 3$. There are no x-intercepts since $f(x) > 0$ for all x.

$$\lim_{x \to \infty} \left(\sqrt{4x^2+9} - 2x\right) = \lim_{x \to \infty} \frac{\left(\sqrt{4x^2+9} - 2x\right)\left(\sqrt{4x^2+9} + 2x\right)}{\sqrt{4x^2+9} + 2x}$$

$$= \lim_{x \to \infty} \frac{(4x^2+9) - 4x^2}{\sqrt{4x^2+9} + 2x} = \lim_{x \to \infty} \frac{9}{\sqrt{4x^2+9} + 2x} = 0$$

and, similarly, $\lim_{x \to -\infty} \left(\sqrt{4x^2+9} + 2x\right) = \lim_{x \to -\infty} \dfrac{9}{\sqrt{4x^2+9} - 2x} = 0,$



so $y = \pm 2x$ are slant asymptotes. f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ with local minimum $f(0) = 3$. $f''(x) > 0$ for all x, so f is CU on $\mathbb{R}$.

54. $y = f(x) = \sqrt{x^2 + 4x} = \sqrt{x(x+4)}$. $x(x+4) \geq 0 \quad \Leftrightarrow \quad x \leq -4$ or $x \geq 0$, so $D = (-\infty, -4] \cup [0, \infty)$.

y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Rightarrow \quad x = -4, 0$.

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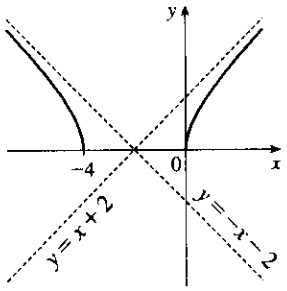
$$\sqrt{x^2+4x} \mp (x+2) = \frac{\sqrt{x^2+4x} \mp (x+2)}{1} \cdot \frac{\sqrt{x^2+4x} \pm (x+2)}{\sqrt{x^2+4x} \pm (x+2)} = \frac{(x^2+4x)-(x^2+4x+4)}{\sqrt{x^2+4x} \pm (x+2)}$$

$$= \frac{-4}{\sqrt{x^2+4x} \pm (x+2)}$$

so $\lim_{x \to \pm\infty} [f(x) \mp (x+2)] = 0$. Thus, the graph of f approaches the slant asymptote $y = x+2$ as $x \to \infty$ and it approaches the slant asymptote $y = -(x+2)$ as $x \to -\infty$. $f'(x) = \dfrac{x+2}{\sqrt{x^2+4x}}$, so $f'(x) < 0$ for $x < -4$ and $f'(x) > 0$ for $x > 0$; that is, f is decreasing on $(-\infty, -4)$ and increasing on $(0, \infty)$. There are no local extreme values.



$$f'(x) = (x+2)(x^2+4x)^{-1/2} \quad \Rightarrow$$

$$\begin{aligned} f''(x) &= (x+2) \cdot \left(-\tfrac{1}{2}\right)(x^2+4x)^{-3/2} \cdot (2x+4) + (x^2+4x)^{-1/2} \\ &= (x^2+4x)^{-3/2}\left[-(x+2)^2 + (x^2+4x)\right] \\ &= -4(x^2+4x)^{-3/2} < 0 \text{ on } D \end{aligned}$$

so f is CD on $(-\infty, -4)$ and $(0, \infty)$. No IP

55. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad y = \pm\dfrac{b}{a}\sqrt{x^2-a^2}$. Now

$$\lim_{x\to\infty}\left[\frac{b}{a}\sqrt{x^2-a^2} - \frac{b}{a}x\right] = \frac{b}{a} \cdot \lim_{x\to\infty}\left(\sqrt{x^2-a^2} - x\right)\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}+x} = \frac{b}{a} \cdot \lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0,$$

which shows that $y = \dfrac{b}{a}x$ is a slant asymptote. Similarly,

$$\lim_{x\to\infty}\left[-\frac{b}{a}\sqrt{x^2-a^2} - \left(-\frac{b}{a}x\right)\right] = -\frac{b}{a} \cdot \lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0, \text{ so } y = -\frac{b}{a}x \text{ is a slant asymptote.}$$

56. $f(x) - x^2 = \dfrac{x^3+1}{x} - x^2 = \dfrac{x^3+1-x^3}{x} = \dfrac{1}{x}$, and $\lim_{x\to\pm\infty}\dfrac{1}{x} = 0$. Therefore, $\lim_{x\to\pm\infty}\left[f(x) - x^2\right] = 0$, and so the graph of f is asymptotic to that of $y = x^2$. For purposes of differentiation, we will use $f(x) = x^2 + 1/x$.

A. $D = \{x \mid x \neq 0\}$ **B.** No y-intercept; to find the x-intercept, we set $y = 0 \quad \Leftrightarrow \quad x = -1$.

C. No symmetry **D.** $\lim_{x\to 0^+}\dfrac{x^3+1}{x} = \infty$ and $\lim_{x\to 0^-}\dfrac{x^3+1}{x} = -\infty$, so $x = 0$ is a vertical asymptote. Also, the graph is asymptotic to the parabola $y = x^2$, as shown above. **E.** $f'(x) = 2x - 1/x^2 > 0 \quad \Leftrightarrow$ $x > \frac{1}{\sqrt[3]{2}}$, so f is increasing on $\left(\frac{1}{\sqrt[3]{2}}, \infty\right)$ and decreasing on $(-\infty, 0)$ and $\left(0, \frac{1}{\sqrt[3]{2}}\right)$. **F.** Local minimum value $f\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{3\sqrt[3]{3}}{2}$, no local maximum **G.** $f''(x) = 2 + 2/x^3 > 0 \quad \Leftrightarrow \quad x < -1$ or $x > 0$, so f is CU on $(-\infty, -1)$ and $(0, \infty)$, and CD on $(-1, 0)$. IP at $(-1, 0)$

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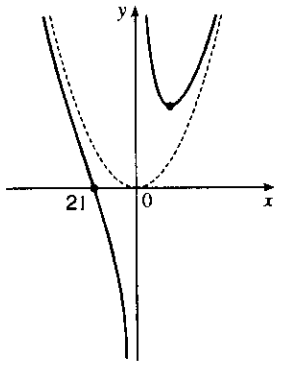

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57. $\lim_{x\to\pm\infty}\left[f(x)-x^3\right]=\lim_{x\to\pm\infty}\dfrac{x^4+1}{x}-\dfrac{x^4}{x}=\lim_{x\to\pm\infty}\dfrac{1}{x}=0$, so the graph of f is asymptotic to that of $y=x^3$.

A. $D=\{x\mid x\neq 0\}$ **B.** No intercept **C.** f is symmetric about the origin. **D.** $\lim_{x\to 0^-}\left(x^3+\dfrac{1}{x}\right)=-\infty$ and $\lim_{x\to 0^+}\left(x^3+\dfrac{1}{x}\right)=\infty$, so $x=0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to that of $y=x^3$. **E.** $f'(x)=3x^2-1/x^2>0 \;\Leftrightarrow\; x^4>\frac{1}{3} \;\Leftrightarrow\; |x|>\frac{1}{\sqrt[4]{3}}$, so f is increasing on $\left(-\infty,-\frac{1}{\sqrt[4]{3}}\right)$ and $\left(\frac{1}{\sqrt[4]{3}},\infty\right)$ and decreasing on $\left(-\frac{1}{\sqrt[4]{3}},0\right)$ and $\left(0,\frac{1}{\sqrt[4]{3}}\right)$. **F.** Local maximum value $f\left(-\frac{1}{\sqrt[4]{3}}\right)=-4\cdot 3^{-5/4}$, local minimum value $f\left(\frac{1}{\sqrt[4]{3}}\right)=4\cdot 3^{-5/4}$

G. $f''(x)=6x+2/x^3>0 \;\Leftrightarrow\; x>0$, so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP

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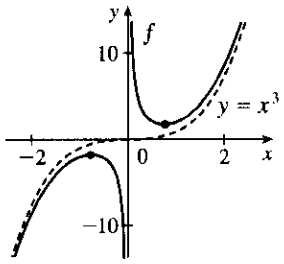
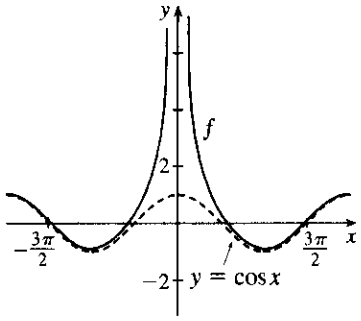
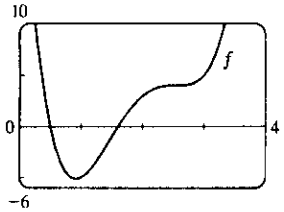


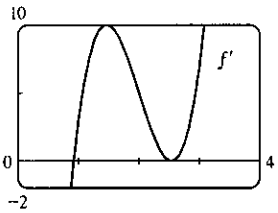
58. $\lim_{x\to\pm\infty}\left[f(x)-\cos x\right]=\lim_{x\to\pm\infty}1/x^2=0$, so the graph of f is asymptotic to that of $\cos x$. The intercepts can only be found approximately. $f(x)=f(-x)$, so f is even. $\lim_{x\to 0}\left(\cos x+\dfrac{1}{x^2}\right)=\infty$, so $x=0$ is a vertical asymptote. We don't need to calculate the derivatives, since we know the asymptotic behavior of the curve.

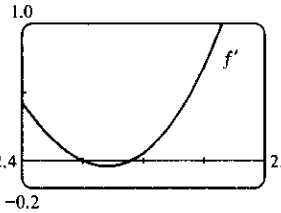


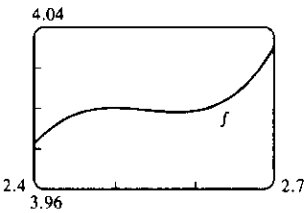
4.6 Graphing with Calculus and Calculators

1. $f(x)=4x^4-32x^3+89x^2-95x+29 \;\Rightarrow\; f'(x)=16x^3-96x^2+178x-95 \;\Rightarrow\; f''(x)=48x^2-192x+178$. $f(x)=0 \;\Leftrightarrow\; x\approx 0.5,\ 1.60$; $f'(x)=0 \;\Leftrightarrow\; x\approx 0.92,\ 2.5,\ 2.58$ and $f''(x)=0 \;\Leftrightarrow\; x\approx 1.46,\ 2.54$.









From the graphs of f', we estimate that $f'<0$ and that f is decreasing on $(-\infty,0.92)$ and $(2.5,2.58)$, and that $f'>0$ and f is increasing on $(0.92,2.5)$ and $(2.58,\infty)$ with local minimum values $f(0.92)\approx -5.12$ and $f(2.58)\approx 3.998$ and local maximum value $f(2.5)=4$. The graphs of f' make it clear that f has a maximum and a minimum near $x=2.5$, shown more clearly in the fourth graph.

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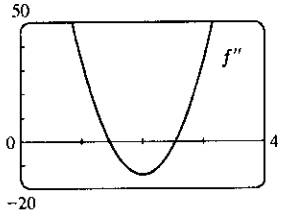
From the graph of f'', we estimate that $f'' > 0$ and that f is CU on $(-\infty, 1.46)$ and $(2.54, \infty)$, and that $f'' < 0$ and f is CD on $(1.46, 2.54)$. There are inflection points at about $(1.46, -1.40)$ and $(2.54, 3.999)$.

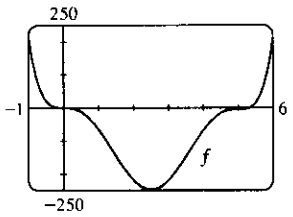


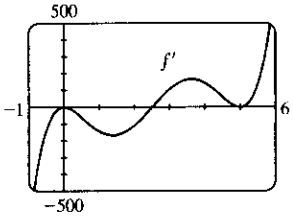
2. $f(x) = x^6 - 15x^5 + 75x^4 - 125x^3 - x \;\Rightarrow$

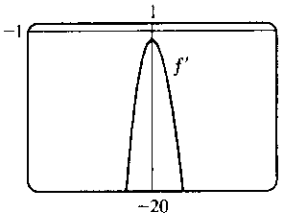
$f'(x) = 6x^5 - 75x^4 + 300x^3 - 375x^2 - 1 \;\Rightarrow\; f''(x) = 30x^4 - 300x^3 + 900x^2 - 750x.$

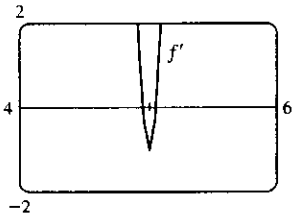
$f(x) = 0 \;\Leftrightarrow\; x = 0$ or $x \approx 5.33$; $f'(x) = 0 \;\Leftrightarrow\; x \approx 2.50, 4.95$, or 5.05;

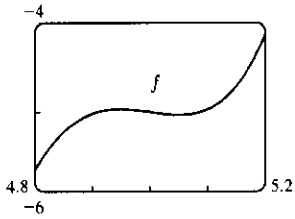
$f''(x) = 0 \;\Leftrightarrow\; x = 0, 5$ or $x \approx 1.38, 3.62$.

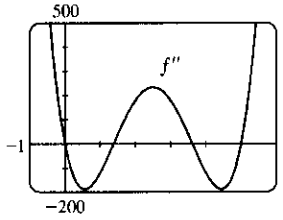






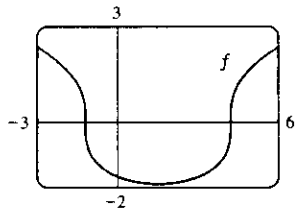


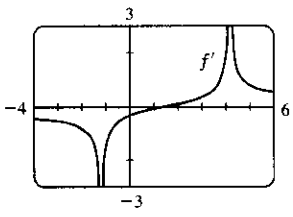


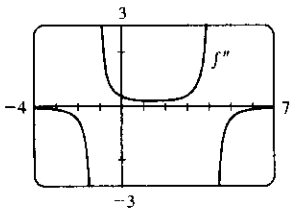


From the graphs of f', we estimate that f is decreasing on $(-\infty, 2.50)$, increasing on $(2.50, 4.95)$, decreasing on $(4.95, 5.05)$, and increasing on $(5.05, \infty)$, with local minimum values $f(2.50) \approx -246.6$ and $f(5.05) \approx -5.03$, and local maximum value $f(4.95) \approx -4.965$ (notice the second graph of f). From the graph of f'', we estimate that f is CU on $(-\infty, 0)$, CD on $(0, 1.38)$, CU on $(1.38, 3.62)$, CD on $(3.62, 5)$, and CU on $(5, \infty)$. There are inflection points at $(0, 0)$ and $(5, -5)$, and at about $(1.38, -126.38)$ and $(3.62, -128.62)$.

3. $f(x) = \sqrt[3]{x^2 - 3x - 5} \;\Rightarrow\; f'(x) = \dfrac{1}{3}\dfrac{2x - 3}{(x^2 - 3x - 5)^{2/3}} \;\Rightarrow\; f''(x) = -\dfrac{2}{9}\dfrac{x^2 - 3x + 24}{(x^2 - 3x - 5)^{5/3}}$







Note: With some CAS's, including Maple, it is necessary to define $f(x) = \dfrac{x^2 - 3x - 5}{|x^2 - 3x - 5|}\,|x^2 - 3x - 5|^{1/3}$, since the CAS does not compute real cube roots of negative numbers. We estimate from the graph of f' that f is

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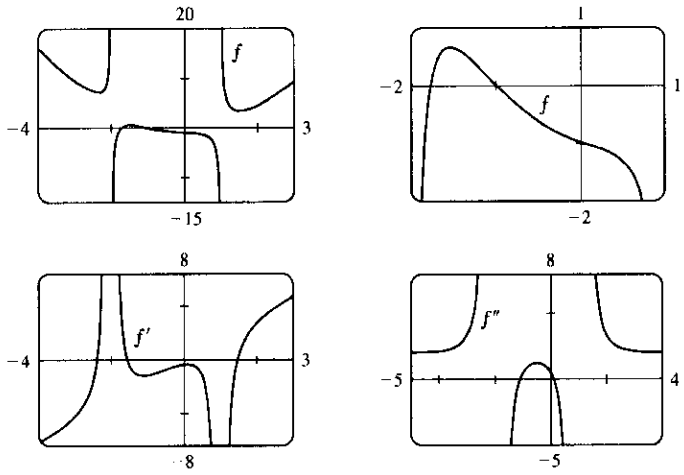
increasing on $(1.5, \infty)$, and decreasing on $(-\infty, 1.5)$. f has no maximum. Minimum value: $f(1.5) \approx -1.9$. From the graph of f'', we estimate that f is CU on $(-1.2, 4.2)$ and CD on $(-\infty, -1.2)$ and $(4.2, \infty)$. IP at $(-1.2, 0)$ and $(4.2, 0)$.

4. $f(x) = \dfrac{x^4 + x^3 - 2x^2 + 2}{x^2 + x - 2} \quad \Rightarrow \quad f'(x) = 2\,\dfrac{x^5 + 2x^4 - 3x^3 - 4x^2 + 2x - 1}{(x^2 + x - 2)^2} \quad \Rightarrow$

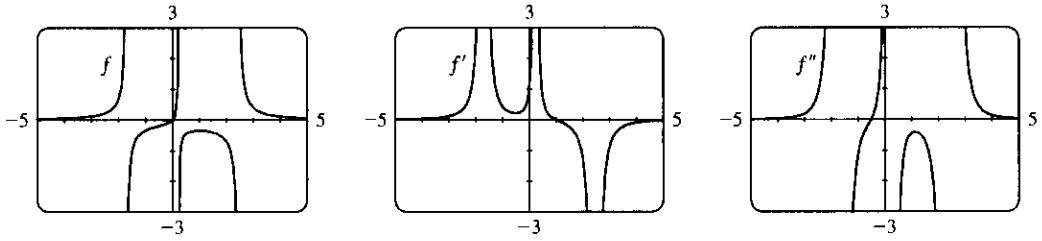
$$f''(x) = 2\,\frac{x^6 + 3x^5 - 3x^4 - 11x^3 + 12x^2 + 18x - 2}{(x^2 + x - 2)^3}$$



We estimate from the graph of f' that f is increasing on $(-2.4, -2)$, $(-2, -1.5)$ and $(1.5, \infty)$ and decreasing on $(-\infty, -2.4)$, $(-1.5, 1)$ and $(1, 1.5)$. Local maximum value: $f(-1.5) \approx 0.7$.
Local minimum values: $f(-2.4) \approx 7.2$, $f(1.5) \approx 3.4$. From the graph of f'', we estimate that f is CU on $(-\infty, -2)$, $(-1.1, 0.1)$ and $(1, \infty)$ and CD on $(-2, -1.1)$ and $(0.1, 1)$.
f has IP at $(-1.1, 0.2)$ and $(0.1, -1.1)$.

5. $f(x) = \dfrac{x}{x^3 - x^2 - 4x + 1} \quad \Rightarrow \quad f'(x) = \dfrac{-2x^3 + x^2 + 1}{(x^3 - x^2 - 4x + 1)^2} \quad \Rightarrow$

$$f''(x) = \frac{2\big(3x^5 - 3x^4 + 5x^3 - 6x^2 + 3x + 4\big)}{(x^3 - x^2 - 4x + 1)^3}$$



We estimate from the graph of f that $y = 0$ is a horizontal asymptote, and that there are vertical asymptotes at $x = -1.7$, $x = 0.24$, and $x = 2.46$. From the graph of f', we estimate that f is increasing on $(-\infty, -1.7)$, $(-1.7, 0.24)$, and $(0.24, 1)$, and that f is decreasing on $(1, 2.46)$ and $(2.46, \infty)$. There is a local maximum value at $f(1) = -\frac{1}{3}$. From the graph of f'', we estimate that f is CU on $(-\infty, -1.7)$, $(-0.506, 0.24)$, and $(2.46, \infty)$, and that f is CD on $(-1.7, -0.506)$ and $(0.24, 2.46)$. There is an inflection point at $(-0.506, -0.192)$.

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27. (a) $f(x) = x^2 - a \quad \Rightarrow \quad f'(x) = 2x$, so Newton's method gives

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = x_n - \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

(b) Using (a) with $a = 1000$ and $x_1 = \sqrt{900} = 30$, we get $x_2 \approx 31.666667$, $x_3 \approx 31.622807$, and $x_4 \approx 31.622777 \approx x_5$. So $\sqrt{1000} \approx 31.622777$.

28. (a) $f(x) = \dfrac{1}{x} - a \quad \Rightarrow \quad f'(x) = -\dfrac{1}{x^2}$, so $x_{n+1} = x_n - \dfrac{1/x_n - a}{-1/x_n^2} = x_n + x_n - ax_n^2 = 2x_n - ax_n^2$.

(b) Using (a) with $a = 1.6894$ and $x_1 = \frac{1}{2} = 0.5$, we get $x_2 = 0.5754$, $x_3 \approx 0.588485$, and $x_4 \approx 0.588789 \approx x_5$. So $1/1.6984 \approx 0.588789$.

29. $f(x) = x^3 - 3x + 6 \quad \Rightarrow \quad f'(x) = 3x^2 - 3$. If $x_1 = 1$, then $f'(x_1) = 0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

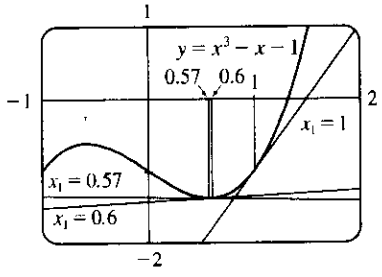
30. $x^3 - x = 1 \quad \Leftrightarrow \quad x^3 - x - 1 = 0$. $f(x) = x^3 - x - 1 \quad \Rightarrow \quad f'(x) = 3x^2 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 - x_n - 1}{3x_n^2 - 1}$.

(a) $x_1 = 1$, $x_2 = 1.5$, $x_3 \approx 1.347826$, $x_4 \approx 1.325200$, $x_5 \approx 1.324718 \approx x_6$

(b) $x_1 = 0.6$, $x_2 = 17.9$, $x_3 \approx 11.946802$, $x_4 \approx 7.985520$, $x_5 \approx 5.356909$, $x_6 \approx 3.624996$, $x_7 \approx 2.505589$, $x_8 \approx 1.820129$, $x_9 \approx 1.461044$, $x_{10} \approx 1.339323$, $x_{11} \approx 1.324913$, $x_{12} \approx 1.324718 \approx x_{13}$

(c) $x_1 = 0.57$, $x_2 \approx -54.165455$, $x_3 \approx -36.114293$, $x_4 \approx -24.082094$, $x_5 \approx -16.063387$, $x_6 \approx -10.721483$, $x_7 \approx -7.165534$, $x_8 \approx -4.801704$, $x_9 \approx -3.233425$, $x_{10} \approx -2.193674$, $x_{11} \approx -1.496867$, $x_{12} \approx -0.997546$, $x_{13} \approx -0.496305$, $x_{14} \approx -2.894162$, $x_{15} \approx -1.967962$, $x_{16} \approx -1.341355$, $x_{17} \approx -0.870187$, $x_{18} \approx -0.249949$, $x_{19} \approx -1.192219$, $x_{20} \approx -0.731952$, $x_{21} \approx 0.355213$, $x_{22} \approx -1.753322$, $x_{23} \approx -1.189420$, $x_{24} \approx -0.729123$, $x_{25} \approx 0.377844$, $x_{26} \approx -1.937872$, $x_{27} \approx -1.320350$, $x_{28} \approx -0.851919$, $x_{29} \approx -0.200959$, $x_{30} \approx -1.119386$, $x_{31} \approx -0.654291$, $x_{32} \approx 1.547010$, $x_{33} \approx 1.360051$, $x_{34} \approx 1.325828$, $x_{35} \approx 1.324719$, $x_{36} \approx 1.324718 \approx x_{37}$.

(d)

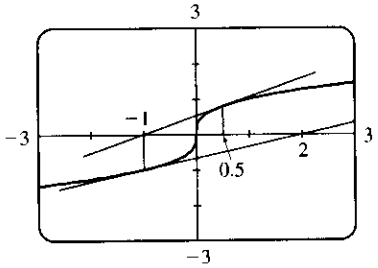


From the figure, we see that the tangent line corresponding to $x_1 = 1$ results in a sequence of approximations that converges quite quickly ($x_5 \approx x_6$). The tangent line corresponding to $x_1 = 0.6$ is close to being horizontal, so x_2 is quite far from the root. But the sequence still converges — just a little more slowly ($x_{12} \approx x_{13}$). Lastly, the tangent line corresponding to $x_1 = 0.57$ is very nearly horizontal, x_2 is farther away from the root, and the sequence takes more iterations to converge ($x_{36} \approx x_{37}$).

31. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

Therefore, each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the root, which is 0. In the figure, we have $x_1 = 0.5$, $x_2 = -2(0.5) = -1$, and $x_3 = -2(-1) = 2$.



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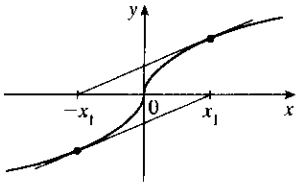
32. According to Newton's Method, for $x_n > 0$,

$x_{n+1} = x_n - \dfrac{\sqrt{x_n}}{1/(2\sqrt{x_n})} = x_n - 2x_n = -x_n$ and for $x_n < 0$,

$x_{n+1} = x_n - \dfrac{-\sqrt{-x_n}}{1/(2\sqrt{-x_n})} = x_n - [-2(-x_n)] = -x_n$. So we can see that after choosing any value x_1 the subsequent values will alternate between $-x_1$ and x_1 and never approach the root.



33. (a) $f(x) = 3x^4 - 28x^3 + 6x^2 + 24x \Rightarrow f'(x) = 12x^3 - 84x^2 + 12x + 24 \Rightarrow$

$f''(x) = 36x^2 - 168x + 12$. Now to solve $f'(x) = 0$, try $x_1 = \frac{1}{2} \Rightarrow x_2 = x_1 - \dfrac{f'(x_1)}{f''(x_1)} = \dfrac{2}{3} \Rightarrow$

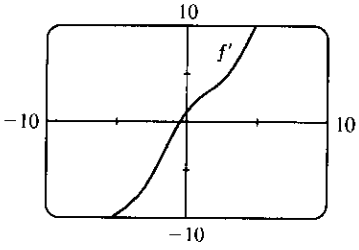
$x_3 \approx 0.6455 \Rightarrow x_4 \approx 0.6452 \Rightarrow x_5 \approx 0.6452$. Now try $x_1 = 6 \Rightarrow x_2 = 7.12 \Rightarrow$

$x_3 \approx 6.8353 \Rightarrow x_4 \approx 6.8102 \Rightarrow x_5 \approx 6.8100$. Finally try $x_1 = -0.5 \Rightarrow x_2 \approx -0.4571 \Rightarrow$

$x_3 \approx -0.4552 \Rightarrow x_4 \approx -0.4552$. Therefore, $x = -0.455$, 6.810 and 0.645 are all critical numbers correct to three decimal places.

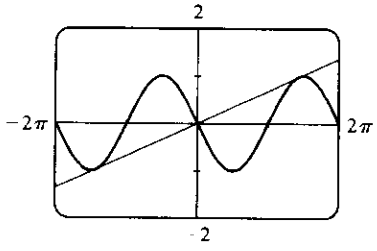
(b) $f(-1) = 13$, $f(7) = -1939$, $f(6.810) \approx -1949.07$, $f(-0.455) \approx -6.912$, $f(0.645) \approx 10.982$. Therefore, $f(6.810) \approx -1949.07$ is the absolute minimum correct to two decimal places.

34. $f(x) = x^2 + \sin x \Rightarrow f'(x) = 2x + \cos x$. $f'(x)$ exists for all x, so to find the minimum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for x_1 is $x_1 = -0.5$. Use $g(x) = 2x + \cos x$ and $g'(x) = 2 - \sin x$ to obtain $x_2 \approx -0.450627$, $x_3 \approx -0.450184 \approx x_4$. Since $f''(x) = 2 - \sin x > 0$ for all x, $f(-0.450184) \approx -0.232466$ is the absolute minimum.



35. $y = x^3 + \cos x \Rightarrow y' = 3x^2 - \sin x \Rightarrow y'' = 6x - \cos x \Rightarrow y''' = 6 + \sin x$. Now to solve $y'' = 0$, try $x_1 = 0$, and then $x_2 = x_1 - \dfrac{y''(x_1)}{y'''(x_1)} = \dfrac{1}{6} \Rightarrow x_3 \approx 0.164419 \approx x_4$. For $x < 0.164419$, $y'' < 0$, and for $x > 0.164419$, $y'' > 0$. Therefore, the point of inflection, correct to six decimal places, is $(0.164419, y(0.164419)) \approx (0.164419, 0.990958)$.

36.



$f(x) = -\sin x \Rightarrow f'(x) = -\cos x$. At $x = a$, the slope of the tangent line is $f'(a) = -\cos a$. The line through the origin and $(a, f(a))$ is $y = \dfrac{-\sin a - 0}{a - 0}x$. If this line is to be tangent to f at $x = a$, then its slope must equal $f'(a)$. Thus, $\dfrac{-\sin a}{a} = -\cos a \Rightarrow \tan a = a$.

To solve this equation using Newton's method, let $g(x) = \tan x - x$, $g'(x) = \sec^2 x - 1$, and $x_{n+1} = x_n - \dfrac{\tan x_n - x_n}{\sec^2 x_n - 1}$ with $x_1 = 4.5$ (estimated from the figure). $x_2 \approx 4.493614$, $x_3 \approx 4.493410$, $x_4 \approx 4.493409 \approx x_5$. Thus, the slope of the line that has the largest slope is $f'(x_5) \approx 0.217234$.

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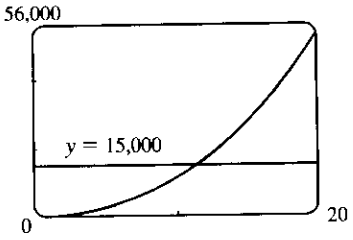
37.



The volume of the silo, in terms of its radius, is $V(r) = \pi r^2(30) + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = 30\pi r^2 + \frac{2}{3}\pi r^3$.

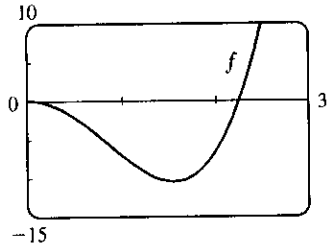
From a graph of V, we see that $V(r) = 15{,}000$ at $r \approx 11$ ft. Now we use Newton's method to solve the equation $V(r) - 15{,}000 = 0$.

$\dfrac{dV}{dr} = 60\pi r + 2\pi r^2$, so $r_{n+1} = r_n - \dfrac{30\pi r_n^2 + \frac{2}{3}\pi r_n^3 - 15{,}000}{60\pi r_n + 2\pi r_n^2}$. Taking $r_1 = 11$, we get $r_2 \approx 11.2853$, $r_3 \approx 11.2807 \approx r_4$. So in order for the silo to hold 15,000 ft^3 of grain, its radius must be about 11.2807 ft.

38. Let the radius of the circle be r. Using $s = r\theta$, we have $5 = r\theta$ and so $r = 5/\theta$. From the Law of Cosines we get

$4^2 = r^2 + r^2 - 2 \cdot r \cdot r \cdot \cos\theta \quad \Leftrightarrow \quad 16 = 2r^2(1 - \cos\theta) = 2(5/\theta)^2(1 - \cos\theta)$.

Multiplying by θ^2 gives $16\theta^2 = 50(1 - \cos\theta)$, so we take $f(\theta) = 16\theta^2 + 50\cos\theta - 50$ and $f'(\theta) = 32\theta - 50\sin\theta$. The formula for Newton's method is $\theta_{n+1} = \theta_n - \dfrac{16\theta_n^2 + 50\cos\theta_n - 50}{32\theta_n - 50\sin\theta_n}$. From the graph of f, we can use $\theta_1 = 2.2$, giving us $\theta_2 \approx 2.2662$, $\theta_3 \approx 2.2622 \approx \theta_4$. So correct to four decimal places, the angle is 2.2622 radians $\approx 130°$.



39. In this case, $A = 18{,}000$, $R = 375$, and $n = 5(12) = 60$. So the formula $A = \dfrac{R}{i}\left[1 - (1+i)^{-n}\right]$ becomes

$18{,}000 = \dfrac{375}{x}\left[1 - (1+x)^{-60}\right] \quad \Leftrightarrow \quad 48x = 1 - (1+x)^{-60}$ [multiply each term by $(1+x)^{60}$] $\Leftrightarrow$

$48x(1+x)^{60} - (1+x)^{60} + 1 = 0$. Let the LHS be called $f(x)$, so that

$$f'(x) = 48x(60)(1+x)^{59} + 48(1+x)^{60} - 60(1+x)^{59}$$
$$= 12(1+x)^{59}\left[4x(60) + 4(1+x) - 5\right] = 12(1+x)^{59}(244x - 1)$$

$x_{n+1} = x_n - \dfrac{48x_n(1+x_n)^{60} - (1+x_n)^{60} + 1}{12(1+x_n)^{59}(244x_n - 1)}$. An interest rate of 1% per month seems like a reasonable estimate for $x = i$. So let $x_1 = 1\% = 0.01$, and we get $x_2 \approx 0.0082202$, $x_3 \approx 0.0076802$, $x_4 \approx 0.0076291$, $x_5 \approx 0.0076286 \approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55% per year, compounded monthly).

40. (a) $p(x) = x^5 - (2+r)x^4 + (1+2r)x^3 - (1-r)x^2 + 2(1-r)x + r - 1 \quad \Rightarrow$

$p'(x) = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1-r)x + 2(1-r)$. So we use

$x_{n+1} = x_n - \dfrac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1-r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1-r)x_n + 2(1-r)}$. We substitute in the value $r \approx 3.04042 \times 10^{-6}$ in order to evaluate the approximations numerically. The libration point L_1 is slightly less than 1 AU from the Sun, so we take $x_1 = 0.95$ as our first approximation, and get $x_2 \approx 0.96682$, $x_3 \approx 0.97770$, $x_4 \approx 0.98451$, $x_5 \approx 0.98830$, $x_6 \approx 0.98976$, $x_7 \approx 0.98998$, $x_8 \approx 0.98999 \approx x_9$. So, to five decimal places, L_1 is located 0.98999 AU from the Sun (or 0.01001 AU from Earth).

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(b) In this case we use Newton's method with the function

$p(x) - 2rx^2 = x^5 - (2+r)x^4 + (1+2r)x^3 - (1+r)x^2 + 2(1-r)x + r - 1 \Rightarrow$

$\left[p(x) - 2rx^2\right]' = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1+r)x + 2(1-r)$. So

$x_{n+1} = x_n - \dfrac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1+r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1+r)x_n + 2(1-r)}$. Again, we substitute $r \approx 3.04042 \times 10^{-6}$. L_2 is slightly more than 1 AU from the Sun and, judging from the result of part (a), probably less than 0.02 AU from Earth. So we take $x_1 = 1.02$ and get $x_2 \approx 1.01422$, $x_3 \approx 1.01118$, $x_4 \approx 1.01018$, $x_5 \approx 1.01008 \approx x_6$. So, to five decimal places, L_2 is located 1.01008 AU from the Sun (or 0.01008 AU from Earth).

4.10 Antiderivatives

1. $f(x) = 6x^2 - 8x + 3 \Rightarrow F(x) = 6\dfrac{x^{2+1}}{2+1} - 8\dfrac{x^{1+1}}{1+1} + 3x + C = 2x^3 - 4x^2 + 3x + C$

Check: $F'(x) = 2 \cdot 3x^2 - 4 \cdot 2x + 3 + 0 = 6x^2 - 8x + 3 = f(x)$

2. $f(x) = 4 + x^2 - 5x^3 \Rightarrow F(x) = 4x + \frac{1}{3}x^3 - \frac{5}{4}x^4 + C$

3. $f(x) = 1 - x^3 + 5x^5 - 3x^7 \Rightarrow F(x) = x - \dfrac{x^{3+1}}{3+1} + 5\dfrac{x^{5+1}}{5+1} - 3\dfrac{x^{7+1}}{7+1} + C = x - \frac{1}{4}x^4 + \frac{5}{6}x^6 - \frac{3}{8}x^8 + C$

4. $f(x) = x^{20} + 4x^{10} + 8 \Rightarrow F(x) = \frac{1}{21}x^{21} + \frac{4}{11}x^{11} + 8x + C$

5. $f(x) = 5x^{1/4} - 7x^{3/4} \Rightarrow F(x) = 5\dfrac{x^{1/4+1}}{\frac{1}{4}+1} - 7\dfrac{x^{3/4+1}}{\frac{3}{4}+1} + C = 5\dfrac{x^{5/4}}{5/4} - 7\dfrac{x^{7/4}}{7/4} + C = 4x^{5/4} - 4x^{7/4} + C$

6. $f(x) = 2x + 3x^{1.7} \Rightarrow F(x) = x^2 + \frac{3}{2.7}x^{2.7} + C = x^2 + \frac{10}{9}x^{2.7} + C$

7. $f(x) = 6\sqrt{x} - \sqrt[6]{x} = 6x^{1/2} - x^{1/6} \Rightarrow$

$F(x) = 6\,\dfrac{x^{1/2+1}}{\frac{1}{2}+1} - \dfrac{x^{1/6+1}}{\frac{1}{6}+1} + C = 6\,\dfrac{x^{3/2}}{3/2} - \dfrac{x^{7/6}}{7/6} + C = 4x^{3/2} - \frac{6}{7}x^{7/6} + C$

8. $f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4} = x^{3/4} + x^{4/3} \Rightarrow F(x) = \dfrac{x^{7/4}}{7/4} + \dfrac{x^{7/3}}{7/3} + C = \frac{4}{7}x^{7/4} + \frac{3}{7}x^{7/3} + C$

9. $f(x) = \dfrac{10}{x^9} = 10x^{-9}$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} \dfrac{10x^{-8}}{-8} + C_1 = -\dfrac{5}{4x^8} + C_1 & \text{if } x < 0 \\ -\dfrac{5}{4x^8} + C_2 & \text{if } x > 0 \end{cases}$

See Example 1 for a similar problem.

10. $g(x) = \dfrac{5 - 4x^3 + 2x^6}{x^6} = 5x^{-6} - 4x^{-3} + 2$ has domain $(-\infty, 0) \cup (0, \infty)$, so

$G(x) = \begin{cases} 5\dfrac{x^{-5}}{-5} - 4\dfrac{x^{-2}}{-2} + 2x + C_1 = -\dfrac{1}{x^5} + \dfrac{2}{x^2} + 2x + C_1 & \text{if } x < 0 \\ -\dfrac{1}{x^5} + \dfrac{2}{x^2} + 2x + C_2 & \text{if } x > 0 \end{cases}$

11. $f(u) = \dfrac{u^4 + 3\sqrt{u}}{u^2} = \dfrac{u^4}{u^2} + \dfrac{3u^{1/2}}{u^2} = u^2 + 3u^{-3/2} \Rightarrow$

$F(u) = \dfrac{u^3}{3} + 3\,\dfrac{u^{-3/2+1}}{-3/2+1} + C = \dfrac{1}{3}u^3 + 3\dfrac{u^{-1/2}}{-1/2} + C = \dfrac{1}{3}u^3 - \dfrac{6}{\sqrt{u}} + C$

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12. $f(x) = x^{2/3} + 2x^{-1/3}$ has domain $(-\infty, 0) \cup (0, \infty)$, so

$$F(x) = \begin{cases} \dfrac{x^{5/3}}{5/3} + \dfrac{2x^{2/3}}{2/3} + C_1 = \frac{3}{5}x^{5/3} + 3x^{2/3} + C_1 & \text{if } x > 0 \\ \frac{3}{5}x^{5/3} + 3x^{2/3} + C_2 & \text{if } x < 0 \end{cases}$$

13. $h(x) = x^3 + 5\sin x \;\Rightarrow\; H(x) = \frac{1}{4}x^4 + 5(-\cos x) + C = \frac{1}{4}x^4 - 5\cos x + C$

14. $f(t) = 3\cos t - 4\sin t \;\Rightarrow\; F(t) = 3(\sin t) - 4(-\cos t) + C = 3\sin t + 4\cos t + C$

15. $f(t) = 4\sqrt{t} - \sec t \tan t \;\Rightarrow\; F(t) = \frac{4}{3/2}t^{3/2} - \sec t + C = \frac{8}{3}t^{3/2} - \sec t + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

16. $f(\theta) = 6\theta^2 - 7\sec^2\theta \;\Rightarrow\; F(\theta) = 2\theta^3 - 7\tan\theta + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

17. $f''(x) = 6x + 12x^2 \;\Rightarrow\; f'(x) = 6 \cdot \dfrac{x^2}{2} + 12 \cdot \dfrac{x^3}{3} + C = 3x^2 + 4x^3 + C \;\Rightarrow$

$f(x) = 3 \cdot \dfrac{x^3}{3} + 4 \cdot \dfrac{x^4}{4} + Cx + D = x^3 + x^4 + Cx + D$ [C and D are just arbitrary constants]

18. $f''(x) = 2 + x^3 + x^6 \;\Rightarrow\; f'(x) = 2x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + C \;\Rightarrow\; f(x) = x^2 + \frac{1}{20}x^5 + \frac{1}{56}x^8 + Cx + D$

19. $f''(x) = 1 + x^{4/5} \;\Rightarrow\; f'(x) = x + \frac{5}{9}x^{9/5} + C \;\Rightarrow$

$f(x) = \frac{1}{2}x^2 + \frac{5}{9} \cdot \frac{5}{14}x^{14/5} + Cx + D = \frac{1}{2}x^2 + \frac{25}{126}x^{14/5} + Cx + D$

20. $f''(x) = \cos x \;\Rightarrow\; f'(x) = \sin x + C \;\Rightarrow\; f(x) = -\cos x + Cx + D$

21. $f'''(t) = 60t^2 \;\Rightarrow\; f''(t) = 20t^3 + C \;\Rightarrow\; f'(t) = 5t^4 + Ct + D \;\Rightarrow\; f(t) = t^5 + \frac{1}{2}Ct^2 + Dt + E$

22. $f'''(t) = t - \sqrt{t} \;\Rightarrow\; f''(t) = \frac{1}{2}t^2 - \frac{2}{3}t^{3/2} + C \;\Rightarrow\; f'(t) = \frac{1}{6}t^3 - \frac{4}{15}t^{5/2} + Ct + D \;\Rightarrow$

$f(t) = \frac{1}{24}t^4 - \frac{8}{105}t^{7/2} + \frac{1}{2}Ct^2 + Dt + E$

23. $f'(x) = 1 - 6x \;\Rightarrow\; f(x) = x - 3x^2 + C$. $f(0) = C$ and $f(0) = 8 \;\Rightarrow\; C = 8$, so $f(x) = x - 3x^2 + 8$.

24. $f'(x) = 8x^3 + 12x + 3 \;\Rightarrow\; f(x) = 2x^4 + 6x^2 + 3x + C$. $f(1) = 11 + C$ and $f(1) = 6 \;\Rightarrow$

$11 + C = 6 \;\Rightarrow\; C = -5$, so $f(x) = 2x^4 + 6x^2 + 3x - 5$.

25. $f'(x) = \sqrt{x}(6 + 5x) = 6x^{1/2} + 5x^{3/2} \;\Rightarrow\; f(x) = 4x^{3/2} + 2x^{5/2} + C$.

$f(1) = 6 + C$ and $f(1) = 10 \;\Rightarrow\; C = 4$, so $f(x) = 4x^{3/2} + 2x^{5/2} + 4$.

26. $f'(x) = 2x - 3/x^4 = 2x - 3x^{-4} \;\Rightarrow\; f(x) = x^2 + x^{-3} + C$ because we're given that $x > 0$.

$f(1) = 2 + C$ and $f(1) = 3 \;\Rightarrow\; C = 1$, so $f(x) = x^2 + 1/x^3 + 1$.

27. $f'(t) = 2\cos t + \sec^2 t \;\Rightarrow\; f(t) = 2\sin t + \tan t + C$ because $-\pi/2 < t < \pi/2$.

$f\left(\frac{\pi}{3}\right) = 2\left(\sqrt{3}/2\right) + \sqrt{3} + C = 2\sqrt{3} + C$ and $f\left(\frac{\pi}{3}\right) = 4 \;\Rightarrow\; C = 4 - 2\sqrt{3}$, so

$f(t) = 2\sin t + \tan t + 4 - 2\sqrt{3}$.

28. $f'(x) = 3x^{-2} \;\Rightarrow\; f(x) = \begin{cases} -3/x + C_1 & \text{if } x > 0 \\ -3/x + C_2 & \text{if } x < 0 \end{cases}$ $\quad f(1) = -3 + C_1 = 0 \;\Rightarrow\; C_1 = 3$,

$f(-1) = 3 + C_2 = 0 \;\Rightarrow\; C_2 = -3$. So $f(x) = \begin{cases} -3/x + 3 & \text{if } x > 0 \\ -3/x - 3 & \text{if } x < 0 \end{cases}$

29. $f''(x) = 24x^2 + 2x + 10 \;\Rightarrow\; f'(x) = 8x^3 + x^2 + 10x + C$. $f'(1) = 8 + 1 + 10 + C$ and $f'(1) = -3 \;\Rightarrow$

$19 + C = -3 \;\Rightarrow\; C = -22$, so $f'(x) = 8x^3 + x^2 + 10x - 22$ and hence,

$f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + D$. $f(1) = 2 + \frac{1}{3} + 5 - 22 + D$ and $f(1) = 5 \;\Rightarrow\; D = 22 - \frac{7}{3} = \frac{59}{3}$,

so $f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + \frac{59}{3}$.

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30. $f''(x) = 4 - 6x - 40x^3 \Rightarrow f'(x) = 4x - 3x^2 - 10x^4 + C$. $f'(0) = C$ and $f'(0) = 1 \Rightarrow C = 1$, so $f'(x) = 4x - 3x^2 - 10x^4 + 1$ and hence, $f(x) = 2x^2 - x^3 - 2x^5 + x + D$. $f(0) = D$ and $f(0) = 2 \Rightarrow D = 2$, so $f(x) = 2x^2 - x^3 - 2x^5 + x + 2$.

31. $f''(\theta) = \sin\theta + \cos\theta \Rightarrow f'(\theta) = -\cos\theta + \sin\theta + C$. $f'(0) = -1 + C$ and $f'(0) = 4 \Rightarrow C = 5$, so $f'(\theta) = -\cos\theta + \sin\theta + 5$ and hence, $f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$. $f(0) = -1 + D$ and $f(0) = 3 \Rightarrow D = 4$, so $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$.

32. $f''(t) = 3/\sqrt{t} = 3t^{-1/2} \Rightarrow f'(t) = 6t^{1/2} + C$. $f'(4) = 12 + C$ and $f'(4) = 7 \Rightarrow C = -5$, so $f'(t) = 6t^{1/2} - 5$ and hence, $f(t) = 4t^{3/2} - 5t + D$. $f(4) = 32 - 20 + D$ and $f(4) = 20 \Rightarrow D = 8$, so $f(t) = 4t^{3/2} - 5t + 8$.

33. $f''(x) = 2 - 12x \Rightarrow f'(x) = 2x - 6x^2 + C \Rightarrow f(x) = x^2 - 2x^3 + Cx + D$. $f(0) = D$ and $f(0) = 9 \Rightarrow D = 9$. $f(2) = 4 - 16 + 2C + 9 = 2C - 3$ and $f(2) = 15 \Rightarrow 2C = 18 \Rightarrow C = 9$, so $f(x) = x^2 - 2x^3 + 9x + 9$.

34. $f''(x) = 20x^3 + 12x^2 + 4 \Rightarrow f'(x) = 5x^4 + 4x^3 + 4x + C \Rightarrow f(x) = x^5 + x^4 + 2x^2 + Cx + D$. $f(0) = D$ and $f(0) = 8 \Rightarrow D = 8$. $f(1) = 1 + 1 + 2 + C + 8 = C + 12$ and $f(1) = 5 \Rightarrow C = -7$, so $f(x) = x^5 + x^4 + 2x^2 - 7x + 8$.

35. $f''(x) = 2 + \cos x \Rightarrow f'(x) = 2x + \sin x + C \Rightarrow f(x) = x^2 - \cos x + Cx + D$. $f(0) = -1 + D$ and $f(0) = -1 \Rightarrow D = 0$. $f\left(\frac{\pi}{2}\right) = \pi^2/4 + \left(\frac{\pi}{2}\right)C$ and $f\left(\frac{\pi}{2}\right) = 0 \Rightarrow \left(\frac{\pi}{2}\right)C = -\pi^2/4 \Rightarrow C = -\frac{\pi}{2}$, so $f(x) = x^2 - \cos x - \left(\frac{\pi}{2}\right)x$.

36. $f'''(x) = \sin x \Rightarrow f''(x) = -\cos x + C \Rightarrow 1 = f''(0) = -1 + C \Rightarrow C = 2$, so $f''(x) = -\cos x + 2 \Rightarrow f'(x) = -\sin x + 2x + D \Rightarrow 1 = f'(0) = D \Rightarrow f'(x) = -\sin x + 2x + 1 \Rightarrow f(x) = \cos x + x^2 + x + E \Rightarrow 1 = f(0) = 1 + E \Rightarrow E = 0$, so $f(x) = \cos x + x^2 + x$.

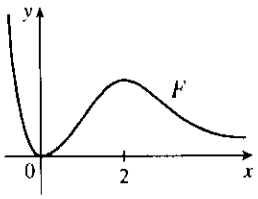
37. Given $f'(x) = 2x + 1$, we have $f(x) = x^2 + x + C$. Since f passes through $(1, 6)$, $f(1) = 6 \Rightarrow 1^2 + 1 + C = 6 \Rightarrow C = 4$. Therefore, $f(x) = x^2 + x + 4$ and $f(2) = 2^2 + 2 + 4 = 10$.

38. $f'(x) = x^3 \Rightarrow f(x) = \frac{1}{4}x^4 + C$. $x + y = 0 \Rightarrow y = -x \Rightarrow m = -1$. Now $m = f'(x) \Rightarrow -1 = x^3 \Rightarrow x = -1 \Rightarrow y = 1$ (from the equation of the tangent line), so $(-1, 1)$ is a point on the graph of f. From f, $1 = \frac{1}{4}(-1)^4 + C \Rightarrow C = \frac{3}{4}$. Therefore, the function is $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$.

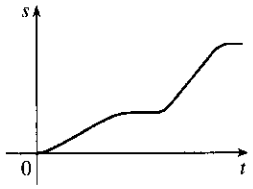
39. b is the antiderivative of f. For small x, f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x, so only b can be f's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

40. We know right away that c cannot be f's antiderivative, since the slope of c is not zero at the x-value where $f = 0$. Now f is positive when a is increasing and negative when a is decreasing, so a is the antiderivative of f.

41. The graph of F will have a minimum at 0 and a maximum at 2, since $f = F'$ goes from negative to positive at $x = 0$, and from positive to negative at $x = 2$.



42. The position function is the antiderivative of the velocity function, so its graph has to be horizontal where the velocity function is equal to 0.



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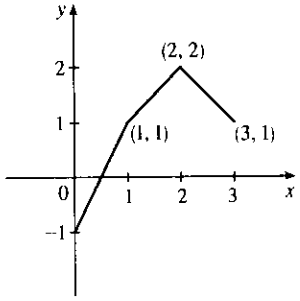
43.



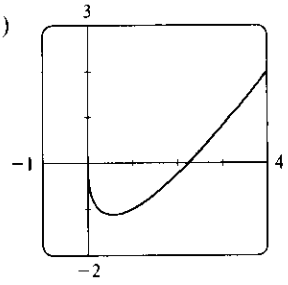
$$f'(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 < x < 2 \\ -1 & \text{if } 2 < x \le 3 \end{cases} \Rightarrow f(x) = \begin{cases} 2x + C & \text{if } 0 \le x < 1 \\ x + D & \text{if } 1 < x < 2 \\ -x + E & \text{if } 2 < x \le 3 \end{cases}$$

$f(0) = -1 \Rightarrow 2(0) + C = -1 \Rightarrow C = -1$. Starting at the point $(0, -1)$ and moving to the right on a line with slope 2 gets us to the point $(1, 1)$. The slope for $1 < x < 2$ is 1, so we get to the point $(2, 2)$. Here we have used the fact that f is continuous. We can include the point $x = 1$ on either the first or the second part of f. The line connecting $(1, 1)$ to $(2, 2)$ is $y = x$, so $D = 0$. The slope for $2 < x \le 3$ is -1, so we get to $(3, 1)$. $f(3) = 1 \Rightarrow -3 + E = 1 \Rightarrow E = 4$. Thus,

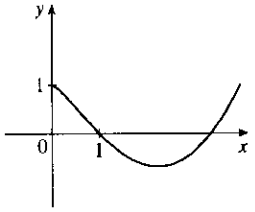
$$f(x) = \begin{cases} 2x - 1 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x < 2 \\ -x + 4 & \text{if } 2 \le x \le 3 \end{cases}$$

Note that $f'(x)$ does not exist at $x = 1$ or at $x = 2$.

44. (a)

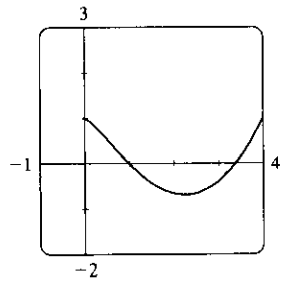


(b) Since $F(0) = 1$, we can start our graph at $(0, 1)$. f has a minimum at about $x = 0.5$, so its derivative is zero there. f is decreasing on $(0, 0.5)$, so its derivative is negative and hence, F is CD on $(0, 0.5)$ and has an IP at $x \approx 0.5$. On $(0.5, 2.2)$, f is negative and increasing (f' is positive), so F is decreasing and CU. On $(2.2, \infty)$, f is positive and increasing, so F is increasing and CU.



(c) $f(x) = 2x - 3\sqrt{x} \Rightarrow$
$F(x) = x^2 - 3 \cdot \frac{2}{3}x^{3/2} + C$. $F(0) = C$ and
$F(0) = 1 \Rightarrow C = 1$, so
$F(x) = x^2 - 2x^{3/2} + 1$.

(d)



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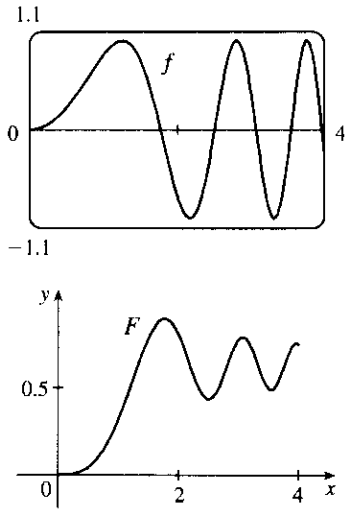
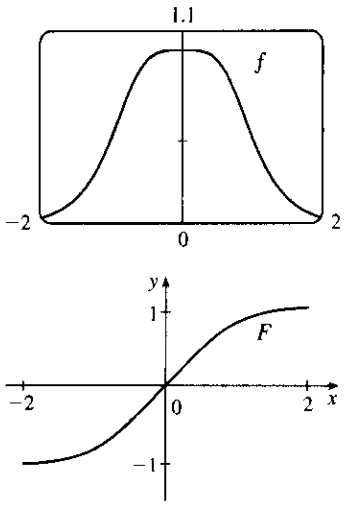
45. $f(x) = \sin(x^2),\ 0 \le x \le 4$



46. $f(x) = 1/(x^4 + 1)$



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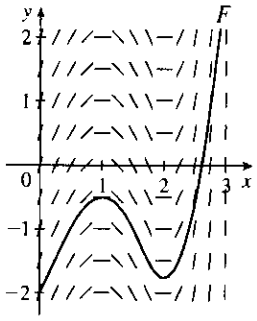

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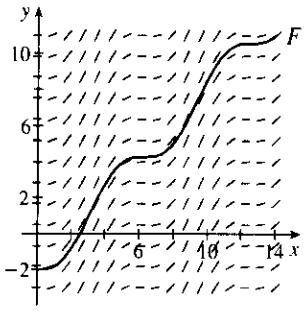
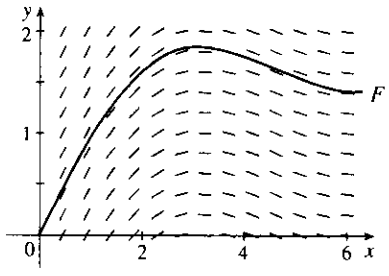


49.

x	$f(x)$
0	1
0.5	0.959
1.0	0.841
1.5	0.665
2.0	0.455
2.5	0.239
3.0	0.047

x	$f(x)$
3.5	−0.100
4.0	−0.189
4.5	−0.217
5.0	−0.192
5.5	−0.128
6.0	−0.047

We compute slopes [values of $f(x) = (\sin x)/x$ for $0 < x < 2\pi$] as in the table [$\lim_{x \to 0^+} f(x) = 1$] and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0, 0)$.



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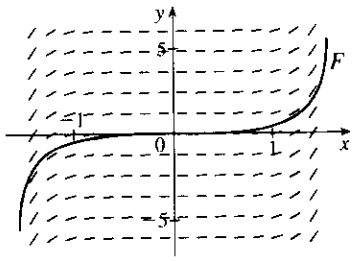
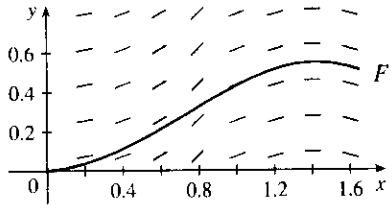
50.

x	$f(x)$
0	0
± 0.2	0.041
± 0.4	0.169
± 0.6	0.410
± 0.8	0.824
± 1.0	1.557
± 1.2	3.087
± 1.4	8.117
± 1.5	21.152

We compute slopes [values of $f(x) = x \tan x$ for $-\pi/2 < x < \pi/2$] as in the table and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0, 0)$ and extending in both directions. Note that if f is an even function, then the antiderivative F that passes through the origin is an odd function.

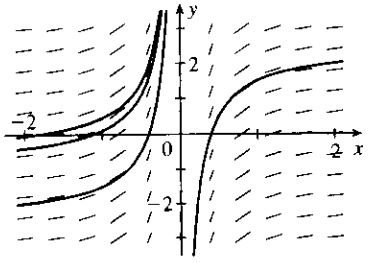


51.



Remember that the given table values of f are the slopes of F at any x. For example, at $x = 1.4$, the slope of F is $f(1.4) = 0$.

52. (a)



(b) The general antiderivative of $f(x) = x^{-2}$ is

$$F(x) = \begin{cases} -1/x + C_1 & \text{if } x < 0 \\ -1/x + C_2 & \text{if } x > 0 \end{cases}$$ since $f(x)$ is not defined at $x = 0$. The graph of the general antiderivatives of $f(x)$ looks like the graph in part (a), as expected.

53. $v(t) = s'(t) = \sin t - \cos t \Rightarrow s(t) = -\cos t - \sin t + C$. $s(0) = -1 + C$ and $s(0) = 0 \Rightarrow C = 1$, so $s(t) = -\cos t - \sin t + 1$.

54. $v(t) = s'(t) = 1.5\sqrt{t} \Rightarrow s(t) = t^{3/2} + C$. $s(4) = 8 + C$ and $s(4) = 10 \Rightarrow C = 2$, so $s(t) = t^{3/2} + 2$.

55. $a(t) = v'(t) = t - 2 \Rightarrow v(t) = \frac{1}{2}t^2 - 2t + C$. $v(0) = C$ and $v(0) = 3 \Rightarrow C = 3$, so $v(t) = \frac{1}{2}t^2 - 2t + 3$ and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + D$. $s(0) = D$ and $s(0) = 1 \Rightarrow D = 1$, and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + 1$.

56. $a(t) = v'(t) = \cos t + \sin t \Rightarrow v(t) = \sin t - \cos t + C \Rightarrow 5 = v(0) = -1 + C \Rightarrow C = 6$, so $v(t) = \sin t - \cos t + 6 \Rightarrow s(t) = -\cos t - \sin t + 6t + D \Rightarrow 0 = s(0) = -1 + D \Rightarrow D = 1$, so $s(t) = -\cos t - \sin t + 6t + 1$.

57. $a(t) = v'(t) = 10\sin t + 3\cos t \Rightarrow v(t) = -10\cos t + 3\sin t + C \Rightarrow$ $s(t) = -10\sin t - 3\cos t + Ct + D$. $s(0) = -3 + D = 0$ and $s(2\pi) = -3 + 2\pi C + D = 12 \Rightarrow D = 3$ and $C = \frac{6}{\pi}$. Thus, $s(t) = -10\sin t - 3\cos t + \frac{6}{\pi}t + 3$.

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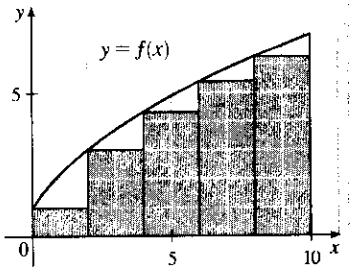
5 □ INTEGRALS

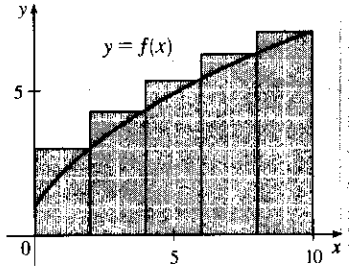
5.1 Areas and Distances

1. (a) Since f is *increasing*, we can obtain a *lower* estimate by using *left* endpoints. We are instructed to use five rectangles, so $n = 5$.



$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{b-a}{n} = \tfrac{10-0}{5} = 2]$$
$$= f(x_0)\cdot 2 + f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2$$
$$= 2\,[f(0) + f(2) + f(4) + f(6) + f(8)]$$
$$\approx 2(1 + 3 + 4.3 + 5.4 + 6.3) = 2(20) = 40$$

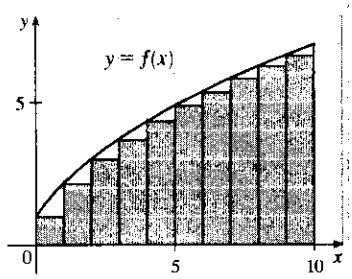
Since f is *increasing*, we can obtain an *upper* estimate by using *right* endpoints.



$$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x$$
$$= 2\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$
$$= 2\,[f(2) + f(4) + f(6) + f(8) + f(10)]$$
$$\approx 2(3 + 4.3 + 5.4 + 6.3 + 7) = 2(26) = 52$$

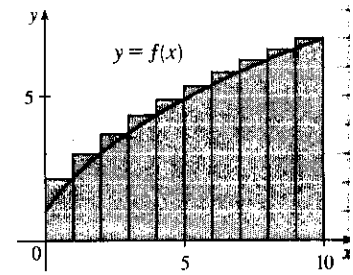
Comparing R_5 to L_5, we see that we have added the area of the rightmost upper rectangle, $f(10)\cdot 2$, to the sum and subtracted the area of the leftmost lower rectangle, $f(0)\cdot 2$, from the sum.

(b) $L_{10} = \sum_{i=1}^{10} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{10-0}{10} = 1]$



$$= 1\,[f(x_0) + f(x_1) + \cdots + f(x_9)]$$
$$= f(0) + f(1) + \cdots + f(9)$$
$$\approx 1 + 2.1 + 3 + 3.7 + 4.3 + 4.9 + 5.4 + 5.8 + 6.3 + 6.7$$
$$= 43.2$$

$$R_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x = f(1) + f(2) + \cdots + f(10)$$
$$= L_{10} + 1\cdot f(10) - 1\cdot f(0) \quad \left[\begin{array}{l}\text{add rightmost upper rectangle,}\\ \text{subtract leftmost lower rectangle}\end{array}\right]$$
$$= 43.2 + 7 - 1 = 49.2$$



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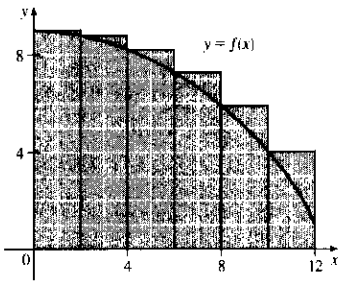
2. (a) (i) $L_6 = \sum_{i=1}^{6} f(x_{i-1})\Delta x \quad [\Delta x = \frac{12-0}{6} = 2]$

$= 2[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$

$= 2[f(0) + f(2) + f(4) + f(6) + f(8) + f(10)]$

$\approx 2(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1)$

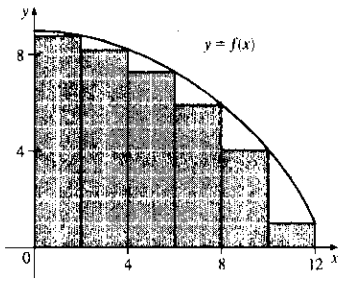
$= 2(43.3) = 86.6$



(ii) $R_6 = L_6 + 2 \cdot f(12) - 2 \cdot f(0)$

$\approx 86.6 + 2(1) - 2(9) = 70.6$

[Add area of rightmost lower rectangle and subtract area of leftmost upper rectangle.]

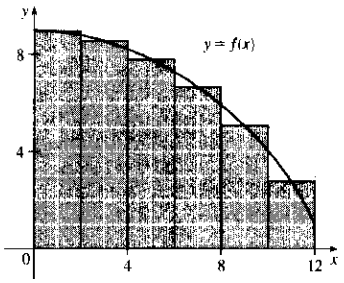


(iii) $M_6 = \sum_{i=1}^{6} f(x_i^*)\,\Delta x$

$= 2[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$

$\approx 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8)$

$= 2(39.7) = 79.4$



(b) Since f is *decreasing*, we obtain an *overestimate* by using *left* endpoints; that is, L_6.

(c) Since f is *decreasing*, we obtain an *underestimate* by using *right* endpoints; that is, R_6.

(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6.

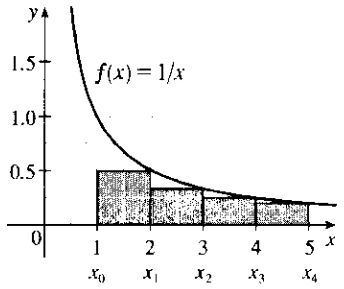
3. (a) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x \quad [\Delta x = \frac{5-1}{4} = 1]$

$= f(x_1) \cdot 1 + f(x_2) \cdot 1 + f(x_3) \cdot 1 + f(x_4) \cdot 1$

$= f(2) + f(3) + f(4) + f(5)$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} = 1.28\overline{3}$

Since f is *decreasing* on $[1, 5]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4.

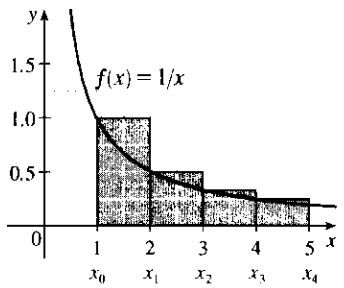


(b) $L_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x$

$= f(1) + f(2) + f(3) + f(4)$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} = 2.08\overline{3}$

L_4 is an overestimate. Alternatively, we could just add the area of the leftmost upper rectangle and subtract the area of the rightmost lower rectangle; that is, $L_4 = R_4 + f(1) \cdot 1 - f(5) \cdot 1$.



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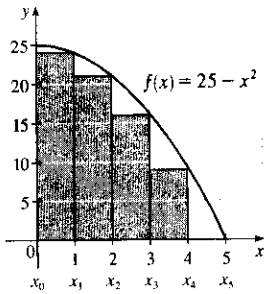
4. (a) $R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x \quad [\Delta x = \frac{5-0}{5} = 1]$

$= f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$

$= f(1) + f(2) + f(3) + f(4) + f(5)$

$= 24 + 21 + 16 + 9 + 0 = 70$

Since f is decreasing on $[0, 5]$, R_5 is an underestimate.

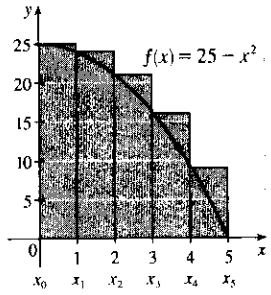


(b) $L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x$

$= f(0) + f(1) + f(2) + f(3) + f(4)$

$= 25 + 24 + 21 + 16 + 9 = 95$

L_5 is an overestimate.



5. (a) $f(x) = 1 + x^2$ and $\Delta x = \dfrac{2-(-1)}{3} = 1 \quad \Rightarrow$

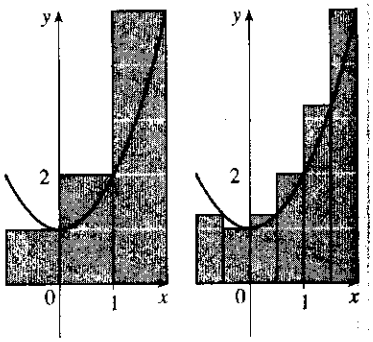
$R_3 = 1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) = 1\cdot 1 + 1\cdot 2 + 1\cdot 5 = 8.$

$\Delta x = \dfrac{2-(-1)}{6} = 0.5 \quad \Rightarrow$

$R_6 = 0.5[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$

$= 0.5(1.25 + 1 + 1.25 + 2 + 3.25 + 5)$

$= 0.5(13.75) = 6.875$

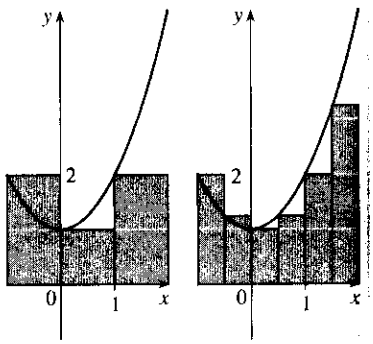


(b) $L_3 = 1\cdot f(-1) + 1\cdot f(0) + 1\cdot f(1) = 1\cdot 2 + 1\cdot 1 + 1\cdot 2 = 5$

$L_6 = 0.5[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)]$

$= 0.5(2 + 1.25 + 1 + 1.25 + 2 + 3.25)$

$= 0.5(10.75) = 5.375$



(c) $M_3 = 1\cdot f(-0.5) + 1\cdot f(0.5) + 1\cdot f(1.5)$

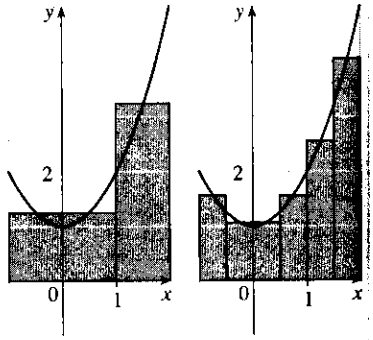
$= 1\cdot 1.25 + 1\cdot 1.25 + 1\cdot 3.25 = 5.75$

$M_6 = 0.5[f(-0.75) + f(-0.25) + f(0.25)$

$\quad + f(0.75) + f(1.25) + f(1.75)]$

$= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625)$

$= 0.5(11.875) = 5.9375$



(d) M_6 appears to be the best estimate.

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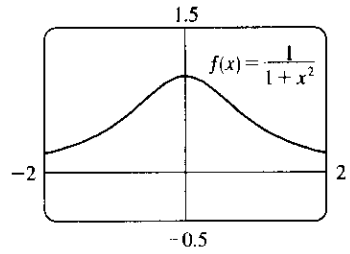
6. (a)



(b) $f(x) = 1/(1+x^2)$ and $\Delta x = \frac{2-(-2)}{4} = 1 \Rightarrow$

(i) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x$

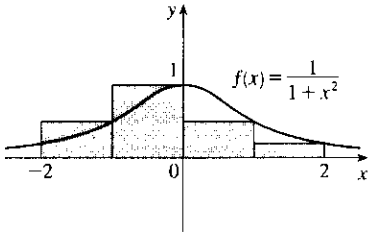
$= f(-1)\cdot 1 + f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1$

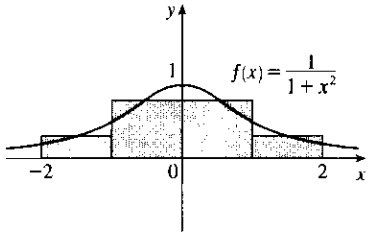
$= \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{5} = \frac{11}{5} = 2.2$

(ii) $M_4 = \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x \quad [\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)]$

$= f(-1.5)\cdot 1 + f(-0.5)\cdot 1 + f(0.5)\cdot 1 + f(1.5)\cdot 1$

$= \frac{4}{13} + \frac{4}{5} + \frac{4}{5} + \frac{4}{13} = \frac{144}{65} \approx 2.2154$





(c) $n = 8$, so $\Delta x = \frac{2-(-2)}{8} = \frac{1}{2}$.

$R_8 = \frac{1}{2}[f(-1.5) + f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$

$= \frac{1}{2}\left[\frac{4}{13} + \frac{1}{2} + \frac{4}{5} + 1 + \frac{4}{5} + \frac{1}{2} + \frac{4}{13} + \frac{1}{5}\right] = \frac{287}{130} \approx 2.2077$

$M_8 = \frac{1}{2}[f(-1.75) + f(-1.25) + f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)]$

$= \frac{1}{2}\left[2\left(\frac{16}{65} + \frac{16}{41} + \frac{16}{25} + \frac{16}{17}\right)\right] \approx 2.2176$

7. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

1 Let SUM = 0, X_MIN = 0, X_MAX = π, N = 10 (or 30 or 50, depending on which sum we are calculating), DELTA_X = (X_MAX − X_MIN)/N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.

2 Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT > X_MAX.

2a Add sin (RIGHT_ENDPOINT) to SUM.

2h Add DELTA_X to RIGHT_ENDPOINT.

At the end of this procedure, (DELTA_X) · (SUM) is equal to the answer we are looking for. We find that

$R_{10} = \frac{\pi}{10}\sum_{i=1}^{10} \sin\left(\frac{i\pi}{10}\right) \approx 1.9835$, $R_{30} = \frac{\pi}{30}\sum_{i=1}^{30} \sin\left(\frac{i\pi}{30}\right) \approx 1.9982$, and $R_{50} = \frac{\pi}{50}\sum_{i=1}^{50} \sin\left(\frac{i\pi}{50}\right) \approx 1.9993$.

It appears that the exact area is 2.

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Shown below is program SUMRIGHT and its output from a TI-83 Plus calculator. To generalize the program, we have input (rather than assigned) values for Xmin, Xmax, and N. Also, the function, $\sin x$, is assigned to Y_1, enabling us to evaluate any right sum merely by changing Y_1 and running the program.

```
PROGRAM:SUMRIGHT
:0→S
:Prompt Xmin
:Prompt Xmax
:Prompt N
:(Xmax-Xmin)/N→D
:Xmin+D→R
:For(I,1,N)
:S+Y1(R)→S
:R+D→R
:End
:D*S→Z
:Disp Z
```

```
prgmSUMRIGHT
Xmin=?0
Xmax=?π
N=?10
      1.983523537
             Done
```

8. We can use the algorithm from Exercise 7 with X_MIN $= 1$, X_MAX $= 2$, and $1/(\text{RIGHT_ENDPOINT})^2$ instead of sin (RIGHT_ENDPOINT) in step 2a. We find that $R_{10} = \dfrac{1}{10}\displaystyle\sum_{i=1}^{10} \frac{1}{(1+i/10)^2} \approx 0.4640$, $R_{30} = \dfrac{1}{30}\displaystyle\sum_{i=1}^{30} \frac{1}{(1+i/30)^2} \approx 0.4877$, and $R_{50} = \dfrac{1}{50}\displaystyle\sum_{i=1}^{50} \frac{\mathrm{I}}{(1+i/50)^2} \approx 0.4926$. It appears that the exact area is $\frac{1}{2}$.

9. In Maple, we have to perform a number of steps before getting a numerical answer. After loading the student package [command: `with(student);`] we use the command `left_sum:=leftsum(x^(1/2),x=1..4,10` [or `30`, or `50`]`);` which gives us the expression in summation notation. To get a numerical approximation to the sum, we use `evalf(left_sum);`. Mathematica does not have a special command for these sums, so we must type them in manually. For example, the first left sum is given by `(3/10)*Sum[Sqrt[1+3(i-1)/10],{i,1,10}]`, and we use the `N` command on the resulting output to get a numerical approximation.

In Derive, we use the `LEFT_RIEMANN` command to get the left sums, but must define the right sums ourselves. (We can define a new function using `LEFT_RIEMANN` with k ranging from 1 to n instead of from 0 to $n-1$.)

(a) With $f(x) = \sqrt{x}$, $1 \le x \le 4$, the left sums are of the form $L_n = \dfrac{3}{n}\displaystyle\sum_{i=1}^{n} \sqrt{1 + \frac{3(i-1)}{n}}$. Specifically, $L_{10} \approx 4.5148$, $L_{30} \approx 4.6165$, and $L_{50} \approx 4.6366$. The right sums are of the form $R_n = \dfrac{3}{n}\displaystyle\sum_{i=1}^{n} \sqrt{1 + \frac{3i}{n}}$. Specifically, $R_{10} \approx 4.8148$, $R_{30} \approx 4.7165$, and $R_{50} \approx 4.6966$.

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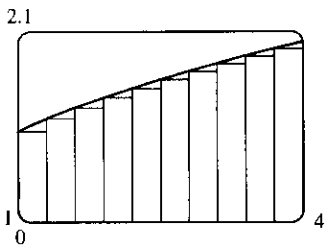
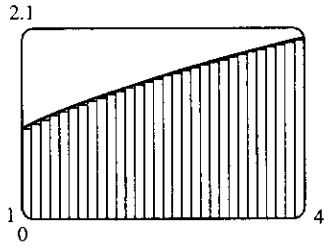
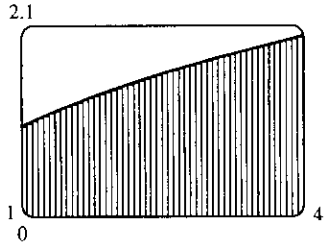
(b) In Maple, we use the `leftbox` and `rightbox` commands (with the same arguments as `leftsum` and `rightsum` above) to generate the graphs.



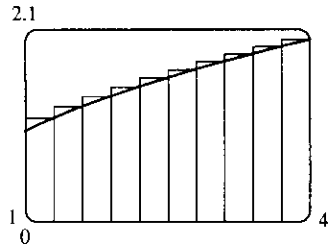
left endpoints, $n = 10$



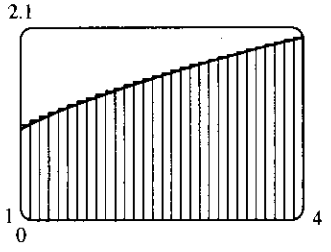
left endpoints, $n = 30$



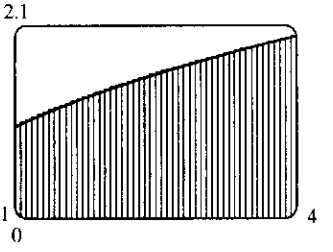
left endpoints, $n = 50$



right endpoints, $n = 10$



right endpoints, $n = 30$



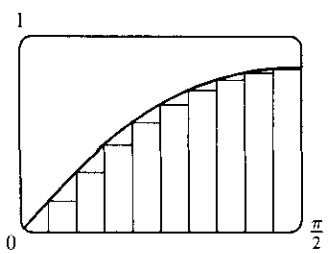
right endpoints, $n = 50$

(c) We know that since $\sqrt{x}$ is an increasing function on $(1, 4)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $4.637 > 4.6$ and the right sum with $n = 50$ is about $4.697 < 4.7$, we conclude that $4.6 < L_{50} <$ exact area $< R_{50} < 4.7$, so the exact area is between 4.6 and 4.7.

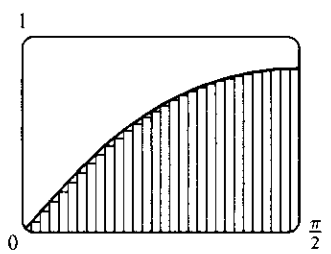
10. See the solution to Exercise 9 for the CAS commands for evaluating the sums.

(a) With $f(x) = \sin(\sin x)$, $0 \le x \le \frac{\pi}{2}$, the left sums are of the form $L_n = \dfrac{\pi}{2n} \displaystyle\sum_{i=1}^{n} \sin\left(\sin \frac{\pi\,(i-1)}{2n}\right)$. In particular, $L_{10} \approx 0.8251$, $L_{30} \approx 0.8710$, and $L_{50} \approx 0.8799$. The right sums are of the form $R_n = \dfrac{\pi}{2n} \displaystyle\sum_{i=1}^{n} \sin\left(\sin \frac{\pi i}{2n}\right)$. In particular, $R_{10} \approx 0.9573$, $R_{30} \approx 0.9150$, and $R_{50} \approx 0.9064$.

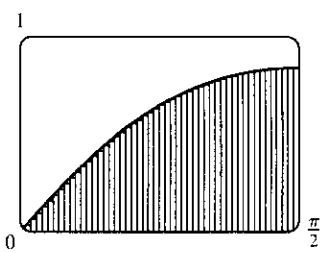
(b) In Maple, we use the `leftbox` and `rightbox` commands (with the same arguments as `leftsum` and `rightsum` above) to generate the graphs.



left endpoints, $n = 10$



left endpoints, $n = 30$



left endpoints, $n = 50$

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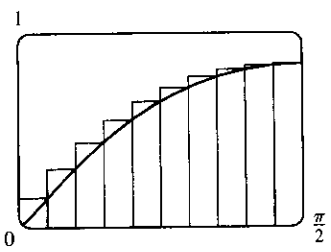
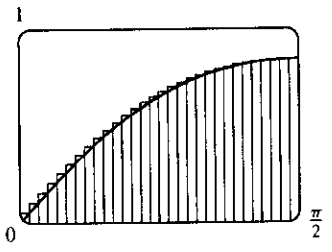
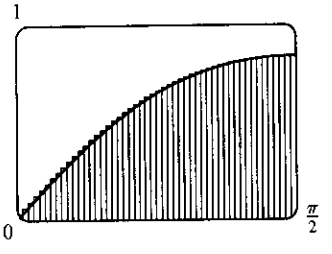


right endpoints, $n = 10$



right endpoints, $n = 30$



right endpoints, $n = 50$

(c) We know that since $\sin(\sin x)$ is an increasing function on $\left(0, \frac{\pi}{2}\right)$ [this is true because its derivative, $-\cos(\sin x)(-\cos x)$, is positive on that interval], all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $0.8799 > 0.87$ and the right sum with $n = 50$ is about $0.9064 < 0.91$, we conclude that $0.87 < L_{50} <$ exact area $< R_{50} < 0.91$, so the exact area is between 0.87 and 0.91.

11. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$$L_6 = (0 \text{ ft/s})(0.5 \text{ s}) + (6.2)(0.5) + (10.8)(0.5) + (14.9)(0.5) + (18.1)(0.5) + (19.4)(0.5)$$
$$= 0.5(69.4) = 34.7 \text{ ft}$$

$$R_6 = 0.5(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 0.5(89.6) = 44.8 \text{ ft}$$

12. (a) $d \approx L_5 = (30 \text{ ft/s})(12 \text{ s}) + 28 \cdot 12 + 25 \cdot 12 + 22 \cdot 12 + 24 \cdot 12$
$$= (30 + 28 + 25 + 22 + 24) \cdot 12 = 129 \cdot 12 = 1548 \text{ ft}$$

(b) $d \approx R_5 = (28 + 25 + 22 + 24 + 27) \cdot 12 = 126 \cdot 12 = 1512 \text{ ft}$

(c) The estimates are neither lower nor upper estimates since v is neither an increasing nor a decreasing function of t.

13. Lower estimate for oil leakage: $R_5 = (7.6 + 6.8 + 6.2 + 5.7 + 5.3)(2) = (31.6)(2) = 63.2$ L.
Upper estimate for oil leakage: $L_5 = (8.7 + 7.6 + 6.8 + 6.2 + 5.7)(2) = (35)(2) = 70$ L.

14. We can find an upper estimate by using the final velocity for each time interval. Thus, the distance d traveled after 62 seconds can be approximated by

$$d = \sum_{i=1}^{6} v(t_i)\Delta t_i = (185 \text{ ft/s})(10 \text{ s}) + 319 \cdot 5 + 447 \cdot 5 + 742 \cdot 12 + 1325 \cdot 27 + 1445 \cdot 3 = 54{,}694 \text{ ft}$$

15. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$M_6 = 1[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5)]$$
$$\approx 55 + 40 + 28 + 18 + 10 + 4 = 155 \text{ ft}$$

For a very rough check on the above calculation, we can draw a line from $(0, 70)$ to $(6, 0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

16. For an increasing function, using left endpoints gives us an underestimate and using right endpoints results in an overestimate. We will use M_6 to get an estimate. $\Delta t = \frac{30 - 0}{6} = 5 \text{ s} = \frac{5}{3600} \text{ h} = \frac{1}{720} \text{ h}$.

$$M_6 = \tfrac{1}{720}[v(2.5) + v(7.5) + v(12.5) + v(17.5) + v(22.5) + v(27.5)]$$
$$= \tfrac{1}{720}(31.25 + 66 + 88 + 103.5 + 113.75 + 119.25) = \tfrac{1}{720}(521.75) \approx 0.725 \text{ km}$$

For a very rough check on the above calculation, we can draw a line from $(0, 0)$ to $(30, 120)$ and calculate the area

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of the triangle: $\frac{1}{2}(30)(120) = 1800$. Divide by 3600 to get 0.5, which is clearly an underestimate, making our midpoint estimate of 0.725 seem reasonable. Of course, answers will vary due to different readings of the graph.

17. $f(x) = \sqrt[4]{x}$, $1 \le x \le 16$. $\Delta x = (16-1)/n = 15/n$ and $x_i = 1 + i\,\Delta x = 1 + 15i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[4]{1 + \tfrac{15i}{n}} \cdot \tfrac{15}{n}.$$

18. $f(x) = 1 + x^4$, $2 \le x \le 5$. $\Delta x = (5-2)/n = 3/n$ and $x_i = 2 + i\,\Delta x = 2 + 3i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + \left(2 + \frac{3i}{n}\right)^4\right] \cdot \frac{3}{n}.$$

19. $f(x) = x\cos x$, $0 \le x \le \frac{\pi}{2}$. $\Delta x = (\frac{\pi}{2} - 0)/n = \frac{\pi}{2}/n$ and $x_i = 0 + i\,\Delta x = \frac{\pi}{2}i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i\pi}{2n} \cos\left(\frac{i\pi}{2n}\right) \cdot \frac{\pi}{2n}.$$

20. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}$ can be interpreted as the area of the region lying under the graph of $y = (5+x)^{10}$ on the interval $[0, 2]$, since for $y = (5+x)^{10}$ on $[0, 2]$ with $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$, $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$, and $x_i^* = x_i$, the expression for the area is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(5 + \dfrac{2i}{n}\right)^{10} \dfrac{2}{n}$. Note that the answer is not unique. We could use $y = x^{10}$ on $[5, 7]$ or, in general, $y = ((5-n) + x)^{10}$ on $[n, n+2]$.

21. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$, since for $y = \tan x$ on $\left[0, \frac{\pi}{4}\right]$ with $\Delta x = \dfrac{\pi/4 - 0}{n} = \dfrac{\pi}{4n}$, $x_i = 0 + i\,\Delta x = \dfrac{i\pi}{4n}$, and $x_i^* = x_i$, the expression for the area is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\dfrac{i\pi}{4n}\right) \dfrac{\pi}{4n}$. Note that this answer is not unique, since the expression for the area is the same for the function $y = \tan(x - k\pi)$ on the interval $\left[k\pi, k\pi + \frac{\pi}{4}\right]$, where k is any integer.

22. (a) $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{i}{n}$. $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\dfrac{i}{n}\right)^3 \cdot \dfrac{1}{n}$.

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{i^3}{n^3} \cdot \dfrac{1}{n} = \lim_{n\to\infty} \dfrac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \dfrac{1}{n^4} \left[\dfrac{n(n+1)}{2}\right]^2 = \lim_{n\to\infty} \dfrac{(n+1)^2}{4n^2} = \dfrac{1}{4} \lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^2 = \dfrac{1}{4}$

23. (a) $y = f(x) = x^5$. $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^5 \cdot \frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{32i^5}{n^5} \cdot \frac{2}{n} = \lim_{n\to\infty} \frac{64}{n^6} \sum_{i=1}^{n} i^5.$$

(b) $\sum_{i=1}^{n} i^5 \overset{\text{CAS}}{=} \dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$

(c) $\lim_{n\to\infty} \dfrac{64}{n^6} \cdot \dfrac{n^2(n+1)^2(2n^2+2n-1)}{12} = \dfrac{64}{12} \lim_{n\to\infty} \dfrac{(n^2+2n+1)(2n^2+2n-1)}{n^2 \cdot n^2}$

$$= \frac{16}{3} \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(2 + \frac{2}{n} - \frac{1}{n^2}\right) = \tfrac{16}{3} \cdot 1 \cdot 2 = \tfrac{32}{3}$$

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24. (a) $y = f(x) = x^4 + 5x^2 + x,\ 2 \le x \le 7 \ \Rightarrow\ \Delta x = \dfrac{7-2}{n} = \dfrac{5}{n},\ x_i = 2 + i\,\Delta x = 2 + \dfrac{5i}{n} \ \Rightarrow$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left[\left(2 + \frac{5i}{n}\right)^4 + 5\left(2 + \frac{5i}{n}\right)^2 + \left(2 + \frac{5i}{n}\right) \right]$$

(b) $R_n = \dfrac{5}{n} \cdot \dfrac{4723n^4 + 7845n^3 + 3475n^2 - 125}{6n^3}$

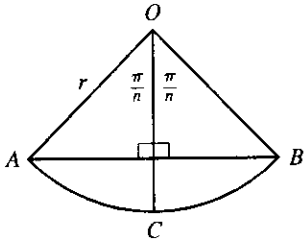
(c) $A = \lim\limits_{n\to\infty} R_n = \dfrac{23{,}615}{6} = 3935.8\overline{3}$

25. $y = f(x) = \cos x.\ \Delta x = \dfrac{b-0}{n} = \dfrac{b}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{bi}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{bi}{n}\right) \cdot \frac{b}{n} \overset{\text{CAS}}{=} \lim_{n\to\infty} \left[\frac{b \sin\left(b\left(\frac{1}{2n} + 1\right)\right)}{2n \sin\left(\frac{b}{2n}\right)} - \frac{b}{2n} \right] \overset{\text{CAS}}{=} \sin b$$

If $b = \frac{\pi}{2}$, then $A = \sin\frac{\pi}{2} = 1$. The value 1 is the *exact* area, whereas the value obtained in Example 3(b), 1.006, is just an *approximation*.

26. (a)



The diagram shows one of the n congruent triangles, $\triangle AOB$, with central angle $2\pi/n$. O is the center of the circle and AB is one of the sides of the polygon. Radius OC is drawn so as to bisect $\angle AOB$. It follows that OC intersects AB at right angles and bisects AB. Thus, $\triangle AOB$ is divided into two right triangles with legs of length $\frac{1}{2}(AB) = r\sin(\pi/n)$ and $r\cos(\pi/n)$.

$\triangle AOB$ has area $2 \cdot \frac{1}{2}[r\sin(\pi/n)][r\cos(\pi/n)] = r^2\sin(\pi/n)\cos(\pi/n) = \frac{1}{2}r^2\sin(2\pi/n)$, so $A_n = n \cdot \text{area}(\triangle AOB) = \frac{1}{2}nr^2\sin(2\pi/n)$.

(b) To use Equation 3.5.2, $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$, we need to have the same expression in the denominator as we have in the argument of the sine function—in this case, $2\pi/n$.

$\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} \frac{1}{2}nr^2\sin(2\pi/n) = \lim\limits_{n\to\infty} \frac{1}{2}nr^2 \dfrac{\sin(2\pi/n)}{2\pi/n} \cdot \dfrac{2\pi}{n} = \lim\limits_{n\to\infty} \dfrac{\sin(2\pi/n)}{2\pi/n}\pi r^2$. Let $\theta = \dfrac{2\pi}{n}$.

Then as $n \to \infty$, $\theta \to 0$, so $\lim\limits_{n\to\infty} \dfrac{\sin(2\pi/n)}{2\pi/n}\pi r^2 = \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}\pi r^2 = (1)\,\pi r^2 = \pi r^2$.

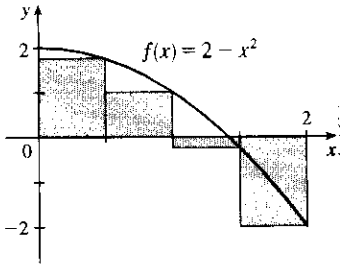
5.2 The Definite Integral

1. $R_4 = \sum\limits_{i=1}^{4} f(x_i)\,\Delta x \quad [x_i^* = x_i \text{ is a right endpoint and } \Delta x = 0.5]$

$= 0.5\,[f(0.5) + f(1) + f(1.5) + f(2)] \quad [f(x) = 2 - x^2]$

$= 0.5\,[1.75 + 1 + (-0.25) + (-2)]$

$= 0.5(0.5) = 0.25$



The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.

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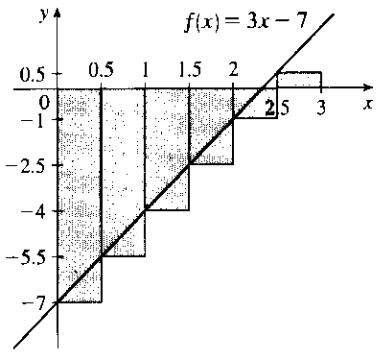
2. $L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x$ $[x_i^* = x_{i-1}$ is a left endpoint and $\Delta x = 0.5]$

$= 0.5\,[f(0) + f(0.5) + f(1)$
$+ f(1.5) + f(2) + f(2.5)]$ $[f(x) = 3x - 7]$

$= 0.5[-7 + (-5.5) + (-4) + (-2.5) + (-1) + 0.5]$

$= 0.5(-19.5) = -9.75$

The Riemann sum represents the area of the rectangle above the x-axis minus the area of the five rectangles below the x-axis.

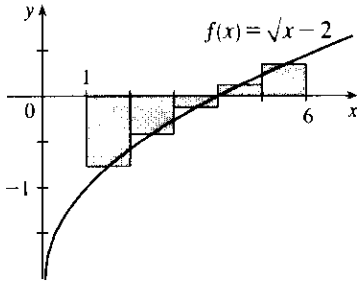


3. $M_5 = \sum_{i=1}^{5} f(\bar{x}_i)\,\Delta x$ $[x_i^* = \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ is a midpoint and $\Delta x = 1]$

$= 1\,[f(1.5) + f(2.5) + f(3.5)$
$+ f(4.5) + f(5.5)]$ $[f(x) = \sqrt{x} - 2]$

≈ -0.856759

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis.

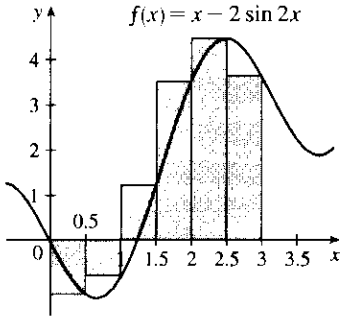


4. (a) $R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$ $[x_i^* = x_i$ is a right endpoint and $\Delta x = 0.5]$

$= 0.5\,[f(0.5) + f(1) + f(1.5) + f(2)$
$+ f(2.5) + f(3)]$ $[f(x) = x - 2\sin 2x]$

≈ 5.353254

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

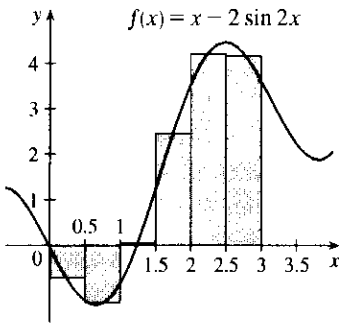


(b) $M_6 = \sum_{i=1}^{6} f(\bar{x}_i)\,\Delta x$ $[x_i^* = \bar{x}_i$ is a midpoint and $\Delta x = 0.5]$

$= 0.5[f(0.25) + f(0.75) + f(1.25) + f(1.75)$
$+ f(2.25) + f(2.75)]$ $[f(x) = x - 2\sin 2x]$

≈ 4.458461

The Riemann sum represents the sum of the areas of the four rectangles ahove the x-axis minus the sum of the areas of the two rectangles below the x-axis.



5. $\Delta x = (b - a)/n = (8 - 0)/4 = 8/4 = 2$.

(a) Using the right endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_i)\,\Delta x = 2[f(2) + f(4) + f(6) + f(8)] \approx 2[1 + 2 + (-2) + 1] = 4.$$

(b) Using the left endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_{i-1})\,\Delta x = 2[f(0) + f(2) + f(4) + f(6)] \approx 2[2 + 1 + 2 + (-2)] = 6.$$

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(c) Using the midpoint of each subinterval to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2[f(1)+f(3)+f(5)+f(7)] \approx 2[3+2+1+(-1)] = 10.$$

6. (a) Using the right endpoints to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\sum_{i=1}^{6} g(x_i)\,\Delta x = 1[g(-2)+g(-1)+g(0)+g(1)+g(2)+g(3)]$$
$$\approx 1-0.5-1.5-1.5-0.5+2.5 = -0.5$$

(b) Using the left endpoints to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\sum_{i=1}^{6} g(x_{i-1})\,\Delta x = 1[g(-3)+g(-2)+g(-1)+g(0)+g(1)+g(2)]$$
$$\approx 2+1-0.5-1.5-1.5-0.5 = -1$$

(c) Using the midpoint of each subinterval to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\sum_{i=1}^{6} g(\overline{x}_i)\,\Delta x = 1[g(-2.5)+g(-1.5)+g(-0.5)+g(0.5)+g(1.5)+g(2.5)]$$
$$\approx 1.5+0-1-1.75-1+0.5 = -1.75$$

7. Since f is increasing, $L_5 \le \int_0^{25} f(x)\,dx \le R_5$.

$$\text{Lower estimate} = L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x = 5[f(0)+f(5)+f(10)+f(15)+f(20)]$$
$$= 5(-42-37-25-6+15) = 5(-95) = -475$$

$$\text{Upper estimate} = R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x = 5[f(5)+f(10)+f(15)+f(20)+f(25)]$$
$$= 5(-37-25-6+15+36) = 5(-17) = -85$$

8. (a) Using the right endpoints to approximate $\int_0^6 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_i)\,\Delta x = 2[f(2)+f(4)+f(6)] = 2(8.3+2.3-10.5) = 0.2$$

(b) Using the left endpoints to approximate $\int_0^6 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_{i-1})\,\Delta x = 2[f(0)+f(2)+f(4)] = 2(9.3+8.3+2.3) = 39.8$$

(c) Using the midpoint of each interval to approximate $\int_0^6 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(\overline{x}_i)\,\Delta x = 2[f(1)+f(3)+f(5)] = 2(9.0+6.5-7.6) = 15.8.$$

The estimate using the right endpoints must be less than $\int_0^6 f(x)\,dx$, since if we take x_i^* to be the right endpoint x_i of each interval, then $f(x_i) \le f(x)$ for all x on $[x_{i-1}, x_i]$, which implies that $f(x_i)\,\Delta x \le \int_{x_{i-1}}^{x_i} f(x)\,dx$, and so the sum $\sum_{i=1}^{3}[f(x_i)\,\Delta x] \le \sum_{i=1}^{3}\left[\int_{x_{i-1}}^{x_i} f(x)\,dx\right] = \int_0^6 f(x)\,dx$. Similarly, if we take x_i^* to be the left endpoint x_{i-1} of each interval, then $f(x_{i-1}) \ge f(x)$ for all x on $[x_{i-1}, x_i]$, and so $\sum_{i=1}^{3}[f(x_{i-1})\,\Delta x] \ge \int_0^6 f(x)\,dx$. We cannot say anything about the midpoint estimate.

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9. $\Delta x = (10-2)/4 = 2$, so the endpoints are 2, 4, 6, 8, and 10, and the midpoints are 3, 5, 7, and 9. The Midpoint Rule gives $\int_2^{10} \sqrt{x^3+1}\,dx \approx \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2\left(\sqrt{3^3+1}+\sqrt{5^3+1}+\sqrt{7^3+1}+\sqrt{9^3+1}\right) \approx 124.1644.$

10. $\Delta x = (\pi-0)/6 = \frac{\pi}{6}$, so the endpoints are 0, $\frac{\pi}{6}$, $\frac{2\pi}{6}$, $\frac{3\pi}{6}$, $\frac{4\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{6\pi}{6}$, and the midpoints are $\frac{\pi}{12}$, $\frac{3\pi}{12}$, $\frac{5\pi}{12}$, $\frac{7\pi}{12}$, $\frac{9\pi}{12}$, and $\frac{11\pi}{12}$. The Midpoint Rule gives

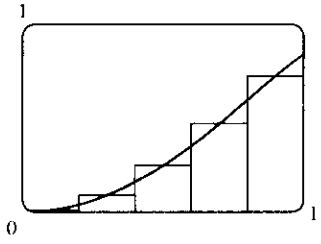
$$\int_0^{\pi} \sec(x/3)\,dx \approx \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x = \tfrac{\pi}{6}\left(\sec\tfrac{\pi}{36}+\sec\tfrac{3\pi}{36}+\sec\tfrac{5\pi}{36}+\sec\tfrac{7\pi}{36}+\sec\tfrac{9\pi}{36}+\sec\tfrac{11\pi}{36}\right) \approx 3.9379.$$

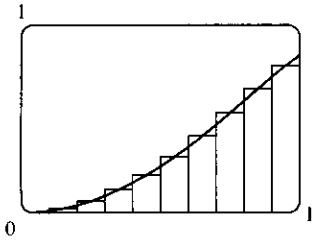
11. $\Delta x = (1-0)/5 = 0.2$, so the endpoints are 0, 0.2, 0.4, 0.6, 0.8, and 1, and the midpoints are 0.1, 0.3, 0.5, 0.7, and 0.9. The Midpoint Rule gives

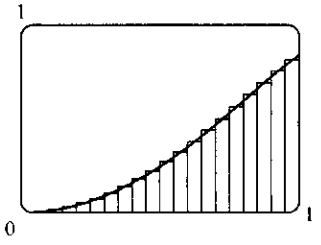
$$\int_0^1 \sin(x^2)\,dx \approx \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = 0.2\left[\sin(0.1)^2+\sin(0.3)^2+\sin(0.5)^2+\sin(0.7)^2+\sin(0.9)^2\right] \approx 0.3084.$$

12. $\Delta x = \dfrac{5-1}{4} = 1$, so the endpoints are 1, 2, 3, 4, and 5, and the midpoints are 1.5, 2.5, 3.5, and 4.5. The Midpoint Rule gives $\displaystyle\int_1^5 \frac{x-1}{x+1}\,dx \approx \sum_{i=1}^{4} f(\overline{x}_i)\Delta x = 1\left[\frac{1.5-1}{1.5+1}+\frac{2.5-1}{2.5+1}+\frac{3.5-1}{3.5+1}+\frac{4.5-1}{4.5+1}\right] \approx 1.8205.$

13. In Maple, we use the command `with(student);` to load the sum and box commands, then `m:=middlesum(sin(x^2),x=0..1,5);` which gives us the sum in summation notation, then `M:=evalf(m);` which gives $M_5 \approx 0.30843908$, confirming the result of Exercise 11. The command `middlebox(sin(x^2),x=0..1,5)` generates the graph. Repeating for $n = 10$ and $n = 20$ gives $M_{10} \approx 0.30981629$ and $M_{20} \approx 0.31015563$.







14. See the solution to Exercise 5.1.7 for a possible algorithm to calculate the sums. With $\Delta x = (1-0)/100 = 0.01$ and subinterval endpoints 1, 1.01, 1.02, ..., 1.99, 2, we calculate that the left Riemann sum is $L_{100} = \sum_{i=1}^{100} \sin(x_{i-1}^2)\,\Delta x \approx 0.30607$, and the right Riemann sum is $R_{100} = \sum_{i=1}^{100} \sin\left(x_i^2\right)\,\Delta x \approx 0.31448$. Since $f(x) = \sin(x^2)$ is an increasing function, we must have $L_{100} \le \int_0^1 \sin(x^2)\,dx \le R_{100}$, so $0.306 < L_{100} \le \int_0^1 \sin(x^2)\,dx \le R_{100} < 0.315$. Therefore, the approximate value $0.3084 \approx 0.31$ in Exercise 11 must be accurate to two decimal places.

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15. We'll create the table of values to approximate $\int_0^\pi \sin x\,dx$ by using the program in the solution to Exercise 5.1.7 with $Y_1 = \sin x$, $\text{Xmin} = 0$, $\text{Xmax} = \pi$, and $n = 5$, 10, 50, and 100.

The values of R_n appear to be approaching 2.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

16. $\int_0^2 \sqrt{1+x^4}\,dx$ with $n = 5$, 10, 50, and 100.

n	L_n	R_n
5	3.080614	4.329856
10	3.354110	3.978731
50	3.591540	3.716464
100	3.622383	3.684845

The value of the integral lies between 3.622 and 3.685. Note that $f(x) = \sqrt{1+x^4}$ is increasing on $(0, 2)$. We cannot make a similar statement for $\int_{-1}^2 \sqrt{1+x^4}\,dx$ since f is decreasing on $(-1, 0)$.

17. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n x_i \sin x_i\,\Delta x = \int_0^\pi x \sin x\,dx$.

18. On $[1, 5]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n \dfrac{x_i}{1+x_i}\,\Delta x = \displaystyle\int_1^5 \frac{x}{1+x}\,dx$.

19. On $[1, 8]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n \sqrt{2x_i^* + (x_i^*)^2}\,\Delta x = \int_1^8 \sqrt{2x+x^2}\,dx$.

20. On $[0, 2]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n [4 - 3(x_i^*)^2 + 6(x_i^*)^5]\,\Delta x = \int_0^2 (4 - 3x^2 + 6x^5)\,dx$.

21. Note that $\Delta x = \dfrac{5-(-1)}{n} = \dfrac{6}{n}$ and $x_i = -1 + i\,\Delta x = -1 + \dfrac{6i}{n}$.

$$\begin{aligned}
\int_{-1}^5 (1+3x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^n \left[1 + 3\left(-1 + \frac{6i}{n}\right)\right]\frac{6}{n} \\
&= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^n \left[-2 + \frac{18i}{n}\right] = \lim_{n\to\infty} \frac{6}{n}\left[\sum_{i=1}^n (-2) + \sum_{i=1}^n \frac{18i}{n}\right] \\
&= \lim_{n\to\infty} \frac{6}{n}\left[-2n + \frac{18}{n}\sum_{i=1}^n i\right] = \lim_{n\to\infty} \frac{6}{n}\left[-2n + \frac{18}{n}\cdot\frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[-12 + \frac{108}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \left[-12 + 54\frac{n+1}{n}\right] \\
&= \lim_{n\to\infty} \left[-12 + 54\left(1 + \frac{1}{n}\right)\right] = -12 + 54\cdot 1 = 42
\end{aligned}$$

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22. $\displaystyle\int_1^4 (x^2+2x-5)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \qquad [\Delta x = 3/n \text{ and } x_i = 1+3i/n]$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n}\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}+2+\frac{6i}{n}-5\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n}\left(\frac{9}{n^2}\cdot i^2 + \frac{12}{n}\cdot i - 2\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\frac{9}{n^2}\sum_{i=1}^{n} i^2 + \frac{12}{n}\sum_{i=1}^{n} i - \sum_{i=1}^{n} 2\right]$$

$$= \lim_{n\to\infty}\left(\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{36}{n^2}\cdot\frac{n(n+1)}{2} - \frac{6}{n}\cdot n\right)$$

$$= \lim_{n\to\infty}\left(\frac{9}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} + 18\cdot\frac{n+1}{n} - 6\right)$$

$$= \lim_{n\to\infty}\left[\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 18\left(1+\frac{1}{n}\right) - 6\right] = \tfrac{9}{2}\cdot 1\cdot 2 + 18\cdot 1 - 6 = 21$$

23. Note that $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$\int_0^2 (2-x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(2-\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^{n} 2 - \frac{4}{n^2}\sum_{i=1}^{n} i^2\right] = \lim_{n\to\infty}\frac{2}{n}\left(2n - \frac{4}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left[4 - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to\infty}\left(4 - \frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty}\left[4 - \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 4 - \tfrac{4}{3}\cdot 1\cdot 2 = \tfrac{4}{3}$$

24. $\displaystyle\int_0^5 (1+2x^3)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \qquad [\Delta x = 5/n \text{ and } x_i = 5i/n]$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+2\cdot\frac{125i^3}{n^3}\right)\left(\frac{5}{n}\right) = \lim_{n\to\infty}\frac{5}{n}\left[\sum_{i=1}^{n} 1 + \frac{250}{n^3}\sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to\infty}\frac{5}{n}\left(1\cdot n + \frac{250}{n^3}\sum_{i=1}^{n} i^3\right) = \lim_{n\to\infty}\left[5 + \frac{1250}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty}\left[5 + 312.5\cdot\frac{(n+1)^2}{n^2}\right] = \lim_{n\to\infty}\left[5 + 312.5\left(1+\frac{1}{n}\right)^2\right]$$

$$= 5 + 312.5 = 317.5$$

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25. Note that $\Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}$ and $x_i = 1 + i\,\Delta x = 1 + i(1/n) = 1 + i/n$.

$$\int_1^2 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n+i}{n}\right)^3$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n^3+3n^2i+3ni^2+i^3) = \lim_{n\to\infty}\frac{1}{n^4}\left[\sum_{i=1}^{n}n^3+\sum_{i=1}^{n}3n^2i+\sum_{i=1}^{n}3ni^2+\sum_{i=1}^{n}i^3\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[n\cdot n^3+3n^2\sum_{i=1}^{n}i+3n\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}i^3\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{3}{n^2}\cdot\frac{n(n+1)}{2}+\frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

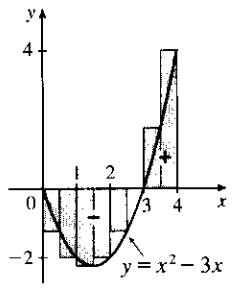
$$= \lim_{n\to\infty}\left[1+\frac{3}{2}\cdot\frac{n+1}{n}+\frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}+\frac{1}{4}\cdot\frac{(n+1)^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{3}{2}\left(1+\frac{1}{n}\right)+\frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{1}{4}\left(1+\frac{1}{n}\right)^2\right] = 1+\frac{3}{2}+\frac{1}{2}\cdot 2+\frac{1}{4} = 3.75$$

26. (a) $\Delta x = (4-0)/8 = 0.5$ and $x_i^* = x_i = 0.5i$.

$$\int_0^4 (x^2-3x)\,dx \approx \sum_{i=1}^{8} f(x_i^*)\,\Delta x$$
$$= 0.5\{[0.5^2-3(0.5)]+[1.0^2-3(1.0)]+\cdots$$
$$+[3.5^2-3(3.5)]+[4.0^2-3(4.0)]\}$$
$$= \tfrac{1}{2}\left(-\tfrac{5}{4}-2-\tfrac{9}{4}-2-\tfrac{5}{4}+0+\tfrac{7}{4}+4\right) = -1.5$$

(b)



(c) $$\int_0^4 (x^2-3x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^2-3\left(\frac{4i}{n}\right)\right]\left(\frac{4}{n}\right)$$

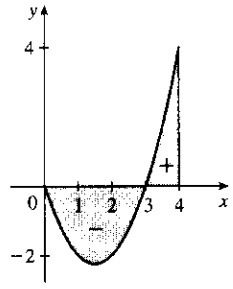
$$= \lim_{n\to\infty}\frac{4}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n}i^2-\frac{12}{n}\sum_{i=1}^{n}i\right]$$

$$= \lim_{n\to\infty}\left[\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{48}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-24\left(1+\frac{1}{n}\right)\right]$$

$$= \tfrac{32}{3}\cdot 2-24 = -\tfrac{8}{3}$$

(d) $\int_0^4 (x^2-3x)\,dx = A_1 - A_2$, where A_1 is the area marked + and A_2 is the area marked −.



27. $$\int_a^b x\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a+\frac{b-a}{n}i\right] = \lim_{n\to\infty}\left[\frac{a(b-a)}{n}\sum_{i=1}^{n}1+\frac{(b-a)^2}{n^2}\sum_{i=1}^{n}i\right]$$

$$= \lim_{n\to\infty}\left[\frac{a(b-a)}{n}n+\frac{(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2}\right] = a(b-a)+\lim_{n\to\infty}\frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)$$
$$= a(b-a)+\tfrac{1}{2}(b-a)^2 = (b-a)\left(a+\tfrac{1}{2}b-\tfrac{1}{2}a\right) = (b-a)\tfrac{1}{2}(b+a) = \tfrac{1}{2}(b^2-a^2)$$

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28. $\displaystyle\int_a^b x^2\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a+\frac{b-a}{n}i\right]^2 = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a^2+2a\frac{b-a}{n}i+\frac{(b-a)^2}{n^2}i^2\right]$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\sum_{i=1}^{n}i^2+\frac{2a(b-a)^2}{n^2}\sum_{i=1}^{n}i+\frac{a^2(b-a)}{n}\sum_{i=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{2a(b-a)^2}{n^2}\frac{n(n+1)}{2}+\frac{a^2(b-a)}{n}n\right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{6}\cdot 1\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+a(b-a)^2\cdot 1\cdot\left(1+\frac{1}{n}\right)+a^2(b-a)\right]$$

$$= \frac{(b-a)^3}{3}+a(b-a)^2+a^2(b-a) = \frac{b^3-3ab^2+3a^2b-a^3}{3}+ab^2-2a^2b+a^3+a^2b-a^3$$

$$= \frac{b^3}{3}-\frac{a^3}{3}-ab^2+a^2b+ab^2-a^2b = \frac{b^3-a^3}{3}$$

29. $f(x)=\dfrac{x}{1+x^5}$, $a=2$, $b=6$, and $\Delta x=\dfrac{6-2}{n}=\dfrac{4}{n}$. Using Equation 3, we get $x_i^*=x_i=2+i\,\Delta x=2+\dfrac{4i}{n}$,

so $\displaystyle\int_2^6\frac{x}{1+x^5}\,dx=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}\cdot\frac{4}{n}$.

30. $f(x)=x^2\sin x$, $[a,b]=[0,2\pi]$, $\Delta x=\dfrac{2\pi-0}{n}=\dfrac{2\pi}{n}$, and $x_i=a+i\,\Delta x=\dfrac{2\pi i}{n}$, so

$$\int_0^{2\pi}x^2\sin x\,dx=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2\pi i}{n}\right)^2\sin\left(\frac{2\pi i}{n}\right)\right]\cdot\frac{2\pi}{n}.$$

31. $\Delta x=(\pi-0)/n=\pi/n$ and $x_i^*=x_i=\pi i/n$.

$$\int_0^{\pi}\sin 5x\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}(\sin 5x_i)\left(\frac{\pi}{n}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n}\overset{\text{CAS}}{=}\pi\lim_{n\to\infty}\frac{1}{n}\cot\left(\frac{5\pi}{2n}\right)\overset{\text{CAS}}{=}\pi\left(\frac{2}{5\pi}\right)=\frac{2}{5}$$

32. $\Delta x=(10-2)/n=8/n$ and $x_i^*=x_i=2+8i/n$.

$$\int_2^{10}x^6\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6\left(\frac{8}{n}\right)=8\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6$$

$$\overset{\text{CAS}}{=}8\lim_{n\to\infty}\frac{1}{n}\cdot\frac{64\left(58{,}593n^6+164{,}052n^5+131{,}208n^4-27{,}776n^2+2048\right)}{21n^5}$$

$$\overset{\text{CAS}}{=}8\left(\tfrac{1{,}249{,}984}{7}\right)=\tfrac{9{,}999{,}872}{7}\approx 1{,}428{,}553.1$$

33. (a) Think of $\int_0^2 f(x)\,dx$ as the area of a trapezoid with bases 1 and 3 and height 2. The area of a trapezoid is $A=\frac{1}{2}(b+B)h$, so $\int_0^2 f(x)\,dx=\frac{1}{2}(1+3)2=4$.

(b) $\int_0^5 f(x)\,dx=\int_0^2 f(x)\,dx+\int_2^3 f(x)\,dx+\int_3^5 f(x)\,dx$

$$\begin{matrix}&&\text{trapezoid}&&\text{rectangle}&&\text{triangle}\\ &=&\frac{1}{2}(1+3)2&+&3\cdot 1&+&\frac{1}{2}\cdot 2\cdot 3\end{matrix}\quad=4+3+3=10$$

(c) $\int_5^7 f(x)\,dx$ is the negative of the area of the triangle with base 2 and height 3. $\int_5^7 f(x)\,dx=-\frac{1}{2}\cdot 2\cdot 3=-3$.

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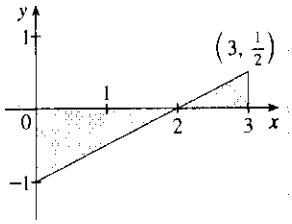
(d) $\int_7^9 f(x)\,dx$ is the negative of the area of a trapezoid with bases 3 and 2 and height 2, so it equals $-\frac{1}{2}(B+b)h = -\frac{1}{2}(3+2)2 = -5$. Thus,
$\int_0^9 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx + \int_7^9 f(x)\,dx = 10 + (-3) + (-5) = 2.$

34. (a) $\int_0^2 g(x)\,dx = \frac{1}{2}\cdot 4\cdot 2 = 4$ (area of a triangle)

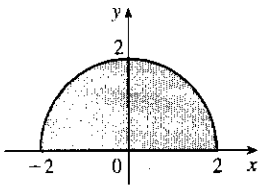
(b) $\int_2^6 g(x)\,dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ (negative of the area of a semicircle)

(c) $\int_6^7 g(x)\,dx = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$ (area of a triangle)
$\int_0^7 g(x)\,dx = \int_0^2 g(x)\,dx + \int_2^6 g(x)\,dx + \int_6^7 g(x)\,dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$

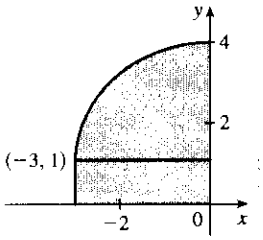
35. $\int_0^3 (\frac{1}{2}x - 1)\,dx$ can be interpreted as the area of the triangle above the x-axis minus the area of the triangle below the x-axis; that is,
$\frac{1}{2}(1)(\frac{1}{2}) - \frac{1}{2}(2)(1) = \frac{1}{4} - 1 = -\frac{3}{4}$.



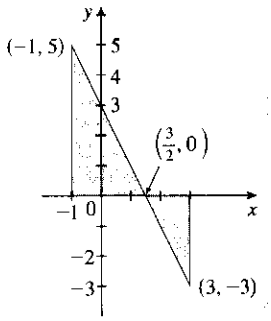
36. $\int_{-2}^2 \sqrt{4-x^2}\,dx$ can be interpreted as the area under the graph of $f(x) = \sqrt{4-x^2}$ between $x = -2$ and $x = 2$. This is equal to half the area of the circle with radius 2, so $\int_{-2}^2 \sqrt{4-x^2}\,dx = \frac{1}{2}\pi\cdot 2^2 = 2\pi$.



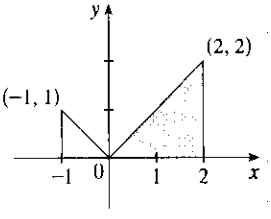
37. $\int_{-3}^0 (1+\sqrt{9-x^2})\,dx$ can be interpreted as the area under the graph of $f(x) = 1 + \sqrt{9-x^2}$ between $x = -3$ and $x = 0$. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so
$\int_{-3}^0 (1+\sqrt{9-x^2})\,dx = \frac{1}{4}\pi\cdot 3^2 + 1\cdot 3 = 3 + \frac{9}{4}\pi$.



38. $\int_{-1}^3 (3-2x)\,dx$ can be interpreted as the area of the triangle above the x-axis minus the area of the triangle below the x-axis; that is,
$\frac{1}{2}(\frac{5}{2})(5) - \frac{1}{2}(\frac{3}{2})(3) = \frac{25}{4} - \frac{9}{4} = 4$.



39. $\int_{-1}^2 |x|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$.



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inflection point to the right of the origin are about $(4.4934, 1.6556)$. Alternatively, we could graph $S''(x)$ and estimate the first positive x-value at which it changes sign.

(d) It seems from the graph that the function has horizontal asymptotes at $y \approx 1.5$, with $\lim_{x\to\pm\infty} \mathrm{Si}(x) \approx \pm 1.5$ respectively. Using the limit command, we get $\lim_{x\to\infty} \mathrm{Si}(x) = \frac{\pi}{2}$. Since $\mathrm{Si}(x)$ is an odd function, $\lim_{x\to-\infty} \mathrm{Si}(x) = -\frac{\pi}{2}$. So $\mathrm{Si}(x)$ has the horizontal asymptotes $y = \pm\frac{\pi}{2}$.

(e) We use the `fsolve` command in Maple (or `FindRoot` in Mathematica) to find that the solution is $x \approx 1.1$. Or, as in Exercise 49(c), we graph $y = \mathrm{Si}(x)$ and $y = 1$ on the same screen to see where they intersect.

51. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7$, and 9. g has local maxima at $x = 1$ and 5 (since $f = g'$ changes from positive to negative there) and local minima at $x = 3$ and 7. There is no local maximum or minimum at $x = 9$, since f is not defined for $x > 9$.

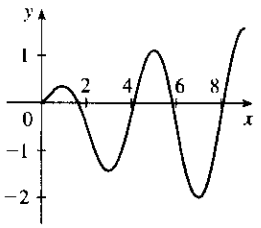
(b) We can see from the graph that $\left|\int_0^1 f\,dt\right| < \left|\int_1^3 f\,dt\right| < \left|\int_3^5 f\,dt\right| < \left|\int_5^7 f\,dt\right| < \left|\int_7^9 f\,dt\right|$.

So $g(1) = \left|\int_0^1 f\,dt\right|$, $g(5) = \int_0^5 f\,dt = g(1) - \left|\int_1^3 f\,dt\right| + \left|\int_3^5 f\,dt\right|$, and

$g(9) = \int_0^9 f\,dt = g(5) - \left|\int_5^7 f\,dt\right| + \left|\int_7^9 f\,dt\right|$. Thus, $g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on (approximately) $(\frac{1}{2}, 2)$, $(4, 6)$ and $(8, 9)$. So g is concave downward on these intervals.

(d)



52. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 2, 4, 6, 8$, and 10. g has local maxima at $x = 2$ and 6 (since $f = g'$ changes from positive to negative there) and local minima at $x = 4$ and 8. There is no local maximum or minimum at $x = 10$, since f is not defined for $x > 10$.

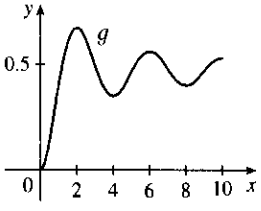
(b) We can see from the graph that $\left|\int_0^2 f\,dt\right| > \left|\int_2^4 f\,dt\right| > \left|\int_4^6 f\,dt\right| > \left|\int_6^8 f\,dt\right| > \left|\int_8^{10} f\,dt\right|$.

So $g(2) = \left|\int_0^2 f\,dt\right|$, $g(6) = \int_0^6 f\,dt = g(2) - \left|\int_2^4 f\,dt\right| + \left|\int_4^6 f\,dt\right|$, and

$g(10) = \int_0^{10} f\,dt = g(6) - \left|\int_6^8 f\,dt\right| + \left|\int_8^{10} f\,dt\right|$. Thus, $g(2) > g(6) > g(10)$, and so the absolute maximum of $g(x)$ occurs at $x = 2$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on $(1, 3)$, $(5, 7)$ and $(9, 10)$. So g is concave downward on these intervals.

(d)



53. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

54. $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right) = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x}\,dx = \left[\frac{2x^{3/2}}{3}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$

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55. Suppose $h < 0$. Since f is continuous on $[x+h, x]$, the Extreme Value Theorem says that there are numbers u and v in $[x+h, x]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x+h, x]$. By Property 8 of integrals, $m(-h) \le \int_{x+h}^{x} f(t)\,dt \le M(-h)$; that is, $f(u)(-h) \le -\int_{x}^{x+h} f(t)\,dt \le f(v)(-h)$. Since $-h > 0$, we can divide this inequality by $-h$: $f(u) \le \dfrac{1}{h}\displaystyle\int_{x}^{x+h} f(t)\,dt \le f(v)$. By Equation 2, $\dfrac{g(x+h)-g(x)}{h} = \dfrac{1}{h}\displaystyle\int_{x}^{x+h} f(t)\,dt$ for $h \neq 0$, and hence $f(u) \le \dfrac{g(x+h)-g(x)}{h} \le f(v)$, which is Equation 3 in the case where $h < 0$.

56. $\dfrac{d}{dx}\displaystyle\int_{g(x)}^{h(x)} f(t)\,dt = \frac{d}{dx}\left[\int_{g(x)}^{a} f(t)\,dt + \int_{a}^{h(x)} f(t)\,dt\right]$ (where a is in the domain of f)

$$= \frac{d}{dx}\left[-\int_{a}^{g(x)} f(t)\,dt\right] + \frac{d}{dx}\left[\int_{a}^{h(x)} f(t)\,dt\right] = -f(g(x))\,g'(x) + f(h(x))\,h'(x)$$

$$= f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

57. (a) Let $f(x) = \sqrt{x} \;\Rightarrow\; f'(x) = 1/(2\sqrt{x}) > 0$ for $x > 0 \;\Rightarrow\; f$ is increasing on $(0, \infty)$. If $x \ge 0$, then $x^3 \ge 0$, so $1 + x^3 \ge 1$ and since f is increasing, this means that $f(1+x^3) \ge f(1) \;\Rightarrow\; \sqrt{1+x^3} \ge 1$ for $x \ge 0$. Next let $g(t) = t^2 - t \;\Rightarrow\; g'(t) = 2t - 1 \;\Rightarrow\; g'(t) > 0$ when $t \ge 1$. Thus, g is increasing on $(1, \infty)$. And since $g(1) = 0$, $g(t) \ge 0$ when $t \ge 1$. Now let $t = \sqrt{1+x^3}$, where $x \ge 0$. $\sqrt{1+x^3} \ge 1$ (from above) $\;\Rightarrow\; t \ge 1 \;\Rightarrow\; g(t) \ge 0 \;\Rightarrow\; (1+x^3) - \sqrt{1+x^3} \ge 0$ for $x \ge 0$. Therefore, $1 \le \sqrt{1+x^3} \le 1 + x^3$ for $x \ge 0$.

(b) From part (a) and Property 7: $\int_0^1 1\,dx \le \int_0^1 \sqrt{1+x^3}\,dx \le \int_0^1 (1+x^3)\,dx \;\Leftrightarrow$ $[x]_0^1 \le \int_0^1 \sqrt{1+x^3}\,dx \le \left[x + \frac{1}{4}x^4\right]_0^1 \;\Leftrightarrow\; 1 \le \int_0^1 \sqrt{1+x^3}\,dx \le 1 + \frac{1}{4} = 1.25$.

58. (a) If $x < 0$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$.

If $0 \le x \le 1$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \left[\frac{1}{2}t^2\right]_0^x = \frac{1}{2}x^2$.

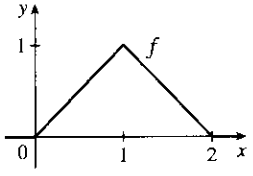
If $1 < x \le 2$, then

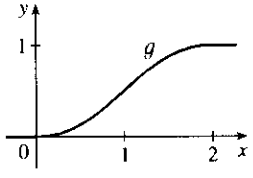
$$g(x) = \int_0^x f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt$$
$$= g(1) + \int_1^x (2-t)\,dt = \tfrac{1}{2}(1)^2 + \left[2t - \tfrac{1}{2}t^2\right]_1^x$$
$$= \tfrac{1}{2} + \left(2x - \tfrac{1}{2}x^2\right) - \left(2 - \tfrac{1}{2}\right) = 2x - \tfrac{1}{2}x^2 - 1.$$

If $x > 2$, then $g(x) = \int_0^x f(t)\,dt = g(2) + \int_2^x 0\,dt = 1 + 0 = 1$. So

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \le x \le 1 \\ 2x - \frac{1}{2}x^2 - 1 & \text{if } 1 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

(b)





(c) f is not differentiable at its corners at $x = 0$, 1, and 2. f is differentiable on $(-\infty, 0)$, $(0, 1)$, $(1, 2)$ and $(2, \infty)$. g is differentiable on $(-\infty, \infty)$.

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59. Using FTC1, we differentiate both sides of $6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ to get $\frac{f(x)}{x^2} = 2\,\frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f(x) = x^{3/2}$.

To find a, we substitute $x = a$ in the original equation to obtain $6 + \int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a} \quad \Rightarrow \quad 6 + 0 = 2\sqrt{a} \quad \Rightarrow$

$3 = \sqrt{a} \quad \Rightarrow \quad a = 9$.

60. By the Fundamental Theorem of Calculus, $\int_1^2 (h')'(u)\,du = h'(2) - h'(1) = 5 - 2 = 3$. The other information is unnecessary.

61. (a) Let $F(t) = \int_0^t f(s)\,ds$. Then, by FTC1, $F'(t) = f(t) =$ rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0, t]$.

(b) $C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = \frac{A + F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0, t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t) = -\frac{1}{t^2}\left[A + \int_0^t f(s)\,ds\right] + \frac{1}{t}f(t)$.

$C'(t) = 0 \quad \Rightarrow \quad t\,f(t) = A + \int_0^t f(s)\,ds \quad \Rightarrow \quad f(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = C(t)$.

62. (a) $C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds$. Using FTC1 and the Product Rule, we have

$C'(t) = \frac{1}{t}[f(t) + g(t)] - \frac{1}{t^2}\int_0^t [f(s) + g(s)]\,ds$. Set $C'(t) = 0$:

$\frac{1}{t}[f(t) + g(t)] - \frac{1}{t^2}\int_0^t [f(s) + g(s)]\,ds = 0 \quad \Rightarrow \quad [f(t) + g(t)] - \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds = 0 \quad \Rightarrow$

$[f(t) + g(t)] - C(t) = 0 \quad \Rightarrow \quad C(t) = f(t) + g(t)$.

(b) For $0 \le t \le 30$, we have $D(t) = \int_0^t \left(\frac{V}{15} - \frac{V}{450}s\right)ds = \left[\frac{V}{15}s - \frac{V}{900}s^2\right]_0^t = \frac{V}{15}t - \frac{V}{900}t^2$.

So $D(t) = V \quad \Rightarrow \quad \frac{V}{15}t - \frac{V}{900}t^2 = V \quad \Rightarrow \quad 60t - t^2 = 900 \quad \Rightarrow \quad t^2 - 60t + 900 = 0 \quad \Rightarrow$

$(t - 30)^2 = 0 \quad \Rightarrow \quad t = 30$. So the length of time T is 30 months.

(c) $C(t) = \frac{1}{t}\int_0^t \left(\frac{V}{15} - \frac{V}{450}s + \frac{V}{12{,}900}s^2\right)ds = \frac{1}{t}\left[\frac{V}{15}s - \frac{V}{900}s^2 + \frac{V}{38{,}700}s^3\right]_0^t$

$= \frac{1}{t}\left(\frac{V}{15}t - \frac{V}{900}t^2 + \frac{V}{38{,}700}t^3\right) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2 \quad \Rightarrow$

$C'(t) = -\frac{V}{900} + \frac{V}{19{,}350}t = 0$ when $\frac{1}{19{,}350}t = \frac{1}{900} \quad \Rightarrow \quad t = 21.5$.

$C(21.5) = \frac{V}{15} - \frac{V}{900}(21.5) + \frac{V}{38{,}700}(21.5)^2 \approx 0.05472V$, $C(0) = \frac{V}{15} \approx 0.06667V$, and

$C(30) = \frac{V}{15} - \frac{V}{900}(30) + \frac{V}{38{,}700}(30)^2 \approx 0.05659V$, so the absolute minimum is $C(21.5) \approx 0.05472V$.

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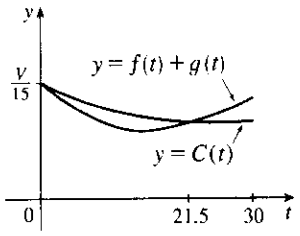
(d) As in part (c), we have $C(t) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2$, so $C(t) = f(t) + g(t)$

$$\Leftrightarrow \quad \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2 = \frac{V}{15} - \frac{V}{450}t + \frac{V}{12{,}900}t^2 \quad \Leftrightarrow$$

$$t^2\left(\frac{1}{12{,}900} - \frac{1}{38{,}700}\right) = t\left(\frac{1}{450} - \frac{1}{900}\right) \quad \Leftrightarrow$$

$t = \dfrac{1/900}{2/38{,}700} = \dfrac{43}{2} = 21.5$. This is the value of t that we obtained as the critical number of C in part (c), so we have verified the result of (a) in this case.



63. $\displaystyle\int_1^9 \frac{1}{2x}\,dx = \frac{1}{2}\int_1^9 \frac{1}{x}\,dx = \tfrac{1}{2}\Big[\ln|x|\Big]_1^9 = \tfrac{1}{2}(\ln 9 - \ln 1) = \tfrac{1}{2}\ln 9 - 0 = \ln 9^{1/2} = \ln 3$

64. $\displaystyle\int_0^1 10^x\,dx = \left[\frac{10^x}{\ln 10}\right]_0^1 = \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{9}{\ln 10}$

65. $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}}\,dt = 6\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-t^2}}\,dt = 6\Big[\sin^{-1}t\Big]_{1/2}^{\sqrt{3}/2} = 6\Big[\sin^{-1}\left(\tfrac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\tfrac{1}{2}\right)\Big]$

$= 6\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right) = 6\left(\tfrac{\pi}{6}\right) = \pi$

66. $\displaystyle\int_0^1 \frac{4}{t^2+1}\,dt = 4\int_0^1 \frac{1}{1+t^2}\,dt = 4\Big[\tan^{-1}t\Big]_0^1 = 4\big(\tan^{-1}1 - \tan^{-1}0\big) = 4\left(\tfrac{\pi}{4} - 0\right) = \pi$

67. $\int_{-1}^1 e^{u+1}\,du = \Big[e^{u+1}\Big]_{-1}^1 = e^2 - e^0 = e^2 - 1$ [or start with $e^{u+1} = e^u e^1$]

68. $\displaystyle\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \left(4u^{-3} + u^{-1}\right)du = \left[\frac{4}{-2}u^{-2} + \ln|u|\right]_1^2 = \left[\frac{-2}{u^2} + \ln u\right]_1^2$

$= \left(-\tfrac{1}{2} + \ln 2\right) - (-2 + \ln 1) = \tfrac{3}{2} + \ln 2$

5.4 Indefinite Integrals and the Net Change Theorem

1. $\dfrac{d}{dx}\left[\sqrt{x^2+1} + C\right] = \dfrac{d}{dx}\left[\left(x^2+1\right)^{1/2} + C\right] = \tfrac{1}{2}\left(x^2+1\right)^{-1/2}\cdot 2x = \dfrac{x}{\sqrt{x^2+1}}$

2. $\dfrac{d}{dx}\left[x\sin x + \cos x + C\right] = x\cos x + (\sin x)\cdot 1 - \sin x = x\cos x$

3. $\dfrac{d}{dx}\left[\dfrac{x}{a^2\sqrt{a^2-x^2}} + C\right] = \dfrac{1}{a^2}\dfrac{\sqrt{a^2-x^2} - x(-x)/\sqrt{a^2-x^2}}{a^2-x^2} = \dfrac{1}{a^2}\dfrac{\left(a^2-x^2\right)+x^2}{\left(a^2-x^2\right)^{3/2}} = \dfrac{1}{\sqrt{\left(a^2-x^2\right)^3}}$

4. $\dfrac{d}{dx}\left[-\dfrac{\sqrt{x^2+a^2}}{a^2x} + C\right] = -\dfrac{1}{a^2}\dfrac{d}{dx}\left[\dfrac{\sqrt{x^2+a^2}}{x}\right] = -\dfrac{x\left(x/\sqrt{x^2+a^2}\right) - \sqrt{x^2+a^2}\cdot 1}{a^2x^2}$

$= -\dfrac{x^2 - \left(x^2+a^2\right)}{a^2x^2\sqrt{x^2+a^2}} = \dfrac{1}{x^2\sqrt{x^2+a^2}}$

5. $\displaystyle\int x^{-3/4}\,dx = \frac{x^{-3/4+1}}{-3/4+1} + C = \frac{x^{1/4}}{1/4} + C = 4x^{1/4} + C$

6. $\displaystyle\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx = \frac{x^{4/3}}{4/3} + C = \tfrac{3}{4}x^{4/3} + C$

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7. $\displaystyle\int (x^3+6x+1)\,dx = \frac{x^4}{4}+6\frac{x^2}{2}+x+C = \tfrac{1}{4}x^4+3x^2+x+C$

8. $\displaystyle\int x(1+2x^4)\,dx = \int (x+2x^5)\,dx = \frac{x^2}{2}+2\frac{x^6}{6}+C = \tfrac{1}{2}x^2+\tfrac{1}{3}x^6+C$

9. $\displaystyle\int (1-t)(2+t^2)\,dt = \int (2-2t+t^2-t^3)\,dt = 2t-2\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+C = 2t-t^2+\tfrac{1}{3}t^3-\tfrac{1}{4}t^4+C$

10. $\displaystyle\int \left(u^2+1+\frac{1}{u^2}\right)du = \int (u^2+1+u^{-2})\,du = \frac{u^3}{3}+u+\frac{u^{-1}}{-1}+C = \tfrac{1}{3}u^3+u-\frac{1}{u}+C$

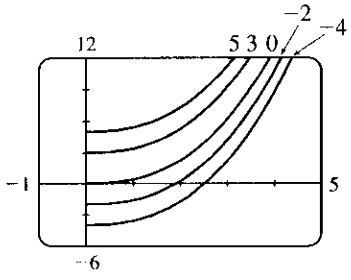
11. $\displaystyle\int (2-\sqrt{x}\,)^2\,dx = \int (4-4\sqrt{x}+x)\,dx = 4x-4\,\frac{x^{3/2}}{3/2}+\frac{x^2}{2}+C = 4x-\tfrac{8}{3}x^{3/2}+\frac{1}{2}x^2+C$

12. $\displaystyle\int (\sin\theta+3\cos\theta)\,d\theta = -\cos\theta+3\sin\theta+C$

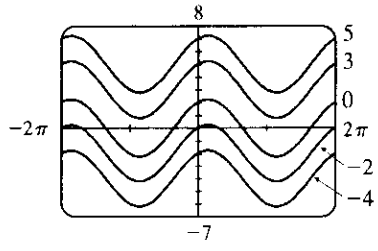
13. $\displaystyle\int \frac{\sin x}{1-\sin^2 x}\,dx = \int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\,dx = \int \sec x\,\tan x\,dx = \sec x+C$

14. $\displaystyle\int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\sin x\cos x}{\sin x}\,dx = \int 2\cos x\,dx = 2\sin x+C$

15. $\int x\sqrt{x}\,dx = \int x^{3/2}\,dx = \frac{2}{5}x^{5/2}+C.$
The members of the family in the figure correspond to $C = 5, 3, 0, -2,$ and -4.



16. $\int (\cos x-2\sin x)\,dx = \sin x+2\cos x+C.$
The members of the family in the figure correspond to $C = 5, 3, 0, -2,$ and -4.



17. $\displaystyle\int_0^2 (6x^2-4x+5)\,dx = \left[6\cdot\tfrac{1}{3}x^3-4\cdot\tfrac{1}{2}x^2+5x\right]_0^2 = \left[2x^3-2x^2+5x\right]_0^2 = (16-8+10)-0 = 18$

18. $\displaystyle\int_1^3 (1+2x-4x^3)\,dx = \left[x+2\cdot\tfrac{1}{2}x^2-4\cdot\tfrac{1}{4}x^4\right]_1^3 = \left[x+x^2-x^4\right]_1^3$
$= (3+9-81)-(1+1-1) = -69-1 = -70$

19. $\displaystyle\int_{-3}^0 (5y^4-6y^2+14)\,dy = \left[5(\tfrac{1}{5}y^5)-6(\tfrac{1}{3}y^3)+14y\right]_{-3}^0 = \left[y^5-2y^3+14y\right]_{-3}^0$
$= 0-(-243+54-42) = 231$

20. $\displaystyle\int_{-2}^0 (u^5-u^3+u^2)\,du = \left[\tfrac{1}{6}u^6-\tfrac{1}{4}u^4+\tfrac{1}{3}u^3\right]_{-2}^0 = 0-\left(\tfrac{32}{3}-4-\tfrac{8}{3}\right) = -4$

21. $\displaystyle\int_{-2}^2 (3u+1)^2\,du = \int_{-2}^2 (9u^2+6u+1)\,du = \left[9\cdot\tfrac{1}{3}u^3+6\cdot\tfrac{1}{2}u^2+u\right]_{-2}^2 = \left[3u^3+3u^2+u\right]_{-2}^2$
$= (24+12+2)-(-24+12-2) = 38-(-14) = 52$

22. $\displaystyle\int_0^4 (2v+5)(3v-1)\,dv = \int_0^4 (6v^2+13v-5)\,dv = \left[6\cdot\tfrac{1}{3}v^3+13\cdot\tfrac{1}{2}v^2-5v\right]_0^4 = \left[2v^3+\tfrac{13}{2}v^2-5v\right]_0^4$
$= (128+104-20)-0 = 212$

23. $\displaystyle\int_1^4 \sqrt{t}\,(1+t)\,dt = \int_1^4 (t^{1/2}+t^{3/2})\,dt = \left[\tfrac{2}{3}t^{3/2}+\tfrac{2}{5}t^{5/2}\right]_1^4 = \left(\tfrac{16}{3}+\tfrac{64}{5}\right)-\left(\tfrac{2}{3}+\tfrac{2}{5}\right) = \tfrac{14}{3}+\tfrac{62}{5} = \tfrac{256}{15}$

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24. $\int_0^9 \sqrt{2t}\,dt = \int_0^9 \sqrt{2}\,t^{1/2}\,dt = \left[\sqrt{2}\cdot\frac{2}{3}t^{3/2}\right]_0^9 = \sqrt{2}\cdot\frac{2}{3}\cdot 27 - 0 = 18\sqrt{2}$

25. $\int_{-2}^{-1}\left(4y^3+\frac{2}{y^3}\right)dy = \left[4\cdot\frac{1}{4}y^4 + 2\cdot\frac{1}{-2}y^{-2}\right]_{-2}^{-1} = \left[y^4 - \frac{1}{y^2}\right]_{-2}^{-1} = (1-1) - \left(16-\frac{1}{4}\right) = -\frac{63}{4}$

26. $\int_1^2 \frac{y+5y^7}{y^3}\,dy = \int_1^2 (y^{-2}+5y^4)\,dy = \left[-y^{-1}+5\cdot\frac{1}{5}y^5\right]_1^2 = \left[-\frac{1}{y}+y^5\right]_1^2 = \left(-\frac{1}{2}+32\right) - (-1+1) = \frac{63}{2}$

27. $\int_0^1 x(\sqrt[3]{x}+\sqrt[4]{x})\,dx = \int_0^1 (x^{4/3}+x^{5/4})\,dx = \left[\frac{3}{7}x^{7/3}+\frac{4}{9}x^{9/4}\right]_0^1 = \left(\frac{3}{7}+\frac{4}{9}\right) - 0 = \frac{55}{63}$

28. $\int_1^2 \left(x+\frac{1}{x}\right)^2 dx = \int_1^2 \left(x^2+2+x^{-2}\right)dx = \left[\frac{x^3}{3}+2x+\frac{x^{-1}}{-1}\right]_1^2 = \left[\frac{x^3}{3}+2x-\frac{1}{x}\right]_1^2$
$= \left(\frac{8}{3}+4-\frac{1}{2}\right) - \left(\frac{1}{3}+2-1\right) = \frac{29}{6}$

29. $\int_1^4 \sqrt{5/x}\,dx = \sqrt{5}\int_1^4 x^{-1/2}\,dx = \sqrt{5}\left[2\sqrt{x}\,\right]_1^4 = \sqrt{5}\,(2\cdot 2 - 2\cdot 1) = 2\sqrt{5}$

30. $\int_1^9 \frac{3x-2}{\sqrt{x}}\,dx = \int_1^9 (3x^{1/2}-2x^{-1/2})\,dx = \left[3\cdot\frac{2}{3}x^{3/2} - 2\cdot 2x^{1/2}\right]_1^9 = \left[2x^{3/2}-4x^{1/2}\right]_1^9$
$= (54-12) - (2-4) = 44$

31. $\int_0^{\pi}(4\sin\theta - 3\cos\theta)\,d\theta = [-4\cos\theta - 3\sin\theta]_0^{\pi} = (4-0) - (-4-0) = 8$

32. $\int_{\pi/4}^{\pi/3}\sec\theta\tan\theta\,d\theta = \left[\sec\theta\right]_{\pi/4}^{\pi/3} = \sec\frac{\pi}{3} - \sec\frac{\pi}{4} = 2-\sqrt{2}$

33. $\int_0^{\pi/4}\frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4}\left(\frac{1}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}\right)d\theta = \int_0^{\pi/4}\left(\sec^2\theta+1\right)d\theta$
$= [\tan\theta+\theta]_0^{\pi/4} = \left(\tan\frac{\pi}{4}+\frac{\pi}{4}\right) - (0+0) = 1+\frac{\pi}{4}$

34. $\int_0^{\pi/3}\frac{\sin\theta+\sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\pi/3}\frac{\sin\theta\,(1+\tan^2\theta)}{\sec^2\theta}\,d\theta = \int_0^{\pi/3}\frac{\sin\theta\sec^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\pi/3}\sin\theta\,d\theta$
$= \left[-\cos\theta\right]_0^{\pi/3} = -\frac{1}{2} - (-1) = \frac{1}{2}$

35. $\int_1^{64}\frac{1+\sqrt[3]{x}}{\sqrt{x}}\,dx = \int_1^{64}\left(\frac{1}{x^{1/2}}+\frac{x^{1/3}}{x^{1/2}}\right)dx = \int_1^{64}(x^{-1/2}+x^{(1/3)-(1/2)})\,dx = \int_1^{64}(x^{-1/2}+x^{-1/6})\,dx$
$= \left[2x^{1/2}+\frac{6}{5}x^{5/6}\right]_1^{64} = \left(16+\frac{192}{5}\right) - \left(2+\frac{6}{5}\right) = 14+\frac{186}{5} = \frac{256}{5}$

36. $\int_0^1 (1+x^2)^3\,dx = \int_0^1 (1+3x^2+3x^4+x^6)\,dx = \left[x+x^3+\frac{3}{5}x^5+\frac{1}{7}x^7\right]_0^1 = \left(1+1+\frac{3}{5}+\frac{1}{7}\right) - 0 = \frac{96}{35}$

37. $\int_0^1 \left(\sqrt[4]{x^5}+\sqrt[5]{x^4}\right)dx = \int_0^1 \left(x^{5/4}+x^{4/5}\right)dx = \left[\frac{x^{9/4}}{9/4}+\frac{x^{9/5}}{9/5}\right]_0^1 = \left[\frac{4}{9}x^{9/4}+\frac{5}{9}x^{9/5}\right]_0^1 = \frac{4}{9}+\frac{5}{9}-0 = 1$

38. $\int_1^8 \frac{x-1}{\sqrt[3]{x^2}}\,dx = \int_1^8 \left(x^{1/3}-x^{-2/3}\right)dx = \left[\frac{x^{4/3}}{4/3}-\frac{x^{1/3}}{1/3}\right]_1^8 = \left[\frac{3}{4}x^{4/3}-3x^{1/3}\right]_1^8$
$= \left(\frac{3}{4}\cdot 16 - 3\cdot 2\right) - \left(\frac{3}{4}-3\right) = \frac{33}{4}$

39. $\int_{-1}^2 (x-2|x|)\,dx = \int_{-1}^0 [x-2(-x)]\,dx + \int_0^2 [x-2(x)]\,dx = \int_{-1}^0 3x\,dx + \int_0^2 (-x)\,dx = 3\left[\frac{1}{2}x^2\right]_{-1}^0 - \left[\frac{1}{2}x^2\right]_0^2$
$= 3\left(0-\frac{1}{2}\right) - (2-0) = -\frac{7}{2} = -3.5$

40. $\int_0^{3\pi/2}|\sin x|\,dx = \int_0^{\pi}\sin x\,dx + \int_{\pi}^{3\pi/2}(-\sin x)\,dx = [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{3\pi/2}$
$= [1-(-1)] + [0-(-1)] = 2+1 = 3$

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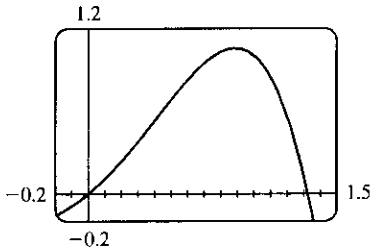
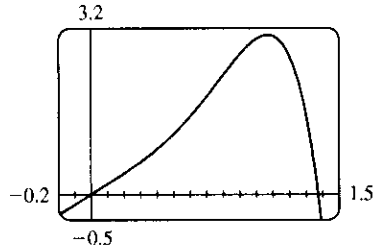
41. The graph shows that $y = x + x^2 - x^4$ has x-intercepts at $x = 0$ and at $x = a \approx 1.32$. So the area of the region that lies under the curve and above the x-axis is

$$\begin{aligned}\int_0^a (x + x^2 - x^4)\,dx &= \left[\tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^a \\ &= \left(\tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 - \tfrac{1}{5}a^5\right) - 0 \\ &\approx 0.84\end{aligned}$$



42. The graph shows that $y = 2x + 3x^4 - 2x^6$ has x-intercepts at $x = 0$ and at $x = a \approx 1.37$. So the area of the region that lies under the curve and above the x-axis is

$$\begin{aligned}\int_0^a (2x + 3x^4 - 2x^6)\,dx &= \left[x^2 + \tfrac{3}{5}x^5 - \tfrac{2}{7}x^7\right]_0^a \\ &= \left(a^2 + \tfrac{3}{5}a^5 - \tfrac{2}{7}a^7\right) - 0 \\ &\approx 2.18\end{aligned}$$



43. $A = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{1}{3}y^3\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$

44. $y = \sqrt[4]{x} \quad \Rightarrow \quad x = y^4$, so $A = \int_0^1 y^4\,dy = \left[\frac{1}{5}y^5\right]_0^1 = \frac{1}{5}$.

45. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t. We know from the Net Change Theorem that $\int_5^{10} w'(t)\,dt = w(10) - w(5)$, so the integral represents the increase in the child's weight (in pounds) between the ages of 5 and 10.

46. $\int_a^b I(t)\,dt = \int_a^b Q'(t)\,dt = Q(b) - Q(a)$ by the Net Change Theorem, so it represents the change in the charge Q from time $t = a$ to $t = b$.

47. Since $r(t)$ is the rate at which oil leaks, we can write $r(t) = -V'(t)$, where $V(t)$ is the volume of oil at time t. [Note that the minus sign is needed because V is decreasing, so $V'(t)$ is negative, but $r(t)$ is positive.] Thus, by the Net Change Theorem, $\int_0^{120} r(t)\,dt = -\int_0^{120} V'(t)\,dt = -[V(120) - V(0)] = V(0) - V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).

48. By the Net Change Theorem, $\int_0^{15} n'(t)\,dt = n(15) - n(0) = n(15) - 100$ represents the increase in the bee population in 15 weeks. So $100 + \int_0^{15} n'(t)\,dt = n(15)$ represents the total bee population after 15 weeks.

49. By the Net Change Theorem, $\int_{1000}^{5000} R'(x)\,dx = R(5000) - R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

50. The slope of the trail is the rate of change of the elevation E, so $f(x) = E'(x)$. By the Net Change Theorem, $\int_3^5 f(x)\,dx = \int_3^5 E'(x)\,dx = E(5) - E(3)$ is the change in the elevation E between $x = 3$ miles and $x = 5$ miles from the start of the trail.

51. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x. Since $f(x)$ is measured in newtons and x is measured in meters, the units for $\int_0^{100} f(x)\,dx$ are newton-meters. (A newton-meter is abbreviated N-m and is called a joule.)

52. The units for $a(x)$ are pounds per foot and the units for x are feet, so the units for da/dx are pounds per foot per foot, denoted (lb/ft)/ft. The unit of measurement for $\int_2^8 a(x)\,dx$ is the product of pounds per foot and feet; that is, pounds.

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53. (a) displacement $= \int_0^3 (3t-5)\,dt = \left[\frac{3}{2}t^2 - 5t\right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2}$ m

(b) distance traveled $= \int_0^3 |3t-5|\,dt = \int_0^{5/3} (5-3t)\,dt + \int_{5/3}^3 (3t-5)\,dt$

$= \left[5t - \frac{3}{2}t^2\right]_0^{5/3} + \left[\frac{3}{2}t^2 - 5t\right]_{5/3}^3 = \frac{25}{3} - \frac{3}{2}\cdot\frac{25}{9} + \frac{27}{2} - 15 - \left(\frac{3}{2}\cdot\frac{25}{9} - \frac{25}{3}\right) = \frac{41}{6}$ m

54. (a) displacement $= \int_1^6 (t^2-2t-8)\,dt = \left[\frac{1}{3}t^3 - t^2 - 8t\right]_1^6 = (72-36-48) - \left(\frac{1}{3} - 1 - 8\right) = -\frac{10}{3}$ m

(b) distance traveled $= \int_1^6 |t^2-2t-8|\,dt = \int_1^6 |(t-4)(t+2)|\,dt$

$= \int_1^4 (-t^2+2t+8)\,dt + \int_4^6 (t^2-2t-8)\,dt = \left[-\frac{1}{3}t^3 + t^2 + 8t\right]_1^4 + \left[\frac{1}{3}t^3 - t^2 - 8t\right]_4^6$

$= \left(-\frac{64}{3} + 16 + 32\right) - \left(-\frac{1}{3} + 1 + 8\right) + (72 - 36 - 48) - \left(\frac{64}{3} - 16 - 32\right) = \frac{98}{3}$ m

55. (a) $v'(t) = a(t) = t+4 \;\Rightarrow\; v(t) = \frac{1}{2}t^2 + 4t + C \;\Rightarrow\; v(0) = C = 5 \;\Rightarrow\; v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s

(b) distance traveled $= \int_0^{10} |v(t)|\,dt = \int_0^{10} \left|\frac{1}{2}t^2 + 4t + 5\right| dt = \int_0^{10} \left(\frac{1}{2}t^2 + 4t + 5\right) dt$

$= \left[\frac{1}{6}t^3 + 2t^2 + 5t\right]_0^{10} = \frac{500}{3} + 200 + 50 = 416\frac{2}{3}$ m

56. (a) $v'(t) = a(t) = 2t+3 \;\Rightarrow\; v(t) = t^2 + 3t + C \;\Rightarrow\; v(0) = C = -4 \;\Rightarrow\; v(t) = t^2 + 3t - 4$

(b) distance traveled $= \int_0^3 |t^2+3t-4|\,dt = \int_0^3 |(t+4)(t-1)|\,dt$

$= \int_0^1 (-t^2-3t+4)\,dt + \int_1^3 (t^2+3t-4)\,dt$

$= \left[-\frac{1}{3}t^3 - \frac{3}{2}t^2 + 4t\right]_0^1 + \left[\frac{1}{3}t^3 + \frac{3}{2}t^2 - 4t\right]_1^3$

$= \left(-\frac{1}{3} - \frac{3}{2} + 4\right) + \left(9 + \frac{27}{2} - 12\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right) = \frac{89}{6}$ m

57. Since $m'(x) = \rho(x)$, $m = \int_0^4 \rho(x)\,dx = \int_0^4 (9 + 2\sqrt{x}\,)\,dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3}$ kg.

58. By the Net Change Theorem, the amount of water that flows from the tank is

$\int_0^{10} r(t)\,dt = \int_0^{10} (200-4t)\,dt = \left[200t - 2t^2\right]_0^{10} = (2000 - 200) - 0 = 1800$ liters.

59. Let s be the position of the car. We know from Equation 2 that $s(100) - s(0) = \int_0^{100} v(t)\,dt$. We use the Midpoint Rule for $0 \le t \le 100$ with $n = 5$. Note that the length of each of the five time intervals is 20 seconds $= \frac{20}{3600}$ hour $= \frac{1}{180}$ hour. So the distance traveled is

$$\int_0^{100} v(t)\,dt \approx \tfrac{1}{180}[v(10) + v(30) + v(50) + v(70) + v(90)]$$

$$= \tfrac{1}{180}(38 + 58 + 51 + 53 + 47)$$

$$= \tfrac{247}{180} \approx 1.4 \text{ miles}$$

60. (a) By the Net Change Theorem, the total amount spewed into the atmosphere is $Q(6) - Q(0) = \int_0^6 r(t)\,dt = Q(6)$ since $Q(0) = 0$. The rate $r(t)$ is positive, so Q is an increasing function. Thus, an upper estimate for $Q(6)$ is R_6 and a lower estimate for $Q(6)$ is L_6. $\Delta t = \dfrac{b-a}{n} = \dfrac{6-0}{6} = 1$.

$R_6 = \sum_{i=1}^{6} r(t_i)\,\Delta t = 10 + 24 + 36 + 46 + 54 + 60 = 230$ tonnes.

$L_6 = \sum_{i=1}^{6} r(t_{i-1})\,\Delta t = R_6 + r(0) - r(6) = 230 + 2 - 60 = 172$ tonnes.

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(b) $\Delta t = \dfrac{b-a}{n} = \dfrac{6-0}{3} = 2$.

$Q(6) \approx M_3 = 2[r(1) + r(3) + r(5)] = 2(10 + 36 + 54) = 2(100) = 200$ tonnes.

61. From the Net Change Theorem, the increase in cost if the production level is raised from 2000 yards to 4000 yards is $C(4000) - C(2000) = \int_{2000}^{4000} C'(x)\,dx$.

$$\int_{2000}^{4000} C'(x)\,dx = \int_{2000}^{4000} \left(3 - 0.01x + 0.000006x^2\right) dx$$

$$= \left[3x - 0.005x^2 + 0.000002x^3\right]_{2000}^{4000} = 60{,}000 - 2{,}000 = \$58{,}000$$

62. By the Net Change Theorem, the amount of water after four days is

$$\begin{aligned} 25{,}000 + \textstyle\int_0^4 r(t)\,dt &\approx 25{,}000 + M_4 \\ &= 25{,}000 + \tfrac{4-0}{4}\left[r(0.5) + r(1.5) + r(2.5) + r(3.5)\right] \\ &\approx 25{,}000 + [1500 + 1770 + 740 + (-690)] = 28{,}320 \text{ liters} \end{aligned}$$

63. (a) We can find the area between the Lorenz curve and the line $y = x$ by subtracting the area under $y = L(x)$ from the area under $y = x$. Thus,

$$\text{coefficient of inequality} = \frac{\text{area between Lorenz curve and line } y = x}{\text{area under line } y = x} = \frac{\int_0^1 [x - L(x)]\,dx}{\int_0^1 x\,dx}$$

$$= \frac{\int_0^1 [x - L(x)]\,dx}{[x^2/2]_0^1} = \frac{\int_0^1 [x - L(x)]\,dx}{1/2} = 2\int_0^1 [x - L(x)]\,dx$$

(b) $L(x) = \frac{5}{12}x^2 + \frac{7}{12}x \quad \Rightarrow \quad L(50\%) = L\left(\frac{1}{2}\right) = \frac{5}{48} + \frac{7}{24} = \frac{19}{48} = 0.39583\overline{3}$, so the bottom 50% of the households receive at most about 40% of the income. Using the result in part (a),

$$\begin{aligned} \text{coefficient of inequality} &= 2\textstyle\int_0^1 [x - L(x)]\,dx = 2\int_0^1 \left(x - \frac{5}{12}x^2 - \frac{7}{12}x\right) dx \\ &= 2\textstyle\int_0^1 \left(\frac{5}{12}x - \frac{5}{12}x^2\right) dx = 2\int_0^1 \frac{5}{12}\left(x - x^2\right) dx \\ &= \tfrac{5}{6}\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{5}{6}\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{5}{6}\left(\tfrac{1}{6}\right) = \tfrac{5}{36} \end{aligned}$$

64. (a) From Exercise 4.1.66(a), $v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$.

(b) $h(125) - h(0) = \int_0^{125} v(t)\,dt = \left[0.000365t^4 - 0.03851t^3 + 12.490845t^2 - 21.26872t\right]_0^{125} \approx 206{,}407$ ft

65. $$\int_1^e \frac{x^2 + x + 1}{x}\,dx = \int_1^e \left(x + 1 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + x + \ln|x|\right]_1^e$$

$$= \left(\tfrac{1}{2}e^2 + e + \ln e\right) - \left(\tfrac{1}{2} + 1 + \ln 1\right) = \tfrac{1}{2}e^2 + e - \tfrac{1}{2}$$

66. $$\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + 2x + \ln|x|\right]_4^9$$

$$= \tfrac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4) = \tfrac{85}{2} + \ln\tfrac{9}{4}$$

67. $$\int \left(x^2 + 1 + \frac{1}{x^2 + 1}\right) dx = \frac{x^3}{3} + x + \tan^{-1} x + C$$

68. $B = 3A \quad \Rightarrow \quad \int_0^b e^x\,dx = 3\int_0^a e^x\,dx \quad \Rightarrow \quad [e^x]_0^b = 3\,[e^x]_0^a \quad \Rightarrow \quad e^b - 1 = 3(e^a - 1) \quad \Rightarrow$

$e^b = 3e^a - 2 \quad \Rightarrow \quad b = \ln(3e^a - 2)$

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5.5 The Substitution Rule

1. Let $u = 3x$. Then $du = 3\,dx$, so $dx = \frac{1}{3}du$. Thus,

$\int \cos 3x\,dx = \int \cos u\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin 3x + C$. Don't forget that it is often very easy to check an indefinite integration by differentiating your answer. In this case,

$\frac{d}{dx}\left(\frac{1}{3}\sin 3x + C\right) = \frac{1}{3}(\cos 3x)\cdot 3 = \cos 3x$, the desired result.

2. Let $u = 4 + x^2$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int x\left(4+x^2\right)^{10}\,dx = \int u^{10}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\cdot\tfrac{1}{11}u^{11} + C = \tfrac{1}{22}\left(4+x^2\right)^{11} + C.$$

3. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so

$$\int x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}\cdot\tfrac{2}{3}u^{3/2} + C = \tfrac{2}{9}(x^3+1)^{3/2} + C.$$

4. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$ and $\dfrac{1}{\sqrt{x}}\,dx = 2\,du$, so

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u\,(2\,du) = 2(-\cos u) + C = -2\cos\sqrt{x} + C.$$

5. Let $u = 1 + 2x$. Then $du = 2\,dx$ and $dx = \frac{1}{2}\,du$, so

$$\int \frac{4}{(1+2x)^3}\,dx = 4\int u^{-3}\left(\tfrac{1}{2}\,du\right) = 2\frac{u^{-2}}{-2} + C = -\frac{1}{u^2} + C = -\frac{1}{(1+2x)^2} + C.$$

6. Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and $\sin\theta\,d\theta = -du$, so

$$\int \cos^4\theta\sin\theta\,d\theta = \int u^4\,(-du) = -\tfrac{1}{5}u^5 + C = -\tfrac{1}{5}\cos^5\theta + C.$$

7. Let $u = x^2 + 3$. Then $du = 2x\,dx$, so $\int 2x(x^2+3)^4\,dx = \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}(x^2+3)^5 + C$.

8. Let $u = x^3 + 5$. Then $du = 3x^2dx$ and $x^2dx = \frac{1}{3}\,du$, so

$$\int x^2(x^3+5)^9\,dx = \int u^9\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\cdot\tfrac{1}{10}u^{10} + C = \tfrac{1}{30}(x^3+5)^{10} + C.$$

9. Let $u = 3x - 2$. Then $du = 3\,dx$ and $dx = \frac{1}{3}\,du$, so

$$\int (3x-2)^{20}\,dx = \int u^{20}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\cdot\tfrac{1}{21}u^{21} + C = \tfrac{1}{63}(3x-2)^{21} + C.$$

10. Let $u = 2 - x$. Then $du = -dx$ and $dx = -du$, so

$$\int (2-x)^6\,dx = \int u^6(-du) = -\tfrac{1}{7}u^7 + C = -\tfrac{1}{7}(2-x)^7 + C.$$

11. Let $u = 1 + x + 2x^2$. Then $du = (1+4x)\,dx$, so

$$\int \frac{1+4x}{\sqrt{1+x+2x^2}}\,dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+x+2x^2} + C.$$

12. Let $u = x^2 + 1$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int \frac{x}{(x^2+1)^2}\,dx = \int u^{-2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\cdot\frac{-1}{u} + C = \frac{-1}{2u} + C = \frac{-1}{2(x^2+1)} + C.$$

13. Let $u = 2y + 1$. Then $du = 2\,dy$ and $dy = \frac{1}{2}\,du$, so

$$\int \frac{3}{(2y+1)^5}\,dy = \int 3u^{-5}\left(\tfrac{1}{2}\,du\right) = \frac{3}{2}\cdot\frac{1}{-4}u^{-4} + C = \frac{-3}{8(2y+1)^4} + C.$$

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14. Let $u = 5t + 4$. Then $du = 5\,dt$ and $dt = \frac{1}{5}\,du$, so

$$\int \frac{1}{(5t+4)^{2.7}}\,dt = \int u^{-2.7}\left(\tfrac{1}{5}\,du\right) = \frac{1}{5}\cdot\frac{1}{-1.7}u^{-1.7} + C = \frac{-1}{8.5}u^{-1.7} + C = \frac{-2}{17(5t+4)^{1.7}} + C.$$

15. Let $u = 4 - t$. Then $du = -dt$ and $dt = -du$, so

$$\int \sqrt{4-t}\,dt = \int u^{1/2}\,(-du) = -\tfrac{2}{3}u^{3/2} + C = -\tfrac{2}{3}(4-t)^{3/2} + C.$$

16. Let $u = 2y^4 - 1$. Then $du = 8y^3\,dy$ and $y^3\,dy = \frac{1}{8}\,du$, so

$$\int y^3\sqrt{2y^4-1}\,dy = \int u^{1/2}\left(\tfrac{1}{8}\,du\right) = \tfrac{1}{8}\cdot\tfrac{2}{3}u^{3/2} + C = \tfrac{1}{12}(2y^4-1)^{3/2} + C.$$

17. Let $u = \pi t$. Then $du = \pi\,dt$ and $dt = \frac{1}{\pi}\,du$, so

$$\int \sin \pi t\,dt = \int \sin u\left(\tfrac{1}{\pi}\,du\right) = \tfrac{1}{\pi}(-\cos u) + C = -\tfrac{1}{\pi}\cos \pi t + C.$$

18. Let $u = 2\theta$. Then $du = 2\,d\theta$ and $d\theta = \frac{1}{2}\,du$, so

$$\int \sec 2\theta\,\tan 2\theta\,d\theta = \int \sec u\tan u\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\sec u + C = \tfrac{1}{2}\sec 2\theta + C.$$

19. Let $u = \sqrt{t}$. Then $du = \dfrac{dt}{2\sqrt{t}}$ and $\dfrac{1}{\sqrt{t}}\,dt = 2\,du$, so

$$\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt = \int \cos u\,(2\,du) = 2\sin u + C = 2\sin\sqrt{t} + C.$$

20. Let $u = 1 + x^{3/2}$. Then $du = \frac{3}{2}x^{1/2}\,dx$ and $\sqrt{x}\,dx = \frac{2}{3}\,du$, so

$$\int \sqrt{x}\sin(1+x^{3/2})\,dx = \int \sin u\left(\tfrac{2}{3}\,du\right) = \tfrac{2}{3}\cdot(-\cos u) + C = -\tfrac{2}{3}\cos(1+x^{3/2}) + C.$$

21. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$, so $\int \cos\theta\,\sin^6\theta\,d\theta = \int u^6\,du = \frac{1}{7}u^7 + C = \frac{1}{7}\sin^7\theta + C$.

22. Let $u = 1 + \tan\theta$. Then $du = \sec^2\theta\,d\theta$, so

$$\int (1+\tan\theta)^5\sec^2\theta\,d\theta = \int u^5\,du = \tfrac{1}{6}u^6 + C = \tfrac{1}{6}(1+\tan\theta)^6 + C.$$

23. Let $u = 1 + z^3$. Then $du = 3z^2\,dz$ and $z^2\,dz = \frac{1}{3}\,du$, so

$$\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz = \int u^{-1/3}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\cdot\tfrac{3}{2}u^{2/3} + C = \tfrac{1}{2}(1+z^3)^{2/3} + C.$$

24. Let $u = ax^2 + 2bx + c$. Then $du = 2(ax+b)\,dx$ and $(ax+b)\,dx = \frac{1}{2}\,du$, so

$$\int \frac{(ax+b)\,dx}{\sqrt{ax^2+2bx+c}} = \int \frac{\frac{1}{2}\,du}{\sqrt{u}} = \tfrac{1}{2}\int u^{-1/2}\,du = u^{1/2} + C = \sqrt{ax^2+2bx+c} + C.$$

25. Let $u = \cot x$. Then $du = -\csc^2 x\,dx$ and $\csc^2 x\,dx = -du$, so

$$\int \sqrt{\cot x}\csc^2 x\,dx = \int \sqrt{u}\,(-du) = -\frac{u^{3/2}}{3/2} + C = -\tfrac{2}{3}(\cot x)^{3/2} + C.$$

26. Let $u = \dfrac{\pi}{x}$. Then $du = -\dfrac{\pi}{x^2}\,dx$ and $\dfrac{1}{x^2}\,dx = -\dfrac{1}{\pi}\,du$, so

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \cos u\left(-\frac{1}{\pi}\,du\right) = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\frac{\pi}{x} + C.$$

27. Let $u = \sec x$. Then $du = \sec x\,\tan x\,dx$, so

$$\int \sec^3 x\,\tan x\,dx = \int \sec^2 x\,(\sec x\,\tan x)\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sec^3 x + C.$$

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28. Let $u = x^3 + 1$. Then $x^3 = u - 1$ and $du = 3x^2\,dx$, so

$$\int \sqrt[3]{x^3+1}\,x^5\,dx = \int \sqrt[3]{x^3+1}\cdot x^3\cdot x^2\,dx = \int u^{1/3}(u-1)\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\int (u^{4/3}-u^{1/3})\,du$$
$$= \tfrac{1}{3}\left(\tfrac{3}{7}u^{7/3} - \tfrac{3}{4}u^{4/3}\right) + C = \tfrac{1}{7}(x^3+1)^{7/3} - \tfrac{1}{4}(x^3+1)^{4/3} + C$$

29. Let $u = b + cx^{a+1}$. Then $du = (a+1)cx^a\,dx$, so

$$\int x^a\sqrt{b+cx^{a+1}}\,dx = \int u^{1/2}\frac{1}{(a+1)c}\,du = \frac{1}{(a+1)c}\left(\tfrac{2}{3}u^{3/2}\right) + C = \frac{2}{3c(a+1)}\left(b+cx^{a+1}\right)^{3/2} + C.$$

30. Let $u = \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so

$$\int \sin t\,\sec^2(\cos t)\,dt = \int \sec^2 u\cdot(-du) = -\tan u + C = -\tan(\cos t) + C.$$

31. Let $u = x + 2$. Then $du = dx$, so

$$\int \frac{x}{\sqrt[4]{x+2}}\,dx = \int \frac{u-2}{\sqrt[4]{u}}\,du = \int \left(u^{3/4} - 2u^{-1/4}\right)du = \tfrac{4}{7}u^{7/4} - 2\cdot\tfrac{4}{3}u^{3/4} + C$$
$$= \tfrac{4}{7}(x+2)^{7/4} - \tfrac{8}{3}(x+2)^{3/4} + C$$

32. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

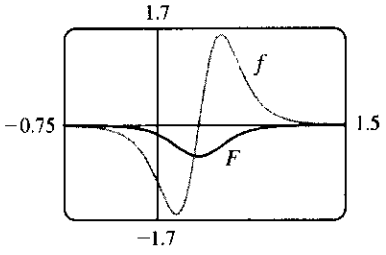
$$\int \frac{x^2}{\sqrt{1-x}}\,dx = \int \frac{(1-u)^2}{\sqrt{u}}\,(-du) = -\int \frac{1-2u+u^2}{\sqrt{u}}\,du = -\int \left(u^{-1/2} - 2u^{1/2} + u^{3/2}\right)du$$
$$= -\left(2u^{1/2} - 2\cdot\tfrac{2}{3}u^{3/2} + \tfrac{2}{5}u^{5/2}\right) + C = -2\sqrt{1-x} + \tfrac{4}{3}(1-x)^{3/2} - \tfrac{2}{5}(1-x)^{5/2} + C$$

In Exercises 33–36, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

33. $f(x) = \dfrac{3x-1}{(3x^2-2x+1)^4}$.

$u = 3x^2 - 2x + 1 \quad\Rightarrow\quad du = (6x-2)\,dx = 2(3x-1)\,dx$, so

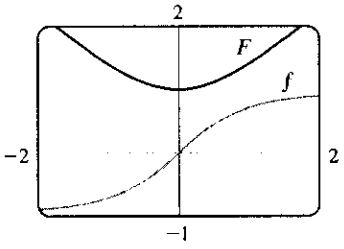
$$\int \frac{3x-1}{(3x^2-2x+1)^4}\,dx = \int \frac{1}{u^4}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-4}\,du$$
$$= -\frac{1}{6}u^{-3} + C = -\frac{1}{6(3x^2-2x+1)^3} + C$$



Notice that at $x = \frac{1}{3}$, f changes from negative to positive, and F has a local minimum.

34. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. $u = x^2 + 1 \quad\Rightarrow\quad du = 2x\,dx$, so

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-1/2}\,du$$
$$= u^{1/2} + C = \sqrt{x^2+1} + C.$$



Note that at $x = 0$, f changes from negative to positive and F has a local minimum.

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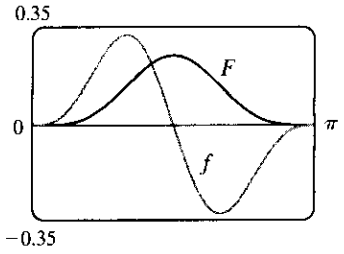
35. $f(x) = \sin^3 x \cos x$. $u = \sin x \quad \Rightarrow \quad du = \cos x\,dx$, so

$$\int \sin^3 x \cos x\,dx = \int u^3\,du = \tfrac{1}{4}u^4 + C = \tfrac{1}{4}\sin^4 x + C$$

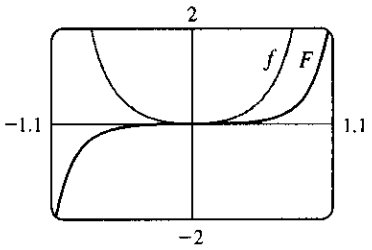
Note that at $x = \frac{\pi}{2}$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period π, so at $x = 0$ and at $x = \pi$, f changes from negative to positive and F has local minima.



36. $f(\theta) = \tan^2\theta \sec^2\theta$. $u = \tan\theta \quad \Rightarrow \quad du = \sec^2\theta\,d\theta$, so

$$\int \tan^2\theta \sec^2\theta\,d\theta = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3\theta + C$$

Note that f is positive and F is increasing. At $x = 0$, $f = 0$ and F has a horizontal tangent.



37. Let $u = x - 1$, so $du = dx$. When $x = 0$, $u = -1$; when $x = 2$, $u = 1$. Thus, $\int_0^2 (x-1)^{25}\,dx = \int_{-1}^1 u^{25}\,du = 0$ by Theorem 6(b), since $f(u) = u^{25}$ is an odd function.

38. Let $u = 4 + 3x$, so $du = 3\,dx$. When $x = 0$, $u = 4$; when $x = 7$, $u = 25$. Thus,

$$\int_0^7 \sqrt{4+3x}\,dx = \int_4^{25} \sqrt{u}\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_4^{25} = \frac{2}{9}(25^{3/2} - 4^{3/2}) = \frac{2}{9}(125 - 8) = \frac{234}{9} = 26$$

39. Let $u = 1 + 2x^3$, so $du = 6x^2\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 3$. Thus,

$$\int_0^1 x^2(1+2x^3)^5\,dx = \int_1^3 u^5\left(\tfrac{1}{6}\,du\right) = \tfrac{1}{6}\left[\tfrac{1}{6}u^6\right]_1^3 = \tfrac{1}{36}(3^6 - 1^6) = \tfrac{1}{36}(729 - 1) = \tfrac{728}{36} = \tfrac{182}{9}$$

40. Let $u = x^2$, so $du = 2x\,dx$. When $x = 0$, $u = 0$; when $x = \sqrt{\pi}$, $u = \pi$. Thus,

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \int_0^{\pi} \cos u\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}[\sin u]_0^{\pi} = \tfrac{1}{2}(\sin\pi - \sin 0) = \tfrac{1}{2}(0 - 0) = 0.$$

41. Let $u = t/4$, so $du = \frac{1}{4}\,dt$. When $t = 0$, $u = 0$; when $t = \pi$, $u = \pi/4$. Thus,

$$\int_0^{\pi} \sec^2(t/4)\,dt = \int_0^{\pi/4} \sec^2 u\,(4\,du) = 4\big[\tan u\big]_0^{\pi/4} = 4\big(\tan\tfrac{\pi}{4} - \tan 0\big) = 4(1 - 0) = 4.$$

42. Let $u = \pi t$, so $du = \pi\,dt$. When $t = \frac{1}{6}$, $u = \frac{\pi}{6}$; when $t = \frac{1}{2}$, $u = \frac{\pi}{2}$. Thus,

$$\int_{1/6}^{1/2} \csc\pi t\,\cot\pi t\,dt = \int_{\pi/6}^{\pi/2} \csc u\,\cot u\left(\tfrac{1}{\pi}\,du\right) = \tfrac{1}{\pi}\big[-\csc u\big]_{\pi/6}^{\pi/2} = -\tfrac{1}{\pi}(1 - 2) = \tfrac{1}{\pi}.$$

43. $\int_{-\pi/6}^{\pi/6} \tan^3\theta\,d\theta = 0$ by Theorem 6(b), since $f(\theta) = \tan^3\theta$ is an odd function.

44. $\displaystyle\int_0^2 \frac{dx}{(2x-3)^2}$ does not exist since $\displaystyle f(x) = \frac{1}{(2x-3)^2}$ has an infinite discontinuity at $x = \frac{3}{2}$.

45. Let $u = \cos\theta$, so $du = -\sin\theta\,d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$. Thus,

$$\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}\,d\theta = \int_1^{1/2} \frac{-du}{u^2} = \int_{1/2}^1 u^{-2}\,du = \left[-\frac{1}{u}\right]_{1/2}^1 = -1 - (-2) = 1.$$

46. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2\sin x}{1+x^6}\,dx = 0$ by Theorem 6(b), since $\displaystyle f(x) = \frac{x^2\sin x}{1+x^6}$ is an odd function.

47. Let $u = 1 + 2x$, so $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Thus,

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} u^{-2/3}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{1}{2}\cdot 3u^{1/3}\right]_1^{27} = \tfrac{3}{2}(3 - 1) = 3.$$

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48. Let $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$$\int_0^{\pi/2} \cos x \sin(\sin x)\,dx = \int_0^1 \sin u\,du = \left[-\cos u\right]_0^1 = -(\cos 1 - 1) = 1 - \cos 1.$$

49. Let $u = x - 1$, so $u + 1 = x$ and $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Thus,

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 (u^{3/2} + u^{1/2})\,du = \left[\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{2}{5} + \tfrac{2}{3} = \tfrac{16}{15}.$$

50. Let $u = 1 + 2x$, so $x = \frac{1}{2}(u - 1)$ and $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 4$, $u = 9$. Thus,

$$\int_0^4 \frac{x\,dx}{\sqrt{1+2x}} = \int_1^9 \frac{\frac{1}{2}(u-1)}{\sqrt{u}}\frac{du}{2} = \tfrac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})\,du = \tfrac{1}{4}\left[\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9$$

$$= \tfrac{1}{4}\cdot\tfrac{2}{3}\left[u^{3/2} - 3u^{1/2}\right]_1^9 = \tfrac{1}{6}[(27 - 9) - (1 - 3)] = \tfrac{20}{6} = \tfrac{10}{3}$$

51. $\displaystyle\int_0^4 \frac{dx}{(x-2)^3}$ does not exist since $f(x) = \dfrac{1}{(x-2)^3}$ has an infinite discontinuity at $x = 2$.

52. Assume $a > 0$. Let $u = a^2 - x^2$, so $du = -2x\,dx$. When $x = 0$, $u = a^2$; when $x = a$, $u = 0$. Thus,

$$\int_0^a x\sqrt{a^2 - x^2}\,dx = \int_{a^2}^0 u^{1/2}\left(-\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^{a^2} u^{1/2}\,du = \tfrac{1}{2}\cdot\left[\tfrac{2}{3}u^{3/2}\right]_0^{a^2} = \tfrac{1}{3}a^3.$$

53. Let $u = x^2 + a^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. When $x = 0$, $u = a^2$; when $x = a$, $u = 2a^2$. Thus,

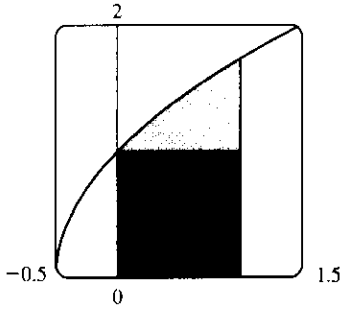
$$\int_0^a x\sqrt{x^2 + a^2}\,dx = \int_{a^2}^{2a^2} u^{1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_{a^2}^{2a^2} = \left[\tfrac{1}{3}u^{3/2}\right]_{a^2}^{2a^2}$$

$$= \tfrac{1}{3}\left[(2a^2)^{3/2} - (a^2)^{3/2}\right] = \tfrac{1}{3}\left(2\sqrt{2} - 1\right)a^3$$

54. $\int_{-a}^a x\sqrt{x^2 + a^2}\,dx = 0$ by Theorem 6(b), since $f(x) = x\sqrt{x^2 + a^2}$ is an odd function.

55. From the graph, it appears that the area under the curve is about $1 + (\text{a little more than } \frac{1}{2}\cdot 1\cdot 0.7)$, or about 1.4. The exact area is given by $A = \int_0^1 \sqrt{2x+1}\,dx$. Let $u = 2x + 1$, so $du = 2\,dx$. The limits change to $2\cdot 0 + 1 = 1$ and $2\cdot 1 + 1 = 3$, and

$$A = \int_1^3 \sqrt{u}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_1^3 = \tfrac{1}{3}\left(3\sqrt{3} - 1\right) = \sqrt{3} - \tfrac{1}{3} \approx 1.399.$$

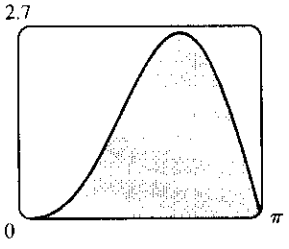


56. From the graph, it appears that the area under the curve is almost $\frac{1}{2}\cdot\pi\cdot 2.6$, or about 4. The exact area is given by

$$A = \int_0^{\pi}(2\sin x - \sin 2x)dx = -2\left[\cos x\right]_0^{\pi} - \int_0^{\pi}\sin 2x\,dx$$

$$= -2(-1 - 1) - 0 = 4$$

Note: $\int_0^{\pi}\sin 2x\,dx = 0$ since it is clear from the graph of $y = \sin 2x$ that $\int_{\pi/2}^{\pi}\sin 2x\,dx = -\int_0^{\pi/2}\sin 2x\,dx$.



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57. First write the integral as a sum of two integrals:

$I = \int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx = I_1 + I_2 = \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \int_{-2}^{2} 3\sqrt{4-x^2}\,dx$. $I_1 = 0$ by Theorem 6(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. We interpret I_2 as three times the area of a semicircle with radius 2, so $I = 0 + 3 \cdot \frac{1}{2}(\pi \cdot 2^2) = 6\pi$.

58. Let $u = x^2$. Then $du = 2x\,dx$ and the limits are unchanged ($0^2 = 0$ and $1^2 = 1$), so $I = \int_0^1 x\sqrt{1-x^4}\,dx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$. But this integral can be interpreted as the area of a quarter-circle with radius 1. So $I = \frac{1}{2} \cdot \frac{1}{4}(\pi \cdot 1^2) = \frac{1}{8}\pi$.

59. The volume of inhaled air in the lungs at time t is

$$V(t) = \int_0^t f(u)\,du = \int_0^t \tfrac{1}{2}\sin\left(\tfrac{2\pi}{5}u\right)du = \int_0^{2\pi t/5} \tfrac{1}{2}\sin v\left(\tfrac{5}{2\pi}\,dv\right) \quad [\text{substitute } v = \tfrac{2\pi}{5}u,\ dv = \tfrac{2\pi}{5}\,du]$$

$$= \tfrac{5}{4\pi}\left[-\cos v\right]_0^{2\pi t/5} = \tfrac{5}{4\pi}\left[-\cos\left(\tfrac{2\pi}{5}t\right) + 1\right] = \tfrac{5}{4\pi}\left[1 - \cos\left(\tfrac{2\pi}{5}t\right)\right] \text{ liters}$$

60. Number of calculators $= x(4) - x(2) = \int_2^4 5000\left[1 - 100(t+10)^{-2}\right]dt$

$$= 5000\left[t + 100(t+10)^{-1}\right]_2^4 = 5000\left[\left(4 + \tfrac{100}{14}\right) - \left(2 + \tfrac{100}{12}\right)\right] \approx 4048$$

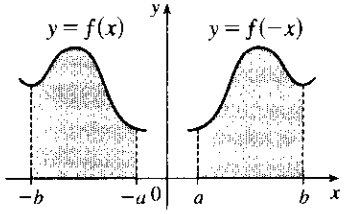
61. Let $u = 2x$. Then $du = 2\,dx$, so $\int_0^2 f(2x)\,dx = \int_0^4 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}(10) = 5$.

62. Let $u = x^2$. Then $du = 2x\,dx$, so $\int_0^3 x f\left(x^2\right)dx = \int_0^9 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}(4) = 2$.

63. (a) Let $u = -x$. Then $du = -dx$, so

$$\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u)\,du = \int_{-b}^{-a} f(x)\,dx$$

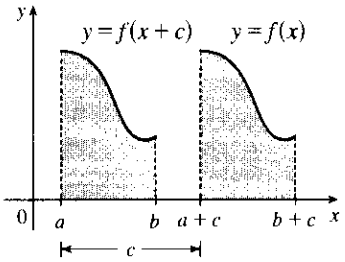
From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f, and the limits of integration, about the y-axis.



(b) Let $u = x + c$. Then $du = dx$, so

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du = \int_{a+c}^{b+c} f(x)\,dx$$

From the diagram, we see that the equality follows from the fact that we are translating the graph of f, and the limits of integration, by a distance c.



64. The area under the graph of $y = \sin\sqrt{x}$ from 0 to 4 is $A_1 = \int_0^4 \sin\sqrt{x}\,dx$. The area under the graph of $y = 2x\sin x$ from 0 to 2 is $A_2 = \int_0^2 2x\sin x\,dx$ $[u = x^2,\ du = 2xdx,\ \sqrt{u} = x \text{ for } 0 \le x \le 2]$ $= \int_0^4 \sin\sqrt{u}\,du$. Since the integration variable is immaterial, $A_1 = A_2$.

65. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

$$\int_0^1 x^a(1-x)^b\,dx = \int_1^0 (1-u)^a u^b(-du) = \int_0^1 u^b(1-u)^a\,du = \int_0^1 x^b(1-x)^a\,dx.$$

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66. Let $u = \pi - x$. Then $du = -dx$. When $x = \pi$, $u = 0$ and when $x = 0$, $u = \pi$. So

$$\int_0^\pi x f(\sin x)\,dx = -\int_\pi^0 (\pi - u) f(\sin(\pi - u))\,du = \int_0^\pi (\pi - u) f(\sin u)\,du$$
$$= \pi \int_0^\pi f(\sin u)\,du - \int_0^\pi u f(\sin u)\,du = \pi \int_0^\pi f(\sin x)\,dx - \int_0^\pi x f(\sin x)\,dx$$

$$\Rightarrow \quad 2\int_0^\pi x f(\sin x)\,dx = \pi \int_0^\pi f(\sin x)\,dx \quad \Rightarrow \quad \int_0^\pi x f(\sin x)\,dx = \tfrac{\pi}{2} \int_0^\pi f(\sin x)\,dx.$$

67. Let $u = 5 - 3x$. Then $du = -3\,dx$ and $dx = -\frac{1}{3}\,du$, so

$$\int \frac{dx}{5 - 3x} = \int \frac{1}{u}\left(-\tfrac{1}{3}\,du\right) = -\tfrac{1}{3}\ln|u| + C = -\tfrac{1}{3}\ln|5 - 3x| + C.$$

68. Let $u = x^2 + 1$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int \frac{x}{x^2 + 1}\,dx = \int \frac{\frac{1}{2}\,du}{u} = \tfrac{1}{2}\ln|u| + C = \tfrac{1}{2}\ln\left|x^2 + 1\right| + C = \tfrac{1}{2}\ln\left(x^2 + 1\right) + C \quad [\text{since } x^2 + 1 > 0]$$

or $\ln(x^2 + 1)^{1/2} + C = \ln\sqrt{x^2 + 1} + C$.

69. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\ln x)^3 + C$.

70. Let $u = \tan^{-1} x$. Then $du = \dfrac{dx}{1 + x^2}$, so $\displaystyle\int \frac{\tan^{-1} x}{1 + x^2}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\tan^{-1} x)^2}{2} + C$.

71. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1 + e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1 + e^x)^{3/2} + C$.

Or: Let $u = \sqrt{1 + e^x}$. Then $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$, so

$\int e^x\sqrt{1 + e^x}\,dx = \int u \cdot 2u\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(1 + e^x)^{3/2} + C$.

72. Let $u = \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so

$\int e^{\cos t}\sin t\,dt = \int e^u\,(-du) = -e^u + C = -e^{\cos t} + C$.

73. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$.

74. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x + 1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x + 1) + C$.

75. $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$\int \cot x\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin x| + C$.

76. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\sin x\,dx = -du$, so

$$\int \frac{\sin x}{1 + \cos^2 x}\,dx = \int \frac{-du}{1 + u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

77. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so

$$\int \frac{1 + x}{1 + x^2}\,dx = \int \frac{1}{1 + x^2}\,dx + \int \frac{x}{1 + x^2}\,dx = \tan^{-1} x + \int \frac{\frac{1}{2}du}{u} = \tan^{-1} x + \tfrac{1}{2}\ln|u| + C$$
$$= \tan^{-1} x + \tfrac{1}{2}\ln\left|1 + x^2\right| + C = \tan^{-1} x + \tfrac{1}{2}\ln\left(1 + x^2\right) + C \ [\text{since } 1 + x^2 > 0].$$

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78. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{1+x^4}\,dx = \int \frac{\frac{1}{2}\,du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(x^2) + C$.

79. Let $u = 1/x$, so $du = -1/x^2\,dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/2} e^u\,(-du) = -\left[e^u\right]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

80. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\left[e^u\right]_0^{-1} = -\tfrac{1}{2}\left(e^{-1} - e^0\right) = \tfrac{1}{2}(1 - 1/e).$$

81. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2 - 1) = 2.$$

82. Let $u = \sin^{-1}x$, so $du = \dfrac{dx}{\sqrt{1-x^2}}$. When $x = 0$, $u = 0$; when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Thus,

$$\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \left[\frac{u^2}{2}\right]_0^{\pi/6} = \frac{\pi^2}{72}.$$

83. $\dfrac{x\sin x}{1+\cos^2 x} = x \cdot \dfrac{\sin x}{2 - \sin^2 x} = xf(\sin x)$, where $f(t) = \dfrac{t}{2-t^2}$. By Exercise 66,

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u = \cos x$. Then $du = -\sin x\,dx$. When $x = \pi$, $u = -1$ and when $x = 0$, $u = 1$. So

$$\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\int_1^{-1} \frac{du}{1+u^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{du}{1+u^2} = \frac{\pi}{2}\left[\tan^{-1}u\right]_{-1}^{1}$$

$$= \frac{\pi}{2}\left[\tan^{-1}1 - \tan^{-1}(-1)\right] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$

5 Review

CONCEPT CHECK

1. (a) $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is an expression for a Riemann sum of a function f.
x_i^* is a point in the ith subinterval $[x_{i-1}, x_i]$ and Δx is the length of the subintervals.

(b) See Figure 1 in Section 5.2.

(c) In Section 5.2, see Figure 3 and the paragraph beside it.

2. (a) See Definition 5.2.2.

(b) See Figure 2 in Section 5.2.

(c) In Section 5.2, see Figure 4 and the paragraph above it.

3. See the Fundamental Theorem of Calculus after Example 8 in Section 5.3.

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4. (a) See the Net Change Theorem after Example 5 in Section 5.4.

(b) $\int_{t_1}^{t_2} r(t)\,dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2.

5. (a) $\int_{60}^{120} v(t)\,dt$ represents the change in position of the particle from $t = 60$ to $t = 120$ seconds.

(b) $\int_{60}^{120} |v(t)|\,dt$ represents the total distance traveled by the particle from $t = 60$ to 120 seconds.

(c) $\int_{60}^{120} a(t)\,dt$ represents the change in the velocity of the particle from $t = 60$ to $t = 120$ seconds.

6. (a) $\int f(x)\,dx$ is the family of functions $\{F \mid F' = f\}$. Any two such functions differ by a constant.

(b) The connection is given by the Evaluation Theorem: $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$ if f is continuous.

7. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it at the end of Section 5.3.

8. See the Substitution Rule (5.5.4). This says that it is permissible to operate with the dx after an integral sign as if it were a differential.

TRUE-FALSE QUIZ

1. True by Property 2 of the Integral in Section 5.2.

2. False. Try $a = 0$, $b = 2$, $f(x) = g(x) = 1$ as a counterexample.

3. True by Property 3 of the Integral in Section 5.2.

4. False. You can't take a variable outside the integral sign. For example, using $f(x) = 1$ on $[0, 1]$,

$$\int_0^1 x\,f(x)\,dx = \int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2} \text{ (a constant) while } x\int_0^1 1\,dx = x\,[x]_0^1 = x \cdot 1 = x \text{ (a variable).}$$

5. False. For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2}\,dx = \int_0^1 x\,dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2\,dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

6. True by the Net Change Theorem.

7. True by Comparison Property 7 of the Integral in Section 5.2.

8. False. For example, let $a = 0$, $b = 1$, $f(x) = 3$, $g(x) = x$. $f(x) > g(x)$ for each x in $(0, 1)$, but $f'(x) = 0 < 1 = g'(x)$ for $x \in (0, 1)$.

9. True. The integrand is an odd function that is continuous on $[-1, 1]$, so the result follows from Theorem 5.5.6(b).

10. True.
$$\begin{aligned}\int_{-5}^{5}(ax^2 + bx + c)\,dx &= \int_{-5}^{5}(ax^2 + c)\,dx + \int_{-5}^{5} bx\,dx \\ &= 2\int_0^5 (ax^2 + c)\,dx \text{ [by 5.5.6(a)]} \quad + 0 \text{ [by 5.5.6(b)]}\end{aligned}$$

11. False. The function $f(x) = 1/x^4$ is not bounded on the interval $[-2, 1]$. It has an infinite discontinuity at $x = 0$, so it is not integrable on the interval. (If the integral were to exist, a positive value would be expected, by Comparison Property 6 of Integrals.)

12. False. See the remarks and Figure 4 before Example 1 in Section 5.2, and notice that $y = x - x^3 < 0$ for $1 < x \le 2$.

13. False. For example, the function $y = |x|$ is continuous on $\mathbb{R}$, but has no derivative at $x = 0$.

14. True by FTC1.

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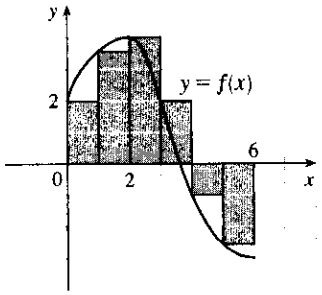
EXERCISES

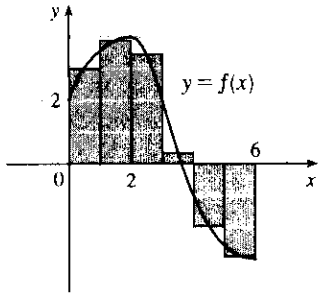
1. (a)



$$L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$
$$= f(x_0)\cdot 1 + f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$$
$$\approx 2 + 3.5 + 4 + 2 + (-1) + (-2.5) = 8$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

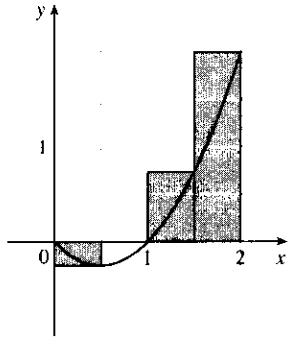
(b)



$$M_6 = \sum_{i=1}^{6} f(\bar{x}_i)\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$
$$= f(\bar{x}_1)\cdot 1 + f(\bar{x}_2)\cdot 1 + f(\bar{x}_3)\cdot 1 + f(\bar{x}_4)\cdot 1 + f(\bar{x}_5)\cdot 1 + f(\bar{x}_6)\cdot 1$$
$$= f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)$$
$$\approx 3 + 3.9 + 3.4 + 0.3 + (-2) + (-2.9) = 5.7$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

2. (a)



$f(x) = x^2 - x$ and $\Delta x = \frac{2-0}{4} = 0.5 \Rightarrow$

$$R_4 = 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) + 0.5f(2)$$
$$= 0.5(-0.25 + 0 + 0.75 + 2) = 1.25$$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the area of the rectangle below the x-axis. (The second rectangle vanishes.)

(b) $\int_0^2 (x^2 - x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \quad [\Delta x = 2/n \text{ and } x_i = 2i/n]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right]$$
$$= \lim_{n\to\infty} \left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2}\cdot\frac{n(n+1)}{2}\right]$$
$$= \lim_{n\to\infty} \left[\frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} - 2\cdot\frac{n+1}{n}\right]$$
$$= \lim_{n\to\infty} \left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 2\left(1+\frac{1}{n}\right)\right] = \tfrac{4}{3}\cdot 1\cdot 2 - 2\cdot 1 = \tfrac{2}{3}$$

(c) $\int_0^2 (x^2 - x)\,dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^2 = \left(\frac{8}{3} - 2\right) = \frac{2}{3}$

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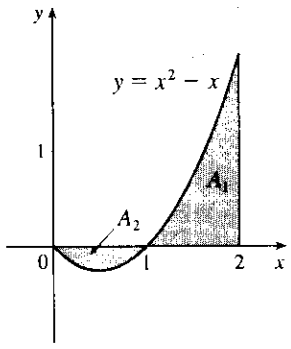
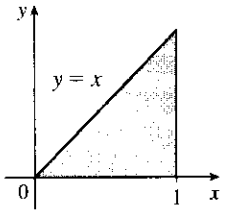
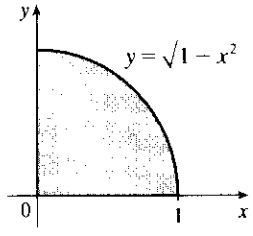
(d) 

$\int_0^2 (x^2 - x)\,dx = A_1 - A_2$, where A_1 and A_2 are the areas shown in the diagram.

3. $\int_0^1 \left(x + \sqrt{1-x^2}\,\right)dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1-x^2}\,dx = I_1 + I_2$.





I_1 can be interpreted as the area of the triangle shown in the figure and I_2 can be interpreted as the area of the quarter-circle. Area $= \frac{1}{2}(1)(1) + \frac{1}{4}(\pi)(1)^2 = \frac{1}{2} + \frac{\pi}{4}$.

4. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sin x_i\,\Delta x = \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -(-1) - (-1) = 2$.

5. $\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx \quad \Rightarrow \quad 10 = 7 + \int_4^6 f(x)\,dx \quad \Rightarrow \quad \int_4^6 f(x)\,dx = 10 - 7 = 3$

6. (a) $\int_1^5 \left(x + 2x^5\right)dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x \qquad \left[\Delta x = \dfrac{5-1}{n} = \dfrac{4}{n},\ x_i = 1 + \dfrac{4i}{n}\right]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + \frac{4i}{n}\right) + 2\left(1 + \frac{4i}{n}\right)^5\right] \cdot \frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{1305n^4 + 3126n^3 + 2080n^2 - 256}{n^3} \cdot \frac{4}{n} = 5220$$

(b) $\int_1^5 \left(x + 2x^5\right)dx = \left[\frac{1}{2}x^2 + \frac{2}{6}x^6\right]_1^5 = \left(\frac{25}{2} + \frac{15{,}625}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) = 12 + 5208 = 5220$

7. First note that either a or b must be the graph of $\int_0^x f(t)\,dt$, since $\int_0^0 f(t)\,dt = 0$, and $c(0) \neq 0$. Now notice that $b > 0$ when c is increasing, and that $c > 0$ when a is increasing. It follows that c is the graph of $f(x)$, b is the graph of $f'(x)$, and a is the graph of $\int_0^x f(t)\,dt$.

8. (a) By FTC2, we have $\displaystyle\int_0^{\pi/2} \frac{d}{dx}\left(\sin\frac{x}{2}\cos\frac{x}{3}\right)dx = \left[\sin\frac{x}{2}\cos\frac{x}{3}\right]_0^{\pi/2} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - 0 \cdot 1 = \frac{\sqrt{6}}{4}$.

(b) $\dfrac{d}{dx}\displaystyle\int_0^{\pi/2} \sin\frac{x}{2}\cos\frac{x}{3}\,dx = 0$, since the definite integral is a constant.

(c) $\dfrac{d}{dx}\displaystyle\int_x^{\pi/2} \sin\frac{t}{2}\cos\frac{t}{3}\,dt = \frac{d}{dx}\left(-\int_{\pi/2}^{x} \sin\frac{t}{2}\cos\frac{t}{3}\,dt\right) = -\frac{d}{dx}\int_{\pi/2}^{x} \sin\frac{t}{2}\cos\frac{t}{2}\,dt = -\sin\frac{x}{2}\cos\frac{x}{3}$, by FTC1.

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9. $\int_1^2 (8x^3 + 3x^2)\,dx = \left[8 \cdot \frac{1}{4}x^4 + 3 \cdot \frac{1}{3}x^3\right]_1^2 = \left[2x^4 + x^3\right]_1^2 = (2 \cdot 2^4 + 2^3) - (2 + 1) = 40 - 3 = 37$

10. $\displaystyle\int_0^T (x^4 - 8x + 7)\,dx = \left[\tfrac{1}{5}x^5 - 4x^2 + 7x\right]_0^T = \left(\tfrac{1}{5}T^5 - 4T^2 + 7T\right) - 0 = \tfrac{1}{5}T^5 - 4T^2 + 7T$

11. $\int_0^1 (1 - x^9)\,dx = \left[x - \frac{1}{10}x^{10}\right]_0^1 = (1 - \frac{1}{10}) - 0 = \frac{9}{10}$

12. Let $u = 1 - x$, so $du = -dx$ and $dx = -du$. When $x = 0$, $u = 1$; when $x = 1$, $u = 0$. Thus,
$\int_0^1 (1 - x)^9\,dx = \int_1^0 u^9(-du) = \int_0^1 u^9\,du = \frac{1}{10}\left[u^{10}\right]_0^1 = \frac{1}{10}(1 - 0) = \frac{1}{10}$.

13. $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du = \int_1^9 (u^{-1/2} - 2u)\,du = \left[2u^{1/2} - u^2\right]_1^9 = (6 - 81) - (2 - 1) = -76$

14. $\int_0^1 (\sqrt[4]{u} + 1)^2\,du = \int_0^1 (u^{1/2} + 2u^{1/4} + 1)\,du = \left[\frac{2}{3}u^{3/2} + \frac{8}{5}u^{5/4} + u\right]_0^1 = (\frac{2}{3} + \frac{8}{5} + 1) - 0 = \frac{49}{15}$

15. Let $u = y^2 + 1$, so $du = 2y\,dy$ and $y\,dy = \frac{1}{2}\,du$. When $y = 0$, $u = 1$; when $y = 1$, $u = 2$. Thus,
$\int_0^1 y(y^2 + 1)^5\,dy = \int_1^2 u^5(\frac{1}{2}\,du) = \frac{1}{2}\left[\frac{1}{6}u^6\right]_1^2 = \frac{1}{12}(64 - 1) = \frac{63}{12} = \frac{21}{4}$.

16. Let $u = 1 + y^3$, so $du = 3y^2\,dy$ and $y^2\,dy = \frac{1}{3}\,du$. When $y = 0$, $u = 1$; when $y = 2$, $u = 9$. Thus,
$\int_0^2 y^2\sqrt{1 + y^3}\,dy = \int_1^9 u^{1/2}(\frac{1}{3}\,du) = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{2}{9}(27 - 1) = \frac{52}{9}$.

17. $\displaystyle\int_1^5 \frac{dt}{(t-4)^2}$ does not exist because the function $f(t) = \frac{1}{(t-4)^2}$ has an infinite discontinuity at $t = 4$; that is, f is discontinuous on the interval $[1, 5]$.

18. Let $u = 3\pi t$, so $du = 3\pi\,dt$. When $t = 0$, $u = 1$; when $t = 1$, $u = 3\pi$. Thus,
$\displaystyle\int_0^1 \sin(3\pi t)\,dt = \int_0^{3\pi} \sin u\left(\frac{1}{3\pi}\,du\right) = \frac{1}{3\pi}\left[-\cos u\right]_0^{3\pi} = -\frac{1}{3\pi}(-1 - 1) = \frac{2}{3\pi}$.

19. Let $u = v^3$, so $du = 3v^2\,dv$. When $v = 0$, $u = 0$; when $v = 1$, $u = 1$. Thus,
$\int_0^1 v^2\cos(v^3)\,dv = \int_0^1 \cos u\,(\frac{1}{3}\,du) = \frac{1}{3}\left[\sin u\right]_0^1 = \frac{1}{3}(\sin 1 - 0) = \frac{1}{3}\sin 1$.

20. $\displaystyle\int_{-1}^1 \frac{\sin x}{1 + x^2}\,dx = 0$ by Theorem 5.5.6(b), since $f(x) = \frac{\sin x}{1 + x^2}$ is an odd function.

21. Let $u = x^2 + 4x$. Then $du = (2x + 4)\,dx = 2(x + 2)\,dx$, so
$\displaystyle\int \frac{x + 2}{\sqrt{x^2 + 4x}}\,dx = \int u^{-1/2}(\tfrac{1}{2}\,du) = \tfrac{1}{2} \cdot 2u^{1/2} + C = \sqrt{u} + C = \sqrt{x^2 + 4x} + C$.

22. Let $u = 3t$. Then $du = 3\,dt$, so $\int \csc^2 3t\,dt = \int \csc^2 u\,(\frac{1}{3}\,du) = \frac{1}{3}(-\cot u) + C = -\frac{1}{3}\cot 3t + C$.

23. Let $u = \sin \pi t$. Then $du = \pi\,\cos \pi t\,dt$, so $\int \sin \pi t\,\cos \pi t\,dt = \int u(\frac{1}{\pi}\,du) = \frac{1}{\pi} \cdot \frac{1}{2}u^2 + C = \frac{1}{2\pi}(\sin \pi t)^2 + C$.

24. Let $u = \cos x$. Then $du = -\sin x\,dx$, so
$\int \sin x\,\cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C$.

25. Let $u = 2\theta$. Then $du = 2\,d\theta$, so

$$\int_0^{\pi/8} \sec 2\theta \tan 2\theta\,d\theta = \int_0^{\pi/4} \sec u \tan u\,(\tfrac{1}{2}\,du) = \tfrac{1}{2}\left[\sec u\right]_0^{\pi/4} = \tfrac{1}{2}\left(\sec \tfrac{\pi}{4} - \sec 0\right)$$

$$= \tfrac{1}{2}\left(\sqrt{2} - 1\right) = \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}$$

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26. Let $u = 1 + \tan t$, so $du = \sec^2 t\, dt$. When $t = 0$, $u = 1$; when $t = \frac{\pi}{4}$, $u = 2$. Thus,

$$\int_0^{\pi/4} (1+\tan t)^3 \sec^2 t\, dt = \int_1^2 u^3\, du = \left[\tfrac{1}{4}u^4\right]_1^2 = \tfrac{1}{4}(2^4 - 1^4) = \tfrac{1}{4}(16-1) = \tfrac{15}{4}.$$

27. Since $x^2 - 4 < 0$ for $0 \le x < 2$ and $x^2 - 4 > 0$ for $2 < x \le 3$, we have $|x^2 - 4| = -(x^2 - 4) = 4 - x^2$ for $0 \le x < 2$ and $|x^2 - 4| = x^2 - 4$ for $2 < x \le 3$. Thus,

$$\begin{aligned}\int_0^3 |x^2 - 4|\, dx &= \int_0^2 (4 - x^2)\, dx + \int_2^3 (x^2 - 4)\, dx = \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^3 \\ &= \left(8 - \tfrac{8}{3}\right) - 0 + (9 - 12) - \left(\tfrac{8}{3} - 8\right) = \tfrac{16}{3} - 3 + \tfrac{16}{3} = \tfrac{32}{3} - \tfrac{9}{3} = \tfrac{23}{3}\end{aligned}$$

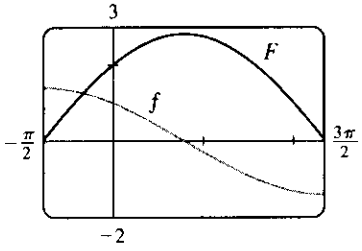
28. Since $\sqrt{x} - 1 < 0$ for $0 \le x < 1$ and $\sqrt{x} - 1 > 0$ for $1 < x \le 4$, we have $|\sqrt{x} - 1| = -(\sqrt{x} - 1) = 1 - \sqrt{x}$ for $0 \le x < 1$ and $|\sqrt{x} - 1| = \sqrt{x} - 1$ for $1 < x \le 4$. Thus,

$$\begin{aligned}\int_0^4 |\sqrt{x} - 1|\, dx &= \int_0^1 (1 - \sqrt{x})\, dx + \int_1^4 (\sqrt{x} - 1)\, dx = \left[x - \tfrac{2}{3}x^{3/2}\right]_0^1 + \left[\tfrac{2}{3}x^{3/2} - x\right]_1^4 \\ &= \left(1 - \tfrac{2}{3}\right) - 0 + \left(\tfrac{16}{3} - 4\right) - \left(\tfrac{2}{3} - 1\right) = \tfrac{1}{3} + \tfrac{16}{3} - 4 + \tfrac{1}{3} = 6 - 4 = 2\end{aligned}$$

In Exercises 29 and 30, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

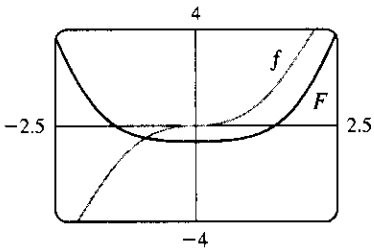
29. Let $u = 1 + \sin x$. Then $du = \cos x\, dx$, so

$$\int \frac{\cos x\, dx}{\sqrt{1+\sin x}} = \int u^{-1/2}\, du = 2u^{1/2} + C = 2\sqrt{1+\sin x} + C.$$



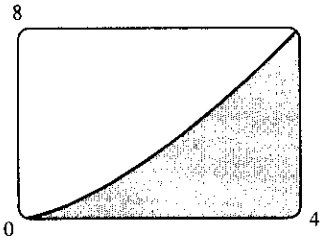
30. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x\, dx = \frac{1}{2}\, du$, so

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}}\, dx &= \int \frac{(u-1)}{\sqrt{u}} \left(\tfrac{1}{2}\, du\right) = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right) du \\ &= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C \\ &= \tfrac{1}{3}\left(x^2+1\right)^{3/2} - \left(x^2+1\right)^{1/2} + C \\ &= \tfrac{1}{3}\left(x^2+1\right)^{1/2}\left[\left(x^2+1\right) - 3\right] + C \\ &= \tfrac{1}{3}\sqrt{x^2+1}\left(x^2 - 2\right) + C\end{aligned}$$



31. From the graph, it appears that the area under the curve $y = x\sqrt{x}$ between $x = 0$ and $x = 4$ is somewhat less than half the area of an 8×4 rectangle, so perhaps about 13 or 14. To find the exact value, we evaluate

$$\int_0^4 x\sqrt{x}\, dx = \int_0^4 x^{3/2}\, dx = \left[\tfrac{2}{5}x^{5/2}\right]_0^4 = \tfrac{2}{5}(4)^{5/2} = \tfrac{64}{5} = 12.8.$$



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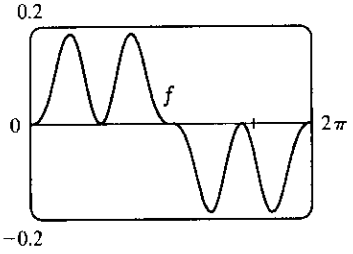
32. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x \sin^3 x\,dx$ is equal to 0.



To evaluate the integral, we write the integral as

$I = \int_0^{2\pi} \cos^2 x\,(1-\cos^2 x)\sin x\,dx$ and let $u = \cos x \;\Rightarrow$

$du = -\sin x\,dx$. Thus, $I = \int_1^1 u^2(1-u^2)(-du) = 0$.

33. By FTC1, $F(x) = \int_1^x \sqrt{1+t^4}\,dt \;\Rightarrow\; F'(x) = \sqrt{1+x^4}$.

34. $F(x) = \int_\pi^x \tan(s^2)\,ds \;\Rightarrow\; F'(x) = \tan(x^2)$

35. $g(x) = \displaystyle\int_0^{x^3} \frac{t\,dt}{\sqrt{1+t^3}}$. Let $y = g(u)$ and $u = x^3$.

Then $g'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{u}{\sqrt{1+u^3}}3x^2 = \dfrac{x^3}{\sqrt{1+x^9}}3x^2 = \dfrac{3x^5}{\sqrt{1+x^9}}$.

36. Let $u = \cos x$. Then $\dfrac{du}{dx} = -\sin x$. Also, $\dfrac{dg}{dx} = \dfrac{dg}{du}\dfrac{du}{dx}$, so

$$\frac{d}{dx}\int_1^{\cos x} \sqrt[3]{1-t^2}\,dt = \frac{d}{du}\int_1^u \sqrt[3]{1-t^2}\,dt \cdot \frac{du}{dx} = \sqrt[3]{1-u^2}\,(-\sin x)$$

$$= \sqrt[3]{1-\cos^2 x}\,(-\sin x) = -\sin x\,\sqrt[3]{\sin^2 x} = -(\sin x)^{5/3}$$

37. $y = \displaystyle\int_{\sqrt{x}}^{x} \frac{\cos\theta}{\theta}\,d\theta = \int_1^x \frac{\cos\theta}{\theta}\,d\theta + \int_{\sqrt{x}}^1 \frac{\cos\theta}{\theta}\,d\theta = \int_1^x \frac{\cos\theta}{\theta}\,d\theta - \int_1^{\sqrt{x}} \frac{\cos\theta}{\theta}\,d\theta \;\Rightarrow$

$y' = \dfrac{\cos x}{x} - \dfrac{\cos\sqrt{x}}{\sqrt{x}}\dfrac{1}{2\sqrt{x}} = \dfrac{2\cos x - \cos\sqrt{x}}{2x}$

38. $y = \int_{2x}^{3x+1} \sin(t^4)\,dt = \int_{2x}^0 \sin(t^4)\,dt + \int_0^{3x+1} \sin(t^4)\,dt = \int_0^{3x+1} \sin(t^4)\,dt - \int_0^{2x} \sin(t^4)\,dt \;\Rightarrow$

$y' = \sin\left[(3x+1)^4\right]\cdot\dfrac{d}{dx}(3x+1) - \sin\left[(2x)^4\right]\cdot\dfrac{d}{dx}(2x) = 3\sin\left[(3x+1)^4\right] - 2\sin\left[(2x)^4\right]$

39. If $1 \le x \le 3$, then $\sqrt{1^2+3} \le \sqrt{x^2+3} \le \sqrt{3^2+3} \;\Rightarrow\; 2 \le \sqrt{x^2+3} \le 2\sqrt{3}$, so

$2(3-1) \le \int_1^3 \sqrt{x^2+3}\,dx \le 2\sqrt{3}(3-1)$; that is, $4 \le \int_1^3 \sqrt{x^2+3}\,dx \le 4\sqrt{3}$.

40. If $3 \le x \le 5$, then $4 \le x+1 \le 6$ and $\dfrac{1}{6} \le \dfrac{1}{x+1} \le \dfrac{1}{4}$, so $\dfrac{1}{6}(5-3) \le \displaystyle\int_3^5 \frac{1}{x+1}\,dx \le \frac{1}{4}(5-3)$; that is,

$\dfrac{1}{3} \le \displaystyle\int_3^5 \frac{1}{x+1}\,dx \le \frac{1}{2}$.

41. $0 \le x \le 1 \;\Rightarrow\; 0 \le \cos x \le 1 \;\Rightarrow\; x^2\cos x \le x^2 \;\Rightarrow$

$\int_0^1 x^2\cos x\,dx \le \int_0^1 x^2\,dx = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}$ [Property 7].

42. On the interval $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$, x is increasing and $\sin x$ is decreasing, so $\dfrac{\sin x}{x}$ is decreasing. Therefore, the largest value of $\dfrac{\sin x}{x}$ on $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$ is $\dfrac{\sin(\pi/4)}{\pi/4} = \dfrac{\sqrt{2}/2}{\pi/4} = \dfrac{2\sqrt{2}}{\pi}$. By Property 8 with $M = \dfrac{2\sqrt{2}}{\pi}$ we get

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \le \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

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43. Let $f(x) = \sqrt{1+x^3}$ on $[0,1]$. The Midpoint Rule with $n = 5$ gives

$$\begin{aligned}\int_0^1 \sqrt{1+x^3}\,dx &\approx \tfrac{1}{5}[f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9)]\\ &= \tfrac{1}{5}\left[\sqrt{1+(0.1)^3}+\sqrt{1+(0.3)^3}+\cdots+\sqrt{1+(0.9)^3}\right] \approx 1.110\end{aligned}$$

44. (a) displacement $= \int_0^5 (t^2-t)\,dt = \left[\frac{1}{3}t^3-\frac{1}{2}t^2\right]_0^5 = \frac{125}{3}-\frac{25}{2} = \frac{175}{6} = 29.1\overline{6}$ meters

(b) distance traveled $= \int_0^5 |t^2-t|\,dt = \int_0^5 |t(t-1)|\,dt = \int_0^1 (t-t^2)\,dt + \int_1^5 (t^2-t)\,dt$

$$\begin{aligned}&= \left[\tfrac{1}{2}t^2-\tfrac{1}{3}t^3\right]_0^1 + \left[\tfrac{1}{3}t^3-\tfrac{1}{2}t^2\right]_1^5\\ &= \tfrac{1}{2}-\tfrac{1}{3}-0+\left(\tfrac{125}{3}-\tfrac{25}{2}\right)-\left(\tfrac{1}{3}-\tfrac{1}{2}\right) = \tfrac{177}{6} = 29.5 \text{ meters}\end{aligned}$$

45. Note that $r(t) = b'(t)$, where $b(t) =$ the number of barrels of oil consumed up to time t. So, by the Net Change Theorem, $\int_0^3 r(t)\,dt = b(3)-b(0)$ represents the number of barrels of oil consumed from Jan. 1, 2000, through Jan. 1, 2003.

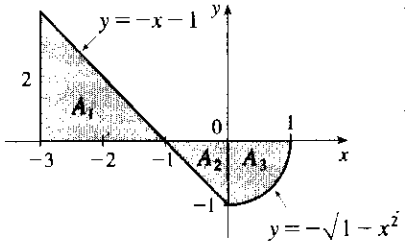
46. Distance covered $= \int_0^{5.0} v(t)\,dt \approx M_5 = \frac{5.0-0}{5}[v(0.5)+v(1.5)+v(2.5)+v(3.5)+v(4.5)]$

$$= 1(4.67+8.86+10.22+10.67+10.81) = 45.23 \text{ m}$$

47. We use the Midpoint Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$. The increase in the bee population was

$$\begin{aligned}\int_0^{24} r(t)\,dt &\approx M_6 = 4[r(2)+r(6)+r(10)+r(14)+r(18)+r(22)]\\ &\approx 4[50+1000+7000+8550+1350+150]\\ &= 4(18{,}100) = 72{,}400\end{aligned}$$

48. $A_1 = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$, $A_2 = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$, and since $y = -\sqrt{1-x^2}$ for $0 \le x \le 1$ represents a quarter-circle with radius 1, $A_3 = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$. So $\int_{-3}^1 f(x)\,dx = A_1 - A_2 - A_3 = 2-\frac{1}{2}-\frac{\pi}{4} = \frac{1}{4}(6-\pi)$.



49. By the Fundamental Theorem of Calculus, we know that $F(x) = \int_a^x t^2\sin(t^2)\,dt$ is an antiderivative of $f(x) = x^2\sin(x^2)$. This integral cannot be expressed in any simpler form. Since $\int_a^a f\,dt = 0$ for any a, we can take $a = 1$, and then $F(1) = 0$, as required. So $F(x) = \int_1^x t^2\sin(t^2)\,dt$ is the desired function.

50. (a) C is increasing on those intervals where C' is positive. By the Fundamental Theorem of Calculus, $C'(x) = \dfrac{d}{dx}\left[\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt\right] = \cos\left(\frac{\pi}{2}x^2\right)$. This is positive when $\frac{\pi}{2}x^2$ is in the interval $\left(\left(2n-\frac{1}{2}\right)\pi, \left(2n+\frac{1}{2}\right)\pi\right)$, n any integer. This implies that $\left(2n-\frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n+\frac{1}{2}\right)\pi \;\Leftrightarrow$ $0 \le |x| \le 1$ or $\sqrt{4n-1} < |x| < \sqrt{4n+1}$, n any positive integer. So C is increasing on the intervals $[-1,1]$, $\left[\sqrt{3},\sqrt{5}\right]$, $\left[-\sqrt{5},-\sqrt{3}\right]$, $\left[\sqrt{7},3\right]$, $\left[-3,-\sqrt{7}\right]$, ...

(b) C is concave upward on those intervals where $C'' > 0$. We differentiate C' to find C'': $C'(x) = \cos\left(\frac{\pi}{2}x^2\right)$ $\Rightarrow$ $C''(x) = -\sin\left(\frac{\pi}{2}x^2\right)\left(\frac{\pi}{2}\cdot 2x\right) = -\pi x\sin\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, this is positive where

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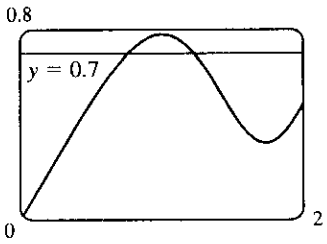
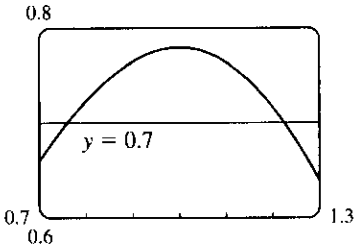
$(2n-1)\pi < \frac{\pi}{2}x^2 < 2n\pi$, n any positive integer $\Leftrightarrow$ $\sqrt{2(2n-1)} < x < 2\sqrt{n}$, n any positive integer.
Since there is a factor of $-x$ in C'', the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{2(2n+1)}, -2\sqrt{n}\right)$, n any nonnegative integer. That is, C is concave upward on $(-\sqrt{2}, 0)$, $(\sqrt{2}, 2)$, $(-\sqrt{6}, -2)$, $(\sqrt{6}, 2\sqrt{2})$, ...

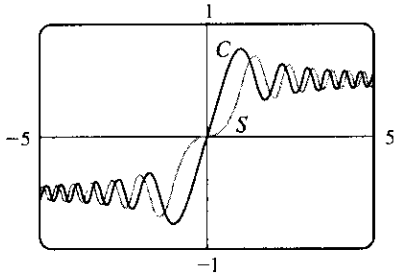
(c)





From the graphs, we can determine that $\int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt = 0.7$ at $x \approx 0.76$ and $x \approx 1.22$.

(d)



The graphs of $S(x)$ and $C(x)$ have similar shapes, except that S's flattens out near the origin, while C's does not. Note that for $x > 0$, C is increasing where S is concave up, and C is decreasing where S is concave down. Similarly, S is increasing where C is concave down, and S is decreasing where C is concave up. For $x < 0$, these relationships are reversed; that is, C is increasing where S is concave down, and S is increasing where C is concave up. See Example 5.3.3 and Exercise 5.3.49 for a discussion of $S(x)$.

51. $\displaystyle\int_0^x f(t)\,dt = x\sin x + \int_0^x \frac{f(t)}{1+t^2}\,dt \;\Rightarrow\; f(x) = x\cos x + \sin x + \frac{f(x)}{1+x^2}$ (by differentiation) $\Rightarrow$

$\displaystyle f(x)\left(1 - \frac{1}{1+x^2}\right) = x\cos x + \sin x \;\Rightarrow\; f(x)\left(\frac{x^2}{1+x^2}\right) = x\cos x + \sin x \;\Rightarrow$

$\displaystyle f(x) = \frac{1+x^2}{x^2}(x\cos x + \sin x)$

52. From the given equation, $\int_a^x f(t)\,dt = \sin x - \frac{1}{2}$. Differentiating both sides using FTC1 gives $f(x) = \cos x$. We put $x = a$ into the first equation to get $0 = \sin a - \frac{1}{2}$, so $a = \frac{\pi}{6}$ satisfies the given equation.

53. Let $u = f(x)$ and $du = f'(x)\,dx$. So $2\int_a^b f(x)f'(x)\,dx = 2\int_{f(a)}^{f(b)} u\,du = \left[u^2\right]_{f(a)}^{f(b)} = [f(b)]^2 - [f(a)]^2$.

54. Let $F(x) = \displaystyle\int_2^x \sqrt{1+t^3}\,dt$. Then $F'(2) = \lim_{h\to 0}\frac{F(2+h)-F(2)}{h} = \lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt$, and

$F'(x) = \sqrt{1+x^3}$, so $\displaystyle\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt = F'(2) = \sqrt{1+2^3} = \sqrt{9} = 3$.

55. Let $u = 1 - x$. Then $du = -dx$, so $\int_0^1 f(1-x)\,dx = \int_1^0 f(u)(-du) = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$.

56. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right] = \lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^9 = \int_0^1 x^9\,dx = \left[\frac{x^{10}}{10}\right]_0^1 = \frac{1}{10}$

The limit is based on Riemann sums using right endpoints and subintervals of equal length.

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PROBLEMS PLUS

1. Differentiating both sides of the equation $x \sin \pi x = \int_0^{x^2} f(t)\,dt$ (using FTC1 and the Chain Rule for the right side) gives $\sin \pi x + \pi x \cos \pi x = 2x f(x^2)$. Letting $x = 2$ so that $f(x^2) = f(4)$, we obtain $\sin 2\pi + 2\pi \cos 2\pi = 4f(4)$, so $f(4) = \frac{1}{4}(0 + 2\pi \cdot 1) = \frac{\pi}{2}$.

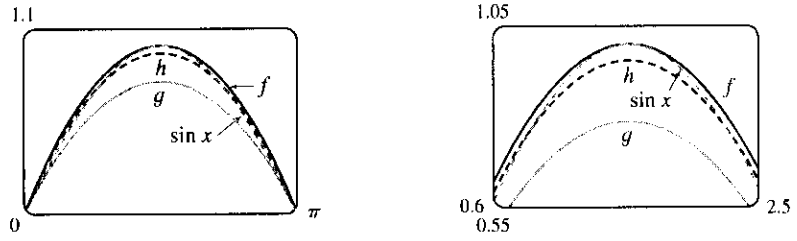
2. (a) Let $f(x) = ax^2 + bx + c$. $f(0) = f(\pi) = 0$, so we know that $f(0) = 0 \;\Rightarrow\; c = 0$, and $f(\pi) = 0 \;\Leftrightarrow\; a\pi^2 + b\pi = 0 \;\Leftrightarrow\; b = -a\pi$. So $f(x) = ax^2 - a\pi x$. Now we want the maximum value of $f(x)$ on $[0, \pi]$ to be the same as that of $\sin x$, that is, 1. So we find the value of x at which f has a maximum by differentiating and setting $f'(x) = 0 \;\Leftrightarrow\; 2ax - a\pi = 0 \;\Leftrightarrow\; x = \frac{\pi}{2}$. Now $f\left(\frac{\pi}{2}\right) = a\left(\frac{\pi}{2}\right)^2 - a\pi\left(\frac{\pi}{2}\right) = -\frac{1}{4}\pi^2 a$. We set this equal to 1, in order to find a: $-\frac{1}{4}\pi^2 a = 1 \;\Leftrightarrow\; a = -\frac{4}{\pi^2} \;\Rightarrow\; b = -a\pi = \frac{4}{\pi}$. Thus, the desired function is $f(x) = -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x$.

Alternate solution (without calculus): Use $f(x) = ax(x - \pi)$.

(b) Once again, $g(0) = g(\pi) = 0 \;\Rightarrow\; g(x) = ax^2 - a\pi x$. We want $g'(0) = \left[\frac{d}{dx}(\sin x)\right]_0 = \cos 0 = 1$. We calculated $g'(x)$ in part (a), so we set $g'(0) = 1 \;\Leftrightarrow\; 2a(0) - a\pi = 1 \;\Leftrightarrow\; a = -\frac{1}{\pi}$. We also want $g'(\pi) = \left[\frac{d}{dx}(\sin x)\right]_\pi = \cos \pi = -1$, so we check that $g'(\pi) = -1$ with $a = -\frac{1}{\pi}$: $g'(\pi) = 2\left(-\frac{1}{\pi}\right)\pi - \left(-\frac{1}{\pi}\right)\pi = -1$. Thus, the desired function is $g(x) = -\frac{1}{\pi}x^2 + x$.

(c) Again, $h(x) = ax^2 - a\pi x$. Now we want the area under the curves of $h(x)$ and $\sin x$ to be the same; that is, $\int_0^\pi h(x)\,dx = \int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = (1) - (-1) = 2$. We integrate h between 0 and π and set the result equal to 2: $\int_0^\pi \left(ax^2 - a\pi x\right)dx = \left[\frac{1}{3}ax^3 - \frac{1}{2}a\pi x^2\right]_0^\pi = \frac{1}{3}a\pi^3 - \frac{1}{2}a\pi^3 = -\frac{1}{6}a\pi^3 = 2 \;\Leftrightarrow\; a = -12/\pi^3$. Thus, the desired function is $h(x) = -\dfrac{12}{\pi^3}x^2 + \dfrac{12}{\pi^2}x$.

(d)



3. For $1 \le x \le 2$, we have $x^4 \le 2^4 = 16$, so $1 + x^4 \le 17$ and $\dfrac{1}{1+x^4} \ge \dfrac{1}{17}$. Thus, $\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx \ge \int_1^2 \frac{1}{17}\,dx = \frac{1}{17}$. Also $1 + x^4 > x^4$ for $1 \le x \le 2$, so $\dfrac{1}{1+x^4} < \dfrac{1}{x^4}$ and $\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx < \int_1^2 x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_1^2 = -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}$. Thus, we have the estimate $\dfrac{1}{17} \le \displaystyle\int_1^2 \frac{1}{1+x^4}\,dx \le \frac{7}{24}$.

4. By FTC2, $\int_0^1 f'(x)\,dx = f(1) - f(0) = 1 - 0 = 1$.

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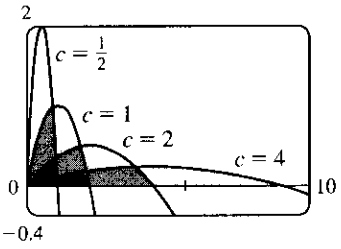
5. Such a function cannot exist. $f'(x) > 3$ for all x means that f is differentiable (and hence continuous) for all x. So by FTC2, $\int_1^4 f'(x)\,dx = f(4) - f(1) = 7 - (-1) = 8$. However, if $f'(x) > 3$ for all x, then $\int_1^4 f'(x)\,dx \geq 3 \cdot (4-1) = 9$ by Comparison Property 8 in Section 5.2.

Another solution: By the Mean Value Theorem, there exists a number $c \in (1, 4)$ such that $f'(c) = \dfrac{f(4) - f(1)}{4-1} = \dfrac{7-(-1)}{3} = \dfrac{8}{3} \quad \Rightarrow \quad 8 = 3f'(c)$. But $f'(x) > 3 \quad \Rightarrow \quad 3f'(c) > 9$, so such a function cannot exist.

6. (a)

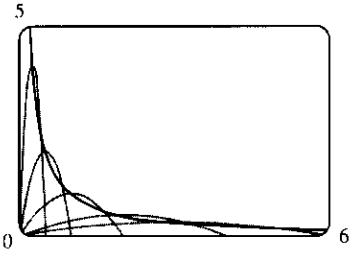


From the graph of $f(x) = \dfrac{2cx - x^2}{c^3}$, it appears that the areas are equal; that is, the area enclosed is independent of c.

(b) We first find the x-intercepts of the curve, to determine the limits of integration: $y = 0 \quad \Leftrightarrow \quad 2cx - x^2 = 0 \quad \Leftrightarrow \quad x = 0$ or $x = 2c$. Now we integrate the function between these limits to find the enclosed area:

$$A = \int_0^{2c} \frac{2cx - x^2}{c^3}\,dx = \frac{1}{c^3}\left[cx^2 - \tfrac{1}{3}x^3\right]_0^{2c} = \frac{1}{c^3}\left[c(2c)^2 - \tfrac{1}{3}(2c)^3\right] = \frac{1}{c^3}\left[4c^3 - \tfrac{8}{3}c^3\right] = \tfrac{4}{3}, \text{ a constant.}$$

(c)



The vertices of the family of parabolas seem to determine a branch of a hyperbola.

(d) For a particular c, the vertex is the point where the maximum occurs. We have seen that the x-intercepts are 0 and $2c$, so by symmetry, the maximum occurs at $x = c$, and its value is $\dfrac{2c(c) - c^2}{c^3} = \dfrac{1}{c}$. So we are interested in the curve consisting of all points of the form $\left(c, \dfrac{1}{c}\right)$, $c > 0$. This is the part of the hyperbola $y = 1/x$ lying in the first quadrant.

7. $f(x) = \displaystyle\int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\,dt$, where $g(x) = \int_0^{\cos x} \left[1 + \sin\left(t^2\right)\right] dt$. Using FTC1 and the Chain Rule (twice) we have $f'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\,g'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\left[1 + \sin\left(\cos^2 x\right)\right](-\sin x)$. Now $g\left(\frac{\pi}{2}\right) = \int_0^0 \left[1 + \sin\left(t^2\right)\right] dt = 0$, so $f'\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{1+0}}(1 + \sin 0)(-1) = 1 \cdot 1 \cdot (-1) = -1$.

8. If $f(x) = \int_0^x x^2 \sin\left(t^2\right) dt = x^2 \int_0^x \sin\left(t^2\right) dt$, then $f'(x) = x^2 \sin\left(x^2\right) + 2x \int_0^x \sin\left(t^2\right) dt$, by the Product Rule and FTC1.

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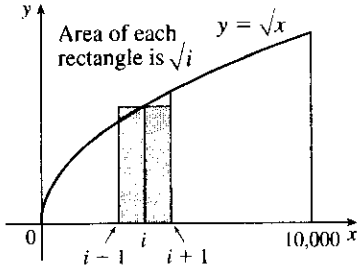
9. $f(x) = 2 + x - x^2 = (-x+2)(x+1) = 0 \quad \Leftrightarrow \quad x = 2$ or $x = -1$. $f(x) \geq 0$ for $x \in [-1, 2]$ and $f(x) < 0$ everywhere else. The integral $\int_a^b (2 + x - x^2)\,dx$ has a maximum on the interval where the integrand is positive, which is $[-1, 2]$. So $a = -1$, $b = 2$. (Any larger interval gives a smaller integral since $f(x) < 0$ outside $[-1, 2]$. Any smaller interval also gives a smaller integral since $f(x) \geq 0$ in $[-1, 2]$.)

10. This sum can be interpreted as a Riemann sum, with the right endpoints of the subintervals as sample points and with $a = 0$, $b = 10{,}000$, and $f(x) = \sqrt{x}$. So we approximate

$$\sum_{i=1}^{10{,}000} \sqrt{i} \approx \lim_{n\to\infty} \frac{10{,}000}{n} \sum_{i=1}^{n} \sqrt{\frac{10{,}000i}{n}} = \int_0^{10{,}000} \sqrt{x}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^{10{,}000} = \tfrac{2}{3}\,(1{,}000{,}000) \approx 666{,}667.$$

Alternate method: We can use graphical methods as follows:



From the figure we see that $\int_{i-1}^{i} \sqrt{x}\,dx < \sqrt{i} < \int_{i}^{i+1} \sqrt{x}\,dx$, so $\int_0^{10{,}000} \sqrt{x}\,dx < \sum_{i=1}^{10{,}000} \sqrt{i} < \int_1^{10{,}001} \sqrt{x}\,dx$. Since $\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$, we get $\int_0^{10{,}000} \sqrt{x}\,dx = 666{,}666.\overline{6}$ and $\int_1^{10{,}001} \sqrt{x}\,dx = \frac{2}{3}\left[(10{,}001)^{3/2} - 1\right] \approx 666{,}766$.

Hence, $666{,}666.\overline{6} < \sum_{i=1}^{10{,}000} \sqrt{i} < 666{,}766$. We can estimate the sum by averaging these bounds:

$\sum_{i=1}^{10{,}000} \approx \frac{666{,}666.\overline{6} + 666{,}766}{2} \approx 666{,}716$. The actual value is about 666,716.46.

11. (a) We can split the integral $\int_0^n \llbracket x \rrbracket\,dx$ into the sum $\sum_{i=1}^{n} \left[\int_{i-1}^{i} \llbracket x \rrbracket\,dx\right]$. But on each of the intervals $[i-1, i)$ of integration, $\llbracket x \rrbracket$ is a constant function, namely $i - 1$. So the ith integral in the sum is equal to $(i-1)[i-(i-1)] = (i-1)$. So the original integral is equal to $\sum_{i=1}^{n} (i-1) = \sum_{i=1}^{n-1} i = \dfrac{(n-1)n}{2}$.

(b) We can write $\int_a^b \llbracket x \rrbracket\,dx = \int_0^b \llbracket x \rrbracket\,dx - \int_0^a \llbracket x \rrbracket\,dx$.

Now $\int_0^b \llbracket x \rrbracket\,dx = \int_0^{\llbracket b \rrbracket} \llbracket x \rrbracket\,dx + \int_{\llbracket b \rrbracket}^b \llbracket x \rrbracket\,dx$. The first of these integrals is equal to $\frac{1}{2}(\llbracket b \rrbracket - 1)\,\llbracket b \rrbracket$, by part (a), and since $\llbracket x \rrbracket = \llbracket b \rrbracket$ on $[\llbracket b \rrbracket, b]$, the second integral is just $\llbracket b \rrbracket\,(b - \llbracket b \rrbracket)$. So

$\int_0^b \llbracket x \rrbracket\,dx = \frac{1}{2}(\llbracket b \rrbracket - 1)\,\llbracket b \rrbracket + \llbracket b \rrbracket\,(b - \llbracket b \rrbracket) = \frac{1}{2}\,\llbracket b \rrbracket\,(2b - \llbracket b \rrbracket - 1)$ and similarly

$\int_0^a \llbracket x \rrbracket\,dx = \frac{1}{2}\,\llbracket a \rrbracket\,(2a - \llbracket a \rrbracket - 1)$. Therefore, $\int_a^b \llbracket x \rrbracket\,dx = \frac{1}{2}\,\llbracket b \rrbracket\,(2b - \llbracket b \rrbracket - 1) - \frac{1}{2}\,\llbracket a \rrbracket\,(2a - \llbracket a \rrbracket - 1)$.

12. By FTC1, $\dfrac{d}{dx} \int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}\,du\right) dt = \int_1^{\sin x} \sqrt{1+u^4}\,du$. Again using FTC1,

$$\frac{d^2}{dx^2} \int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}\,du\right) dt = \frac{d}{dx} \int_1^{\sin x} \sqrt{1+u^4}\,du = \sqrt{1+\sin^4 x}\cos x.$$

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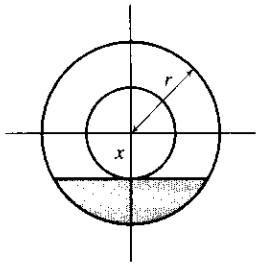
13. Differentiating the equation $\int_0^x f(t)\,dt = [f(x)]^2$ using FTC1 gives $f(x) = 2f(x)f'(x) \;\Rightarrow$ $f(x)[2f'(x) - 1] = 0$, so $f(x) = 0$ or $f'(x) = \frac{1}{2}$. $f'(x) = \frac{1}{2} \;\Rightarrow\; f(x) = \frac{1}{2}x + C$. To find C we substitute into the original equation to get $\int_0^x \left(\frac{1}{2}t + C\right) dt = \left(\frac{1}{2}x + C\right)^2 \;\Leftrightarrow\; \frac{1}{4}x^2 + Cx = \frac{1}{4}x^2 + Cx + C^2$. It follows that $C = 0$, so $f(x) = \frac{1}{2}x$. Therefore, $f(x) = 0$ or $f(x) = \frac{1}{2}x$.

14.



Let x be the distance between the center of the disk and the surface of the liquid. The wetted circular region has area $\pi r^2 - \pi x^2$ while the unexposed wetted region (shaded in the diagram) has area $2\int_x^r \sqrt{r^2 - t^2}\,dt$, so the exposed wetted region has area $A(x) = \pi r^2 - \pi x^2 - 2\int_x^r \sqrt{r^2 - t^2}\,dt$, $0 \le x \le r$. By FTC1, we have $A'(x) = -2\pi x + 2\sqrt{r^2 - x^2}$.

Now $A'(x) > 0 \;\Rightarrow\; -2\pi x + 2\sqrt{r^2 - x^2} > 0 \;\Rightarrow\; \sqrt{r^2 - x^2} > \pi x \;\Rightarrow\; r^2 - x^2 > \pi^2 x^2 \;\Rightarrow$ $r^2 > \pi^2 x^2 + x^2 \;\Rightarrow\; r^2 > x^2(\pi^2 + 1) \;\Rightarrow\; x^2 < \dfrac{r^2}{\pi^2 + 1} \;\Rightarrow\; x < \dfrac{r}{\sqrt{\pi^2 + 1}}$, and we'll call this value x^*.

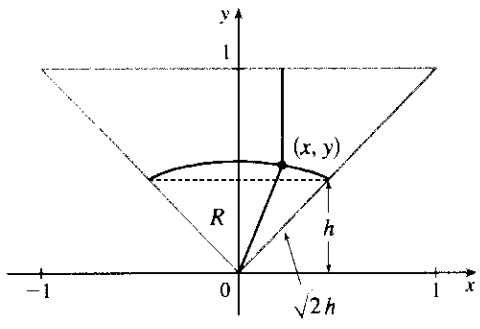
Since $A'(x) > 0$ for $0 < x < x^*$ and $A'(x) < 0$ for $x^* < x < r$, we have an absolute maximum when $x = x^*$.

15. Note that $\dfrac{d}{dx}\left(\int_0^x \left[\int_0^u f(t)\,dt\right] du\right) = \int_0^x f(t)\,dt$ by FTC1, while

$$\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\left[\int_0^x f(u)u\,du\right]$$
$$= \int_0^x f(u)\,du + xf(x) - f(x)x = \int_0^x f(u)\,du$$

Hence, $\int_0^x f(u)(x-u)\,du = \int_0^x \left[\int_0^u f(t)\,dt\right] du + C$. Setting $x = 0$ gives $C = 0$.

16.



We restrict our attention to the triangle shown. A point in this triangle is closer to the side shown than to any other side, so if we find the area of the region R consisting of all points in the triangle that are closer to the center than to that side, we can multiply this area by 4 to find the total area. We find the equation of the set of points which are equidistant from the center and the side: the distance of the point (x, y) from the side is $1 - y$, and its distance from the center is $\sqrt{x^2 + y^2}$.

So the distances are equal if $\sqrt{x^2 + y^2} = 1 - y \;\Leftrightarrow\; x^2 + y^2 = 1 - 2y + y^2 \;\Leftrightarrow\; y = \frac{1}{2}(1 - x^2)$. Note that the area we are interested in is equal to the area of a triangle plus a crescent-shaped area. To find these areas, we

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have to find the y-coordinate h of the horizontal line separating them. From the diagram, $1-h=\sqrt{2}h \quad \Leftrightarrow$ $h=\frac{1}{1+\sqrt{2}}=\sqrt{2}-1$. We calculate the areas in terms of h, and substitute afterward.

The area of the triangle is $\frac{1}{2}(2h)(h)=h^2$, and the area of the crescent-shaped section is $\int_{-h}^{h}\left[\frac{1}{2}\left(1-x^2\right)-h\right]dx=2\int_0^h\left(\frac{1}{2}-h-\frac{1}{2}x^2\right)dx=2\left[\left(\frac{1}{2}-h\right)x-\frac{1}{6}x^3\right]_0^h=h-2h^2-\frac{1}{3}h^3$. So the area of the whole region is

$$4\left[\left(h-2h^2-\tfrac{1}{3}h^3\right)+h^2\right]=4h\left(1-h-\tfrac{1}{3}h^2\right)=4\left(\sqrt{2}-1\right)\left[1-\left(\sqrt{2}-1\right)-\tfrac{1}{3}\left(\sqrt{2}-1\right)^2\right]$$

$$=4\left(\sqrt{2}-1\right)\left(1-\tfrac{1}{3}\sqrt{2}\right)=\tfrac{4}{3}\left(4\sqrt{2}-5\right)$$

17.
$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)\\
&\quad=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}}+\sqrt{\frac{n}{n+2}}+\cdots+\sqrt{\frac{n}{n+n}}\right)\\
&\quad=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}}+\frac{1}{\sqrt{1+2/n}}+\cdots+\frac{1}{\sqrt{1+1}}\right)\\
&\quad=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{i}{n}\right)\qquad\left[\text{where } f(x)=\frac{1}{\sqrt{1+x}}\right]\\
&\quad=\int_0^1\frac{1}{\sqrt{1+x}}\,dx=\left[2\sqrt{1+x}\,\right]_0^1=2\left(\sqrt{2}-1\right)
\end{aligned}$$

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Chapter 6

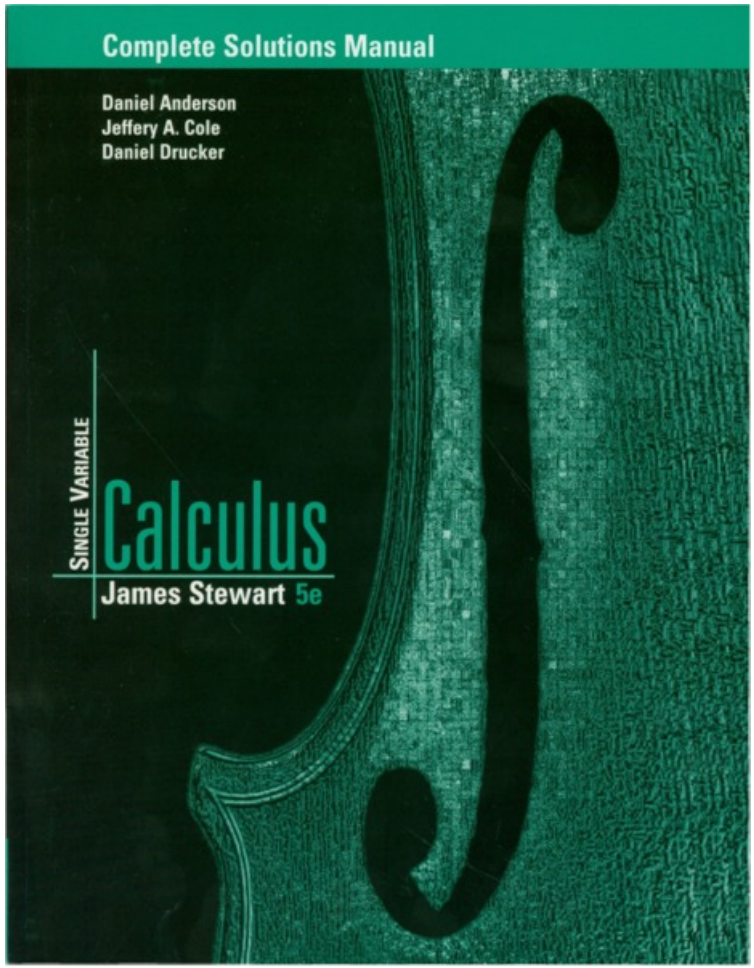
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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6 □ APPLICATIONS OF INTEGRATION

6.1 Areas between Curves

1. $A = \int_{x=0}^{x=4} (y_T - y_B)\,dx = \int_0^4 \left[(5x - x^2) - x\right] dx = \int_0^4 (4x - x^2)\,dx$

$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \left(32 - \frac{64}{3}\right) - (0) = \frac{32}{3}$

2. $A = \int_0^6 \left[2x - (x^2 - 4x)\right] dx = \int_0^6 (6x - x^2)\,dx = \left[3x^2 - \frac{1}{3}x^3\right]_0^6 = 108 - 72 = 36$

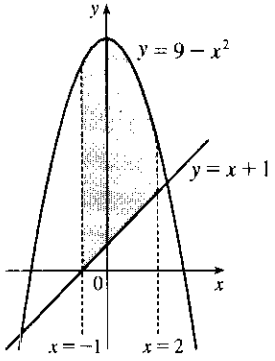
3. $A = \int_{y=0}^{y=1} (x_R - x_L)\,dy = \int_0^1 \left[\sqrt{y} - (y^2 - 1)\right] dy = \int_0^1 \left(y^{1/2} - y^2 + 1\right) dy$

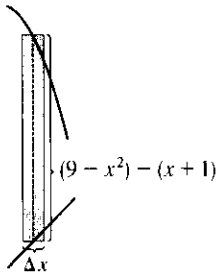
$= \left[\frac{2}{3}y^{3/2} - \frac{1}{3}y^3 + y\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3} + 1\right) - (0) = \frac{4}{3}$

4. $A = \int_0^3 \left[(2y - y^2) - (y^2 - 4y)\right] dy = \int_0^3 (-2y^2 + 6y)\,dy$

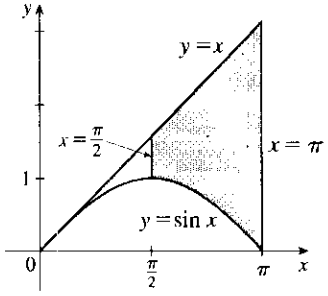
$= \left[-\frac{2}{3}y^3 + 3y^2\right]_0^3 = (-18 + 27) - 0 = 9$

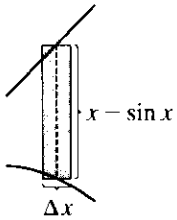
5.
$$A = \int_{-1}^{2} \left[(9 - x^2) - (x + 1)\right] dx$$
$$= \int_{-1}^{2} (8 - x - x^2)\,dx$$
$$= \left[8x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left(16 - 2 - \tfrac{8}{3}\right) - \left(-8 - \tfrac{1}{2} + \tfrac{1}{3}\right)$$
$$= 22 - 3 + \tfrac{1}{2} = \tfrac{39}{2}$$





6.
$$A = \int_{\pi/2}^{\pi} (x - \sin x)\,dx$$
$$= \left[\frac{x^2}{2} + \cos x\right]_{\pi/2}^{\pi}$$
$$= \left(\frac{\pi^2}{2} - 1\right) - \left(\frac{\pi^2}{8} + 0\right)$$
$$= \frac{3\pi^2}{8} - 1$$





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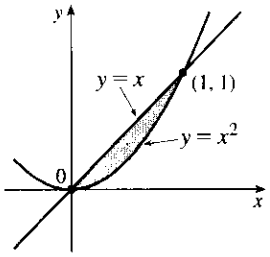
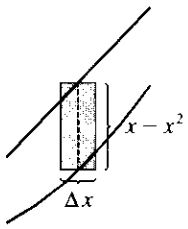
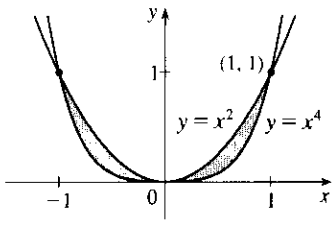
7. The curves intersect when $x = x^2 \Rightarrow x^2 - x = 0 \Leftrightarrow x(x-1) = 0 \Leftrightarrow x = 0, 1$.

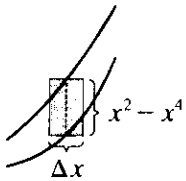
$$\begin{aligned} A &= \int_0^1 (x - x^2)\,dx \\ &= \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 \\ &= \tfrac{1}{2} - \tfrac{1}{3} \\ &= \tfrac{1}{6} \end{aligned}$$





8. $$\begin{aligned} A &= \int_{-1}^{1} (x^2 - x^4)\,dx \\ &= 2\int_0^1 (x^2 - x^4)\,dx \\ &= 2\left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^1 \\ &= 2\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{4}{15} \end{aligned}$$

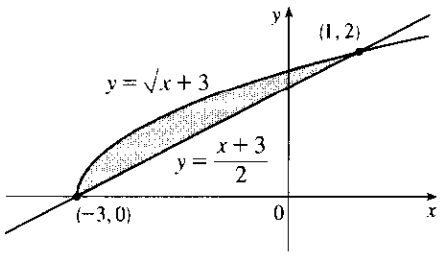


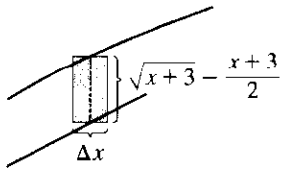


9. First find the points of intersection: $\sqrt{x+3} = \dfrac{x+3}{2} \Rightarrow \left(\sqrt{x+3}\right)^2 = \left(\dfrac{x+3}{2}\right)^2 \Rightarrow x + 3 = \tfrac{1}{4}(x+3)^2$

$\Rightarrow 4(x+3) - (x+3)^2 = 0 \Rightarrow (x+3)[4 - (x+3)] = 0 \Rightarrow (x+3)(1-x) = 0 \Rightarrow x = -3$ or 1. So

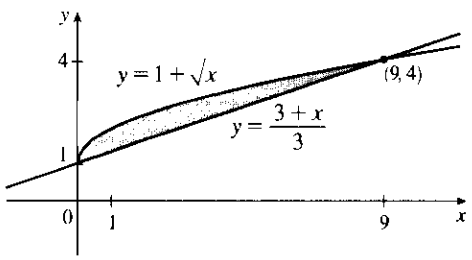
$$A = \int_{-3}^{1} \left(\sqrt{x+3} - \frac{x+3}{2}\right) dx = \left[\tfrac{2}{3}(x+3)^{3/2} - \frac{(x+3)^2}{4}\right]_{-3}^{1} = \left(\tfrac{16}{3} - 4\right) - (0 - 0) = \tfrac{4}{3}$$

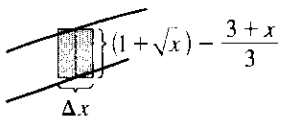




10. $1 + \sqrt{x} = \dfrac{3+x}{3} = 1 + \dfrac{x}{3} \Rightarrow \sqrt{x} = \dfrac{x}{3} \Rightarrow x = \dfrac{x^2}{9} \Rightarrow 9x - x^2 = 0 \Rightarrow x(9-x) = 0 \Rightarrow x = 0$ or 9, so

$$\begin{aligned} A &= \int_0^9 \left[\left(1 + \sqrt{x}\right) - \left(\frac{3+x}{3}\right)\right] dx = \int_0^9 \left[\left(1 + \sqrt{x}\right) - \left(1 + \frac{x}{3}\right)\right] dx \\ &= \int_0^9 \left(\sqrt{x} - \tfrac{1}{3}x\right) dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{6}x^2\right]_0^9 = 18 - \tfrac{27}{2} = \tfrac{9}{2} \end{aligned}$$





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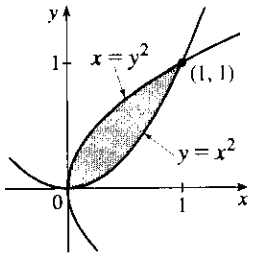
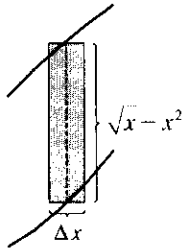
11. $A = \int_0^1 (\sqrt{x} - x^2)\,dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1$

$= \frac{2}{3} - \frac{1}{3}$

$= \frac{1}{3}$

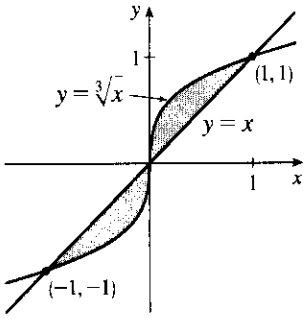


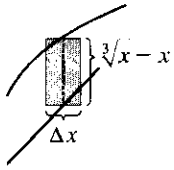


12. $x = \sqrt[3]{x} \;\Rightarrow\; x^3 = x \;\Rightarrow\; x^3 - x = 0 \;\Rightarrow\; x(x^2 - 1) = 0 \;\Rightarrow\; x(x+1)(x-1) = 0 \;\Rightarrow$ $x = -1, 0$, or 1, so

$$A = \int_{-1}^{1} \left|\sqrt[3]{x} - x\right| dx = \int_{-1}^{0} \left(x - \sqrt[3]{x}\right) dx + \int_0^1 \left(\sqrt[3]{x} - x\right) dx = 2\int_0^1 \left(x^{1/3} - x\right) dx \qquad \text{[by symmetry]}$$

$$= 2\left[\tfrac{3}{4}x^{4/3} - \tfrac{1}{2}x^2\right]_0^1 = 2\left(\tfrac{3}{4} - \tfrac{1}{2}\right) = \tfrac{1}{2}$$

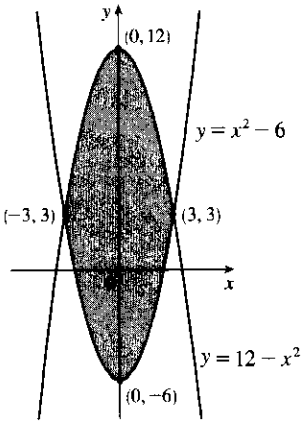


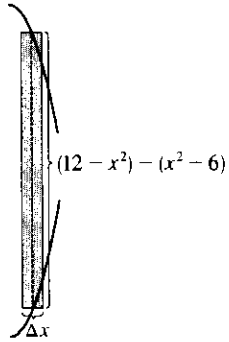


13. $12 - x^2 = x^2 - 6 \;\Leftrightarrow\; 2x^2 = 18 \;\Leftrightarrow\; x^2 = 9 \;\Leftrightarrow\; x^2 = \pm 3$, so

$$A = \int_{-3}^{3} \left[(12 - x^2) - (x^2 - 6)\right] dx = 2\int_0^3 (18 - 2x^2)\,dx \quad \text{[by symmetry]}$$

$$= 2\left[18x - \tfrac{2}{3}x^3\right]_0^3 = 2\left[(54 - 18) - 0\right] = 2(36) = 72$$





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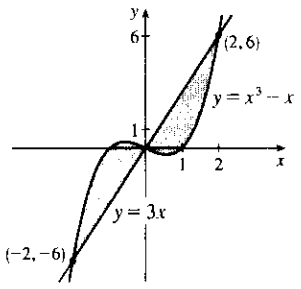
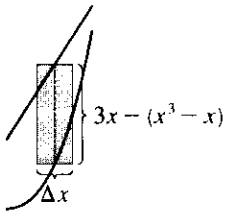
14. $x^3 - x = 3x \;\Rightarrow\; x^3 - 4x = 0 \;\Rightarrow\; x(x^2 - 4) = 0 \;\Rightarrow\; x(x+2)(x-2) = 0 \;\Rightarrow\; x = 0, -2, \text{or } 2.$

By symmetry,

$$A = \int_{-2}^{2} |3x - (x^3 - x)|\, dx = 2\int_0^2 [3x - (x^3 - x)]\, dx = 2\int_0^2 (4x - x^3)\, dx = 2\left[2x^2 - \tfrac{1}{4}x^4\right]_0^2$$

$$= 2(8 - 4) = 8$$

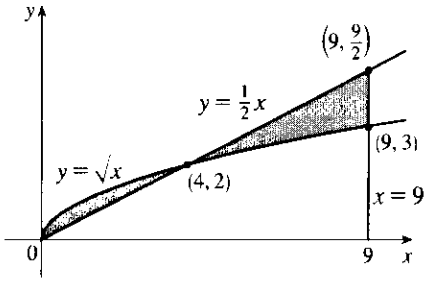


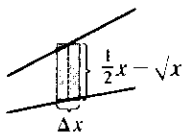


15. $\frac{1}{2}x = \sqrt{x} \;\Rightarrow\; \frac{1}{4}x^2 = x \;\Rightarrow\; x^2 - 4x = 0 \;\Rightarrow\; x(x-4) = 0 \;\Rightarrow\; x = 0 \text{ or } 4, \text{ so}$

$$A = \int_0^4 \left(\sqrt{x} - \tfrac{1}{2}x\right) dx + \int_4^9 \left(\tfrac{1}{2}x - \sqrt{x}\right) dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^2\right]_0^4 + \left[\tfrac{1}{4}x^2 - \tfrac{2}{3}x^{3/2}\right]_4^9$$

$$= \left[\left(\tfrac{16}{3} - 4\right) - 0\right] + \left[\left(\tfrac{81}{4} - 18\right) - \left(4 - \tfrac{16}{3}\right)\right] = \tfrac{81}{4} + \tfrac{32}{3} - 26 = \tfrac{59}{12}$$



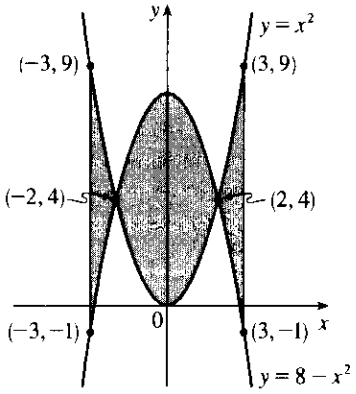


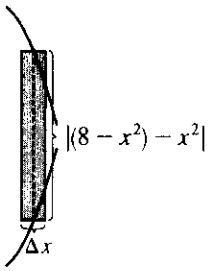
For $4 < x < 9$

16. $A = \int_{-3}^{3} \left|(8 - x^2) - x^2\right| dx = 2\int_0^3 |8 - 2x^2|\, dx = 2\int_0^2 (8 - 2x^2)\, dx + 2\int_2^3 (2x^2 - 8)\, dx$

$$= 2\left[8x - \tfrac{2}{3}x^3\right]_0^2 + 2\left[\tfrac{2}{3}x^3 - 8x\right]_2^3 = 2\left[\left(16 - \tfrac{16}{3}\right) - 0\right] + 2\left[(18 - 24) - \left(\tfrac{16}{3} - 16\right)\right]$$

$$= 32 - \tfrac{32}{3} + 20 - \tfrac{32}{3} = 52 - \tfrac{64}{3} = \tfrac{92}{3}$$





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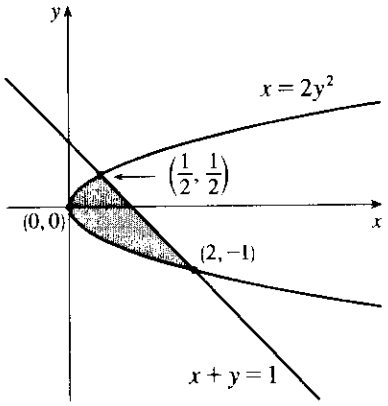
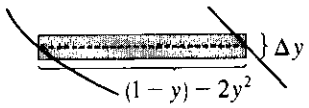
17. $2y^2 = 1 - y \quad \Leftrightarrow \quad 2y^2 + y - 1 = 0 \quad \Leftrightarrow \quad (2y-1)(y+1) = 0 \quad \Leftrightarrow \quad y = \frac{1}{2}$ or -1, so $x = \frac{1}{2}$ or 2 and

$$A = \int_{-1}^{1/2} \left[(1-y) - 2y^2\right] dy = \int_{-1}^{1/2} \left(1 - y - 2y^2\right) dy = \left[y - \tfrac{1}{2}y^2 - \tfrac{2}{3}y^3\right]_{-1}^{1/2}$$

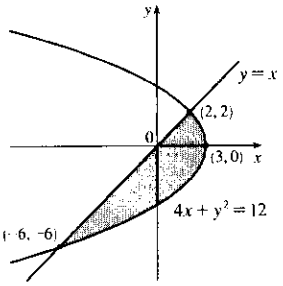
$$= \left(\tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{12}\right) - \left(-1 - \tfrac{1}{2} + \tfrac{2}{3}\right) = \tfrac{7}{24} - \left(-\tfrac{5}{6}\right) = \tfrac{7}{24} + \tfrac{20}{24} = \tfrac{27}{24} = \tfrac{9}{8}$$

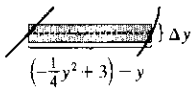




18. $4x + x^2 = 12 \quad \Leftrightarrow \quad (x+6)(x-2) = 0 \quad \Leftrightarrow \quad x = -6$ or $x = 2$, so $y = -6$ or $y = 2$ and

$$A = \int_{-6}^{2} \left[\left(-\tfrac{1}{4}y^2 + 3\right) - y\right] dy = \left[-\tfrac{1}{12}y^3 - \tfrac{1}{2}y^2 + 3y\right]_{-6}^{2} = \left(-\tfrac{2}{3} - 2 + 6\right) - (18 - 18 - 18) = 22 - \tfrac{2}{3} = \tfrac{64}{3}.$$





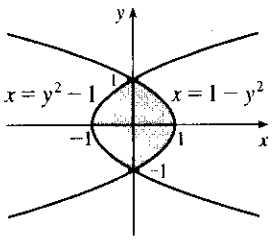
19. The curves intersect when $1 - y^2 = y^2 - 1 \quad \Leftrightarrow \quad 2 = 2y^2 \quad \Leftrightarrow \quad y^2 = 1 \quad \Leftrightarrow \quad y = \pm 1$.

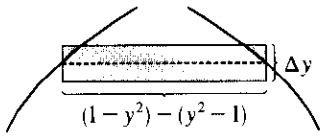
$$A = \int_{-1}^{1} \left[(1-y^2) - (y^2-1)\right] dy$$

$$= \int_{-1}^{1} 2(1-y^2)\, dy$$

$$= 2 \cdot 2 \int_{0}^{1} (1-y^2)\, dy$$

$$= 4\left[y - \tfrac{1}{3}y^3\right]_0^1 = 4\left(1 - \tfrac{1}{3}\right) = \tfrac{8}{3}$$





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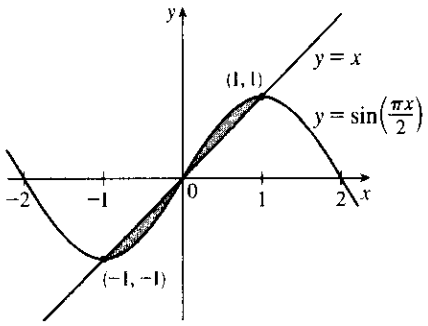
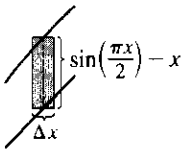
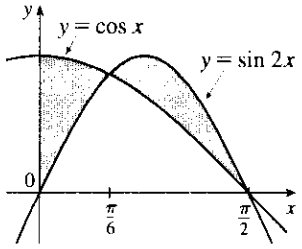
20. $A = 2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x\right] dx = 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right]_0^1 = 2\left[\left(0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi} - 0\right)\right] = \frac{4}{\pi} - 1$



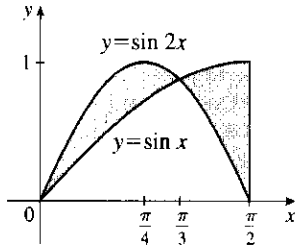


21. Notice that $\cos x = \sin 2x = 2\sin x\cos x \quad \Leftrightarrow$
$2\sin x\cos x - \cos x = 0 \quad \Leftrightarrow \quad \cos x\,(2\sin x - 1) = 0 \quad \Leftrightarrow$
$2\sin x = 1$ or $\cos x = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{6}$ or $\frac{\pi}{2}$.



$$A = \int_0^{\pi/6} (\cos x - \sin 2x)\,dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\,dx$$
$$= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} - \left(0 + \tfrac{1}{2}\cdot 1\right) + \left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\right) = \tfrac{1}{2}$$

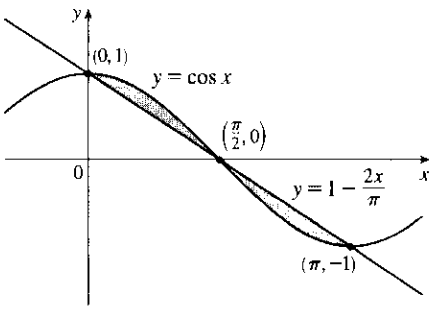
22. $\sin x = \sin 2x = 2\sin x\cos x$ when $\sin x = 0$ and when $\cos x = \frac{1}{2}$; that is, when $x = 0$ or $\frac{\pi}{3}$.

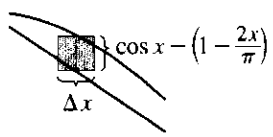


$$A = \int_0^{\pi/3} (\sin 2x - \sin x)\,dx + \int_{\pi/3}^{\pi/2} (\sin x - \sin 2x)\,dx$$
$$= \left[-\tfrac{1}{2}\cos 2x + \cos x\right]_0^{\pi/3} + \left[\tfrac{1}{2}\cos 2x - \cos x\right]_{\pi/3}^{\pi/2}$$
$$= \left[-\tfrac{1}{2}\left(-\tfrac{1}{2}\right) + \tfrac{1}{2}\right] - \left(-\tfrac{1}{2} + 1\right)$$
$$+ \left(-\tfrac{1}{2} - 0\right) - \left[\tfrac{1}{2}\left(-\tfrac{1}{2}\right) - \tfrac{1}{2}\right] = \tfrac{1}{2}$$

23. From the graph, we see that the curves intersect at $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$. By symmetry,

$$A = \int_0^{\pi} \left|\cos x - \left(1 - \frac{2x}{\pi}\right)\right| dx = 2\int_0^{\pi/2} \left[\cos x - \left(1 - \frac{2x}{\pi}\right)\right] dx = 2\int_0^{\pi/2} \left(\cos x - 1 + \frac{2x}{\pi}\right) dx$$
$$= 2\left[\sin x - x + \tfrac{1}{\pi}x^2\right]_0^{\pi/2} = 2\left[\left(1 - \tfrac{\pi}{2} + \tfrac{1}{\pi}\cdot\tfrac{\pi^2}{4}\right) - 0\right] = 2\left(1 - \tfrac{\pi}{2} + \tfrac{\pi}{4}\right) = 2 - \tfrac{\pi}{2}$$





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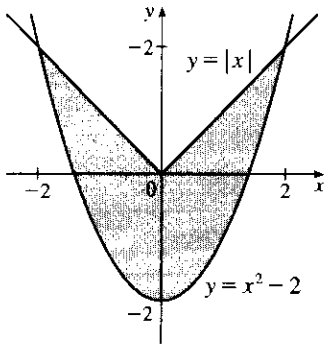
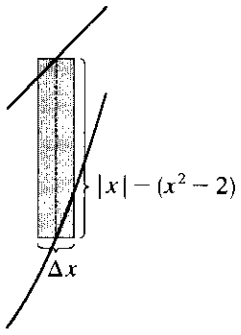
24. For $x > 0$, $x = x^2 - 2 \Rightarrow 0 = x^2 - x - 2 \Rightarrow 0 = (x-2)(x+1) \Rightarrow x = 2$. By symmetry,

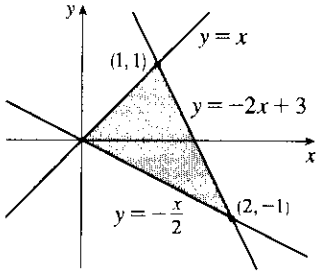
$$\int_{-2}^{2} \left[|x| - (x^2 - 2)\right] dx = 2\int_0^2 \left[x - (x^2 - 2)\right] dx = 2\int_0^2 (x - x^2 + 2)\,dx = 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + 2x\right]_0^2$$
$$= 2\left(2 - \tfrac{8}{3} + 4\right) = \tfrac{20}{3}$$

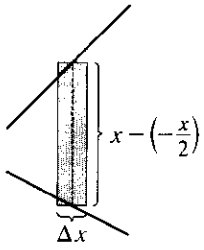




25. Graph the three functions $y = x$, $y = -\frac{1}{2}x$, and $y = -2x + 3$; then determine the points of intersection: $(0, 0)$, $(1, 1)$, and $(2, -1)$.

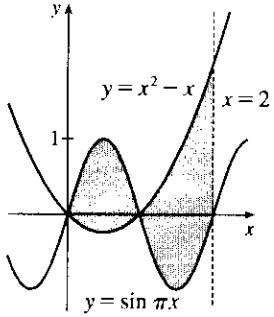
$$A = \int_0^1 \left[x - \left(-\tfrac{1}{2}x\right)\right] dx + \int_1^2 \left[(3 - 2x) - \left(-\tfrac{1}{2}x\right)\right] dx = \int_0^1 \tfrac{3}{2}x\,dx + \int_1^2 \left(3 - \tfrac{3}{2}x\right) dx$$
$$= \left[\tfrac{3}{4}x^2\right]_0^1 + \left[3x - \tfrac{3}{4}x^2\right]_1^2 = \left(\tfrac{3}{4} - 0\right) + (6 - 3) - \left(3 - \tfrac{3}{4}\right) = \tfrac{3}{2}$$

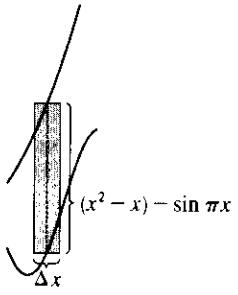




26. $A = \int_0^1 \left[\sin \pi x - (x^2 - x)\right] dx + \int_1^2 \left[(x^2 - x) - \sin \pi x\right] dx$

$$= \left[-\tfrac{1}{\pi}\cos \pi x - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right]_0^1 + \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{\pi}\cos \pi x\right]_1^2$$
$$= \left(\tfrac{1}{\pi} - \tfrac{1}{3} + \tfrac{1}{2}\right) - \left(-\tfrac{1}{\pi}\right) + \left(\tfrac{8}{3} - 2 + \tfrac{1}{\pi}\right) - \left(\tfrac{1}{3} - \tfrac{1}{2} - \tfrac{1}{\pi}\right)$$
$$= \tfrac{4}{\pi} + 1$$





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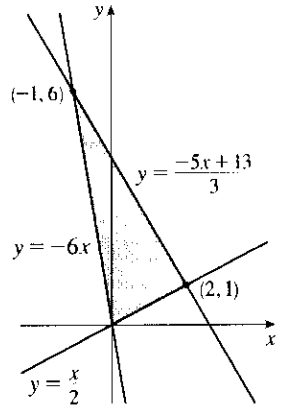
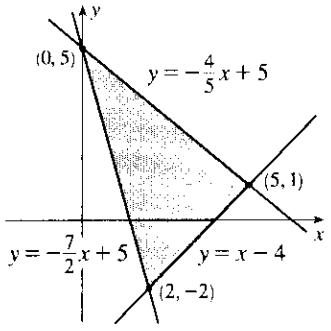
27. An equation of the line through $(0,0)$ and $(2,1)$ is $y = \frac{1}{2}x$; through $(0,0)$ and $(-1,6)$ is $y = -6x$; through $(2,1)$ and $(-1,6)$ is $y = -\frac{5}{3}x + \frac{13}{3}$.



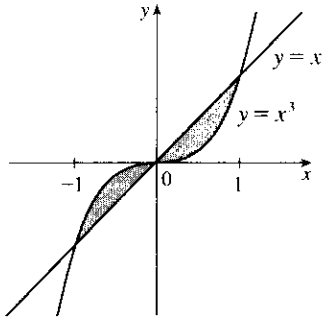
$$\begin{aligned}
A &= \int_{-1}^{0} \left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - (-6x)\right] dx + \int_{0}^{2} \left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - \tfrac{1}{2}x\right] dx \\
&= \int_{-1}^{0} \left(\tfrac{13}{3}x + \tfrac{13}{3}\right) dx + \int_{0}^{2} \left(-\tfrac{13}{6}x + \tfrac{13}{3}\right) dx \\
&= \tfrac{13}{3} \int_{-1}^{0} (x+1)\, dx + \tfrac{13}{3} \int_{0}^{2} \left(-\tfrac{1}{2}x + 1\right) dx \\
&= \tfrac{13}{3} \left[\tfrac{1}{2}x^2 + x\right]_{-1}^{0} + \tfrac{13}{3} \left[-\tfrac{1}{4}x^2 + x\right]_{0}^{2} \\
&= \tfrac{13}{3} \left[0 - \left(\tfrac{1}{2} - 1\right)\right] + \tfrac{13}{3}[(-1+2) - 0] = \tfrac{13}{3} \cdot \tfrac{1}{2} + \tfrac{13}{3} \cdot 1 = \tfrac{13}{2}
\end{aligned}$$

28.



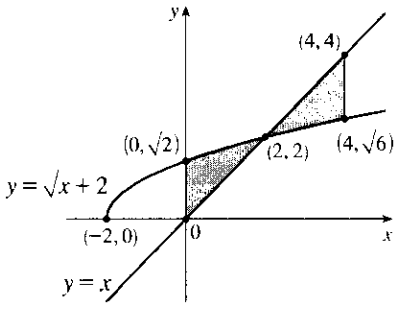
$$\begin{aligned}
A &= \int_{0}^{2} \left[\left(-\tfrac{4}{5}x + 5\right) - \left(-\tfrac{7}{2}x + 5\right)\right] dx + \int_{2}^{5} \left[\left(-\tfrac{4}{5}x + 5\right) - (x - 4)\right] dx \\
&= \int_{0}^{2} \tfrac{27}{10}x\, dx + \int_{2}^{5} \left(-\tfrac{9}{5}x + 9\right) dx \\
&= \left[\tfrac{27}{20}x^2\right]_{0}^{2} + \left[-\tfrac{9}{10}x^2 + 9x\right]_{2}^{5} \\
&= \left(\tfrac{27}{5} - 0\right) + \left(-\tfrac{45}{2} + 45\right) - \left(-\tfrac{18}{5} + 18\right) = \tfrac{27}{2}
\end{aligned}$$

29.



$$\begin{aligned}
A &= \int_{-1}^{1} |x^3 - x|\, dx \\
&= 2 \int_{0}^{1} (x - x^3)\, dx \qquad \text{[by symmetry]} \\
&= 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right]_{0}^{1} \\
&= 2\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{1}{2}
\end{aligned}$$

30. The curves intersect when $\sqrt{x+2} = x \;\Rightarrow\; x + 2 = x^2 \;\Rightarrow\; x^2 - x - 2 = 0 \;\Rightarrow\; (x-2)(x+1) = 0 \;\Rightarrow\; x = -1$ or 2. [-1 is extraneous]



$$\begin{aligned}
A &= \int_{0}^{4} \left|\sqrt{x+2} - x\right| dx \\
&= \int_{0}^{2} \left(\sqrt{x+2} - x\right) dx + \int_{2}^{4} \left(x - \sqrt{x+2}\right) dx \\
&= \left[\tfrac{2}{3}(x+2)^{3/2} - \tfrac{1}{2}x^2\right]_{0}^{2} + \left[\tfrac{1}{2}x^2 - \tfrac{2}{3}(x+2)^{3/2}\right]_{2}^{4} \\
&= \left(\tfrac{16}{3} - 2\right) - \left(\tfrac{2}{3}\left(2\sqrt{2}\right) - 0\right) + \left(8 - \tfrac{2}{3}\left(6\sqrt{6}\right)\right) - \left(2 - \tfrac{16}{3}\right) \\
&= 4 + \tfrac{32}{3} - \tfrac{4}{3}\sqrt{2} - 4\sqrt{6} = \tfrac{44}{3} - 4\sqrt{6} - \tfrac{4}{3}\sqrt{2}
\end{aligned}$$

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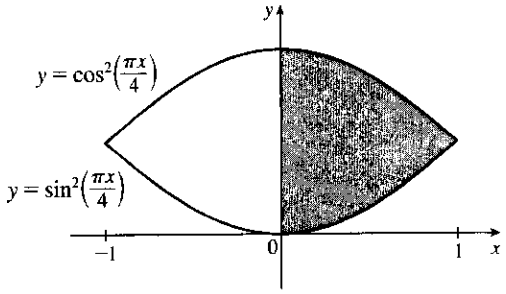
31. Let $f(x) = \cos^2\left(\frac{\pi x}{4}\right) - \sin^2\left(\frac{\pi x}{4}\right)$ and $\Delta x = \frac{1-0}{4}$.

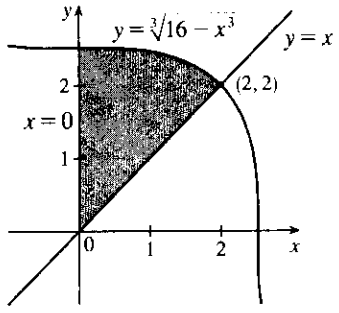


The shaded area is given by

$$\begin{aligned} A = \int_0^1 f(x)\,dx &\approx M_4 \\ &= \tfrac{1}{4}\left[f\left(\tfrac{1}{8}\right) + f\left(\tfrac{3}{8}\right) + f\left(\tfrac{5}{8}\right) + f\left(\tfrac{7}{8}\right)\right] \\ &\approx 0.6407 \end{aligned}$$

32. The curves intersect when $\sqrt[3]{16 - x^3} = x \;\Rightarrow$

$16 - x^3 = x^3 \;\Rightarrow\; 2x^3 = 16 \;\Rightarrow\; x^3 = 8 \;\Rightarrow\; x = 2.$

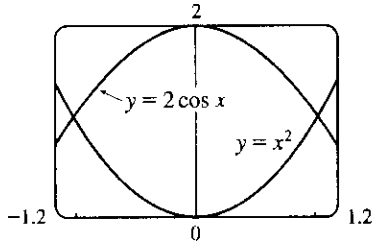


Let $f(x) = \sqrt[3]{16 - x^3} - x$ and $\Delta x = \frac{2-0}{4}$.

The shaded area is given by

$$\begin{aligned} A = \int_0^2 f(x)\,dx &\approx M_4 \\ &= \tfrac{2}{4}\left[f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right)\right] \\ &\approx 2.8144 \end{aligned}$$

33.

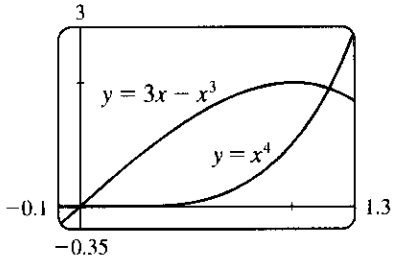


From the graph, we see that the curves intersect at $x = \pm a \approx \pm 1.02$, with $2\cos x > x^2$ on $(-a, a)$. So the area of the region bounded by the curves is

$$A = \int_{-a}^{a} \left(2\cos x - x^2\right) dx = 2\int_0^a \left(2\cos x - x^2\right) dx$$

$$= 2\left[2\sin x - \tfrac{1}{3}x^3\right]_0^a \approx 2.70$$

34.

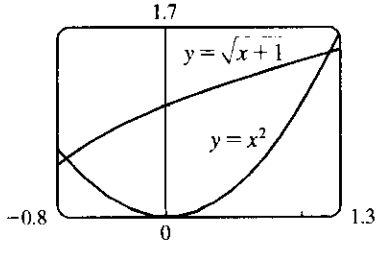


From the graph, we see that the curves intersect at $x = 0$ and at $x = a \approx 1.17$, with $3x - x^3 > x^4$ on $(0, a)$. So the area of the region bounded by the curves is

$$A = \int_0^a \left[\left(3x - x^3\right) - x^4\right] dx = \left[\tfrac{3}{2}x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5\right]_0^a$$

$$\approx 1.15$$

35.



From the graph, we see that the curves intersect at $x = a \approx -0.72$ and at $x = b \approx 1.22$, with $\sqrt{x+1} > x^2$ on (a, b). So the area between the curves is

$$A = \int_a^b \left(\sqrt{x+1} - x^2\right) dx = \left[\tfrac{2}{3}(x+1)^{3/2} - \tfrac{1}{3}x^3\right]_a^b$$

$$\approx 1.38$$

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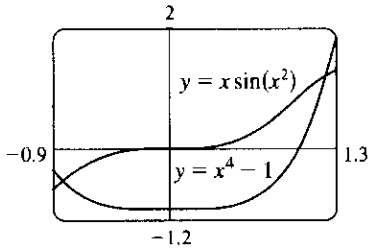
36.



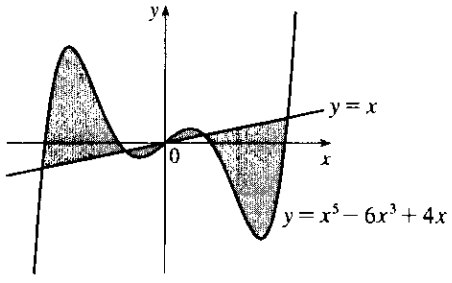
From the graph, we see that the curves intersect at $x = a \approx -0.83$ and $x = b \approx 1.22$, with $x\sin(x^2) > x^4 - 1$ on (a, b). So the area of the region bounded by the curves is

$$A = \int_a^b \left[x\sin(x^2) - (x^4 - 1)\right] dx$$
$$= \left[-\tfrac{1}{2}\cos(x^2) - \tfrac{1}{5}x^5 + x\right]_a^b \approx 1.78$$

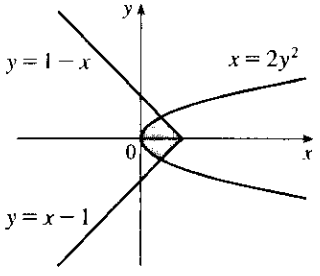
37. As the figure illustrates, the curves $y = x$ and $y = x^5 - 6x^3 + 4x$ enclose a four-part region symmetric about the origin (since $x^5 - 6x^3 + 4x$ and x are odd functions of x). The curves intersect at values of x where $x^5 - 6x^3 + 4x = x$; that is, where $x(x^4 - 6x^2 + 3) = 0$. That happens at $x = 0$ and where $x^2 = \dfrac{6 \pm \sqrt{36 - 12}}{2} = 3 \pm \sqrt{6}$; that is, at $x = -\sqrt{3 + \sqrt{6}}$, $-\sqrt{3 - \sqrt{6}}$, 0, $\sqrt{3 - \sqrt{6}}$, and $\sqrt{3 + \sqrt{6}}$.

The exact area is

$$2\int_0^{\sqrt{3+\sqrt{6}}} \left|(x^5 - 6x^3 + 4x) - x\right| dx = 2\int_0^{\sqrt{3+\sqrt{6}}} \left|x^5 - 6x^3 + 3x\right| dx$$
$$= 2\int_0^{\sqrt{3-\sqrt{6}}} \left(x^5 - 6x^3 + 3x\right) dx + 2\int_{\sqrt{3-\sqrt{6}}}^{\sqrt{3+\sqrt{6}}} \left(-x^5 + 6x^3 - 3x\right) dx$$
$$\overset{\text{CAS}}{=} 12\sqrt{6} - 9$$



38. The inequality $x \geq 2y^2$ describes the region that lies on, or to the right of, the parabola $x = 2y^2$. The inequality $x \leq 1 - |y|$ describes the region that lies on, or to the left of, the curve $x = 1 - |y| = \begin{cases} 1 - y & \text{if } y \geq 0 \\ 1 + y & \text{if } y < 0 \end{cases}$.



So the given region is the shaded region that lies between the curves.

The graphs of $x = 1 - y$ and $x = 2y^2$ intersect when $1 - y = 2y^2 \quad \Leftrightarrow$ $2y^2 + y - 1 = 0 \quad \Leftrightarrow \quad (2y - 1)(y + 1) = 0 \quad \Rightarrow \quad y = \frac{1}{2}$ (for $y \geq 0$).

By symmetry,

$$A = 2\int_0^{1/2} \left[(1 - y) - 2y^2\right] dy = 2\left[-\tfrac{2}{3}y^3 - \tfrac{1}{2}y^2 + y\right]_0^{1/2} = 2\left[\left(-\tfrac{1}{12} - \tfrac{1}{8} + \tfrac{1}{2}\right) - 0\right] = 2\left(\tfrac{7}{24}\right) = \tfrac{7}{12}.$$

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37. We see from the graph in Exercise 6.1.35 that the x-coordinates of the points of intersection are $x = a \approx -0.72$ and $x = b \approx 1.22$, with $\sqrt{x+1} > x^2$ on the interval (a, b), so the volume of revolution is

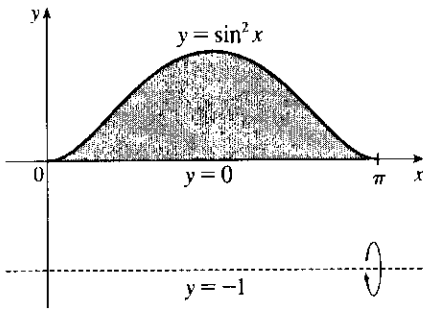
$$\pi \int_a^b \left[\left(\sqrt{x+1}\right)^2 - \left(x^2\right)^2 \right] dx = \pi \int_a^b \left(x + 1 - x^4\right) dx = \pi \left[\tfrac{1}{2}x^2 + x - \tfrac{1}{5}x^5 \right]_a^b \approx 5.80.$$

38. We see from the graph in Exercise 6.1.34 that the x-coordinates of the points of intersection are $x = 0$ and $x = a \approx 1.17$, with $3x - x^3 > x^4$ on the interval $(0, a)$, so the volume of revolution is

$$\begin{aligned} \pi \int_0^a \left[\left(3x - x^3\right)^2 - \left(x^4\right)^2 \right] dx &= \pi \int_0^a \left(9x^2 - 6x^4 + x^6 - x^8\right) dx \\ &= \pi \left[3x^3 - \tfrac{6}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{9}x^9 \right]_0^a \approx 6.74 \end{aligned}$$

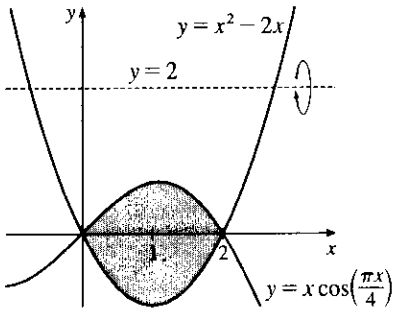
39. $V = \pi \displaystyle\int_0^{\pi} \left\{ \left[\sin^2 x - (-1)\right]^2 - \left[0 - (-1)\right]^2 \right\} dx$

$\overset{\text{CAS}}{=} \dfrac{11}{8}\pi^2$



40. $V = \pi \displaystyle\int_0^2 \left\{ \left[2 - (x^2 - 2x)\right]^2 - \left[2 - x\cos(\pi x/4)\right]^2 \right\} dx$

$\overset{\text{CAS}}{=} \dfrac{4(19\pi^2 + 120\pi - 210)}{15\pi}$



41. $\pi \int_0^{\pi/2} \cos^2 x\, dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ of the xy-plane about the x-axis.

42. $\pi \int_2^5 y\, dy = \pi \int_2^5 \left(\sqrt{y}\right)^2 dy$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 2 \le y \le 5, 0 \le x \le \sqrt{y}\right\}$ of the xy-plane about the y-axis.

43. $\pi \displaystyle\int_0^1 \left(y^4 - y^8\right) dy = \pi \int_0^1 \left[\left(y^2\right)^2 - \left(y^4\right)^2 \right] dy$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le y \le 1, y^4 \le x \le y^2\right\}$ of the xy-plane about the y-axis.

44. $\pi \int_0^{\pi/2} \left[(1 + \cos x)^2 - 1^2\right] dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 1 \le y \le 1 + \cos x\right\}$ of the xy-plane about the x-axis.

Or: The solid could be obtained by rotating the region $\mathcal{R}' = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ about the line $y = -1$.

45. There are 10 subintervals over the 15-cm length, so we'll use $n = 10/2 = 5$ for the Midpoint Rule.

$$\begin{aligned} V = \textstyle\int_0^{15} A(x)\, dx \approx M_5 &= \tfrac{15-0}{5}\left[A(1.5) + A(4.5) + A(7.5) + A(10.5) + A(13.5)\right] \\ &= 3(18 + 79 + 106 + 128 + 39) = 3 \cdot 370 = 1110 \text{ cm}^3 \end{aligned}$$

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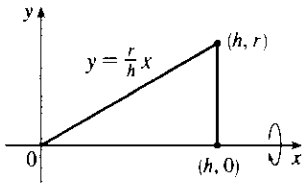
46. $V = \int_0^{10} A(x)\,dx \approx M_5 = \frac{10-0}{5}[A(1) + A(3) + A(5) + A(7) + A(9)]$

$= 2(0.65 + 0.61 + 0.59 + 0.55 + 0.50) = 2(2.90) = 5.80 \text{ m}^3$

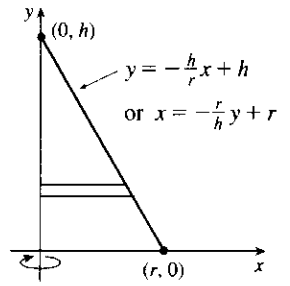
47. We'll form a right circular cone with height h and base radius r by revolving the line $y = \frac{r}{h}x$ about the x-axis.



$$V = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2}x^2\,dx = \pi \frac{r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$

$$= \pi \frac{r^2}{h^2}\left(\frac{1}{3}h^3\right) = \frac{1}{3}\pi r^2 h$$

Another solution: Revolve $x = -\frac{r}{h}y + r$ about the y-axis.



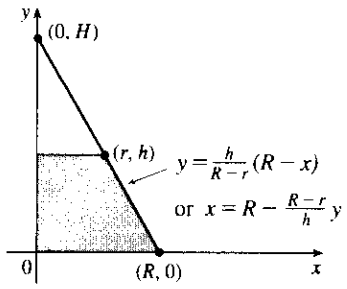
$$V = \pi \int_0^h \left(-\frac{r}{h}y + r\right)^2 dy \overset{*}{=} \pi \int_0^h \left[\frac{r^2}{h^2}y^2 - \frac{2r^2}{h}y + r^2\right] dy$$

$$= \pi\left[\frac{r^2}{3h^2}y^3 - \frac{r^2}{h}y^2 + r^2 y\right]_0^h = \pi\left(\tfrac{1}{3}r^2h - r^2h + r^2h\right) = \tfrac{1}{3}\pi r^2 h$$

* Or use substitution with $u = r - \frac{r}{h}y$ and $du = -\frac{r}{h}\,dy$ to get

$$\pi \int_r^0 u^2\left(-\frac{h}{r}\,du\right) = -\pi\frac{h}{r}\left[\frac{1}{3}u^3\right]_r^0 = -\pi\frac{h}{r}\left(-\frac{1}{3}r^3\right) = \frac{1}{3}\pi r^2 h.$$

48.



$$V = \pi \int_0^h \left(R - \frac{R-r}{h}y\right)^2 dy$$

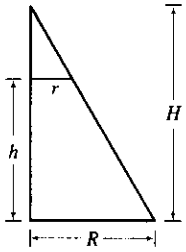
$$= \pi \int_0^h \left[R^2 - \frac{2R(R-r)}{h}y + \left(\frac{R-r}{h}\right)^2 y^2\right] dy$$

$$= \pi\left[R^2 y - \frac{R(R-r)}{h}y^2 + \frac{1}{3}\left(\frac{R-r}{h}\right)^2 y^3\right]_0^h$$

$$= \pi\left[R^2h - R(R-r)h + \tfrac{1}{3}(R-r)^2 h\right]$$

$$= \tfrac{1}{3}\pi h\left[3Rr + (R^2 - 2Rr + r^2)\right] = \tfrac{1}{3}\pi h(R^2 + Rr + r^2)$$

Another solution: $\frac{H}{R} = \frac{H-h}{r}$ by similar triangles. Therefore,



$Hr = HR - hR \;\Rightarrow\; hR = H(R-r) \;\Rightarrow\; H = \frac{hR}{R-r}$. Now

$V = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2(H-h)$ [by Exercise 47]

$$= \tfrac{1}{3}\pi R^2 \frac{hR}{R-r} - \tfrac{1}{3}\pi r^2 \frac{rh}{R-r} \qquad \left[H - h = \frac{rH}{R} = \frac{rhR}{R(R-r)}\right]$$

$$= \frac{1}{3}\pi h \frac{R^3 - r^3}{R-r} = \tfrac{1}{3}\pi h(R^2 + Rr + r^2)$$

$$= \tfrac{1}{3}\left[\pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)}\right]h = \tfrac{1}{3}(A_1 + A_2 + \sqrt{A_1 A_2})h$$

where A_1 and A_2 are the areas of the bases of the frustum. (See Exercise 50 for a related result.)

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49. $x^2 + y^2 = r^2 \quad \Leftrightarrow \quad x^2 = r^2 - y^2$

$$V = \pi \int_{r-h}^{r} (r^2 - y^2)\,dy = \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^{r}$$

$$= \pi \left\{ \left[r^3 - \frac{r^3}{3} \right] - \left[r^2(r-h) - \frac{(r-h)^3}{3} \right] \right\}$$

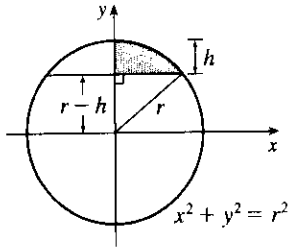
$$= \pi \left\{ \tfrac{2}{3} r^3 - \tfrac{1}{3}(r-h)\left[3r^2 - (r-h)^2 \right] \right\}$$

$$= \tfrac{1}{3}\pi \left\{ 2r^3 - (r-h)\left[3r^2 - \left(r^2 - 2rh + h^2 \right) \right] \right\}$$

$$= \tfrac{1}{3}\pi \left\{ 2r^3 - (r-h)\left[2r^2 + 2rh - h^2 \right] \right\}$$

$$= \tfrac{1}{3}\pi \left(2r^3 - 2r^3 - 2r^2 h + rh^2 + 2r^2 h + 2rh^2 - h^3 \right)$$

$$= \tfrac{1}{3}\pi \left(3rh^2 - h^3 \right) = \tfrac{1}{3}\pi h^2 (3r - h), \text{ or, equivalently, } \pi h^2 \left(r - \frac{h}{3} \right)$$



50. An equation of the line is $x = \dfrac{\Delta x}{\Delta y}\, y + (x\text{-intercept}) = \dfrac{a/2 - b/2}{h - 0}\, y + \dfrac{b}{2} = \dfrac{a-b}{2h}\, y + \dfrac{b}{2}$.

$$V = \int_0^h A(y)\,dy = \int_0^h (2x)^2\,dy$$

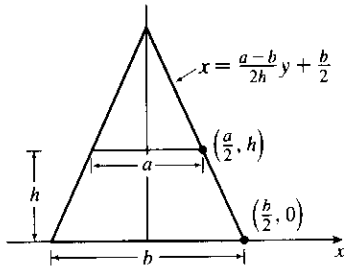
$$= \int_0^h \left[2\left(\frac{a-b}{2h} y + \frac{b}{2} \right) \right]^2 dy = \int_0^h \left[\frac{a-b}{h} y + b \right]^2 dy$$

$$= \int_0^h \left[\frac{(a-b)^2}{h^2} y^2 + \frac{2b(a-b)}{h} y + b^2 \right] dy$$

$$= \left[\frac{(a-b)^2}{3h^2} y^3 + \frac{b(a-b)}{h} y^2 + b^2 y \right]_0^h$$

$$= \tfrac{1}{3}(a-b)^2 h + b(a-b)h + b^2 h = \tfrac{1}{3}\left(a^2 - 2ab + b^2 + 3ab \right) h$$

$$= \tfrac{1}{3}\left(a^2 + ab + b^2 \right) h$$



[Note that this can be written as $\tfrac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2} \right) h$, as in Exercise 48.]

If $a = b$, we get a rectangular solid with volume $b^2 h$. If $a = 0$, we get a square pyramid with volume $\tfrac{1}{3} b^2 h$.

51. For a cross-section at height y, we see from similar triangles that $\dfrac{\alpha/2}{b/2} = \dfrac{h-y}{h}$, so $\alpha = b\left(1 - \dfrac{y}{h} \right)$.

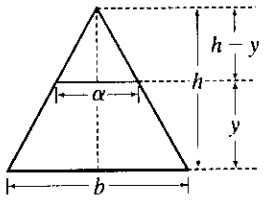
Similarly, for cross-sections having $2b$ as their base and β replacing α, $\beta = 2b\left(1 - \dfrac{y}{h} \right)$. So

$$V = \int_0^h A(y)\,dy = \int_0^h \left[b\left(1 - \frac{y}{h} \right) \right] \left[2b\left(1 - \frac{y}{h} \right) \right] dy$$

$$= \int_0^h 2b^2 \left(1 - \frac{y}{h} \right)^2 dy = 2b^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2} \right) dy$$

$$= 2b^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2} \right]_0^h = 2b^2 \left[h - h + \tfrac{1}{3} h \right]$$

$$= \tfrac{2}{3} b^2 h \quad [\, = \tfrac{1}{3} Bh \text{ where } B \text{ is the area of the base, as with any pyramid.}]$$



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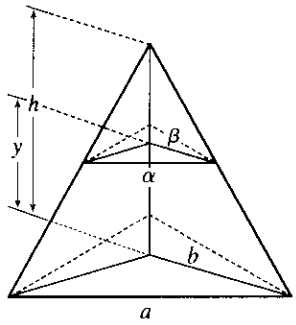
52. Consider the triangle consisting of two vertices of the base and the center of the base. This triangle is similar to the corresponding triangle at a height y, so $a/b = \alpha/\beta \;\Rightarrow\; \alpha = a\beta/b$. Also by similar triangles, $b/h = \beta/(h-y)$ $\Rightarrow\; \beta = b(h-y)/h$. These two equations imply that $\alpha = a(1-y/h)$, and since the cross-section is an equilateral triangle, it has area

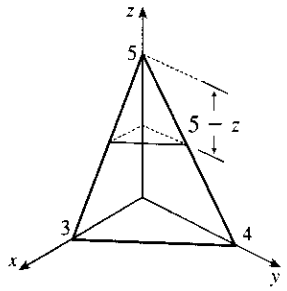


$$A(y) = \frac{1}{2}\cdot\alpha\cdot\frac{\sqrt{3}}{2}\alpha = \frac{a^2(1-y/h)^2}{4}\sqrt{3}, \text{ so}$$

$$V = \int_0^h A(y)\,dy = \frac{a^2\sqrt{3}}{4}\int_0^h \left(1-\frac{y}{h}\right)^2 dy$$

$$= \frac{a^2\sqrt{3}}{4}\left[-\frac{h}{3}\left(1-\frac{y}{h}\right)^3\right]_0^h = -\frac{\sqrt{3}}{12}a^2h(-1) = \frac{\sqrt{3}}{12}a^2h$$

53. A cross-section at height z is a triangle similar to the base, so we'll multiply the legs of the base triangle, 3 and 4, by a proportionality factor of $(5-z)/5$. Thus, the triangle at height z has area



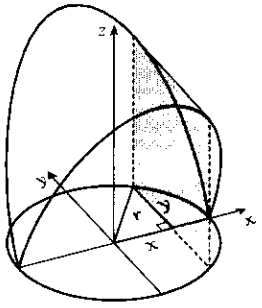
$$A(z) = \frac{1}{2}\cdot 3\left(\frac{5-z}{5}\right)\cdot 4\left(\frac{5-z}{5}\right) = 6\left(1-\frac{z}{5}\right)^2, \text{ so}$$

$$V = \int_0^5 A(z)\,dz = 6\int_0^5 (1-z/5)^2\,dz$$

$$= 6\int_1^0 u^2(-5\,du) \quad [u = 1-z/5,\ du = -\tfrac{1}{5}dz]$$

$$= -30\left[\tfrac{1}{3}u^3\right]_1^0 = -30\left(-\tfrac{1}{3}\right) = 10\ \text{cm}^3$$

54. A cross-section is shaded in the diagram.



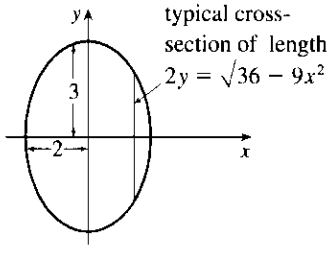
$$A(x) = (2y)^2 = \left(2\sqrt{r^2-x^2}\,\right)^2, \text{ so}$$

$$V = \int_{-r}^{r} A(x)\,dx = 2\int_0^r 4(r^2-x^2)\,dx$$

$$= 8\left[r^2x - \tfrac{1}{3}x^3\right]_0^r$$

$$= 8\left(\tfrac{2}{3}r^3\right) = \tfrac{16}{3}r^3$$

55. If l is a leg of the isosceles right triangle and $2y$ is the hypotenuse, then $l^2 + l^2 = (2y)^2 \;\Rightarrow\; 2l^2 = 4y^2 \;\Rightarrow\; l^2 = 2y^2$.



$$V = \int_{-2}^{2} A(x)\,dx = 2\int_0^2 A(x)\,dx = 2\int_0^2 \tfrac{1}{2}(l)(l)\,dx = 2\int_0^2 y^2\,dx$$

$$= 2\int_0^2 \tfrac{1}{4}(36-9x^2)\,dx = \tfrac{9}{2}\int_0^2 (4-x^2)\,dx$$

$$= \tfrac{9}{2}\left[4x - \tfrac{1}{3}x^3\right]_0^2 = \tfrac{9}{2}\left(8-\tfrac{8}{3}\right) = 24$$

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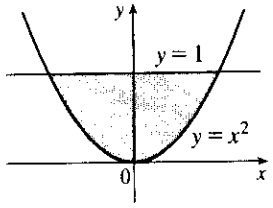
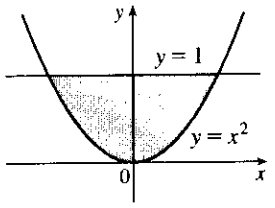
56. The cross-section of the base corresponding to the coordinate y has length $2x = 2\sqrt{y}$. The corresponding equilateral triangle with side s has area $A(y) = s^2\left(\frac{\sqrt{3}}{4}\right) = (2x)^2\left(\frac{\sqrt{3}}{4}\right) = \left(2\sqrt{y}\right)^2\left(\frac{\sqrt{3}}{4}\right) = y\sqrt{3}$. Therefore, $V = \int_0^1 A(y)\,dy = \int_0^1 y\sqrt{3}\,dy = \sqrt{3}\left[\frac{1}{2}y^2\right]_0^1 = \frac{\sqrt{3}}{2}$.



57. The cross-section of the base corresponding to the coordinate y has length $2x = 2\sqrt{y}$. The square has area $A(y) = \left(2\sqrt{y}\right)^2 = 4y$, so $V = \int_0^1 A(y)\,dy = \int_0^1 4y\,dy = \left[2y^2\right]_0^1 = 2$.

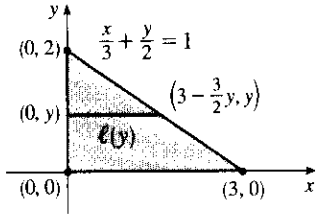


58. A typical cross-section perpendicular to the y-axis in the base has length $\ell(y) = 3 - \frac{3}{2}y$. This length is the diameter of a cross-sectional semicircle in S, so

$$V = \int_0^2 A(y)\,dy = \int_0^2 \frac{\pi}{2}\left[\frac{\ell(y)}{2}\right]^2 dy = \frac{\pi}{8}\int_0^2 \left(3 - \tfrac{3}{2}y\right)^2 dy$$

$$= \tfrac{\pi}{8}\textstyle\int_3^0 u^2\left(-\frac{2}{3}\,du\right) \quad \left[u = 3 - \frac{3}{2}y,\ du = -\frac{3}{2}\,dy\right]$$

$$= -\tfrac{\pi}{12}\left[\tfrac{1}{3}u^3\right]_3^0 = -\tfrac{\pi}{12}(-9) = \tfrac{3\pi}{4}$$

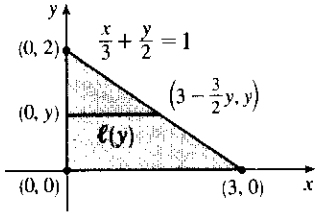


59. A typical cross-section perpendicular to the y-axis in the base has length $\ell(y) = 3 - \frac{3}{2}y$. This length is the leg of an isosceles right triangle, so

$$A(y) = \tfrac{1}{2}\left[\ell(y)\right]^2 \quad \left[\tfrac{1}{2}bh \text{ with base} = \text{height}\right]$$

$$= \tfrac{1}{2}\left[3\left(1 - \tfrac{1}{2}y\right)\right]^2 = \tfrac{9}{2}\left(1 - \tfrac{1}{2}y\right)^2$$

Thus,



$$V = \textstyle\int_0^2 A(y)\,dy = \frac{9}{2}\int_1^0 u^2(-2\,du) \quad \left[u = 1 - \frac{1}{2}y,\ du = -\frac{1}{2}\,dy\right]$$

$$= -9\left[\tfrac{1}{3}u^3\right]_1^0 = -9\left(-\tfrac{1}{3}\right) = 3$$

60. (a) $V = \int_{-r}^{r} A(x)\,dx = 2\int_0^r A(x)\,dx = 2\int_0^r \frac{1}{2}h\left(2\sqrt{r^2 - x^2}\right)dx = 2h\int_0^r \sqrt{r^2 - x^2}\,dx$

(b) Observe that the integral represents one quarter of the area of a circle of radius r, so $V = 2h \cdot \frac{1}{4}\pi r^2 = \frac{1}{2}\pi h r^2$.

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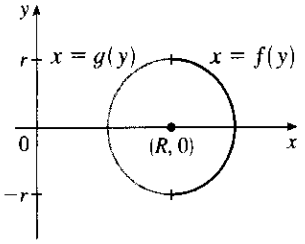
61. (a) The torus is obtained by rotating the circle $(x-R)^2+y^2=r^2$ about the y-axis. Solving for x, we see that the right half of the circle is given by $x=R+\sqrt{r^2-y^2}=f(y)$ and the left half by $x=R-\sqrt{r^2-y^2}=g(y)$. So



$$V=\pi\int_{-r}^{r}\{[f(y)]^2-[g(y)]^2\}\,dy$$

$$=2\pi\int_0^r\left[\left(R^2+2R\sqrt{r^2-y^2}+r^2-y^2\right)-\left(R^2-2R\sqrt{r^2-y^2}+r^2-y^2\right)\right]dy$$

$$=2\pi\int_0^r 4R\sqrt{r^2-y^2}\,dy=8\pi R\int_0^r\sqrt{r^2-y^2}\,dy$$

(b) Observe that the integral represents a quarter of the area of a circle with radius r, so $8\pi R\int_0^r\sqrt{r^2-y^2}\,dy=8\pi R\cdot\frac{1}{4}\pi r^2=2\pi^2r^2R$.

62. The cross-sections perpendicular to the y-axis in Figure 17 are rectangles. The rectangle corresponding to the coordinate y has a base of length $2\sqrt{16-y^2}$ in the xy-plane and a height of $\frac{1}{\sqrt{3}}y$, since $\angle BAC=30^\circ$ and $|BC|=\frac{1}{\sqrt{3}}|AB|$. Thus, $A(y)=\frac{2}{\sqrt{3}}y\sqrt{16-y^2}$ and

$$V=\int_0^4 A(y)\,dy=\tfrac{2}{\sqrt{3}}\int_0^4\sqrt{16-y^2}\,y\,dy$$

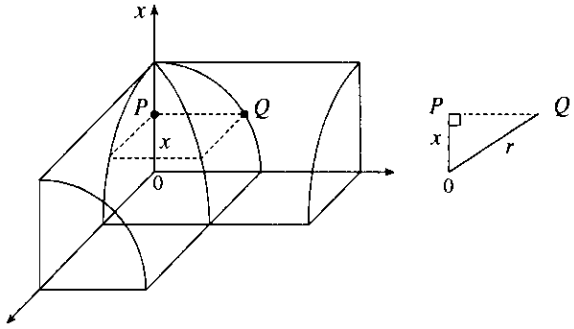
$$=\tfrac{2}{\sqrt{3}}\int_{16}^0 u^{1/2}\left(-\tfrac{1}{2}\,du\right)\qquad[\text{Put } u=16-y^2,\ \text{so } du=-2y\,dy]$$

$$=\tfrac{1}{\sqrt{3}}\int_0^{16}u^{1/2}\,du=\tfrac{1}{\sqrt{3}}\,\tfrac{2}{3}\left[u^{3/2}\right]_0^{16}=\tfrac{2}{3\sqrt{3}}(64)=\tfrac{128}{3\sqrt{3}}$$

63. (a) Volume$(S_1)=\int_0^h A(z)\,dz=$ Volume(S_2) since the cross-sectional area $A(z)$ at height z is the same for both solids.

(b) By Cavalieri's Principle, the volume of the cylinder in the figure is the same as that of a right circular cylinder with radius r and height h, that is, πr^2h.

64. Each cross-section of the solid S in a plane perpendicular to the x-axis is a square (since the edges of the cut lie on the cylinders, which are perpendicular). One-quarter of this square and one-eighth of S are shown. The area of this quarter-square is $|PQ|^2=r^2-x^2$. Therefore, $A(x)=4\left(r^2-x^2\right)$ and the volume of S is



$$V=\int_{-r}^{r}A(x)\,dx=4\int_{-r}^{r}\left(r^2-x^2\right)dx$$

$$=8\int_0^r\left(r^2-x^2\right)dx=8\left[r^2x-\tfrac{1}{3}x^3\right]_0^r=\tfrac{16}{3}r^3$$

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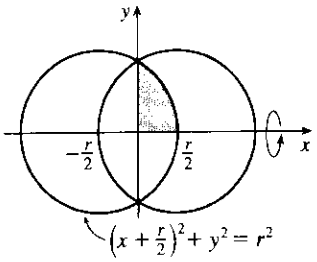
65. The volume is obtained by rotating the area common to two circles of radius r, as shown. The volume of the right half is



$$V_{\text{right}} = \pi \int_0^{r/2} y^2\,dx = \pi \int_0^{r/2} \left[r^2 - \left(\tfrac{1}{2}r + x\right)^2\right] dx$$

$$= \pi\left[r^2 x - \tfrac{1}{3}\left(\tfrac{1}{2}r + x\right)^3\right]_0^{r/2} = \pi\left[\left(\tfrac{1}{2}r^3 - \tfrac{1}{3}r^3\right) - \left(0 - \tfrac{1}{24}r^3\right)\right] = \tfrac{5}{24}\pi r^3$$

So by symmetry, the total volume is twice this, or $\frac{5}{12}\pi r^3$.

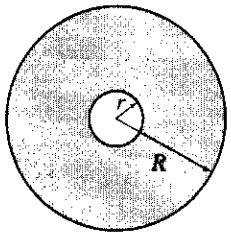
Another solution: We observe that the volume is the twice the volume of a cap of a sphere, so we can use the formula from Exercise 49 with $h = \frac{1}{2}r$: $V = 2 \cdot \frac{1}{3}\pi h^2(3r - h) = \frac{2}{3}\pi\left(\frac{1}{2}r\right)^2\left(3r - \frac{1}{2}r\right) = \frac{5}{12}\pi r^3$.

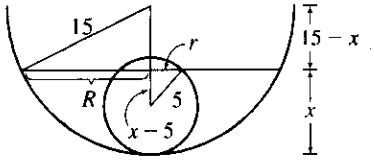
66. We consider two cases: one in which the ball is not completely submerged and the other in which it is.

Case 1: $0 \le h \le 10$ The ball will not be completely submerged, and so a cross-section of the water parallel to the surface will be the shaded area shown in the first diagram. We can find the area of the cross-section at height x above the bottom of the bowl by using the Pythagorean Theorem: $R^2 = 15^2 - (15 - x)^2$ and $r^2 = 5^2 - (x - 5)^2$, so $A(x) = \pi\left(R^2 - r^2\right) = 20\pi x$. The volume of water when it has depth h is then

$V(h) = \int_0^h A(x)\,dx = \int_0^h 20\pi x\,dx = \left[10\pi x^2\right]_0^h = 10\pi h^2$ cm^3, $0 \le h \le 10$.

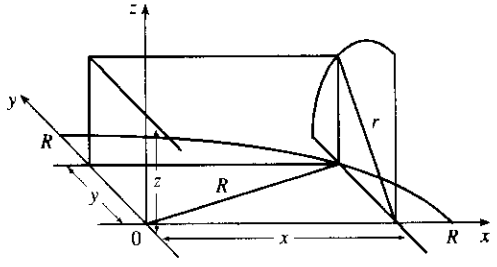
Case 2: $10 < h \le 15$ In this case we can find the volume by simply subtracting the volume displaced by the ball from the total volume inside the bowl underneath the surface of the water. The total volume underneath the surface is just the volume of a cap of the bowl, so we use the formula from Exercise 49: $V_{\text{cap}}(h) = \frac{1}{3}\pi h^2(45 - h)$. The volume of the small sphere is $V_{\text{ball}} = \frac{4}{3}\pi(5)^3 = \frac{500}{3}\pi$, so the total volume is





$$V_{\text{cap}} - V_{\text{ball}} = \tfrac{1}{3}\pi\left(45h^2 - h^3 - 500\right) \text{ cm}^3.$$

67. Take the x-axis to be the axis of the cylindrical hole of radius r. A quarter of the cross-section through y, perpendicular to the y-axis, is the rectangle shown. Using the Pythagorean Theorem twice, we see that the dimensions of this rectangle are $x = \sqrt{R^2 - y^2}$ and $z = \sqrt{r^2 - y^2}$, so $\frac{1}{4}A(y) = xz = \sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}$, and



$$V = \int_{-r}^{r} A(y)\,dy = \int_{-r}^{r} 4\sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}\,dy$$

$$= 8\int_0^r \sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}\,dy$$

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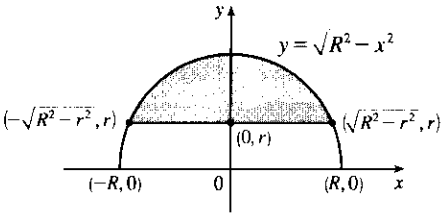
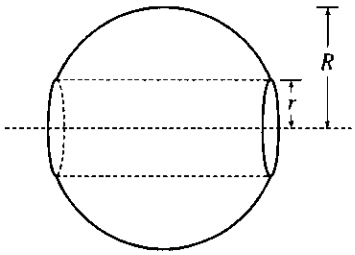
68. The line $y = r$ intersects the semicircle $y = \sqrt{R^2 - x^2}$ when $r = \sqrt{R^2 - x^2} \;\Rightarrow\; r^2 = R^2 - x^2 \;\Rightarrow\; x^2 = R^2 - r^2 \;\Rightarrow\; x = \pm\sqrt{R^2 - r^2}$. Rotating the shaded region about the x-axis gives us

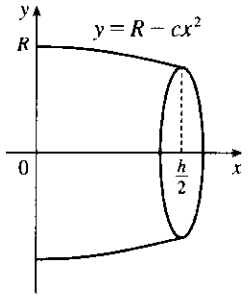
$$\begin{aligned} V &= \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi\left[\left(\sqrt{R^2-x^2}\right)^2 - r^2\right] dx \\ &= 2\pi \int_0^{\sqrt{R^2-r^2}} \left(R^2 - x^2 - r^2\right) dx \qquad \text{[by symmetry]} \\ &= 2\pi \int_0^{\sqrt{R^2-r^2}} \left[\left(R^2 - r^2\right) - x^2\right] dx = 2\pi\left[\left(R^2 - r^2\right)x - \tfrac{1}{3}x^3\right]_0^{\sqrt{R^2-r^2}} \\ &= 2\pi\left[\left(R^2 - r^2\right)^{3/2} - \tfrac{1}{3}\left(R^2 - r^2\right)^{3/2}\right] \\ &= 2\pi \cdot \tfrac{2}{3}\left(R^2 - r^2\right)^{3/2} = \tfrac{4\pi}{3}\left(R^2 - r^2\right)^{3/2} \end{aligned}$$

Our answer makes sense in limiting cases. As $r \to 0$, $V \to \frac{4}{3}\pi R^3$, which is the volume of the full sphere. As $r \to R$, $V \to 0$, which makes sense because the hole's radius is approaching that of the sphere.





69. (a) The radius of the barrel is the same at each end by symmetry, since the function $y = R - cx^2$ is even. Since the barrel is obtained by rotating the graph of the function y about the x-axis, this radius is equal to the value of y at $x = \frac{1}{2}h$, which is $R - c\left(\frac{1}{2}h\right)^2 = R - d = r$.



(b) The barrel is symmetric about the y-axis, so its volume is twice the volume of that part of the barrel for $x > 0$. Also, the barrel is a volume of rotation, so

$$\begin{aligned} V &= 2\int_0^{h/2} \pi y^2\, dx = 2\pi \int_0^{h/2} \left(R - cx^2\right)^2 dx = 2\pi\left[R^2x - \tfrac{2}{3}Rcx^3 + \tfrac{1}{5}c^2x^5\right]_0^{h/2} \\ &= 2\pi\left(\tfrac{1}{2}R^2h - \tfrac{1}{12}Rch^3 + \tfrac{1}{160}c^2h^5\right) \end{aligned}$$

Trying to make this look more like the expression we want, we rewrite it as $V = \frac{1}{3}\pi h\left[2R^2 + \left(R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4\right)\right]$. But

$$R^2 - \tfrac{1}{2}Rch^2 + \tfrac{3}{80}c^2h^4 = \left(R - \tfrac{1}{4}ch^2\right)^2 - \tfrac{1}{40}c^2h^4 = (R - d)^2 - \tfrac{2}{5}\left(\tfrac{1}{4}ch^2\right)^2 = r^2 - \tfrac{2}{5}d^2.$$

Substituting this back into V, we see that $V = \frac{1}{3}\pi h\left(2R^2 + r^2 - \frac{2}{5}d^2\right)$, as required.

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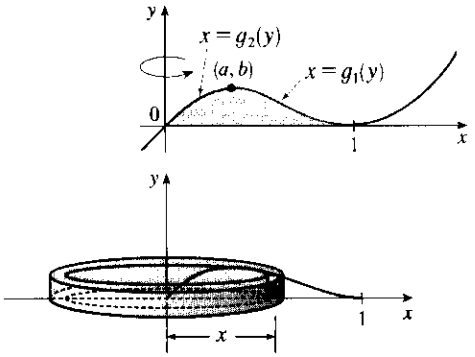
70. It suffices to consider the case where $\mathcal{R}$ is bounded by the curves $y = f(x)$ and $y = g(x)$ for $a \le x \le b$, where $g(x) \le f(x)$ for all x in $[a, b]$, since other regions can be decomposed into subregions of this type. We are concerned with the volume obtained when $\mathcal{R}$ is rotated about the line $y = -k$, which is equal to

$$V_2 = \pi \int_a^b \left([f(x) + k]^2 - [g(x) + k]^2 \right) dx = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx + 2\pi k \int_a^b [f(x) - g(x)]\, dx$$
$$= V_1 + 2\pi k A$$

6.3 Volumes by Cylindrical Shells

1.

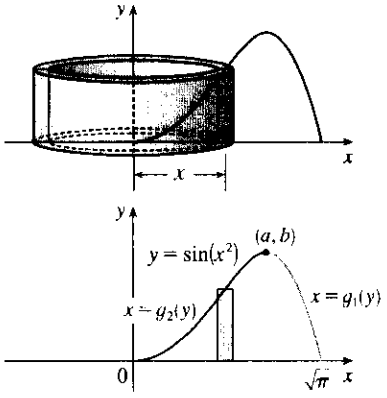


If we were to use the "washer" method, we would first have to locate the local maximum point (a, b) of $y = x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x-1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V = \pi \int_0^b \left\{ [g_1(y)]^2 - [g_2(y)]^2 \right\} dy$.

Using shells, we find that a typical approximating shell has radius x, so its circumference is $2\pi x$. Its height is y, that is, $x(x-1)^2$. So the total volume is

$$V = \int_0^1 2\pi x \left[x(x-1)^2 \right] dx = 2\pi \int_0^1 \left(x^4 - 2x^3 + x^2 \right) dx = 2\pi \left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3} \right]_0^1 = \frac{\pi}{15}$$

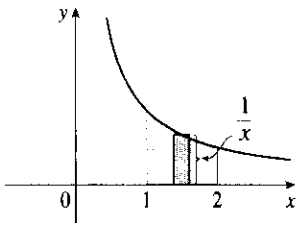
2.

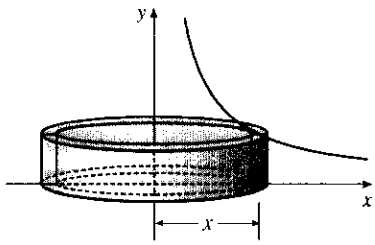


A typical cylindrical shell has circumference $2\pi x$ and height $\sin(x^2)$. $V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\, dx$. Let $u = x^2$. Then $du = 2x\, dx$, so $V = \pi \int_0^{\pi} \sin u\, du = \pi[-\cos u]_0^{\pi} = \pi[1 - (-1)] = 2\pi$.
For slicing, we would first have to locate the local maximum point (a, b) of $y = \sin(x^2)$ using the methods of Chapter 4. Then we would have to solve the equation $y = \sin(x^2)$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the second figure. Finally we would find the volume using $V = \pi \int_0^b \left\{ [g_1(y)]^2 - [g_2(y)]^2 \right\} dy$. Using shells is definitely preferable to slicing.

3. $V = \displaystyle\int_1^2 2\pi x \cdot \frac{1}{x}\, dx = 2\pi \int_1^2 1\, dx$

$= 2\pi\, [x]_1^2 = 2\pi(2 - 1) = 2\pi$





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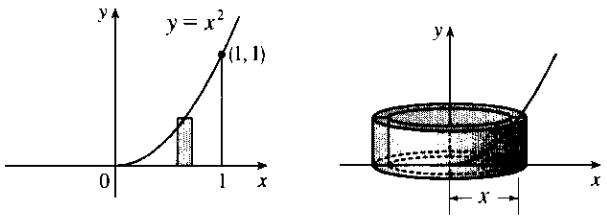
4. $V = \int_0^1 2\pi x \cdot x^2\,dx = 2\pi \int_0^1 x^3\,dx$

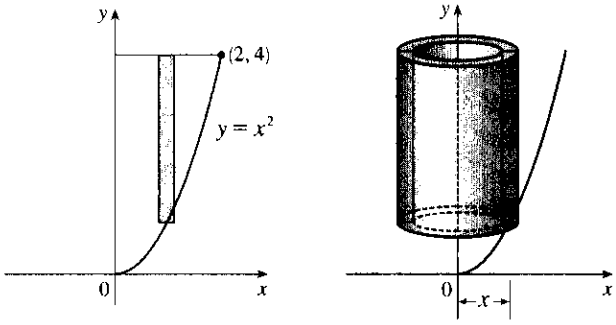
$= 2\pi\left[\frac{1}{4}x^4\right]_0^1 = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$



5. $V = \int_0^2 2\pi x\,(4 - x^2)\,dx = 2\pi \int_0^2 (4x - x^3)\,dx$

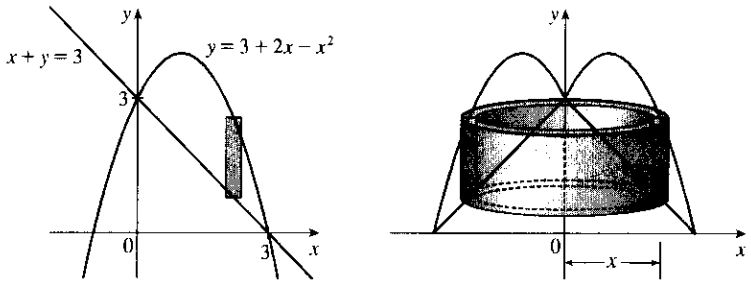
$= 2\pi\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 2\pi\,(8 - 4)$

$= 8\pi$



6. $V = 2\pi \int_0^3 \{x[(3 + 2x - x^2) - (3 - x)]\}\,dx = 2\pi \int_0^3 \left[x(3x - x^2)\right]\,dx$

$= 2\pi \int_0^3 (3x^2 - x^3)\,dx = 2\pi\left[x^3 - \frac{1}{4}x^4\right]_0^3 = 2\pi\left(27 - \frac{81}{4}\right) = 2\pi\left(\frac{27}{4}\right) = \frac{27\pi}{2}$



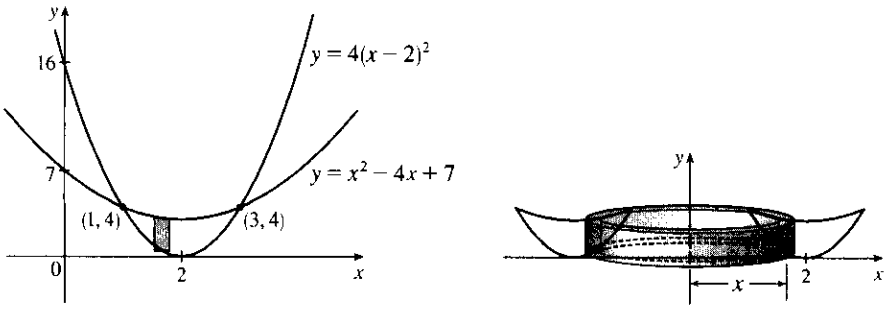
7. The curves intersect when $4(x - 2)^2 = x^2 - 4x + 7 \;\Leftrightarrow\; 4x^2 - 16x + 16 = x^2 - 4x + 7 \;\Leftrightarrow$

$3x^2 - 12x + 9 = 0 \;\Leftrightarrow\; 3(x^2 - 4x + 3) = 0 \;\Leftrightarrow\; 3(x - 1)(x - 3) = 0$, so $x = 1$ or 3.

$V = 2\pi \int_1^3 \{x\left[(x^2 - 4x + 7) - 4(x - 2)^2\right]\}\,dx = 2\pi \int_1^3 \left[x(x^2 - 4x + 7 - 4x^2 + 16x - 16)\right]\,dx$

$= 2\pi \int_1^3 \left[x(-3x^2 + 12x - 9)\right]\,dx = 2\pi(-3) \int_1^3 (x^3 - 4x^2 + 3x)\,dx = -6\pi\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_1^3$

$= -6\pi\left[\left(\frac{81}{4} - 36 + \frac{27}{2}\right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right)\right] = -6\pi\left(20 - 36 + 12 + \frac{4}{3}\right) = -6\pi\left(-\frac{8}{3}\right) = 16\pi$



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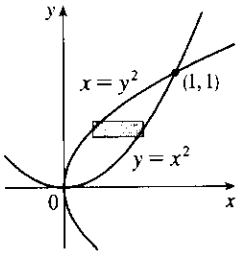
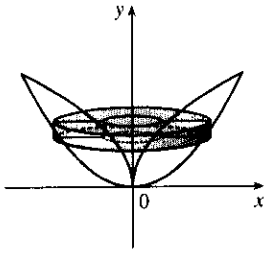
8. By slicing:

$$V = \int_0^1 \pi\left[\left(\sqrt{y}\right)^2 - \left(y^2\right)^2\right] dy = \pi \int_0^1 \left(y - y^4\right) dy = \pi\left[\tfrac{1}{2}y^2 - \tfrac{1}{5}y^5\right]_0^1 = \pi\left(\tfrac{1}{2} - \tfrac{1}{5}\right) = \tfrac{3\pi}{10}$$



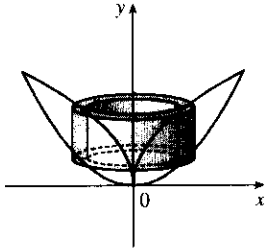


By cylindrical shells:

$$V = \int_0^1 2\pi x\left(\sqrt{x} - x^2\right) dx = 2\pi \int_0^1 \left(x^{3/2} - x^3\right) dx$$

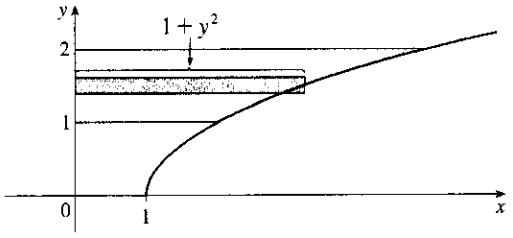
$$= 2\pi\left[\tfrac{2}{5}x^{5/2} - \tfrac{1}{4}x^4\right]_0^1 = 2\pi\left(\tfrac{2}{5} - \tfrac{1}{4}\right)$$

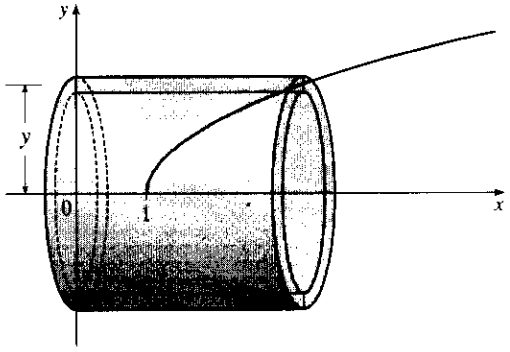
$$= 2\pi\left(\tfrac{3}{20}\right) = \tfrac{3\pi}{10}$$



9. $V = \int_1^2 2\pi y\left(1 + y^2\right) dy = 2\pi \int_1^2 \left(y + y^3\right) dy = 2\pi\left[\tfrac{1}{2}y^2 + \tfrac{1}{4}y^4\right]_1^2$

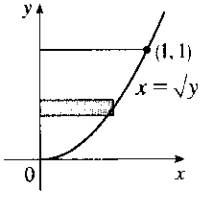
$$= 2\pi\left[(2 + 4) - \left(\tfrac{1}{2} + \tfrac{1}{4}\right)\right] = 2\pi\left(\tfrac{21}{4}\right) = \tfrac{21\pi}{2}$$

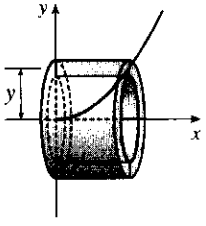




10. $V = \int_0^1 2\pi y\sqrt{y}\, dy = 2\pi \int_0^1 y^{3/2}\, dy$

$$= 2\pi\left[\tfrac{2}{5}y^{5/2}\right]_0^1 = \tfrac{4\pi}{5}$$





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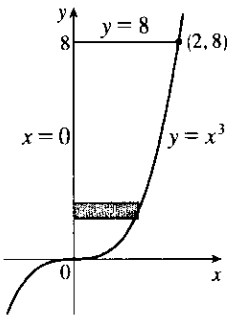
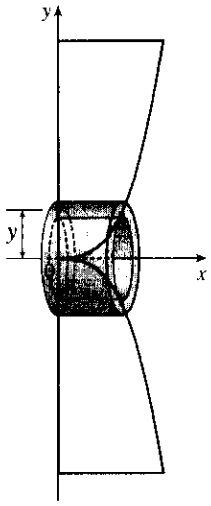
11. $V = 2\pi \int_0^8 [y(\sqrt[3]{y} - 0)]\, dy$

$= 2\pi \int_0^8 y^{4/3}\, dy = 2\pi \left[\frac{3}{7} y^{7/3}\right]_0^8$

$= \frac{6\pi}{7}(8^{7/3}) = \frac{6\pi}{7}(2^7) = \frac{768\pi}{7}$



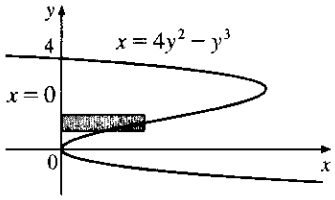


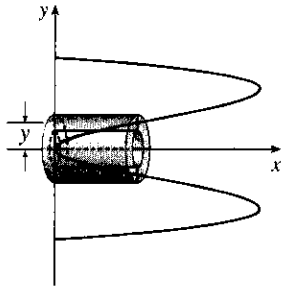
12. $V = 2\pi \int_0^4 [y(4y^2 - y^3)]\, dy$

$= 2\pi \int_0^4 (4y^3 - y^4)\, dy$

$= 2\pi \left[y^4 - \frac{1}{5} y^5\right]_0^4 = 2\pi\left(256 - \frac{1024}{5}\right)$

$= 2\pi\left(\frac{256}{5}\right) = \frac{512\pi}{5}$





13. The curves intersect when $4x^2 = 6 - 2x \;\Leftrightarrow\; 2x^2 + x - 3 = 0 \;\Leftrightarrow\; (2x+3)(x-1) = 0 \;\Leftrightarrow\; x = -\frac{3}{2}$ or 1.

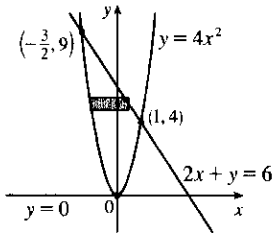
Solving the equations for x gives us $y = 4x^2 \;\Rightarrow\; x = \pm\frac{1}{2}\sqrt{y}$ and $2x + y = 6 \;\Rightarrow\; x = -\frac{1}{2}y + 3$.

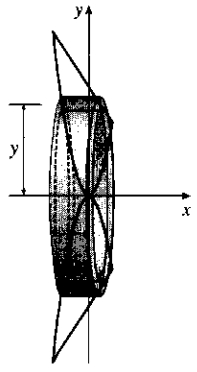
$V = 2\pi \int_0^4 \left\{y\left[\left(\frac{1}{2}\sqrt{y}\right) - \left(-\frac{1}{2}\sqrt{y}\right)\right]\right\} dy + 2\pi \int_4^9 \left\{y\left[\left(-\frac{1}{2}y + 3\right) - \left(-\frac{1}{2}\sqrt{y}\right)\right]\right\} dy$

$= 2\pi \int_0^4 (y\sqrt{y})\, dy + 2\pi \int_4^9 \left(-\frac{1}{2}y^2 + 3y + \frac{1}{2}y^{3/2}\right) dy = 2\pi\left[\frac{2}{5}y^{5/2}\right]_0^4 + 2\pi\left[-\frac{1}{6}y^3 + \frac{3}{2}y^2 + \frac{1}{5}y^{5/2}\right]_4^9$

$= 2\pi\left(\frac{2}{5} \cdot 32\right) + 2\pi\left[\left(-\frac{243}{2} + \frac{243}{2} + \frac{243}{5}\right) - \left(-\frac{32}{3} + 24 + \frac{32}{5}\right)\right]$

$= \frac{128}{5}\pi + 2\pi\left(\frac{433}{15}\right) = \frac{1250}{15}\pi = \frac{250}{3}\pi$





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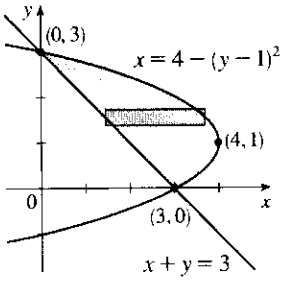
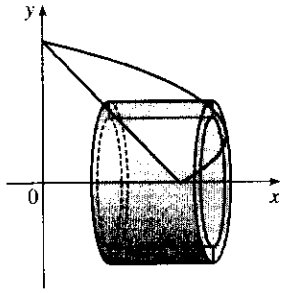
14. $V = \int_0^3 2\pi y\left[4-(y-1)^2-(3-y)\right]dy$

$= 2\pi\int_0^3 y(-y^2+3y)\,dy$

$= 2\pi\int_0^3 (-y^3+3y^2)\,dy = 2\pi\left[-\frac{1}{4}y^4+y^3\right]_0^3$

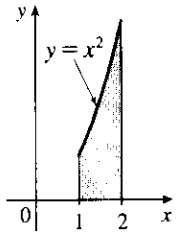
$= 2\pi\left(-\frac{81}{4}+27\right) = 2\pi\left(\frac{27}{4}\right) = \frac{27\pi}{2}$

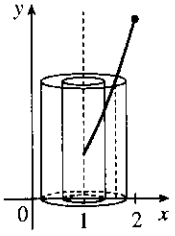




15. $V = \int_1^2 2\pi(x-1)x^2\,dx = 2\pi\left[\frac{1}{4}x^4-\frac{1}{3}x^3\right]_1^2$

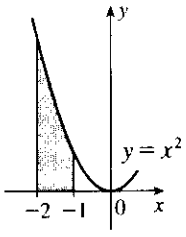
$= 2\pi\left[\left(4-\frac{8}{3}\right)-\left(\frac{1}{4}-\frac{1}{3}\right)\right] = \frac{17}{6}\pi$

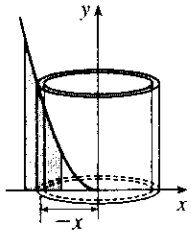




16. $V = \int_{-2}^{-1} 2\pi(-x)\cdot x^2\,dx = 2\pi\left[-\frac{1}{4}x^4\right]_{-2}^{-1}$

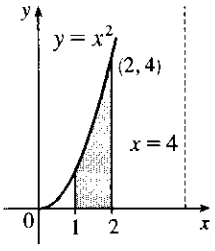
$= 2\pi\left[\left(-\frac{1}{4}\right)-(-4)\right] = \frac{15}{2}\pi$

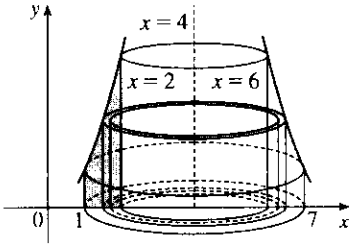




17. $V = \int_1^2 2\pi(4-x)x^2\,dx = 2\pi\left[\frac{4}{3}x^3-\frac{1}{4}x^4\right]_1^2$

$= 2\pi\left[\left(\frac{32}{3}-4\right)-\left(\frac{4}{3}-\frac{1}{4}\right)\right] = \frac{67}{6}\pi$





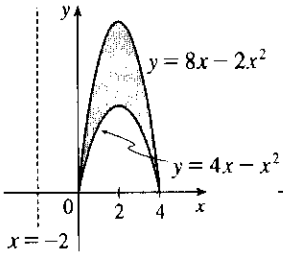
18. $V = \int_0^4 2\pi[x-(-2)][(8x-2x^2)-(4x-x^2)]\,dx$

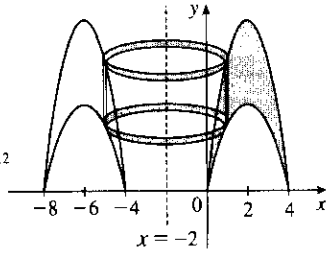
$= \int_0^4 2\pi(2+x)(4x-x^2)\,dx$

$= 2\pi\int_0^4 (8x+2x^2-x^3)\,dx$

$= 2\pi\left[4x^2+\frac{2}{3}x^3-\frac{1}{4}x^4\right]_0^4$

$= 2\pi\left(64+\frac{128}{3}-64\right) = \frac{256}{3}\pi$





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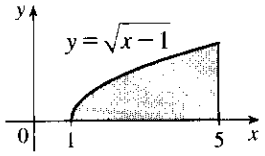
19. $V = \int_0^2 2\pi(3-y)(5-x)dy$

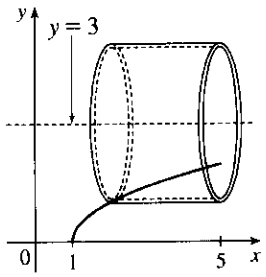
$= \int_0^2 2\pi(3-y)\left(5-y^2-1\right)dy$

$= \int_0^2 2\pi\left(12-4y-3y^2+y^3\right)dy$

$= 2\pi\left[12y-2y^2-y^3+\tfrac{1}{4}y^4\right]_0^2$

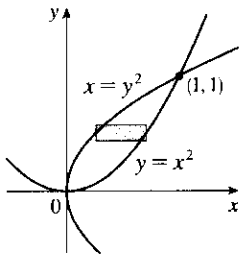
$= 2\pi(24-8-8+4) = 24\pi$

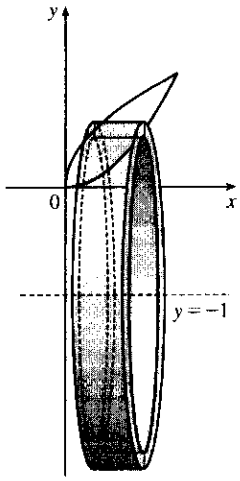




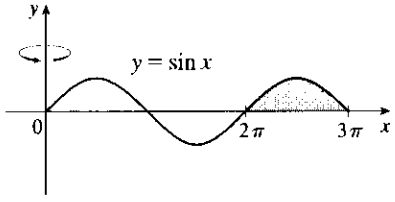
20. $V = \int_0^1 2\pi(y+1)\left(\sqrt{y}-y^2\right)dy = 2\pi\int_0^1 \left(y^{3/2}+y^{1/2}-y^3-y^2\right)dy$

$= 2\pi\left[\tfrac{2}{5}y^{5/2}+\tfrac{2}{3}y^{3/2}-\tfrac{1}{4}y^4-\tfrac{1}{3}y^3\right]_0^1 = 2\pi\left(\tfrac{2}{5}+\tfrac{2}{3}-\tfrac{1}{4}-\tfrac{1}{3}\right) = 2\pi\left(\tfrac{29}{60}\right) = \tfrac{29\pi}{30}$

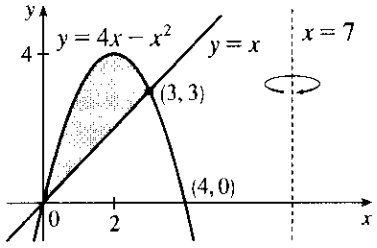




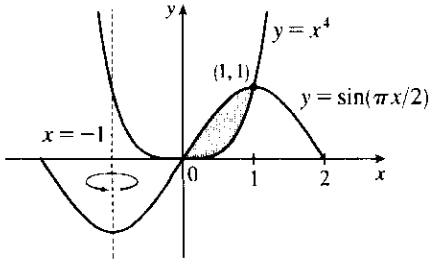
21. $V = \int_{2\pi}^{3\pi} 2\pi x \sin x\, dx$



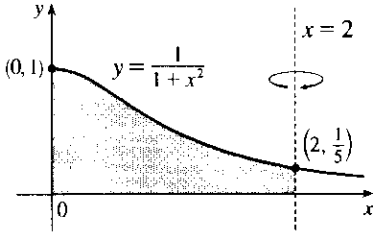
22. $V = \int_0^3 2\pi(7-x)\left[\left(4x-x^2\right)-x\right]dx$



23. $V = \int_0^1 2\pi[x-(-1)]\left(\sin\tfrac{\pi}{2}x-x^4\right)dx$



24. $V = \int_0^2 2\pi(2-x)\left(\dfrac{1}{1+x^2}\right)dx$



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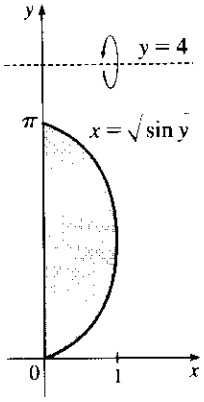
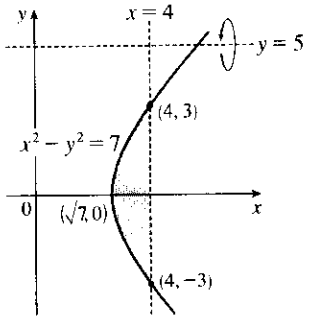
25. $V = \int_0^{\pi} 2\pi(4-y)\sqrt{\sin y}\,dy$



26. $V = \int_{-3}^{3} 2\pi(5-y)\left(4-\sqrt{y^2+7}\right)dy$



27. $\Delta x = \dfrac{\pi/4 - 0}{4} = \dfrac{\pi}{16}$.

$V = \int_0^{\pi/4} 2\pi x \tan x\,dx \approx 2\pi \cdot \frac{\pi}{16}\left(\frac{\pi}{32}\tan\frac{\pi}{32} + \frac{3\pi}{32}\tan\frac{3\pi}{32} + \frac{5\pi}{32}\tan\frac{5\pi}{32} + \frac{7\pi}{32}\tan\frac{7\pi}{32}\right) \approx 1.142$

28. $\Delta x = \frac{12-2}{5} = 2$, $n = 5$ and $x_i^* = 2 + (2i+1)$, where $i = 0, 1, 2, 3, 4$. The values of $f(x)$ are taken directly from the diagram.

$$V = \int_2^{12} 2\pi x f(x)\,dx \approx 2\pi[3f(3) + 5f(5) + 7f(7) + 9f(9) + 11f(11)] \cdot 2$$

$$\approx 2\pi[3(2) + 5(4) + 7(4) + 9(2) + 11(1)]2 = 332\pi$$

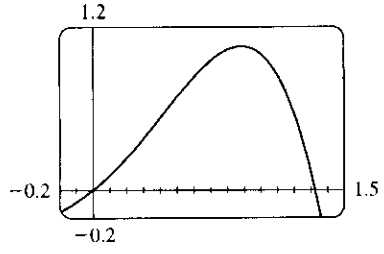
29. $\int_0^3 2\pi x^5\,dx = 2\pi \int_0^3 x(x^4)\,dx$. The solid is obtained by rotating the region $0 \le y \le x^4$, $0 \le x \le 3$ about the y-axis using cylindrical shells.

30. $2\pi \displaystyle\int_0^2 \frac{y}{1+y^2}\,dy = 2\pi \int_0^2 y\left(\frac{1}{1+y^2}\right)dy$. The solid is obtained by rotating the region $0 \le x \le \dfrac{1}{1+y^2}$, $0 \le y \le 2$ about the x-axis using cylindrical shells.

31. $\int_0^1 2\pi(3-y)(1-y^2)\,dy$. The solid is obtained by rotating the region bounded by (i) $x = 1 - y^2$, $x = 0$, and $y = 0$ or (ii) $x = y^2$, $x = 1$, and $y = 0$ about the line $y = 3$ using cylindrical shells.

32. $\int_0^{\pi/4} 2\pi(\pi - x)(\cos x - \sin x)\,dx$. The solid is obtained by rotating tbe region bounded by (i) $0 \le y \le \cos x - \sin x$, $0 \le x \le \frac{\pi}{4}$ or (ii) $\sin x \le y \le \cos x$, $0 \le x \le \frac{\pi}{4}$ about the line $x = \pi$ using cylindrical shells.

33.



From the graph, the curves intersect at $x = 0$ and at $x = a \approx 1.32$, with $x + x^2 - x^4 > 0$ on the interval $(0, a)$. So the volume of the solid obtained by rotating the region about the y-axis is

$$V = 2\pi \int_0^a \left[x(x + x^2 - x^4)\right]dx = 2\pi \int_0^a (x^2 + x^3 - x^5)\,dx$$

$$= 2\pi\left[\tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^a \approx 4.05$$

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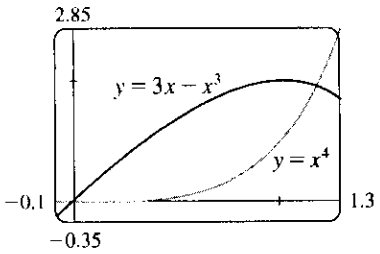
34.

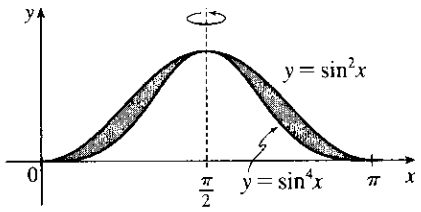


From the graph, the curves intersect at $x = 0$ and at $x = a \approx 1.17$, with $3x - x^3 > x^4$ on the interval $(0, a)$. So the volume of the solid obtained by rotating the region about the y-axis is

$$V = 2\pi \int_0^a \{x[(3x - x^3) - x^4]\}\, dx = 2\pi \int_0^a (3x^2 - x^4 - x^5)\, dx$$
$$= 2\pi \left[x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6\right]_0^a \approx 4.62$$

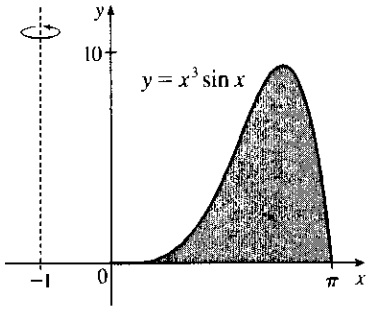
35. $V = 2\pi \displaystyle\int_0^{\pi/2} \left[\left(\tfrac{\pi}{2} - x\right)\left(\sin^2 x - \sin^4 x\right)\right] dx$

$\overset{\text{CAS}}{=} \tfrac{1}{32}\pi^3$



36. $V = 2\pi \displaystyle\int_0^{\pi} \{[x - (-1)](x^3 \sin x)\}\, dx \overset{\text{CAS}}{=} 2\pi(\pi^4 + \pi^3 - 12\pi^2 - 6\pi + 48)$

$= 2\pi^5 + 2\pi^4 - 24\pi^3 - 12\pi^2 + 96\pi$



37. Use disks:

$$V = \int_{-2}^{1} \pi\left(x^2 + x - 2\right)^2\, dx = \pi \int_{-2}^{1} \left(x^4 + 2x^3 - 3x^2 - 4x + 4\right) dx$$
$$= \pi\left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 - x^3 - 2x^2 + 4x\right]_{-2}^{1} = \pi\left[\left(\tfrac{1}{5} + \tfrac{1}{2} - 1 - 2 + 4\right) - \left(-\tfrac{32}{5} + 8 + 8 - 8 - 8\right)\right]$$
$$= \pi\left(\tfrac{33}{5} + \tfrac{3}{2}\right) = \tfrac{81}{10}\pi$$

38. Use shells:

$$V = \int_1^2 2\pi x\left(-x^2 + 3x - 2\right) dx = 2\pi \int_1^2 \left(-x^3 + 3x^2 - 2x\right) dx$$
$$= 2\pi\left[-\tfrac{1}{4}x^4 + x^3 - x^2\right]_1^2 = 2\pi\left[(-4 + 8 - 4) - \left(-\tfrac{1}{4} + 1 - 1\right)\right] = \tfrac{\pi}{2}$$

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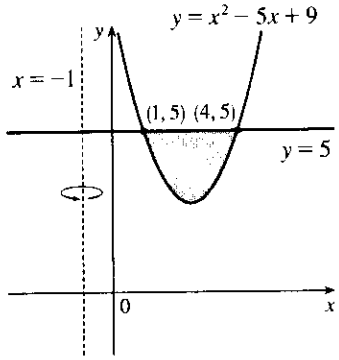
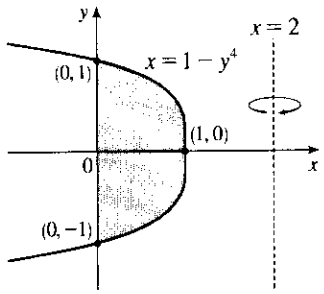
39. Use shells:

$$V = \int_1^4 2\pi\,[x-(-1)]\left[5-\left(x^2-5x+9\right)\right]dx$$
$$= 2\pi\int_1^4 (x+1)\left(-x^2+5x-4\right)dx$$
$$= 2\pi\int_1^4 \left(-x^3+4x^2+x-4\right)dx = 2\pi\left[-\tfrac{1}{4}x^4+\tfrac{4}{3}x^3+\tfrac{1}{2}x^2-4x\right]_1^4$$
$$= 2\pi\left[\left(-64+\tfrac{256}{3}+8-16\right)-\left(-\tfrac{1}{4}+\tfrac{4}{3}+\tfrac{1}{2}-4\right)\right]$$
$$= 2\pi\left(\tfrac{63}{4}\right) = \tfrac{63\pi}{2}$$



40. Use washers:

$$V = \int_{-1}^1 \pi\left\{[2-0]^2-\left[2-\left(1-y^4\right)\right]^2\right\}dy$$
$$= 2\pi\int_0^1 \left[4-\left(1+y^4\right)^2\right]dy \qquad \text{[by symmetry]}$$
$$= 2\pi\int_0^1 \left[4-\left(1+2y^4+y^8\right)\right]dy = 2\pi\int_0^1 \left(3-2y^4-y^8\right)dy$$
$$= 2\pi\left[3y-\tfrac{2}{5}y^5-\tfrac{1}{9}y^9\right]_0^1 = 2\pi\left(3-\tfrac{2}{5}-\tfrac{1}{9}\right) = 2\pi\left(\tfrac{112}{45}\right) = \tfrac{224\pi}{45}$$



41. Use disks: $V = \pi\displaystyle\int_0^2 \left[\sqrt{1-(y-1)^2}\right]^2 dy = \pi\int_0^2 \left(2y-y^2\right)dy = \pi\left[y^2-\tfrac{1}{3}y^3\right]_0^2 = \pi\left(4-\frac{8}{3}\right) = \frac{4}{3}\pi$

42. Using shells, we have

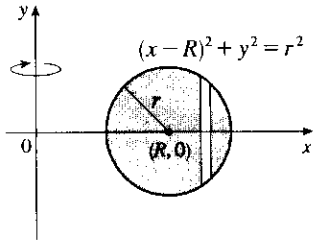
$$V = \int_0^2 2\pi y\left[\sqrt{1-(y-1)^2}-\left(-\sqrt{1-(y-1)^2}\right)\right]dy$$
$$= 2\pi\int_0^2 y\cdot 2\sqrt{1-(y-1)^2}\,dy = 4\pi\int_{-1}^1 (u+1)\sqrt{1-u^2}\,du \qquad [\text{let } u = y-1]$$
$$= 4\pi\int_{-1}^1 u\sqrt{1-u^2}\,du + 4\pi\int_{-1}^1 \sqrt{1-u^2}\,du$$

The first definite integral equals zero because its integrand is an odd function. The second is the area of a semicircle of radius 1, that is, $\frac{\pi}{2}$. Thus, $V = 4\pi\cdot 0 + 4\pi\cdot\frac{\pi}{2} = 2\pi^2$.

43. $V = 2\displaystyle\int_0^r 2\pi x\sqrt{r^2-x^2}\,dx = -2\pi\int_0^r \left(r^2-x^2\right)^{1/2}(-2x)\,dx = \left[-2\pi\cdot\tfrac{2}{3}\left(r^2-x^2\right)^{3/2}\right]_0^r$
$= -\frac{4}{3}\pi\left(0-r^3\right) = \frac{4}{3}\pi r^3$

44. $V = \displaystyle\int_{R-r}^{R+r} 2\pi x\cdot 2\sqrt{r^2-(x-R)^2}\,dx$
$= \displaystyle\int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \qquad [\text{let } u = x-R]$
$= 4\pi R\displaystyle\int_{-r}^{r}\sqrt{r^2-u^2}\,du + 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du$

The first integral is the area of a semicircle of radius r, that is, $\frac{1}{2}\pi r^2$, and the second is zero since the integrand is an odd function. Thus, $V = 4\pi R\left(\frac{1}{2}\pi r^2\right) + 4\pi\cdot 0 = 2\pi R r^2$.



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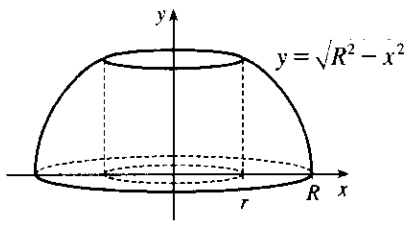
45. $V = 2\pi \int_0^r x\left(-\frac{h}{r}x + h\right) dx = 2\pi h \int_0^r \left(-\frac{x^2}{r} + x\right) dx = 2\pi h\left[-\frac{x^3}{3r} + \frac{x^2}{2}\right]_0^r = 2\pi h\,\frac{r^2}{6} = \frac{\pi r^2 h}{3}$

46. By symmetry, the volume of a napkin ring obtained by drilling a hole of radius r through a sphere with radius R is twice the volume obtained by rotating the area above the x-axis and below the curve $y = \sqrt{R^2 - x^2}$ (the equation of the top half of the cross-section of the sphere), between $x = r$ and $x = R$, about the y-axis.



This volume is equal to

$$2\int_{\text{inner radius}}^{\text{outer radius}} 2\pi r h\,dx = 2\cdot 2\pi \int_r^R x\sqrt{R^2 - x^2}\,dx = 4\pi\left[-\tfrac{1}{3}\left(R^2 - x^2\right)^{3/2}\right]_r^R = \tfrac{4}{3}\pi\left(R^2 - r^2\right)^{3/2}$$

But by the Pythagorean Theorem, $R^2 - r^2 = \left(\frac{1}{2}h\right)^2$, so the volume of the napkin ring is $\frac{4}{3}\pi\left(\frac{1}{2}h\right)^3 = \frac{1}{6}\pi h^3$, which is independent of both R and r; that is, the amount of wood in a napkin ring of height h is the same regardless of the size of the sphere used. Note that most of this calculation has been done already, but with more difficulty, in Exercise 6.2.68.

Another solution: The height of the missing cap is the radius of the sphere minus half the height of the cut-out cylinder, that is, $R - \frac{1}{2}h$. Using Exercise 6.2.49,

$$V_{\text{napkin ring}} = V_{\text{sphere}} - V_{\text{cylinder}} - 2V_{\text{cap}} = \tfrac{4}{3}\pi R^3 - \pi r^2 h - 2\cdot\tfrac{\pi}{3}\left(R - \tfrac{1}{2}h\right)^2\left[3R - \left(R - \tfrac{1}{2}h\right)\right] = \tfrac{1}{6}\pi h^3$$

6.4 Work

1. By Equation 2, $W = Fd = (900)(8) = 7200$ J.

2. $F = mg = (60)(9.8) = 588$ N; $W = Fd = 588 \cdot 2 = 1176$ J

3. $W = \int_a^b f(x)\,dx = \int_0^9 \frac{10}{(1+x)^2}\,dx = 10\int_1^{10} \frac{1}{u^2}\,du \qquad [u = 1 + x,\ du = dx]$

$= 10\left[-\frac{1}{u}\right]_1^{10} = 10\left(-\frac{1}{10} + 1\right) = 9$ ft-lb

4. $W = \int_1^2 \cos\left(\frac{1}{3}\pi x\right) dx = \frac{3}{\pi}\left[\sin\left(\frac{1}{3}\pi x\right)\right]_1^2 = \frac{3}{\pi}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = 0$ N-m $= 0$ J.

Interpretation: From $x = 1$ to $x = \frac{3}{2}$, the force does work equal to $\int_1^{3/2} \cos\left(\frac{1}{3}\pi x\right)dx = \frac{3}{\pi}\left(1 - \frac{\sqrt{3}}{2}\right)$ J in accelerating the particle and increasing its kinetic energy. From $x = \frac{3}{2}$ to $x = 2$, the force opposes the motion of the particle, decreasing its kinetic energy. This is negative work, equal in magnitude but opposite in sign to the work done from $x = 1$ to $x = \frac{3}{2}$.

5. The force function is given by $F(x)$ (in newtons) and the work (in joules) is the area under the curve, given by $\int_0^8 F(x)\,dx = \int_0^4 F(x)\,dx + \int_4^8 F(x)\,dx = \frac{1}{2}(4)(30) + (4)(30) = 180$ J.

6. $W = \int_4^{20} f(x)\,dx \approx M_4 = \Delta x[f(6) + f(10) + f(14) + f(18)] = \frac{20-4}{4}[5.8 + 8.8 + 8.2 + 5.2] = 4(28) = 112$ J

7. $10 = f(x) = kx = \frac{1}{3}k$ [4 inches $= \frac{1}{3}$ foot], so $k = 30$ lb/ft and $f(x) = 30x$. Now 6 inches $= \frac{1}{2}$ foot, so $W = \int_0^{1/2} 30x\,dx = \left[15x^2\right]_0^{1/2} = \frac{15}{4}$ ft-lb.

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8. $25 = f(x) = kx = k(0.1)$ [10 cm = 0.1 m], so $k = 250$ N/m and $f(x) = 250x$. Now 5 cm = 0.05 m, so $W = \int_0^{0.05} 250x\,dx = \left[125x^2\right]_0^{0.05} = 125(0.0025) = 0.3125 \approx 0.31$ J.

9. If $\int_0^{0.12} kx\,dx = 2$ J, then $2 = \left[\frac{1}{2}kx^2\right]_0^{0.12} = \frac{1}{2}k(0.0144) = 0.0072k$ and $k = \frac{2}{0.0072} = \frac{2500}{9} \approx 277.78$ N/m. Thus, the work needed to stretch the spring from 35 cm to 40 cm is $\int_{0.05}^{0.10} \frac{2500}{9}x\,dx = \left[\frac{1250}{9}x^2\right]_{1/20}^{1/10} = \frac{1250}{9}\left(\frac{1}{100} - \frac{1}{400}\right) = \frac{25}{24} \approx 1.04$ J.

10. If $12 = \int_0^1 kx\,dx = \left[\frac{1}{2}kx^2\right]_0^1 = \frac{1}{2}k$, then $k = 24$ lb/ft and the work required is $\int_0^{3/4} 24x\,dx = \left[12x^2\right]_0^{3/4} = 12 \cdot \frac{9}{16} = \frac{27}{4} = 6.75$ ft-lb.

11. $f(x) = kx$, so $30 = \frac{2500}{9}x$ and $x = \frac{270}{2500}$ m $= 10.8$ cm

12. Let L be the natural length of the spring in meters. Then $6 = \int_{0.10-L}^{0.12-L} kx\,dx = \left[\frac{1}{2}kx^2\right]_{0.10-L}^{0.12-L} = \frac{1}{2}k\left[(0.12-L)^2 - (0.10-L)^2\right]$ and $10 = \int_{0.12-L}^{0.14-L} kx\,dx = \left[\frac{1}{2}kx^2\right]_{0.12-L}^{0.14-L} = \frac{1}{2}k\left[(0.14-L)^2 - (0.12-L)^2\right]$. Simplifying gives us $12 = k(0.0044 - 0.04L)$ and $20 = k(0.0052 - 0.04L)$. Subtracting the first equation from the second gives $8 = 0.0008k$, so $k = 10{,}000$. Now the second equation becomes $20 = 52 - 400L$, so $L = \frac{32}{400}$ m $= 8$ cm.

In Exercises 13–20, n is the number of subintervals of length Δx, and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

13. (a) The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\,\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\,\Delta x$ ft-lb. The total work is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}x_i^*\,\Delta x = \int_0^{50} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{50} = \frac{2500}{4} = 625 \text{ ft-lb}$$

Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

(b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is $W_1 = \int_0^{25} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{25} = \frac{625}{4}$ ft-lb. The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is $W_2 = \int_{25}^{50} \frac{1}{2} \cdot 25\,dx = \frac{25}{2}[x]_{25}^{50} = \frac{625}{2}$ ft-lb. The total work done in pulling half the rope to the top of the building is $W = W_1 + W_2 = \frac{625}{2} + \frac{625}{4} = \frac{3}{4} \cdot 625 = \frac{1875}{4}$ ft-lb.

14. *Assumptions*: 1. After lifting, the chain is L-shaped, with 4 m of the chain lying along the ground.

2. The chain slides effortlessly and without friction along the ground while its end is lifted.

3. The weight density of the chain is constant throughout its length and therefore equals

$(8 \text{ kg/m})(9.8 \text{ m/s}^2) = 78.4 \text{ N/m}.$

The part of the chain x m from the lifted end is raised $6 - x$ m if $0 \le x \le 6$ m, and it is lifted 0 m if $x > 6$ m. Thus, the work needed is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} (6 - x_i^*) \cdot 78.4\,\Delta x = \int_0^6 (6-x)78.4\,dx = 78.4\left[6x - \frac{1}{2}x^2\right]_0^6 = (78.4)(18) = 1411.2 \text{ J}.$$

15. The work needed to lift the cable is $\lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^*\,\Delta x = \int_0^{500} 2x\,dx = \left[x^2\right]_0^{500} = 250{,}000$ ft-lb. The work needed to lift the coal is 800 lb $\cdot$ 500 ft $= 400{,}000$ ft-lb. Thus, the total work required is $250{,}000 + 400{,}000 = 650{,}000$ ft-lb.

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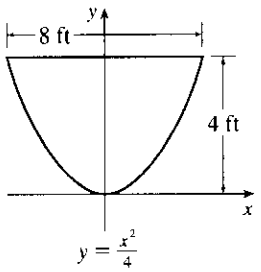
28. The work needed to raise the elevator alone is $1600 \text{ lb} \times 30 \text{ ft} = 48{,}000$ ft-lb. The work needed to raise the bottom 170 ft of cable is $170 \text{ ft} \times 10 \text{ lb/ft} \times 30 \text{ ft} = 51{,}000$ ft-lb. The work needed to raise the top 30 ft of cable is $\int_0^{30} 10x\,dx = \left[5x^2\right]_0^{30} = 5 \cdot 900 = 4500$ ft-lb. Adding these, we see that the total work needed is $48{,}000 + 51{,}000 + 4{,}500 = 103{,}500$ ft-lb.

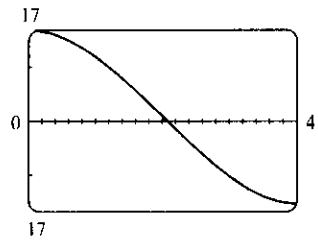
29. (a) The parabola has equation $y = ax^2$ with vertex at the origin and passing through $(4, 4)$. $4 = a \cdot 4^2 \;\Rightarrow\; a = \frac{1}{4} \;\Rightarrow\; y = \frac{1}{4}x^2 \;\Rightarrow\; x^2 = 4y \;\Rightarrow\; x = 2\sqrt{y}$. Each circular disk has radius $2\sqrt{y}$ and is moved $4 - y$ ft.



$$W = \int_0^4 \pi \left(2\sqrt{y}\right)^2 62.5\,(4 - y)\,dy = 250\pi \int_0^4 y(4 - y)\,dy$$

$$= 250\pi\left[2y^2 - \tfrac{1}{3}y^3\right]_0^4 = 250\pi\left(32 - \tfrac{64}{3}\right) = \tfrac{8000\pi}{3} \approx 8378 \text{ ft-lb}$$

(b) In part (a) we knew the final water level (0) but not the amount of work done. Here we use the same equation, except with the work fixed, and the lower limit of integration (that is, the final water level — call it h) unknown: $W = 4000 \;\Leftrightarrow\; 250\pi\left[2y^2 - \frac{1}{3}y^3\right]_h^4 = 4000 \;\Leftrightarrow$



$\frac{16}{\pi} = \left[\left(32 - \frac{64}{3}\right) - \left(2h^2 - \frac{1}{3}h^3\right)\right] \;\Leftrightarrow\; h^3 - 6h^2 + 32 - \frac{48}{\pi} = 0.$

We graph the function $f(h) = h^3 - 6h^2 + 32 - \frac{48}{\pi}$ on the interval $[0, 4]$ to see where it is 0. From the graph, $f(h) = 0$ for $h \approx 2.1$. So the depth of water remaining is about 2.1 ft.

30. $f_{\text{ave}} = \frac{1}{10 - 0}\int_0^{10} t\,\sin(t^2)\,dt = \frac{1}{10}\int_0^{100} \sin u\left(\frac{1}{2}\,du\right) \qquad [u = t^2,\ du = 2t\,dt]$

$$= \frac{1}{20}\Big[-\cos u\Big]_0^{100} = \frac{1}{20}(-\cos 100 + \cos 0) = \frac{1}{20}(1 - \cos 100) \approx 0.007$$

31. $\lim_{h\to 0} f_{\text{ave}} = \lim_{h\to 0} \dfrac{1}{(x + h) - x}\displaystyle\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} \frac{F(x + h) - F(x)}{h}$, where $F(x) = \int_a^x f(t)\,dt$. But we recognize this limit as being $F'(x)$ by the definition of a derivative. Therefore, $\lim_{h\to 0} f_{\text{ave}} = F'(x) = f(x)$ by FTC1.

32. (a) $\mathcal{R}_1$ is the region below the graph of $y = x^2$ and above the x-axis between $x = 0$ and $x = b$, and $\mathcal{R}_2$ is the region to the left of the graph of $x = \sqrt{y}$ and to the right of the y-axis between $y = 0$ and $y = b^2$. So the area of $\mathcal{R}_1$ is $A_1 = \int_0^b x^2\,dx = \left[\frac{1}{3}x^3\right]_0^b = \frac{1}{3}b^3$, and the area of $\mathcal{R}_2$ is $A_2 = \int_0^{b^2} \sqrt{y}\,dy = \left[\frac{2}{3}y^{3/2}\right]_0^{b^2} = \frac{2}{3}b^3$. So there is no solution to $A_1 = A_2$ for $b \neq 0$.

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(b) Using disks, we calculate the volume of rotation of $\mathcal{R}_1$ about the x-axis to be $V_{1,x} = \pi \int_0^b \left(x^2\right)^2 dx = \frac{1}{5}\pi b^5$.

Using cylindrical shells, we calculate the volume of rotation of $\mathcal{R}_1$ about the y-axis to be

$V_{1,y} = 2\pi \int_0^b x\left(x^2\right) dx = 2\pi\left[\frac{1}{4}x^4\right]_0^b = \frac{1}{2}\pi b^4$. So $V_{1,x} = V_{1,y} \quad \Leftrightarrow \quad \frac{1}{5}\pi b^5 = \frac{1}{2}\pi b^4 \quad \Leftrightarrow \quad 2b = 5 \quad \Leftrightarrow$

$b = \frac{5}{2}$. So the volumes of rotation about the x- and y-axes are the same for $b = \frac{5}{2}$.

(c) We use cylindrical shells to calculate the volume of rotation of $\mathcal{R}_2$ about the x-axis:

$\mathcal{R}_{2,x} = 2\pi \int_0^{b^2} y\left(\sqrt{y}\right) dy = 2\pi\left[\frac{2}{5}y^{5/2}\right]_0^{b^2} = \frac{4}{5}\pi b^5$. We already know the volume of rotation of $\mathcal{R}_1$ about the x-axis from part (b), and $\mathcal{R}_{1,x} = \mathcal{R}_{2,x} \quad \Leftrightarrow \quad \frac{1}{5}\pi b^5 = \frac{4}{5}\pi b^5$, which has no solution for $b \neq 0$.

(d) We use disks to calculate the volume of rotation of $\mathcal{R}_2$ about the y-axis:

$\mathcal{R}_{2,y} = \pi \int_0^{b^2} \left(\sqrt{y}\right)^2 dy = \pi\left[\frac{1}{2}y^2\right]_0^{b^2} = \frac{1}{2}\pi b^4$. We know the volume of rotation of $\mathcal{R}_1$ about the y-axis from part (b), and $\mathcal{R}_{1,y} = \mathcal{R}_{2,y} \quad \Leftrightarrow \quad \frac{1}{2}\pi b^4 = \frac{1}{2}\pi b^4$. But this equation is true for all b, so the volumes of rotation about the y-axis are equal for all values of b.

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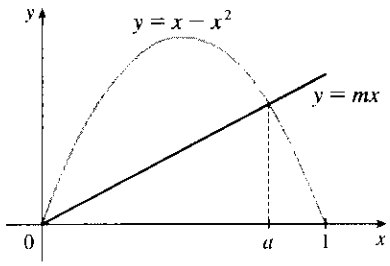
PROBLEMS PLUS

1. (a) The area under the graph of f from 0 to t is equal to $\int_0^t f(x)\,dx$, so the requirement is that $\int_0^t f(x)\,dx = t^3$ for all t. We differentiate both sides of this equation with respect to t (with the help of FTC1) to get $f(t) = 3t^2$. This function is positive and continuous, as required.

(b) The volume generated from $x = 0$ to $x = b$ is $\int_0^b \pi[f(x)]^2\,dx$. Hence, we are given that $b^2 = \int_0^b \pi[f(x)]^2\,dx$ for all $b > 0$. Differentiating both sides of this equation with respect to b using the Fundamental Theorem of Calculus gives $2b = \pi[f(b)]^2 \quad \Rightarrow \quad f(b) = \sqrt{2b/\pi}$, since f is positive. Therefore, $f(x) = \sqrt{2x/\pi}$.

2. The total area of the region bounded by the parabola $y = x - x^2 = x(1-x)$ and the x-axis is

$\int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{6}$.

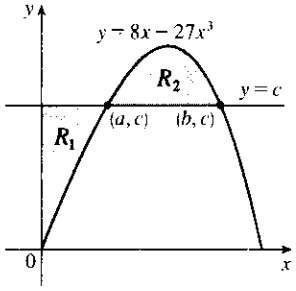


Let the slope of the line we are looking for be m. Then the area above this line but below the parabola is $\int_0^a \left[(x - x^2) - mx\right]dx$, where a is the x-coordinate of the point of intersection of the line and the parabola. We find the point of intersection by solving the equation $x - x^2 = mx \quad \Leftrightarrow$ $1 - x = m \quad \Leftrightarrow \quad x = 1 - m$. So the value of a is $1 - m$, and

$$\int_0^{1-m} \left[(x - x^2) - mx\right]dx = \int_0^{1-m} \left[(1-m)x - x^2\right]dx = \left[\tfrac{1}{2}(1-m)x^2 - \tfrac{1}{3}x^3\right]_0^{1-m}$$
$$= \tfrac{1}{2}(1-m)(1-m)^2 - \tfrac{1}{3}(1-m)^3 = \tfrac{1}{6}(1-m)^3$$

We want this to be half of $\frac{1}{6}$, so $\frac{1}{6}(1-m)^3 = \frac{1}{12} \quad \Leftrightarrow \quad (1-m)^3 = \frac{6}{12} \quad \Leftrightarrow$ $1 - m = \sqrt[3]{\frac{1}{2}} \quad \Leftrightarrow \quad m = 1 - \frac{1}{\sqrt[3]{2}}$. So tbe slope of the required line is $1 - \frac{1}{\sqrt[3]{2}} \approx 0.206$.

3. Let a and b be the x-coordinates of the points wbere the line intersects the curve. From the figure, $R_1 = R_2 \quad \Rightarrow$



$$\int_0^a \left[c - (8x - 27x^3)\right]dx = \int_a^b \left[(8x - 27x^3) - c\right]dx$$

$$\left[cx - 4x^2 + \tfrac{27}{4}x^4\right]_0^a = \left[4x^2 - \tfrac{27}{4}x^4 - cx\right]_a^b$$

$$ac - 4a^2 + \tfrac{27}{4}a^4 = \left(4b^2 - \tfrac{27}{4}b^4 - bc\right) - \left(4a^2 - \tfrac{27}{4}a^4 - ac\right)$$

$$0 = 4b^2 - \tfrac{27}{4}b^4 - bc = 4b^2 - \tfrac{27}{4}b^4 - b\left(8b - 27b^3\right)$$

$$= 4b^2 - \tfrac{27}{4}b^4 - 8b^2 + 27b^4 = \tfrac{81}{4}b^4 - 4b^2$$

$$= b^2\left(\tfrac{81}{4}b^2 - 4\right)$$

So for $b > 0$, $b^2 = \frac{16}{81} \quad \Rightarrow \quad b = \frac{4}{9}$. Thus, $c = 8b - 27b^3 = 8\left(\frac{4}{9}\right) - 27\left(\frac{64}{729}\right) = \frac{32}{9} - \frac{64}{27} = \frac{32}{27}$.

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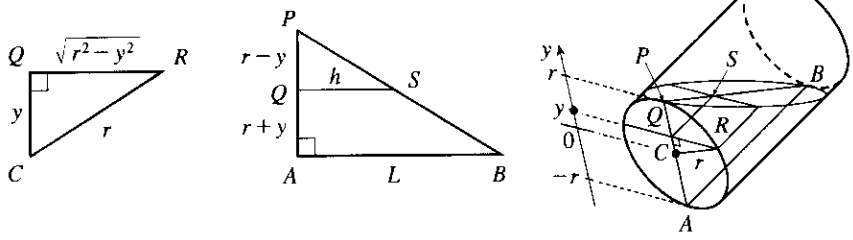
4. (a) Take slices perpendicular to the line through the center C of the bottom of the glass and the point P where the top surface of the water meets the bottom of the glass.



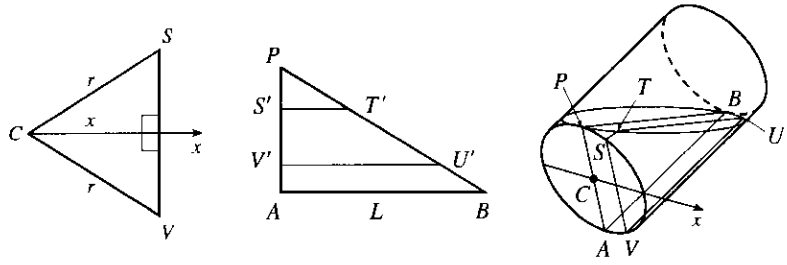
A typical rectangular cross-section y units above the axis of the glass has width $2\,|QR| = 2\sqrt{r^2-y^2}$ and length $h = |QS| = \dfrac{L}{2r}\,(r-y)$. [Triangles PQS and PAB are similar, so $\dfrac{h}{L} = \dfrac{|PQ|}{|PA|} = \dfrac{r-y}{2r}$.] Thus,

$$V = \int_{-r}^{r} 2\sqrt{r^2-y^2}\cdot\frac{L}{2r}\,(r-y)\,dy = L\int_{-r}^{r}\left(1-\frac{y}{r}\right)\sqrt{r^2-y^2}\,dy$$

$$= L\int_{-r}^{r}\sqrt{r^2-y^2}\,dy - \frac{L}{r}\int_{-r}^{r} y\sqrt{r^2-y^2}\,dy$$

$$= L\cdot\frac{\pi r^2}{2} - \frac{L}{r}\cdot 0 \qquad \left[\begin{array}{c}\text{the first integral is the area of a semicircle of radius } r,\\ \text{and the second has an odd integrand}\end{array}\right] \qquad = \frac{\pi r^2 L}{2}$$

(b) Slice parallel to the plane through the axis of the glass and the point of contact P. (This is the plane determined by P, B, and C in the figure.) $STUV$ is a typical trapezoidal slice. With respect to an x-axis with origin at C as shown, if S and V have x-coordinate x, then $|SV| = 2\sqrt{r^2-x^2}$. Projecting the trapezoid $STUV$ onto the plane of the triangle PAB (call the projection $S'T'U'V'$), we see that $|AP| = 2r$, $|SV| = 2\sqrt{r^2-x^2}$, and $|S'P| = |V'A| = \frac{1}{2}(|AP| - |SV|) = r - \sqrt{r^2-x^2}$.



By similar triangles, $\dfrac{|ST|}{|S'P|} = \dfrac{|AB|}{|AP|}$, so $|ST| = \left(r - \sqrt{r^2-x^2}\right)\cdot\dfrac{L}{2r}$. In the same way, we find that $\dfrac{|VU|}{|V'P|} = \dfrac{|AB|}{|AP|}$, so $|VU| = |V'P|\cdot\dfrac{L}{2r} = (|AP| - |V'A|)\cdot\dfrac{L}{2r} = \left(r + \sqrt{r^2-x^2}\right)\cdot\dfrac{L}{2r}$.

The area $A(x)$ of the trapezoid $STUV$ is $\frac{1}{2}|SV|\cdot(|ST| + |VU|)$; that is,

$A(x) = \frac{1}{2}\cdot 2\sqrt{r^2-x^2}\cdot\left[\left(r-\sqrt{r^2-x^2}\right)\cdot\dfrac{L}{2r} + \left(r+\sqrt{r^2-x^2}\right)\cdot\dfrac{L}{2r}\right] = L\sqrt{r^2-x^2}$. Thus,

$V = \int_{-r}^{r} A(x)\,dx = L\int_{-r}^{r}\sqrt{r^2-x^2}\,dx = L\cdot\dfrac{\pi r^2}{2} = \dfrac{\pi r^2 L}{2}$.

(c) See the computation of V in part (a) or part (b).

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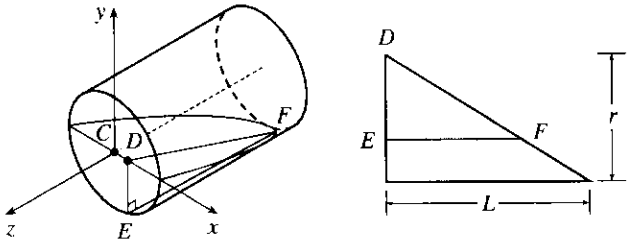
(d) The volume of the water is exactly half the volume of the cylindrical glass, so $V = \frac{1}{2}\pi r^2 L$.

(e)



Choose x-, y-, and z-axes as shown in the figure. Then slices perpendicular to the x-axis are triangular, slices perpendicular to the y-axis are rectangular, and slices perpendicular to the z-axis are segments of circles. Using triangular slices, we find that the area $A(x)$ of a typical slice DEF, where D has x-coordinate x, is given by $A(x) = \frac{1}{2}|DE| \cdot |EF| = \frac{1}{2}|DE| \cdot \left(\frac{L}{r}|DE|\right) = \frac{L}{2r}|DE|^2 = \frac{L}{2r}\left(r^2 - x^2\right)$. Thus,

$$V = \int_{-r}^{r} A(x)\,dx = \frac{L}{2r}\int_{-r}^{r}\left(r^2 - x^2\right)dx = \frac{L}{r}\int_0^r \left(r^2 - x^2\right)dx = \frac{L}{r}\left[r^2 x - \frac{x^3}{3}\right]_0^r$$

$$= \frac{L}{r}\left(r^3 - \frac{r^3}{3}\right) = \frac{L}{r}\cdot\tfrac{2}{3}r^3 = \tfrac{2}{3}r^2 L \quad \text{[This is } 2/(3\pi) \approx 0.21 \text{ of the volume of the glass.]}$$

5. (a) $V = \pi h^2(r - h/3) = \frac{1}{3}\pi h^2(3r - h)$. See the solution to Exercise 6.2.49.

(b) The smaller segment has height $h = 1 - x$ and so by part (a) its volume is $V = \frac{1}{3}\pi(1-x)^2\,[3(1) - (1-x)] = \frac{1}{3}\pi(x-1)^2(x+2)$. This volume must be $\frac{1}{3}$ of the total volume of the sphere, which is $\frac{4}{3}\pi(1)^3$. So $\frac{1}{3}\pi(x-1)^2(x+2) = \frac{1}{3}\left(\frac{4}{3}\pi\right) \Rightarrow (x^2 - 2x + 1)(x+2) = \frac{4}{3} \Rightarrow x^3 - 3x + 2 = \frac{4}{3} \Rightarrow 3x^3 - 9x + 2 = 0$. Using Newton's method with $f(x) = 3x^3 - 9x + 2$, $f'(x) = 9x^2 - 9$, we get $x_{n+1} = x_n - \dfrac{3x_n^3 - 9x_n + 2}{9x_n^2 - 9}$. Taking $x_1 = 0$, we get $x_2 \approx 0.2222$, and $x_3 \approx 0.2261 \approx x_4$, so, correct to four decimal places, $x \approx 0.2261$.

(c) With $r = 0.5$ and $s = 0.75$, the equation $x^3 - 3rx^2 + 4r^3 s = 0$ becomes $x^3 - 3(0.5)x^2 + 4(0.5)^3(0.75) = 0 \Rightarrow x^3 - \frac{3}{2}x^2 + 4\left(\frac{1}{8}\right)\frac{3}{4} = 0 \Rightarrow 8x^3 - 12x^2 + 3 = 0$. We use Newton's method with $f(x) = 8x^3 - 12x^2 + 3$, $f'(x) = 24x^2 - 24x$, so $x_{n+1} = x_n - \dfrac{8x_n^3 - 12x_n^2 + 3}{24x_n^2 - 24x_n}$. Take $x_1 = 0.5$. Then $x_2 \approx 0.6667$, and $x_3 \approx 0.6736 \approx x_4$. So to four decimal places the depth is 0.6736 m.

(d) (i) From part (a) with $r = 5$ in., the volume of water in the howl is $V = \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi h^2(15 - h) = 5\pi h^2 - \frac{1}{3}\pi h^3$. We are given that $\dfrac{dV}{dt} = 0.2$ m^3/s and we want to find $\dfrac{dh}{dt}$ when $h = 3$. Now $\dfrac{dV}{dt} = 10\pi h\dfrac{dh}{dt} - \pi h^2\dfrac{dh}{dt}$, so $\dfrac{dh}{dt} = \dfrac{0.2}{\pi(10h - h^2)}$. When $h = 3$, we have $\dfrac{dh}{dt} = \dfrac{0.2}{\pi(10\cdot 3 - 3^2)} = \dfrac{1}{105\pi} \approx 0.003$ in/s.

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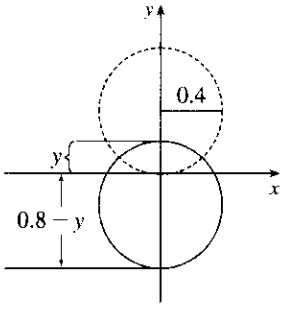
(ii) From part (a), the volume of water required to fill the bowl from the instant that the water is 4 in. deep is $V = \frac{1}{2} \cdot \frac{4}{3}\pi(5)^3 - \frac{1}{3}\pi(4)^2(15-4) = \frac{2}{3} \cdot 125\pi - \frac{16}{3} \cdot 11\pi = \frac{74}{3}\pi$. To find the time required to fill the bowl we divide this volume by the rate: Time $= \frac{74\pi/3}{0.2} = \frac{370\pi}{3} \approx 387$ s ≈ 6.5 min

6. (a) The volume above the surface is $\int_0^{L-h} A(y)\,dy = \int_{-h}^{L-h} A(y)\,dy - \int_{-h}^{0} A(y)\,dy$. So the proportion of volume above the surface is $\dfrac{\int_0^{L-h} A(y)\,dy}{\int_{-h}^{L-h} A(y)\,dy} = \dfrac{\int_{-h}^{L-h} A(y)\,dy - \int_{-h}^{0} A(y)\,dy}{\int_{-h}^{L-h} A(y)\,dy}$. Now by Archimedes' Principle, we have $F = W \quad \Rightarrow \quad \rho_f g \int_{-h}^{0} A(y)\,dy = \rho_0 g \int_{-h}^{L-h} A(y)\,dy$, so $\int_{-h}^{0} A(y)\,dy = (\rho_0/\rho_f)\int_{-h}^{L-h} A(y)\,dy$. Therefore, $\dfrac{\int_0^{L-h} A(y)\,dy}{\int_{-h}^{L-h} A(y)\,dy} = \dfrac{\int_{-h}^{L-h} A(y)\,dy - (\rho_0/\rho_f)\int_{-h}^{L-h} A(y)\,dy}{\int_{-h}^{L-h} A(y)\,dy} = \dfrac{\rho_f - \rho_0}{\rho_f}$, so the percentage of volume above the surface is $100\left(\dfrac{\rho_f - \rho_0}{\rho_f}\right)\%$.

(b) For an iceberg, the percentage of volume above the surface is $100\left(\frac{1030-917}{1030}\right)\% \approx 11\%$.

(c) No, the water does not overflow. Let V_i be the volume of the ice cube, and let V_w be the volume of the water which results from the melting. Then by the formula derived in part (a), the volume of ice above the surface of the water is $\left[(\rho_f - \rho_0)/\rho_f\right]V_i$, so the volume below the surface is $V_i - \left[(\rho_f - \rho_0)/\rho_f\right]V_i = (\rho_0/\rho_f)V_i$. Now the mass of the ice cube is the same as the mass of the water which is created when it melts, namely $m = \rho_0 V_i = \rho_f V_w \quad \Rightarrow \quad V_w = (\rho_0/\rho_f)V_i$. So when the ice cube melts, the volume of the resulting water is the same as the underwater volume of the ice cube, and so the water does not overflow.

(d)



Tbe figure shows the instant when the height of the exposed part of the ball is y. Using the formula in Problem 5(a) with $r = 0.4$ and $h = 0.8 - y$, we see that the volume of the submerged part of the sphere is $\frac{1}{3}\pi(0.8-y)^2[1.2-(0.8-y)]$, so its weight is $1000g \cdot \frac{1}{3}\pi s^2(1.2-s)$, where $s = 0.8 - y$. Then the work done to submerge the sphere is

$$W = \int_0^{0.8} g\tfrac{1000}{3}\pi s^2(1.2-s)\,ds = g\tfrac{1000}{3}\pi \int_0^{0.8} \left(1.2s^2 - s^3\right) ds$$

$$= g\tfrac{1000}{3}\pi\left[0.4s^3 - \tfrac{1}{4}s^4\right]_0^{0.8} = g\tfrac{1000}{3}\pi(0.2048 - 0.1024)$$

$$= 9.8\,\tfrac{1000}{3}\,\pi(0.1024) \approx 1.05 \times 10^3 \text{ joules}$$

7. We are given that the rate of change of the volume of water is $\dfrac{dV}{dt} = -kA(x)$, where k is some positive constant and $A(x)$ is the area of the surface when the water has depth x. Now we are concerned with the rate of change of the depth of the water with respect to time, that is, $\dfrac{dx}{dt}$. But by the Chain Rule, $\dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt}$, so the first equation can be written $\dfrac{dV}{dx}\dfrac{dx}{dt} = -kA(x)$ $(\star)$. Also, we know that the total volume of water up to a depth x is

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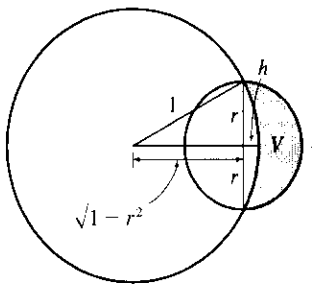
$V(x) = \int_0^x A(s)\,ds$, where $A(s)$ is the area of a cross-section of the water at a depth s. Differentiating this equation with respect to x, we get $dV/dx = A(x)$. Substituting this into equation $\star$, we get $A(x)(dx/dt) = -kA(x) \quad \Rightarrow \quad dx/dt = -k$, a constant.

8. A typical sphere of radius r is shown in the figure. We wish to maximize the shaded volume V, which can be thought of as the volume of a hemisphere of radius r minus the volume of the spherical cap with height $h = 1 - \sqrt{1-r^2}$ and radius 1.



$$V = \tfrac{1}{2}\cdot\tfrac{4}{3}\pi r^3 - \tfrac{1}{3}\pi\big(1-\sqrt{1-r^2}\,\big)^2\big[3(1) - \big(1-\sqrt{1-r^2}\,\big)\big] \text{ [by Problem 5(a)]}$$
$$= \tfrac{1}{3}\pi\big[2r^3 - \big(2-2\sqrt{1-r^2}-r^2\big)\big(2+\sqrt{1-r^2}\,\big)\big]$$
$$= \tfrac{1}{3}\pi\big[2r^3 - 2 + \big(r^2+2\big)\sqrt{1-r^2}\,\big]$$

$$V' = \tfrac{1}{3}\pi\left[6r^2 + \frac{(r^2+2)(-r)}{\sqrt{1-r^2}} + \sqrt{1-r^2}(2r)\right] = \tfrac{1}{3}\pi\left[\frac{6r^2\sqrt{1-r^2} - r(r^2+2) + 2r(1-r^2)}{\sqrt{1-r^2}}\right]$$
$$= \tfrac{1}{3}\pi\left(\frac{6r^2\sqrt{1-r^2} - 3r^3}{\sqrt{1-r^2}}\right) = \frac{\pi r^2\big(2\sqrt{1-r^2} - r\big)}{\sqrt{1-r^2}}$$

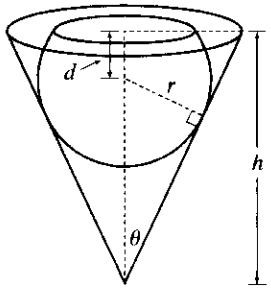
$V'(r) = 0 \quad \Leftrightarrow \quad 2\sqrt{1-r^2} = r \quad \Leftrightarrow \quad 4 - 4r^2 = r^2 \quad \Leftrightarrow \quad r^2 = \frac{4}{5} \quad \Leftrightarrow \quad r = \frac{2}{\sqrt{5}} \approx 0.89$. Since $V'(r) > 0$ for $0 < r < \frac{2}{\sqrt{5}}$ and $V'(r) < 0$ for $\frac{2}{\sqrt{5}} < r < 1$, we know that V attains a maximum at $r = \frac{2}{\sqrt{5}}$.

9. We must find expressions for the areas A and B, and then set them equal and see what this says about the curve C. If $P = \big(a, 2a^2\big)$, then area A is just $\int_0^a \big(2x^2 - x^2\big)\,dx = \int_0^a x^2\,dx = \frac{1}{3}a^3$. To find area B, we use y as the variable of integration. So we find the equation of the middle curve as a function of y: $y = 2x^2 \quad \Leftrightarrow \quad x = \sqrt{y/2}$, since we are concerned with the first quadrant only. We can express area B as

$\int_0^{2a^2}\Big[\sqrt{y/2} - C(y)\Big]\,dy = \Big[\frac{4}{3}(y/2)^{3/2}\Big]_0^{2a^2} - \int_0^{2a^2} C(y)\,dy = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\,dy$, where $C(y)$ is the function with graph C. Setting $A = B$, we get $\frac{1}{3}a^3 = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\,dy \quad \Leftrightarrow \quad \int_0^{2a^2} C(y)\,dy = a^3$. Now we differentiate this equation with respect to a using the Chain Rule and the Fundamental Theorem:

$C\big(2a^2\big)(4a) = 3a^2 \quad \Rightarrow \quad C(y) = \frac{3}{4}\sqrt{y/2}$, where $y = 2a^2$. Now we can solve for y: $x = \frac{3}{4}\sqrt{y/2} \quad \Rightarrow$ $x^2 = \frac{9}{16}(y/2) \quad \Rightarrow \quad y = \frac{32}{9}x^2$.

10. We want to find the volume of that part of the sphere which is below the surface of the water. As we can see from the diagram, this region is a cap of a sphere with radius r and height $r + d$. If we can find an expression for d in terms of h, r and θ, then we can determine the volume of the region [see Problem 5(a)], and then differentiate with respect to r to find the maximum. We see that



$$\sin\theta = \frac{r}{h-d} \quad \Leftrightarrow \quad h - d = \frac{r}{\sin\theta} \quad \Leftrightarrow \quad d = h - r\csc\theta.$$

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Now we can use the formula from Problem 5(a) to find the volume of water displaced:

$$V = \tfrac{1}{3}\pi h^2(3r - h) = \tfrac{1}{3}\pi(r + d)^2\,[3r - (r + d)] = \tfrac{1}{3}\pi(r + h - r\csc\theta)^2(2r - h + r\csc\theta)$$

$$= \tfrac{\pi}{3}[r(1 - \csc\theta) + h]^2[r(2 + \csc\theta) - h]$$

Now we differentiate with respect to r:

$$dV/dr = \tfrac{\pi}{3}\left([r(1 - \csc\theta) + h]^2(2 + \csc\theta) + 2[r(1 - \csc\theta) + h](1 - \csc\theta)[r(2 + \csc\theta) - h]\right)$$

$$= \tfrac{\pi}{3}[r(1 - \csc\theta) + h]([r(1 - \csc\theta) + h](2 + \csc\theta) + 2(1 - \csc\theta)[r(2 + \csc\theta) - h])$$

$$= \tfrac{\pi}{3}[r(1 - \csc\theta) + h](3(2 + \csc\theta)(1 - \csc\theta)r + [(2 + \csc\theta) - 2(1 - \csc\theta)]h)$$

$$= \tfrac{\pi}{3}[r(1 - \csc\theta) + h][3(2 + \csc\theta)(1 - \csc\theta)r + 3h\csc\theta]$$

This is 0 when $r = \dfrac{h}{\csc\theta - 1}$ and when $r = \dfrac{h\csc\theta}{(\csc\theta + 2)(\csc\theta - 1)}$. Now since $V\left(\dfrac{h}{\csc\theta - 1}\right) = 0$ (the first factor vanishes; this corresponds to $d = -r$), the maximum volume of water is displaced when $r = \dfrac{h\csc\theta}{(\csc\theta - 1)(\csc\theta + 2)}$. (Our intuition tells us that a maximum value does exist, and it must occur at a critical number.) Multiplying numerator and denominator by $\sin^2\theta$, we get an alternative form of the answer:

$$r = \frac{h\sin\theta}{\sin\theta + \cos 2\theta}.$$

11. (a) Stacking disks along the y-axis gives us $V = \int_0^h \pi\,[f(y)]^2\,dy$.

(b) Using the Chain Rule, $\dfrac{dV}{dt} = \dfrac{dV}{dh}\cdot\dfrac{dh}{dt} = \pi\,[f(h)]^2\,\dfrac{dh}{dt}$.

(c) $kA\sqrt{h} = \pi[f(h)]^2\,\dfrac{dh}{dt}$. Set $\dfrac{dh}{dt} = C$: $\pi[f(h)]^2\,C = kA\sqrt{h} \;\Rightarrow\; [f(h)]^2 = \dfrac{kA}{\pi C}\sqrt{h} \;\Rightarrow$

$f(h) = \sqrt{\dfrac{kA}{\pi C}}\,h^{1/4}$; that is, $f(y) = \sqrt{\dfrac{kA}{\pi C}}\,y^{1/4}$. The advantage of having $\dfrac{dh}{dt} = C$ is that the markings on the container are equally spaced.

12. (a) We first use the cylindrical shell method to express the volume V in terms of h, r, and ω:

$$V = \int_0^r 2\pi xy\,dx = \int_0^r 2\pi x\left[h + \frac{\omega^2x^2}{2g}\right]dx = 2\pi\int_0^r\left(hx + \frac{\omega^2x^3}{2g}\right)dx$$

$$= 2\pi\left[\frac{hx^2}{2} + \frac{\omega^2x^4}{8g}\right]_0^r = 2\pi\left[\frac{hr^2}{2} + \frac{\omega^2r^4}{8g}\right] = \pi hr^2 + \frac{\pi\omega^2r^4}{4g} \;\Rightarrow$$

$$h = \frac{V - (\pi\omega^2r^4)/(4g)}{\pi r^2} = \frac{4gV - \pi\omega^2r^4}{4\pi gr^2}.$$

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(b) The surface touches the bottom when $h = 0 \;\Rightarrow\; 4gV - \pi\omega^2 r^4 = 0 \;\Rightarrow\; \omega^2 = \dfrac{4gV}{\pi r^4} \;\Rightarrow\; \omega = \dfrac{2\sqrt{gV}}{\sqrt{\pi}r^2}$.

To spill over the top, $y(r) > L \;\Leftrightarrow$

$$L < h + \frac{\omega^2 r^2}{2g} = \frac{4gV - \pi\omega^2 r^4}{4\pi g r^2} + \frac{\omega^2 r^2}{2g} = \frac{4gV}{4\pi g r^2} - \frac{\pi\omega^2 r^2}{4\pi g r^2} + \frac{\omega^2 r^2}{2g}$$

$$= \frac{V}{\pi r^2} - \frac{\omega^2 r^2}{4g} + \frac{\omega^2 r^2}{2g} = \frac{V}{\pi r^2} + \frac{\omega^2 r^2}{4g} \quad \Leftrightarrow$$

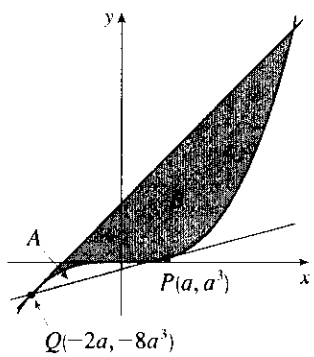
$\dfrac{\omega^2 r^2}{4g} > L - \dfrac{V}{\pi r^2} = \dfrac{\pi r^2 L - V}{\pi r^2} \;\Leftrightarrow\; \omega^2 > \dfrac{4g(\pi r^2 L - V)}{\pi r^4}$. So for spillage, the angular speed should be $\omega > \dfrac{2\sqrt{g(\pi r^2 L - V)}}{r^2\sqrt{\pi}}$.

(c) (i) Here we have $r = 2, L = 7, h = 7 - 5 = 2$. When $x = 1, y = 7 - 4 = 3$. Therefore, $3 = 2 + \dfrac{\omega^2 \cdot 1^2}{2 \cdot 32}$

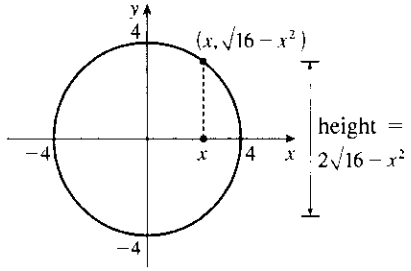
$\Rightarrow\; 1 = \dfrac{\omega^2}{2 \cdot 32} \;\Rightarrow\; \omega^2 = 64 \;\Rightarrow\; \omega = 8$ rad/s. $V = \pi(2)(2)^2 + \dfrac{\pi \cdot 8^2 \cdot 2^4}{4g} = 8\pi + 8\pi = 16\pi$ ft^2.

(ii) At the wall, $x = 2$, so $y = 2 + \dfrac{8^2 \cdot 2^2}{2 \cdot 32} = 6$ and the surface is $7 - 6 = 1$ ft below the top of the tank.

13. We assume that P lies in the region of positive x. Since $y = x^3$ is an odd function, this assumption will not affect the result of the calculation. Let $P = (a, a^3)$. The slope of the tangent to the curve $y = x^3$ at P is $3a^2$, and so the equation of the tangent is $y - a^3 = 3a^2(x - a) \;\Leftrightarrow\; y = 3a^2x - 2a^3$. We solve this simultaneously with $y = x^3$ to find the other point of intersection: $x^3 = 3a^2x - 2a^3 \;\Leftrightarrow\; (x - a)^2(x + 2a) = 0$. So $Q = (-2a, -8a^3)$ is the other point of intersection. The equation of the tangent at Q is $y - (-8a^3) = 12a^2[x - (-2a)] \;\Leftrightarrow\; y = 12a^2x + 16a^3$. By symmetry, this tangent will intersect the curve again at $x = -2(-2a) = 4a$. The curve lies above the first tangent, and below the second, so we are looking for a relationship between $A = \int_{-2a}^{a} \left[x^3 - \left(3a^2x - 2a^3\right)\right] dx$ and $B = \int_{-2a}^{4a} \left[\left(12a^2x + 16a^3\right) - x^3\right] dx$. We calculate $A = \left[\frac{1}{4}x^4 - \frac{3}{2}a^2x^2 + 2a^3x\right]_{-2a}^{a} = \frac{3}{4}a^4 - \left(-6a^4\right) = \frac{27}{4}a^4$, and $B = \left[6a^2x^2 + 16a^3x - \frac{1}{4}x^4\right]_{-2a}^{4a} = 96a^4 - \left(-12a^4\right) = 108a^4$. We see that $B = 16A = 2^4A$. This is because our calculation of area B was essentially the same as that of area A, with a replaced by $-2a$, so if we replace a with $-2a$ in our expression for A, we get $\frac{27}{4}(-2a)^4 = 108a^4 = B$.



14. (a) Place the round flat tortilla on an xy-coordinate system as shown in the first figure. An equation of the circle is $x^2 + y^2 = 4^2$ and the height of a cross-section is $2\sqrt{16 - x^2}$.



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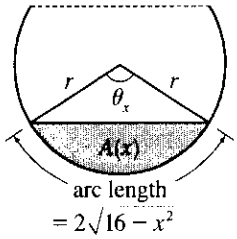
Now look at a cross-section with central angle θ_x as shown in the second figure (r is the radius of the circular cylinder). The filled area $A(x)$ is equal to the area $A_1(x)$ of the sector minus the area $A_2(x)$ of the triangle.



$$\begin{aligned} A(x) &= A_1(x) - A_2(x) \\ &= \tfrac{1}{2}r^2\theta_x - \tfrac{1}{2}r^2 \sin\theta_x \qquad \text{[area formulas from trigonometry]} \\ &= \tfrac{1}{2}r(r\theta_x) - \tfrac{1}{2}r^2 \sin\left(\frac{s}{r}\right) \qquad \text{[arc length } s = r\theta_x \quad \Rightarrow \quad \theta_x = s/r\text{]} \\ &= \tfrac{1}{2}r \cdot 2\sqrt{16-x^2} - \tfrac{1}{2}r^2 \sin\left(\frac{2\sqrt{16-x^2}}{r}\right) \qquad [s = 2\sqrt{16-x^2}\,] \\ &= r\sqrt{16-x^2} - \tfrac{1}{2}r^2 \sin\left(\frac{2}{r}\sqrt{16-x^2}\right) \quad (\star) \end{aligned}$$

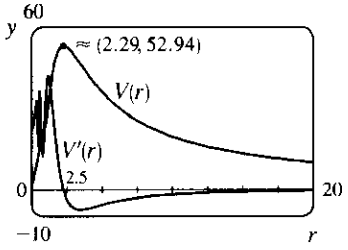
Note that the central angle θ_x will be small near the ends of the tortilla; that is, when $|x| \approx 4$. But near the center of the tortilla (when $|x| \approx 0$), the central angle θ_x may exceed $180°$. Thus, the sine of θ_x will be negative and the second term in $(\star)$ will be positive (actually adding area to the area of the sector). The volume of the taco can be found by integrating the cross-sectional areas from $x = -4$ to $x = 4$. Thus,

$$V(x) = \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \left[r\sqrt{16-x^2} - \tfrac{1}{2}r^2 \sin\left(\frac{2}{r}\sqrt{16-x^2}\right)\right] dx$$

(b) To find the value of r that maximizes the volume of the taco, we can define the function

$$V(r) = \int_{-4}^{4} \left[r\sqrt{16-x^2} - \tfrac{1}{2}r^2 \sin\left(\frac{2}{r}\sqrt{16-x^2}\right)\right] dx$$

The third figure shows a graph of $y = V(r)$ and $y = V'(r)$. The maximum volume of about 52.94 occurs when $r \approx 2.2912$.



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Chapter 7

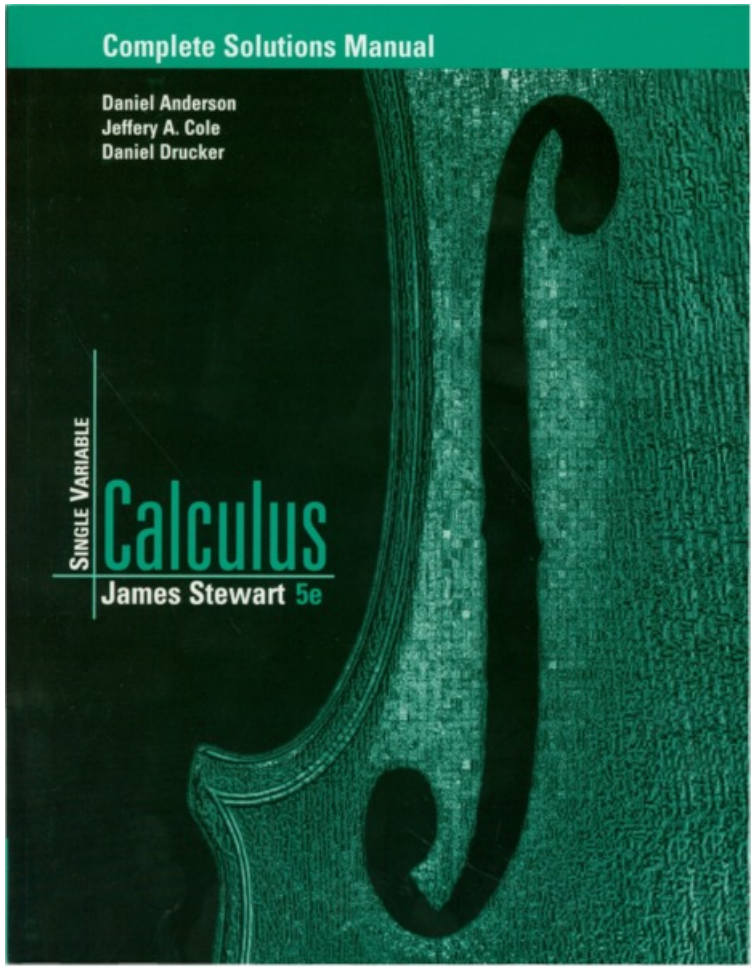
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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7 INVERSE FUNCTIONS: Exponential, Logarithmic, and Inverse Trigonometric Functions

7.1 Inverse Functions

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

(b) See the steps in (5).

(c) Reflect the graph of f about the line $y = x$.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

4. f is one-to-one since for any two different domain values, there are different range values.

5. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.

6. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

7. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

8. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.

9. The graph of $f(x) = \frac{1}{2}(x + 5)$ is a line with slope $\frac{1}{2}$. It passes the Horizontal Line Test, so f is one-to-one.
Algebraic solution: If $x_1 \neq x_2$, then $x_1 + 5 \neq x_2 + 5 \quad \Rightarrow \quad \frac{1}{2}(x_1 + 5) \neq \frac{1}{2}(x_2 + 5) \quad \Rightarrow \quad f(x_1) \neq f(x_2)$, so f is one-to-one.

10. The graph of $f(x) = 1 + 4x - x^2$ is a parabola with axis of symmetry $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$. Pick any x-values equidistant from 2 to find two equal function values. For example, $f(1) = 4$ and $f(3) = 4$, so f is not 1-1.

11. $x_1 \neq x_2 \quad \Rightarrow \quad \sqrt{x_1} \neq \sqrt{x_2} \quad \Rightarrow \quad g(x_1) \neq g(x_2)$, so g is 1-1.

12. $g(x) = |x| \quad \Rightarrow \quad g(-1) = 1 = g(1)$, so g is not one-to-one.

13. $h(x) = x^4 + 5 \quad \Rightarrow \quad h(1) = 6 = h(-1)$, so h is not 1-1.

14. $x_1 \neq x_2 \quad \Rightarrow \quad x_1^4 \neq x_2^4 \quad [\text{since } x \geq 0] \quad \Rightarrow \quad x_1^4 + 5 \neq x_2^4 + 5 \quad \Rightarrow \quad h(x_1) \neq h(x_2)$, so h is 1-1.

15. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

16. f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

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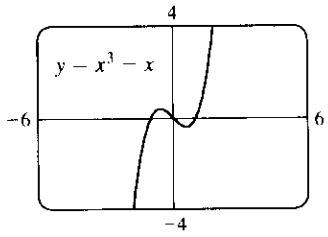
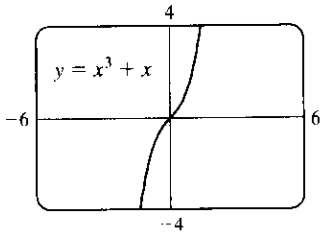
17. f does not pass the Horizontal Line Test, so f is not 1-1.



18. f passes the Horizontal Line Test, so f is 1-1.



19. Since $f(2) = 9$ and f is 1-1, we know that $f^{-1}(9) = 2$. Remember, if the point $(2, 9)$ is on the graph of f, then the point $(9, 2)$ is on the graph of f^{-1}.

20. $f(x) = x + \cos x \;\Rightarrow\; f'(x) = 1 - \sin x \geq 0$, with equality only if $x = \frac{\pi}{2} + 2n\pi$. So f is increasing on $\mathbb{R}$, and hence, 1-1. By inspection, $f(0) = 0 + \cos 0 = 1$, so $f^{-1}(1) = 0$.

21. $h(x) = x + \sqrt{x} \;\Rightarrow\; h'(x) = 1 + 1/(2\sqrt{x}) > 0$ on $(0, \infty)$. So h is increasing and hence, 1-1. By inspection, $h(4) = 4 + \sqrt{4} = 6$, so $h^{-1}(6) = 4$.

22. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of f = $[-3, 3]$ = Range of f^{-1}. Range of f = $[-2, 2]$ = Domain of f^{-1}.

(c) Since $f(-2) = 1$, $f^{-1}(1) = -2$.

23. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \;\Rightarrow\; F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \;\Rightarrow\; \frac{9}{5}C + 32 \geq -459.67 \;\Rightarrow\; \frac{9}{5}C \geq -491.67 \;\Rightarrow\; C \geq -273.15$, the domain of the inverse function.

24. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} \;\Rightarrow\; 1 - \dfrac{v^2}{c^2} = \dfrac{m_0^2}{m^2} \;\Rightarrow\; \dfrac{v^2}{c^2} = 1 - \dfrac{m_0^2}{m^2} \;\Rightarrow\; v^2 = c^2\left(1 - \dfrac{m_0^2}{m^2}\right) \;\Rightarrow$

$v = c\sqrt{1 - \dfrac{m_0^2}{m^2}}$. This formula gives us the speed v of the particle in terms of its mass m, that is, $v = f^{-1}(m)$.

25. $y = f(x) = 3 - 2x \;\Rightarrow\; 2x = 3 - y \;\Rightarrow\; x = \dfrac{3 - y}{2}$. Interchange x and y: $y = \dfrac{3 - x}{2}$. So $f^{-1}(x) = \dfrac{3 - x}{2}$.

26. $f(x) = \dfrac{4x - 1}{2x + 3} \;\Rightarrow\; y = \dfrac{4x - 1}{2x + 3} \;\Rightarrow\; y(2x + 3) = 4x - 1 \;\Rightarrow\; 2xy + 3y = 4x - 1 \;\Rightarrow$

$3y + 1 = 4x - 2xy \;\Rightarrow\; 3y + 1 = (4 - 2y)x \;\Rightarrow\; x = \dfrac{3y + 1}{4 - 2y}$. Interchange x and y: $y = \dfrac{3x + 1}{4 - 2x}$.

So $f^{-1}(x) = \dfrac{3x + 1}{4 - 2x}$.

27. $f(x) = \sqrt{10 - 3x} \;\Rightarrow\; y = \sqrt{10 - 3x} \;\; (y \geq 0) \;\Rightarrow\; y^2 = 10 - 3x \;\Rightarrow\; 3x = 10 - y^2 \;\Rightarrow$ $x = -\frac{1}{3}y^2 + \frac{10}{3}$. Interchange x and y: $y = -\frac{1}{3}x^2 + \frac{10}{3}$. So $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{10}{3}$. Note that the domain of f^{-1} is $x \geq 0$.

28. $y = f(x) = 2x^3 + 3 \;\Rightarrow\; y - 3 = 2x^3 \;\Rightarrow\; \dfrac{y - 3}{2} = x^3 \;\Rightarrow\; x = \sqrt[3]{\dfrac{y - 3}{2}}$.

Interchange x and y: $y = \sqrt[3]{\dfrac{x - 3}{2}}$. So $f^{-1}(x) = \sqrt[3]{\dfrac{x - 3}{2}}$.

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29. For $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$, the domain is $x \geq 0$. $f(0) = 1$ and as x increases, y decreases. As $x \to \infty$, $\dfrac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \dfrac{1/\sqrt{x}}{1/\sqrt{x}} = \dfrac{1/\sqrt{x}-1}{1/\sqrt{x}+1} \to \dfrac{-1}{1} = -1$, so the range of f is $-1 < y \leq 1$. Thus, the domain of f^{-1} is $-1 < x \leq 1$.

$y = \dfrac{1-\sqrt{x}}{1+\sqrt{x}} \quad \Rightarrow \quad y(1+\sqrt{x}) = 1-\sqrt{x} \quad \Rightarrow \quad y + y\sqrt{x} = 1-\sqrt{x} \quad \Rightarrow \quad \sqrt{x} + y\sqrt{x} = 1-y \quad \Rightarrow$

$\sqrt{x}(1+y) = 1-y \quad \Rightarrow \quad \sqrt{x} = \dfrac{1-y}{1+y} \quad \Rightarrow \quad x = \left(\dfrac{1-y}{1+y}\right)^2$. Interchange x and y: $y = \left(\dfrac{1-x}{1+x}\right)^2$. So

$f^{-1}(x) = \left(\dfrac{1-x}{1+x}\right)^2$ with $-1 < x \leq 1$.

30. $y = f(x) = 2x^2 - 8x, x \geq 2 \quad \Rightarrow \quad 2x^2 - 8x - y = 0, x \geq 2 \quad \Rightarrow$

$x = \dfrac{8+\sqrt{64+8y}}{4}$ $\left[\begin{array}{l}\text{quadratic formula with} \\ a=2, b=-8, \text{and } c=-y\end{array}\right]$ $= \dfrac{8+2\sqrt{16+2y}}{4} = 2 + \frac{1}{2}\sqrt{16+2y}$. Interchange x and y: $y = 2 + \frac{1}{2}\sqrt{16+2x}$. So $f^{-1}(x) = 2 + \frac{1}{2}\sqrt{16+2x}$.

Alternate solution (by completing the square): $y = 2x^2 - 8x, x \geq 2 \quad \Rightarrow \quad x^2 - 4x = y/2, x \geq 2 \quad \Rightarrow$

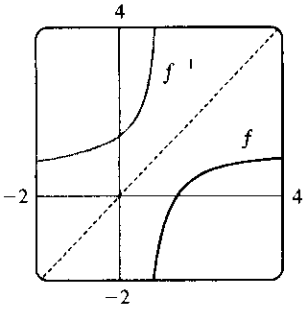
$(x-2)^2 = x^2 - 4x + 4 = \dfrac{y}{2} + 4 = \dfrac{y+8}{2} = \dfrac{2y+16}{4}, x \geq 2 \quad \Rightarrow \quad x - 2 = +\sqrt{\dfrac{2y+16}{4}} \quad \Rightarrow$

$x = 2 + \frac{1}{2}\sqrt{2y+16}$. Interchange x and y: $y = 2 + \frac{1}{2}\sqrt{2x+16}$. So $f^{-1}(x) = 2 + \frac{1}{2}\sqrt{2x+16}$.

31. $y = f(x) = 1 - \dfrac{2}{x^2} \quad \Rightarrow \quad 1-y = \dfrac{2}{x^2} \quad \Rightarrow \quad x^2 = \dfrac{2}{1-y} \quad \Rightarrow$

$x = \sqrt{\dfrac{2}{1-y}}$, since $x > 0$. Interchange x and y: $y = \sqrt{\dfrac{2}{1-x}}$.

So $f^{-1}(x) = \sqrt{\dfrac{2}{1-x}}$.

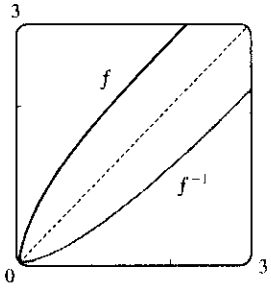


32. $y = f(x) = \sqrt{x^2+2x}, x > 0 \quad \Rightarrow \quad y > 0$ and $y^2 = x^2 + 2x \quad \Rightarrow$

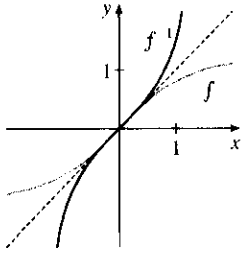
$x^2 + 2x - y^2 = 0$. Now we use the quadratic formula:

$x = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-y^2)}}{2 \cdot 1} = -1 \pm \sqrt{1+y^2}$. But $x > 0$, so the negative root is inadmissible. Interchange x and y: $y = -1 + \sqrt{1+x^2}$.

So $f^{-1}(x) = -1 + \sqrt{1+x^2}, x > 0$.



33. The function f is one-to-one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y = x$.



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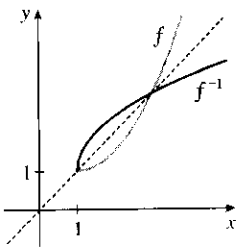
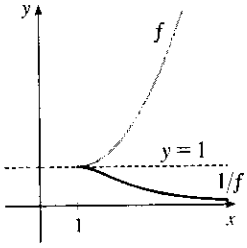
34. The function f is one-to-one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y = x$. For the graph of $1/f$, the y-coordinates are simply the reciprocals of f. For example, if $f(5) = 9$, then $1/f(5) = \frac{1}{9}$. If we draw the horizontal line $y = 1$, we see that the only place where the graphs intersect is on that line.





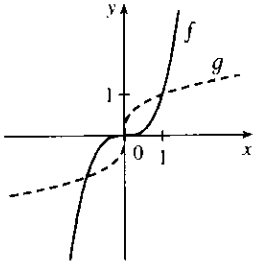
35. (a) $x_1 \neq x_2 \Rightarrow x_1^3 \neq x_2^3 \Rightarrow f(x_1) \neq f(x_2)$, so f is one-to-one.

(b) $f'(x) = 3x^2$ and $f(2) = 8 \Rightarrow g(8) = 2$, so $g'(8) = 1/f'(g(8)) = 1/f'(2) = \frac{1}{12}$.

(c) $y = x^3 \Rightarrow x = y^{1/3}$. Interchanging x and y gives $y = x^{1/3}$, so $f^{-1}(x) = x^{1/3}$. Domain(g) = range$(f) = \mathbb{R}$.
Range(g) = domain$(f) = \mathbb{R}$.

(d) $g(x) = x^{1/3} \Rightarrow g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(8) = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$ as in part (b).

(e)



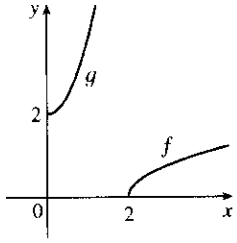
36. (a) $x_1 \neq x_2 \Rightarrow x_1 - 2 \neq x_2 - 2 \Rightarrow \sqrt{x_1 - 2} \neq \sqrt{x_2 - 2} \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $f(6) = 2$, so $g(2) = 6$. Also $f'(x) = \dfrac{1}{2\sqrt{x-2}}$, so $g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(6)} = \dfrac{1}{1/4} = 4$.

(c) $y = \sqrt{x-2} \Rightarrow y^2 = x - 2 \Rightarrow x = y^2 + 2$.
Interchange x and y: $y = x^2 + 2$. So $g(x) = x^2 + 2$.
Domain $= [0, \infty)$, range $= [2, \infty)$.

(d) $g(x) = x^2 + 2 \Rightarrow g'(x) = 2x \Rightarrow g'(2) = 4$.

(e)



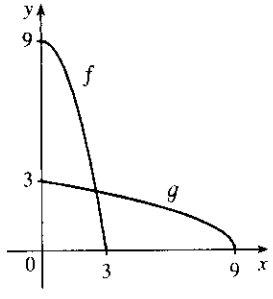
37. (a) Since $x \geq 0$, $x_1 \neq x_2 \Rightarrow x_1^2 \neq x_2^2 \Rightarrow 9 - x_1^2 \neq 9 - x_2^2 \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $f'(x) = -2x$ and $f(1) = 8 \Rightarrow g(8) = 1$, so $g'(8) = \dfrac{1}{f'(g(8))} = \dfrac{1}{f'(1)} = \dfrac{1}{(-2)} = -\dfrac{1}{2}$.

(c) $y = 9 - x^2 \Rightarrow x^2 = 9 - y \Rightarrow x = \sqrt{9-y}$. Interchange x and y: $y = \sqrt{9-x}$, so $f^{-1}(x) = \sqrt{9-x}$.
Domain (g) = range $(f) = [0, 9]$.
Range (g) = domain $(f) = [0, 3]$.

(d) $g'(x) = -1/(2\sqrt{9-x}) \Rightarrow g'(8) = -\frac{1}{2}$ as in part (b).

(e)



38. (a) $x_1 \neq x_2 \Rightarrow x_1 - 1 \neq x_2 - 1 \Rightarrow \dfrac{1}{x_1 - 1} \neq \dfrac{1}{x_2 - 1} \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $g(2) = \frac{3}{2}$ since $f\left(\frac{3}{2}\right) = 2$. Also $f'(x) = -1/(x-1)^2$, so $g'(2) = 1/f'\left(\frac{3}{2}\right) = \frac{1}{-4} = -\frac{1}{4}$.

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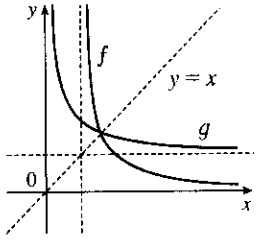
(c) $y = 1/(x-1) \Rightarrow x - 1 = 1/y \Rightarrow x = 1 + 1/y$. Interchange x and y: $y = 1 + 1/x$. So $g(x) = 1 + 1/x$, $x > 0$ (since $y > 1$).

Domain $= (0, \infty)$, range $= (1, \infty)$.

(d) $g'(x) = -1/x^2$, so $g'(2) = -\frac{1}{4}$.

(e)



39. $f(0) = 1 \Rightarrow f^{-1}(1) = 0$, and $f(x) = x^3 + x + 1 \Rightarrow f'(x) = 3x^2 + 1$ and $f'(0) = 1$. Thus,

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{1} = 1.$$

40. $f(1) = 2 \Rightarrow f^{-1}(2) = 1$, and $f(x) = x^5 - x^3 + 2x \Rightarrow f'(x) = 5x^4 - 3x^2 + 2$ and $f'(1) = 4$. Thus,

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{4}.$$

41. $f(0) = 3 \Rightarrow f^{-1}(3) = 0$, and $f(x) = 3 + x^2 + \tan(\pi x/2) \Rightarrow f'(x) = 2x + \frac{\pi}{2}\sec^2(\pi x/2)$ and $f'(0) = \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$. Thus, $(f^{-1})'(3) = 1/f'\big(f^{-1}(3)\big) = 1/f'(0) = 2/\pi$.

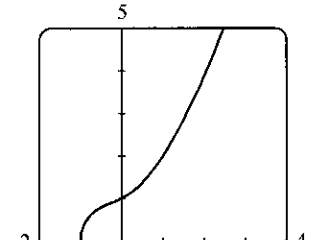
42. $f(1) = 2 \Rightarrow f^{-1}(2) = 1$, and $f(x) = \sqrt{x^3 + x^2 + x + 1} \Rightarrow f'(x) = \dfrac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}}$ and $f'(1) = \dfrac{3 + 2 + 1}{2\sqrt{1 + 1 + 1 + 1}} = \dfrac{3}{2}$. Thus, $(f^{-1})'(2) = 1/f'\big(f^{-1}(2)\big) = 1/f'(1) = \frac{2}{3}$.

43. $f(4) = 5 \Rightarrow g(5) = 4$. Thus, $g'(5) = \dfrac{1}{f'(g(5))} = \dfrac{1}{f'(4)} = \dfrac{1}{2/3} = \dfrac{3}{2}$.

44. $f(3) = 2 \Rightarrow g(2) = 3$. Thus, $g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(3)} = 9$. Hence, $G(x) = \dfrac{1}{g(x)} \Rightarrow$

$$G'(x) = -\frac{g'(x)}{[g(x)]^2} \Rightarrow G'(2) = -\frac{g'(2)}{[g(2)]^2} = -\frac{9}{(3)^2} = -1.$$

45. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1. Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y. Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:



$$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$$

where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$. Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\dfrac{1}{6}\dfrac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$, where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

46. Since $\sin(2n\pi) = 0$, $h(x) = \sin x$ is not one-to-one. $h'(x) = \cos x > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so h is increasing and hence 1-1 on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Let $y = f^{-1}(x) = \sin^{-1} x$ so that $\sin y = x$. Differentiating $\sin y = x$ implicitly with respect to x gives us $\cos y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos y}$. Now $\cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \pm\sqrt{1 - \sin^2 y}$, but since $\cos y > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - \sin^2 y}} = \dfrac{1}{\sqrt{1 - x^2}}$.

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47. (a) If the point (x, y) is on the graph of $y = f(x)$, then the point $(x - c, y)$ is that point shifted c units to the left. Since f is 1-1, the point (y, x) is on the graph of $y = f^{-1}(x)$ and the point corresponding to $(x - c, y)$ on the graph of f is $(y, x - c)$ on the graph of f^{-1}. Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x) = f^{-1}(x) - c$.

(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y = x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x) = f(cx)$ can be expressed as $h^{-1}(x) = (1/c)\, f^{-1}(x)$.

48. (a) We know that $g'(x) = \dfrac{1}{f'(g(x))}$. Thus,

$$g''(x) = -\frac{f''(g(x)) \cdot g'(x)}{[f'(g(x))]^2} = -\frac{f''(g(x)) \cdot [1/f'(g'(x))]}{[f'(g(x))]^2} = -\frac{f''(g(x))}{f'(g(x))[f'(g(x))]^2} = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

(b) f is increasing $\Rightarrow$ $f'(g(x)) > 0$ $\Rightarrow$ $[f'(g(x))]^3 > 0$. f is concave upward $\Rightarrow$ $f''(g(x)) > 0$. So $g''(x) = -\dfrac{f''(g(x))}{[f'(g(x))]^3} < 0$, which implies that g (f's inverse) is concave downward.

7.2 Exponential Functions and Their Derivatives

1. (a) $f(x) = a^x$, $a > 0$

(b) $\mathbb{R}$

(c) $(0, \infty)$

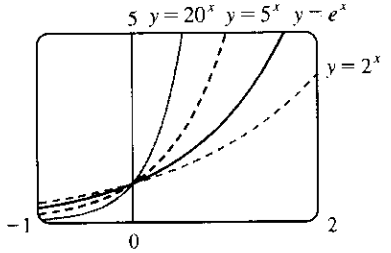
(d) See Figures 6(c), 6(b), and 6(a), respectively.

2. (a) The number e is the value of a such that the slope of the tangent line at $x = 0$ on the graph of $y = a^x$ is exactly 1.

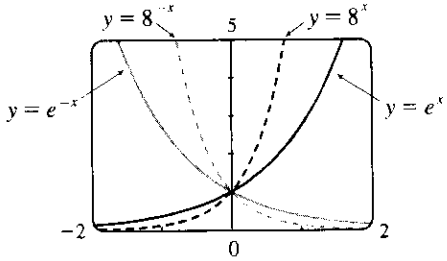
(b) $e \approx 2.71828$

(c) $f(x) = e^x$

3. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



4. The graph of e^{-x} is the reflection of the graph of e^x about the y-axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y-axis. The graph of 8^x increases more quickly than that of e^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



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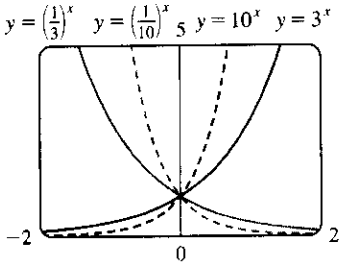
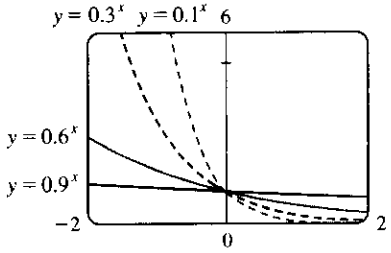
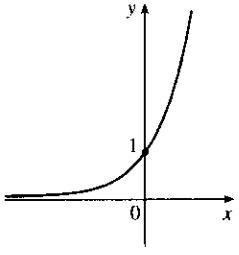
5. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 [$(\frac{1}{3})^x$ and $(\frac{1}{10})^x$] are decreasing. The graph of $(\frac{1}{3})^x$ is the reflection of that of 3^x about the y-axis, and the graph of $(\frac{1}{10})^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



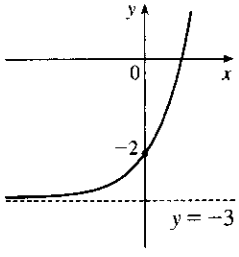
6. Each of the graphs approaches ∞ as $x \to -\infty$, and each approaches 0 as $x \to \infty$. The smaller the base, the faster the function grows as $x \to -\infty$, and the faster it approaches 0 as $x \to \infty$.



7. We start with the graph of $y = 4^x$ (Figure 3) and then shift 3 units downward. This shift doesn't affect the domain, but the range of $y = 4^x - 3$ is $(-3, \infty)$. There is a horizontal asymptote of $y = -3$.

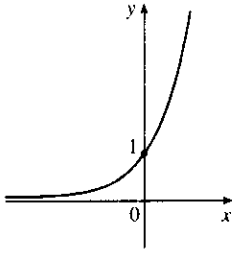


$y = 4^x$

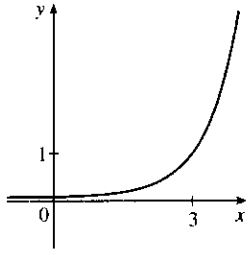


$y = 4^x - 3$

8. We start with the graph of $y = 4^x$ (Figure 3) and then shift 3 units to the right. There is a horizontal asymptote of $y = 0$.

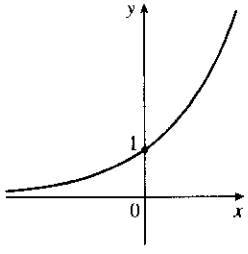


$y = 4^x$

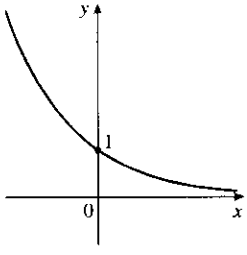


$y = 4^{x-3}$

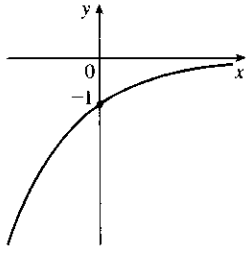
9. We start with the graph of $y = 2^x$ (Figure 3), reflect it about the y-axis, and then about the x-axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -2^{-x}$. In each graph, $y = 0$ is the horizontal asymptote.



$y = 2^x$



$y = 2^{-x}$



$y = -2^{-x}$

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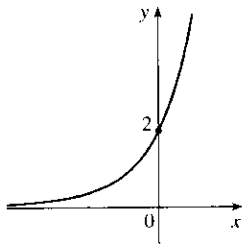
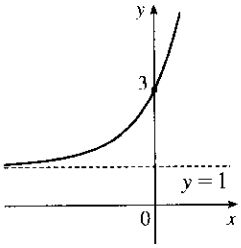
10. We start with the graph of $y = e^x$ (Figure 12), vertically stretch by a factor of 2, and then shift 1 unit upward. There is a horizontal asymptote of $y = 1$.

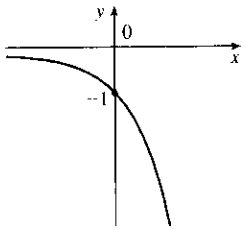


$y = 2e^x$

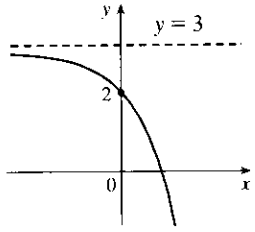


$y = 1 + 2e^x$

11. We start with the graph of $y = e^x$ (Figure 12), reflect it about the x-axis, and then shift 3 units upward. Note the horizontal asymptote of $y = 3$.

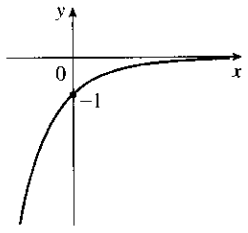


$y = -e^x$

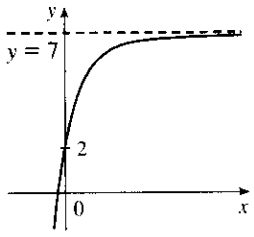


$y = 3 - e^x$

12. We start with the graph of $y = e^x$ (Figure 12), reflect it about the y-axis, and then about the x-axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -e^{-x}$. Now shift this graph 1 unit upward, vertically stretch by a factor of 5, and then shift 2 units upward.



$y = -e^{-x}$



$y = 2 + 5(1 - e^{-x})$

13. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

14. (a) This reflection consists of first reflecting the graph about the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

(b) This reflection consists of first reflecting the graph about the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

15. (a) The denominator $1 + e^x$ is never equal to zero because $e^x > 0$, so the domain of $f(x) = 1/(1 + e^x)$ is $\mathbb{R}$.

(b) $1 - e^x = 0 \quad \Leftrightarrow \quad e^x = 1 \quad \Leftrightarrow \quad x = 0$, so the domain of $f(x) = 1/(1 - e^x)$ is $(-\infty, 0) \cup (0, \infty)$.

16. (a) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^{-t})$ also has domain $\mathbb{R}$.

(b) The function $g(t) = \sqrt{1 - 2^t}$ has domain $\{t \mid 1 - 2^t \geq 0\} = \{t \mid 2^t \leq 1\} = \{t \mid t \leq 0\} = (-\infty, 0]$.

17. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1 \quad [C = \frac{6}{a}]$ and $24 = Ca^3 \quad \Rightarrow \quad 24 = \left(\frac{6}{a}\right)a^3 \quad \Rightarrow$

$4 = a^2 \quad \Rightarrow \quad a = 2 \quad [\text{since } a > 0]$ and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

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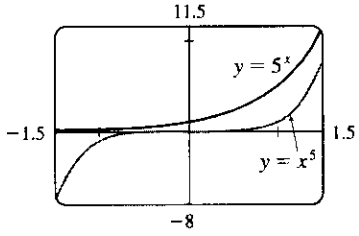
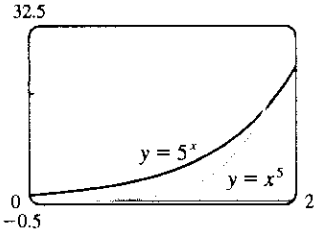
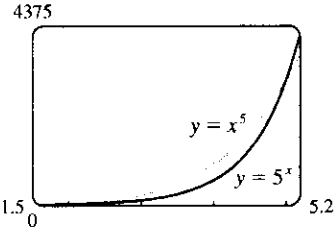
18. Given the y-intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $\left(2, \frac{2}{9}\right)$ gives us $\frac{2}{9} = 2a^2 \Rightarrow \frac{1}{9} = a^2 \Rightarrow a = \frac{1}{3}$ [since $a > 0$]. The function is $f(x) = 2\left(\frac{1}{3}\right)^x$ or $f(x) = 2(3)^{-x}$.

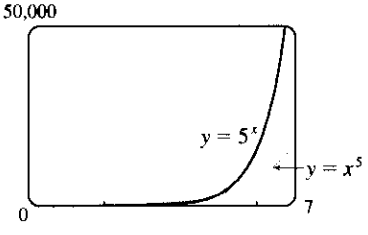
19. 2 ft = 24 in, $f(24) = 24^2$ in = 576 in = 48 ft. $g(24) = 2^{24}$ in $= 2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

20. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.

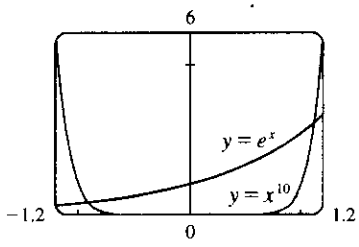


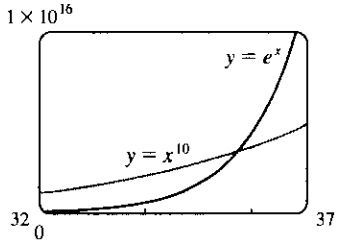




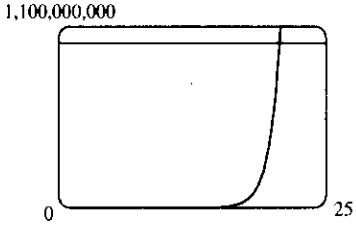


21. The graph of g finally surpasses that of f at $x \approx 35.8$.





22. We graph $y = e^x$ and $y = 1{,}000{,}000{,}000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so $e^x > 1 \times 10^9$ for $x > 20.723$.



23. $\lim_{x \to \infty} (1.001)^x = \infty$ by (3), since $1.001 > 1$.

24. Let $t = -x^2$. As $x \to \infty$, $t \to -\infty$. So $\lim_{x \to \infty} e^{-x^2} = \lim_{t \to -\infty} e^t = 0$ by (11).

25. Divide numerator and denominator by e^{3x}: $\lim_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x \to \infty} \dfrac{1 - e^{-6x}}{1 + e^{-6x}} = \dfrac{1 - 0}{1 + 0} = 1$

26. If we let $t = \tan x$, then as $x \to (\pi/2)^+$, $t \to -\infty$. Thus, $\lim_{x \to (\pi/2)^+} e^{\tan x} = \lim_{t \to -\infty} e^t = 0$.

27. Let $t = 3/(2 - x)$. As $x \to 2^+$, $t \to -\infty$. So $\lim_{x \to 2^+} e^{3/(2-x)} = \lim_{t \to -\infty} e^t = 0$ by (11).

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28. Let $t = 3/(2-x)$. As $x \to 2^-$, $t \to \infty$. So $\lim_{x\to 2^-} e^{3/(2-x)} = \lim_{t\to\infty} e^t = \infty$ by (11).

29. By the Product Rule, $f(x) = x^2e^x \quad \Rightarrow \quad f'(x) = x^2\,\dfrac{d}{dx}\left(e^x\right) + e^x\,\dfrac{d}{dx}\left(x^2\right) = x^2e^x + e^x(2x) = xe^x(x+2)$.

30. By the Quotient Rule, $y = \dfrac{e^x}{1+x} \quad \Rightarrow \quad y' = \dfrac{(1+x)\,e^x - e^x(1)}{(1+x)^2} = \dfrac{e^x + xe^x - e^x}{(x+1)^2} = \dfrac{xe^x}{(x+1)^2}$.

31. By (10), $y = e^{ax^3} \quad \Rightarrow \quad y' = e^{ax^3}\,\dfrac{d}{dx}\left(ax^3\right) = 3ax^2e^{ax^3}$.

32. $y = e^u(\cos u + cu) \quad \Rightarrow \quad y' = e^u(-\sin u + c) + (\cos u + cu)e^u = e^u(\cos u - \sin u + cu + c)$

33. $f(u) = e^{1/u} \quad \Rightarrow \quad f'(u) = e^{1/u}\cdot\dfrac{d}{du}\left(\dfrac{1}{u}\right) = e^{1/u}\left(\dfrac{-1}{u^2}\right) = \left(\dfrac{-1}{u^2}\right)e^{1/u}$

34. By the Product Rule, $g(x) = \sqrt{x}\,e^x = x^{1/2}e^x \quad \Rightarrow \quad g'(x) = x^{1/2}(e^x) + e^x\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2}x^{-1/2}e^x(2x+1)$.

35. By (10), $F(t) = e^{t\sin 2t} \quad \Rightarrow$
$F'(t) = e^{t\sin 2t}(t\sin 2t)' = e^{t\sin 2t}(t\cdot 2\cos 2t + \sin 2t\cdot 1) = e^{t\sin 2t}(2t\cos 2t + \sin 2t)$

36. $y = e^{k\tan\sqrt{x}} \quad \Rightarrow \quad y' = e^{k\tan\sqrt{x}}\cdot\dfrac{d}{dx}(k\tan\sqrt{x}\,) = e^{k\tan\sqrt{x}}\left(k\sec^2\sqrt{x}\cdot\tfrac{1}{2}x^{-1/2}\right) = \dfrac{k\sec^2\sqrt{x}}{2\sqrt{x}}\,e^{k\tan\sqrt{x}}$

37. $y = \sqrt{1+2e^{3x}} \quad \Rightarrow \quad y' = \dfrac{1}{2}(1+2e^{3x})^{-1/2}\,\dfrac{d}{dx}\,(1+2e^{3x}) = \dfrac{1}{2\sqrt{1+2e^{3x}}}(2e^{3x}\cdot 3) = \dfrac{3e^{3x}}{\sqrt{1+2e^{3x}}}$

38. $y = \cos\left(e^{\pi x}\right) \quad \Rightarrow \quad y' = -\sin(e^{\pi x})\cdot e^{\pi x}\cdot\pi = -\pi e^{\pi x}\sin(e^{\pi x})$

39. $y = e^{e^x} \quad \Rightarrow \quad y' = e^{e^x}\cdot\dfrac{d}{dx}\left(e^x\right) = e^{e^x}\cdot e^x \;\text{ or }\; e^{e^x+x}$

40. $y = \sqrt{1+xe^{-2x}} \quad \Rightarrow \quad y' = \frac{1}{2}\left(1+xe^{-2x}\right)^{-1/2}\left[x\left(-2e^{-2x}\right) + e^{-2x}\right] = \dfrac{e^{-2x}(-2x+1)}{2\sqrt{1+xe^{-2x}}}$

41. By the Quotient Rule, $y = \dfrac{ae^x + b}{ce^x + d} \quad \Rightarrow$
$y' = \dfrac{(ce^x+d)(ae^x) - (ae^x+b)(ce^x)}{(ce^x+d)^2} = \dfrac{(ace^x + ad - ace^x - bc)e^x}{(ce^x+d)^2} = \dfrac{(ad-bc)e^x}{(ce^x+d)^2}$.

42. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \quad \Rightarrow \quad y' = \dfrac{\left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right) - \left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right)}{\left(e^x - e^{-x}\right)^2}$

$$= \frac{\left(e^{2x} - 2 + e^{-2x}\right) - \left(e^{2x} + 2 + e^{-2x}\right)}{\left(e^x - e^{-x}\right)^2} = -\frac{4}{\left(e^x - e^{-x}\right)^2}$$

43. $y = e^{2x}\cos\pi x \quad \Rightarrow \quad y' = e^{2x}(-\pi\sin\pi x) + (\cos\pi x)(2e^{2x}) = e^{2x}(2\cos\pi x - \pi\sin\pi x)$.
At $(0,1)$, $y' = 1(2-0) = 2$, so an equation of the tangent line is $y - 1 = 2(x-0)$, or $y = 2x + 1$.

44. $y = \dfrac{e^x}{x} \quad \Rightarrow \quad y' = \dfrac{x\cdot e^x - e^x\cdot 1}{x^2} = \dfrac{e^x(x-1)}{x^2}$. At $(1,e)$, $y' = 0$, and an equation of the tangent line is $y - e = 0(x-1)$, or $y = e$.

45. $\dfrac{d}{dx}\left(e^{x^2y}\right) = \dfrac{d}{dx}\,(x+y) \quad \Rightarrow \quad e^{x^2y}(x^2y' + y\cdot 2x) = 1 + y' \quad \Rightarrow \quad x^2e^{x^2y}y' + 2xye^{x^2y} = 1 + y' \quad \Rightarrow$

$x^2e^{x^2y}y' - y' = 1 - 2xye^{x^2y} \quad \Rightarrow \quad y'(x^2e^{x^2y} - 1) = 1 - 2xye^{x^2y} \quad \Rightarrow \quad y' = \dfrac{1 - 2xye^{x^2y}}{x^2e^{x^2y} - 1}$

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46. $xe^y + ye^x = 1 \;\Rightarrow\; xe^y y' + e^y \cdot 1 + ye^x + e^x y' = 0 \;\Rightarrow\; y'(xe^y + e^x) = -e^y - ye^x \;\Rightarrow$

$y' = -\dfrac{e^y + ye^x}{xe^y + e^x}$. At $(0,1)$, $y' = -\dfrac{e + 1 \cdot 1}{0 + 1} = -(e+1)$, so an equation for the tangent line is

$y - 1 = -(e+1)(x - 0)$, or $y = -(e+1)x + 1$.

47. $y = e^x + e^{-x/2} \;\Rightarrow\; y' = e^x - \frac{1}{2}e^{-x/2} \;\Rightarrow\; y'' = e^x + \frac{1}{4}e^{-x/2}$, so

$2y'' - y' - y = 2\left(e^x + \frac{1}{4}e^{-x/2}\right) - \left(e^x - \frac{1}{2}e^{-x/2}\right) - \left(e^x + e^{-x/2}\right) = 0.$

48. $y = Ae^{-x} + Bxe^{-x} \;\Rightarrow\; y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B - A)e^{-x} - Bxe^{-x} \;\Rightarrow$

$y'' = (A - B)e^{-x} - Be^{-x} + Bxe^{-x} = (A - 2B)e^{-x} + Bxe^{-x}$, so

$y'' + 2y' + y = (A - 2B)e^{-x} + Bxe^{-x} + 2\left[(B - A)e^{-x} - Bxe^{-x}\right] + Ae^{-x} + Bxe^{-x} = 0.$

49. $y = e^{rx} \;\Rightarrow\; y' = re^{rx} \;\Rightarrow\; y'' = r^2 e^{rx}$, so if $y = e^{rx}$ satisfies the differential equation $y'' + 6y' + 8y = 0$, then $r^2 e^{rx} + 6re^{rx} + 8e^{rx} = 0$; that is, $e^{rx}(r^2 + 6r + 8) = 0$. Since $e^{rx} > 0$ for all x, we must have $r^2 + 6r + 8 = 0$, or $(r+2)(r+4) = 0$, so $r = -2$ or -4.

50. $y = e^{\lambda x} \;\Rightarrow\; y' = \lambda e^{\lambda x} \;\Rightarrow\; y'' = \lambda^2 e^{\lambda x}$. Thus, $y + y' = y'' \;\Leftrightarrow\; e^{\lambda x} + \lambda e^{\lambda x} = \lambda^2 e^{\lambda x} \;\Leftrightarrow$

$e^{\lambda x}(\lambda^2 - \lambda - 1) = 0 \;\Leftrightarrow\; \lambda = \frac{1 \pm \sqrt{5}}{2}$, since $e^{\lambda x} \neq 0$.

51. $f(x) = e^{2x} \;\Rightarrow\; f'(x) = 2e^{2x} \;\Rightarrow\; f''(x) = 2 \cdot 2e^{2x} = 2^2 e^{2x} \;\Rightarrow$

$f'''(x) = 2^2 \cdot 2e^{2x} = 2^3 e^{2x} \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = 2^n e^{2x}$

52. $f(x) = xe^{-x} \;\Rightarrow\; f'(x) = x(-e^{-x}) + e^{-x} = (1 - x)e^{-x} \;\Rightarrow$

$f''(x) = (1 - x)(-e^{-x}) + e^{-x}(-1) = (x - 2)e^{-x} \;\Rightarrow\; f'''(x) = (x - 2)(-e^{-x}) + e^{-x} = (3 - x)e^{-x} \;\Rightarrow$

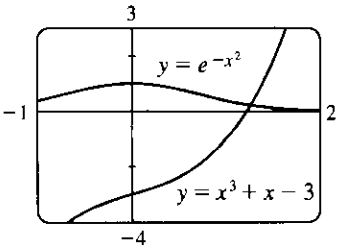
$f^{(4)}(x) = (3 - x)(-e^{-x}) + e^{-x}(-1) = (x - 4)e^{-x} \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = (-1)^n (x - n)e^{-x}$.

So $D^{1000} xe^{-x} = (x - 1000)e^{-x}$.

53. (a) $f(x) = e^x + x$ is continuous on $\mathbb{R}$ and $f(-1) = e^{-1} - 1 < 0 < 1 = f(0)$, so by the Intermediate Value Theorem, $e^x + x = 0$ has a root in $(-1, 0)$.

(b) $f(x) = e^x + x \;\Rightarrow\; f'(x) = e^x + 1$, so $x_{n+1} = x_n - \dfrac{e^{x_n} + x_n}{e^{x_n} + 1}$. Using $x_1 = -0.5$, we get $x_2 \approx -0.566311$, $x_3 \approx -0.567143 \approx x_4$, so the root is -0.567143 to six decimal places.

54.



From the graph, it appears that the curves intersect at about $x \approx 1.2$ or 1.3. We use Newton's Method with $f(x) = x^3 + x - 3 - e^{-x^2}$, so $f'(x) = 3x^2 + 1 + 2xe^{-x^2}$, and the formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. We take $x_1 = 1.2$, and the formula gives $x_2 \approx 1.252462$, $x_3 \approx 1.251045$, and $x_4 \approx x_5 \approx 1.251044$. So the root of the equation, correct to six decimal places, is $x = 1.251044$.

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55. (a) $m(t) = 24 \cdot 2^{-t/25} \Rightarrow$

$$m(40) = 24 \cdot 2^{-40/25} \approx 7.92 \text{ mg}$$

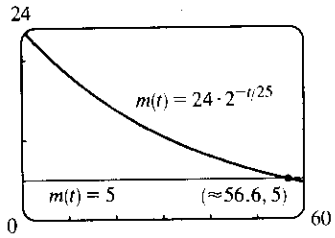
(b) $m'(t) = 24 \dfrac{d}{dt}\left(2^{-t/25}\right)$

$$\approx 24(0.69)2^{-t/25} \frac{d}{dt}\left(-\frac{t}{25}\right) \qquad [(7) \text{ and } (10)]$$

$$= 24(0.69)\left(-\tfrac{1}{25}\right) 2^{-t/25}$$

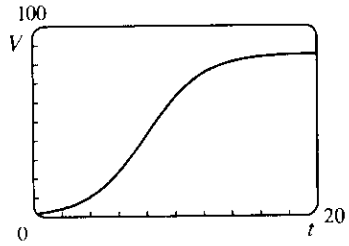
so $m'(40) \approx -\frac{24}{25}(0.69)2^{-40/25} \approx -0.22$ mg/yr.

(c)



From the graph, we can determine that $m(t) = 5 \Rightarrow t \approx 56.6$ yr.

56.



From the graph, we estimate that the most rapid increase in the percentage of households in the United States with at least one VCR occurs at about $t = 8$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. We'll use $V(t) = \dfrac{a}{1 + be^{ct}}$, and substitute $a = 85$, $b = 53$, and $c = -0.5$ later.

$$V'(t) = -\frac{a(bce^{ct})}{(1 + be^{ct})^2} \quad \text{[by the Reciprocal Rule]} \quad \text{and}$$

$$V''(t) = -abc \cdot \frac{(1 + be^{ct})^2 \cdot ce^{ct} - e^{ct} \cdot 2(1 + be^{ct}) \cdot bce^{ct}}{\left[(1 + be^{ct})^2\right]^2}$$

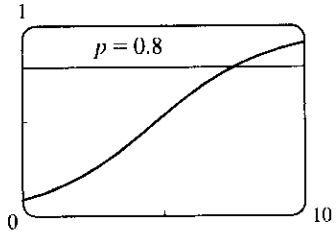
$$= \frac{-abc \cdot ce^{ct}(1 + be^{ct})[(1 + be^{ct}) - 2be^{ct}]}{(1 + be^{ct})^4} = \frac{-abc^2 e^{ct}(1 - be^{ct})}{(1 + be^{ct})^3}$$

So $V''(t) = 0 \Leftrightarrow 1 = be^{ct} \Leftrightarrow e^{ct} = 1/b$. Now graph $y = e^{-0.5t}$ and $y = \frac{1}{53}$. These graphs intersect at $t \approx 7.94$ years, which corresponds to roughly midyear 1988. [Alternatively, we could use the rootfinder on a calculator to solve $e^{-0.5t} = \frac{1}{53}$. Or, if you have already studied logarithms, you can solve $e^{ct} = 1/b$ as follows: $ct = \ln(1/b) \Leftrightarrow t = (1/c)\ln(1/b) = -2 \ln \frac{1}{53} \approx 7.94$ years.

57. (a) $\lim\limits_{t \to \infty} p(t) = \lim\limits_{t \to \infty} \dfrac{1}{1 + ae^{-kt}} = \dfrac{1}{1 + a \cdot 0} = 1$, since $k > 0 \Rightarrow -kt \to -\infty \Rightarrow e^{-kt} \to 0$.

(b) $p(t) = \left(1 + ae^{-kt}\right)^{-1} \Rightarrow \dfrac{dp}{dt} = -\left(1 + ae^{-kt}\right)^{-2}\left(-kae^{-kt}\right) = \dfrac{kae^{-kt}}{\left(1 + ae^{-kt}\right)^2}$

(c)



From the graph of $p(t) = \left(1 + 10e^{-0.5t}\right)^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

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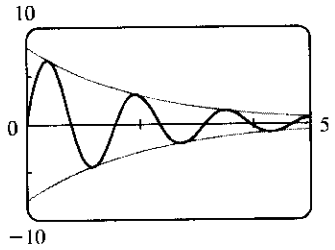
58. (a)



The displacement function is squeezed between the other two functions. This is because $-1 \le \sin 4t \le 1 \Rightarrow$ $-8e^{-t/2} \le 8e^{-t/2}\sin 4t \le 8e^{-t/2}$.

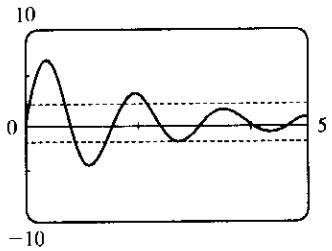
(b) The maximum value of the displacement is about 6.6 cm, occurring at $t \approx 0.36$ s. It occurs just before the graph of the displacement function touches the graph of $8e^{-t/2}$ (when $t = \frac{\pi}{8} \approx 0.39$).

(c) The velocity of the object is the derivative of its displacement function, that is,

$$\frac{d}{dt}\left(8e^{-t/2}\sin 4t\right) = 8\left[e^{-t/2}\cos 4t(4) + \sin 4t\left(-\tfrac{1}{2}\right)e^{-t/2}\right].$$

If the displacement is zero, then we must have $\sin 4t = 0$ (since the exponential term in the displacement function is always positive). The first time that $\sin 4t = 0$ after $t = 0$ occurs at $t = \frac{\pi}{4}$. Substituting this into our expression for the velocity, and noting that the second term vanishes, we get $v\left(\frac{\pi}{4}\right) = 8e^{-\pi/8}\cos\left(4 \cdot \frac{\pi}{4}\right)\cdot 4 = -32e^{-\pi/8} \approx -21.6$ cm/s.

(d)

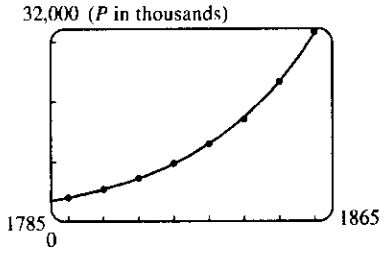


The graph indicates that the displacement is less than 2 cm from equilibrium whenever t is larger than about 2.8.

59. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a = 100.0124369$ and $b = 0.000045145933$. We can change this model to one with base e and exponent $\ln b$ [$b^t = e^{t\ln b}$ from precalculus mathematics or from Section 7.3]: $Q = ae^{t\ln b} = 100.012437e^{-10.005531t}$.

(b) Use $Q'(t) = ab^t \ln b$ or the calculator command `nDeriv(Y1,X,.04)` with $Y_1 = ab^x$ to get $Q'(0.04) \approx -670.63\ \mu$A. The result of Example 2 in Section 2.1 was $-670\ \mu$A.

60. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \dfrac{7240 - 5308}{1810 - 1800} = 193.2$.

So $P'(1800) \approx (m_1 + m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192 - 17{,}063}{1850 - 1840} = 612.9$, $m_2 = \dfrac{31{,}443 - 23{,}192}{1860 - 1850} = 825.1$.

So $P'(1850) \approx (m_1 + m_2)/2 = 719$ thousand people/year.

(c) Use the calculator command `nDeriv(Y1,X,year)` with $Y_1 = ab^x$ to get $P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

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61. $f(x) = x - e^x \Rightarrow f'(x) = 1 - e^x = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0$. Now $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, so the absolute maximum value is $f(0) = 0 - 1 = -1$.

62. $g(x) = \dfrac{e^x}{x} \Rightarrow g'(x) = \dfrac{xe^x - e^x}{x^2} = 0 \Leftrightarrow e^x(x-1) = 0 \Rightarrow x = 1$. Now $g'(x) > 0 \Leftrightarrow \dfrac{xe^x - e^x}{x^2} > 0 \Leftrightarrow x - 1 > 0 \Leftrightarrow x > 1$ and $g'(x) < 0 \Leftrightarrow \dfrac{xe^x - e^x}{x^2} < 0 \Leftrightarrow x - 1 < 0 \Leftrightarrow x < 1$. Thus there is an absolute minimum value of $g(1) = e$ at $x = 1$.

63. (a) $f(x) = xe^x \Rightarrow f'(x) = e^x + xe^x = e^x(1+x) > 0 \Leftrightarrow 1 + x > 0 \Leftrightarrow x > -1$, so f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

(b) $f''(x) = e^x(1+x) + e^x = e^x(2+x) > 0 \Leftrightarrow 2 + x > 0 \Leftrightarrow x > -2$, so f is CU on $(-2, \infty)$ and CD on $(-\infty, -2)$.

(c) f has an inflection point at $\left(-2, -2e^{-2}\right)$.

64. (a) $f(x) = x^2e^x \Rightarrow f'(x) = 2xe^x + x^2e^x = (x^2 + 2x)e^x$. $f'(x) > 0 \Leftrightarrow x(x+2) > 0 \Leftrightarrow x < -2$ or $x > 0$, $f'(x) < 0 \Leftrightarrow -2 < x < 0$, so f is increasing on $(-\infty, -2)$ and $(0, \infty)$ and decreasing on $(-2, 0)$.

(b) $f''(x) = (2x+2)e^x + (x^2+2x)e^x = (x^2+4x+2)e^x = 0 \Leftrightarrow x^2 + 4x + 2 = 0 \Leftrightarrow x = -2 \pm \sqrt{2}$ $f''(x) > 0$ when $x > -2 + \sqrt{2}$ or $x < -2 - \sqrt{2}$, so f is CU on $\left(-\infty, -2 - \sqrt{2}\right)$ and $\left(-2 + \sqrt{2}, \infty\right)$ and CD on $\left(-2 - \sqrt{2}, -2 + \sqrt{2}\right)$.

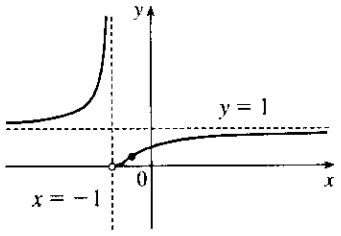
(c) f has inflection points at $\left(-2 + \sqrt{2}, \left(6 - 4\sqrt{2}\right)e^{-2+\sqrt{2}}\right)$ and $\left(-2 - \sqrt{2}, \left(6 + 4\sqrt{2}\right)e^{-2-\sqrt{2}}\right)$.

65. $y = f(x) = e^{-1/(x+1)}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** No x-intercept; y-intercept $= f(0) = e^{-1}$ **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \to 0$, so $y = 1$ is a HA. $\lim\limits_{x \to -1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \to -\infty$, $\lim\limits_{x \to -1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \to \infty$, so $x = -1$ is a VA. **E.** $f'(x) = e^{-1/(x+1)}/(x+1)^2 \Rightarrow f'(x) > 0$ for all x except 1, so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{e^{-1/(x+1)}}{(x+1)^4} + \dfrac{e^{-1/(x+1)}(-2)}{(x+1)^3} = -\dfrac{e^{-1/(x+1)}(2x+1)}{(x+1)^4}$

$\Rightarrow f''(x) > 0 \Leftrightarrow 2x + 1 < 0 \Leftrightarrow x < -\frac{1}{2}$, so f is CU on $(-\infty, -1)$ and $\left(-1, -\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2}, \infty\right)$. f has an IP at $\left(-\frac{1}{2}, e^{-2}\right)$.

H.



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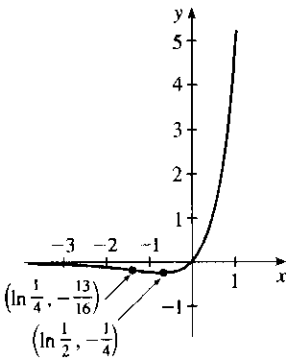
66. $y = f(x) = e^{2x} - e^{x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$;
x-intercepts: $f(x) = 0 \;\Rightarrow\; e^{2x} = e^{x} \;\Rightarrow\; e^{x} = 1 \;\Rightarrow\; x = 0$.
C. No symmetry **D.** $\lim_{x \to -\infty} e^{2x} - e^{x} = 0$, so $y = 0$ is a HA. No VA.
E. $f'(x) = 2e^{2x} - e^{x} = e^{x}(2e^{x} - 1)$, so $f'(x) > 0 \;\Leftrightarrow\; e^{x} > \frac{1}{2}$ (*)
$\Leftrightarrow\; x > \ln \frac{1}{2} = -\ln 2$ and $f'(x) < 0 \;\Leftrightarrow\; e^{x} < \frac{1}{2} \;\Leftrightarrow\; x < \ln \frac{1}{2}$, so f is decreasing on $(-\infty, \ln \frac{1}{2})$ and increasing on $(\ln \frac{1}{2}, \infty)$. **F.** Local minimum value $f(\ln \frac{1}{2}) = e^{2\ln(1/2)} - e^{\ln(1/2)} = (\frac{1}{2})^2 - \frac{1}{2} = -\frac{1}{4}$
G. $f''(x) = 4e^{2x} - e^{x} = e^{x}(4e^{x} - 1)$, so $f''(x) > 0 \;\Leftrightarrow$
$e^{x} > \frac{1}{4} \;\Leftrightarrow\; x > \ln \frac{1}{4}$ and $f''(x) < 0 \;\Leftrightarrow\; x < \ln \frac{1}{4}$.
Thus, f is CD on $(-\infty, \ln \frac{1}{4})$ and CU on $(\ln \frac{1}{4}, \infty)$. f has an IP at $\left(\ln \frac{1}{4}, (\frac{1}{4})^2 - \frac{1}{4}\right) = (\ln \frac{1}{4}, -\frac{3}{16})$.

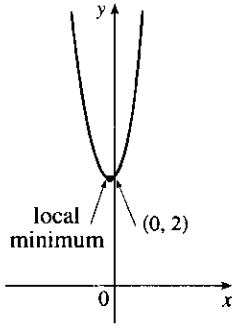
H.



(*) If you have not yet learned about logarithms, graph $y = e^{x}$ and $y = \frac{1}{2}$, find the point of intersection, and use decimal approximations for the rest of the solution.

67. $y = f(x) = e^{3x} + e^{-2x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$;
no x-intercept **C.** No symmetry **D.** No asymptotes
E. $f'(x) = 3e^{3x} - 2e^{-2x}$, so $f'(x) > 0 \;\Leftrightarrow\; 3e^{3x} > 2e^{-2x}$
[multiply by e^{2x}] $\Leftrightarrow\; e^{5x} > \frac{2}{3}$ (*) $\Leftrightarrow\; 5x > \ln \frac{2}{3} \;\Leftrightarrow$
$x > \frac{1}{5} \ln \frac{2}{3} \approx -0.081$. Similarly, $f'(x) < 0 \;\Leftrightarrow\; x < \frac{1}{5} \ln \frac{2}{3}$.
f is decreasing on $(-\infty, \frac{1}{5} \ln \frac{2}{3})$ and increasing on $(\frac{1}{5} \ln \frac{2}{3}, \infty)$.

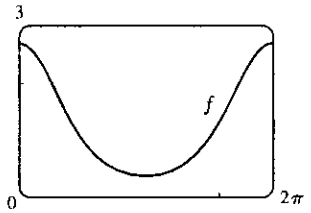
H.

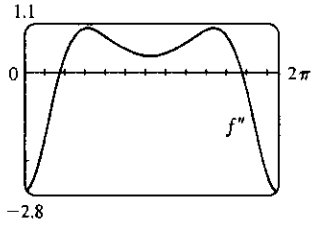


F. Local minimum value $f(\frac{1}{5} \ln \frac{2}{3}) = (\frac{2}{3})^{3/5} + (\frac{2}{3})^{-2/5} \approx 1.96$; no local maximum.
G. $f''(x) = 9e^{3x} + 4e^{-2x}$, so $f''(x) > 0$ for all x, and f is CU on $(-\infty, \infty)$. No IP

(*) If you have not yet learned about logarithms, graph $y = e^{5x}$ and $y = \frac{2}{3}$, find the point of intersection, and use decimal approximations for the rest of the solution.

68. The function $f(x) = e^{\cos x}$ is periodic with period 2π, so we consider it only on the interval $[0, 2\pi]$. We see that it has local maxima of about $f(0) \approx 2.72$ and $f(2\pi) \approx 2.72$, and a local minimum of about $f(3.14) \approx 0.37$. To find the exact values, we calculate $f'(x) = -\sin x e^{\cos x}$. This is 0 when $-\sin x = 0 \;\Leftrightarrow\; x = 0, \pi$ or 2π (since we are only considering $x \in [0, 2\pi]$). Also $f'(x) > 0 \;\Leftrightarrow\; \sin x < 0 \;\Leftrightarrow\; 0 < x < \pi$. So $f(0) = f(2\pi) = e$





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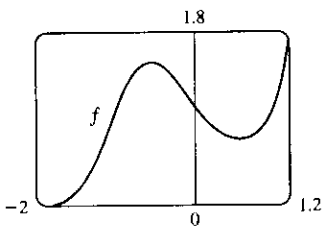
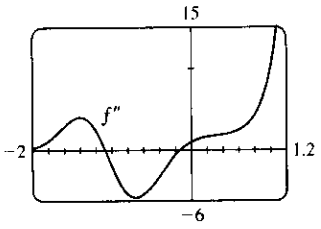
(both maxima) and $f(\pi) = e^{\cos \pi} = 1/e$ (minimum). To find the inflection points, we calculate and graph $f''(x) = \dfrac{d}{dx}\left(-\sin x\, e^{\cos x}\right) = -\cos x\, e^{\cos x} - \sin x(e^{\cos x})(-\sin x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$. From the graph of $f''(x)$, we see that f has inflection points at $x \approx 0.90$ and at $x \approx 5.38$. These x-coordinates correspond to inflection points $(0.90, 1.86)$ and $(5.38, 1.86)$.

69. $f(x) = e^{x^3 - x} \to 0$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$. From the graph, it appears that f has a local minimum of about $f(0.58) = 0.68$, and a local maximum of about $f(-0.58) = 1.47$.





To find the exact values, we calculate $f'(x) = (3x^2 - 1)e^{x^3 - x}$, which is 0 when $3x^2 - 1 = 0 \quad \Leftrightarrow \quad x = \pm\frac{1}{\sqrt{3}}$. The negative root corresponds to the local maximum $f\left(-\frac{1}{\sqrt{3}}\right) = e^{(-1/\sqrt{3})^3 - (-1/\sqrt{3})} = e^{2\sqrt{3}/9}$, and the positive root corresponds to the local minimum $f\left(\frac{1}{\sqrt{3}}\right) = e^{(1/\sqrt{3})^3 - (1/\sqrt{3})} = e^{-2\sqrt{3}/9}$. To estimate the inflection points, we calculate and graph $f''(x) = \dfrac{d}{dx}\left[(3x^2 - 1)e^{x^3 - x}\right] = (3x^2 - 1)e^{x^3 - x}(3x^2 - 1) + e^{x^3 - x}(6x) = e^{x^3 - x}\left(9x^4 - 6x^2 + 6x + 1\right)$. From the graph, it appears that $f''(x)$ changes sign (and thus f has inflection points) at $x \approx -0.15$ and $x \approx -1.09$. From the graph of f, we see that these x-values correspond to inflection points at about $(-0.15, 1.15)$ and $(-1.09, 0.82)$.

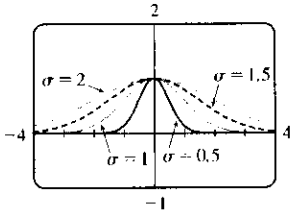
70. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \quad \Rightarrow$ $f'(x) = e^{-x^2/(2\sigma^2)}\left(-x/\sigma^2\right)$. $f'(x) = 0 \quad \Leftrightarrow \quad x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find

$$f''(x) = -\frac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}\left(-x/\sigma^2\right)\right] = \frac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}\left(1 - x^2/\sigma^2\right).$$

$f''(x) = 0 \quad \Leftrightarrow \quad x^2 = \sigma^2 \quad \Leftrightarrow \quad x = \pm\sigma$. $f''(x) < 0 \quad \Leftrightarrow \quad x^2 < \sigma^2 \quad \Leftrightarrow \quad -\sigma < x < \sigma$. So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $\left(\pm\sigma, e^{-1/2}\right)$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

71. Let $u = -3x$. Then $du = -3\,dx$, so

$$\int_0^5 e^{-3x}\,dx = -\tfrac{1}{3}\int_0^{-15} e^u\,du = -\tfrac{1}{3}\left[e^u\right]_0^{-15} = -\tfrac{1}{3}\left(e^{-15} - e^0\right) = \tfrac{1}{3}\left(1 - e^{-15}\right).$$

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72. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\left[e^u\right]_0^{-1} = -\tfrac{1}{2}\left(e^{-1} - e^0\right) = \tfrac{1}{2}(1 - 1/e).$$

73. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

74. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so $\int \sec^2 x\,e^{\tan x}\,dx = \int e^u\,du = e^u + C = e^{\tan x} + C$.

75. $\displaystyle\int \frac{e^x + 1}{e^x}\,dx = \int (1 + e^{-x})\,dx = x - e^{-x} + C$

76. Let $u = \dfrac{1}{x}$. Then $du = -\dfrac{1}{x^2}\,dx$, so $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{1/x} + C$.

77. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$, so $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$.

78. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x \sin(e^x)\,dx = \int \sin u\,du = -\cos u + C = -\cos(e^x) + C$.

79. Area $= \int_0^1 \left(e^{3x} - e^x\right)dx = \left[\frac{1}{3}e^{3x} - e^x\right]_0^1 = \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3} - 1\right) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644$

80. $f''(x) = 3e^x + 5\sin x \;\Rightarrow\; f'(x) = 3e^x - 5\cos x + C \;\Rightarrow\; 2 = f'(0) = 3 - 5 + C \;\Rightarrow\; C = 4$, so $f'(x) = 3e^x - 5\cos x + 4 \;\Rightarrow\; f(x) = 3e^x - 5\sin x + 4x + D \;\Rightarrow\; 1 = f(0) = 3 + D \;\Rightarrow\; D = -2$, so $f(x) = 3e^x - 5\sin x + 4x - 2$.

81. $V = \int_0^1 \pi(e^x)^2\,dx = \pi\int_0^1 e^{2x}\,dx = \frac{1}{2}\pi\left[e^{2x}\right]_0^1 = \frac{\pi}{2}\left(e^2 - 1\right)$

82. $V = \int_0^1 2\pi x e^{-x^2}\,dx$. Let $u = x^2$. Thus $du = 2x\,dx$, so $V = \pi\int_0^1 e^{-u}\,du = \pi\left[-e^{-u}\right]_0^1 = \pi(1 - 1/e)$.

83. We use Theorem 7.1.7. Note that $f(0) = 3 + 0 + e^0 = 4$, so $f^{-1}(4) = 0$. Also $f'(x) = 1 + e^x$. Therefore,

$$\left(f^{-1}\right)'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}.$$

84. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form $\displaystyle\lim_{x\to\pi}\frac{f(x) - f(\pi)}{x - \pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos\pi)e^{\sin\pi} = -1\cdot e^0 = -1$.

85. (a) Let $f(x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 = 0$, and for $x \geq 0$, we have $f'(x) = e^x - 1 \geq 0$. Now, since $f(0) = 0$ and f is increasing on $[0,\infty)$, $f(x) \geq 0$ for $x \geq 0 \;\Rightarrow\; e^x - 1 - x \geq 0 \;\Rightarrow\; e^x \geq 1 + x$.

(b) For $0 \leq x \leq 1$, $x^2 \leq x$, so $e^{x^2} \leq e^x$ [since e^x is increasing]. Hence [from (a)] $1 + x^2 \leq e^{x^2} \leq e^x$. So $\frac{4}{3} = \int_0^1 (1 + x^2)\,dx \leq \int_0^1 e^{x^2}\,dx \leq \int_0^1 e^x\,dx = e - 1 < e \;\Rightarrow\; \frac{4}{3} \leq \int_0^1 e^{x^2}\,dx \leq e$.

86. (a) Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus, $f'(x) = e^x - 1 - x$, which is positive for $x \geq 0$ by Exercise 85(a). Thus $f(x)$ is increasing on $(0,\infty)$, so on that interval, $0 = f(0) \leq f(x) = e^x - 1 - x - \frac{1}{2}x^2 \;\Rightarrow\; e^x \geq 1 + x + \frac{1}{2}x^2$.

(b) Using the same argument as in Exercise 85(b), from part (a) we have $1 + x^2 + \frac{1}{2}x^4 \leq e^{x^2} \leq e^x$ [for $0 \leq x \leq 1$] $\;\Rightarrow\; \int_0^1 \left(1 + x^2 + \frac{1}{2}x^4\right)dx \leq \int_0^1 e^{x^2}\,dx \leq \int_0^1 e^x\,dx \;\Rightarrow\; \frac{43}{30} \leq \int_0^1 e^{x^2}\,dx \leq e - 1$.

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87. (a) By Exercise 85(a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^k}{k!}$ for $x \geq 0$.

Let $f(x) = e^x - 1 - x - \dfrac{x^2}{2!} - \cdots - \dfrac{x^k}{k!} - \dfrac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \dfrac{x^k}{k!} \geq 0$ by assumption. Hence $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that $0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \dfrac{x^k}{k!} - \dfrac{x^{k+1}}{(k+1)!}$, and hence $e^x \geq 1 + x + \cdots + \dfrac{x^k}{k!} + \dfrac{x^{k+1}}{(k+1)!}$ for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}$ for every positive integer n, by mathematical induction.

(b) Taking $n = 4$ and $x = 1$ in (a), we have $e = e^1 \geq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\overline{3} > 2.7$.

(c) $e^x \geq 1 + x + \cdots + \dfrac{x^k}{k!} + \dfrac{x^{k+1}}{(k+1)!} \quad \Rightarrow \quad \dfrac{e^x}{x^k} \geq \dfrac{1}{x^k} + \dfrac{1}{x^{k-1}} + \cdots + \dfrac{1}{k!} + \dfrac{x}{(k+1)!} \geq \dfrac{x}{(k+1)!}$.

But $\lim\limits_{x \to \infty} \dfrac{x}{(k+1)!} = \infty$, so $\lim\limits_{x \to \infty} \dfrac{e^x}{x^k} = \infty$.

7.3 Logarithmic Functions

1. (a) It is defined as the inverse of the exponential function with base a, that is, $\log_a x = y \quad \Leftrightarrow \quad a^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 1.

2. (a) The natural logarithm is the logarithm with base e, denoted $\ln x$.

(b) The common logarithm is the logarithm with base 10, denoted $\log x$.

(c) See Figure 3.

3. (a) $\log_{10} 1000 = 3$ because $10^3 = 1000$. *Or:* $\log_{10} 1000 = \log_{10} 10^3 = 3$ by (2).

(b) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$. *Or:* $\log_{16} 4 = \log_{16} 16^{1/2} = \frac{1}{2}$ by (2).

4. (a) By (6), $\ln e^{-100} = -100$. (b) $\log_3 81 = 4$ since $3^4 = 81$.

5. (a) $\log_5 \frac{1}{25} = \log_5 5^{-2} = -2$ by (2). (b) $e^{\ln 15} = 15$ by (6).

6. (a) $\log_{10} 0.1 = -1$ since $10^{-1} = 0.1$.

(b) $\log_8 320 - \log_8 5 = \log_8 \frac{320}{5} = \log_8 64 = 2$ since $8^2 = 64$.

7. (a) $\log_{12} 3 + \log_{12} 48 = \log_{12}(3 \cdot 48) = \log_{12} 144 = 2$ since $12^2 = 144$.

(b) $\log_2 5 - \log_2 90 + 2 \log_2 3 = \log_2 5 + \log_2 3^2 - \log_2 90 = \log_2(5 \cdot 9) - \log_2 90$
$= \log_2\left(\frac{45}{90}\right) = \log_2\left(\frac{1}{2}\right) = -1$ since $2^{-1} = \frac{1}{2}$.

8. (a) $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 15} = 15$ [*Or:* $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 3} \cdot 2^{\log_2 5} = 3 \cdot 5 = 15$]

(b) $e^{3 \ln 2} = e^{\ln(2^3)} = e^{\ln 8} = 8$ [*Or:* $e^{3 \ln 2} = \left(e^{\ln 2}\right)^3 = 2^3 = 8$]

9. $\log_2\left(\dfrac{x^3 y}{z^2}\right) = \log_2(x^3 y) - \log_2 z^2 = \log_2 x^3 + \log_2 y - \log_2 z^2 = 3 \log_2 x + \log_2 y - 2 \log_2 z$

(assuming that the variables are positive)

10. $\ln \sqrt{a(b^2 + c^2)} = \ln(a(b^2 + c^2))^{1/2} = \frac{1}{2} \ln(a(b^2 + c^2)) = \frac{1}{2}\left[\ln a + \ln(b^2 + c^2)\right]$
$= \frac{1}{2} \ln a + \frac{1}{2} \ln(b^2 + c^2)$

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11. $\ln(uv)^{10} = 10\ln(uv) = 10(\ln u + \ln v) = 10\ln u + 10\ln v$

12. $\ln\dfrac{3x^2}{(x+1)^5} = \ln 3x^2 - \ln(x+1)^5 = \ln 3 + \ln x^2 - 5\ln(x+1) = \ln 3 + 2\ln x - 5\ln(x+1)$

13. $\log_{10} a - \log_{10} b + \log_{10} c = \log_{10}\dfrac{a}{b} + \log_{10} c = \log_{10}\left(\dfrac{a}{b}\cdot c\right) = \log_{10}\dfrac{ac}{b}$

14. $\ln(x+y) + \ln(x-y) - 2\ln z = \ln((x+y)(x-y)) - \ln z^2 = \ln(x^2-y^2) - \ln z^2 = \ln\dfrac{x^2-y^2}{z^2}$

15. $2\ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln\frac{16}{2} = \ln 8$

16. $\ln 3 + \frac{1}{3}\ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln 2 = \ln(3\cdot 2) = \ln 6$

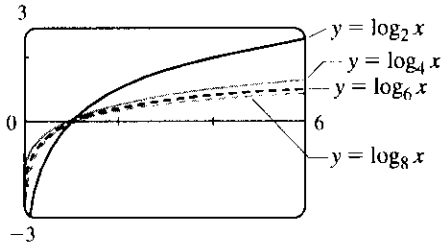
17. $\frac{1}{2}\ln x - 5\ln(x^2+1) = \ln x^{1/2} - \ln(x^2+1)^5 = \ln\dfrac{\sqrt{x}}{(x^2+1)^5}$

18. $\ln x + a\ln y - b\ln z = \ln x + \ln y^a - \ln z^b = \ln(x\cdot y^a) - \ln z^b = \ln\left(xy^a/z^b\right)$

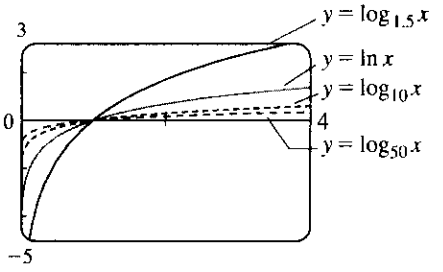
19. (a) $\log_{12} e = \dfrac{\ln e}{\ln 12} = \dfrac{1}{\ln 12} \approx 0.402430$ (b) $\log_6 13.54 = \dfrac{\ln 13.54}{\ln 6} \approx 1.454240$

(c) $\log_2 \pi = \dfrac{\ln \pi}{\ln 2} \approx 1.651496$

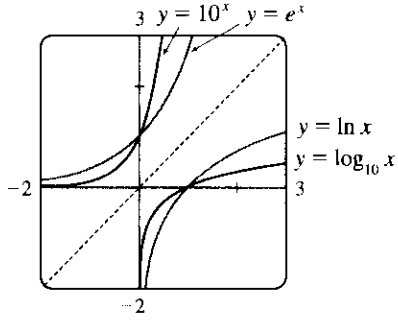
20. To graph the functions, we use $\log_2 x = \dfrac{\ln x}{\ln 2}$, $\log_4 x = \dfrac{\ln x}{\ln 4}$, etc. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y-axis (as $x \to 0^+$).



21. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.



22. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10} x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



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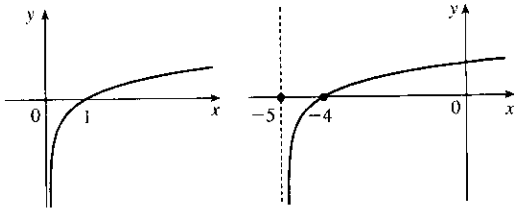
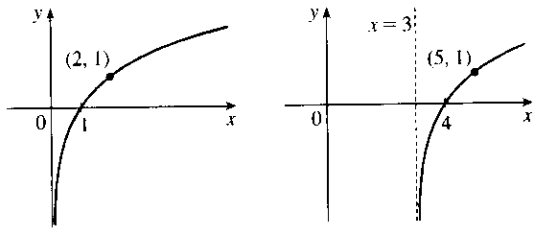
23. Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x + 5)$. Note the vertical asymptote of $x = -5$.

$y = \log_{10} x$ $\qquad$ $y = \log_{10}(x + 5)$



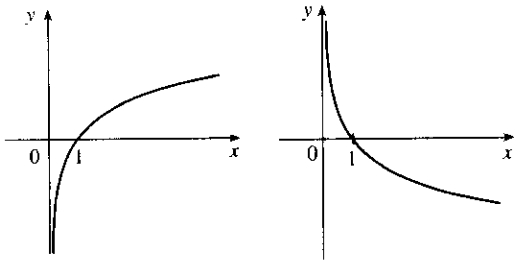
24. $\log_2(x - 3)$: Start with the graph of $y = \log_2 x$ and shift 3 units to the right.

$y = \log_2 x$ $\qquad$ $y = \log_2(x - 3)$



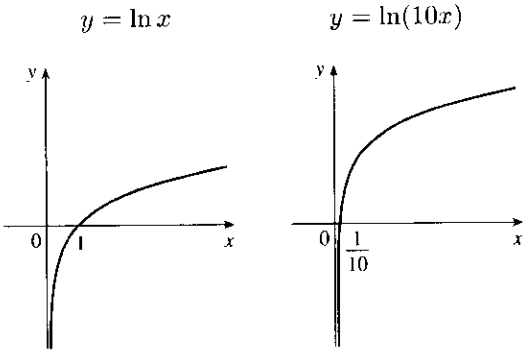
25. Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

$y = \ln x$ $\qquad$ $y = -\ln x$



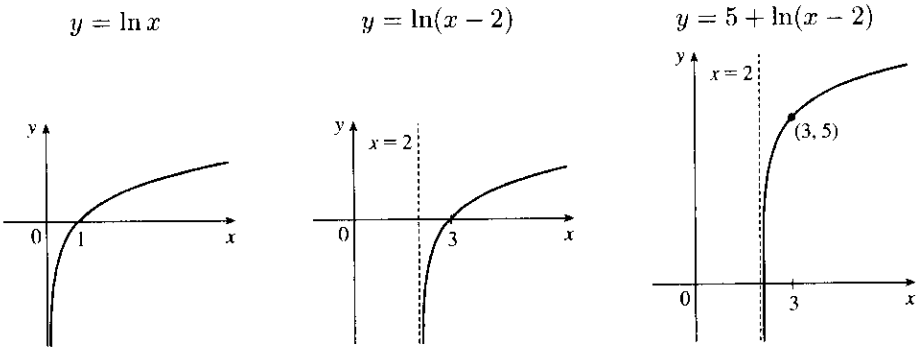
26. $y = \ln(10x)$: Start with the graph of $y = \ln x$ and compress horizontally by a factor of 10. *Or*: $y = \ln(10x) = \ln 10 + \ln x$, so we could start with $y = \ln x$ and shift $\ln 10$ units upward.

$y = \ln x$ $\qquad$ $y = \ln(10x)$



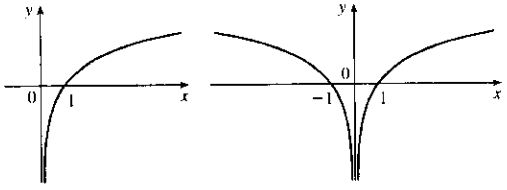
27. $y = 5 + \ln(x - 2)$: Start with the graph of $y = \ln x$, shift 2 units to the right and then shift 5 units upward.

$y = \ln x$ $\qquad$ $y = \ln(x - 2)$ $\qquad$ $y = 5 + \ln(x - 2)$



28. Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln|x|$ is that reflection in addition to the original portion.

$y = \ln x$ $\qquad$ $y = \ln|x|$



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29. (a) $2\ln x = 1 \quad \Rightarrow \quad \ln x = \frac{1}{2} \quad \Rightarrow \quad x = e^{1/2} = \sqrt{e}$

(b) $e^{-x} = 5 \quad \Rightarrow \quad -x = \ln 5 \quad \Rightarrow \quad x = -\ln 5$

30. (a) $e^{2x+3} - 7 = 0 \quad \Rightarrow \quad e^{2x+3} = 7 \quad \Rightarrow \quad 2x + 3 = \ln 7 \quad \Rightarrow \quad 2x = \ln 7 - 3 \quad \Rightarrow \quad x = \frac{1}{2}(\ln 7 - 3)$

(b) $\ln(5 - 2x) = -3 \quad \Rightarrow \quad 5 - 2x = e^{-3} \quad \Rightarrow \quad 2x = 5 - e^{-3} \quad \Rightarrow \quad x = \frac{1}{2}(5 - e^{-3})$

31. (a) $5^{x-3} = 10 \quad \Leftrightarrow \quad \log_{10} 5^{x-3} = \log_{10} 10 \quad \Leftrightarrow \quad (x-3)\log_{10} 5 = 1 \quad \Leftrightarrow \quad x - 3 = 1/\log_{10} 5 \quad \Leftrightarrow$
$x = 3 + 1/\log_{10} 5$

(b) $\log_{10}(x+1) = 4 \quad \Leftrightarrow \quad x + 1 = 10^4 \quad \Leftrightarrow \quad x = 10{,}000 - 1 = 9999$

32. (a) $e^{3x+1} = k \quad \Leftrightarrow \quad 3x + 1 = \ln k \quad \Leftrightarrow \quad x = \frac{1}{3}(\ln k - 1)$

(b) $\log_2(mx) = c \quad \Leftrightarrow \quad mx = 2^c \quad \Leftrightarrow \quad x = 2^c/m$

33. $\ln(\ln x) = 1 \quad \Leftrightarrow \quad e^{\ln(\ln x)} = e^1 \quad \Leftrightarrow \quad \ln x = e^1 = e \quad \Leftrightarrow \quad e^{\ln x} = e^e \quad \Leftrightarrow \quad x = e^e$

34. $e^{e^x} = 10 \quad \Leftrightarrow \quad \ln\left(e^{e^x}\right) = \ln 10 \quad \Leftrightarrow \quad e^x \ln e = e^x = \ln 10 \quad \Leftrightarrow \quad \ln e^x = \ln(\ln 10) \quad \Leftrightarrow \quad x = \ln(\ln 10)$

35. $2\ln x = \ln 2 + \ln(3x - 4) \quad \Rightarrow \quad \ln x^2 = \ln[2(3x - 4)] \quad \Rightarrow \quad \ln x^2 = \ln(6x - 8) \quad \Rightarrow \quad x^2 = 6x - 8 \quad \Rightarrow$
$x^2 - 6x + 8 = 0 \quad \Rightarrow \quad (x-2)(x-4) = 0 \quad \Rightarrow \quad x = 2$ or $x = 4$, both are valid solutions.

36. $\ln(2x+1) = 2 - \ln x \quad \Rightarrow \quad \ln x + \ln(2x+1) = \ln e^2 \quad \Rightarrow \quad \ln[x(2x+1)] = \ln e^2 \quad \Rightarrow \quad 2x^2 + x = e^2 \quad \Rightarrow$
$2x^2 + x - e^2 = 0 \quad \Rightarrow \quad x = \dfrac{-1 + \sqrt{1 + 8e^2}}{4}$ [since $x > 0$].

37. $e^{ax} = Ce^{bx} \quad \Leftrightarrow \quad \ln e^{ax} = \ln[C(e^{bx})] \quad \Leftrightarrow \quad ax = \ln C + bx + \ln e^{bx} \quad \Leftrightarrow \quad ax = \ln C + bx \quad \Leftrightarrow$
$ax - bx = \ln C \quad \Leftrightarrow \quad (a-b)x = \ln C \quad \Leftrightarrow \quad x = \dfrac{\ln C}{a-b}$

38. $7e^x - e^{2x} = 12 \quad \Leftrightarrow \quad (e^x)^2 - 7e^x + 12 = 0 \quad \Leftrightarrow \quad (e^x - 3)(e^x - 4) = 0$, so we have either $e^x = 3 \quad \Leftrightarrow$
$x = \ln 3$, or $e^x = 4 \quad \Leftrightarrow \quad x = \ln 4$.

39. $e^{2+5x} = 100 \quad \Rightarrow \quad \ln\left(e^{2+5x}\right) = \ln 100 \quad \Rightarrow \quad 2 + 5x = \ln 100 \quad \Rightarrow \quad 5x = \ln 100 - 2 \quad \Rightarrow$
$x = \frac{1}{5}(\ln 100 - 2) \approx 0.5210$

40. $\ln(1 + \sqrt{x}) = 2 \quad \Rightarrow \quad 1 + \sqrt{x} = e^2 \quad \Rightarrow \quad \sqrt{x} = e^2 - 1 \quad \Rightarrow \quad x = (e^2 - 1)^2 \approx 40.8200$

41. $\ln(e^x - 2) = 3 \quad \Rightarrow \quad e^x - 2 = e^3 \quad \Rightarrow \quad e^x = e^3 + 2 \quad \Rightarrow \quad x = \ln(e^3 + 2) \approx 3.0949$

42. $3^{1/(x-4)} = 7 \quad \Rightarrow \quad \ln 3^{1/(x-4)} = \ln 7 \quad \Rightarrow \quad \dfrac{1}{x-4}\ln 3 = \ln 7 \quad \Rightarrow \quad x - 4 = \dfrac{\ln 3}{\ln 7} \quad \Rightarrow$
$x = 4 + \dfrac{\ln 3}{\ln 7} \approx 4.5646$

43. (a) $e^x < 10 \quad \Rightarrow \quad \ln e^x < \ln 10 \quad \Rightarrow \quad x < \ln 10 \quad \Rightarrow \quad x \in (-\infty, \ln 10)$

(b) $\ln x > -1 \quad \Rightarrow \quad e^{\ln x} > e^{-1} \quad \Rightarrow \quad x > e^{-1} \quad \Rightarrow \quad x \in (1/e, \infty)$

44. (a) $2 < \ln x < 9 \quad \Rightarrow \quad e^2 < e^{\ln x} < e^9 \quad \Rightarrow \quad e^2 < x < e^9 \quad \Rightarrow \quad x \in \left(e^2, e^9\right)$

(b) $e^{2-3x} > 4 \quad \Rightarrow \quad \ln e^{2-3x} > \ln 4 \quad \Rightarrow \quad 2 - 3x > \ln 4 \quad \Rightarrow \quad -3x > \ln 4 - 2 \quad \Rightarrow$
$x < -\frac{1}{3}(\ln 4 - 2) \quad \Rightarrow \quad x \in \left(-\infty, \frac{1}{3}(2 - \ln 4)\right)$

45. 3 ft = 36 in, so we need x such that $\log_2 x = 36 \quad \Leftrightarrow \quad x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is
$68{,}719{,}476{,}736 \text{ in} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7 \text{ mi}.$

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46. (a) $v(t) = ce^{-kt} \Rightarrow a(t) = v'(t) = -kce^{-kt} = -kv(t)$

(b) $v(0) = ce^0 = c$, so c is the initial velocity.

(c) $v(t) = ce^{-kt} = c/2 \Rightarrow e^{-kt} = \frac{1}{2} \Rightarrow -kt = \ln\frac{1}{2} = -\ln 2 \Rightarrow t = (\ln 2)/k$

47. If I is the intensity of the 1989 San Francisco earthquake, then $\log_{10}(I/S) = 7.1 \Rightarrow$ $\log_{10}(16I/S) = \log_{10} 16 + \log_{10}(I/S) = \log_{10} 16 + 7.1 \approx 8.3$.

48. Let I_1 and I_2 be the intensities of the music and the mower. Then $10\log_{10}\left(\frac{I_1}{I_0}\right) = 120$ and $10\log_{10}\left(\frac{I_2}{I_0}\right) = 106$, so $\log_{10}\left(\frac{I_1}{I_2}\right) = \log_{10}\left(\frac{I_1/I_0}{I_2/I_0}\right) = \log_{10}\left(\frac{I_1}{I_0}\right) - \log_{10}\left(\frac{I_2}{I_0}\right) = 12 - 10.6 = 1.4 \Rightarrow \frac{I_1}{I_2} = 10^{1.4} \approx 25$.

49. (a) $n = 100 \cdot 2^{t/3} \Rightarrow \frac{n}{100} = 2^{t/3} \Rightarrow \log_2\left(\frac{n}{100}\right) = \frac{t}{3} \Rightarrow t = 3\log_2\left(\frac{n}{100}\right)$. Using formula (7), we can write this as $t = 3 \cdot \dfrac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n = 50{,}000 \Rightarrow t = 3\log_2 \dfrac{50{,}000}{100} = 3\log_2 500 = 3\left(\dfrac{\ln 500}{\ln 2}\right) \approx 26.9$ hours

50. (a) $Q = Q_0\left(1 - e^{-t/a}\right) \Rightarrow \frac{Q}{Q_0} = 1 - e^{-t/a} \Rightarrow e^{-t/a} = 1 - \frac{Q}{Q_0} \Rightarrow -\frac{t}{a} = \ln\left(1 - \frac{Q}{Q_0}\right) \Rightarrow$ $t = -a\ln(1 - Q/Q_0)$. This gives us the time t necessary to obtain a given charge Q.

(b) $Q = 0.9Q_0$ and $a = 2 \Rightarrow t = -2\ln\left(1 - 0.9(Q_0/Q_0)\right) = -2\ln 0.1 \approx 4.6$ seconds.

51. Let $t = 2 - x$. As $x \to 2^-$, $t \to 0^+$. $\lim\limits_{x\to 2^-} \ln(2 - x) = \lim\limits_{t\to 0^+} \ln t = -\infty$ by (8).

52. Let $t = x^2 - 5x + 6$. As $x \to 3^+$, $t = (x-2)(x-3) \to 0^+$. $\lim\limits_{x\to 3^+} \log_{10}\left(x^2 - 5x + 6\right) = \lim\limits_{t\to 0^+} \log_{10} t = -\infty$ by (4).

53. $\lim\limits_{x\to 0} \ln(\cos x) = \ln 1 = 0$. [$\ln(\cos x)$ is continuous at $x = 0$ since it is the composite of two continuous functions.]

54. $\lim\limits_{x\to 0^+} \ln(\sin x) = -\infty$ since $\sin x \to 0^+$ as $x \to 0^+$.

55. $\lim\limits_{x\to\infty} [\ln(1 + x^2) - \ln(1 + x)] = \lim\limits_{x\to\infty} \ln \dfrac{1 + x^2}{1 + x} = \ln\left(\lim\limits_{x\to\infty} \dfrac{1 + x^2}{1 + x}\right) = \ln\left(\lim\limits_{x\to\infty} \dfrac{\frac{1}{x} + x}{\frac{1}{x} + 1}\right) = \infty$, since the limit in parentheses is ∞.

56. $\lim\limits_{x\to\infty} [\ln(2 + x) - \ln(1 + x)] = \lim\limits_{x\to\infty} \ln\left(\dfrac{2 + x}{1 + x}\right) = \lim\limits_{x\to\infty} \ln\left(\dfrac{2/x + 1}{1/x + 1}\right) = \ln\dfrac{1}{1} = \ln 1 = 0$

57. The domain of $f(x) = \log_2(5x - 3)$ is $\{x \mid 5x - 3 > 0\} = \left\{x \mid x > \frac{3}{5}\right\} = \left(\frac{3}{5}, \infty\right)$. Since $5x - 3$ takes on all positive values for x in $\left(\frac{3}{5}, \infty\right)$, the range of f is $\mathbb{R}$.

58. The domain of $G(t) = \ln(e^t - 2)$ is $\left\{t \mid e^t - 2 > 0\right\} = \left\{t \mid e^t > 2\right\} = \{t \mid t > \ln 2\} = (\ln 2, \infty)$. Since $e^t - 2$ takes on all positive values for t in $(\ln 2, \infty)$, the range of G is $\mathbb{R}$.

59. (a) For $f(x) = \sqrt{3 - e^{2x}}$, we must have $3 - e^{2x} \geq 0 \Rightarrow e^{2x} \leq 3 \Rightarrow 2x \leq \ln 3 \Rightarrow x \leq \frac{1}{2}\ln 3$. Thus, the domain of f is $(-\infty, \frac{1}{2}\ln 3]$.

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(b) $y = f(x) = \sqrt{3 - e^{2x}}$ [note that $y \geq 0$] $\Rightarrow$ $y^2 = 3 - e^{2x}$ $\Rightarrow$ $e^{2x} = 3 - y^2$ $\Rightarrow$ $2x = \ln(3 - y^2)$ $\Rightarrow$ $x = \frac{1}{2}\ln(3 - y^2)$. Interchange x and y: $y = \frac{1}{2}\ln(3 - x^2)$. So $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$. For the domain of f^{-1}, we must have $3 - x^2 > 0$ $\Rightarrow$ $x^2 < 3$ $\Rightarrow$ $|x| < \sqrt{3}$ $\Rightarrow$ $-\sqrt{3} < x < \sqrt{3}$ $\Rightarrow$ $0 \leq x < \sqrt{3}$ since $x \geq 0$. Note that the domain of f^{-1}, $[0, \sqrt{3}\,)$, equals the range of f.

60. (a) For $f(x) = \ln(2 + \ln x)$, we must have $2 + \ln x > 0$ $\Rightarrow$ $\ln x > -2$ $\Rightarrow$ $x > e^{-2}$. Thus, the domain of f is (e^{-2}, ∞).

(b) $y = f(x) = \ln(2 + \ln x)$ $\Rightarrow$ $e^y = 2 + \ln x$ $\Rightarrow$ $\ln x = e^y - 2$ $\Rightarrow$ $x = e^{e^y - 2}$. Interchange x and y: $y = e^{e^x - 2}$. So $f^{-1}(x) = e^{e^x - 2}$. The domain of f^{-1}, as well as the range of f, is $\mathbb{R}$.

61. $y = \ln(x + 3)$ $\Rightarrow$ $e^y = e^{\ln(x+3)} = x + 3$ $\Rightarrow$ $x = e^y - 3$.
Interchange x and y: the inverse function is $y = e^x - 3$.

62. $y = 2^{10^x}$ $\Rightarrow$ $\log_2 y = 10^x$ $\Rightarrow$ $\log_{10}(\log_2 y) = x$.
Interchange x and y: $y = \log_{10}(\log_2 x)$ is the inverse function.

63. $f(x) = e^{x^3}$ $\Rightarrow$ $y = e^{x^3}$ $\Rightarrow$ $\ln y = x^3$ $\Rightarrow$ $x = \sqrt[3]{\ln y}$. Interchange x and y: $y = \sqrt[3]{\ln x}$.
So $f^{-1}(x) = \sqrt[3]{\ln x}$.

64. $y = (\ln x)^2$, $x \geq 1$, $\ln x = \sqrt{y}$ $\Rightarrow$ $x = e^{\sqrt{y}}$. Interchange x and y: $y = e^{\sqrt{x}}$ is the inverse function.

65. $y = \dfrac{10^x}{10^x + 1}$ $\Rightarrow$ $10^x y + y = 10^x$ $\Rightarrow$ $10^x(1 - y) = y$ $\Rightarrow$ $10^x = \dfrac{y}{1 - y}$ $\Rightarrow$ $x = \log_{10}\left(\dfrac{y}{1 - y}\right)$.

Interchange x and y: $y = \log_{10}\left(\dfrac{x}{1 - x}\right)$ is the inverse function.

66. $y = \dfrac{1 + e^x}{1 - e^x}$ $\Rightarrow$ $y - ye^x = 1 + e^x$ $\Rightarrow$ $e^x(y + 1) = y - 1$ $\Rightarrow$ $e^x = \dfrac{y - 1}{y + 1}$ $\Rightarrow$ $x = \ln\left(\dfrac{y - 1}{y + 1}\right)$.

Interchange x and y: $y = \ln\left(\dfrac{x - 1}{x + 1}\right)$ is the inverse function.

67. $f(x) = e^{3x} - e^x$ $\Rightarrow$ $f'(x) = 3e^{3x} - e^x$. Thus, $f'(x) > 0$ $\Leftrightarrow$ $3e^{3x} > e^x$ $\Leftrightarrow$ $\dfrac{3e^{3x}}{e^x} > \dfrac{e^x}{e^x}$ $\Leftrightarrow$ $3e^{2x} > 1$ $\Leftrightarrow$ $e^{2x} > \frac{1}{3}$ $\Leftrightarrow$ $2x > \ln\left(\frac{1}{3}\right) = -\ln 3$ $\Leftrightarrow$ $x > -\frac{1}{2}\ln 3$, so f is increasing on $\left(-\frac{1}{2}\ln 3, \infty\right)$.

68. $y = 2e^x - e^{-3x}$ $\Rightarrow$ $y' = 2e^x + 3e^{-3x}$ $\Rightarrow$ $y'' = 2e^x - 9e^{-3x}$. Thus, $y'' < 0$ $\Leftrightarrow$ $2e^x < 9e^{-3x}$ $\Leftrightarrow$ $e^{4x} < \frac{9}{2}$ $\Leftrightarrow$ $4x < \ln\frac{9}{2}$ $\Leftrightarrow$ $x < \frac{1}{4}\ln\frac{9}{2}$, so f is concave downward on $\left(-\infty, \frac{1}{4}\ln\frac{9}{2}\right)$.

69. (a) We have to show that $-f(x) = f(-x)$.

$$-f(x) = -\ln\left(x + \sqrt{x^2 + 1}\right) = \ln\left(\left(x + \sqrt{x^2 + 1}\right)^{-1}\right) = \ln\frac{1}{x + \sqrt{x^2 + 1}}$$

$$= \ln\left(\frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{x - \sqrt{x^2 + 1}}{x - \sqrt{x^2 + 1}}\right) = \ln\frac{x - \sqrt{x^2 + 1}}{x^2 - x^2 - 1}$$

$$= \ln\left(\sqrt{x^2 + 1} - x\right) = f(-x)$$

Thus, f is an odd function.

(b) Let $y = \ln\left(x + \sqrt{x^2 + 1}\,\right)$. Then $e^y = x + \sqrt{x^2 + 1}$ $\Leftrightarrow$ $(e^y - x)^2 = x^2 + 1$ $\Leftrightarrow$ $e^{2y} - 2xe^y + x^2 = x^2 + 1$ $\Leftrightarrow$ $2xe^y = e^{2y} - 1$ $\Leftrightarrow$ $x = \dfrac{e^{2y} - 1}{2e^y} = \frac{1}{2}(e^y - e^{-y})$. Thus, the inverse function is $f^{-1}(x) = \frac{1}{2}(e^x - e^{-x})$.

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70. Let (a, e^{-a}) be the point where the tangent meets the curve. The tangent has slope $-e^{-a}$ and is perpendicular to the line $2x - y = 8$, which has slope 2. So $-e^{-a} = -\frac{1}{2} \Rightarrow e^{-a} = \frac{1}{2} \Rightarrow e^{a} = 2 \Rightarrow a = \ln(e^a) = \ln 2$. Thus, the point on the curve is $(\ln 2, \frac{1}{2})$ and the equation of the tangent is $y - \frac{1}{2} = -\frac{1}{2}(x - \ln 2)$ or $x + 2y = 1 + \ln 2$.

71. $x^{1/\ln x} = 2 \Rightarrow \ln(x^{1/\ln x}) = \ln(2) \Rightarrow \dfrac{1}{\ln x} \cdot \ln x = \ln 2 \Rightarrow 1 = \ln 2$, a contradiction, so the given equation has no solution. The function $f(x) = x^{1/\ln x} = (e^{\ln x})^{1/\ln x} = e^1 = e$ for all $x > 0$, so the function $f(x) = x^{1/\ln x}$ is the constant function $f(x) = e$.

72. (a) $\lim\limits_{x\to\infty} x^{\ln x} = \lim\limits_{x\to\infty} \left(e^{\ln x}\right)^{\ln x} = \lim\limits_{x\to\infty} e^{(\ln x)^2} = \infty$ since $(\ln x)^2 \to \infty$ as $x \to \infty$.

(b) $\lim\limits_{x\to 0^+} x^{-\ln x} = \lim\limits_{x\to 0^+} \left(e^{\ln x}\right)^{-\ln x} = \lim\limits_{x\to 0^+} e^{-(\ln x)^2} = 0$ since $-(\ln x)^2 \to -\infty$ as $x \to 0^+$.

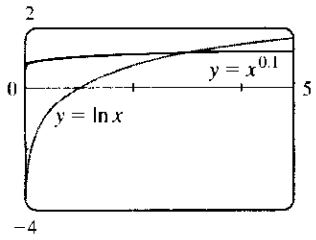
(c) $\lim\limits_{x\to 0^+} x^{1/x} = \lim\limits_{x\to 0^+} \left(e^{\ln x}\right)^{1/x} = \lim\limits_{x\to 0^+} e^{(\ln x)/x} = 0$ since $\dfrac{\ln x}{x} \to -\infty$ as $x \to 0^+$. Note that as $x \to 0^+$, $\ln x$ is a large negative number and x is a small positive number, so $(\ln x)/x \to -\infty$.

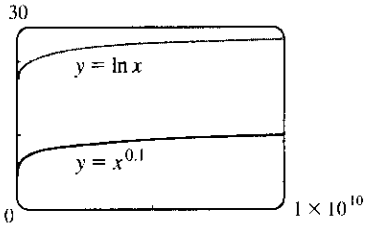
(d) $\lim\limits_{x\to\infty} (\ln 2x)^{-\ln x} = \lim\limits_{x\to\infty} \left[e^{\ln(\ln 2x)}\right]^{-\ln x} = \lim\limits_{x\to\infty} e^{-\ln x \ln(\ln 2x)} = 0$ since $-\ln x \ln(\ln 2x) \to -\infty$ as $x \to \infty$.

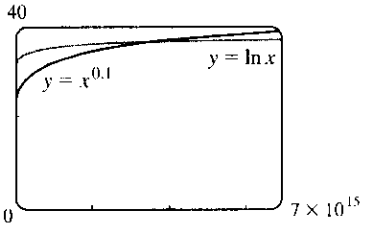
73. (a) Let $\varepsilon > 0$ be given. We need N such that $|a^x - 0| < \varepsilon$ when $x < N$. But $a^x < \varepsilon \Leftrightarrow x < \log_a \varepsilon$. Let $N = \log_a \varepsilon$. Then $x < N \Rightarrow x < \log_a \varepsilon \Rightarrow |a^x - 0| = a^x < \varepsilon$, so $\lim\limits_{x\to-\infty} a^x = 0$.

(b) Let $M > 0$ be given. We need N such that $a^x > M$ when $x > N$. But $a^x > M \Leftrightarrow x > \log_a M$. Let $N = \log_a M$. Then $x > N \Rightarrow x > \log_a M \Rightarrow a^x > M$, so $\lim\limits_{x\to\infty} a^x = \infty$.

74. (a)

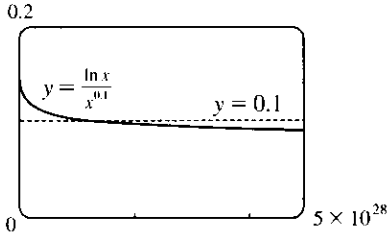






From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

(b)



(c) From the graph at left, it seems that $\dfrac{\ln x}{x^{0.1}} < 0.1$ whenever $x > 1.3 \times 10^{28}$ (approximately). So we can take $N = 1.3 \times 10^{28}$, or any larger number.

75. $\ln(x^2 - 2x - 2) \le 0 \Rightarrow 0 < x^2 - 2x - 2 \le 1$. Now $x^2 - 2x - 2 \le 1$ gives $x^2 - 2x - 3 \le 0$ and hence $(x - 3)(x + 1) \le 0$. So $-1 \le x \le 3$. Now $0 < x^2 - 2x - 2 \Rightarrow x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$. Therefore, $\ln(x^2 - 2x - 2) \le 0 \Leftrightarrow -1 \le x < 1 - \sqrt{3}$ or $1 + \sqrt{3} < x \le 3$.

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76. (a) The primes less than 25 are 2, 3, 5, 7, 11, 13, 17, 19, and 23. There are 9 of them, so $\pi(25) = 9$. We use the sieve of Eratosthenes, and arrive at the figure at right. There are 25 numbers left over, so $\pi(100) = 25$.

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	57	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	~~79~~	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
91	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

(b) Let $f(n) = \dfrac{\pi(n)}{n/\ln n}$. We compute $f(100) = \dfrac{25}{100/\ln 100} \approx 1.15$, $f(1000) \approx 1.16$, $f(10^4) \approx 1.13$, $f(10^5) \approx 1.10$, $f(10^6) \approx 1.08$, and $f(10^7) \approx 1.07$.

(c) By the Prime Number Theorem, the number of primes less than a billion, that is, $\pi(10^9)$, should be close to $10^9/\ln 10^9 \approx 48{,}254{,}942$. In fact, $\pi(10^9) = 50{,}847{,}543$, so our estimate is off by about 5.1%. Do not attempt this calculation at home.

7.4 Derivatives of Logarithmic Functions

1. The differentiation formula for logarithmic functions, $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$, is simplest when $a = e$ because $\ln e = 1$.

2. $f(x) = \ln(x^2 + 10) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2 + 10} \dfrac{d}{dx}(x^2 + 10) = \dfrac{2x}{x^2 + 10}$

3. $f(\theta) = \ln(\cos\theta) \quad \Rightarrow \quad f'(\theta) = \dfrac{1}{\cos\theta} \dfrac{d}{d\theta}(\cos\theta) = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

4. $f(x) = \cos(\ln x) \quad \Rightarrow \quad f'(x) = -\sin(\ln x) \cdot \dfrac{1}{x} = \dfrac{-\sin(\ln x)}{x}$

5. $f(x) = \log_2(1 - 3x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(1 - 3x)\ln 2} \dfrac{d}{dx}(1 - 3x) = \dfrac{-3}{(1 - 3x)\ln 2}$ or $\dfrac{3}{(3x - 1)\ln 2}$

6. $f(x) = \log_{10}\left(\dfrac{x}{x - 1}\right) = \log_{10} x - \log_{10}(x - 1) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x \ln 10} - \dfrac{1}{(x - 1)\ln 10}$ or $-\dfrac{1}{x(x - 1)\ln 10}$

7. $f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \quad \Rightarrow \quad f'(x) = \frac{1}{5}(\ln x)^{-4/5} \dfrac{d}{dx}(\ln x) = \dfrac{1}{5(\ln x)^{4/5}} \cdot \dfrac{1}{x} = \dfrac{1}{5x\sqrt[5]{(\ln x)^4}}$

8. $f(x) = \ln \sqrt[5]{x} = \ln x^{1/5} = \frac{1}{5}\ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{5} \cdot \dfrac{1}{x} = \dfrac{1}{5x}$

9. $f(x) = \sqrt{x}\ln x \quad \Rightarrow \quad f'(x) = \sqrt{x}\left(\dfrac{1}{x}\right) + (\ln x) \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{\sqrt{x}} + \dfrac{\ln x}{2\sqrt{x}} = \dfrac{2 + \ln x}{2\sqrt{x}}$

10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t} \quad \Rightarrow$

$$f'(t) = \frac{(1 - \ln t)(1/t) - (1 + \ln t)(-1/t)}{(1 - \ln t)^2} = \frac{(1/t)[(1 - \ln t) + (1 + \ln t)]}{(1 - \ln t)^2} = \frac{2}{t(1 - \ln t)^2}$$

11. $F(t) = \ln\dfrac{(2t + 1)^3}{(3t - 1)^4} = \ln(2t + 1)^3 - \ln(3t - 1)^4 = 3\ln(2t + 1) - 4\ln(3t - 1) \quad \Rightarrow$

$F'(t) = 3 \cdot \dfrac{1}{2t + 1} \cdot 2 - 4 \cdot \dfrac{1}{3t - 1} \cdot 3 = \dfrac{6}{2t + 1} - \dfrac{12}{3t - 1}$, or combined, $\dfrac{-6(t + 3)}{(2t + 1)(3t - 1)}$.

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12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\,\right) \Rightarrow$

$$h'(x) = \frac{1}{x + \sqrt{x^2 - 1}}\left(1 + \frac{x}{\sqrt{x^2 - 1}}\right) = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

13. $g(x) = \ln\dfrac{a - x}{a + x} = \ln(a - x) - \ln(a + x) \Rightarrow$

$$g'(x) = \frac{1}{a - x}(-1) - \frac{1}{a + x} = \frac{-(a + x) - (a - x)}{(a - x)(a + x)} = \frac{-2a}{a^2 - x^2}$$

14. $F(y) = y\ln(1 + e^y) \Rightarrow F'(y) = y \cdot \dfrac{1}{1 + e^y} \cdot e^y + \ln(1 + e^y) \cdot 1 = \dfrac{ye^y}{1 + e^y} + \ln(1 + e^y)$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)} \Rightarrow$

$$f'(u) = \frac{[1 + \ln(2u)] \cdot \frac{1}{u} - \ln u \cdot \frac{1}{2u} \cdot 2}{[1 + \ln(2u)]^2} = \frac{\frac{1}{u}[1 + \ln(2u) - \ln u]}{[1 + \ln(2u)]^2}$$

$$= \frac{1 + (\ln 2 + \ln u) - \ln u}{u\,[1 + \ln(2u)]^2} = \frac{1 + \ln 2}{u\,[1 + \ln(2u)]^2}$$

16. $y = \ln(x^4 \sin^2 x) = \ln x^4 + \ln(\sin x)^2 = 4\,\ln x + 2\,\ln\sin x \Rightarrow y' = 4 \cdot \dfrac{1}{x} + 2 \cdot \dfrac{1}{\sin x} \cdot \cos x = \dfrac{4}{x} + 2\,\cot x$

17. $h(t) = t^3 - 3^t \Rightarrow h'(t) = 3t^2 - 3^t \ln 3$

18. $y = 10^{\tan\theta} \Rightarrow y' = 10^{\tan\theta}(\ln 10)(\sec^2\theta)$

19. $y = \ln\left|2 - x - 5x^2\right| \Rightarrow y' = \dfrac{1}{2 - x - 5x^2} \cdot (-1 - 10x) = \dfrac{-10x - 1}{2 - x - 5x^2}$ or $\dfrac{10x + 1}{5x^2 + x - 2}$

20. $G(u) = \ln\sqrt{\dfrac{3u + 2}{3u - 2}} = \frac{1}{2}[\ln(3u + 2) - \ln(3u - 2)] \Rightarrow G'(u) = \dfrac{1}{2}\left(\dfrac{3}{3u + 2} - \dfrac{3}{3u - 2}\right) = \dfrac{-6}{9u^2 - 4}$

21. $y = \ln\left(e^{-x} + xe^{-x}\right) = \ln\left(e^{-x}(1 + x)\right) = \ln\left(e^{-x}\right) + \ln(1 + x) = -x + \ln(1 + x) \Rightarrow$

$$y' = -1 + \frac{1}{1 + x} = \frac{-1 - x + 1}{1 + x} = -\frac{x}{1 + x}$$

22. $y = [\ln(1 + e^x)]^2 \Rightarrow y' = 2\,[\ln(1 + e^x)] \cdot \dfrac{1}{1 + e^x} \cdot e^x = \dfrac{2e^x \ln(1 + e^x)}{1 + e^x}$

23. Using Formula 7 and the Chain Rule, $y = 5^{-1/x} \Rightarrow y' = 5^{-1/x}(\ln 5)\left[-1 \cdot \left(-x^{-2}\right)\right] = 5^{-1/x}(\ln 5)/x^2$

24. $y = 2^{3^{x^2}} \Rightarrow y' = 2^{3^{x^2}}(\ln 2)\,\dfrac{d}{dx}\left(3^{x^2}\right) = 2^{3^{x^2}}(\ln 2)3^{x^2}(\ln 3)(2x)$

25. $y = x\ln x \Rightarrow y' = x(1/x) + (\ln x) \cdot 1 = 1 + \ln x \Rightarrow y'' = 1/x$

26. $y = \dfrac{\ln x}{x^2} \Rightarrow y' = \dfrac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{x(1 - 2\ln x)}{x^4} = \dfrac{1 - 2\ln x}{x^3} \Rightarrow$

$$y'' = \frac{x^3(-2/x) - (1 - 2\ln x)(3x^2)}{(x^3)^2} = \frac{x^2(-2 - 3 + 6\ln x)}{x^6} = \frac{6\ln x - 5}{x^4}$$

27. $y = \log_{10} x \Rightarrow y' = \dfrac{1}{x\ln 10} = \dfrac{1}{\ln 10}\left(\dfrac{1}{x}\right) \Rightarrow y'' = \dfrac{1}{\ln 10}\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x^2\ln 10}$

28. $y = \ln(\sec x + \tan x) \Rightarrow y' = \dfrac{\sec x\,\tan x + \sec^2 x}{\sec x + \tan x} = \sec x \Rightarrow y'' = \sec x\,\tan x$

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29. $f(x) = \dfrac{x}{1-\ln(x-1)} \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot\dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$
$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$\text{Dom}(f) = \{x \mid x-1>0 \quad \text{and} \quad 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \quad \text{and} \quad \ln(x-1)\neq 1\}$
$= \{x \mid x>1 \quad \text{and} \quad x-1\neq e^1\} = \{x \mid x>1 \quad \text{and} \quad x\neq 1+e\} = (1,1+e)\cup(1+e,\infty)$

30. $f(x) = \dfrac{1}{1+\ln x} \Rightarrow f'(x) = -\dfrac{1/x}{(1+\ln x)^2}$ [Reciprocal Rule] $= -\dfrac{1}{x(1+\ln x)^2}$.

$\text{Dom}(f) = \{x \mid x>0 \quad \text{and} \quad \ln x\neq -1\} = \{x \mid x>0 \quad \text{and} \quad x\neq 1/e\} = (0,1/e)\cup(1/e,\infty)$.

31. $f(x) = x^2\ln(1-x^2) \Rightarrow f'(x) = 2x\ln(1-x^2) + \dfrac{x^2(-2x)}{1-x^2} = 2x\ln(1-x^2) - \dfrac{2x^3}{1-x^2}$.

$\text{Dom}(f) = \{x \mid 1-x^2>0\} = \{x \mid |x|<1\} = (-1,1)$.

32. $f(x) = \ln\ln\ln x \Rightarrow f'(x) = \dfrac{1}{\ln\ln x}\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x}$.

$\text{Dom}(f) = \{x \mid \ln\ln x>0\} = \{x \mid \ln x>1\} = \{x \mid x>e\} = (e,\infty)$.

33. $f(x) = \dfrac{x}{\ln x} \Rightarrow f'(x) = \dfrac{\ln x - x(1/x)}{(\ln x)^2} = \dfrac{\ln x-1}{(\ln x)^2} \Rightarrow f'(e) = \dfrac{1-1}{1^2} = 0$

34. $f(x) = x^2\ln x \Rightarrow f'(x) = 2x\ln x + x^2\left(\dfrac{1}{x}\right) = 2x\ln x + x \Rightarrow f'(1) = 2\ln 1 + 1 = 1$

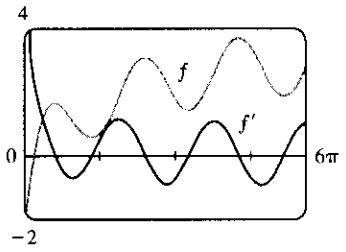
35. $y = f(x) = \ln\ln x \Rightarrow f'(x) = \dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) \Rightarrow f'(e) = \dfrac{1}{e}$, so an equation of the tangent line at $(e,0)$ is

$y-0 = \dfrac{1}{e}(x-e)$, or $y = \dfrac{1}{e}x - 1$, or $x - ey = e$.

36. $y = \ln(x^3-7) \Rightarrow y' = \dfrac{1}{x^3-7}\cdot 3x^2 \Rightarrow y'(2) = \dfrac{12}{8-7} = 12$, so an equation of a tangent line at $(2,0)$ is

$y-0 = 12(x-2)$ or $y = 12x-24$.

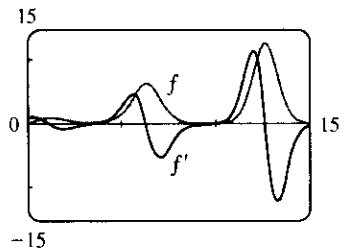
37. $f(x) = \sin x + \ln x \Rightarrow f'(x) = \cos x + 1/x$. This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



38. $f(x) = x^{\cos x} = e^{\ln x\cos x} \Rightarrow$

$$f'(x) = e^{\ln x\cos x}\left[\ln x(-\sin x) + \cos x\left(\frac{1}{x}\right)\right]$$
$$= x^{\cos x}\left[\frac{\cos x}{x} - \sin x\ln x\right]$$

This is reasonable, because the graph shows that f increases when $f'(x)$ is positive.



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39. $y = (2x+1)^5(x^4-3)^6 \;\Rightarrow\; \ln y = \ln\left((2x+1)^5(x^4-3)^6\right) \;\Rightarrow$

$\ln y = 5\ln(2x+1) + 6\ln(x^4-3) \;\Rightarrow\; \dfrac{1}{y}\,y' = 5\cdot\dfrac{1}{2x+1}\cdot 2 + 6\cdot\dfrac{1}{x^4-3}\cdot 4x^3 \;\Rightarrow$

$y' = y\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4-3}\right) = (2x+1)^5(x^4-3)^6\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4-3}\right).$

[The answer could be simplified to $y' = 2(2x+1)^4(x^4-3)^5(29x^4+12x^3-15)$, but this is unnecessary.]

40. $y = \sqrt{x}\,e^{x^2}(x^2+1)^{10} \;\Rightarrow\; \ln y = \ln\sqrt{x} + \ln e^{x^2} + \ln(x^2+1)^{10} \;\Rightarrow\; \ln y = \frac{1}{2}\ln x + x^2 + 10\ln(x^2+1)$

$\Rightarrow\; \dfrac{1}{y}\,y' = \dfrac{1}{2}\cdot\dfrac{1}{x} + 2x + 10\cdot\dfrac{1}{x^2+1}\cdot 2x \;\Rightarrow\; y' = \sqrt{x}\,e^{x^2}(x^2+1)^{10}\left(\dfrac{1}{2x} + 2x + \dfrac{20x}{x^2+1}\right)$

41. $y = \dfrac{\sin^2 x\,\tan^4 x}{(x^2+1)^2} \;\Rightarrow\; \ln y = \ln(\sin^2 x\,\tan^4 x) - \ln(x^2+1)^2 \;\Rightarrow$

$\ln y = \ln(\sin x)^2 + \ln(\tan x)^4 - \ln(x^2+1)^2 \;\Rightarrow\; \ln y = 2\,\ln|\sin x| + 4\,\ln|\tan x| - 2\,\ln(x^2+1) \;\Rightarrow$

$\dfrac{1}{y}\,y' = 2\cdot\dfrac{1}{\sin x}\cdot\cos x + 4\cdot\dfrac{1}{\tan x}\cdot\sec^2 x - 2\cdot\dfrac{1}{x^2+1}\cdot 2x \;\Rightarrow$

$y' = \dfrac{\sin^2 x\,\tan^4 x}{(x^2+1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2+1}\right)$

42. $y = \sqrt[4]{\dfrac{x^2+1}{x^2-1}} \;\Rightarrow\; \ln y = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1) \;\Rightarrow\; \dfrac{1}{y}y' = \dfrac{1}{4}\cdot\dfrac{1}{x^2+1}\cdot 2x - \dfrac{1}{4}\cdot\dfrac{1}{x^2-1}\cdot 2x \;\Rightarrow$

$y' = \sqrt[4]{\dfrac{x^2+1}{x^2-1}}\cdot\dfrac{1}{2}\left(\dfrac{x}{x^2+1} - \dfrac{x}{x^2-1}\right) = \dfrac{1}{2}\sqrt[4]{\dfrac{x^2+1}{x^2-1}}\left(\dfrac{-2x}{x^4-1}\right) = \dfrac{x}{1-x^4}\sqrt[4]{\dfrac{x^2+1}{x^2-1}}$

43. $y = x^x \;\Rightarrow\; \ln y = \ln x^x \;\Rightarrow\; \ln y = x\ln x \;\Rightarrow\; y'/y = x(1/x) + (\ln x)\cdot 1 \;\Rightarrow$

$y' = y(1+\ln x) \;\Rightarrow\; y' = x^x(1+\ln x)$

44. $y = x^{1/x} \;\Rightarrow\; \ln y = \dfrac{1}{x}\ln x \;\Rightarrow\; \dfrac{y'}{y} = \dfrac{1}{x}\left(\dfrac{1}{x}\right) + (\ln x)\left(-\dfrac{1}{x^2}\right) \;\Rightarrow\; y' = x^{1/x}\dfrac{1-\ln x}{x^2}$

45. $y = x^{\sin x} \;\Rightarrow\; \ln y = \ln x^{\sin x} \;\Rightarrow\; \ln y = \sin x\ln x \;\Rightarrow\; \dfrac{y'}{y} = (\sin x)\cdot\dfrac{1}{x} + (\ln x)(\cos x) \;\Rightarrow$

$y' = y\left(\dfrac{\sin x}{x} + \ln x\cos x\right) \;\Rightarrow\; y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x\,\cos x\right)$

46. $y = (\sin x)^x \;\Rightarrow\; \ln y = x\ln(\sin x) \;\Rightarrow\; \dfrac{y'}{y} = x\cdot\dfrac{1}{\sin x}\cdot\cos x + [\ln(\sin x)]\cdot 1 \;\Rightarrow$

$y' = (\sin x)^x\,[x\cot x + \ln(\sin x)]$

47. $y = (\ln x)^x \;\Rightarrow\; \ln y = \ln(\ln x)^x \;\Rightarrow\; \ln y = x\ln\ln x \;\Rightarrow\; \dfrac{y'}{y} = x\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x} + (\ln\ln x)\cdot 1 \;\Rightarrow$

$y' = y\left(\dfrac{x}{x\ln x} + \ln\ln x\right) \;\Rightarrow\; y' = (\ln x)^x\left(\dfrac{1}{\ln x} + \ln\ln x\right)$

48. $y = x^{\ln x} \;\Rightarrow\; \ln y = \ln x\ln x = (\ln x)^2 \;\Rightarrow\; \dfrac{y'}{y} = 2\ln x\left(\dfrac{1}{x}\right) \;\Rightarrow\; y' = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$

49. $y = x^{e^x} \;\Rightarrow\; \ln y = e^x\ln x \;\Rightarrow\; \dfrac{y'}{y} = e^x\cdot\dfrac{1}{x} + (\ln x)\cdot e^x \;\Rightarrow\; y' = x^{e^x}e^x\left(\ln x + \dfrac{1}{x}\right)$

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50. $y = (\ln x)^{\cos x} \Rightarrow \ln y = \cos x \ln(\ln x) \Rightarrow \dfrac{y'}{y} = \cos x \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x} + (\ln \ln x)(-\sin x) \Rightarrow$

$y' = (\ln x)^{\cos x}\left(\dfrac{\cos x}{x \ln x} - \sin x \ln \ln x\right)$

51. $y = \ln(x^2 + y^2) \Rightarrow y' = \dfrac{1}{x^2+y^2}\dfrac{d}{dx}(x^2+y^2) \Rightarrow y' = \dfrac{2x+2yy'}{x^2+y^2} \Rightarrow x^2y' + y^2y' = 2x + 2yy'$

$\Rightarrow x^2y' + y^2y' - 2yy' = 2x \Rightarrow (x^2+y^2-2y)y' = 2x \Rightarrow y' = \dfrac{2x}{x^2+y^2-2y}$

52. $x^y = y^x \Rightarrow y \ln x = x \ln y \Rightarrow y \cdot \dfrac{1}{x} + (\ln x) \cdot y' = x \cdot \dfrac{1}{y} \cdot y' + \ln y \Rightarrow y' \ln x - \dfrac{x}{y}y' = \ln y - \dfrac{y}{x} \Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

53. $f(x) = \ln(x-1) \Rightarrow f'(x) = 1/(x-1) = (x-1)^{-1} \Rightarrow f''(x) = -(x-1)^{-2} \Rightarrow$

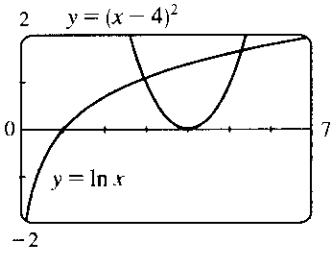
$f'''(x) = 2(x-1)^{-3} \Rightarrow f^{(4)}(x) = -2 \cdot 3(x-1)^{-4} \Rightarrow \cdots \Rightarrow$

$f^{(n)}(x) = (-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1)(x-1)^{-n} = (-1)^{n-1}\dfrac{(n-1)!}{(x-1)^n}$

54. $y = x^8 \ln x$, so $D^9y = D^8y' = D^8\left(8x^7 \ln x + x^7\right)$. But the eighth derivative of x^7 is 0, so we now have

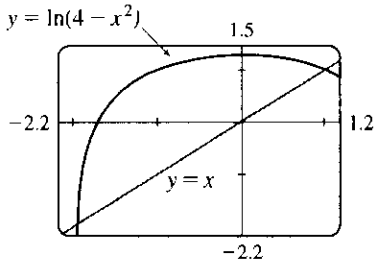
$$D^8\left(8x^7 \ln x\right) = D^7\left(8 \cdot 7x^6 \ln x + 8x^6\right) = D^7\left(8 \cdot 7x^6 \ln x\right)$$
$$= D^6\left(8 \cdot 7 \cdot 6x^5 \ln x\right) = \cdots = D\left(8!\, x^0 \ln x\right) = 8!/x.$$

55.



From the graph, it appears that the curves $y = (x-4)^2$ and $y = \ln x$ intersect just to the left of $x = 3$ and to the right of $x = 5$, at about $x = 5.3$. Let $f(x) = \ln x - (x-4)^2$. Then $f'(x) = 1/x - 2(x-4)$, so Newton's Method says that $x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - \dfrac{\ln x_n - (x_n-4)^2}{1/x_n - 2(x_n-4)}$. Taking $x_0 = 3$, we get $x_1 \approx 2.957738$, $x_2 \approx 2.958516 \approx x_3$, so the first root is 2.958516, to six decimal places. Taking $x_0 = 5$, we get $x_1 \approx 5.290755$, $x_2 \approx 5.290718 \approx x_3$, so the second (and final) root is 5.290718, to six decimal places.

56.



We use Newton's Method with $f(x) = \ln(4-x^2) - x$ and $f'(x) = \dfrac{1}{4-x^2}(-2x) - 1 = -1 - \dfrac{2x}{4-x^2}$. The formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. From the graphs it seems that the roots occur at approximately $x = -1.9$ and $x = 1.1$. However, if we use $x_1 = -1.9$ as an initial approximation to the first root, we get $x_2 \approx -2.009611$, and $f(x) = \ln(x-2)^2 - x$ is undefined at this point, making it impossible to calculate x_3. We must use a more accurate first estimate, such as $x_1 = -1.95$. With this approximation, we get $x_1 = -1.95$, $x_2 \approx -1.1967495$, $x_3 \approx -1.964760$, $x_4 \approx x_5 \approx -1.964636$. Calculating the second root gives $x_1 = 1.1$, $x_2 \approx 1.058649$, $x_3 \approx 1.058007$, $x_4 \approx x_5 \approx 1.058006$. So, correct to six decimal places, the two roots of the equation $\ln(4-x^2) = x$ are $x = -1.964636$ and $x = 1.058006$.

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57. $f(x) = \dfrac{\ln x}{\sqrt{x}} \Rightarrow f'(x) = \dfrac{\sqrt{x}\,(1/x) - (\ln x)[1/(2\sqrt{x})]}{x} = \dfrac{2 - \ln x}{2x^{3/2}} \Rightarrow$

$f''(x) = \dfrac{2x^{3/2}(-1/x) - (2 - \ln x)(3x^{1/2})}{4x^3} = \dfrac{3\ln x - 8}{4x^{5/2}} > 0 \Leftrightarrow \ln x > \frac{8}{3} \Leftrightarrow x > e^{8/3}$, so f is CU on $\left(e^{8/3}, \infty\right)$ and CD on $\left(0, e^{8/3}\right)$. The inflection point is $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$.

58. $f(x) = x \ln x$, $f'(x) = \ln x + 1 = 0$ when $\ln x = -1 \Leftrightarrow x = e^{-1}$. $f'(x) > 0 \Leftrightarrow \ln x + 1 > 0 \Leftrightarrow \ln x > -1 \Leftrightarrow x > 1/e$. $f'(x) < 0 \Leftrightarrow \ln x + 1 < 0 \Leftrightarrow x < 1/e$. Therefore, there is an absolute minimum value of $f(1/e) = (1/e)\ln(1/e) = -1/e$.

59. $y = f(x) = \ln(\sin x)$

A. $D = \{x \text{ in } \mathbb{R} \mid \sin x > 0\} = \bigcup_{n=-\infty}^{\infty} (2n\pi, (2n+1)\pi)$

$= \cdots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$

B. No y-intercept; x-intercepts: $f(x) = 0 \Leftrightarrow \ln(\sin x) = 0 \Leftrightarrow \sin x = e^0 = 1 \Leftrightarrow x = 2n\pi + \frac{\pi}{2}$ for each integer n. **C.** f is periodic with period 2π. **D.** $\lim_{x \to (2n\pi)^+} f(x) = -\infty$ and $\lim_{x \to [(2n+1)\pi]^-} f(x) = -\infty$, so the lines $x = n\pi$ are VAs for all integers n. **E.** $f'(x) = \frac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n, and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$ for each integer n. **F.** Local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 0$, no local minimum. **G.** $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n. No IP

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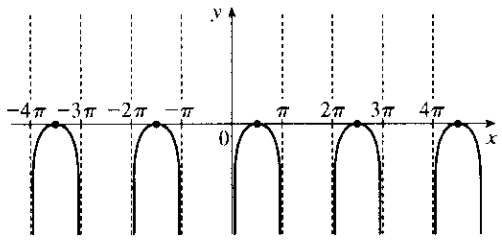

60. $y = \ln(\tan^2 x)$ **A.** $D = \{x \mid x \neq n\pi/2\}$ **B.** x-intercepts $n\pi + \frac{\pi}{4}$, no y-intercept. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. Also $f(x + \pi) = f(x)$, so f is periodic with period π, and we consider parts D–G only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **D.** $\lim_{x \to 0} \ln(\tan^2 x) = -\infty$ and $\lim_{x \to (\pi/2)^-} \ln(\tan^2 x) = \infty$,

$\lim_{x \to -(-\pi/2)^+} \ln(\tan^2 x) = \infty$, so $x = 0$, $x = \pm\frac{\pi}{2}$ are VA. **E.** $f'(x) = \dfrac{2\tan x \sec^2 x}{\tan^2 x} = 2\dfrac{\sec^2 x}{\tan x} > 0 \Leftrightarrow$

$\tan x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$, so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(-\frac{\pi}{2}, 0\right)$. **F.** No maximum or minimum

G. $f'(x) = \dfrac{2}{\sin x \cos x} = \dfrac{4}{\sin 2x} \Rightarrow f''(x) = \dfrac{-8\cos 2x}{\sin^2 2x} < 0$

$\Leftrightarrow \cos 2x > 0 \Leftrightarrow -\frac{\pi}{4} < x < \frac{\pi}{4}$, so f is CD on $\left(-\frac{\pi}{4}, 0\right)$ and $\left(0, \frac{\pi}{4}\right)$ and CU on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. IP are $\left(\pm\frac{\pi}{4}, 0\right)$.

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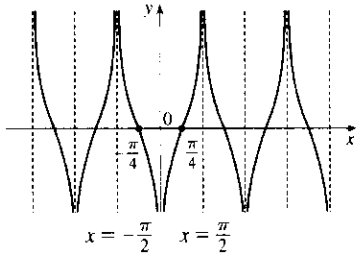

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61. $y = f(x) = \ln(1 + x^2)$ **A.** $D = \mathbb{R}$ **B.** Both intercepts are 0. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim_{x \to \pm\infty} \ln(1 + x^2) = \infty$, no asymptotes. **E.** $f'(x) = \dfrac{2x}{1 + x^2} > 0 \quad \Leftrightarrow$ $x > 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

F. $f(0) = 0$ is a local and absolute minimum.

G. $f''(x) = \dfrac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \dfrac{2(1 - x^2)}{(1 + x^2)^2} > 0 \quad \Leftrightarrow$ $|x| < 1$, so f is CU on $(-1, 1)$, CD on $(-\infty, -1)$ and $(1, \infty)$. IP $(1, \ln 2)$ and $(-1, \ln 2)$.

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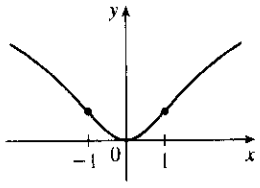

62. $y = f(x) = \ln(x^2 - 3x + 2) = \ln[(x - 1)(x - 2)]$

A. $D = \{x \text{ in } \mathbb{R}: x^2 - 3x + 2 > 0\} = (-\infty, 1) \cup (2, \infty)$.

B. y-intercept: $f(0) = \ln 2$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x^2 - 3x + 2 = e^0 \quad \Leftrightarrow \quad x^2 - 3x + 1 = 0 \quad \Leftrightarrow$ $x = \dfrac{3 \pm \sqrt{5}}{2} \quad \Rightarrow \quad x \approx 0.38, 2.62$ **C.** No symmetry **D.** $\lim_{x \to 1^-} f(x) = \lim_{x \to 2^+} f(x) = -\infty$, so $x = 1$ and $x = 2$ are VAs. No HA. **E.** $f'(x) = \dfrac{2x - 3}{x^2 - 3x + 2} = \dfrac{2(x - 3/2)}{(x - 1)(x - 2)}$, so $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 2$. Thus, f is decreasing on $(-\infty, 1)$ and increasing on $(2, \infty)$. **F.** No extreme values

G.
$$f''(x) = \frac{(x^2 - 3x + 2) \cdot 2 - (2x - 3)^2}{(x^2 - 3x + 2)^2} = \frac{2x^2 - 6x + 4 - 4x^2 + 12x - 9}{(x^2 - 3x + 2)^2} = \frac{-2x^2 + 6x - 5}{(x^2 - 3x + 2)^2}$$

The numerator is negative for all x and the denominator is positive, so $f''(x) < 0$ for all x in the domain of f. Thus, f is CD on $(-\infty, 1)$ and $(2, \infty)$. No IP

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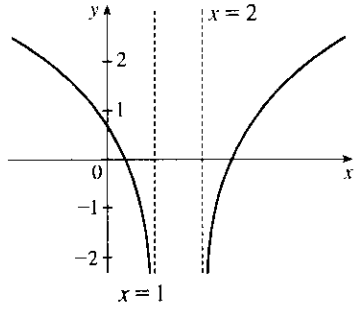
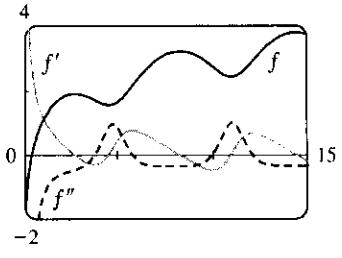


63. We use the CAS to calculate $f'(x) = \dfrac{2 + \sin x + x \cos x}{2x + x \sin x}$ and $f''(x) = \dfrac{2x^2 \sin x + 4 \sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4 \sin x - 5)}$. From the graphs, it seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8$, 5.7, 10.0 and 12.0.



Looking back at the graph of $f(x) = \ln(2x + x \sin x)$, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.

64. We see that if $c \leq 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \quad \Rightarrow \quad |x| > \sqrt{-c}$, and $\lim_{x \to \sqrt{-c}^+} f(x) = \lim_{x \to -\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \to -\infty$ as $y \to 0$. Thus, for $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$,

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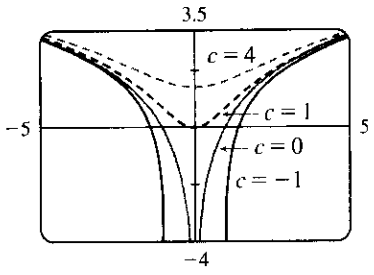
$\lim_{x\to 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there is no asymptote. To find the maxima, minima, and inflection points, we differentiate: $f(x) = \ln(x^2 + c) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2 + c}\,(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get $f''(x) = \dfrac{1}{x^2 + c}\,(2) + 2x\left[-\left(x^2 + c\right)^{-2}(2x)\right] = \dfrac{2(c - x^2)}{(x^2 + c)^2}$. Now if $c \le 0$, this is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2 \;\Leftrightarrow\; x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $\pm\sqrt{c}$, and as c increases, the inflection points get further apart.



65. $\displaystyle\int_2^4 \frac{3}{x}\,dx = 3\int_2^4 \frac{1}{x}\,dx = 3\Big[\ln|x|\Big]_2^4 = 3(\ln 4 - \ln 2) = 3\ln\frac{4}{2} = 3\ln 2$

66. $\displaystyle\int_1^2 \frac{4 + u^2}{u^3}\,du = \int_1^2 \left(4u^{-3} + u^{-1}\right)du = \left[\frac{4}{-2}u^{-2} + \ln|u|\right]_1^2 = \left[\frac{-2}{u^2} + \ln u\right]_1^2$
$= \left(-\frac{1}{2} + \ln 2\right) - (-2 + \ln 1) = \frac{3}{2} + \ln 2$

67. $\displaystyle\int_1^2 \frac{dt}{8 - 3t} = \left[-\frac{1}{3}\ln|8 - 3t|\right]_1^2 = -\frac{1}{3}\ln 2 - \left(-\frac{1}{3}\ln 5\right) = \frac{1}{3}(\ln 5 - \ln 2) = \frac{1}{3}\ln\frac{5}{2}$

Or: Let $u = 8 - 3t$. Then $du = -3\,dt$, so

$$\int_1^2 \frac{dt}{8 - 3t} = \int_5^2 \frac{-\frac{1}{3}\,du}{u} = \left[-\frac{1}{3}\ln|u|\right]_5^2 = -\frac{1}{3}\ln 2 - \left(-\frac{1}{3}\ln 5\right) = \frac{1}{3}(\ln 5 - \ln 2) = \frac{1}{3}\ln\frac{5}{2}.$$

68. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right)dx = \left[\tfrac{1}{2}x^2 + 2x + \ln x\right]_4^9 = \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4)$
$= \frac{85}{2} + \ln\frac{9}{4}$

69. $\displaystyle\int_1^e \frac{x^2 + x + 1}{x}\,dx = \int_1^e \left(x + 1 + \frac{1}{x}\right)dx = \left[\tfrac{1}{2}x^2 + x + \ln x\right]_1^e = \left(\tfrac{1}{2}e^2 + e + 1\right) - \left(\tfrac{1}{2} + 1 + 0\right)$
$= \frac{1}{2}e^2 + e - \frac{1}{2}$

70. Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$, so $\displaystyle\int_e^6 \frac{dx}{x\ln x} = \int_1^{\ln 6} \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^{\ln 6} = \ln\ln 6 - \ln 1 = \ln\ln 6$

71. Let $u = 6x - x^3$. Then $du = (6 - 3x^2)\,dx = 3(2 - x^2)\,dx$, so

$$\int \frac{2 - x^2}{6x - x^3}\,dx = \int \frac{\frac{1}{3}\,du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\left|6x - x^3\right| + C.$$

72. Let $u = 2 + \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x}{2 + \sin x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|2 + \sin x| + C = \ln(2 + \sin x) + C \quad [\text{since } 2 + \sin x > 0].$$

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73. Let $u = \ln x$. Then $du = \dfrac{dx}{x} \quad \Rightarrow \quad \displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C.$

74. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x+1) + C.$

75. $\displaystyle\int_1^2 10^t\,dt = \left[\frac{10^t}{\ln 10}\right]_1^2 = \frac{10^2}{\ln 10} - \frac{10^1}{\ln 10} = \frac{100-10}{\ln 10} = \frac{90}{\ln 10}$

76. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int x2^{x^2}\,dx = \tfrac{1}{2}\int 2^u\,du = \frac{1}{2}\frac{2^u}{\ln 2} + C = \frac{1}{2\ln 2}2^{x^2} + C.$

77. (a) $\dfrac{d}{dx}\left(\ln|\sin x| + C\right) = \dfrac{1}{\sin x}\cos x = \cot x$

(b) Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$

78. Let $u = x - 2$. Then the area is

$$A = -\int_{-4}^{-1} \frac{2}{x-2}\,dx = -2\int_{-6}^{-3} \frac{du}{u} = \Big[-2\ln|u|\Big]_{-6}^{-3} = -2\ln 3 + 2\ln 6 = 2\ln 2 \approx 1.386.$$

79. The cross-sectional area is $\pi\left(1/\sqrt{x+1}\right)^2 = \pi/(x+1)$. Therefore, the volume is

$$\int_0^1 \frac{\pi}{x+1}\,dx = \pi[\ln(x+1)]_0^1 = \pi(\ln 2 - \ln 1) = \pi\ln 2.$$

80. Using cylindrical shells, we get $V = \displaystyle\int_0^3 \frac{2\pi x}{x^2+1}\,dx = \pi\left[\ln(1+x^2)\right]_0^3 = \pi\ln 10.$

81. $W = \int_{V_1}^{V_2} P\,dV = \displaystyle\int_{600}^{1000} \frac{C}{V}\,dV = C\int_{600}^{1000} \frac{1}{V}\,dV = C\left[\ln|V|\right]_{600}^{1000}$

$= C(\ln 1000 - \ln 600) = C\ln\tfrac{1000}{600} = C\ln\tfrac{5}{3}$

Initially, $PV = C$, where $P = 150$ kPa and $V = 600$ cm^3, so $C = (150)(600) = 90{,}000$. Thus,

$W = 90{,}000\ln\tfrac{5}{3} \approx 45{,}974$ kPa $\cdot$ cm^3 $= 45{,}974(10^3$ Pa$)(10^{-6}$ m$^3) = 45{,}974$ Pa$\cdot$m$^3 = 45{,}974$ N$\cdot$m

[Pa $=$ N/m^2] $= 45{,}974$ J

82. $f''(x) = x^{-2}, x > 0 \quad \Rightarrow \quad f'(x) = -1/x + C \quad \Rightarrow \quad f(x) = -\ln x + Cx + D.$ $0 = f(1) = C + D$ and

$0 = f(2) = -\ln 2 + 2C + D = -\ln 2 + 2C - C = -\ln 2 + C \quad \Rightarrow \quad C = \ln 2$ and $D = -\ln 2$. So

$f(x) = -\ln x + (\ln 2)x - \ln 2.$

83. $f(x) = 2x + \ln x \quad \Rightarrow \quad f'(x) = 2 + 1/x.$ If $g = f^{-1}$, then $f(1) = 2 \quad \Rightarrow \quad g(2) = 1$, so

$g'(2) = 1/f'(g(2)) = 1/f'(1) = \tfrac{1}{3}.$

84. $f(x) = e^x + \ln x \quad \Rightarrow \quad f'(x) = e^x + 1/x.$ $h = f^{-1}$ and $f(1) = e \quad \Rightarrow \quad h(e) = 1$, so

$h'(e) = 1/f'(1) = 1/(e+1).$

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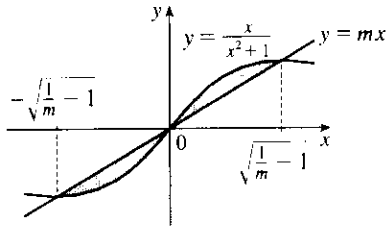
85. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2+1) = mx \Rightarrow x = 0$ or $mx^2 + m - 1 = 0 \Rightarrow x = 0$ or $x = \dfrac{\pm\sqrt{-4(m)(m-1)}}{2m} = \pm\sqrt{\dfrac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2+1)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2+1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is



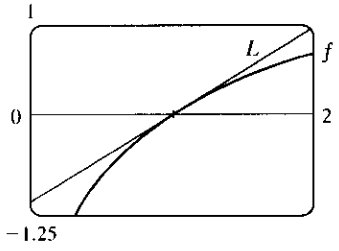
$$2\int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx\right] dx = 2\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}}$$

$$= \left[\ln\left(\frac{1}{m} - 1 + 1\right) - m\left(\frac{1}{m} - 1\right)\right] - (\ln 1 - 0)$$

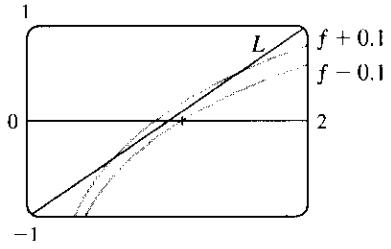
$$= \ln\left(\frac{1}{m}\right) + m - 1 = m - \ln m - 1$$

86. (a) Let $f(x) = \ln x \Rightarrow f'(x) = 1/x \Rightarrow f''(x) = -1/x^2$. The linear approximation to $\ln x$ near 1 is $\ln x \approx f(1) + f'(1)(x-1) = \ln 1 + \frac{1}{1}(x-1) = x - 1$.

(b)



(c)



From the graph, it appears that the linear approximation is accurate to within 0.1 for x between about 0.62 and 1.51.

87. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x\to 0} \dfrac{f(x)}{x} = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = f'(0) = 1$.

88. Let $m = n/x$. Then $n = xm$, and as $n \to \infty$, $m \to \infty$.

Therefore, $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = \lim\limits_{m\to\infty} \left(1 + \dfrac{1}{m}\right)^{mx} = \left[\lim\limits_{m\to\infty} \left(1 + \dfrac{1}{m}\right)^m\right]^x = e^x$ by Equation 9.

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7.2* The Natural Logarithmic Function

1. $\ln \dfrac{x^3 y}{z^2} = \ln x^3 y - \ln z^2 = \ln x^3 + \ln y - \ln z^2 = 3 \ln x + \ln y - 2 \ln z$

2. $\ln \sqrt{a(b^2 + c^2)} = \ln(a(b^2 + c^2))^{1/2} = \frac{1}{2} \ln(a(b^2 + c^2)) = \frac{1}{2}\left[\ln a + \ln(b^2 + c^2)\right]$
$= \frac{1}{2} \ln a + \frac{1}{2} \ln(b^2 + c^2)$

3. $\ln(uv)^{10} = 10 \ln(uv) = 10(\ln u + \ln v) = 10 \ln u + 10 \ln v$

4. $\ln \dfrac{3x^2}{(x+1)^5} = \ln 3x^2 - \ln(x+1)^5 = \ln 3 + \ln x^2 - 5 \ln(x+1) = \ln 3 + 2 \ln x - 5 \ln(x+1)$

5. $2 \ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln \frac{16}{2} = \ln 8$

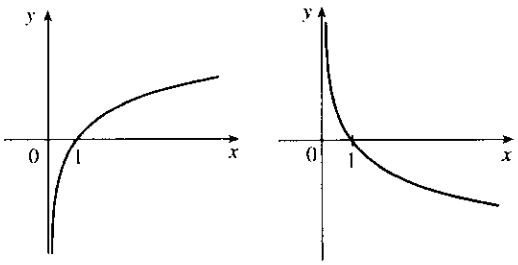
6. $\ln 3 + \frac{1}{3} \ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln 2 = \ln(3 \cdot 2) = \ln 6$

7. $\frac{1}{2} \ln x - 5 \ln(x^2 + 1) = \ln x^{1/2} - \ln(x^2 + 1)^5 = \ln \dfrac{\sqrt{x}}{(x^2 + 1)^5}$

8. $\ln x + a \ln y - b \ln z = \ln x + \ln y^a - \ln z^b = \ln(x \cdot y^a) - \ln z^b = \ln(xy^a / z^b)$

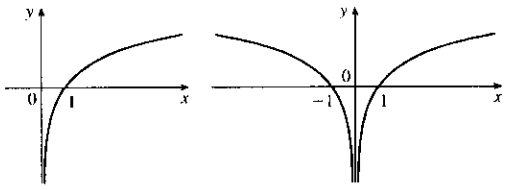
9. Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

$y = \ln x$ $\qquad$ $y = -\ln x$

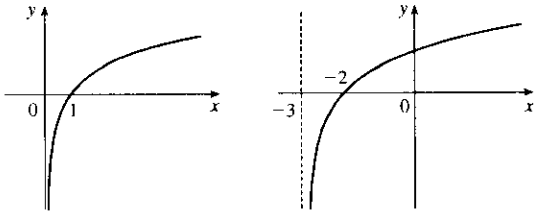


10. Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln |x|$ is that reflection in addition to the original portion.

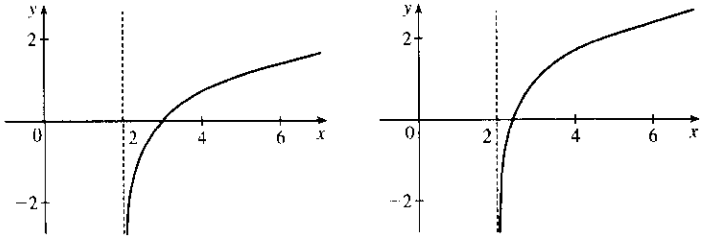
$y = \ln x$ $\qquad$ $y = \ln |x|$



11. $y = \ln x$ $\qquad$ $y = \ln(x + 3)$



12. $y = \ln(x - 2)$ $\qquad$ $y = 1 + \ln(x - 2)$



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13. $f(x) = \sqrt{x}\ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{2\sqrt{x}}\ln x + \sqrt{x}\left(\dfrac{1}{x}\right) = \dfrac{\ln x + 2}{2\sqrt{x}}$

14. $f(x) = \ln(x^2 + 10) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2+10}\dfrac{d}{dx}\left(x^2+10\right) = \dfrac{2x}{x^2+10}$

15. $f(\theta) = \ln(\cos\theta) \quad \Rightarrow \quad f'(\theta) = \dfrac{1}{\cos\theta}\dfrac{d}{d\theta}(\cos\theta) = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

16. $f(x) = \cos(\ln x) \quad \Rightarrow \quad f'(x) = -\sin(\ln x)\cdot\dfrac{1}{x} = \dfrac{-\sin(\ln x)}{x}$

17. $f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \quad \Rightarrow \quad f'(x) = \frac{1}{5}(\ln x)^{-4/5}\dfrac{d}{dx}(\ln x) = \dfrac{1}{5(\ln x)^{4/5}}\cdot\dfrac{1}{x} = \dfrac{1}{5x\sqrt[5]{(\ln x)^4}}$

18. $f(x) = \ln\sqrt[5]{x} = \ln x^{1/5} = \frac{1}{5}\ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{5}\cdot\dfrac{1}{x} = \dfrac{1}{5x}$

19. $g(x) = \ln\dfrac{a-x}{a+x} = \ln(a-x) - \ln(a+x) \quad \Rightarrow$

$$g'(x) = \frac{1}{a-x}(-1) - \frac{1}{a+x} = \frac{-(a+x)-(a-x)}{(a-x)(a+x)} = \frac{-2a}{a^2-x^2}$$

20. $h(x) = \ln\left(x + \sqrt{x^2-1}\right) \quad \Rightarrow$

$$h'(x) = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

21. $f(u) = \dfrac{\ln u}{1+\ln(2u)} \quad \Rightarrow$

$$f'(u) = \frac{[1+\ln(2u)]\cdot\frac{1}{u} - \ln u\cdot\frac{1}{2u}\cdot 2}{[1+\ln(2u)]^2} = \frac{\frac{1}{u}[1+\ln(2u)-\ln u]}{[1+\ln(2u)]^2}$$

$$= \frac{1+(\ln 2+\ln u)-\ln u}{u[1+\ln(2u)]^2} = \frac{1+\ln 2}{u[1+\ln(2u)]^2}$$

22. $f(t) = \dfrac{1+\ln t}{1-\ln t} \quad \Rightarrow$

$$f'(t) = \frac{(1-\ln t)(1/t)-(1+\ln t)(-1/t)}{(1-\ln t)^2} = \frac{(1/t)[(1-\ln t)+(1+\ln t)]}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$$

23. $F(t) = \ln\dfrac{(2t+1)^3}{(3t-1)^4} = \ln(2t+1)^3 - \ln(3t-1)^4 = 3\ln(2t+1) - 4\ln(3t-1) \quad \Rightarrow$

$F'(t) = 3\cdot\dfrac{1}{2t+1}\cdot 2 - 4\cdot\dfrac{1}{3t-1}\cdot 3 = \dfrac{6}{2t+1} - \dfrac{12}{3t-1}$, or combined, $\dfrac{-6(t+3)}{(2t+1)(3t-1)}$.

24. $y = \ln(x^4\sin^2 x) = \ln x^4 + \ln(\sin x)^2 = 4\ln x + 2\ln\sin x \quad \Rightarrow \quad y' = 4\cdot\dfrac{1}{x} + 2\cdot\dfrac{1}{\sin x}\cdot\cos x = \dfrac{4}{x} + 2\cot x$

25. $y = \ln|2-x-5x^2| \quad \Rightarrow \quad y' = \dfrac{1}{2-x-5x^2}\cdot(-1-10x) = \dfrac{-10x-1}{2-x-5x^2}$ or $\dfrac{10x+1}{5x^2+x-2}$

26. $G(u) = \ln\sqrt{\dfrac{3u+2}{3u-2}} = \frac{1}{2}[\ln(3u+2)-\ln(3u-2)] \quad \Rightarrow \quad G'(u) = \dfrac{1}{2}\left(\dfrac{3}{3u+2} - \dfrac{3}{3u-2}\right) = \dfrac{-6}{9u^2-4}$

27. $y = \ln\left(\dfrac{x+1}{x-1}\right)^{3/5} = \frac{3}{5}[\ln(x+1)-\ln(x-1)] \quad \Rightarrow \quad y' = \dfrac{3}{5}\left(\dfrac{1}{x+1} - \dfrac{1}{x-1}\right) = \dfrac{-6}{5(x^2-1)}$

28. $y = (\ln\tan x)^2 \quad \Rightarrow \quad y' = 2(\ln\tan x)\cdot\dfrac{1}{\tan x}\cdot\sec^2 x = \dfrac{2(\ln\tan x)\sec^2 x}{\tan x}$

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29. $y = \tan[\ln(ax+b)] \Rightarrow y' = \sec^2[\ln(ax+b)] \cdot \dfrac{1}{ax+b} \cdot a = \sec^2[\ln(ax+b)]\,\dfrac{a}{ax+b}$

30. $y = \ln|\tan 2x| \Rightarrow y' = \dfrac{2\sec^2 2x}{\tan 2x}$

31. $y = \ln\ln x \Rightarrow y' = \dfrac{1}{\ln x}\,\dfrac{d}{dx}(\ln x) = \dfrac{1}{\ln x}\cdot\dfrac{1}{x} = \dfrac{1}{x\ln x} \Rightarrow$

$$y'' = -\frac{\frac{d}{dx}(x\ln x)}{(x\ln x)^2} \quad \text{[Reciprocal Rule]} \quad = -\frac{x\cdot\frac{1}{x}+\ln x\cdot 1}{(x\ln x)^2} = -\frac{1+\ln x}{(x\ln x)^2}$$

32. $y = \dfrac{\ln x}{x^2} \Rightarrow y' = \dfrac{x^2(1/x)-(\ln x)(2x)}{(x^2)^2} = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3} \Rightarrow$

$$y'' = \frac{x^3(-2/x)-(1-2\ln x)(3x^2)}{(x^3)^2} = \frac{x^2(-2-3+6\ln x)}{x^6} = \frac{6\ln x-5}{x^4}$$

33. $f(x) = \dfrac{x}{1-\ln(x-1)} \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot\frac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\frac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$
$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$\text{Dom}(f) = \{x \mid x-1>0 \text{ and } 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \text{ and } \ln(x-1)\neq 1\}$
$= \{x \mid x>1 \text{ and } x-1\neq e^1\} = \{x \mid x>1 \text{ and } x\neq 1+e\} = (1,1+e)\cup(1+e,\infty)$

34. $f(x) = \dfrac{1}{1+\ln x} \Rightarrow f'(x) = -\dfrac{1/x}{(1+\ln x)^2}$ [Reciprocal Rule] $= -\dfrac{1}{x(1+\ln x)^2}$.

$\text{Dom}(f) = \{x \mid x>0 \text{ and } \ln x\neq -1\} = \{x \mid x>0 \text{ and } x\neq 1/e\} = (0,1/e)\cup(1/e,\infty)$.

35. $f(x) = \sqrt{1-\ln x}$ is defined $\Leftrightarrow$ $x>0$ [so that $\ln x$ is defined] and $1-\ln x\geq 0$
$\Leftrightarrow$ $x>0$ and $\ln x\leq 1$ $\Leftrightarrow$ $0<x\leq e$, so the domain of f is $(0,e]$. Now

$$f'(x) = \frac{1}{2}(1-\ln x)^{-1/2}\cdot\frac{d}{dx}(1-\ln x) = \frac{1}{2\sqrt{1-\ln x}}\cdot\left(-\frac{1}{x}\right) = \frac{-1}{2x\sqrt{1-\ln x}}.$$

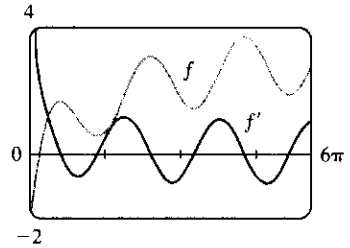
36. $f(x) = \ln\ln\ln x \Rightarrow f'(x) = \dfrac{1}{\ln\ln x}\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x}$.

$\text{Dom}(f) = \{x \mid \ln\ln x>0\} = \{x \mid \ln x>1\} = \{x \mid x>e\} = (e,\infty)$.

37. $f(x) = \dfrac{x}{\ln x} \Rightarrow f'(x) = \dfrac{\ln x - x(1/x)}{(\ln x)^2} = \dfrac{\ln x-1}{(\ln x)^2} \Rightarrow f'(e) = \dfrac{1-1}{1^2} = 0$

38. $f(t) = t\ln(4+3t) \Rightarrow f'(t) = t\cdot\dfrac{1}{4+3t}\cdot 3 + \ln(4+3t) = \dfrac{3t}{4+3t} + \ln(4+3t)$,
so $f'(-1) = \frac{-3}{1} + \ln 1 = -3+0 = -3$.

39. $f(x) = \sin x + \ln x \Rightarrow f'(x) = \cos x + 1/x$. This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



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26. If we let $t = \tan x$, then as $x \to (\pi/2)^+$, $t \to -\infty$. Thus, $\lim\limits_{x \to (\pi/2)^+} e^{\tan x} = \lim\limits_{t \to -\infty} e^t = 0$.

27. Divide numerator and denominator by e^{3x}: $\lim\limits_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim\limits_{x \to \infty} \dfrac{1 - e^{-6x}}{1 + e^{-6x}} = \dfrac{1 - 0}{1 + 0} = 1$

28. Divide numerator and denominator by e^{-3x}: $\lim\limits_{x \to -\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim\limits_{x \to -\infty} \dfrac{e^{6x} - 1}{e^{6x} + 1} = \dfrac{0 - 1}{0 + 1} = -1$

29. Let $t = 3/(2 - x)$. As $x \to 2^+$, $t \to -\infty$. So $\lim\limits_{x \to 2^+} e^{3/(2-x)} = \lim\limits_{t \to -\infty} e^t = 0$ by (6).

30. Let $t = 3/(2 - x)$. As $x \to 2^-$, $t \to \infty$. So $\lim\limits_{x \to 2^-} e^{3/(2-x)} = \lim\limits_{t \to \infty} e^t = \infty$ by (6).

31. By the Product Rule, $f(x) = x^2 e^x \;\Rightarrow\; f'(x) = x^2 \dfrac{d}{dx}(e^x) + e^x \dfrac{d}{dx}(x^2) = x^2 e^x + e^x(2x) = xe^x(x + 2)$.

32. By the Quotient Rule, $y = \dfrac{e^x}{1 + x} \;\Rightarrow\; y' = \dfrac{(1 + x)\,e^x - e^x(1)}{(1 + x)^2} = \dfrac{e^x + xe^x - e^x}{(x + 1)^2} = \dfrac{xe^x}{(x + 1)^2}$.

33. By (9), $y = e^{ax^3} \;\Rightarrow\; y' = e^{ax^3} \dfrac{d}{dx}(ax^3) = 3ax^2 e^{ax^3}$.

34. $y = e^u(\cos u + cu) \;\Rightarrow\; y' = e^u(-\sin u + c) + (\cos u + cu)e^u = e^u(\cos u - \sin u + cu + c)$

35. $f(u) = e^{1/u} \;\Rightarrow\; f'(u) = e^{1/u} \cdot \dfrac{d}{du}\left(\dfrac{1}{u}\right) = e^{1/u}\left(\dfrac{-1}{u^2}\right) = \left(\dfrac{-1}{u^2}\right) e^{1/u}$

36. $y = e^x \ln x \;\Rightarrow\; y' = e^x\left(\dfrac{1}{x}\right) + (\ln x)(e^x) = e^x\left(\ln x + \dfrac{1}{x}\right)$

37. By (9), $F(t) = e^{t \sin 2t} \;\Rightarrow$
$F'(t) = e^{t \sin 2t}(t \sin 2t)' = e^{t \sin 2t}(t \cdot 2\cos 2t + \sin 2t \cdot 1) = e^{t \sin 2t}(2t \cos 2t + \sin 2t)$

38. $y = e^{k \tan \sqrt{x}} \;\Rightarrow\; y' = e^{k \tan \sqrt{x}} \cdot \dfrac{d}{dx}(k \tan \sqrt{x}) = e^{k \tan \sqrt{x}}\left(k \sec^2 \sqrt{x} \cdot \tfrac{1}{2} x^{-1/2}\right) = \dfrac{k \sec^2 \sqrt{x}}{2\sqrt{x}}\, e^{k \tan \sqrt{x}}$

39. $y = \sqrt{1 + 2e^{3x}} \;\Rightarrow\; y' = \dfrac{1}{2}(1 + 2e^{3x})^{-1/2} \dfrac{d}{dx}(1 + 2e^{3x}) = \dfrac{1}{2\sqrt{1 + 2e^{3x}}}(2e^{3x} \cdot 3) = \dfrac{3e^{3x}}{\sqrt{1 + 2e^{3x}}}$

40. $y = \cos(e^{\pi x}) \;\Rightarrow\; y' = -\sin(e^{\pi x}) \cdot e^{\pi x} \cdot \pi = -\pi e^{\pi x} \sin(e^{\pi x})$

41. $y = e^{e^x} \;\Rightarrow\; y' = e^{e^x} \cdot \dfrac{d}{dx}(e^x) = e^{e^x} \cdot e^x$ or $e^{e^x + x}$

42. $y = \sqrt{1 + xe^{-2x}} \;\Rightarrow\; y' = \tfrac{1}{2}(1 + xe^{-2x})^{-1/2}\left[x(-2e^{-2x}) + e^{-2x}\right] = \dfrac{e^{-2x}(-2x + 1)}{2\sqrt{1 + xe^{-2x}}}$

43. By the Quotient Rule, $y = \dfrac{ae^x + b}{ce^x + d} \;\Rightarrow$

$$y' = \frac{(ce^x + d)(ae^x) - (ae^x + b)(ce^x)}{(ce^x + d)^2} = \frac{(ace^x + ad - ace^x - bc)e^x}{(ce^x + d)^2} = \frac{(ad - bc)e^x}{(ce^x + d)^2}.$$

44. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \;\Rightarrow\; y' = \dfrac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$

$$= \frac{(e^{2x} - 2 + e^{-2x}) - (e^{2x} + 2 + e^{-2x})}{(e^x - e^{-x})^2} = -\frac{4}{(e^x - e^{-x})^2}$$

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45. $y = e^{2x}\cos\pi x \;\Rightarrow\; y' = e^{2x}(-\pi\sin\pi x) + (\cos\pi x)(2e^{2x}) = e^{2x}(2\cos\pi x - \pi\sin\pi x)$.
At $(0,1)$, $y' = 1(2-0) = 2$, so an equation of the tangent line is $y - 1 = 2(x-0)$, or $y = 2x + 1$.

46. $y = \dfrac{e^x}{x} \;\Rightarrow\; y' = \dfrac{x\cdot e^x - e^x\cdot 1}{x^2} = \dfrac{e^x(x-1)}{x^2}$. At $(1,e)$, $y' = 0$, and an equation of the tangent line is $y - e = 0(x-1)$, or $y = e$.

47. $\dfrac{d}{dx}\left(e^{x^2y}\right) = \dfrac{d}{dx}(x+y) \;\Rightarrow\; e^{x^2y}(x^2y' + y\cdot 2x) = 1 + y' \;\Rightarrow\; x^2e^{x^2y}y' + 2xye^{x^2y} = 1 + y' \;\Rightarrow$
$x^2e^{x^2y}y' - y' = 1 - 2xye^{x^2y} \;\Rightarrow\; y'(x^2e^{x^2y} - 1) = 1 - 2xye^{x^2y} \;\Rightarrow\; y' = \dfrac{1 - 2xye^{x^2y}}{x^2e^{x^2y} - 1}$

48. $y = Ae^{-x} + Bxe^{-x} \;\Rightarrow\; y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B-A)e^{-x} - Bxe^{-x} \;\Rightarrow$
$y'' = (A-B)e^{-x} - Be^{-x} + Bxe^{-x} = (A-2B)e^{-x} + Bxe^{-x}$, so
$y'' + 2y' + y = (A-2B)e^{-x} + Bxe^{-x} + 2\left[(B-A)e^{-x} - Bxe^{-x}\right] + Ae^{-x} + Bxe^{-x} = 0$.

49. $y = e^{rx} \;\Rightarrow\; y' = re^{rx} \;\Rightarrow\; y'' = r^2e^{rx}$, so if $y = e^{rx}$ satisfies the differential equation $y'' + 6y' + 8y = 0$, then $r^2e^{rx} + 6re^{rx} + 8e^{rx} = 0$; that is, $e^{rx}(r^2 + 6r + 8) = 0$. Since $e^{rx} > 0$ for all x, we must have $r^2 + 6r + 8 = 0$, or $(r+2)(r+4) = 0$, so $r = -2$ or -4.

50. $y = e^{\lambda x} \;\Rightarrow\; y' = \lambda e^{\lambda x} \;\Rightarrow\; y'' = \lambda^2e^{\lambda x}$. Thus, $y + y' = y'' \;\Leftrightarrow\; e^{\lambda x} + \lambda e^{\lambda x} = \lambda^2e^{\lambda x} \;\Leftrightarrow$
$e^{\lambda x}\left(\lambda^2 - \lambda - 1\right) = 0 \;\Leftrightarrow\; \lambda = \frac{1\pm\sqrt{5}}{2}$, since $e^{\lambda x} \neq 0$.

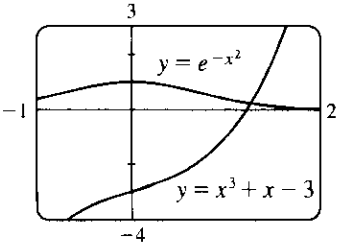
51. $f(x) = e^{2x} \;\Rightarrow\; f'(x) = 2e^{2x} \;\Rightarrow\; f''(x) = 2\cdot 2e^{2x} = 2^2e^{2x} \;\Rightarrow$
$f'''(x) = 2^2\cdot 2e^{2x} = 2^3e^{2x} \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = 2^ne^{2x}$

52. $f(x) = xe^{-x} \;\Rightarrow\; f'(x) = x(-e^{-x}) + e^{-x} = (1-x)e^{-x} \;\Rightarrow$
$f''(x) = (1-x)(-e^{-x}) + e^{-x}(-1) = (x-2)e^{-x} \;\Rightarrow\; f'''(x) = (x-2)(-e^{-x}) + e^{-x} = (3-x)e^{-x} \;\Rightarrow$
$f^{(4)}(x) = (3-x)(-e^{-x}) + e^{-x}(-1) = (x-4)e^{-x} \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = (-1)^n(x-n)e^{-x}$.
So $D^{1000}xe^{-x} = (x-1000)e^{-x}$.

53. (a) $f(x) = e^x + x$ is continuous on $\mathbb{R}$ and $f(-1) = e^{-1} - 1 < 0 < 1 = f(0)$, so by the Intermediate Value Theorem, $e^x + x = 0$ has a root in $(-1,0)$.

(b) $f(x) = e^x + x \;\Rightarrow\; f'(x) = e^x + 1$, so $x_{n+1} = x_n - \dfrac{e^{x_n} + x_n}{e^{x_n} + 1}$. Using $x_1 = -0.5$, we get $x_2 \approx -0.566311$, $x_3 \approx -0.567143 \approx x_4$, so the root is -0.567143 to six decimal places.

54.



From the graph, it appears that the curves intersect at about $x \approx 1.2$ or 1.3. We use Newton's Method with $f(x) = x^3 + x - 3 - e^{-x^2}$, so $f'(x) = 3x^2 + 1 + 2xe^{-x^2}$, and the formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. We take $x_1 = 1.2$, and the formula gives $x_2 \approx 1.252462$, $x_3 \approx 1.251045$, and $x_4 \approx x_5 \approx 1.251044$. So the root of the equation, correct to six decimal places, is $x = 1.251044$.

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55. (a) $m(t) = 24 \cdot e^{-(\ln 2)t/25} = 24 \cdot 2^{-t/25} \quad \Rightarrow \quad m(40) = 24 \cdot 2^{-40/25} \approx 7.92$ mg.

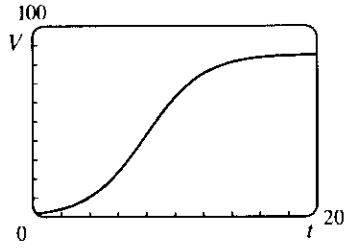
(b) $m'(t) = 24\frac{d}{dt}\left[e^{-(\ln 2)t/25}\right] = 24 \cdot e^{-(\ln 2)t/25}\left(-\frac{\ln 2}{25}\right)$, so

$m'(40) = 24e^{-(\ln 2)(40)/25}\left(-\frac{\ln 2}{25}\right) \approx -0.22$ mg/yr

(c) $m(t) = 5 \quad \Rightarrow \quad 24e^{-(\ln 2)t/25} = 5 \quad \Rightarrow \quad e^{-(\ln 2)t/25} = \frac{5}{24} \quad \Rightarrow \quad -(\ln 2)t/25 = \ln\frac{5}{24} \quad \Rightarrow$

$$t = -25\,\frac{\ln\frac{5}{24}}{\ln 2} \approx 56.6 \text{ yr}$$

56.



From the graph, we estimate that the most rapid increase in the percentage of households in the United States with at least one VCR occurs at about $t = 8$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. We'll use $V(t) = \dfrac{a}{1 + be^{ct}}$, and substitute $a = 85$, $b = 53$, and $c = -0.5$ later.

$$V'(t) = -\frac{a(bce^{ct})}{(1 + be^{ct})^2} \quad \text{[by the Reciprocal Rule]} \quad \text{and}$$

$$V''(t) = -abc \cdot \frac{(1 + be^{ct})^2 \cdot ce^{ct} - e^{ct} \cdot 2(1 + be^{ct}) \cdot bce^{ct}}{\left[(1 + be^{ct})^2\right]^2}$$

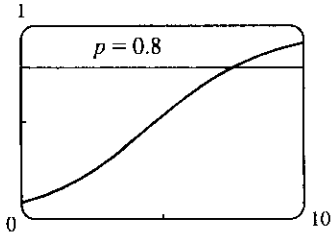
$$= \frac{-abc \cdot ce^{ct}(1 + be^{ct})[(1 + be^{ct}) - 2be^{ct}]}{(1 + be^{ct})^4} = \frac{-abc^2e^{ct}(1 - be^{ct})}{(1 + be^{ct})^3}$$

So $V''(t) = 0 \quad \Leftrightarrow \quad 1 = be^{ct} \quad \Leftrightarrow \quad e^{ct} = 1/b$. Now graph $y = e^{-0.5t}$ and $y = \frac{1}{53}$. These graphs intersect at $t \approx 7.94$ years, which corresponds to roughly midyear 1988. [Alternatively, we could use the rootfinder on a calculator to solve $e^{-0.5t} = \frac{1}{53}$. Or, if you have already studied logarithms, you can solve $e^{ct} = 1/b$ as follows: $ct = \ln(1/b) \quad \Leftrightarrow \quad t = (1/c)\ln(1/b) = -2\ln\frac{1}{53} \approx 7.94$ years.

57. (a) $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \dfrac{1}{1 + ae^{-kt}} = \dfrac{1}{1 + a \cdot 0} = 1$, since $k > 0 \quad \Rightarrow \quad -kt \to -\infty \quad \Rightarrow \quad e^{-kt} \to 0$.

(b) $p(t) = \left(1 + ae^{-kt}\right)^{-1} \quad \Rightarrow \quad \dfrac{dp}{dt} = -\left(1 + ae^{-kt}\right)^{-2}\left(-kae^{-kt}\right) = \dfrac{kae^{-kt}}{(1 + ae^{-kt})^2}$

(c)



From the graph of $p(t) = \left(1 + 10e^{-0.5t}\right)^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

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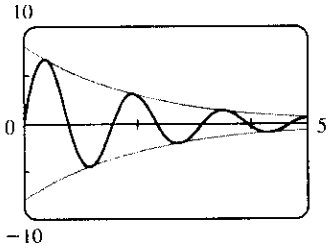
58. (a)



The displacement function is squeezed between the other two functions. This is because $-1 \le \sin 4t \le 1 \Rightarrow -8e^{-t/2} \le 8e^{-t/2}\sin 4t \le 8e^{-t/2}$.

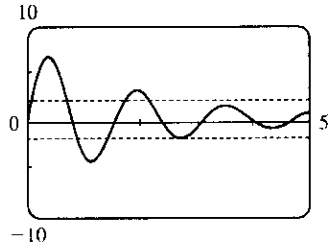
(b) The maximum value of the displacement is about 6.6 cm, occurring at $t \approx 0.36$ s. It occurs just before the graph of the displacement function touches the graph of $8e^{-t/2}$ (when $t = \frac{\pi}{8} \approx 0.39$).

(c) The velocity of the object is the derivative of its displacement function, that is,

$$\frac{d}{dt}\left(8e^{-t/2}\sin 4t\right) = 8\left[e^{-t/2}\cos 4t(4) + \sin 4t\left(-\tfrac{1}{2}\right)e^{-t/2}\right].$$

If the displacement is zero, then we must have $\sin 4t = 0$ (since the exponential term in the displacement function is always positive). The first time that $\sin 4t = 0$ after $t = 0$ occurs at $t = \frac{\pi}{4}$. Substituting this into our expression for the velocity, and noting that the second term vanishes, we get $v\left(\frac{\pi}{4}\right) = 8e^{-\pi/8}\cos\left(4 \cdot \frac{\pi}{4}\right)\cdot 4 = -32e^{-\pi/8} \approx -21.6$ cm/s.

(d)



The graph indicates that the displacement is less than 2 cm from equilibrium whenever t is larger than about 2.8.

59. $f(x) = x - e^x \Rightarrow f'(x) = 1 - e^x = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0$. Now $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, so the absolute maximum value is $f(0) = 0 - 1 = -1$.

60. $g(x) = \dfrac{e^x}{x} \Rightarrow g'(x) = \dfrac{xe^x - e^x}{x^2} = 0 \Leftrightarrow e^x(x-1) = 0 \Rightarrow x = 1$. Now $g'(x) > 0 \Leftrightarrow \dfrac{xe^x - e^x}{x^2} > 0 \Leftrightarrow x - 1 > 0 \Leftrightarrow x > 1$ and $g'(x) < 0 \Leftrightarrow \dfrac{xe^x - e^x}{x^2} < 0 \Leftrightarrow x - 1 < 0 \Leftrightarrow x < 1$. Thus there is an absolute minimum value of $g(1) = e$ at $x = 1$.

61. $y = xe^{3x} \Rightarrow y' = xe^{3x}\cdot 3 + e^{3x}\cdot 1 = (3x+1)e^{3x} \Rightarrow y'' = (3x+1)e^{3x}\cdot 3 + e^{3x}\cdot 3 = (9x+6)e^{3x}$. The curve is concave upward at $x \Leftrightarrow y'' > 0$ at $x \Leftrightarrow 9x + 6 > 0 \Leftrightarrow x > -\frac{2}{3}$. Thus, the curve is concave upward on $\left(-\frac{2}{3}, \infty\right)$.

62. $f(x) = x^2e^{-x} \Rightarrow f'(x) = x^2(-e^{-x}) + e^{-x}\cdot 2x = (2x - x^2)e^{-x}$, so $f'(x) > 0 \Leftrightarrow 2x - x^2 > 0 \Leftrightarrow x(2-x) > 0 \Leftrightarrow 0 < x < 2$, so f is increasing on $(0, 2)$.

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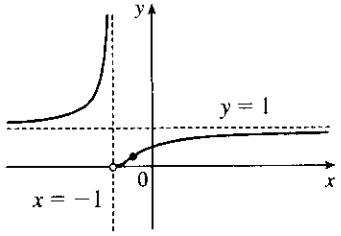
63. $y = f(x) = e^{-1/(x+1)}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** No x-intercept; y-intercept $= f(0) = e^{-1}$ **C.** No symmetry **D.** $\lim_{x \to \pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \to 0$, so $y = 1$ is a HA. $\lim_{x \to -1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \to -\infty$, $\lim_{x \to -1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \to \infty$, so $x = -1$ is a VA. **E.** $f'(x) = e^{-1/(x+1)}/(x+1)^2 \Rightarrow f'(x) > 0$ for all x except 1, so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values **H.**

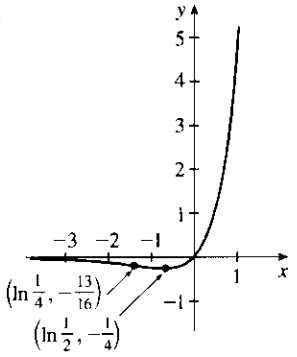


G. $f''(x) = \dfrac{e^{-1/(x+1)}}{(x+1)^4} + \dfrac{e^{-1/(x+1)}(-2)}{(x+1)^3} = -\dfrac{e^{-1/(x+1)}(2x+1)}{(x+1)^4}$

$\Rightarrow f''(x) > 0 \Leftrightarrow 2x + 1 < 0 \Leftrightarrow x < -\frac{1}{2}$, so f is CU on $(-\infty, -1)$ and $\left(-1, -\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2}, \infty\right)$. f has an IP at $\left(-\frac{1}{2}, e^{-2}\right)$.

64. $y = f(x) = e^{2x} - e^x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow e^{2x} = e^x \Rightarrow e^x = 1 \Rightarrow x = 0$. **C.** No symmetry **D.** $\lim_{x \to -\infty} e^{2x} - e^x = 0$, so $y = 0$ is a HA. No VA.

H.



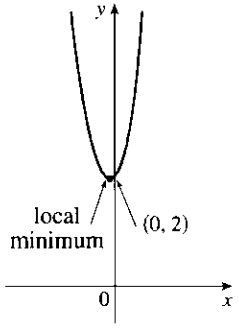
E. $f'(x) = 2e^{2x} - e^x = e^x(2e^x - 1)$, so $f'(x) > 0 \Leftrightarrow e^x > \frac{1}{2} \Leftrightarrow x > \ln \frac{1}{2} = -\ln 2$ and $f'(x) < 0 \Leftrightarrow e^x < \frac{1}{2} \Leftrightarrow x < \ln \frac{1}{2}$, so f is decreasing on $\left(-\infty, \ln \frac{1}{2}\right)$ and increasing on $\left(\ln \frac{1}{2}, \infty\right)$. **F.** Local minimum value $f\left(\ln \frac{1}{2}\right) = e^{2\ln(1/2)} - e^{\ln(1/2)} = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$

G. $f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1)$, so $f''(x) > 0 \Leftrightarrow e^x > \frac{1}{4} \Leftrightarrow x > \ln \frac{1}{4}$ and $f''(x) < 0 \Leftrightarrow x < \ln \frac{1}{4}$.

Thus, f is CD on $\left(-\infty, \ln \frac{1}{4}\right)$ and CU on $\left(\ln \frac{1}{4}, \infty\right)$. f has an IP at $\left(\ln \frac{1}{4}, \left(\frac{1}{4}\right)^2 - \frac{1}{4}\right) = \left(\ln \frac{1}{4}, -\frac{3}{16}\right)$.

65. $y = f(x) = e^{3x} + e^{-2x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$; no x-intercept **C.** No symmetry **D.** No asymptotes

H.



E. $f'(x) = 3e^{3x} - 2e^{-2x}$, so $f'(x) > 0 \Leftrightarrow 3e^{3x} > 2e^{-2x}$ [multiply by e^{2x}] $\Leftrightarrow e^{5x} > \frac{2}{3} \Leftrightarrow 5x > \ln \frac{2}{3} \Leftrightarrow x > \frac{1}{5} \ln \frac{2}{3} \approx -0.081$. Similarly, $f'(x) < 0 \Leftrightarrow x < \frac{1}{5} \ln \frac{2}{3}$. f is decreasing on $\left(-\infty, \frac{1}{5} \ln \frac{2}{3}\right)$ and increasing on $\left(\frac{1}{5} \ln \frac{2}{3}, \infty\right)$.

F. Local minimum value $f\left(\frac{1}{5} \ln \frac{2}{3}\right) = \left(\frac{2}{3}\right)^{3/5} + \left(\frac{2}{3}\right)^{-2/5} \approx 1.96$; no local maximum.

G. $f''(x) = 9e^{3x} + 4e^{-2x}$, so $f''(x) > 0$ for all x, and f is CU on $(-\infty, \infty)$. No IP

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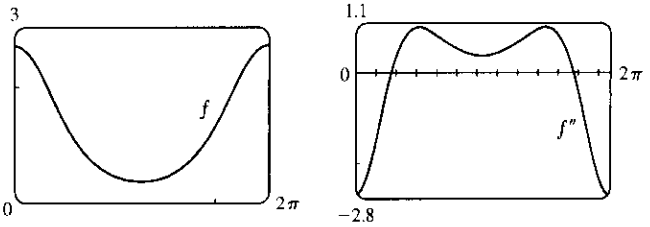
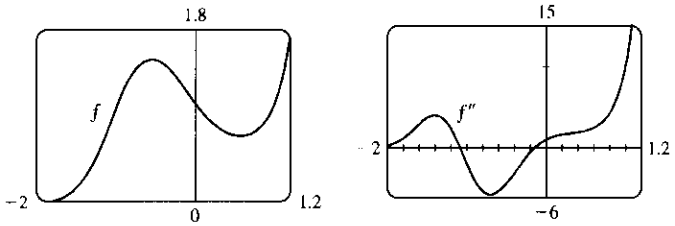
66. The function $f(x) = e^{\cos x}$ is periodic with period 2π, so we consider it only on the interval $[0, 2\pi]$. We see that it has local maxima of about $f(0) \approx 2.72$ and $f(2\pi) \approx 2.72$, and a local minimum of about $f(3.14) \approx 0.37$. To find the exact values, we calculate $f'(x) = -\sin x e^{\cos x}$. This is 0 when $-\sin x = 0 \quad \Leftrightarrow \quad x = 0, \pi$ or 2π (since we are only considering $x \in [0, 2\pi]$). Also $f'(x) > 0 \quad \Leftrightarrow \quad \sin x < 0 \quad \Leftrightarrow \quad 0 < x < \pi$. So $f(0) = f(2\pi) = e$ (both maxima) and $f(\pi) = e^{\cos \pi} = 1/e$ (minimum). To find the inflection points, we calculate and graph $f''(x) = \dfrac{d}{dx}\left(-\sin x\, e^{\cos x}\right) = -\cos x\, e^{\cos x} - \sin x\left(e^{\cos x}\right)(-\sin x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$. From the graph of $f''(x)$, we see that f has inflection points at $x \approx 0.90$ and at $x \approx 5.38$. These x-coordinates correspond to inflection points $(0.90, 1.86)$ and $(5.38, 1.86)$.



67. $f(x) = e^{x^3 - x} \to 0$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$. From the graph, it appears that f has a local minimum of about $f(0.58) = 0.68$, and a local maximum of about $f(-0.58) = 1.47$. To find the exact values, we calculate $f'(x) = (3x^2 - 1)e^{x^3 - x}$, which is 0 when $3x^2 - 1 = 0 \quad \Leftrightarrow \quad x = \pm\frac{1}{\sqrt{3}}$. The negative root corresponds to the local maximum $f\left(-\frac{1}{\sqrt{3}}\right) = e^{(-1/\sqrt{3})^3 - (-1/\sqrt{3})} = e^{2\sqrt{3}/9}$, and the positive root corresponds to the local minimum $f\left(\frac{1}{\sqrt{3}}\right) = e^{(1/\sqrt{3})^3 - (1/\sqrt{3})} = e^{-2\sqrt{3}/9}$. To estimate the inflection points, we calculate and graph $f''(x) = \dfrac{d}{dx}\left[(3x^2 - 1)e^{x^3 - x}\right] = (3x^2 - 1)e^{x^3 - x}(3x^2 - 1) + e^{x^3 - x}(6x) = e^{x^3 - x}(9x^4 - 6x^2 + 6x + 1)$. From the graph, it appears that $f''(x)$ changes sign (and thus f has inflection points) at $x \approx -0.15$ and $x \approx -1.09$. From the graph of f, we see that these x-values correspond to inflection points at about $(-0.15, 1.15)$ and $(-1.09, 0.82)$.



68. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \quad \Rightarrow$ $f'(x) = e^{-x^2/(2\sigma^2)}\left(-x/\sigma^2\right)$. $f'(x) = 0 \quad \Leftrightarrow \quad x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find

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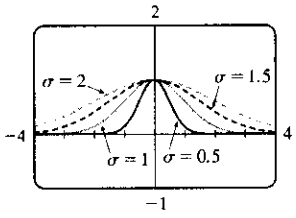
$$f''(x) = -\frac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}\left(-x/\sigma^2\right)\right] = \frac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}\left(1 - x^2/\sigma^2\right).$$

$f''(x) = 0 \;\Leftrightarrow\; x^2 = \sigma^2 \;\Leftrightarrow\; x = \pm\sigma$. $f''(x) < 0 \;\Leftrightarrow\; x^2 < \sigma^2 \;\Leftrightarrow\; -\sigma < x < \sigma$. So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $\left(\pm\sigma, e^{-1/2}\right)$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

69. Let $u = -3x$. Then $du = -3\,dx$, so

$\int_0^5 e^{-3x}\,dx = -\frac{1}{3}\int_0^{-15} e^u\,du = -\frac{1}{3}[e^u]_0^{-15} = -\frac{1}{3}\left(e^{-15} - e^0\right) = \frac{1}{3}\left(1 - e^{-15}\right)$.

70. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\left[e^u\right]_0^{-1} = -\frac{1}{2}\left(e^{-1} - e^0\right) = \frac{1}{2}(1 - 1/e)$.

71. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1 + e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1 + e^x)^{3/2} + C$.

72. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so $\int \sec^2 x\,e^{\tan x}\,dx = \int e^u\,du = e^u + C = e^{\tan x} + C$.

73. $\displaystyle\int \frac{e^x + 1}{e^x}\,dx = \int (1 + e^{-x})\,dx = x - e^{-x} + C$

74. Let $u = \dfrac{1}{x}$. Then $du = -\dfrac{1}{x^2}\,dx$, so $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{1/x} + C$.

75. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$, so $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$.

76. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x \sin(e^x)\,dx = \int \sin u\,du = -\cos u + C = -\cos(e^x) + C$.

77. Area $= \int_0^1 \left(e^{3x} - e^x\right)dx = \left[\frac{1}{3}e^{3x} - e^x\right]_0^1 = \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3} - 1\right) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644$

78. $f''(x) = 3e^x + 5\sin x \;\Rightarrow\; f'(x) = 3e^x - 5\cos x + C \;\Rightarrow\; 2 = f'(0) = 3 - 5 + C \;\Rightarrow\; C = 4$, so $f'(x) = 3e^x - 5\cos x + 4 \;\Rightarrow\; f(x) = 3e^x - 5\sin x + 4x + D \;\Rightarrow\; 1 = f(0) = 3 + D \;\Rightarrow\; D = -2$, so $f(x) = 3e^x - 5\sin x + 4x - 2$.

79. $V = \int_0^1 \pi(e^x)^2\,dx = \pi\int_0^1 e^{2x}\,dx = \frac{1}{2}\pi\left[e^{2x}\right]_0^1 = \frac{\pi}{2}\left(e^2 - 1\right)$

80. $V = \int_0^1 2\pi x e^{-x^2}\,dx$. Let $u = x^2$. Thus $du = 2x\,dx$, so $V = \pi\int_0^1 e^{-u}\,du = \pi\left[-e^{-u}\right]_0^1 = \pi(1 - 1/e)$.

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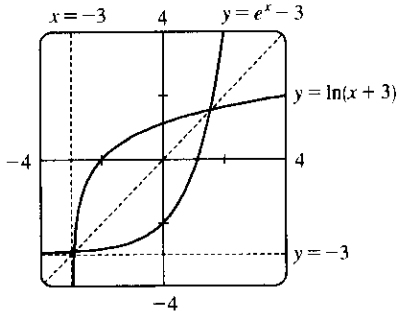
81. $y = \ln(x+3) \quad \Rightarrow \quad e^y = x+3 \quad \Rightarrow$

$x = e^y - 3$. Interchanging x and y, we get

$y = e^x - 3$, so $f^{-1}(x) = e^x - 3$.

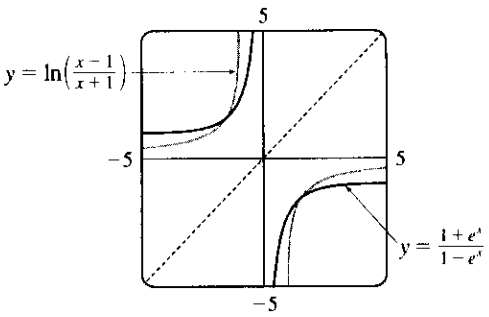


82. $y = \dfrac{1+e^x}{1-e^x} \quad \Rightarrow \quad y - ye^x = 1 + e^x \quad \Rightarrow$

$e^x(y+1) = y - 1 \quad \Rightarrow \quad e^x = \dfrac{y-1}{y+1} \quad \Rightarrow$

$x = \ln\left(\dfrac{y-1}{y+1}\right)$. Interchange x and y:

$y = \ln\left(\dfrac{x-1}{x+1}\right)$ is the inverse function.



83. We use Theorem 7.1.7. Note that $f(0) = 3 + 0 + e^0 = 4$, so $f^{-1}(4) = 0$. Also $f'(x) = 1 + e^x$. Therefore,

$$\left(f^{-1}\right)'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}.$$

84. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form $\lim_{x \to \pi} \dfrac{f(x) - f(\pi)}{x - \pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos \pi)e^{\sin \pi} = -1 \cdot e^0 = -1$.

85. Using the second law of logarithms and Equation 5, we have $\ln(e^x/e^y) = \ln e^x - \ln e^y = x - y = \ln(e^{x-y})$. Since ln is a one-to-one function, it follows that $e^x/e^y = e^{x-y}$.

86. Using the third law of logarithms and Equation 5, we have $\ln e^{rx} = rx = r \ln e^x = \ln (e^x)^r$. Since ln is a one-to-one function, it follows that $e^{rx} = (e^x)^r$.

87. (a) Let $f(x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 = 0$, and for $x \geq 0$, we have $f'(x) = e^x - 1 \geq 0$. Now, since $f(0) = 0$ and f is increasing on $[0, \infty)$, $f(x) \geq 0$ for $x \geq 0 \quad \Rightarrow \quad e^x - 1 - x \geq 0 \quad \Rightarrow \quad e^x \geq 1 + x$.

(b) For $0 \leq x \leq 1$, $x^2 \leq x$, so $e^{x^2} \leq e^x$ [since e^x is increasing]. Hence [from (a)] $1 + x^2 \leq e^{x^2} \leq e^x$. So $\frac{4}{3} = \int_0^1 (1+x^2)\,dx \leq \int_0^1 e^{x^2}\,dx \leq \int_0^1 e^x\,dx = e - 1 < e \quad \Rightarrow \quad \frac{4}{3} \leq \int_0^1 e^{x^2}\,dx \leq e$.

88. (a) Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus, $f'(x) = e^x - 1 - x$, which is positive for $x \geq 0$ by Exercise 87(a). Thus $f(x)$ is increasing on $(0, \infty)$, so on that interval, $0 = f(0) \leq f(x) = e^x - 1 - x - \frac{1}{2}x^2 \quad \Rightarrow$ $e^x \geq 1 + x + \frac{1}{2}x^2$.

(b) Using the same argument as in Exercise 87(b), from part (a) we have $1 + x^2 + \frac{1}{2}x^4 \leq e^{x^2} \leq e^x$ [for $0 \leq x \leq 1$] $\quad \Rightarrow \quad \int_0^1 \left(1 + x^2 + \frac{1}{2}x^4\right) dx \leq \int_0^1 e^{x^2}\,dx \leq \int_0^1 e^x\,dx \quad \Rightarrow \quad \frac{43}{30} \leq \int_0^1 e^{x^2}\,dx \leq e - 1$.

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89. (a) By Exercise 87(a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$ for $x \geq 0$.

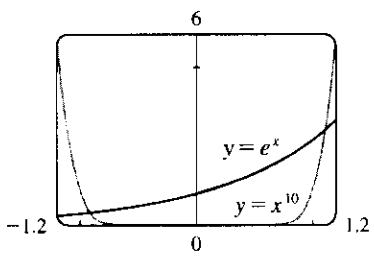
Let $f(x) = e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} \geq 0$ by assumption. Hence $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that $0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}$, and hence $e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}$ for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ for every positive integer n, by mathematical induction.

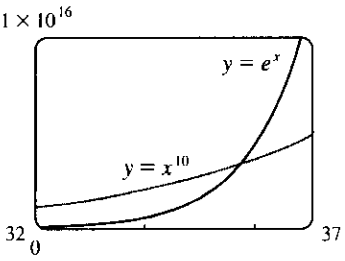
(b) Taking $n = 4$ and $x = 1$ in (a), we have $e = e^1 \geq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\overline{3} > 2.7$.

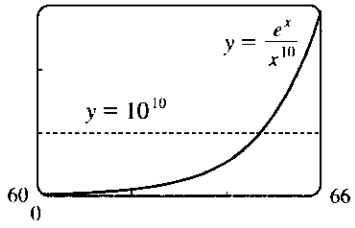
(c) $e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} \quad \Rightarrow \quad \frac{e^x}{x^k} \geq \frac{1}{x^k} + \frac{1}{x^{k-1}} + \cdots + \frac{1}{k!} + \frac{x}{(k+1)!} \geq \frac{x}{(k+1)!}$.

But $\lim_{x \to \infty} \frac{x}{(k+1)!} = \infty$, so $\lim_{x \to \infty} \frac{e^x}{x^k} = \infty$.

90. (a) The grapb of g finally surpasses that of f at $x \approx 35.8$.





(b) 

(c) From the graph in part (b), it seems that $e^x/x^{10} > 10^{10}$ whenever $x > 65$, approximately. So we can take $N \geq 65$.

7.4* General Logarithmic and Exponential Functions

1. (a) $a^x = e^{x \ln a}$

(b) The domain of $f(x) = a^x$ is $\mathbb{R}$.

(c) The range of $f(x) = a^x$ $(a \neq 1)$ is $(0, \infty)$.

(d) (i) See Figure 1. (ii) See Figure 3. (iii) See Figure 2.

2. (a) $\log_a x$ is the number y such that $a^y = x$.

(b) The domain of $f(x) = \log_a x$ is $(0, \infty)$.

(c) The range of $f(x) = \log_a x$ is $\mathbb{R}$.

(d) See Figure 9.

3. $5^{\sqrt{7}} = (e^{\ln 5})^{\sqrt{7}} = e^{\sqrt{7} \ln 5}$

4. $10^{x^2} = (e^{\ln 10})^{x^2} = e^{x^2 \ln 10}$

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5. $(\cos x)^x = (e^{\ln \cos x})^x = e^{x \ln(\cos x)}$.

6. $x^{\cos x} = (e^{\ln x})^{\cos x} = e^{(\cos x)(\ln x)}$

7. (a) $\log_{10} 1000 = 3$ because $10^3 = 1000$.

(b) $\log_2 \frac{1}{16} = -4$ since $2^{-4} = \frac{1}{16}$. [*Or:* $\log_2 \frac{1}{16} = \log_2 2^{-4} = -4$]

8. (a) $\log_{10} 0.1 = -1$ since $10^{-1} = 0.1$.

(b) $\log_8 320 - \log_8 5 = \log_8 \frac{320}{5} = \log_8 64 = 2$ since $8^2 = 64$.

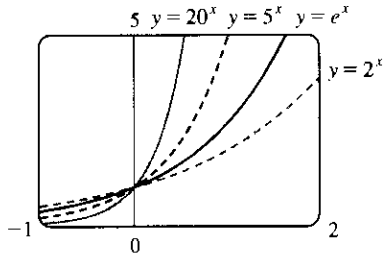
9. (a) $\log_{12} 3 + \log_{12} 48 = \log_{12}(3 \cdot 48) = \log_{12} 144 = 2$ since $12^2 = 144$.

(b) $\log_5 5^{\sqrt{2}} = \sqrt{2}$ by the cancellation property $\log_a a^x = x$.
[*Or:* $\log_5 5^{\sqrt{2}} = \sqrt{2} \log_5 5 = \sqrt{2} \cdot 1 = \sqrt{2}$]

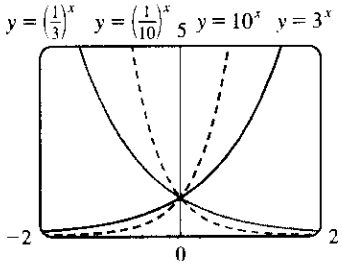
10. (a) $\log_a \frac{1}{a} = -1$ since $a^{-1} = \frac{1}{a}$. [*Or:* $\log_a \frac{1}{a} = \log_a a^{-1} = -1$]

(b) $10^{(\log_{10} 4 + \log_{10} 7)} = 10^{\log_{10} 4} \cdot 10^{\log_{10} 7} = 4 \cdot 7 = 28$
[*Or:* $10^{(\log_{10} 4 + \log_{10} 7)} = 10^{\log_{10}(4 \cdot 7)} = 10^{\log_{10} 28} = 28$]

11. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



12. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 [$\left(\frac{1}{3}\right)^x$ and $\left(\frac{1}{10}\right)^x$] are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



13. (a) $\log_{12} e = \dfrac{\ln e}{\ln 12} = \dfrac{1}{\ln 12} \approx 0.402430$ (b) $\log_6 13.54 = \dfrac{\ln 13.54}{\ln 6} \approx 1.454240$

(c) $\log_2 \pi = \dfrac{\ln \pi}{\ln 2} \approx 1.651496$

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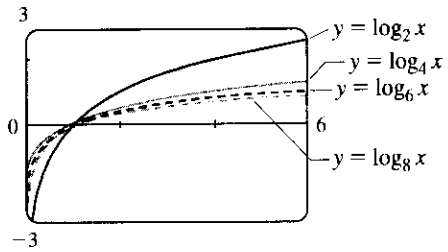
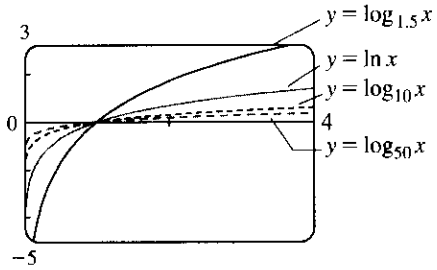
14. To graph the functions, we use $\log_2 x = \dfrac{\ln x}{\ln 2}$, $\log_4 x = \dfrac{\ln x}{\ln 4}$, etc.

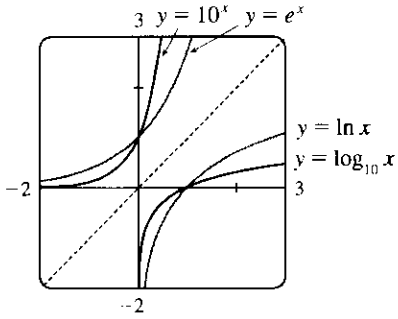
These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y-axis (as $x \to 0^+$).



15. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.



16. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10} x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



17. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1 \quad \left[C = \frac{6}{a}\right]$ and $24 = Ca^3 \quad \Rightarrow \quad 24 = \left(\dfrac{6}{a}\right)a^3 \quad \Rightarrow$ $4 = a^2 \quad \Rightarrow \quad a = 2 \quad$ [since $a > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

18. Given the y-intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $\left(2, \frac{2}{9}\right)$ gives us $\frac{2}{9} = 2a^2 \quad \Rightarrow \quad \frac{1}{9} = a^2$ $\Rightarrow \quad a = \frac{1}{3} \quad$ [since $a > 0$]. The function is $f(x) = 2\left(\frac{1}{3}\right)^x$ or $f(x) = 2(3)^{-x}$.

19. (a) 2 ft = 24 in, $f(24) = 24^2$ in = 576 in = 48 ft. $g(24) = 2^{24}$ in $= 2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

(b) 3 ft = 36 in, so we need x such that $\log_2 x = 36 \quad \Leftrightarrow \quad x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is

$$68{,}719{,}476{,}736 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7 \text{ mi}.$$

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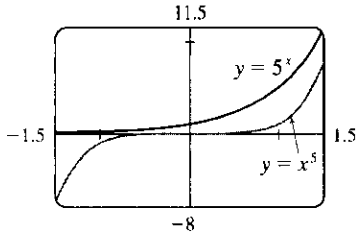
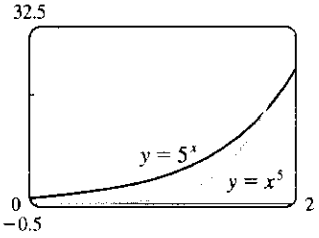
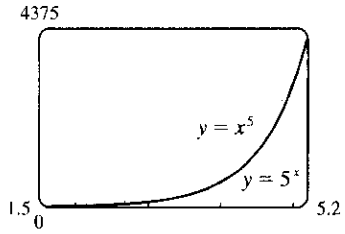
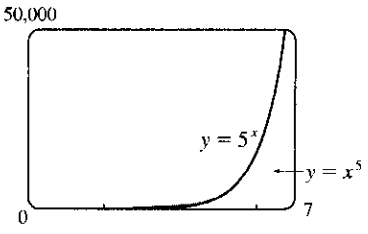
20. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.









21. $\lim_{t\to\infty} 2^{-t^2} = \lim_{u\to-\infty} 2^u \quad [\text{where } u = -t^2] \quad = 0$

22. Let $t = x^2 - 5x + 6$. As $x \to 3^+$, $t = (x-2)(x-3) \to 0^+$. $\lim_{x\to3^+} \log_{10}(x^2 - 5x + 6) = \lim_{t\to0^+} \log_{10} t = -\infty$ by (4) in Section 7.3 .

23. $h(t) = t^3 - 3^t \quad \Rightarrow \quad h'(t) = 3t^2 - 3^t \ln 3$

24. $g(x) = x^4 4^x \quad \Rightarrow \quad g'(x) = x^4 4^x \ln 4 + 4^x \cdot 4x^3 = x^3 4^x (x \ln 4 + 4)$

25. Using Formula 4 and the Chain Rule, $y = 5^{-1/x} \quad \Rightarrow \quad y' = 5^{-1/x}(\ln 5)\left[-1 \cdot \left(-x^{-2}\right)\right] = 5^{-1/x}(\ln 5)/x^2$

26. $y = 10^{\tan\theta} \quad \Rightarrow \quad y' = 10^{\tan\theta}(\ln 10)(\sec^2\theta)$

27. $f(u) = (2^u + 2^{-u})^{10} \quad \Rightarrow$

$$f'(u) = 10(2^u + 2^{-u})^9 \frac{d}{du}(2^u + 2^{-u}) = 10(2^u + 2^{-u})^9 \left[2^u \ln 2 + 2^{-u} \ln 2 \cdot (-1)\right]$$
$$= 10 \ln 2 (2^u + 2^{-u})^9 (2^u - 2^{-u})$$

28. $y = 2^{3^{x^2}} \quad \Rightarrow \quad y' = 2^{3^{x^2}} (\ln 2) \frac{d}{dx}\left(3^{x^2}\right) = 2^{3^{x^2}} (\ln 2) 3^{x^2} (\ln 3)(2x)$

29. $f(x) = \log_3(x^2 - 4) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(x^2 - 4)\ln 3}(2x) = \dfrac{2x}{(x^2 - 4)\ln 3}$

30. $f(x) = \log_{10}\left(\dfrac{x}{x-1}\right) = \log_{10} x - \log_{10}(x-1) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x \ln 10} - \dfrac{1}{(x-1)\ln 10}$ or $-\dfrac{1}{x(x-1)\ln 10}$

31. $y = x^x \quad \Rightarrow \quad \ln y = x \ln x \quad \Rightarrow \quad y'/y = \ln x + x(1/x) \quad \Rightarrow \quad y' = x^x(\ln x + 1)$

32. $y = x^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x} \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = -\dfrac{1}{x^2} \ln x + \dfrac{1}{x}\left(\dfrac{1}{x}\right) \quad \Rightarrow \quad y' = x^{1/x} \dfrac{1 - \ln x}{x^2}$

33. $y = x^{\sin x} \quad \Rightarrow \quad \ln y = \sin x \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = \cos x \ln x + \dfrac{\sin x}{x} \quad \Rightarrow \quad y' = x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$

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34. $y = (\sin x)^x \Rightarrow \ln y = x \ln(\sin x) \Rightarrow y'/y = \ln(\sin x) + x(\cos x)/(\sin x) \Rightarrow$
$y' = (\sin x)^x[\ln(\sin x) + x \cot x]$

35. $y = (\ln x)^x \Rightarrow \ln y = x \ln \ln x \Rightarrow \dfrac{y'}{y} = \ln \ln x + x \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x} \Rightarrow y' = (\ln x)^x \left(\ln \ln x + \dfrac{1}{\ln x} \right)$

36. $y = x^{\ln x} \Rightarrow \ln y = \ln x \ln x = (\ln x)^2 \Rightarrow \dfrac{y'}{y} = 2 \ln x \left(\dfrac{1}{x} \right) \Rightarrow y' = x^{\ln x} \left(\dfrac{2 \ln x}{x} \right)$

37. $y = x^{e^x} \Rightarrow \ln y = e^x \ln x \Rightarrow \dfrac{y'}{y} = e^x \ln x + \dfrac{e^x}{x} \Rightarrow y' = x^{e^x} e^x \left(\ln x + \dfrac{1}{x} \right)$

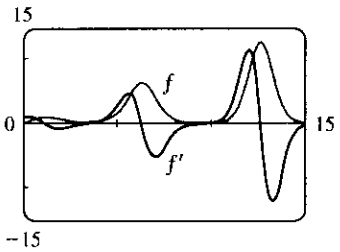
38. $y = (\ln x)^{\cos x} \Rightarrow \ln y = \cos x \ln(\ln x) \Rightarrow \dfrac{y'}{y} = \cos x \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x} + (\ln \ln x)(-\sin x) \Rightarrow$
$y' = (\ln x)^{\cos x} \left(\dfrac{\cos x}{x \ln x} - \sin x \ln \ln x \right)$

39. $y = 10^x \Rightarrow y' = 10^x \ln 10$, so at $(1, 10)$, the slope of the tangent line is $10^1 \ln 10 = 10 \ln 10$, and its equation is $y - 10 = 10 \ln 10(x - 1)$, or $y = (10 \ln 10)x + 10(1 - \ln 10)$.

40. $f(x) = x^{\cos x} = e^{\ln x \cos x} \Rightarrow$

$$f'(x) = e^{\ln x \cos x} \left[\ln x(-\sin x) + \cos x \left(\frac{1}{x} \right) \right]$$
$$= x^{\cos x} \left[\frac{\cos x}{x} - \sin x \ln x \right]$$

This is reasonable, because the graph shows that f increases when $f'(x)$ is positive.



41. $\displaystyle\int_1^2 10^t \, dt = \left[\frac{10^t}{\ln 10} \right]_1^2 = \frac{10^2}{\ln 10} - \frac{10^1}{\ln 10} = \frac{100 - 10}{\ln 10} = \frac{90}{\ln 10}$

42. Let $v = -2u$. Then $dv = -2\,du$ and

$$\int_0^1 4^{-2u} \, du = \int_0^{-2} 4^v \left(-\frac{1}{2} \right) dv = -\frac{1}{2} \left[\frac{4^v}{\ln 4} \right]_0^{-2} = -\frac{1}{2 \ln 4} \left(4^{-2} - 4^0 \right) = -\frac{1}{2 \ln 2^2} \left(\frac{1}{16} - 1 \right)$$
$$= -\frac{1}{4 \ln 2} \left(-\frac{15}{16} \right) = \frac{15}{64 \ln 2}$$

43. $\displaystyle\int \frac{\log_{10} x}{x} \, dx = \int \frac{(\ln x)/(\ln 10)}{x} \, dx = \frac{1}{\ln 10} \int \frac{\ln x}{x} \, dx$. Now put $u = \ln x$, so $du = \dfrac{1}{x} \, dx$, and the expression becomes $\dfrac{1}{\ln 10} \displaystyle\int u \, du = \frac{1}{\ln 10} \left(\tfrac{1}{2} u^2 + C_1 \right) = \frac{1}{2 \ln 10} (\ln x)^2 + C$.

Or: The substitution $u = \log_{10} x$ gives $du = \dfrac{dx}{x \ln 10}$ and we get $\displaystyle\int \frac{\log_{10} x}{x} \, dx = \tfrac{1}{2} \ln 10 (\log_{10} x)^2 + C$.

44. $\displaystyle\int \left(x^5 + 5^x \right) dx = \tfrac{1}{6} x^6 + \frac{1}{\ln 5} 5^x + C$

45. Let $u = \sin \theta$. Then $du = \cos \theta \, d\theta$ and $\displaystyle\int 3^{\sin \theta} \cos \theta \, d\theta = \int 3^u du = \frac{3^u}{\ln 3} + C = \frac{1}{\ln 3} 3^{\sin \theta} + C$.

46. Let $u = 2^x + 1$. Then $du = 2^x \ln 2 \, dx$, so $\displaystyle\int \frac{2^x}{2^x + 1} \, dx = \int \frac{1}{u} \frac{du}{\ln 2} = \frac{1}{\ln 2} \ln |u| + C = \frac{1}{\ln 2} \ln(2^x + 1) + C$.

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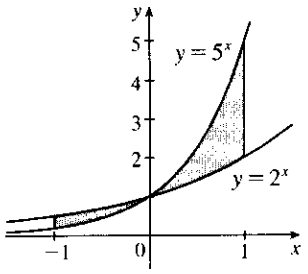
47. $A = \int_{-1}^{0} (2^x - 5^x)\,dx + \int_{0}^{1} (5^x - 2^x)\,dx$

$= \left[\dfrac{2^x}{\ln 2} - \dfrac{5^x}{\ln 5}\right]_{-1}^{0} + \left[\dfrac{5^x}{\ln 5} - \dfrac{2^x}{\ln 2}\right]_{0}^{1}$

$= \left(\dfrac{1}{\ln 2} - \dfrac{1}{\ln 5}\right) - \left(\dfrac{1/2}{\ln 2} - \dfrac{1/5}{\ln 5}\right) + \left(\dfrac{5}{\ln 5} - \dfrac{2}{\ln 2}\right) - \left(\dfrac{1}{\ln 5} - \dfrac{1}{\ln 2}\right)$

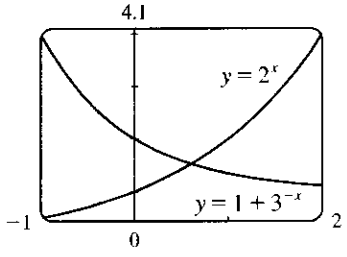
$= \dfrac{16}{5\ln 5} - \dfrac{1}{2\ln 2}$



48. Using disks, the volume is $V = \int_0^1 \pi \left[10^{-x}\right]^2 dx = \pi \int_0^1 10^{-2x}\,dx$. To evaluate the integral, we let $u = -2x \;\Rightarrow\; du = -2\,dx$, $x = 0 \;\Rightarrow\; u = 0$, and $x = 1 \;\Rightarrow\; u = -2$, so we have

$$V = -\frac{\pi}{2}\int_0^{-2} 10^u\,du = -\frac{\pi}{2}\left[\frac{1}{\ln 10}10^u\right]_0^{-2} = -\frac{\pi}{2\ln 10}\left(10^{-2} - 1\right) = \frac{99\pi}{200\ln 10}$$

49. We see that the graphs of $y = 2^x$ and $y = 1 + 3^{-x}$ intersect at $x \approx 0.6$. We let $f(x) = 2^x - 1 - 3^{-x}$ and calculate $f'(x) = 2^x \ln 2 + 3^{-x}\ln 3$, and using the formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$ (Newton's Method), we get $x_1 = 0.6$, $x_2 \approx x_3 \approx 0.600967$. So, correct to six decimal places, the root occurs at $x = 0.600967$.



50. $x^y = y^x \;\Rightarrow\; y\ln x = x\ln y \;\Rightarrow\; y\cdot\dfrac{1}{x} + (\ln x)\cdot y' = x\cdot\dfrac{1}{y}\cdot y' + \ln y \;\Rightarrow\; y'\ln x - \dfrac{x}{y}y' = \ln y - \dfrac{y}{x} \;\Rightarrow\; y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

51. $y = \dfrac{10^x}{10^x + 1} \;\Leftrightarrow\; (10^x + 1)y = 10^x \;\Leftrightarrow\; 10^x\cdot y + y = 10^x \;\Leftrightarrow\; y = 10^x - 10^x y \;\Leftrightarrow$

$y = 10^x(1 - y) \;\Leftrightarrow\; 10^x = \dfrac{y}{1-y} \;\Leftrightarrow\; \log_{10} 10^x = \log_{10}\left(\dfrac{y}{1-y}\right) \;\Leftrightarrow\; x = \log_{10} y - \log_{10}(1-y)$.

Interchange x and y: $y = \log_{10} x - \log_{10}(1-x)$ is the inverse function.

52. $\lim\limits_{x\to 0^+} x^{-\ln x} = \lim\limits_{x\to 0^+} \left(e^{\ln x}\right)^{-\ln x} = \lim\limits_{x\to 0^+} e^{-(\ln x)^2} = 0$ since $-(\ln x)^2 \to -\infty$ as $x \to 0^+$.

53. If I is the intensity of the 1989 San Francisco earthquake, then $\log_{10}(I/S) = 7.1 \;\Rightarrow$ $\log_{10}(16I/S) = \log_{10} 16 + \log_{10}(I/S) = \log_{10} 16 + 7.1 \approx 8.3$.

54. Let I_1 and I_2 be the intensities of the music and the mower. Then $10\log_{10}\left(\dfrac{I_1}{I_0}\right) = 120$ and $10\log_{10}\left(\dfrac{I_2}{I_0}\right) = 106$, so $\log_{10}\left(\dfrac{I_1}{I_2}\right) = \log_{10}\left(\dfrac{I_1/I_0}{I_2/I_0}\right) = \log_{10}\left(\dfrac{I_1}{I_0}\right) - \log_{10}\left(\dfrac{I_2}{I_0}\right) = 12 - 10.6 = 1.4 \;\Rightarrow\; \dfrac{I_1}{I_2} = 10^{1.4} \approx 25$.

55. We find I with the loudness formula from Exercise 55, substituting $I_0 = 10^{-12}$ and $L = 50$:

$50 = 10\log_{10}\dfrac{I}{10^{-12}} \;\Leftrightarrow\; 5 = \log_{10}\dfrac{I}{10^{-12}} \;\Leftrightarrow\; 10^5 = \dfrac{I}{10^{-12}} \;\Leftrightarrow\; I = 10^{-7}$ watt/m^2. Now we differentiate L with respect to I: $L = 10\log_{10}\dfrac{I}{I_0} \;\Rightarrow\; \dfrac{dL}{dI} = 10\dfrac{1}{(I/I_0)\ln 10}\left(\dfrac{1}{I_0}\right) = \dfrac{10}{\ln 10}\left(\dfrac{1}{I}\right)$.

Substituting $I = 10^{-7}$, we get $L'(50) = \dfrac{10}{\ln 10}\left(\dfrac{1}{10^{-7}}\right) = \dfrac{10^8}{\ln 10} \approx 4.34 \times 10^7\ \dfrac{\text{dB}}{\text{watt/m}^2}$.

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56. (a) $I(x) = I_0 a^x \quad \Rightarrow \quad I'(x) = I_0(\ln a)a^x = (I_0 a^x)\ln a = I(x)\ln a$

(b) We substitute $I_0 = 8$, $a = 0.38$ and $x = 20$ into the first expression for $I'(x)$ above: $I'(20) = 8(\ln 0.38)(0.38)^{20} \approx -3.05 \times 10^{-8}$.

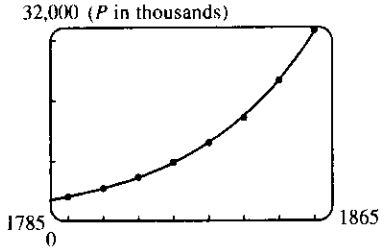
(c) The average value of the function $I(x)$ between $x = 0$ and $x = 20$ is

$$\frac{\int_0^{20} I(x)\,dx}{20-0} = \frac{1}{20}\int_0^{20} 8(0.38)^x\,dx = \frac{2}{5}\left[\frac{(0.38)^x}{\ln 0.38}\right]_0^{20} = \frac{2(0.38^{20}-1)}{5\ln 0.38} \approx 0.41.$$

57. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a = 100.0124369$ and $b = 0.000045145933$. We can change this model to one with base e and exponent $\ln b$ [$b^t = e^{t\ln b}$ from precalculus mathematics or from Section 7.3]: $Q = ae^{t\ln b} = 100.012437e^{-10.005531t}$.

(b) Use $Q'(t) = ab^t \ln b$ or the calculator command `nDeriv(Y1,X,.04)` with $Y_1 = ab^x$ to get $Q'(0.04) \approx -670.63\ \mu$A. The result of Example 2 in Section 2.1 was $-670\ \mu$A.

58. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308-3929}{1800-1790} = 137.9$, $m_2 = \dfrac{7240-5308}{1810-1800} = 193.2$.

So $P'(1800) \approx (m_1+m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192-17{,}063}{1850-1840} = 612.9$, $m_2 = \dfrac{31{,}443-23{,}192}{1860-1850} = 825.1$.

So $P'(1850) \approx (m_1+m_2)/2 = 719$ thousand people/year.

(c) Use the calculator command `nDeriv(Y1,X,year)` with $Y_1 = ab^x$ to get $P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

59. Using Definition 1 and the second law of exponents for e^x, we have

$$a^{x-y} = e^{(x-y)\ln a} = e^{x\ln a - y\ln a} = \frac{e^{x\ln a}}{e^{y\ln a}} = \frac{a^x}{a^y}.$$

60. Using Definition 1, the first law of logarithms, and the first law of exponents for e^x, we have $(ab)^x = e^{x\ln(ab)} = e^{x(\ln a+\ln b)} = e^{x\ln a + x\ln b} = e^{x\ln a}e^{x\ln b} = a^x b^x$.

61. Let $\log_a x = r$ and $\log_a y = s$. Then $a^r = x$ and $a^s = y$.

(a) $xy = a^r a^s = a^{r+s} \quad \Rightarrow \quad \log_a(xy) = r+s = \log_a x + \log_a y$

(b) $\dfrac{x}{y} = \dfrac{a^r}{a^s} = a^{r-s} \quad \Rightarrow \quad \log_a \dfrac{x}{y} = r-s = \log_a x - \log_a y$

(c) $x^y = (a^r)^y = a^{ry} \quad \Rightarrow \quad \log_a(x^y) = ry = y\log_a x$

62. Let $m = n/x$. Then $n = xm$, and as $n \to \infty$, $m \to \infty$.

Therefore, $\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n = \lim\limits_{m\to\infty}\left(1+\dfrac{1}{m}\right)^{mx} = \left[\lim\limits_{m\to\infty}\left(1+\dfrac{1}{m}\right)^m\right]^x = e^x$ by Equation 9.

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7.5 Inverse Trigonometric Functions

1. (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\cos^{-1}(-1) = \pi$ since $\cos\pi = -1$ and π is in $[0, \pi]$.

2. (a) $\arctan(-1) = -\frac{\pi}{4}$ since $\tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\csc^{-1} 2 = \frac{\pi}{6}$ since $\csc\frac{\pi}{6} = 2$ and $\frac{\pi}{6}$ is in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

3. (a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ since $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

4. (a) $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$ since $\sec\frac{\pi}{4} = \sqrt{2}$ and $\frac{\pi}{4}$ is in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

(b) $\arcsin 1 = \frac{\pi}{2}$ since $\sin\frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

5. (a) $\arccos(\cos 2\pi) = \arccos(1) = 0$

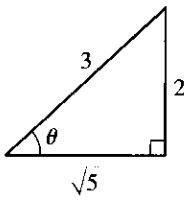
(b) $\tan(\tan^{-1} 5) = 5$

6. (a) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

(b) $\cos\left(\arcsin\frac{1}{2}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

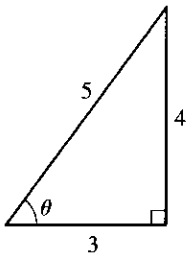
7. Let $\theta = \sin^{-1}\left(\frac{2}{3}\right)$.

Then $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \tan\theta = \frac{2}{\sqrt{5}}$.



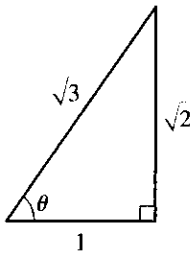
8. Let $\theta = \arccos\frac{3}{5}$.

Then $\csc\left(\arccos\left(\frac{3}{5}\right)\right) = \csc\theta = \frac{5}{4}$.



9. Let $\theta = \tan^{-1}\sqrt{2}$. Then

$$\sin\left(2\tan^{-1}\sqrt{2}\right) = \sin(2\theta) = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{2}}{3}$$



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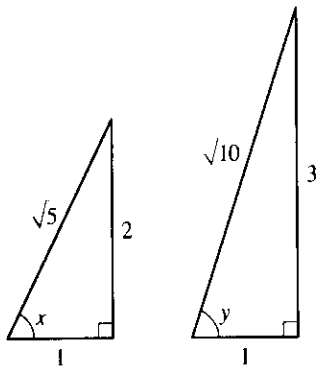
10. Let $x = \tan^{-1} 2$ and $y = \tan^{-1} 3$. Then

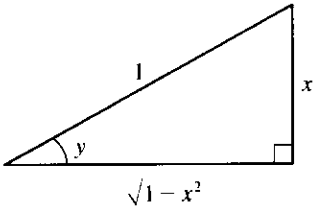
$$\begin{aligned}\cos\left(\tan^{-1} 2 + \tan^{-1} 3\right) &= \cos(x+y) = \cos x \cos y - \sin x \sin y \\ &= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\frac{3}{\sqrt{10}} \\ &= \frac{-5}{\sqrt{50}} = \frac{-5}{5\sqrt{2}} = \frac{-1}{\sqrt{2}}\end{aligned}$$



11. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad \Rightarrow \quad \cos y \ge 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

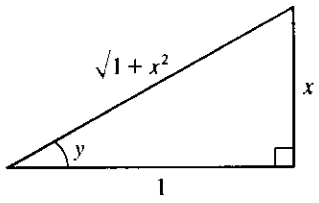
12. Let $y = \sin^{-1} x$. Then $\sin y = x$, so from the triangle we see that

$$\tan(\sin^{-1} x) = \tan y = \frac{x}{\sqrt{1-x^2}}.$$



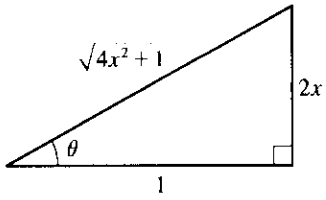
13. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that

$$\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1+x^2}}.$$

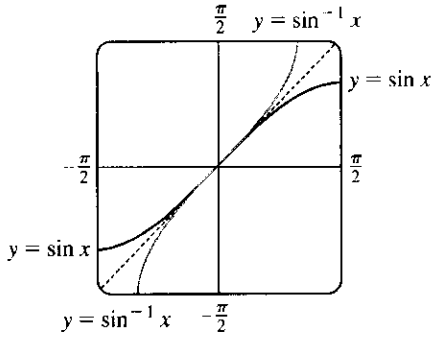


14. Let $\theta = \arctan 2x$. Then $\tan\theta = 2x$, so from the diagram we see that

$$\csc(\arctan 2x) = \csc\theta = \frac{\sqrt{4x^2+1}}{2x}.$$

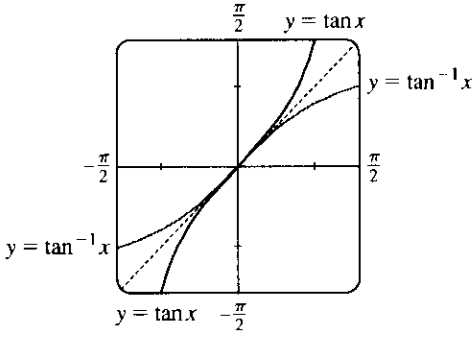


15.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

16.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y = x$.

17. Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \le y \le \pi \quad \Rightarrow \quad -\sin y \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}.$$ [Note that $\sin y \ge 0$ for $0 \le y \le \pi$.]

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18. (a) Let $a = \sin^{-1} x$ and $b = \cos^{-1} x$. Then $\cos a = \sqrt{1 - \sin^2 a} = \sqrt{1 - x^2}$ since $\cos a \geq 0$ for $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$. Similarly, $\sin b = \sqrt{1 - x^2}$. So

$$\sin(\sin^{-1} x + \cos^{-1} x) = \sin(a + b) = \sin a \cos b + \cos a \sin b = x \cdot x + \sqrt{1 - x^2}\,\sqrt{1 - x^2}$$
$$= x^2 + (1 - x^2) = 1$$

But $-\frac{\pi}{2} \leq \sin^{-1} x + \cos^{-1} x \leq \frac{3\pi}{2}$, and so $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

(b) We differentiate $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ with respect to x, and get

$$\frac{1}{\sqrt{1 - x^2}} + \frac{d}{dx}\cos^{-1} x = 0 \quad \Rightarrow \quad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}.$$

19. Let $y = \cot^{-1} x$. Then $\cot y = x \quad \Rightarrow \quad -\csc^2 y \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{1}{\csc^2 y} = -\dfrac{1}{1 + \cot^2 y} = -\dfrac{1}{1 + x^2}$.

20. Let $y = \sec^{-1} x$. Then $\sec y = x$ and $y \in \left(0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$. Differentiate with respect to x:

$\sec y \tan y \left(\dfrac{dy}{dx}\right) = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\sec y \tan y} = \dfrac{1}{\sec y \sqrt{\sec^2 y - 1}} = \dfrac{1}{x\sqrt{x^2 - 1}}$. Note that

$\tan^2 y = \sec^2 y - 1 \quad \Rightarrow \quad \tan y = \sqrt{\sec^2 y - 1}$ since $\tan y > 0$ when $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$.

21. Let $y = \csc^{-1} x$. Then $\csc y = x \quad \Rightarrow \quad -\csc y \cot y \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\csc y \cot y} = -\dfrac{1}{\csc y \sqrt{\csc^2 y - 1}} = -\dfrac{1}{x\sqrt{x^2 - 1}}$. Note that $\cot y \geq 0$ on the domain of $\csc^{-1} x$.

22. $y = \sqrt{\tan^{-1} x} = (\tan^{-1} x)^{1/2} \quad \Rightarrow$

$$y' = \tfrac{1}{2}(\tan^{-1} x)^{-1/2} \cdot \frac{d}{dx}(\tan^{-1} x) = \frac{1}{2\sqrt{\tan^{-1} x}} \cdot \frac{1}{1 + x^2} = \frac{1}{2\sqrt{\tan^{-1} x}\,(1 + x^2)}$$

23. $y = \tan^{-1}\sqrt{x} \quad \Rightarrow \quad y' = \dfrac{1}{1 + (\sqrt{x})^2} \cdot \dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{1 + x}\left(\tfrac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x}\,(1 + x)}$

24. $h(x) = \sqrt{1 - x^2}\arcsin x \quad \Rightarrow$

$$h'(x) = \sqrt{1 - x^2} \cdot \frac{1}{\sqrt{1 - x^2}} + \arcsin x\left[\tfrac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] = 1 - \frac{x \arcsin x}{\sqrt{1 - x^2}}$$

25. $y = \sin^{-1}(2x + 1) \quad \Rightarrow$

$$y' = \frac{1}{\sqrt{1 - (2x + 1)^2}} \cdot \frac{d}{dx}(2x + 1) = \frac{1}{\sqrt{1 - (4x^2 + 4x + 1)}} \cdot 2 = \frac{2}{\sqrt{-4x^2 - 4x}} = \frac{1}{\sqrt{-x^2 - x}}$$

26. $f(x) = x \ln(\arctan x) \quad \Rightarrow$

$$f'(x) = x \cdot \frac{1}{\arctan x} \cdot \frac{1}{1 + x^2} + \ln(\arctan x) \cdot 1 = \frac{x}{(1 + x^2)\arctan x} + \ln(\arctan x)$$

27. $H(x) = (1 + x^2)\arctan x \quad \Rightarrow \quad H'(x) = (1 + x^2)\dfrac{1}{1 + x^2} + (\arctan x)(2x) = 1 + 2x \arctan x$

28. $h(t) = e^{\sec^{-1} t} \quad \Rightarrow \quad h'(t) = e^{\sec^{-1} t}\dfrac{d}{dt}\left(\sec^{-1} t\right) = \dfrac{e^{\sec^{-1} t}}{t\sqrt{t^2 - 1}}$

29. $y = \cos^{-1}(e^{2x}) \quad \Rightarrow \quad y' = -\dfrac{1}{\sqrt{1 - (e^{2x})^2}} \cdot \dfrac{d}{dx}\left(e^{2x}\right) = -\dfrac{2e^{2x}}{\sqrt{1 - e^{4x}}}$

30. $y = x\cos^{-1} x - \sqrt{1 - x^2} \quad \Rightarrow \quad y' = \cos^{-1} x - \dfrac{x}{\sqrt{1 - x^2}} + \dfrac{x}{\sqrt{1 - x^2}} = \cos^{-1} x$

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31. $y = \arctan(\cos\theta) \quad \Rightarrow \quad y' = \dfrac{1}{1+(\cos\theta)^2}(-\sin\theta) = -\dfrac{\sin\theta}{1+\cos^2\theta}$

32. $y = \tan^{-1}\left(x - \sqrt{x^2+1}\,\right) \quad \Rightarrow$

$$y' = \frac{1}{1+\left(x-\sqrt{x^2+1}\,\right)^2}\left(1 - \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{1+x^2-2x\sqrt{x^2+1}+x^2+1}\left(\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1}-x}{2\left(1+x^2-x\sqrt{x^2+1}\,\right)\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}-x}{2\left[\sqrt{x^2+1}\,(1+x^2)-x(x^2+1)\right]}$$

$$= \frac{\sqrt{x^2+1}-x}{2\left[(1+x^2)\left(\sqrt{x^2+1}-x\right)\right]} = \frac{1}{2(1+x^2)}$$

33. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t) \quad \Rightarrow$

$$h'(t) = -\frac{1}{1+t^2} - \frac{1}{1+(1/t)^2}\cdot\frac{d}{dt}\frac{1}{t} = -\frac{1}{1+t^2} - \frac{t^2}{t^2+1}\cdot\left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2} + \frac{1}{t^2+1} = 0.$$

Note that this makes sense because $h(t) = \dfrac{\pi}{2}$ for $t > 0$ and $h(t) = -\dfrac{\pi}{2}$ for $t < 0$.

34. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x-a}{x+a}} = \tan^{-1}\left(\dfrac{x}{a}\right) + \frac{1}{2}\ln(x-a) - \frac{1}{2}\ln(x+a) \quad \Rightarrow$

$$y' = \frac{a}{x^2+a^2} + \frac{1/2}{x-a} - \frac{1/2}{x+a} = \frac{a}{x^2+a^2} + \frac{a}{x^2-a^2} = \frac{2ax^2}{x^4-a^4}$$

35. $y = \arccos\left(\dfrac{b+a\cos x}{a+b\cos x}\right) \quad \Rightarrow$

$$y' = -\frac{1}{\sqrt{1-\left(\dfrac{b+a\cos x}{a+b\cos x}\right)^2}}\,\frac{(a+b\cos x)(-a\sin x)-(b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{1}{\sqrt{a^2+b^2\cos^2 x - b^2 - a^2\cos^2 x}}\,\frac{(a^2-b^2)\sin x}{|a+b\cos x|}$$

$$= \frac{1}{\sqrt{a^2-b^2}\sqrt{1-\cos^2 x}}\,\frac{(a^2-b^2)\sin x}{|a+b\cos x|} = \frac{\sqrt{a^2-b^2}}{|a+b\cos x|}\,\frac{\sin x}{|\sin x|}$$

But $0 \le x \le \pi$, so $|\sin x| = \sin x$. Also $a > b > 0 \quad \Rightarrow \quad b\cos x \ge -b > -a$, so $a + b\cos x > 0$.

Thus $y' = \dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$.

36. $f(x) = \arcsin(e^x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{\sqrt{1-(e^x)^2}}\cdot e^x = \dfrac{e^x}{\sqrt{1-e^{2x}}}$.

$\text{Domain}(f) = \{x \mid -1 \le e^x \le 1\} = \{x \mid 0 < e^x \le 1\} = (-\infty, 0]$.

$\text{Domain}(f') = \{x \mid 1 - e^{2x} > 0\} = \{x \mid e^{2x} < 1\} = \{x \mid 2x < 0\} = (-\infty, 0)$.

37. $g(x) = \cos^{-1}(3-2x) \quad \Rightarrow \quad g'(x) = -\dfrac{1}{\sqrt{1-(3-2x)^2}}(-2) = \dfrac{2}{\sqrt{1-(3-2x)^2}}$.

$\text{Domain}(g) = \{x \mid -1 \le 3-2x \le 1\} = \{x \mid -4 \le -2x \le -2\} = \{x \mid 2 \ge x \ge 1\} = [1, 2]$.

$\text{Domain}(g') = \{x \mid 1-(3-2x)^2 > 0\} = \{x \mid (3-2x)^2 < 1\} = \{x \mid |3-2x| < 1\}$

$= \{x \mid -1 < 3-2x < 1\} = \{x \mid -4 < -2x < -2\} = \{x \mid 2 > x > 1\} = (1, 2)$

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38. $\tan^{-1}(xy) = 1 + x^2y \;\Rightarrow\; \dfrac{1}{1+x^2y^2}(xy' + y\cdot 1) = 0 + x^2y' + 2xy \;\Rightarrow$

$$y'\left(\frac{x}{1+x^2y^2} - x^2\right) = 2xy - \frac{y}{1+x^2y^2} \;\Rightarrow$$

$$y' = \frac{2xy - \dfrac{y}{1+x^2y^2}}{\dfrac{x}{1+x^2y^2} - x^2} = \frac{2xy(1+x^2y^2) - y}{x - x^2(1+x^2y^2)} = \frac{y(-1-2x-2x^3y^2)}{x(1-x-x^3y^2)}.$$

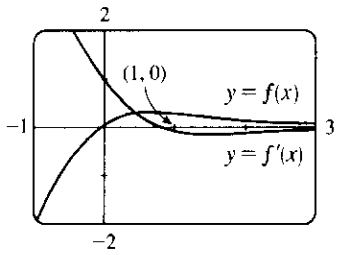
39. $g(x) = x\sin^{-1}\left(\dfrac{x}{4}\right) + \sqrt{16-x^2} \;\Rightarrow\; g'(x) = \sin^{-1}\left(\dfrac{x}{4}\right) + \dfrac{x}{4\sqrt{1-(x/4)^2}} - \dfrac{x}{\sqrt{16-x^2}} = \sin^{-1}\left(\dfrac{x}{4}\right) \;\Rightarrow$

$g'(2) = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

40. $y = 3\arccos\dfrac{x}{2} \;\Rightarrow\; y' = 3\left[-\dfrac{1}{\sqrt{1-(x/2)^2}}\right]\left(\dfrac{1}{2}\right)$, so at $(1,\pi)$, $y' = -\dfrac{3}{2\sqrt{1-\frac{1}{4}}} = -\sqrt{3}$. An equation of the tangent line is $y - \pi = -\sqrt{3}\,(x-1)$, or $y = -\sqrt{3}\,x + \pi + \sqrt{3}$.

41. $f(x) = e^{-x}\arctan x \;\Rightarrow$

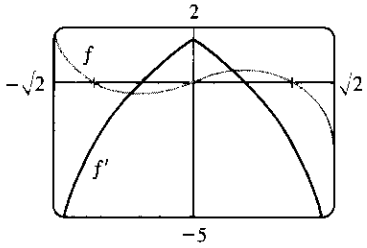
$f'(x) = \dfrac{e^{-x}}{1+x^2} - e^{-x}\arctan x$. The answer is reasonable since f' is positive where f is increasing and f' is negative where f is decreasing.



42. $f(x) = x\arcsin(1-x^2) \;\Rightarrow$

$$f'(x) = x\left[\frac{-2x}{\sqrt{1-(1-x^2)^2}}\right] + \arcsin\left(1-x^2\right)\cdot 1$$

$$= \arcsin\left(1-x^2\right) - \frac{2x^2}{\sqrt{2x^2-x^4}}$$



This is reasonable because the graphs show that f is increasing when f' is positive, and that f has an inflection point when f' changes from increasing to decreasing.

43. $\lim\limits_{x\to -1^+} \sin^{-1} x = \sin^{-1}(-1) = -\frac{\pi}{2}$

44. Let $t = \dfrac{1+x^2}{1+2x^2}$. As $x \to \infty$, $t = \dfrac{1+x^2}{1+2x^2} = \dfrac{1/x^2+1}{1/x^2+2} \to \dfrac{1}{2}$.

$\lim\limits_{x\to\infty} \arccos\left(\dfrac{1+x^2}{1+2x^2}\right) = \lim\limits_{t\to 1/2} \arccos t = \arccos\frac{1}{2} = \frac{\pi}{3}$.

45. Let $t = e^x$. As $x \to \infty$, $t \to \infty$. $\lim\limits_{x\to\infty} \arctan(e^x) = \lim\limits_{t\to\infty} \arctan t = \frac{\pi}{2}$ by (8).

46. Let $t = \ln x$. As $x \to 0^+$, $t \to -\infty$. $\lim\limits_{x\to 0^+} \tan^{-1}(\ln x) = \lim\limits_{t\to -\infty} \tan^{-1} t = -\frac{\pi}{2}$ by (8).

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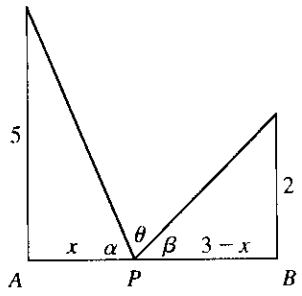
47.



From the figure, $\tan\alpha = \dfrac{5}{x}$ and $\tan\beta = \dfrac{2}{3-x}$. Since

$\alpha + \beta + \theta = 180° = \pi$, $\theta = \pi - \tan^{-1}\left(\dfrac{5}{x}\right) - \tan^{-1}\left(\dfrac{2}{3-x}\right) \Rightarrow$

$$\frac{d\theta}{dx} = -\frac{1}{1+\left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right) - \frac{1}{1+\left(\frac{2}{3-x}\right)^2}\left[\frac{2}{(3-x)^2}\right]$$

$$= \frac{x^2}{x^2+25}\cdot\frac{5}{x^2} - \frac{(3-x)^2}{(3-x)^2+4}\cdot\frac{2}{(3-x)^2}.$$

Now $\dfrac{d\theta}{dx} = 0 \quad\Rightarrow\quad \dfrac{5}{x^2+25} = \dfrac{2}{x^2-6x+13} \quad\Rightarrow\quad 2x^2+50 = 5x^2-30x+65 \quad\Rightarrow$

$3x^2 - 30x + 15 = 0 \quad\Rightarrow\quad x^2 - 10x + 5 = 0 \quad\Rightarrow\quad x = 5 \pm 2\sqrt{5}$. We reject the root with the $+$ sign, since it is larger than 3. $d\theta/dx > 0$ for $x < 5 - 2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5 - 2\sqrt{5}$, so θ is maximized when $|AP| = x = 5 - 2\sqrt{5} \approx 0.53$.

48. Let x be the distance from the observer to the wall. Then, from the given figure,

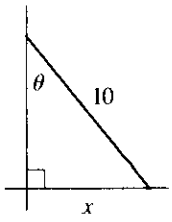
$$\theta = \tan^{-1}\left(\frac{h+d}{x}\right) - \tan^{-1}\left(\frac{d}{x}\right),\ x > 0 \quad\Rightarrow$$

$$\frac{d\theta}{dx} = \frac{1}{1+[(h+d)/x]^2}\left[-\frac{h+d}{x^2}\right] - \frac{1}{1+(d/x)^2}\left[-\frac{d}{x^2}\right] = -\frac{h+d}{x^2+(h+d)^2} + \frac{d}{x^2+d^2}$$

$$= \frac{d[x^2+(h+d)^2] - (h+d)(x^2+d^2)}{[x^2+(h+d)^2](x^2+d^2)} = \frac{h^2d + hd^2 - hx^2}{[x^2+(h+d)^2](x^2+d^2)} = 0 \quad\Leftrightarrow$$

$hx^2 = h^2d + hd^2 \quad\Leftrightarrow\quad x^2 = hd + d^2 \quad\Leftrightarrow\quad x = \sqrt{d(h+d)}$. Since $d\theta/dx > 0$ for all $x < \sqrt{d(h+d)}$ and $d\theta/dx < 0$ for all $x > \sqrt{d(h+d)}$, the absolute maximum occurs when $x = \sqrt{d(h+d)}$.

49.

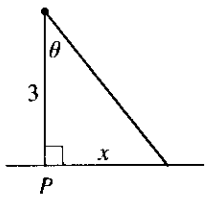


$\dfrac{dx}{dt} = 2$ ft/s, $\sin\theta = \dfrac{x}{10} \quad\Rightarrow\quad \theta = \sin^{-1}\left(\dfrac{x}{10}\right)$, $\dfrac{d\theta}{dx} = \dfrac{1/10}{\sqrt{1-(x/10)^2}}$,

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt} = \frac{1/10}{\sqrt{1-(x/10)^2}}\,(2) \text{ rad/s},$$

$$\left.\frac{d\theta}{dt}\right]_{x=6} = \frac{2/10}{\sqrt{1-(6/10)^2}} \text{ rad/s} = \tfrac{1}{4} \text{ rad/s}$$

50.



$\dfrac{d\theta}{dt} = 4$ rev/min $= 8\pi \cdot 60$ rad/h. From the diagram, we see that $\tan\theta = \dfrac{x}{3}$

$\Rightarrow\quad \theta = \tan^{-1}\left(\dfrac{x}{3}\right)$. Thus, $8\pi \cdot 60 = \dfrac{d\theta}{dt} = \dfrac{d\theta}{dx}\dfrac{dx}{dt} = \dfrac{1/3}{1+(x/3)^2}\dfrac{dx}{dt}$.

So $\dfrac{dx}{dt} = 8\pi \cdot 60 \cdot 3\left[1 + \left(\dfrac{x}{3}\right)^2\right]$ km/h, and at $x = 1$,

$\dfrac{dx}{dt} = 8\pi \cdot 60 \cdot 3\left[1 + \frac{1}{9}\right]$ km/h $= 1600\pi$ km/h.

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51. $y = f(x) = \sin^{-1}(x/(x+1))$ **A.** $D = \{x \mid -1 \le x/(x+1) \le 1\}$. For $x > -1$ we have $-x-1 \le x \le x+1 \Leftrightarrow 2x \ge -1 \Leftrightarrow x \ge -\frac{1}{2}$, so $D = \left[-\frac{1}{2}, \infty\right)$. **B.** Intercepts are 0 **C.** No symmetry **D.** $\lim_{x\to\infty} \sin^{-1}\left(\dfrac{x}{x+1}\right) = \lim_{x\to\infty} \sin^{-1}\left(\dfrac{1}{1+1/x}\right) = \sin^{-1} 1 = \frac{\pi}{2}$, so $y = \frac{\pi}{2}$ is a HA.

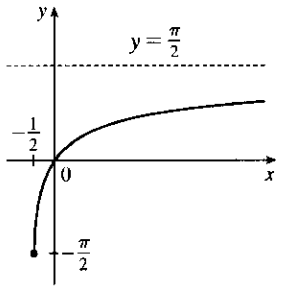
E. $f'(x) = \dfrac{1}{\sqrt{1-[x/(x+1)]^2}} \dfrac{(x+1)-x}{(x+1)^2} = \dfrac{1}{(x+1)\sqrt{2x+1}} > 0$,

so f is increasing on $\left(-\frac{1}{2}, \infty\right)$. **F.** No local maximum or minimum, $f\left(-\frac{1}{2}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$ is an absolute minimum

G. $f''(x) = -\dfrac{\sqrt{2x+1} + (x+1)/\sqrt{2x+1}}{(x+1)^2(2x+1)}$

$= -\dfrac{3x+2}{(x+1)^2(2x+1)^{3/2}} < 0$ on D, so f is CD on $\left(-\frac{1}{2}, \infty\right)$.

H.



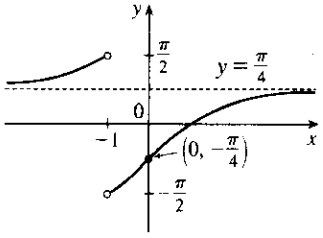
52. $y = f(x) = \tan^{-1}\left(\dfrac{x-1}{x+1}\right)$ **A.** $D = \{x \mid x \ne -1\}$

B. x-intercept $= 1$, y-intercept $= f(0) = \tan^{-1}(-1) = -\frac{\pi}{4}$ **C.** No symmetry

D. $\lim_{x\to\pm\infty} \tan^{-1}\left(\dfrac{x-1}{x+1}\right) = \lim_{x\to\pm\infty} \tan^{-1}\left(\dfrac{1-1/x}{1+1/x}\right) = \tan^{-1} 1 = \frac{\pi}{4}$, so $y = \frac{\pi}{4}$ is a HA. Also

$\lim_{x\to-1^+} \tan^{-1}\left(\dfrac{x-1}{x+1}\right) = -\dfrac{\pi}{2}$ and $\lim_{x\to-1^-} \tan^{-1}\left(\dfrac{x-1}{x+1}\right) = \dfrac{\pi}{2}$.

H.



E. $f'(x) = \dfrac{1}{1+[(x-1)/(x+1)]^2} \dfrac{(x+1)-(x-1)}{(x+1)^2}$

$= \dfrac{2}{(x+1)^2+(x-1)^2} = \dfrac{1}{x^2+1} > 0$

so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = -2x/(x^2+1)^2 > 0 \Leftrightarrow x < 0$, so f is CU on $(-\infty, -1)$ and $(-1, 0)$, and CD on $(0, \infty)$. IP at $\left(0, -\frac{\pi}{4}\right)$

53. $y = f(x) = x - \tan^{-1} x$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim_{x\to\infty} (x - \tan^{-1} x) = \infty$ and $\lim_{x\to-\infty} (x - \tan^{-1} x) = -\infty$, no HA.

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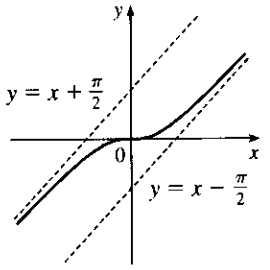
But $f(x) - \left(x - \frac{\pi}{2}\right) = -\tan^{-1} x + \frac{\pi}{2} \to 0$ as $x \to \infty$, and $f(x) - \left(x + \frac{\pi}{2}\right) = -\tan^{-1} x - \frac{\pi}{2} \to 0$ as $x \to -\infty$, so $y = x \pm \frac{\pi}{2}$ are slant asymptotes. **E.** $f'(x) = 1 - \dfrac{1}{x^2+1} = \dfrac{x^2}{x^2+1} > 0$, so f is increasing on $\mathbb{R}$. **F.** No extrema

G. $f''(x) = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2} > 0 \quad \Leftrightarrow \quad x > 0$, so f is CU on $(0, \infty)$, CD on $(-\infty, 0)$. IP at $(0, 0)$.

H.



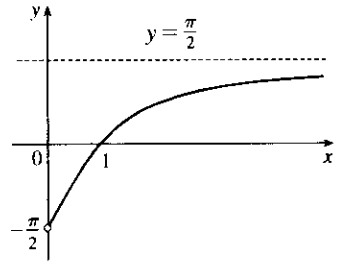
54. $y = \tan^{-1}(\ln x)$ **A.** $D = (0, \infty)$ **B.** No y-intercept, x-intercept when $\tan^{-1}(\ln x) = 0 \quad \Leftrightarrow \quad \ln x = 0 \quad \Leftrightarrow \quad x = 1$. **C.** No symmetry **D.** $\lim_{x \to \infty} \tan^{-1}(\ln x) = \frac{\pi}{2}$, so $y = \frac{\pi}{2}$ is a HA. Also $\lim_{x \to 0^+} \tan^{-1}(\ln x) = -\frac{\pi}{2}$.

E. $f'(x) = \dfrac{1}{x\left[1 + (\ln x)^2\right]} > 0$, so f is increasing on $(0, \infty)$.

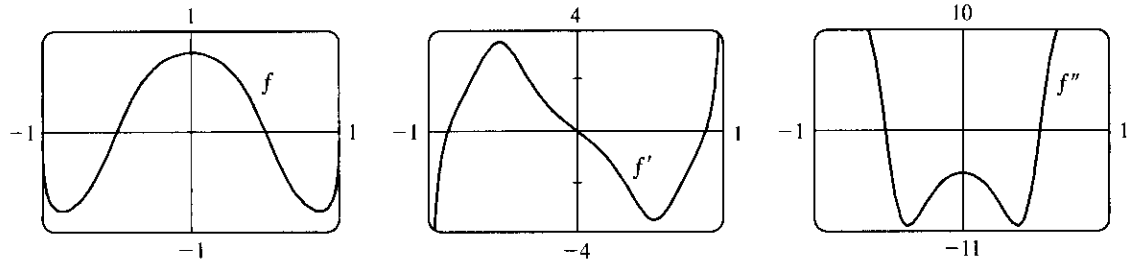
F. No maximum or minimum

G. $f''(x) = \dfrac{-\left[1 + (\ln x)^2 + x(2 \ln x / x)\right]}{x^2\left[1 + (\ln x)^2\right]^2} = -\dfrac{(1 + \ln x)^2}{x^2[1 + (\ln x)^2]^2} < 0$, so f is CD on $(0, \infty)$.

H.



55. $f(x) = \arctan(\cos(3 \arcsin x))$. We use a CAS to compute f' and f'', and to graph f, f', and f'':



From the graph of f', it appears that the only maximum occurs at $x = 0$ and there are minima at $x = \pm 0.87$. From the graph of f'', it appears that there are inflection points at $x = \pm 0.52$.

56. First note that the function $f(x) = x - c \sin^{-1} x$ is only defined on the interval $[-1, 1]$, since $\sin^{-1}$ is only defined on that interval. We differentiate to get $f'(x) = 1 - c/\sqrt{1 - x^2}$. Now if $c \leq 0$, then $f'(x) \geq 1$, so there is no

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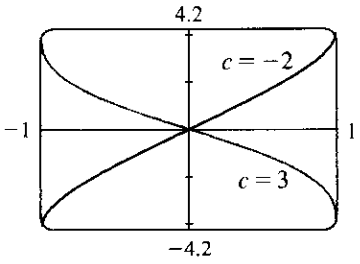
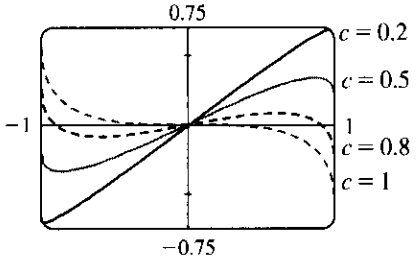
extremum and f is increasing on its domain. If $c > 1$, then $f'(x) < 0$, so there is no local extremum and f is decreasing on its domain, and if $c = 1$, then there is still no extremum, since $f'(x)$ does not change sign at $x = 0$. So we can only have local extrema if $0 < c < 1$. In this case, f is increasing where $f'(x) > 0 \quad \Leftrightarrow \quad \sqrt{1-x^2} > c \quad \Leftrightarrow \quad |x| < \sqrt{1-c^2}$, and decreasing where $\sqrt{1-c^2} < |x| \le 1$. f has a maximum at $x = \sqrt{1-c^2}$ and a minimum at $x = -\sqrt{1-c^2}$.





57. $f(x) = 2x + 5(1-x^2)^{-1/2} = 2x + 5/\sqrt{1-x^2} \quad \Rightarrow \quad F(x) = x^2 + 5\sin^{-1}x + C$

58. $f'(x) = 4 - 3(1+x^2)^{-1} \quad \Rightarrow \quad f(x) = 4x - 3\tan^{-1}x + C \quad \Rightarrow \quad f\left(\frac{\pi}{4}\right) = \pi - 3 + C = 0 \quad \Rightarrow \quad C = 3 - \pi$, so $f(x) = 4x - 3\tan^{-1}x + 3 - \pi$.

59. $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}}\,dt = 6\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-t^2}}\,dt = 6\left[\sin^{-1}t\right]_{1/2}^{\sqrt{3}/2} = 6\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right]$
$= 6\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 6\left(\frac{\pi}{6}\right) = \pi$

60. $\displaystyle\int_0^1 \frac{4}{t^2+1}\,dt = 4\int_0^1 \frac{1}{1+t^2}\,dt = 4\left[\tan^{-1}t\right]_0^1 = 4\left(\tan^{-1}1 - \tan^{-1}0\right) = 4\left(\frac{\pi}{4} - 0\right) = \pi$

61. Let $u = 4x$. Then $du = 4\,dx$, so

$$\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2} = \frac{1}{4}\int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du = \tfrac{1}{4}\left[\tan^{-1}u\right]_0^{\sqrt{3}} = \tfrac{1}{4}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right) = \tfrac{1}{4}\left(\tfrac{\pi}{3} - 0\right) = \tfrac{\pi}{12}.$$

62. Let $u = 2t$. Then $\sqrt{1-4t^2} = \sqrt{1-u^2}$ and $du = 2\,dt$, so

$$\int \frac{dt}{\sqrt{1-4t^2}} = \int \frac{\frac{1}{2}\,du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}(2t) + C.$$

63. Let $u = \sin^{-1}x$. Then $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$, so $\displaystyle\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \left.\frac{u^2}{2}\right]_0^{\pi/6} = \frac{1}{2}\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{72}$.

64. Let $u = -\cos x$. Then $du = \sin x\,dx$, so

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx = \int_{-1}^0 \frac{1}{1+u^2}\,du = \left[\tan^{-1}u\right]_{-1}^0 = \tan^{-1}0 - \tan^{-1}(-1) = 0 - \left(-\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}.$$

65. $\displaystyle\int \frac{x+9}{x^2+9}\,dx = \int \frac{x}{x^2+9}\,dx + 9\int \frac{1}{x^2+9}\,dx = \tfrac{1}{2}\ln(x^2+9) + 3\tan^{-1}\frac{x}{3} + C$

(Let $u = x^2 + 9$ in the first integral; use Equation 14 in the second.)

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66. Let $u = \tan^{-1} x$. Then $du = dx/(1+x^2)$, so $\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\,dx = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}(\tan^{-1} x)^2 + C$.

67. Let $u = t^3$. Then $du = 3t^2\,dt$ and $\displaystyle\int \frac{t^2}{\sqrt{1-t^6}}\,dt = \int \frac{\frac{1}{3}\,du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1} u + C = \frac{1}{3}\sin^{-1}(t^3) + C$.

68. Let $u = \frac{1}{2}x$. Then $du = \frac{1}{2}\,dx \;\Rightarrow$

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{dx}{2x\sqrt{(x/2)^2-1}} = \int \frac{2\,du}{4u\sqrt{u^2-1}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}}$$
$$= \tfrac{1}{2}\sec^{-1} u + C = \tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right) + C$$

69. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}}$ and $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{2\,du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C$.

70. Let $u = e^{2x}$. Then $du = 2e^{2x}\,dx \;\Rightarrow\; \displaystyle\int \frac{e^{2x}\,dx}{\sqrt{1-e^{4x}}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \tfrac{1}{2}\sin^{-1} u + C = \tfrac{1}{2}\sin^{-1}(e^{2x}) + C$

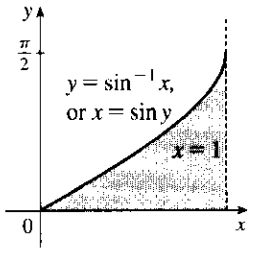
71. Let $u = x/a$. Then $du = dx/a$, so

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dx}{a\sqrt{1-(x/a)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}\frac{x}{a} + C.$$

72. We use the disk method: $A = \displaystyle\int_0^2 \pi\left[\frac{1}{\sqrt{x^2+4}}\right]^2 dx = \pi\int_0^2 \frac{1}{x^2+4}\,dx$. By Formula 14, this is equal to

$\pi\left[\frac{1}{2}\tan^{-1}(x/2)\right]_0^2 = \frac{\pi}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{8}$.

73.



The integral represents the area below the curve $y = \sin^{-1} x$ on the interval $x \in [0, 1]$. The bounding curves are $y = \sin^{-1} x \;\Leftrightarrow\; x = \sin y$, $y = 0$ and $x = 1$. We see that y ranges between $\sin^{-1} 0 = 0$ and $\sin^{-1} 1 = \frac{\pi}{2}$. So we have to integrate the function $x = 1 - \sin y$ between $y = 0$ and $y = \frac{\pi}{2}$:
$\int_0^1 \sin^{-1} x\,dx = \int_0^{\pi/2}(1-\sin y)\,dy = \left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0 + \cos 0) = \frac{\pi}{2} - 1$.

74. Let $a = \arctan x$ and $b = \arctan y$. Then by the addition formula for the tangent (see Reference Page 2 in the textbook), $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - (\tan a)(\tan b)} = \dfrac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\,\tan(\arctan y)} \;\Rightarrow\; \tan(a+b) = \dfrac{x+y}{1-xy}$

$\Rightarrow\; \arctan x + \arctan y = a + b = \arctan\left(\dfrac{x+y}{1-xy}\right)$, since $-\frac{\pi}{2} < \arctan x + \arctan y < \frac{\pi}{2}$.

75. (a) $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\dfrac{\frac{1}{2}+\frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}}\right) = \arctan 1 = \dfrac{\pi}{4}$

(b) $2\arctan\frac{1}{3} + \arctan\frac{1}{7} = \left(\arctan\frac{1}{3} + \arctan\frac{1}{3}\right) + \arctan\frac{1}{7} = \arctan\left(\dfrac{\frac{1}{3}+\frac{1}{3}}{1-\frac{1}{3}\cdot\frac{1}{3}}\right) + \arctan\frac{1}{7}$

$= \arctan\frac{3}{4} + \arctan\frac{1}{7} = \arctan\left(\dfrac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right) = \arctan 1 = \dfrac{\pi}{4}$

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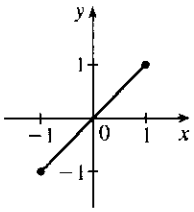
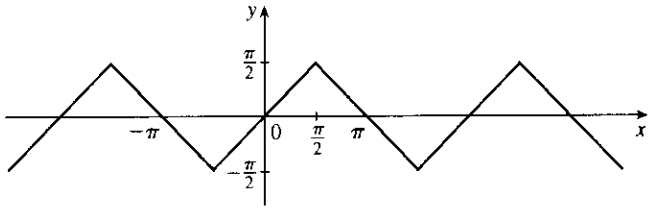
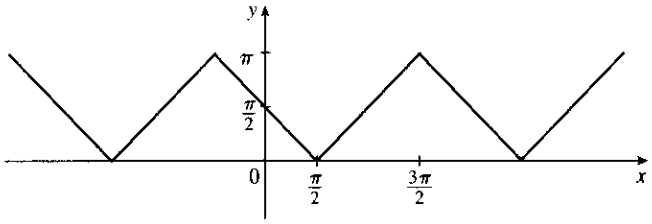
76. (a) $f(x) = \sin(\sin^{-1} x)$ (b) $g(x) = \sin^{-1}(\sin x)$





(c) $g'(x) = \dfrac{d}{dx}\sin^{-1}(\sin x) = \dfrac{1}{\sqrt{1-\sin^2 x}}\cos x = \dfrac{\cos x}{\sqrt{\cos^2 x}} = \dfrac{\cos x}{|\cos x|}$

(d) $h(x) = \cos^{-1}(\sin x)$, so $h'(x) = -\dfrac{\cos x}{\sqrt{1-\sin^2 x}} = \dfrac{\cos x}{|\cos x|}$



77. Let $f(x) = 2\sin^{-1} x - \cos^{-1}(1-2x^2)$. Then

$$f'(x) = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{\sqrt{1-(1-2x^2)^2}} = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{2x\sqrt{1-x^2}} = 0 \qquad [\text{since } x \geq 0]$$

Thus $f'(x) = 0$ for all $x \in [0, 1)$. Thus $f(x) = C$. To find C let $x = 0$. Thus $2\sin^{-1}(0) - \cos^{-1}(1) = 0 = C$. Therefore we see that $f(x) = 2\sin^{-1} x - \cos^{-1}(1-2x^2) = 0 \;\Rightarrow\; 2\sin^{-1} x = \cos^{-1}(1-2x^2)$.

78. Let $f(x) = \sin^{-1}\left(\dfrac{x-1}{x+1}\right) - 2\tan^{-1}\sqrt{x} + \dfrac{\pi}{2}$. Note that the domain of f is $[0, \infty)$. Thus

$$f'(x) = \frac{1}{\sqrt{1-\left(\dfrac{x-1}{x+1}\right)^2}} \frac{(x+1)-(x-1)}{(x+1)^2} - \frac{2}{1+x}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}\,(x+1)} - \frac{1}{\sqrt{x}\,(x+1)} = 0$$

Then $f(x) = C$. To find C, we let $x = 0 \;\Rightarrow\; \sin^{-1}(-1) - 2\tan^{-1}(0) + \frac{\pi}{2} = C \;\Rightarrow\; -\frac{\pi}{2} - 0 + \frac{\pi}{2} = 0 = C$.

Thus, $f(x) = 0 \;\Rightarrow\; \sin^{-1}\left(\dfrac{x-1}{x+1}\right) = 2\tan^{-1}\sqrt{x} - \dfrac{\pi}{2}$.

79. $y = \sec^{-1} x \;\Rightarrow\; \sec y = x \;\Rightarrow\; \sec y \tan y \dfrac{dy}{dx} = 1 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{\sec y \tan y}$. Now

$\tan^2 y = \sec^2 y - 1 = x^2 - 1$, so $\tan y = \pm\sqrt{x^2-1}$. For $y \in \left[0, \frac{\pi}{2}\right)$, $x \geq 1$, so $\sec y = x = |x|$ and $\tan y \geq 0$ $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}} = \dfrac{1}{|x|\sqrt{x^2-1}}$. For $y \in \left(\frac{\pi}{2}, \pi\right]$, $x \leq -1$, so $|x| = -x$ and $\tan y = -\sqrt{x^2-1}$ $\Rightarrow$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x\left(-\sqrt{x^2-1}\right)} = \frac{1}{(-x)\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

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80. (a) Since $|\arctan(1/x)| < \frac{\pi}{2}$, we have $0 \le |x \arctan(1/x)| \le \frac{\pi}{2}|x| \to 0$ as $x \to 0$. So, by the Squeeze Theorem, $\lim_{x\to 0} f(x) = 0 = f(0)$, so f is continuous at 0.

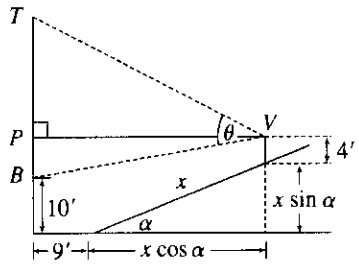
(b) Here $\dfrac{f(x)-f(0)}{x-0} = \dfrac{x\arctan(1/x)-0}{x} = \arctan\left(\dfrac{1}{x}\right)$. So (see Exercise 44 in Section 3.2 for a discussion of left- and right-hand derivatives)

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \arctan\left(\frac{1}{x}\right) = \lim_{y\to -\infty} \arctan y = -\frac{\pi}{2}, \text{ while}$$

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \arctan\left(\frac{1}{x}\right) = \lim_{y\to \infty} \arctan y = \frac{\pi}{2}. \text{ So } f'(0) \text{ does not exist.}$$

APPLIED PROJECT Where to Sit at the Movies

1.



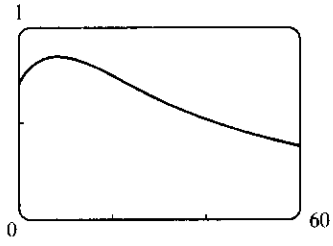
$|VP| = 9 + x\cos\alpha$, $|PT| = 35 - (4 + x\sin\alpha) = 31 - x\sin\alpha$, and

$|PB| = (4 + x\sin\alpha) - 10 = x\sin\alpha - 6$. So using the Pythagorean Theorem, we have

$|VT| = \sqrt{|VP|^2 + |PT|^2} = \sqrt{(9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2} = a$, and

$|VB| = \sqrt{|VP|^2 + |PB|^2} = \sqrt{(9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2} = b$. Using the Law of Cosines on $\triangle VBT$, we get $25^2 = a^2 + b^2 - 2ab\cos\theta \quad\Leftrightarrow\quad \cos\theta = \dfrac{a^2+b^2-625}{2ab} \quad\Leftrightarrow\quad \theta = \arccos\left(\dfrac{a^2+b^2-625}{2ab}\right)$, as required.

2. From the graph of θ, it appears that the value of x which maximizes θ is $x \approx 8.25$ ft. Assuming that the first row is at $x = 0$, the row closest to this value of x is the fourth row, at $x = 9$ ft, and from the graph, the viewing angle in this row seems to be about 0.85 radians, or about 49°.



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3. With a CAS, we type in the definition of θ, substitute in the proper values of a and b in terms of x and $\alpha = 20^\circ = \frac{\pi}{9}$ radians, and then use the differentiation command to find the derivative. We use a numerical root finder and find that the root of the equation $d\theta/dx = 0$ is $x \approx 8.253062$, as approximated in Problem 2.

4. From the graph in Problem 2, it seems that the average value of the function on the interval $[0, 60]$ is about 0.6. We can use a CAS to approximate $\frac{1}{60}\int_0^{60} \theta(x)\,dx \approx 0.625 \approx 36^\circ$. (The calculation is much faster if we reduce the number of digits of accuracy required.) The minimum value is $\theta(60) \approx 0.38$ and, from Problem 2, the maximum value is about 0.85.

7.6 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^0) = 0$ (b) $\cosh 0 = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(1+1) = 1$

2. (a) $\tanh 0 = \dfrac{(e^0 - e^{-0})/2}{(e^0 + e^{-0})/2} = 0$ (b) $\tanh 1 = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}} = \dfrac{e^2 - 1}{e^2 + 1} \approx 0.76159$

3. (a) $\sinh(\ln 2) = \dfrac{e^{\ln 2} - e^{-\ln 2}}{2} = \dfrac{e^{\ln 2} - (e^{\ln 2})^{-1}}{2} = \dfrac{2 - 2^{-1}}{2} = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$

(b) $\sinh 2 = \frac{1}{2}(e^2 - e^{-2}) \approx 3.62686$

4. (a) $\cosh 3 = \frac{1}{2}(e^3 + e^{-3}) \approx 10.06766$ (b) $\cosh(\ln 3) = \dfrac{e^{\ln 3} + e^{-\ln 3}}{2} = \dfrac{3 + \frac{1}{3}}{2} = \dfrac{5}{3}$

5. (a) $\operatorname{sech} 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

6. (a) $\sinh 1 = \frac{1}{2}(e^1 - e^{-1}) \approx 1.17520$

(b) Using Equation 3, we have $\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right) \approx 0.88137$.

7. $\sinh(-x) = \frac{1}{2}\left[e^{-x} - e^{-(-x)}\right] = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^x - e^{-x}) = -\sinh x$

8. $\cosh(-x) = \frac{1}{2}\left[e^{-x} + e^{-(-x)}\right] = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$

9. $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^x) = e^x$

10. $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^{-x}) = e^{-x}$

11.
$$\begin{aligned}\sinh x \cosh y + \cosh x \sinh y &= \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\ &= \tfrac{1}{4}\left[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})\right] \\ &= \tfrac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \tfrac{1}{2}\left[e^{x+y} - e^{-(x+y)}\right] = \sinh(x+y)\end{aligned}$$

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12. $\cosh x \cosh y + \sinh x \sinh y = \left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]\left[\frac{1}{2}\left(e^y + e^{-y}\right)\right] + \left[\frac{1}{2}\left(e^x - e^{-x}\right)\right]\left[\frac{1}{2}\left(e^y - e^{-y}\right)\right]$

$= \frac{1}{4}\left[\left(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}\right) + \left(e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}\right)\right]$

$= \frac{1}{4}\left(2e^{x+y} + 2e^{-x-y}\right) = \frac{1}{2}\left[e^{x+y} + e^{-(x+y)}\right] = \cosh(x+y)$

13. Divide both sides of tbe identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \quad \Leftrightarrow \quad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

14. $\tanh(x+y) = \dfrac{\sinh(x+y)}{\cosh(x+y)} = \dfrac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} = \dfrac{\dfrac{\sinh x \cosh y}{\cosh x \cosh y} + \dfrac{\cosh x \sinh y}{\cosh x \cosh y}}{\dfrac{\cosh x \cosh y}{\cosh x \cosh y} + \dfrac{\sinh x \sinh y}{\cosh x \cosh y}}$

$$= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

15. Putting $y = x$ in the result from Exercise 11, we have

$\sinh 2x = \sinh(x+x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x.$

16. Putting $y = x$ in the result from Exercise 12, we have

$\cosh 2x = \cosh(x+x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x.$

17. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{\left(e^{\ln x} - e^{-\ln x}\right)/2}{\left(e^{\ln x} + e^{-\ln x}\right)/2} = \dfrac{x - \left(e^{\ln x}\right)^{-1}}{x + \left(e^{\ln x}\right)^{-1}} = \dfrac{x - x^{-1}}{x + x^{-1}}$

$$= \frac{x - 1/x}{x + 1/x} = \frac{(x^2 - 1)/x}{(x^2 + 1)/x} = \frac{x^2 - 1}{x^2 + 1}$$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = \dfrac{1 + (\sinh x)/\cosh x}{1 - (\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{\frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right)}{\frac{1}{2}\left(e^x + e^{-x}\right) - \frac{1}{2}\left(e^x - e^{-x}\right)}$

$$= \frac{e^x + e^{-x} + e^x - e^{-x}}{e^x + e^{-x} - e^x + e^{-x}} = \frac{2e^x}{2e^{-x}} = e^{2x}$$

Or: Using the results of Exercises 9 and 10, $\dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^x}{e^{-x}} = e^{2x}$

19. By Exercise 9, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx.$

20. $\sinh x = \frac{3}{4} \quad \Rightarrow \quad \operatorname{csch} x = 1/\sinh x = \frac{4}{3}.$ $\cosh^2 x = \sinh^2 x + 1 = \frac{9}{16} + 1 = \frac{25}{16} \quad \Rightarrow \quad \cosh x = \frac{5}{4}$ (since $\cosh x > 0$). $\operatorname{sech} x = 1/\cosh x = \frac{4}{5}$, $\tanh x = \sinh x/\cosh x = \frac{3/4}{5/4} = \frac{3}{5}$, and $\coth x = 1/\tanh x = \frac{5}{3}$.

21. $\tanh x = \frac{4}{5} > 0$, so $x > 0$. $\coth x = 1/\tanh x = \frac{5}{4}$, $\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} \quad \Rightarrow$

$\operatorname{sech} x = \frac{3}{5}$ (since $\operatorname{sech} x > 0$), $\cosh x = 1/\operatorname{sech} x = \frac{5}{3}$, $\sinh x = \tanh x \cosh x = \frac{4}{5} \cdot \frac{5}{3} = \frac{4}{3}$, and

$\operatorname{csch} x = 1/\sinh x = \frac{3}{4}$.

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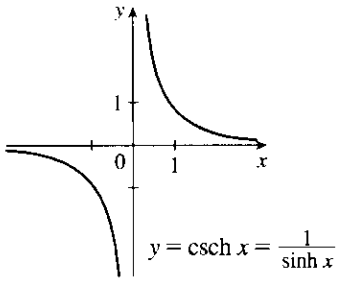
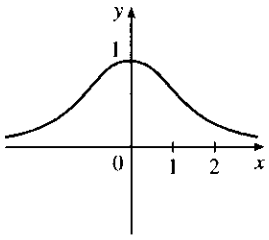
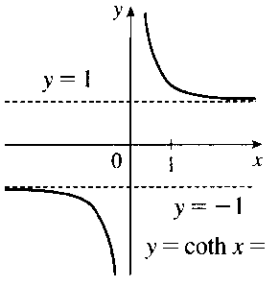
22.



$y = \operatorname{csch} x = \dfrac{1}{\sinh x}$



$y = \operatorname{sech} x = \dfrac{1}{\cosh x}$



$y = \coth x = \dfrac{1}{\tanh x}$

23. (a) $\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \dfrac{1 - e^{-2x}}{1 + e^{-2x}} = \dfrac{1-0}{1+0} = 1$

(b) $\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \dfrac{e^x}{e^x} = \lim_{x\to-\infty} \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{0-1}{0+1} = -1$

(c) $\lim_{x\to\infty} \sinh x = \lim_{x\to\infty} \dfrac{e^x - e^{-x}}{2} = \infty$

(d) $\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \dfrac{e^x - e^{-x}}{2} = -\infty$

(e) $\lim_{x\to\infty} \operatorname{sech} x = \lim_{x\to\infty} \dfrac{2}{e^x + e^{-x}} = 0$

(f) $\lim_{x\to\infty} \coth x = \lim_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \dfrac{1 + e^{-2x}}{1 - e^{-2x}} = \dfrac{1+0}{1-0} = 1$ [*Or:* Use part (a)]

(g) $\lim_{x\to0^+} \coth x = \lim_{x\to0^+} \dfrac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\lim_{x\to0^-} \coth x = \lim_{x\to0^-} \dfrac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\lim_{x\to-\infty} \operatorname{csch} x = \lim_{x\to-\infty} \dfrac{2}{e^x - e^{-x}} = 0$

24. (a) $\dfrac{d}{dx}\cosh x = \dfrac{d}{dx}\left[\tfrac{1}{2}\left(e^x + e^{-x}\right)\right] = \tfrac{1}{2}\left(e^x - e^{-x}\right) = \sinh x$

(b) $\dfrac{d}{dx}\tanh x = \dfrac{d}{dx}\left[\dfrac{\sinh x}{\cosh x}\right] = \dfrac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(c) $\dfrac{d}{dx}\operatorname{csch} x = \dfrac{d}{dx}\left[\dfrac{1}{\sinh x}\right] = -\dfrac{\cosh x}{\sinh^2 x} = -\dfrac{1}{\sinh x} \cdot \dfrac{\cosh x}{\sinh x} = -\operatorname{csch} x \coth x$

(d) $\dfrac{d}{dx}\operatorname{sech} x = \dfrac{d}{dx}\left[\dfrac{1}{\cosh x}\right] = -\dfrac{\sinh x}{\cosh^2 x} = -\dfrac{1}{\cosh x} \cdot \dfrac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$

(e) $\dfrac{d}{dx}\coth x = \dfrac{d}{dx}\left[\dfrac{\cosh x}{\sinh x}\right] = \dfrac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} = \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\dfrac{1}{\sinh^2 x}$

$= -\operatorname{csch}^2 x$

25. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh^2 y - \sinh^2 y = 1 \;\Rightarrow\;$ [with $\cosh y > 0$] $\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$. So by Exercise 9, $e^y = \sinh y + \cosh y = x + \sqrt{1 + x^2} \;\Rightarrow$ $y = \ln\left(x + \sqrt{1 + x^2}\right)$.

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26. Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0$, so $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$. So, by Exercise 9,

$e^y = \cosh y + \sinh y = x + \sqrt{x^2 - 1} \quad \Rightarrow \quad y = \ln\left(x + \sqrt{x^2 - 1}\right)$.

Another method: Write $x = \cosh y = \frac{1}{2}\left(e^y + e^{-y}\right)$ and solve a quadratic, as in Example 3.

27. (a) Let $y = \tanh^{-1} x$. Then $x = \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2} \cdot \dfrac{e^y}{e^y} = \dfrac{e^{2y} - 1}{e^{2y} + 1} \quad \Rightarrow$

$xe^{2y} + x = e^{2y} - 1 \quad \Rightarrow \quad 1 + x = e^{2y} - xe^{2y} \quad \Rightarrow \quad 1 + x = e^{2y}(1 - x) \quad \Rightarrow$

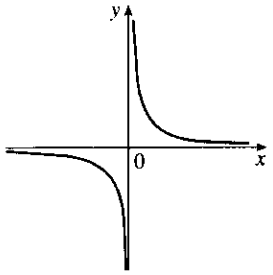
$e^{2y} = \dfrac{1 + x}{1 - x} \quad \Rightarrow \quad 2y = \ln\left(\dfrac{1 + x}{1 - x}\right) \quad \Rightarrow \quad y = \frac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$.

(b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 18 we have

$e^{2y} = \dfrac{1 + \tanh y}{1 - \tanh y} = \dfrac{1 + x}{1 - x} \quad \Rightarrow \quad 2y = \ln\left(\dfrac{1 + x}{1 - x}\right) \quad \Rightarrow \quad y = \frac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$.

28. (a) (i) $y = \operatorname{csch}^{-1} x \quad \Leftrightarrow \quad \operatorname{csch} y = x \quad (x \neq 0)$

(ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 22) about the line $y = x$.



(iii) Let $y = \operatorname{csch}^{-1} x$. Then $x = \operatorname{csch} y = \dfrac{2}{e^y - e^{-y}} \quad \Rightarrow \quad xe^y - xe^{-y} = 2$

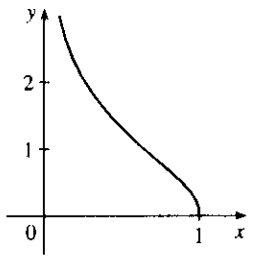
$\Rightarrow \quad x(e^y)^2 - 2e^y - x = 0 \quad \Rightarrow \quad e^y = \dfrac{1 \pm \sqrt{x^2 + 1}}{x}$.

But $e^y > 0$, so for $x > 0$, $e^y = \dfrac{1 + \sqrt{x^2 + 1}}{x}$ and for $x < 0$, $e^y = \dfrac{1 - \sqrt{x^2 + 1}}{x}$.

Thus, $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2 + 1}}{|x|}\right)$.

(b) (i) $y = \operatorname{sech}^{-1} x \quad \Leftrightarrow \quad \operatorname{sech} y = x$ and $y > 0$.

(ii) We sketch the graph of sech^{-1} by reflecting the graph of sech (see Exercise 22) about the line $y = x$.



(iii) Let $y = \operatorname{sech}^{-1} x$, so $x = \operatorname{sech} y = \dfrac{2}{e^y + e^{-y}} \quad \Rightarrow \quad xe^y + xe^{-y} = 2$

$\Rightarrow \quad x(e^y)^2 - 2e^y + x = 0 \quad \Leftrightarrow \quad e^y = \dfrac{1 \pm \sqrt{1 - x^2}}{x}$.

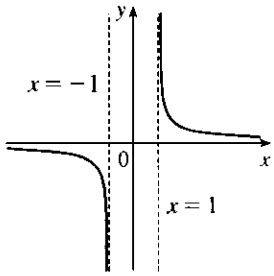
But $y > 0 \quad \Rightarrow \quad e^y > 1$. This rules out the minus sign because $\dfrac{1 - \sqrt{1 - x^2}}{x} > 1 \quad \Leftrightarrow \quad 1 - \sqrt{1 - x^2} > x$

$\Leftrightarrow \quad 1 - x > \sqrt{1 - x^2} \quad \Leftrightarrow \quad 1 - 2x + x^2 > 1 - x^2 \quad \Leftrightarrow \quad x^2 > x \quad \Leftrightarrow \quad x > 1$, but $x = \operatorname{sech} y \leq 1$. Thus,

$e^y = \dfrac{1 + \sqrt{1 - x^2}}{x} \quad \Rightarrow \quad \operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right)$.

(c) (i) $y = \coth^{-1} x \quad \Leftrightarrow \quad \coth y = x$

(ii) We sketch the graph of $\coth^{-1}$ by reflecting the graph of coth (see Exercise 22) about the line $y = x$.



(iii) Let $y = \coth^{-1} x$. Then $x = \coth y = \dfrac{e^y + e^{-y}}{e^y - e^{-y}} \quad \Rightarrow$

$xe^y - xe^{-y} = e^y + e^{-y} \quad \Rightarrow \quad (x - 1)e^y = (x + 1)e^{-y} \quad \Rightarrow$

$e^{2y} = \dfrac{x + 1}{x - 1} \quad \Rightarrow \quad 2y = \ln\dfrac{x + 1}{x - 1} \quad \Rightarrow \quad \coth^{-1} x = \frac{1}{2}\ln\dfrac{x + 1}{x - 1}$

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29. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0 \;\Rightarrow\; \sinh y \dfrac{dy}{dx} = 1 \;\Rightarrow$

$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$ (since $\sinh y \geq 0$ for $y \geq 0$). *Or:* Use Formula 4.

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \;\Rightarrow\; \operatorname{sech}^2 y \dfrac{dy}{dx} = 1 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \;\Rightarrow\; -\operatorname{csch} y \coth y \dfrac{dy}{dx} = 1 \;\Rightarrow\; \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y}$.

By Exercise 13, $\coth y = \pm\sqrt{\operatorname{csch}^2 y + 1} = \pm\sqrt{x^2 + 1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2 + 1}$. If $x < 0$, then $\coth y < 0$, so $\coth y = -\sqrt{x^2 + 1}$. In either case we have

$$\frac{dy}{dx} = -\frac{1}{\operatorname{csch} y \coth y} = -\frac{1}{|x|\sqrt{x^2 + 1}}.$$

(d) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \;\Rightarrow\; -\operatorname{sech} y \tanh y \dfrac{dy}{dx} = 1 \;\Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{sech} y \tanh y} = -\frac{1}{\operatorname{sech} y \sqrt{1 - \operatorname{sech}^2 y}} = -\frac{1}{x\sqrt{1 - x^2}}.$$ (Note that $y > 0$ and so $\tanh y > 0$.)

(e) Let $y = \coth^{-1} x$. Then $\coth y = x \;\Rightarrow\; -\operatorname{csch}^2 y \dfrac{dy}{dx} = 1 \;\Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{csch}^2 y} = \frac{1}{1 - \coth^2 y} = \frac{1}{1 - x^2}$$ by Exercise 13.

30. $f(x) = \tanh 4x \;\Rightarrow\; f'(x) = 4\operatorname{sech}^2 4x$

31. $f(x) = x \cosh x \;\Rightarrow\; f'(x) = x(\cosh x)' + (\cosh x)(x)' = x \sinh x + \cosh x$

32. $g(x) = \sinh^2 x \;\Rightarrow\; g'(x) = 2 \sinh x \cosh x$

33. $h(x) = \sinh(x^2) \;\Rightarrow\; h'(x) = \cosh(x^2) \cdot 2x = 2x \cosh(x^2)$

34. $F(x) = \sinh x \tanh x \;\Rightarrow\; F'(x) = \sinh x \operatorname{sech}^2 x + \tanh x \cosh x$

35. $G(x) = \dfrac{1 - \cosh x}{1 + \cosh x} \;\Rightarrow$

$$G'(x) = \frac{(1 + \cosh x)(-\sinh x) - (1 - \cosh x)(\sinh x)}{(1 + \cosh x)^2}$$
$$= \frac{-\sinh x - \sinh x \cosh x - \sinh x + \sinh x \cosh x}{(1 + \cosh x)^2} = \frac{-2 \sinh x}{(1 + \cosh x)^2}$$

36. $f(t) = e^t \operatorname{sech} t \;\Rightarrow\; f'(t) = e^t(-\operatorname{sech} t \tanh t) + (\operatorname{sech} t)\, e^t = e^t \operatorname{sech} t\,(1 - \tanh t)$

37. $h(t) = \coth\sqrt{1 + t^2} \;\Rightarrow\; h'(t) = -\operatorname{csch}^2\sqrt{1 + t^2} \cdot \frac{1}{2}(1 + t^2)^{-1/2}(2t) = -\dfrac{t \operatorname{csch}^2\sqrt{1 + t^2}}{\sqrt{1 + t^2}}$

38. $f(t) = \ln(\sinh t) \;\Rightarrow\; f'(t) = \dfrac{1}{\sinh t} \cosh t = \coth t$

39. $H(t) = \tanh(e^t) \;\Rightarrow\; H'(t) = \operatorname{sech}^2(e^t) \cdot e^t = e^t \operatorname{sech}^2(e^t)$

40. $y = \sinh(\cosh x) \;\Rightarrow\; y' = \cosh(\cosh x) \cdot \sinh x$

41. $y = e^{\cosh 3x} \;\Rightarrow\; y' = e^{\cosh 3x} \cdot \sinh 3x \cdot 3 = 3e^{\cosh 3x} \sinh 3x$

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42. $y = x^2 \sinh^{-1}(2x) \quad \Rightarrow \quad y' = x^2 \cdot \dfrac{1}{\sqrt{1+(2x)^2}} \cdot 2 + \sinh^{-1}(2x) \cdot 2x = 2x\left[\dfrac{x}{\sqrt{1+4x^2}} + \sinh^{-1}(2x)\right]$

43. $y = \tanh^{-1}\sqrt{x} \quad \Rightarrow \quad y' = \dfrac{1}{1-(\sqrt{x})^2} \cdot \tfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}(1-x)}$

44. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2} = x\tanh^{-1}x + \tfrac{1}{2}\ln(1-x^2) \quad \Rightarrow$

$y' = \tanh^{-1}x + \dfrac{x}{1-x^2} + \dfrac{1}{2}\left(\dfrac{1}{1-x^2}\right)(-2x) = \tanh^{-1}x$

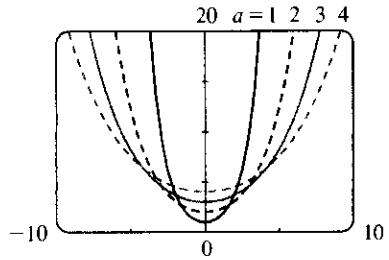
45. $y = x\sinh^{-1}(x/3) - \sqrt{9+x^2} \quad \Rightarrow$

$y' = \sinh^{-1}\left(\dfrac{x}{3}\right) + x\dfrac{1/3}{\sqrt{1+(x/3)^2}} - \dfrac{2x}{2\sqrt{9+x^2}} = \sinh^{-1}\left(\dfrac{x}{3}\right) + \dfrac{x}{\sqrt{9+x^2}} - \dfrac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\dfrac{x}{3}\right)$

46. $y = \operatorname{sech}^{-1}\sqrt{1-x^2} \quad \Rightarrow \quad y' = -\dfrac{1}{\sqrt{1-x^2}\sqrt{1-(1-x^2)}}\dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{x}{(1-x^2)\,|x|}$

47. $y = \coth^{-1}\sqrt{x^2+1} \quad \Rightarrow \quad y' = \dfrac{1}{1-(x^2+1)}\dfrac{2x}{2\sqrt{x^2+1}} = -\dfrac{1}{x\sqrt{x^2+1}}$

48.



For $y = a\cosh(x/a)$ with $a > 0$, we have the y-intercept equal to a. As a increases, the graph flattens.

49. (a) $y = 20\cosh(x/20) - 15 \quad \Rightarrow \quad y' = 20\sinh(x/20) \cdot \frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh\frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan\alpha =$ slope of the line $= \sinh\frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343$ rad $\approx 19.66°$. Thus, the angle between the line and the pole is $\theta = 90° - \alpha \approx 70.34°$.

50. We differentiate the function twice, then substitute into the differential equation: $y = \dfrac{T}{\rho g}\cosh\dfrac{\rho g x}{T} \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{T}{\rho g}\sinh\left(\frac{\rho g x}{T}\right)\frac{\rho g}{T} = \sinh\frac{\rho g x}{T} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \cosh\left(\frac{\rho g x}{T}\right)\frac{\rho g}{T} = \frac{\rho g}{T}\cosh\frac{\rho g x}{T}.$$

We evaluate the two sides separately: $\text{LHS} = \dfrac{d^2y}{dx^2} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$,

$$\text{RHS} = \frac{\rho g}{T}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\rho g}{T}\sqrt{1+\sinh^2\frac{\rho g x}{T}} = \frac{\rho g}{T}\cosh\frac{\rho g x}{T}, \text{ by the identity proved in Example 1(a).}$$

51. (a) $y = A\sinh mx + B\cosh mx \quad \Rightarrow \quad y' = mA\cosh mx + mB\sinh mx \quad \Rightarrow$

$y'' = m^2 A\sinh mx + m^2 B\cosh mx = m^2(A\sinh mx + B\cosh mx) = m^2 y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. So $-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B = -4$. Now $y'(x) = 3A\cosh 3x - 12\sinh 3x \quad \Rightarrow$ $6 = y'(0) = 3A \quad \Rightarrow \quad A = 2$, so $y = 2\sinh 3x - 4\cosh 3x$.

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52. $\lim_{x\to\infty}\dfrac{\sinh x}{e^x}=\lim_{x\to\infty}\dfrac{e^x-e^{-x}}{2e^x}=\lim_{x\to\infty}\dfrac{1-e^{-2x}}{2}=\dfrac{1-0}{2}=\dfrac{1}{2}$

53. The tangent to $y=\cosh x$ has slope 1 when $y'=\sinh x=1 \;\Rightarrow\; x=\sinh^{-1}1=\ln\left(1+\sqrt{2}\right)$, by Equation 3.

Since $\sinh x=1$ and $y=\cosh x=\sqrt{1+\sinh^2 x}$, we have $\cosh x=\sqrt{2}$. The point is $\left(\ln\left(1+\sqrt{2}\right),\sqrt{2}\right)$.

54. $\cosh x=\cosh[\ln(\sec\theta+\tan\theta)]=\frac{1}{2}\left[e^{\ln(\sec\theta+\tan\theta)}+e^{-\ln(\sec\theta+\tan\theta)}\right]$

$$=\frac{1}{2}\left[\sec\theta+\tan\theta+\frac{1}{\sec\theta+\tan\theta}\right]=\frac{1}{2}\left[\sec\theta+\tan\theta+\frac{\sec\theta-\tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}\right]$$

$$=\frac{1}{2}\left[\sec\theta+\tan\theta+\frac{\sec\theta-\tan\theta}{\sec^2\theta-\tan^2\theta}\right]=\tfrac{1}{2}(\sec\theta+\tan\theta+\sec\theta-\tan\theta)=\sec\theta$$

55. Let $u=\cosh x$. Then $du=\sinh x\,dx$, so $\int\sinh x\cosh^2 x\,dx=\int u^2\,du=\frac{1}{3}u^3+C=\frac{1}{3}\cosh^3 x+C$.

56. Let $u=1+4x$. Then $du=4\,dx$, so $\int\sinh(1+4x)\,dx=\frac{1}{4}\int\sinh u\,du=\frac{1}{4}\cosh u+C=\frac{1}{4}\cosh(1+4x)+C$.

57. Let $u=\sqrt{x}$. Then $du=\dfrac{dx}{2\sqrt{x}}$ and $\displaystyle\int\frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx=\int\sinh u\cdot 2\,du=2\cosh u+C=2\cosh\sqrt{x}=C$.

58. Let $u=\cosh x$. Then $du=\sinh x\,dx$, and

$$\int\tanh x\,dx=\int\frac{\sinh x}{\cosh x}\,dx=\int\frac{du}{u}=\ln|u|+C=\ln(\cosh x)+C.$$

59. $\displaystyle\int\frac{\cosh x}{\cosh^2 x-1}\,dx=\int\frac{\cosh x}{\sinh^2 x}\,dx=\int\frac{\cosh x}{\sinh x}\cdot\frac{1}{\sinh x}\,dx=\int\coth x\,\operatorname{csch} x\,dx=-\operatorname{csch} x+C$

60. Let $u=2+\tanh x$. Then $du=\operatorname{sech}^2 x\,dx$, so

$$\int\frac{\operatorname{sech}^2 x}{2+\tanh x}\,dx=\int\frac{1}{u}\,du=\ln|u|+C=\ln|2+\tanh x|+C=\ln(2+\tanh x)+C$$

(since $2+\tanh x>1$).

61. Let $t=3u$. Then $dt=3\,du$ and

$$\int_4^6\frac{1}{\sqrt{t^2-9}}\,dt=\int_{4/3}^2\frac{1}{\sqrt{9u^2-9}}3\,du=\int_{4/3}^2\frac{du}{\sqrt{u^2-1}}=\left[\cosh^{-1}u\right]_{4/3}^2=\cosh^{-1}2-\cosh^{-1}\left(\tfrac{4}{3}\right)\quad\text{or}$$

$$=\left[\cosh^{-1}u\right]_{4/3}^2=\left[\ln\left(u+\sqrt{u^2-1}\right)\right]_{4/3}^2$$

$$=\ln\left(2+\sqrt{3}\right)-\ln\left(\frac{4+\sqrt{7}}{3}\right)=\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)$$

62. Let $u=4t$. Then $du=4\,dt$ and

$$\int_0^1\frac{dt}{\sqrt{16t^2+1}}=\int_0^4\frac{\frac{1}{4}\,du}{\sqrt{u^2+1}}=\frac{1}{4}\left[\sinh^{-1}u\right]_0^4=\frac{1}{4}\left[\ln\left(u+\sqrt{u^2+1}\right)\right]_0^4$$

$$=\tfrac{1}{4}\left[\ln\left(4+\sqrt{17}\right)-\ln 1\right]=\tfrac{1}{4}\ln\left(4+\sqrt{17}\right)$$

63. Let $u=e^x$. Then $du=e^x\,dx$ and $\displaystyle\int\frac{e^x}{1-e^{2x}}\,dx=\int\frac{du}{1-u^2}=\tanh^{-1}u+C=\tanh^{-1}(e^x)+C$

$\left[\text{or }\dfrac{1}{2}\ln\left(\dfrac{1+e^x}{1-e^x}\right)+C\right]$.

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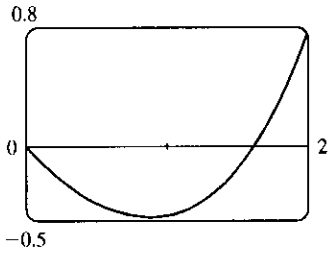
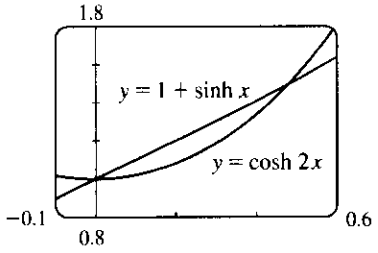
64. We want $\int_0^1 \sinh cx\,dx = 1$. To calculate the integral, we put $u = cx$, so $du = c\,dx$, the upper limit becomes c, and the equation becomes

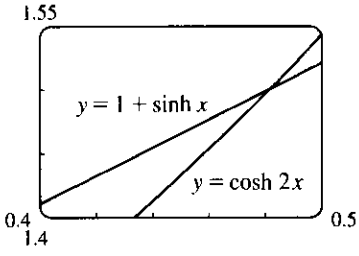
$$\frac{1}{c}\int_0^c \sinh u\,du = 1 \quad \Leftrightarrow \quad \frac{1}{c}[\cosh c - 1] = 1 \quad \Leftrightarrow \quad \cosh c - 1 = c.$$

We plot the function $f(c) = \cosh c - c - 1$, and see that its positive root lies at approximately $c = 1.62$. So the equation $\int_0^1 \sinh cx\,dx = 1$ holds for $c \approx 1.62$.



65. (a) From the graphs, we estimate that the two curves $y = \cosh 2x$ and $y = 1 + \sinh x$ intersect at $x = 0$ and at $x = a \approx 0.481$.





(b) We have found the two roots of the equation $\cosh 2x = 1 + \sinh x$ to be $x = 0$ and $x = a \approx 0.481$. Note from the first graph that $1 + \sinh x > \cosh 2x$ on the interval $(0, a)$, so the area between the two curves is

$$A = \int_0^a (1 + \sinh x - \cosh 2x)\,dx = \left[x + \cosh x - \tfrac{1}{2}\sinh 2x\right]_0^a$$
$$= [a + \cosh a - \tfrac{1}{2}\sinh 2a] - [0 + \cosh 0 - \tfrac{1}{2}\sinh 0] \approx 0.0402$$

66. The area of the triangle with vertices O, P, and $(\cosh t, 0)$ is $\frac{1}{2}\sinh t \cosh t$, and the area under the curve $x^2 - y^2 = 1$, from $x = 1$ to $x = \cosh t$, is $\int_1^{\cosh t} \sqrt{x^2 - 1}\,dx$. Therefore, the area of the shaded region is $A(t) = \frac{1}{2}\sinh t \cosh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\,dx$. So, by FTC1,

$$A'(t) = \tfrac{1}{2}(\cosh^2 t + \sinh^2 t) - \sqrt{\cosh^2 t - 1}\sinh t = \tfrac{1}{2}(\cosh^2 t + \sinh^2 t) - \sqrt{\sinh^2 t}\,\sinh t$$
$$= \tfrac{1}{2}(\cosh^2 t + \sinh^2 t) - \sinh^2 t = \tfrac{1}{2}(\cosh^2 t - \sinh^2 t) = \tfrac{1}{2}(1) = \tfrac{1}{2}$$

Thus $A(t) = \frac{1}{2}t + C$, since $A'(t) = \frac{1}{2}$. To calculate C, we let $t = 0$. Thus, $A(0) = \frac{1}{2}\sinh 0 \cosh 0 - \int_1^{\cosh 0} \sqrt{x^2 - 1}\,dx = \frac{1}{2}(0) + C \quad \Rightarrow \quad C = 0$. Thus $A(t) = \frac{1}{2}t$.

67. If $ae^x + be^{-x} = \alpha\cosh(x + \beta)$ [or $\alpha\sinh(x + \beta)$], then $ae^x + be^{-x} = \frac{\alpha}{2}(e^{x+\beta} \pm e^{-x-\beta}) = \frac{\alpha}{2}(e^x e^\beta \pm e^{-x}e^{-\beta}) = (\frac{\alpha}{2}e^\beta)e^x \pm (\frac{\alpha}{2}e^{-\beta})e^{-x}$. Comparing coefficients of e^x and e^{-x}, we have $a = \frac{\alpha}{2}e^\beta$ **(1)** and $b = \pm\frac{\alpha}{2}e^{-\beta}$ **(2)**. We need to find α and β. Dividing equation **(1)** by equation **(2)** gives us $\dfrac{a}{b} = \pm e^{2\beta} \quad \Rightarrow \quad (*) \quad 2\beta = \ln(\pm\frac{a}{b}) \quad \Rightarrow \quad \beta = \frac{1}{2}\ln(\pm\frac{a}{b})$. Solving equations **(1)** and **(2)** for e^β gives us $e^\beta = \dfrac{2a}{\alpha}$ and $e^\beta = \pm\dfrac{\alpha}{2b}$, so $\dfrac{2a}{\alpha} = \pm\dfrac{\alpha}{2b} \quad \Rightarrow \quad \alpha^2 = \pm 4ab \quad \Rightarrow \quad \alpha = 2\sqrt{\pm ab}$.

$(*)$ If $\dfrac{a}{b} > 0$, we use the $+$ sign and obtain a cosh function, whereas if $\dfrac{a}{b} < 0$, we use the $-$ sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^x + be^{-x} = 2\sqrt{ab}\cosh(x + \frac{1}{2}\ln\frac{a}{b})$, whereas, if a and b have the opposite sign, then $ae^x + be^{-x} = 2\sqrt{-ab}\sinh(x + \frac{1}{2}\ln(-\frac{a}{b}))$.

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7.7 Indeterminate Forms and L'Hospital's Rule

The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{\text{H}}{=}$.

1. (a) $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$.

(b) $\lim_{x\to a}\dfrac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim_{x\to a}\dfrac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim_{x\to a} p(x) = \infty$ and $f(x) \to 0$ through positive values, then $\lim_{x\to a}\dfrac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \to 0$ through negative values, then $\lim_{x\to a}\dfrac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \to 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.]

(e) $\lim_{x\to a}\dfrac{p(x)}{q(x)}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$.

2. (a) $\lim_{x\to a}[f(x)p(x)]$ is an indeterminate form of type $0\cdot\infty$.

(b) When x is near a, $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim_{x\to a}[h(x)p(x)] = \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x\to a}[p(x)q(x)] = \infty$.

3. (a) When x is near a, $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim_{x\to a}[f(x) - p(x)] = -\infty$.

(b) $\lim_{x\to a}[p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x) + q(x)$ is large. Thus, $\lim_{x\to a}[p(x) + q(x)] = \infty$.

4. (a) $\lim_{x\to a}[f(x)]^{g(x)}$ is an indeterminate form of type 0^0.

(b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x)\ln f(x)$. When x is near a, $p(x) \to \infty$ and $\ln f(x) \to -\infty$, so $\ln y \to -\infty$. Therefore, $\lim_{x\to a}[f(x)]^{p(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = 0$, provided f^p is defined.

(c) $\lim_{x\to a}[h(x)]^{p(x)}$ is an indeterminate form of type 1^∞.

(d) $\lim_{x\to a}[p(x)]^{f(x)}$ is an indeterminate form of type ∞^0.

(e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x)\ln p(x)$. When x is near a, $q(x) \to \infty$ and $\ln p(x) \to \infty$, so $\ln y \to \infty$. Therefore, $\lim_{x\to a}[p(x)]^{q(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = \infty$.

(f) $\lim_{x\to a}\sqrt[q(x)]{p(x)} = \lim_{x\to a}[p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0.

5. This limit has the form $\frac{0}{0}$. We can simply factor the numerator to evaluate this limit.

$$\lim_{x\to -1}\frac{x^2-1}{x+1} = \lim_{x\to -1}\frac{(x+1)(x-1)}{x+1} = \lim_{x\to -1}(x-1) = -2$$

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6. $\lim_{x\to -2}\dfrac{x+2}{x^2+3x+2}=\lim_{x\to -2}\dfrac{x+2}{(x+1)(x+2)}=\lim_{x\to -2}\dfrac{1}{x+1}=-1$

7. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1}\dfrac{x^9-1}{x^5-1}\overset{\text{H}}{=}\lim_{x\to 1}\dfrac{9x^8}{5x^4}=\dfrac{9}{5}\lim_{x\to 1}x^4=\dfrac{9}{5}(1)=\dfrac{9}{5}$

8. $\lim_{x\to 1}\dfrac{x^a-1}{x^b-1}\overset{\text{H}}{=}\lim_{x\to 1}\dfrac{ax^{a-1}}{bx^{b-1}}=\dfrac{a}{b}$

9. This limit has the form $\frac{0}{0}$. $\lim_{x\to(\pi/2)^+}\dfrac{\cos x}{1-\sin x}\overset{\text{H}}{=}\lim_{x\to(\pi/2)^+}\dfrac{-\sin x}{-\cos x}=\lim_{x\to(\pi/2)^+}\tan x=-\infty.$

10. $\lim_{x\to 0}\dfrac{x+\tan x}{\sin x}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{1+\sec^2 x}{\cos x}=\dfrac{1+1^2}{1}=2$

11. This limit has the form $\frac{0}{0}$. $\lim_{t\to 0}\dfrac{e^t-1}{t^3}\overset{\text{H}}{=}\lim_{t\to 0}\dfrac{e^t}{3t^2}=\infty$ since $e^t\to 1$ and $3t^2\to 0^+$ as $t\to 0$.

12. $\lim_{t\to 0}\dfrac{e^{3t}-1}{t}\overset{\text{H}}{=}\lim_{t\to 0}\dfrac{3e^{3t}}{1}=3$

13. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\dfrac{\tan px}{\tan qx}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{p\sec^2 px}{q\sec^2 qx}=\dfrac{p(1)^2}{q(1)^2}=\dfrac{p}{q}$

14. $\lim_{\theta\to\pi/2}\dfrac{1-\sin\theta}{\csc\theta}=\dfrac{0}{1}=0$. L'Hospital's Rule does not apply.

15. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty}\dfrac{\ln x}{x}\overset{\text{H}}{=}\lim_{x\to\infty}\dfrac{1/x}{1}=0$

16. $\lim_{x\to\infty}\dfrac{e^x}{x}\overset{\text{H}}{=}\lim_{x\to\infty}\dfrac{e^x}{1}=\lim_{x\to\infty}e^x=\infty$

17. $\lim_{x\to 0^+}[(\ln x)/x]=-\infty$ since $\ln x\to-\infty$ as $x\to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

18. $\lim_{x\to\infty}\dfrac{\ln\ln x}{x}\overset{\text{H}}{=}\lim_{x\to\infty}\dfrac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}=\lim_{x\to\infty}\dfrac{1}{x\ln x}=0$

19. This limit has the form $\frac{0}{0}$. $\lim_{t\to 0}\dfrac{5^t-3^t}{t}\overset{\text{H}}{=}\lim_{t\to 0}\dfrac{5^t\ln 5-3^t\ln 3}{1}=\ln 5-\ln 3=\ln\frac{5}{3}$

20. $\lim_{x\to 1}\dfrac{\ln x}{\sin\pi x}\overset{\text{H}}{=}\lim_{x\to 1}\dfrac{1/x}{\pi\cos\pi x}=\dfrac{1}{\pi(-1)}=-\dfrac{1}{\pi}$

21. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\dfrac{e^x-1-x}{x^2}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{e^x-1}{2x}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{e^x}{2}=\dfrac{1}{2}$

22. $\lim_{x\to 0}\dfrac{e^x-1-x-x^2/2}{x^3}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{e^x-1-x}{3x^2}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{e^x-1}{6x}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{e^x}{6}=\dfrac{1}{6}$

23. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty}\dfrac{e^x}{x^3}\overset{\text{H}}{=}\lim_{x\to\infty}\dfrac{e^x}{3x^2}\overset{\text{H}}{=}\lim_{x\to\infty}\dfrac{e^x}{6x}\overset{\text{H}}{=}\lim_{x\to\infty}\dfrac{e^x}{6}=\infty$

24. $\lim_{x\to 0}\dfrac{\sin x}{\sinh x}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{\cos x}{\cosh x}=\dfrac{1}{1}=1$

25. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\dfrac{\sin^{-1}x}{x}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{1/\sqrt{1-x^2}}{1}=\lim_{x\to 0}\dfrac{1}{\sqrt{1-x^2}}=\dfrac{1}{1}=1$

26. $\lim_{x\to 0}\dfrac{\sin x-x}{x^3}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{\cos x-1}{3x^2}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{-\sin x}{6x}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{-\cos x}{6}=-\dfrac{1}{6}$

27. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{\sin x}{2x}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{\cos x}{2}=\dfrac{1}{2}$

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28. $\lim_{x\to\infty}\dfrac{(\ln x)^2}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\dfrac{2(\ln x)(1/x)}{1} = 2\lim_{x\to\infty}\dfrac{\ln x}{x} \overset{\text{H}}{=} 2\lim_{x\to\infty}\dfrac{1/x}{1} = 2(0) = 0$

29. $\lim_{x\to 0}\dfrac{x+\sin x}{x+\cos x} = \dfrac{0+0}{0+1} = \dfrac{0}{1} = 0$. L'Hospital's Rule does not apply.

30. $\lim_{x\to 0}\dfrac{\cos mx - \cos nx}{x^2} \overset{\text{H}}{=} \lim_{x\to 0}\dfrac{-m\sin mx + n\sin nx}{2x} \overset{\text{H}}{=} \lim_{x\to 0}\dfrac{-m^2\cos mx + n^2\cos nx}{2} = \frac{1}{2}(n^2 - m^2)$

31. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty}\dfrac{x}{\ln(1+2e^x)} \overset{\text{H}}{=} \lim_{x\to\infty}\dfrac{1}{\dfrac{1}{1+2e^x}\cdot 2e^x} = \lim_{x\to\infty}\dfrac{1+2e^x}{2e^x} \overset{\text{H}}{=} \lim_{x\to\infty}\dfrac{2e^x}{2e^x} = 1$

32. $\lim_{x\to 0}\dfrac{x}{\tan^{-1}(4x)} \overset{\text{H}}{=} \lim_{x\to 0}\dfrac{1}{\dfrac{1}{1+(4x)^2}\cdot 4} = \lim_{x\to 0}\dfrac{1+16x^2}{4} = \dfrac{1}{4}$

33. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1}\dfrac{1-x+\ln x}{1+\cos\pi x} \overset{\text{H}}{=} \lim_{x\to 1}\dfrac{-1+1/x}{-\pi\sin\pi x} \overset{\text{H}}{=} \lim_{x\to 1}\dfrac{-1/x^2}{-\pi^2\cos\pi x} = \dfrac{-1}{-\pi^2(-1)} = -\dfrac{1}{\pi^2}$

34. $\lim_{x\to\infty}\dfrac{\sqrt{x^2+2}}{\sqrt{2x^2+1}} = \lim_{x\to\infty}\sqrt{\dfrac{x^2+2}{2x^2+1}} = \sqrt{\lim_{x\to\infty}\dfrac{x^2+2}{2x^2+1}} = \sqrt{\lim_{x\to\infty}\dfrac{1+2/x^2}{2+1/x^2}} = \sqrt{\dfrac{1}{2}}$

35. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1}\dfrac{x^a - ax + a - 1}{(x-1)^2} \overset{\text{H}}{=} \lim_{x\to 1}\dfrac{ax^{a-1} - a}{2(x-1)} \overset{\text{H}}{=} \lim_{x\to 1}\dfrac{a(a-1)x^{a-2}}{2} = \dfrac{a(a-1)}{2}$

36. $\lim_{x\to 0}\dfrac{1-e^{-2x}}{\sec x} = \dfrac{1-1}{1} = 0$. L'Hospital's Rule does not apply.

37. This limit has the form $0\cdot(-\infty)$. We need to write this product as a quotient, but keep in mind that we will have to differentiate both the numerator and the denominator. If we differentiate $\dfrac{1}{\ln x}$, we get a complicated expression that results in a more difficult limit. Instead we write the quotient as $\dfrac{\ln x}{x^{-1/2}}$.

$$\lim_{x\to 0^+}\sqrt{x}\ln x = \lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{1/x}{-\frac{1}{2}x^{-3/2}}\cdot\frac{-2x^{3/2}}{-2x^{3/2}} = \lim_{x\to 0^+}(-2\sqrt{x}) = 0$$

38. $\lim_{x\to-\infty} x^2e^x = \lim_{x\to-\infty}\dfrac{x^2}{e^{-x}} \overset{\text{H}}{=} \lim_{x\to-\infty}\dfrac{2x}{-e^{-x}} \overset{\text{H}}{=} \lim_{x\to-\infty}\dfrac{2}{e^{-x}} = \lim_{x\to-\infty} 2e^x = 0$

39. This limit has the form $\infty\cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to 0}\cot 2x\sin 6x = \lim_{x\to 0}\frac{\sin 6x}{\tan 2x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{6\cos 6x}{2\sec^2 2x} = \frac{6(1)}{2(1)^2} = 3$$

40. $\lim_{x\to 0^+}\sin x\ln x = \lim_{x\to 0^+}\dfrac{\ln x}{\csc x} \overset{\text{H}}{=} \lim_{x\to 0^+}\dfrac{1/x}{-\csc x\cot x} = -\lim_{x\to 0^+}\left(\dfrac{\sin x}{x}\cdot\tan x\right)$

$$= -\left(\lim_{x\to 0^+}\frac{\sin x}{x}\right)\left(\lim_{x\to 0^+}\tan x\right) = -1\cdot 0 = 0$$

41. This limit has the form $\infty\cdot 0$. $\lim_{x\to\infty} x^3e^{-x^2} = \lim_{x\to\infty}\dfrac{x^3}{e^{x^2}} \overset{\text{H}}{=} \lim_{x\to\infty}\dfrac{3x^2}{2xe^{x^2}} = \lim_{x\to\infty}\dfrac{3x}{2e^{x^2}} \overset{\text{H}}{=} \lim_{x\to\infty}\dfrac{3}{4xe^{x^2}} = 0$

42. $\lim_{x\to\pi/4}(1-\tan x)\sec x = (1-1)\sqrt{2} = 0$. L'Hospital's Rule does not apply.

43. This limit has the form $0\cdot(-\infty)$.

$$\lim_{x\to 1^+}\ln x\tan(\pi x/2) = \lim_{x\to 1^+}\frac{\ln x}{\cot(\pi x/2)} \overset{\text{H}}{=} \lim_{x\to 1^+}\frac{1/x}{(-\pi/2)\csc^2(\pi x/2)} = \frac{1}{(-\pi/2)(1)^2} = -\frac{2}{\pi}$$

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44. $\lim_{x\to\infty} x\tan(1/x) = \lim_{x\to\infty} \dfrac{\tan(1/x)}{1/x} \overset{H}{=} \lim_{x\to\infty} \dfrac{\sec^2(1/x)(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \sec^2(1/x) = 1^2 = 1$

45. $\lim_{x\to 0}\left(\dfrac{1}{x} - \csc x\right) = \lim_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right) = \lim_{x\to 0} \dfrac{\sin x - x}{x\sin x}$

$$\overset{H}{=} \lim_{x\to 0} \frac{\cos x - 1}{x\cos x + \sin x} \overset{H}{=} \lim_{x\to 0} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

46. $\lim_{x\to 0}(\csc x - \cot x) = \lim_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right) = \lim_{x\to 0} \dfrac{1-\cos x}{\sin x} \overset{H}{=} \lim_{x\to 0} \dfrac{\sin x}{\cos x} = 0$

47. We will multiply and divide by the conjugate of the expression to change the form of the expression.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right) = \lim_{x\to\infty}\left(\frac{\sqrt{x^2+x}-x}{1}\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right) = \lim_{x\to\infty}\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

As an alternate solution, write $\sqrt{x^2+x} - x$ as $\sqrt{x^2+x} - \sqrt{x^2}$, factor out $\sqrt{x^2}$, rewrite as $(\sqrt{1+1/x}-1)/(1/x)$, and apply l'Hospital's Rule.

48. $\lim_{x\to 1}\left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right) = \lim_{x\to 1}\dfrac{x-1-\ln x}{(x-1)\ln x} \overset{H}{=} \lim_{x\to 1}\dfrac{1-1/x}{(x-1)(1/x)+\ln x}\cdot\dfrac{x}{x}$

$$= \lim_{x\to 1}\frac{x-1}{x-1+x\ln x} \overset{H}{=} \lim_{x\to 1}\frac{1}{1+1+\ln x} = \frac{1}{2+0} = \frac{1}{2}$$

49. The limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$\lim_{x\to\infty}(x - \ln x) = \lim_{x\to\infty} x\left(1 - \dfrac{\ln x}{x}\right) = \infty$ since $\lim_{x\to\infty}\dfrac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty}\dfrac{1/x}{1} = 0.$

50. As $x\to\infty$, $1/x\to 0$, and $e^{1/x}\to 1$. So the limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}\left(xe^{1/x} - x\right) = \lim_{x\to\infty} x\left(e^{1/x} - 1\right) = \lim_{x\to\infty}\frac{e^{1/x}-1}{1/x} \overset{H}{=} \lim_{x\to\infty}\frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} e^{1/x} = e^0 = 1$$

51. $y = x^{x^2} \quad\Rightarrow\quad \ln y = x^2\ln x$, so

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x^2\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x^2} \overset{H}{=} \lim_{x\to 0^+}\frac{1/x}{-2/x^3} = \lim_{x\to 0^+}\left(-\frac{1}{2}x^2\right) = 0 \quad\Rightarrow$$

$$\lim_{x\to 0^+} x^{x^2} = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

52. $y = (\tan 2x)^x \quad\Rightarrow\quad \ln y = x\cdot\ln\tan 2x$, so

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x\cdot\ln\tan 2x = \lim_{x\to 0^+}\frac{\ln\tan 2x}{1/x}$$

$$\overset{H}{=} \lim_{x\to 0^+}\frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2} = \lim_{x\to 0^+}\frac{-2x^2\cos 2x}{\sin 2x\cos^2 2x} = \lim_{x\to 0^+}\frac{2x}{\sin 2x}\cdot\lim_{x\to 0^+}\frac{-x}{\cos 2x} = 1\cdot 0 = 0 \;=$$

$$\lim_{x\to 0^+}(\tan 2x)^x = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

53. $y = (1-2x)^{1/x} \quad\Rightarrow\quad \ln y = \dfrac{1}{x}\ln(1-2x)$, so $\lim_{x\to 0}\ln y = \lim_{x\to 0}\dfrac{\ln(1-2x)}{x} \overset{H}{=} \lim_{x\to 0}\dfrac{-2/(1-2x)}{1} = -2 \quad\Rightarrow$

$\lim_{x\to 0}(1-2x)^{1/x} = \lim_{x\to 0} e^{\ln y} = e^{-2}.$

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54. $y = \left(1 + \frac{a}{x}\right)^{bx} \Rightarrow \ln y = bx \ln\left(1 + \frac{a}{x}\right)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{b\ln(1+a/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{b\left(\frac{1}{1+a/x}\right)\left(-\frac{a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{ab}{1+a/x} = ab \Rightarrow$$

$$\lim_{x\to\infty}\left(1 + \frac{a}{x}\right)^{bx} = \lim_{x\to\infty} e^{\ln y} = e^{ab}.$$

55. $y = \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x \Rightarrow \ln y = x\ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right) \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\left(-\frac{3}{x^2} - \frac{10}{x^3}\right)\Big/\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{3 + \frac{10}{x}}{1 + \frac{3}{x} + \frac{5}{x^2}} = 3,$$

so $\lim_{x\to\infty}\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^3.$

56. $y = x^{(\ln 2)/(1+\ln x)} \Rightarrow \ln y = \frac{\ln 2}{1+\ln x}\ln x \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{(\ln 2)(\ln x)}{1+\ln x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{(\ln 2)(1/x)}{1/x} = \lim_{x\to\infty} \ln 2 = \ln 2,$$

so $\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2.$

57. $y = x^{1/x} \Rightarrow \ln y = (1/x)\ln x \Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0 \Rightarrow$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$$

58. $y = (e^x + x)^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln(e^x + x)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x + 1} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \Rightarrow$$

$$\lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^1 = e.$$

59. $y = \left(\frac{x}{x+1}\right)^x \Rightarrow \ln y = x\ln\left(\frac{x}{x+1}\right) \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right) = \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x - 1/(x+1)}{-1/x^2}$$

$$= \lim_{x\to\infty}\left(-x + \frac{x^2}{x+1}\right) = \lim_{x\to\infty} \frac{-x}{x+1} = -1$$

so $\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^{-1}$

Or: $\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty}\left[\left(\frac{x+1}{x}\right)^{-1}\right]^x = \left[\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x\right]^{-1} = e^{-1}$

60. $y = (\cos 3x)^{5/x} \Rightarrow \ln y = \frac{5}{x}\ln(\cos 3x) \Rightarrow \lim_{x\to 0} \ln y = 5\lim_{x\to 0} \frac{\ln(\cos 3x)}{x} \overset{\text{H}}{=} 5\lim_{x\to 0} \frac{-3\tan 3x}{1} = 0,$

so $\lim_{x\to 0}(\cos 3x)^{5/x} = e^0 = 1.$

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61. $y = (\cos x)^{1/x^2} \quad \Rightarrow \quad \ln y = \dfrac{1}{x^2}\ln\cos x \quad \Rightarrow$

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{-\tan x}{2x} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{-\sec^2 x}{2} = -\frac{1}{2} \quad \Rightarrow$$

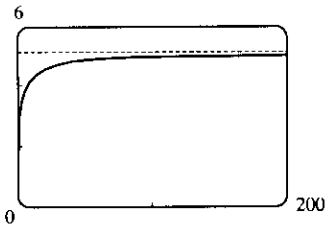
$$\lim_{x\to 0^+}(\cos x)^{1/x^2} = \lim_{x\to 0^+} e^{\ln y} = e^{-1/2} = 1/\sqrt{e}$$

62. $y = \left(\dfrac{2x-3}{2x+5}\right)^{2x+1} \quad \Rightarrow \quad \ln y = (2x+1)\ln\left(\dfrac{2x-3}{2x+5}\right) \quad \Rightarrow$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2/(2x-3)-2/(2x+5)}{-2/(2x+1)^2} = \lim_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty}\frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)} = -8 \quad \Rightarrow \quad \lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1} = e^{-8}$$

63.



From the graph, it appears that $\lim_{x\to\infty} x\,[\ln(x+5)-\ln x] = 5$.

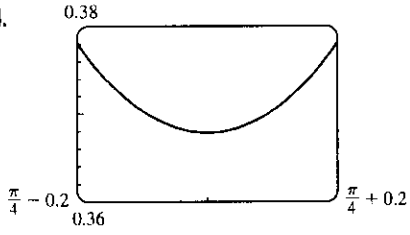
To prove this, we first note that

$\ln(x+5)-\ln x = \ln\dfrac{x+5}{x} = \ln\left(1+\dfrac{5}{x}\right) \to \ln 1 = 0$ as $x\to\infty$. Thus,

$$\lim_{x\to\infty} x\,[\ln(x+5)-\ln x] = \lim_{x\to\infty}\frac{\ln(x+5)-\ln x}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{x+5}-\dfrac{1}{x}}{-1/x^2}$$

$$= \lim_{x\to\infty}\left[\frac{x-(x+5)}{x(x+5)}\cdot\frac{-x^2}{1}\right] = \lim_{x\to\infty}\frac{5x^2}{x^2+5x} = 5$$

64.



From the graph, it appears that $\lim_{x\to\pi/4}(\tan x)^{\tan 2x} \approx 0.368$.

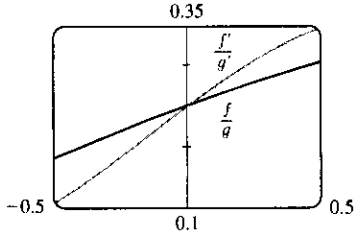
The limit has the form 1^∞. Now $y = (\tan x)^{\tan 2x} \quad \Rightarrow$

$\ln y = \tan 2x\,\ln(\tan x)$, so

$$\lim_{x\to\pi/4}\ln y = \lim_{x\to\pi/4}\frac{\ln(\tan x)}{\cot 2x} \overset{\text{H}}{=} \lim_{x\to\pi/4}\frac{\sec^2 x/\tan x}{-2\csc^2 2x} = \frac{2/1}{-2(1)} = -1$$

$$\Rightarrow \quad \lim_{x\to\pi/4}(\tan x)^{\tan 2x} = \lim_{x\to\pi/4} e^{\ln y} = e^{-1} = 1/e \approx 0.3679.$$

65.

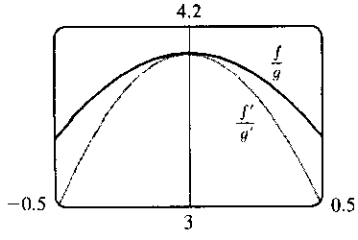


From the graph, it appears that

$\lim_{x\to 0}\dfrac{f(x)}{g(x)} = \lim_{x\to 0}\dfrac{f'(x)}{g'(x)} = 0.25$. We calculate

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{e^x-1}{x^3+4x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{e^x}{3x^2+4} = \frac{1}{4}.$$

66.



From the graph, it appears that $\lim_{x\to 0}\dfrac{f(x)}{g(x)} = \lim_{x\to 0}\dfrac{f'(x)}{g'(x)} = 4$.

We calculate

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{2x\sin x}{\sec x-1} \overset{\text{H}}{=} \lim_{x\to 0}\frac{2(x\cos x+\sin x)}{\sec x\,\tan x}$$

$$\overset{\text{H}}{=} \lim_{x\to 0}\frac{2(-x\sin x+\cos x+\cos x)}{\sec x(\sec^2 x)+\tan x(\sec x\,\tan x)} = \frac{4}{1} = 4$$

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38. $y = (\cos x)^x \quad \Rightarrow \quad \ln y = \ln(\cos x)^x = x \ln \cos x \quad \Rightarrow \quad \dfrac{y'}{y} = x \cdot \dfrac{1}{\cos x} \cdot (-\sin x) + \ln \cos x \cdot 1 \quad \Rightarrow$

$y' = (\cos x)^x (\ln \cos x - x \tan x)$

39. $y = \ln \sin x - \frac{1}{2} \sin^2 x \quad \Rightarrow \quad y' = \dfrac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot 2 \sin x \cdot \cos x = \cot x - \sin x \cos x$

40. $y = \arctan(\arcsin \sqrt{x}\,) \quad \Rightarrow \quad y' = \dfrac{1}{1 + (\arcsin \sqrt{x}\,)^2} \cdot \dfrac{1}{\sqrt{1-x}} \cdot \dfrac{1}{2\sqrt{x}}$

41. $y = \ln \dfrac{1}{x} + \dfrac{1}{\ln x} = \ln x^{-1} + (\ln x)^{-1} = -\ln x + (\ln x)^{-1} \quad \Rightarrow$

$y' = -1 \cdot \dfrac{1}{x} + (-1)(\ln x)^{-2} \cdot \dfrac{1}{x} = -\dfrac{1}{x} - \dfrac{1}{x\,(\ln x)^2}$

42. $xe^y = y - 1 \quad \Rightarrow \quad e^y + xe^y y' = y' \quad \Rightarrow \quad y' = e^y/(1 - xe^y)$

43. $y = \ln(\cosh 3x) \quad \Rightarrow \quad y' = (1/\cosh 3x)(\sinh 3x)(3) = 3 \tanh 3x$

44. $y = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \quad \Rightarrow$

$\ln y = \ln \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} = \ln(x^2+1)^4 - \ln[(2x+1)^3(3x-1)^5]$

$= 4\ln(x^2+1) - [\ln(2x+1)^3 + \ln(3x-1)^5] = 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1) \quad \Rightarrow$

$\dfrac{y'}{y} = 4 \cdot \dfrac{1}{x^2+1} \cdot 2x - 3 \cdot \dfrac{1}{2x+1} \cdot 2 - 5 \cdot \dfrac{1}{3x-1} \cdot 3 \quad \Rightarrow$

$y' = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \left(\dfrac{8x}{x^2+1} - \dfrac{6}{2x+1} - \dfrac{15}{3x-1} \right).$

[The answer could be simplified to $y' = -\dfrac{(x^2+56x+9)(x^2+1)^3}{(2x+1)^4(3x-1)^6}$, but this is unnecessary.]

45. $y = \cosh^{-1}(\sinh x) \quad \Rightarrow \quad y' = (\cosh x)/\sqrt{\sinh^2 x - 1}$

46. $y = x \tanh^{-1} \sqrt{x} \quad \Rightarrow \quad y' = \tanh^{-1} \sqrt{x} + x \dfrac{1}{1 - (\sqrt{x})^2} \dfrac{1}{2\sqrt{x}} = \tanh^{-1} \sqrt{x} + \dfrac{\sqrt{x}}{2(1-x)}$

47. $f(x) = e^{\sin^3(\ln(x^2+1))} \quad \Rightarrow$

$f'(x) = e^{\sin^3(\ln(x^2+1))} \cdot 3\sin^2(\ln(x^2+1)) \cdot \cos(\ln(x^2+1)) \cdot \dfrac{1}{x^2+1} \cdot 2x$

$= \dfrac{6x}{x^2+1} \sin^2(\ln(x^2+1)) \cdot \cos(\ln(x^2+1)) \cdot e^{\sin^3(\ln(x^2+1))}$

48. $\dfrac{d}{dx}\left(\frac{1}{2} \tan^{-1} x + \frac{1}{4} \ln \dfrac{(x+1)^2}{x^2+1} \right) = \dfrac{d}{dx}\left(\frac{1}{2} \tan^{-1} x + \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) \right)$

$= \dfrac{1}{2}\dfrac{1}{x^2+1} + \dfrac{1}{2}\dfrac{1}{x+1} - \dfrac{1}{4}\dfrac{2x}{x^2+1} = \dfrac{1}{2}\left(\dfrac{1}{x^2+1} - \dfrac{x}{x^2+1} + \dfrac{1}{x+1} \right)$

$= \dfrac{1}{2}\left(\dfrac{1-x}{x^2+1} + \dfrac{1}{x+1} \right) = \dfrac{1}{2}\left(\dfrac{1-x^2}{(x^2+1)(1+x)} + \dfrac{x^2+1}{(x^2+1)(1+x)} \right)$

$= \dfrac{1}{2}\dfrac{2}{(x^2+1)(1+x)} = \dfrac{1}{(1+x)(x^2+1)}$

49. $f(x) = e^{g(x)} \quad \Rightarrow \quad f'(x) = e^{g(x)} g'(x)$

50. $f(x) = g(e^x) \quad \Rightarrow \quad f'(x) = g'(e^x) e^x$

51. $f(x) = \ln|g(x)| \quad \Rightarrow \quad f'(x) = \dfrac{1}{g(x)} g'(x) = \dfrac{g'(x)}{g(x)}$

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52. $f(x) = g(\ln x) \quad \Rightarrow \quad f'(x) = g'(\ln x) \cdot \dfrac{1}{x} = \dfrac{g'(\ln x)}{x}$

53. $f(x) = 2^x \quad \Rightarrow \quad f'(x) = 2^x \ln 2 \quad \Rightarrow \quad f''(x) = 2^x (\ln 2)^2 \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad f^{(n)}(x) = 2^x (\ln 2)^n$

54. $f(x) = \ln(2x) = \ln 2 + \ln x \quad \Rightarrow \quad f'(x) = x^{-1}, f''(x) = -x^{-2}, f'''(x) = 2x^{-3}, f^{(4)}(x) = -2 \cdot 3x^{-4}, \ldots,$ $f^{(n)}(x) = (-1)^{n-1}(n-1)!\, x^{-n}$

55. We first show it is true for $n = 1$: $f'(x) = e^x + xe^x = (x+1)e^x$. We now assume it is true for $n = k$: $f^{(k)}(x) = (x+k)e^x$. With this assumption, we must show it is true for $n = k+1$:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] = \frac{d}{dx}\left[(x+k)e^x\right] = e^x + (x+k)e^x = \left[x + (k+1)\right]e^x.$$

Therefore, $f^{(n)}(x) = (x+n)e^x$ by mathematical induction.

56. Using implicit differentiation, $y = x + \arctan y \quad \Rightarrow \quad y' = 1 + \dfrac{1}{1+y^2}y' \quad \Rightarrow \quad y'\left(1 - \dfrac{1}{1+y^2}\right) = 1 \quad \Rightarrow$

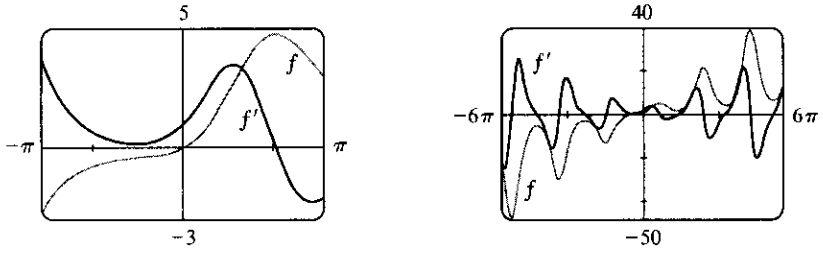
$$y'\left(\frac{y^2}{1+y^2}\right) = 1 \quad \Rightarrow \quad y' = \frac{1+y^2}{y^2} = \frac{1}{y^2} + 1.$$

57. $y = (2+x)e^{-x} \quad \Rightarrow \quad y' = (2+x)(-e^{-x}) + e^{-x} \cdot 1 = e^{-x}[-(2+x)+1] = e^{-x}(-x-1)$. At $(0,2)$, $y' = 1(-1) = -1$, so an equation of the tangent line is $y - 2 = -1(x-0)$, or $y = -x + 2$.

58. $y = f(x) = x \ln x \quad \Rightarrow \quad f'(x) = \ln x + 1$, so the slope of the tangent at (e, e) is $f'(e) = 2$ and an equation is $y - e = 2(x-e)$ or $y = 2x - e$.

59. $y = [\ln(x+4)]^2 \quad \Rightarrow \quad y' = 2\dfrac{\ln(x+4)}{x+4} = 0 \quad \Leftrightarrow \quad \ln(x+4) = 0 \quad \Leftrightarrow \quad x+4 = 1 \quad \Leftrightarrow \quad x = -3$, so the tangent is horizontal at $(-3, 0)$.

60. $f(x) = xe^{\sin x} \quad \Rightarrow \quad f'(x) = x\left[e^{\sin x}(\cos x)\right] + e^{\sin x}(1) = e^{\sin x}(x\cos x + 1)$. As a check on our work, we notice from the graphs that $f'(x) > 0$ when f is increasing. Also, we see in the larger viewing rectangle a certain similarity in the graphs of f and f': the sizes of the oscillations of f and f' are linked.



61. (a) The line $x - 4y = 1$ has slope $\frac{1}{4}$. The tangent to $y = e^x$ has slope $\frac{1}{4}$ when $y' = e^x = \frac{1}{4} \quad \Rightarrow$ $x = \ln\frac{1}{4} = -\ln 4$, so an equation is $y - \frac{1}{4} = \frac{1}{4}(x + \ln 4)$ or $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$.

(b) The slope of the tangent at the point (a, e^a) is $\dfrac{d}{dx}e^x\Big]_{x=a} = e^a$. An equation of the tangent line is thus $y - e^a = e^a(x-a)$. We substitute $x = 0, y = 0$ into this equation, since we want the line to pass through the origin: $0 - e^a = e^a(0-a) \quad \Leftrightarrow \quad -e^a = e^a(-a) \quad \Leftrightarrow \quad a = 1$. So an equation of the tangent is $y - e = e(x-1)$, or $y = ex$.

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62. (a) $\lim_{t\to\infty} C(t) = \lim_{t\to\infty}\left[K\left(e^{-at}-e^{-bt}\right)\right] = K\lim_{t\to\infty}\left(e^{-at}-e^{-bt}\right) = K(0-0) = 0$ because $-at \to -\infty$ and $-bt \to -\infty$ as $t \to \infty$.

(b) $C(t) = K\left(e^{-at}-e^{-bt}\right) \quad\Rightarrow\quad C'(t) = K\left(e^{-at}(-a)-e^{-bt}(-b)\right) = K\left(-ae^{-at}+be^{-bt}\right)$

(c) $C'(t) = 0 \quad\Rightarrow\quad be^{-bt} = ae^{-at} \quad\Rightarrow\quad \dfrac{b}{a} = e^{(-a+b)t} \quad\Rightarrow\quad \ln\dfrac{b}{a} = (b-a)t \quad\Rightarrow\quad t = \dfrac{\ln(b/a)}{b-a}$

63. $\lim_{x\to\infty} e^{-3x} = 0$ since $-3x \to -\infty$ as $x \to \infty$ and $\lim_{t\to-\infty} e^{t} = 0$.

64. $\lim_{x\to 10^-} \ln\left(100-x^2\right) = -\infty$ since as $x \to 10^-$, $\left(100-x^2\right) \to 0^+$.

65. Let $t = 2/(x-3)$. As $x \to 3^-$, $t \to -\infty$. $\lim_{x\to 3^-} e^{2/(x-3)} = \lim_{t\to-\infty} e^{t} = 0$

66. If $y = x^3 - x = x(x^2-1)$, then as $x \to \infty$, $y \to \infty$. $\lim_{x\to\infty} \arctan(x^3-x) = \lim_{y\to\infty} \arctan y = \frac{\pi}{2}$ by (7.5.8).

67. Let $t = \sinh x$. As $x \to 0^+$, $t \to 0^+$. $\lim_{x\to 0^+} \ln(\sinh x) = \lim_{t\to 0^+} \ln t = -\infty$

68. $-1 \le \sin x \le 1 \quad\Rightarrow\quad -e^{-x} \le e^{-x}\sin x \le e^{-x}$. Now $\lim_{x\to\infty}\left(\pm e^{-x}\right) = 0$, so by the Squeeze Theorem, $\lim_{x\to\infty} e^{-x}\sin x = 0$.

69. $\lim_{x\to\infty} \dfrac{(1+2^x)/2^x}{(1-2^x)/2^x} = \lim_{x\to\infty} \dfrac{1/2^x+1}{1/2^x-1} = \dfrac{0+1}{0-1} = -1$

70. Let $t = x/4$, so $x = 4t$. As $x \to \infty$, $t \to \infty$. $\lim_{x\to\infty}\left(1+\dfrac{4}{x}\right)^x = \lim_{t\to\infty}\left(1+\dfrac{1}{t}\right)^{4t} = \left[\lim_{t\to\infty}\left(1+\dfrac{1}{t}\right)^t\right]^4 = e^4$

71. $\lim_{x\to 0} \dfrac{\tan \pi x}{\ln(1+x)} \overset{\text{H}}{=} \lim_{x\to 0} \dfrac{\pi \sec^2 \pi x}{1/(1+x)} = \dfrac{\pi\cdot 1^2}{1/1} = \pi$

72. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2+x} \overset{\text{H}}{=} \lim_{x\to 0} \dfrac{\sin x}{2x+1} = \dfrac{0}{1} = 0$

73. $\lim_{x\to 0} \dfrac{e^{4x}-1-4x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0} \dfrac{4e^{4x}-4}{2x} \overset{\text{H}}{=} \lim_{x\to 0} \dfrac{16e^{4x}}{2} = \lim_{x\to 0} 8e^{4x} = 8\cdot 1 = 8$

74. $\lim_{x\to\infty} \dfrac{e^{4x}-1-4x}{x^2} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{4e^{4x}-4}{2x} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{16e^{4x}}{2} = \lim_{x\to\infty} 8e^{4x} = \infty$

75. $\lim_{x\to\infty} x^3e^{-x} = \lim_{x\to\infty} \dfrac{x^3}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{3x^2}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{6x}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{6}{e^x} = 0$

76. $\lim_{x\to 0^+} x^2\ln x = \lim_{x\to 0^+} \dfrac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim_{x\to 0^+} \dfrac{1/x}{-2/x^3} = \lim_{x\to 0^+}\left(-\tfrac{1}{2}x^2\right) = 0$

77. $\lim_{x\to 1^+}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right) = \lim_{x\to 1^+}\left(\dfrac{x\ln x - x + 1}{(x-1)\ln x}\right) \overset{\text{H}}{=} \lim_{x\to 1^+} \dfrac{x\cdot(1/x)+\ln x - 1}{(x-1)\cdot(1/x)+\ln x}$

$= \lim_{x\to 1^+} \dfrac{\ln x}{1-1/x+\ln x} \overset{\text{H}}{=} \lim_{x\to 1^+} \dfrac{1/x}{1/x^2+1/x} = \dfrac{1}{1+1} = \dfrac{1}{2}$

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78. $y = (\tan x)^{\cos x} \quad \Rightarrow \quad \ln y = \cos x \ln \tan x$, so

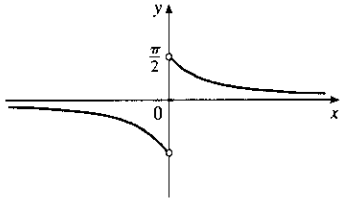
$$\lim_{x\to(\pi/2)^-} \ln y = \lim_{x\to(\pi/2)^-} \frac{\ln \tan x}{\sec x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-} \frac{(1/\tan x)\sec^2 x}{\sec x \tan x} = \lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan^2 x}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0, \text{ so}$$

$$\lim_{x\to(\pi/2)^-} (\tan x)^{\cos x} = \lim_{x\to(\pi/2)^-} e^{\ln y} = e^0 = 1.$$

79. $y = f(x) = \tan^{-1}(1/x)$ **A.** $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim_{x\to\pm\infty} \tan^{-1}(1/x) = \tan^{-1} 0 = 0$, so $y = 0$ is a HA. $\lim_{x\to 0^+} \tan^{-1}(1/x) = \frac{\pi}{2}$ and $\lim_{x\to 0^-} \tan^{-1}(1/x) = -\frac{\pi}{2}$ since $\frac{1}{x} \to \pm\infty$ as $x \to 0^{\pm}$.

H.



E. $f'(x) = \dfrac{1}{1+(1/x)^2}\left(-1/x^2\right) = \dfrac{-1}{x^2+1} \quad \Rightarrow \quad f'(x) < 0$, so f is decreasing on $(-\infty, 0)$ and $(0, \infty)$. **F.** No maximum nor minimum

G. $f''(x) = \dfrac{2x}{(x^2+1)^2} > 0 \quad \Leftrightarrow \quad x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$.

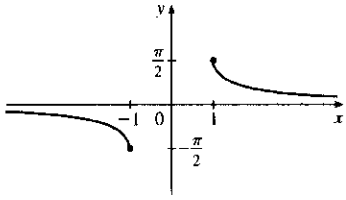
80. $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \le 1/x \le 1\} = (-\infty, -1] \cup [1, \infty)$. **B.** No intercept

C. $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim_{x\to\pm\infty} \sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y = 0$ is a HA.

E. $f'(x) = \dfrac{1}{\sqrt{1-(1/x)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{\sqrt{x^4-x^2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

F. No local extreme value, but $f(1) = \frac{\pi}{2}$ is the absolute maximum value and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum value.

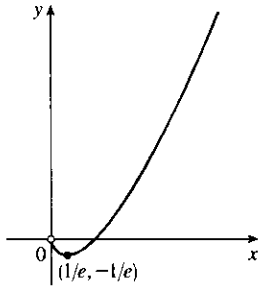
H.



G. $f''(x) = \dfrac{4x^3-2x}{2(x^4-x^2)^{3/2}} = \dfrac{x\left(2x^2-1\right)}{(x^4-x^2)^{3/2}} > 0$ for $x > 1$ and $f''(x) < 0$ for $x < -1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, -1)$. No IP

81. $y = f(x) = x \ln x$ **A.** $D = (0, \infty)$ **B.** No y-intercept; x-intercept 1. **C.** No symmetry **D.** No asymptote [Note that the graph approaches the point $(0, 0)$ as $x \to 0^+$.]

H.



E. $f'(x) = x(1/x) + (\ln x)(1) = 1 + \ln x$, so $f'(x) \to -\infty$ as $x \to 0^+$ and $f'(x) \to \infty$ as $x \to \infty$. $f'(x) = 0 \quad \Leftrightarrow \quad \ln x = -1 \quad \Leftrightarrow$ $x = e^{-1} = 1/e$. $f'(x) > 0$ for $x > 1/e$, so f is decreasing on $(0, 1/e)$ and increasing on $(1/e, \infty)$. **F.** Local minimum: $f(1/e) = -1/e$. No local maximum. **G.** $f''(x) = 1/x$, so $f''(x) > 0$ for $x > 0$. The graph is CU on $(0, \infty)$ and there is no IP.

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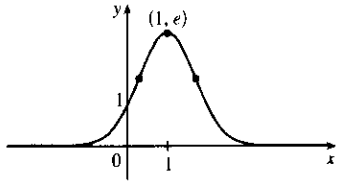
82. $y = f(x) = e^{2x-x^2}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 1; no x-intercept **C.** No symmetry **D.** $\lim_{x\to\pm\infty} e^{2x-x^2} = 0$, so $y = 0$ is a HA. **E.** $y = f(x) = e^{2x-x^2} \Rightarrow f'(x) = 2(1-x)e^{2x-x^2} > 0 \Leftrightarrow x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. **F.** $f(1) = e$ is a local and absolute maximum value.

G. $f''(x) = 2(2x^2 - 4x + 1)e^{2x-x^2} = 0 \Leftrightarrow x = 1 \pm \frac{\sqrt{2}}{2}$.

$f''(x) > 0 \Leftrightarrow x < 1 - \frac{\sqrt{2}}{2}$ or $x > 1 + \frac{\sqrt{2}}{2}$, so f is CU on $\left(-\infty, 1 - \frac{\sqrt{2}}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \infty\right)$, and CD on $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$.

IP at $\left(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e}\right)$

H.



83. $y = f(x) = e^x + e^{-3x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 2; no x-intercept **C.** No symmetry

D. $\lim_{x\to\pm\infty} (e^x + e^{-3x}) = \infty$, no asymptote **E.** $y = f(x) = e^x + e^{-3x} \Rightarrow$

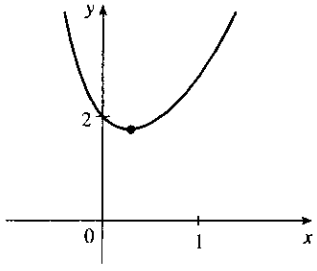
$f'(x) = e^x - 3e^{-3x} = e^{-3x}(e^{4x} - 3) > 0 \Leftrightarrow e^{4x} > 3 \Leftrightarrow$

$4x > \ln 3 \Leftrightarrow x > \frac{1}{4}\ln 3 \approx 0.27$, so f is increasing on $(\frac{1}{4}\ln 3, \infty)$ and decreasing on $(-\infty, \frac{1}{4}\ln 3)$.

F. Absolute minimum value $f(\frac{1}{4}\ln 3) = 3^{1/4} + 3^{-3/4} \approx 1.75$.

G. $f''(x) = e^x + 9e^{-3x} > 0$, so f is CU on $(-\infty, \infty)$. No IP

H.



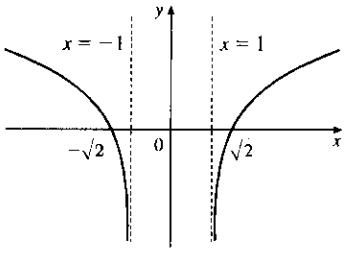
84. $y = f(x) = \ln(x^2 - 1)$ **A.** $D = (-\infty, -1) \cup (1, \infty)$ **B.** No y-intercept; x-intercepts $\pm\sqrt{2}$ **C.** Symmetric about the y-axis **D.** $\lim_{x\to\pm\infty} \ln(x^2 - 1) = \infty$, $\lim_{x\to 1^+} \ln(x^2 - 1) = -\infty$, $\lim_{x\to -1^-} \ln(x^2 - 1) = -\infty$, so $x = 1$ and $x = -1$ are VA. **E.** $y = f(x) = \ln(x^2 - 1) \Rightarrow f'(x) = \dfrac{2x}{x^2 - 1} > 0$ for $x > 1$ and $f'(x) < 0$ for $x < -1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, -1)$.

Note that the domain of f is $|x| > 1$. **F.** No extreme value

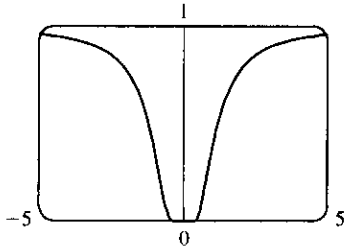
G. $f''(x) = -2\dfrac{x^2 + 1}{(x^2 - 1)^2} < 0$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$.

No IP

H.



85.



From the graph, we estimate the points of inflection to be about $(\pm 0.82, 0.22)$. $f(x) = e^{-1/x^2} \Rightarrow f'(x) = 2x^{-3}e^{-1/x^2} \Rightarrow$

$$f''(x) = 2\left[x^{-3}(2x^{-3})e^{-1/x^2} + e^{-1/x^2}(-3x^{-4})\right]$$
$$= 2x^{-6}e^{-1/x^2}(2 - 3x^2).$$

This is 0 when $2 - 3x^2 = 0 \Leftrightarrow x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $\left(\pm\sqrt{\frac{2}{3}}, e^{-3/2}\right)$.

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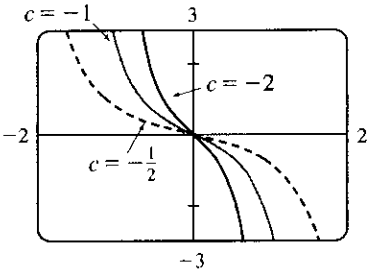
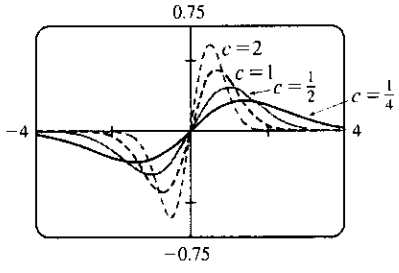
86. We exclude the case $c = 0$, since in that case $f(x) = 0$ for all x. To find the maxima and minima, we differentiate: $f(x) = cxe^{-cx^2} \quad \Rightarrow \quad f'(x) = c\left[xe^{-cx^2}(-2cx) + e^{-cx^2}(1)\right] = ce^{-cx^2}\left(-2cx^2 + 1\right)$. This is 0 where $-2cx^2 + 1 = 0 \quad \Leftrightarrow \quad x = \pm 1/\sqrt{2c}$. So if $c > 0$, there are two maxima or minima, whose x-coordinates approach 0 as c increases. The negative root gives a minimum and the positive root gives a maximum, by the First Derivative Test. By substituting back into the equation, we see that $f\left(\pm 1/\sqrt{2c}\right) = c\left(\pm 1/\sqrt{2c}\right) e^{-c\left(\pm 1/\sqrt{2c}\right)^2} = \pm\sqrt{c/2e}$. So as c increases, the extreme points become more pronounced. Note that if $c > 0$, then $\lim_{x \to \pm\infty} f(x) = 0$. If $c < 0$, then there are no extreme values, and $\lim_{x \to \pm\infty} f(x) = \mp\infty$.

To find the points of inflection, we differentiate again: $f'(x) = ce^{-cx^2}\left(-2cx^2 + 1\right) \quad \Rightarrow$ $f''(x) = c\left[e^{-cx^2}(-4cx) + \left(-2cx^2 + 1\right)\left(-2cxe^{-cx^2}\right)\right] = -2c^2xe^{-cx^2}\left(3 - 2cx^2\right)$. This is 0 at $x = 0$ and where $3 - 2cx^2 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{3/(2c)} \quad \Rightarrow$ IP at $\left(\pm\sqrt{3/(2c)}, \pm\sqrt{3c/2}\, e^{-3/2}\right)$. If $c > 0$ there are three inflection points, and as c increases, the x-coordinates of the nonzero inflection points approach 0. If $c < 0$, there is only one inflection point, the origin.





87. $s(t) = Ae^{-ct}\cos(\omega t + \delta) \quad \Rightarrow$

$$\begin{aligned} v(t) = s'(t) &= A\{e^{-ct}\left[-\omega\sin(\omega t + \delta)\right] + \cos(\omega t + \delta)(-ce^{-ct})\} \\ &= -Ae^{-ct}\left[\omega\sin(\omega t + \delta) + c\cos(\omega t + \delta)\right] \quad \Rightarrow \end{aligned}$$

$$\begin{aligned} a(t) = v'(t) &= -A\{e^{-ct}[\omega^2\cos(\omega t + \delta) - c\omega\sin(\omega t + \delta)] + [\omega\sin(\omega t + \delta) + c\cos(\omega t + \delta)](-ce^{-ct})\} \\ &= -Ae^{-ct}[\omega^2\cos(\omega t + \delta) - c\omega\sin(\omega t + \delta) - c\omega\sin(\omega t + \delta) - c^2\cos(\omega t + \delta)] \\ &= -Ae^{-ct}[(\omega^2 - c^2)\cos(\omega t + \delta) - 2c\omega\sin(\omega t + \delta)] \\ &= Ae^{-ct}[(c^2 - \omega^2)\cos(\omega t + \delta) + 2c\omega\sin(\omega t + \delta)] \end{aligned}$$

88. (a) Let $f(x) = \ln x + x - 3$. Then $f'(x) = 1/x + 1 > 0$ (for $x > 0$) and $f(2) \approx -0.307$ and $f(e) \approx 0.718$. f is differentiable on $(2, e)$, continuous on $[2, e]$ and $f(2) < 0$, $f(e) > 0$. Therefore, by the Intermediate Value Theorem there exists a number c in $(2, e)$ such that $f(c) = 0$. Thus, there is one root. But $f'(x) > 0$ for $x \in (2, e)$, so f is increasing on $(2, e)$, which means that there is exactly one root.

(b) We use Newton's Method with $f(x) = \ln x + x - 3$, $f'(x) = 1/x + 1$, and $x_1 = 2$.

$x_2 = x_1 - \dfrac{\ln x_1 + x_1 - 3}{1/x_1 + 1} = 2 - \dfrac{\ln 2 + 2 - 3}{1/2 + 1} \approx 2.20457$. Similarly, $x_3 \approx 2.20794$, $x_4 = 2.20794$. Thus, the root of the equation, correct to four decimal places, is 2.2079.

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89. Let $P(t) = \dfrac{64}{1+31e^{-0.7944t}} = \dfrac{A}{1+Be^{ct}} = A(1+Be^{ct})^{-1}$, where $A = 64$, $B = 31$, and $c = -0.7944$.

$$P'(t) = -A(1+Be^{ct})^{-2}(Bce^{ct}) = -ABce^{ct}(1+Be^{ct})^{-2}$$

$$P''(t) = -ABce^{ct}\left[-2(1+Be^{ct})^{-3}(Bce^{ct})\right] + (1+Be^{ct})^{-2}(-ABc^2e^{ct})$$

$$= -ABc^2e^{ct}(1+Be^{ct})^{-3}\left[-2Be^{ct} + (1+Be^{ct})\right] = -\frac{ABc^2e^{ct}(1-Be^{ct})}{(1+Be^{ct})^3}$$

The population is increasing most rapidly when its graph changes from CU to CD; that is, when $P''(t) = 0$ in this case. $P''(t) = 0 \;\Rightarrow\; Be^{ct} = 1 \;\Rightarrow\; e^{ct} = \dfrac{1}{B} \;\Rightarrow$ $ct = \ln\dfrac{1}{B} \;\Rightarrow\; t = \dfrac{\ln(1/B)}{c} = \dfrac{\ln(1/31)}{-0.7944} \approx 4.32$ days. Note that $P\left(\dfrac{1}{c}\ln\dfrac{1}{B}\right) = \dfrac{A}{1+Be^{c(1/c)\ln(1/B)}} = \dfrac{A}{1+Be^{\ln(1/B)}} = \dfrac{A}{1+B(1/B)} = \dfrac{A}{1+1} = \dfrac{A}{2}$, one-half the limit of P as $t \to \infty$.

90. Let $t = 4u$. Then $dt = 4\,du$ and

$$\int_0^4 \frac{1}{16+t^2}\,dt = \int_0^1 \frac{1}{16+16u^2}\cdot 4\,du = \frac{1}{4}\int_0^1 \frac{du}{1+u^2} = \frac{1}{4}\left[\tan^{-1}u\right]_0^1$$
$$= \tfrac{1}{4}(\tan^{-1}1 - \tan^{-1}0) = \tfrac{1}{4}\left(\tfrac{\pi}{4} - 0\right) = \tfrac{\pi}{16}$$

91. Let $u = -2y^2$. Then $du = -4y\,dy$ and

$$\int_0^1 ye^{-2y^2}\,dy = \int_0^{-2} e^u\left(-\tfrac{1}{4}\,du\right) = -\tfrac{1}{4}\left[e^u\right]_0^{-2} = -\tfrac{1}{4}(e^{-2}-1) = \tfrac{1}{4}(1-e^{-2}).$$

92. $\displaystyle\int_2^5 \frac{dr}{1+2r} = \tfrac{1}{2}\left[\ln|1+2r|\right]_2^5 = \tfrac{1}{2}(\ln 11 - \ln 5) = \tfrac{1}{2}\ln\tfrac{11}{5}$

93. $$\int_2^4 \frac{1+x-x^2}{x^2}\,dx = \int_2^4 \left(x^{-2} + \frac{1}{x} - 1\right)dx = \left[-\frac{1}{x} + \ln x - x\right]_2^4$$
$$= \left(-\tfrac{1}{4} + \ln 4 - 4\right) - \left(-\tfrac{1}{2} + \ln 2 - 2\right) = \ln 2 - \tfrac{7}{4}$$

94. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx = \int_0^1 \frac{1}{1+u^2}\,du = \left[\tan^{-1}u\right]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \tfrac{\pi}{4} - 0 = \tfrac{\pi}{4}$$

95. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}} \;\Rightarrow\; \displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C.$

96. Let $u = \ln x$. Then $du = \dfrac{dx}{x} \;\Rightarrow\; \displaystyle\int \frac{\cos(\ln x)}{x}\,dx = \int \cos u\,du = \sin u + C = \sin(\ln x) + C.$

97. Let $u = x^2 + 2x$. Then $du = (2x+2)\,dx = 2(x+1)\,dx$ and

$$\int \frac{x+1}{x^2+2x}\,dx = \int \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+2x\right| + C.$$

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98. Let $u = e^{-x}$. Then $du = -e^{-x}\,dx$ and

$$\int \frac{e^{-x}}{1+e^{-2x}}\,dx = \int \frac{-du}{1+u^2} = -\tan^{-1}u + C = -\tan^{-1}(e^{-x}) + C.$$

99. Let $u = \ln(\cos x)$. Then $du = \dfrac{-\sin x}{\cos x}\,dx = -\tan x\,dx \;\Rightarrow$

$$\int \tan x \ln(\cos x)\,dx = -\int u\,du = -\tfrac{1}{2}u^2 + C = -\tfrac{1}{2}[\ln(\cos x)]^2 + C.$$

100. Let $u = x^2$. Then $du = 2x\,dx \;\Rightarrow\; \displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \tfrac{1}{2}\sin^{-1}u + C = \tfrac{1}{2}\sin^{-1}(x^2) + C.$

101. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$ and $\displaystyle\int 2^{\tan\theta}\sec^2\theta\,d\theta = \int 2^u du = \frac{2^u}{\ln 2} + C = \frac{2^{\tan\theta}}{\ln 2} + C.$

102. $\displaystyle\int \sinh au\,du = \frac{1}{a}\cosh au + C$

103. Let $u = 1 + \sec\theta$, so $du = \sec\theta\tan\theta\,d\theta \;\Rightarrow\; \displaystyle\int \frac{\sec\theta\tan\theta}{1+\sec\theta}\,d\theta = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+\sec\theta| + C.$

104. $1 + e^{2x} > e^{2x} \;\Rightarrow\; \sqrt{1+e^{2x}} > \sqrt{e^{2x}} = e^x \;\Rightarrow\; \int_0^1 \sqrt{1+e^{2x}}\,dx \geq \int_0^1 e^x\,dx = e^x]_0^1 = e - 1$

105. $\cos x \leq 1 \;\Rightarrow\; e^x\cos x \leq e^x \;\Rightarrow\; \int_0^1 e^x\cos x\,dx \leq \int_0^1 e^x\,dx = e^x]_0^1 = e - 1$

106. For $0 \leq x \leq 1$, $0 \leq \sin^{-1}x \leq \frac{\pi}{2}$, so $\int_0^1 x\sin^{-1}x\,dx \leq \int_0^1 x\left(\frac{\pi}{2}\right)dx = \frac{\pi}{4}x^2\big]_0^1 = \frac{\pi}{4}$.

107. $\displaystyle f(x) = \int_1^{\sqrt{x}} \frac{e^s}{s}\,ds \;\Rightarrow\; f'(x) = \frac{d}{dx}\int_1^{\sqrt{x}} \frac{e^s}{s}\,ds = \frac{e^{\sqrt{x}}}{\sqrt{x}}\frac{d}{dx}\sqrt{x} = \frac{e^{\sqrt{x}}}{\sqrt{x}}\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x}$

108. $\displaystyle f(x) = \int_{\ln x}^{2x} e^{-t^2}\,dt \;\Rightarrow\; f'(x) = \frac{d}{dx}\int_{\ln x}^{2x} e^{-t^2}\,dt = -\frac{d}{dx}\int_0^{\ln x} e^{-t^2}\,dt + \frac{d}{dx}\int_0^{2x} e^{-t^2}\,dt$

$$= -e^{-(\ln x)^2}\left(\frac{1}{x}\right) + e^{-(2x)^2}(2) = -\frac{e^{-(\ln x)^2}}{x} + 2e^{-4x^2}$$

109. $\displaystyle f_{\text{ave}} = \frac{1}{4-1}\int_1^4 \frac{1}{x}\,dx = \frac{1}{3}\Big[\ln|x|\Big]_1^4 = \frac{1}{3}[\ln 4 - \ln 1] = \frac{1}{3}\ln 4$

110. $A = \int_{-2}^0 \left(e^{-x} - e^x\right)dx + \int_0^1 \left(e^x - e^{-x}\right)dx = \left[-e^{-x} - e^x\right]_{-2}^0 + \left[e^x + e^{-x}\right]_0^1$

$$= [(-1-1) - (-e^2 - e^{-2})] + [(e + e^{-1}) - (1+1)] = e^2 + e + e^{-1} + e^{-2} - 4$$

111. $\displaystyle V = \int_0^1 \frac{2\pi x}{1+x^4}\,dx$ by cylindrical shells. Let $u = x^2 \;\Rightarrow\; du = 2x\,dx$. Then

$$V = \int_0^1 \frac{\pi}{1+u^2}\,du = \pi\left[\tan^{-1}u\right]_0^1 = \pi\left(\tan^{-1}1 - \tan^{-1}0\right) = \pi\left(\frac{\pi}{4}\right) = \frac{\pi^2}{4}.$$

112. $f(x) = x + x^2 + e^x \;\Rightarrow\; f'(x) = 1 + 2x + e^x$ and $f(0) = 1 \;\Rightarrow\; g(1) = 0$, so

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{2}.$$

113. $f(x) = \ln x + \tan^{-1}x \;\Rightarrow\; f(1) = \ln 1 + \tan^{-1}1 = \frac{\pi}{4} \;\Rightarrow\; g\left(\frac{\pi}{4}\right) = 1.$

$$f'(x) = \frac{1}{x} + \frac{1}{1+x^2}, \text{ so } g'\left(\tfrac{\pi}{4}\right) = \frac{1}{f'(1)} = \frac{1}{3/2} = \frac{2}{3}.$$

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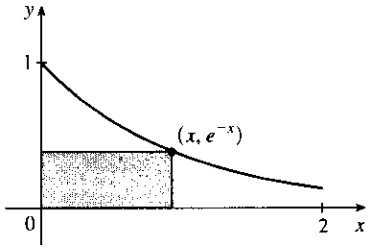
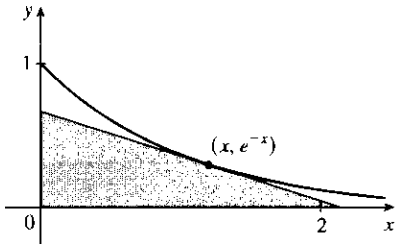
114.



The area of such a rectangle is just the product of its sides, that is, $A(x) = x \cdot e^{-x}$. We want to find the maximum of this function, so we differentiate: $A'(x) = x\left(-e^{-x}\right) + e^{-x}(1) = e^{-x}(1-x)$. This is 0 only at $x = 1$, and changes from positive to negative there, so by the First Derivative Test this gives a local maximum. So the largest area is $A(1) = 1/e$.

115.



We find the equation of a tangent to the curve $y = e^{-x}$, so that we can find the x- and y-intercepts of this tangent, and then we can find the area of the triangle. The slope of the tangent at the point $\left(a, e^{-a}\right)$ is given by $\left.\dfrac{d}{dx}e^{-x}\right]_{x=a} = -e^{-a}$, and so the equation of the tangent is $y - e^{-a} = -e^{-a}(x-a) \quad \Leftrightarrow \quad y = e^{-a}(a - x + 1)$.

The y-intercept of this line is $y = e^{-a}(a - 0 + 1) = e^{-a}(a+1)$. To find the x-intercept we set $y = 0 \quad \Rightarrow \quad e^{-a}(a - x + 1) = 0 \quad \Rightarrow \quad x = a + 1$. So the area of the triangle is $A(a) = \frac{1}{2}\left[e^{-a}(a+1)\right](a+1) = \frac{1}{2}e^{-a}(a+1)^2$. We differentiate this with respect to a: $A'(a) = \frac{1}{2}\left[e^{-a}(2)(a+1) + (a+1)^2 e^{-a}(-1)\right] = \frac{1}{2}e^{-a}\left(1 - a^2\right)$. This is 0 at $a = \pm 1$, and the root $a = 1$ gives a maximum, by the First Derivative Test. So the maximum area of the triangle is $A(1) = \frac{1}{2}e^{-1}(1+1)^2 = 2e^{-1} = 2/e$.

116. Using Formula 5.2.3 with $a = 0$ and $b = 1$, we have $\displaystyle\int_0^1 e^x\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} e^{i/n}$. This series is a geometric series with $a = r = e^{1/n}$, so $\displaystyle\sum_{i=1}^{n} e^{i/n} = e^{1/n}\frac{e^{n/n}-1}{e^{1/n}-1} = e^{1/n}\frac{e-1}{e^{1/n}-1} \quad \Rightarrow$

$$\int_0^1 e^x\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} e^{i/n} = \lim_{n\to\infty}(e-1)e^{1/n}\frac{1/n}{e^{1/n}-1}.$$

As $n \to \infty$, $1/n \to 0^+$, so $e^{1/n} \to e^0 = 1$. Let $t = 1/n$. Then $e^{1/n} - 1 = e^t - 1 \to 0^+$, so l'Hospital's Rule gives $\lim\limits_{t\to 0}\dfrac{t}{e^t-1} = \lim\limits_{t\to 0}\dfrac{1}{e^t} = 1$ and we have

$$\int_0^1 e^x\,dx = \left[\lim_{t\to 0^+}(e-1)e^t\right]\left[\lim_{t\to 0^+}\frac{t}{e^t-1}\right] = e - 1.$$

117. $\displaystyle\lim_{x\to -1}F(x) = \lim_{x\to -1}\frac{b^{x+1}-a^{x+1}}{x+1} \overset{\text{H}}{=} \lim_{x\to -1}\frac{b^{x+1}\ln b - a^{x+1}\ln a}{1} = \ln b - \ln a = F(-1)$, so F is continuous at -1.

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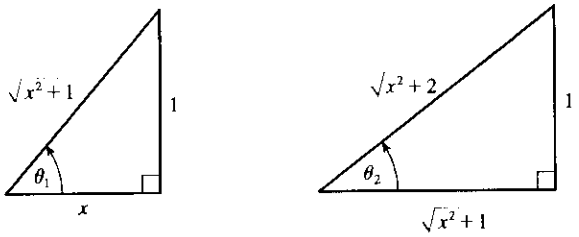
118. Let $\theta_1 = \operatorname{arccot} x$, so $\cot\theta_1 = x = x/1$. So $\sin(\operatorname{arccot} x) = \sin\theta_1 = \dfrac{1}{\sqrt{x^2+1}}$.

Let $\theta_2 = \arctan\left[\dfrac{1}{\sqrt{x^2+1}}\right]$, so $\tan\theta_2 = \dfrac{1}{\sqrt{x^2+1}}$.

Hence, $\cos(\arctan(\sin(\operatorname{arccot} x))) = \cos\theta_2 = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2+2}} = \sqrt{\dfrac{x^2+1}{x^2+2}}$.



119. Differentiating both sides of the given equation, using the Fundamental Theorem for each side, gives

$f(x) = e^{2x} + 2xe^{2x} + e^{-x}f(x)$. So $f(x)(1-e^{-x}) = e^{2x} + 2xe^{2x}$. Hence $f(x) = \dfrac{e^{2x}(1+2x)}{1-e^{-x}}$.

120. (a) Let $f(x) = x - \ln x - 1$, so $f'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x}$. Since $x > 0$, $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. So there is an absolute minimum at $x = 1$ with $f(1) = 0$.
So for $x > 0$, $x \neq 1$, $x - \ln x - 1 = f(x) > f(1) = 0$, and hence $\ln x < x - 1$.

(b) Here let $f(x) = \ln x - \dfrac{x-1}{x} = \ln x - 1 + \dfrac{1}{x}$. So $f'(x) = \dfrac{1}{x} - \dfrac{1}{x^2} = \dfrac{x-1}{x^2}$. As in (a), we see that there is an absolute minimum value at $x = 1$ and that $f(1) = 0$. So for $x > 0$, $x \neq 1$, $\ln x - \dfrac{x-1}{x} = f(x) > f(1) = 0$ and hence $\dfrac{x-1}{x} < \ln x$.

(c) Let $b > a > 0$, so $b/a > 1$. Letting $x = b/a$ in the inequalities in (a) and (b) gives $\dfrac{b-a}{b} = \dfrac{b/a - 1}{b/a} < \ln\dfrac{b}{a} < \dfrac{b}{a} - 1 = \dfrac{b-a}{a}$. Noting that $\ln\dfrac{b}{a} = \ln b - \ln a$, the result follows after dividing through by $b - a$.

(d) Let $f(x) = \ln x$. From the given diagram, we see that (slope of tangent at $x = b$) < (slope of secant line) < (slope of tangent at $x = a$). Since $f'(x) = \dfrac{1}{x}$, we therefore have $\dfrac{1}{b} < \dfrac{\ln b - \ln a}{b-a} < \dfrac{1}{a}$. To make this geometric argument more rigorous, we could use the Mean Value Theorem: For any a and b with $0 < a < b$, there exists some $c \in (a, b)$ for which $f'(c) = \dfrac{1}{c} = \dfrac{\ln b - \ln a}{b-a}$. But $\dfrac{1}{x}$ is a decreasing function on $(0, \infty)$, so $\dfrac{1}{b} < \dfrac{1}{c} = \dfrac{\ln b - \ln a}{b-a} < \ln\dfrac{1}{a}$.

(e) Since $\dfrac{1}{b} < \dfrac{1}{x} < \dfrac{1}{a}$ for $a < x < b$, Property 8 says that $\dfrac{1}{b}(b-a) < \displaystyle\int_a^b \frac{1}{x}\,dx < \frac{1}{a}(b-a) \quad \Rightarrow$

$\dfrac{1}{b}(b-a) < \ln b - \ln a < \dfrac{1}{a}(b-a) \quad \Rightarrow \quad \dfrac{1}{b} < \dfrac{\ln b - \ln a}{b-a} < \dfrac{1}{a}$. (Note from the proof of Property 8 that we are justified in making all of the inequalities strict.)

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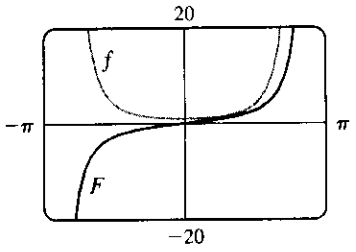
52. $\int \sec^4 \frac{x}{2}\,dx = \int \left(\tan^2 \frac{x}{2} + 1\right) \sec^2 \frac{x}{2}\,dx$

$= \int \left(u^2 + 1\right) 2\,du \quad [u = \tan \frac{x}{2},\ du = \frac{1}{2} \sec^2 \frac{x}{2}\,dx]$

$= \frac{2}{3}u^3 + 2u + C = \frac{2}{3} \tan^3 \frac{x}{2} + 2 \tan \frac{x}{2} + C$



Notice that F is increasing and f is positive on the intervals on which they are defined. Also, F has no horizontal tangent and f is never zero.

53. $f_{\text{ave}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 x \cos^3 x\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 x \left(1 - \sin^2 x\right) \cos x\,dx$

$= \frac{1}{2\pi} \int_0^0 u^2 \left(1 - u^2\right) du \qquad [\text{where } u = \sin x]$

$= 0$

54. (a) Let $u = \cos x$. Then $du = -\sin x\,dx \quad \Rightarrow \quad \int \sin x \cos x\,dx = \int u(-du) = -\frac{1}{2}u^2 + C = -\frac{1}{2} \cos^2 x + C_1$.

(b) Let $u = \sin x$. Then $du = \cos x\,dx \quad \Rightarrow \quad \int \sin x \cos x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2} \sin^2 x + C_2$.

(c) $\int \sin x \cos x\,dx = \int \frac{1}{2} \sin 2x\,dx = -\frac{1}{4} \cos 2x + C_3$

(d) Let $u = \sin x$, $dv = \cos x\,dx$. Then $du = \cos x\,dx$, $v = \sin x$, so

$\int \sin x \cos x\,dx = \sin^2 x - \int \sin x \cos x\,dx$, by Equation 8.1.2, so $\int \sin x \cos x\,dx = \frac{1}{2} \sin^2 x + C_4$.

The answers differ from one another by constants. Since $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$, we find that $-\frac{1}{4} \cos 2x = \frac{1}{2} \sin^2 x - \frac{1}{4} = -\frac{1}{2} \cos^2 x + \frac{1}{4}$.

55. For $0 < x < \frac{\pi}{2}$, we have $0 < \sin x < 1$, so $\sin^3 x < \sin x$. Hence the area is $\int_0^{\pi/2} \left(\sin x - \sin^3 x\right) dx = \int_0^{\pi/2} \sin x \left(1 - \sin^2 x\right) dx = \int_0^{\pi/2} \cos^2 x \sin x\,dx$. Now let $u = \cos x \quad \Rightarrow$ $du = -\sin x\,dx$. Then area $= \int_1^0 u^2 (-du) = \int_0^1 u^2 du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}$.

56. $\sin x > 0$ for $0 < x < \frac{\pi}{2}$, so the sign of $2\sin^2 x - \sin x \quad [\text{which equals } 2\sin x \left(\sin x - \frac{1}{2}\right)]$ is the same as that of $\sin x - \frac{1}{2}$. Thus $2\sin^2 x - \sin x$ is positive on $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ and negative on $\left(0, \frac{\pi}{6}\right)$. The desired area is

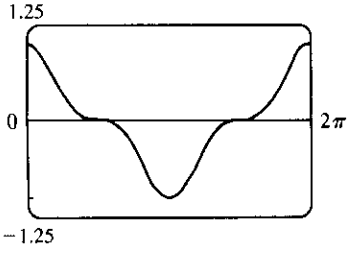
$\int_0^{\pi/6} \left(\sin x - 2\sin^2 x\right) dx + \int_{\pi/6}^{\pi/2} \left(2\sin^2 x - \sin x\right) dx$

$= \int_0^{\pi/6} (\sin x - 1 + \cos 2x)\,dx + \int_{\pi/6}^{\pi/2} (1 - \cos 2x - \sin x)\,dx$

$= \left[-\cos x - x + \frac{1}{2} \sin 2x\right]_0^{\pi/6} + \left[x - \frac{1}{2} \sin 2x + \cos x\right]_{\pi/6}^{\pi/2}$

$= -\frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{4} - (-1) + \frac{\pi}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) = 1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2}$

57.



It seems from the graph that $\int_0^{2\pi} \cos^3 x\,dx = 0$, since the area below the x-axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is $\left[\sin x - \frac{1}{3} \sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$ (n any integer) is 0.

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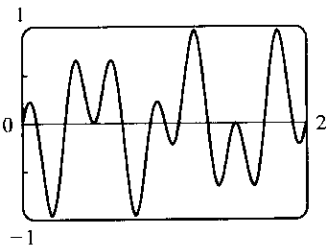
58.



It seems from the graph that $\int_0^2 \sin 2\pi x \cos 5\pi x\,dx = 0$, since each bulge above the x-axis seems to have a corresponding depression below the x-axis. To evaluate the integral, we use a trigonometric identity:

$$\begin{aligned}\int_0^1 \sin 2\pi x \cos 5\pi x\,dx &= \tfrac{1}{2}\int_0^2 [\sin(2\pi x - 5\pi x) + \sin(2\pi x + 5\pi x)]\,dx \\ &= \tfrac{1}{2}\int_0^2 [\sin(-3\pi x) + \sin 7\pi x]\,dx \\ &= \tfrac{1}{2}\left[\tfrac{1}{3\pi}\cos(-3\pi x) - \tfrac{1}{7\pi}\cos 7\pi x\right]_0^2 \\ &= \tfrac{1}{2}\left[\tfrac{1}{3\pi}(1-1) - \tfrac{1}{7\pi}(1-1)\right] = 0\end{aligned}$$

59. $V = \int_{\pi/2}^{\pi} \pi \sin^2 x\,dx = \pi\int_{\pi/2}^{\pi} \tfrac{1}{2}(1-\cos 2x)\,dx = \pi\left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_{\pi/2}^{\pi} = \pi\left(\tfrac{\pi}{2} - 0 - \tfrac{\pi}{4} + 0\right) = \tfrac{\pi^2}{4}$

60. $\text{Volume} = \int_0^{\pi/4} \pi\left(\tan^2 x\right)^2 dx = \pi\int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right)dx = \pi\int_0^{\pi/4} \tan^2 x \sec^2 x\,dx - \pi\int_0^{\pi/4} \tan^2 x\,dx$

$= \pi\int_0^{\pi/4} u^2 du - \pi\int_0^{\pi/4} \left(\sec^2 x - 1\right)dx$ [where $u = \tan x$ and $du = \sec^2 x\,dx$]

$= \pi\left[\tfrac{1}{3}u^3\right]_{x=0}^{\pi/4} - \pi[\tan x - x]_0^{\pi/4} = \pi\left[\tfrac{1}{3}\tan^3 x - \tan x + x\right]_0^{\pi/4} = \pi\left[\tfrac{1}{3} - 1 + \tfrac{\pi}{4}\right] = \pi\left(\tfrac{\pi}{4} - \tfrac{2}{3}\right)$

61. $\text{Volume} = \pi\int_0^{\pi/2} \left[(1+\cos x)^2 - 1^2\right]dx = \pi\int_0^{\pi/2} \left(2\cos x + \cos^2 x\right)dx$

$= \pi\left[2\sin x + \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi\left(2 + \tfrac{\pi}{4}\right) = 2\pi + \tfrac{\pi^2}{4}$

62. $\text{Volume} = \pi\int_0^{\pi/2} \left[1^2 - (1-\cos x)^2\right]dx = \pi\int_0^{\pi/2} \left(2\cos x - \cos^2 x\right)dx$

$= \pi\left[2\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi\left[\left(2 - \tfrac{\pi}{4} - 0\right) - 0\right] = 2\pi - \tfrac{\pi^2}{4}$

63. $s = f(t) = \int_0^t \sin\omega u \cos^2 \omega u\,du$. Let $y = \cos\omega u \;\Rightarrow\; dy = -\omega\sin\omega u\,du$. Then

$s = -\tfrac{1}{\omega}\int_1^{\cos\omega t} y^2 dy = -\tfrac{1}{\omega}\left[\tfrac{1}{3}y^3\right]_1^{\cos\omega t} = \tfrac{1}{3\omega}\left(1 - \cos^3\omega t\right)$.

64. (a) We want to calculate the square root of the average value of $[E(t)]^2 = [155\sin(120\pi t)]^2 = 155^2\sin^2(120\pi t)$. First, we calculate the average value itself, by integrating $[E(t)]^2$ over one cycle (between $t = 0$ and $t = \frac{1}{60}$, since there are 60 cycles per second) and dividing by $\left(\frac{1}{60} - 0\right)$:

$$\begin{aligned}[E(t)]^2_{\text{ave}} &= \tfrac{1}{1/60}\int_0^{1/60} \left[155^2\sin^2(120\pi t)\right]dt = 60\cdot 155^2\int_0^{1/60} \tfrac{1}{2}[1-\cos(240\pi t)]\,dt \\ &= 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[t - \tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[\left(\tfrac{1}{60} - 0\right) - (0-0)\right] = \tfrac{155^2}{2}\end{aligned}$$

The RMS value is just the square root of this quantity, which is $\frac{155}{\sqrt{2}} \approx 110$ V.

(b) $220 = \sqrt{[E(t)]^2_{\text{ave}}} \;\Rightarrow$

$$\begin{aligned}220^2 = [E(t)]^2_{\text{ave}} &= \tfrac{1}{1/60}\int_0^{1/60} A^2\sin^2(120\pi t)\,dt = 60A^2\int_0^{1/60} \tfrac{1}{2}[1-\cos(240\pi t)]\,dt \\ &= 30A^2\left[t - \tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 30A^2\left[\left(\tfrac{1}{60} - 0\right) - (0-0)\right] = \tfrac{1}{2}A^2\end{aligned}$$

Thus, $220^2 = \frac{1}{2}A^2 \;\Rightarrow\; A = 220\sqrt{2} \approx 311$ V.

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65. Just note that the integrand is odd $[f(-x) = -f(x)]$.

Or: If $m \neq n$, calculate

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = \int_{-\pi}^{\pi} \tfrac{1}{2}[\sin(m-n)x + \sin(m+n)x]\, dx$$

$$= \frac{1}{2}\left[-\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$$

If $m = n$, then the first term in each set of brackets is zero.

66. $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \int_{-\pi}^{\pi} \tfrac{1}{2}[\cos(m-n)x - \cos(m+n)x]\, dx$. If $m \neq n$,

this is equal to $\dfrac{1}{2}\left[\dfrac{\sin(m-n)x}{m-n} - \dfrac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$. If $m = n$, we get

$$\int_{-\pi}^{\pi} \tfrac{1}{2}[1 - \cos(m+n)x]\, dx = \left[\tfrac{1}{2}x\right]_{-\pi}^{\pi} - \left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi - 0 = \pi.$$

67. $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{\pi} \tfrac{1}{2}[\cos(m-n)x + \cos(m+n)x]\, dx$. If $m \neq n$,

this is equal to $\dfrac{1}{2}\left[\dfrac{\sin(m-n)x}{m-n} + \dfrac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$. If $m = n$, we get

$$\int_{-\pi}^{\pi} \tfrac{1}{2}[1 + \cos(m+n)x]\, dx = \left[\tfrac{1}{2}x\right]_{-\pi}^{\pi} + \left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi + 0 = \pi.$$

68. $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\, \sin mx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\left(\sum_{n=1}^{m} a_n \sin nx\right)\sin mx\right] dx = \sum_{n=1}^{m} \frac{a_n}{\pi}\int_{-\pi}^{\pi} \sin mx \sin nx\, dx.$

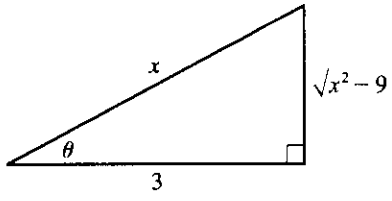
By Exercise 66, every term is zero except the mth one, and that term is $\dfrac{a_m}{\pi} \cdot \pi = a_m$.

8.3 Trigonometric Substitution

1. Let $x = 3\sec\theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\tan\theta\, d\theta$ and

$$\sqrt{x^2 - 9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9(\sec^2\theta - 1)} = \sqrt{9\tan^2\theta}$$

$$= 3|\tan\theta| = 3\tan\theta \text{ for the relevant values of } \theta.$$



$$\int \frac{1}{x^2\sqrt{x^2-9}}\, dx = \int \frac{1}{9\sec^2\theta \cdot 3\tan\theta}\, 3\sec\theta\tan\theta\, d\theta = \tfrac{1}{9}\int \cos\theta\, d\theta = \tfrac{1}{9}\sin\theta + C = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

Note that $-\sec(\theta + \pi) = \sec\theta$, so the figure is sufficient for the case $\pi \leq \theta < \frac{3\pi}{2}$.

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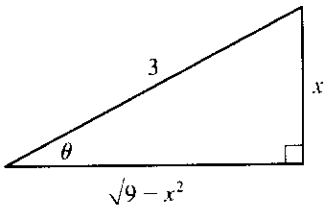
2. Let $x = 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 3\cos\theta\,d\theta$ and

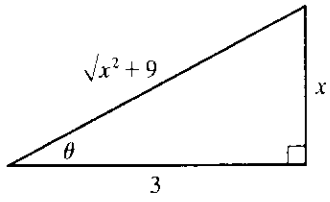
$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta}$$
$$= 3\,|\cos\theta| = 3\cos\theta \text{ for the relevant values of } \theta.$$



$$\int x^3\sqrt{9-x^2}\,dx = \int 3^3\sin^3\theta\cdot 3\cos\theta\cdot 3\cos\theta\,d\theta = 3^5\int \sin^3\theta\cos^2\theta\,d\theta$$
$$= 3^5\int \sin^2\theta\cos^2\theta\sin\theta\,d\theta = 3^5\int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta$$
$$= 3^5\int (1-u^2)u^2\,(-du) \qquad [u=\cos\theta,\ du=-\sin\theta\,d\theta]$$
$$= 3^5\int (u^4-u^2)\,du = 3^5\left(\tfrac{1}{5}u^5-\tfrac{1}{3}u^3\right)+C = 3^5\left(\tfrac{1}{5}\cos^5\theta-\tfrac{1}{3}\cos^3\theta\right)+C$$
$$= 3^5\left[\frac{1}{5}\frac{(9-x^2)^{5/2}}{3^5}-\frac{1}{3}\frac{(9-x^2)^{3/2}}{3^3}\right]+C$$
$$= \tfrac{1}{5}(9-x^2)^{5/2}-3(9-x^2)^{3/2}+C \text{ or } -\tfrac{1}{5}(x^2+6)(9-x)^{3/2}+C$$

3. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9(\tan^2\theta+1)} = \sqrt{9\sec^2\theta}$$
$$= 3\,|\sec\theta| = 3\sec\theta \text{ for the relevant values of } \theta.$$



$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{3^3\tan^3\theta}{3\sec\theta}3\sec^2\theta\,d\theta = 3^3\int \tan^3\theta\sec\theta d\theta = 3^3\int \tan^2\theta\tan\theta\sec\theta\,d\theta$$
$$= 3^3\int (\sec^2\theta-1)\tan\theta\sec\theta\,d\theta = 3^3\int (u^2-1)\,du \qquad [u=\sec\theta,\ du=\sec\theta\tan\theta\,d\theta]$$
$$= 3^3\left(\tfrac{1}{3}u^3-u\right)+C = 3^3\left(\tfrac{1}{3}\sec^3\theta-\sec\theta\right)+C = 3^3\left[\frac{1}{3}\frac{(x^2+9)^{3/2}}{3^3}-\frac{\sqrt{x^2+9}}{3}\right]+C$$
$$= \tfrac{1}{3}(x^2+9)^{3/2}-9\sqrt{x^2+9}+C \quad \text{or} \quad \tfrac{1}{3}(x^2-18)\sqrt{x^2+9}+C$$

4. Let $x = 4\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 4\cos\theta\,d\theta$ and $\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4\,|\cos\theta| = 4\cos\theta$. When $x = 0$, $4\sin\theta = 0 \;\Rightarrow\; \theta = 0$, and when $x = 2\sqrt{3}$, $4\sin\theta = 2\sqrt{3} \;\Rightarrow\; \sin\theta = \frac{\sqrt{3}}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}$. Thus, substitution gives

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = \int_0^{\pi/3} \frac{4^3\sin^3\theta}{4\cos\theta}4\cos\theta\,d\theta = 4^3\int_0^{\pi/3}\sin^3\theta\,d\theta$$
$$= 4^3\int_0^{\pi/3}(1-\cos^2\theta)\sin\theta\,d\theta$$
$$\overset{c}{=} -4^3\int_1^{1/2}(1-u^2)\,du = -64\left[u-\tfrac{1}{3}u^3\right]_1^{1/2}$$
$$= -64\left[\left(\tfrac{1}{2}-\tfrac{1}{24}\right)-\left(1-\tfrac{1}{3}\right)\right] = -64\left(-\tfrac{5}{24}\right) = \tfrac{40}{3}$$

Or: Let $u = 16-x^2$, $x^2 = 16-u$, $du = -2x\,dx$.

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5. Let $t = \sec\theta$, so $dt = \sec\theta\tan\theta\,d\theta$, $t = \sqrt{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right)\right] = \tfrac{1}{2}\left(\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{8} - \tfrac{1}{4}$$

6. Let $x = 2\tan\theta$, so $dx = 2\sec^2\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 2 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Then

$$\int_0^2 x^3\sqrt{x^2+4}\,dx = \int_0^{\pi/4} 2^3\tan^3\theta\cdot 2\sec\theta\cdot 2\sec^2\theta\,d\theta = 2^5\int_0^{\pi/4}\tan^2\theta\sec^2\theta\sec\theta\tan\theta\,d\theta$$

$$= 2^5\int_0^{\pi/4}(\sec^2\theta - 1)\sec^2\theta\sec\theta\tan\theta\,d\theta$$

$$= 2^5\int_1^{\sqrt{2}}(u^2-1)u^2\,du \quad [u = \sec\theta,\ du = \sec\theta\tan\theta\,d\theta]$$

$$= 2^5\int_1^{\sqrt{2}}(u^4-u^2)\,du = 2^5\left[\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3\right]_1^{\sqrt{2}} = 2^5\left[\left(\tfrac{1}{5}\cdot 4\sqrt{2} - \tfrac{1}{3}\cdot 2\sqrt{2}\right) - \left(\tfrac{1}{5} - \tfrac{1}{3}\right)\right]$$

$$= 32\left(\tfrac{2}{15}\sqrt{2} + \tfrac{2}{15}\right) = \tfrac{64}{15}\left(\sqrt{2}+1\right)$$

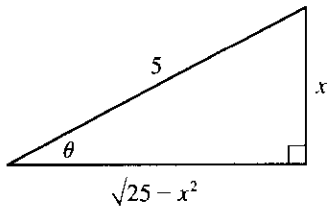
Or: Let $u = x^2+4$, $x^2 = u-4$, $du = 2x\,dx$.

7. Let $x = 5\sin\theta$, so $dx = 5\cos\theta\,d\theta$. Then

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{5^2\sin^2\theta\cdot 5\cos\theta}5\cos\theta\,d\theta$$

$$= \tfrac{1}{25}\int \csc^2\theta\,d\theta = -\tfrac{1}{25}\cot\theta + C$$

$$= -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C$$



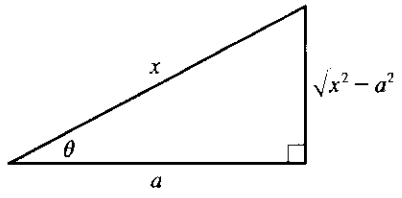
8. Let $x = a\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = a\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-a^2} = a\tan\theta$, so

$$\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx = \int \frac{a\tan\theta}{a^4\sec^4\theta}a\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{a^2}\int \sin^2\theta\cos\theta\,d\theta$$

$$= \frac{1}{3a^2}\sin^3\theta + C = \frac{\left(x^2-a^2\right)^{3/2}}{3a^2x^3} + C$$



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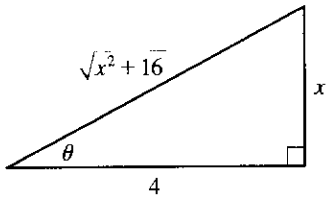
9. Let $x = 4\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 4\sec^2\theta\,d\theta$ and

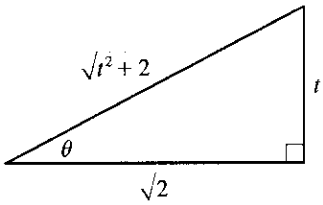


$$\begin{aligned}\sqrt{x^2+16} &= \sqrt{16\tan^2\theta+16} = \sqrt{16(\tan^2\theta+1)}\\ &= \sqrt{16\sec^2\theta} = 4\,|\sec\theta|\\ &= 4\sec\theta \text{ for the relevant values of } \theta.\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+16}} &= \int \frac{4\sec^2\theta\,d\theta}{4\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1\\ &= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C_1 = \ln\left|\sqrt{x^2+16}+x\right| - \ln|4| + C_1\\ &= \ln\left(\sqrt{x^2+16}+x\right) + C, \text{ where } C = C_1 - \ln 4.\end{aligned}$$

(Since $\sqrt{x^2+16}+x > 0$, we don't need the absolute value.)

10. Let $t = \sqrt{2}\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dt = \sqrt{2}\sec^2\theta\,d\theta$ and

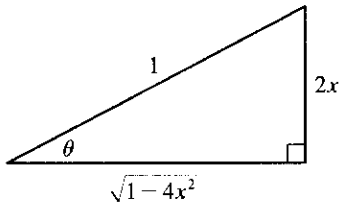


$$\begin{aligned}\sqrt{t^2+2} &= \sqrt{2\tan^2\theta+2} = \sqrt{2(\tan^2\theta+1)} = \sqrt{2\sec^2\theta}\\ &= \sqrt{2}\,|\sec\theta| = \sqrt{2}\sec\theta \text{ for the relevant values of } \theta.\end{aligned}$$

$$\begin{aligned}\int \frac{t^5}{\sqrt{t^2+2}}\,dt &= \int \frac{4\sqrt{2}\tan^5\theta}{\sqrt{2}\sec\theta}\sqrt{2}\sec^2\theta\,d\theta = 4\sqrt{2}\int \tan^5\theta\sec\theta\,d\theta = 4\sqrt{2}\int \left(\sec^2\theta-1\right)^2\sec\theta\tan\theta\,d\theta\\ &= 4\sqrt{2}\int \left(u^2-1\right)^2 du \quad [u=\sec\theta,\ du=\sec\theta\tan\theta\,d\theta] \quad = 4\sqrt{2}\int \left(u^4-2u^2+1\right) du\\ &= 4\sqrt{2}\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right) + C = \frac{4\sqrt{2}}{15}u(3u^4-10u^2+15) + C\\ &= \frac{4\sqrt{2}}{15}\cdot\frac{\sqrt{t^2+2}}{\sqrt{2}}\left[3\cdot\frac{(t^2+2)^2}{2^2} - 10\,\frac{t^2+2}{2} + 15\right] + C\\ &= \tfrac{4}{15}\sqrt{t^2+2}\cdot\tfrac{1}{4}\left[3(t^4+4t^2+4) - 20(t^2+2) + 60\right] + C\\ &= \tfrac{1}{15}\sqrt{t^2+2}\,(3t^4-8t^2+32) + C\end{aligned}$$

11. Let $2x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $x = \frac{1}{2}\sin\theta$,

$dx = \frac{1}{2}\cos\theta\,d\theta$, and $\sqrt{1-4x^2} = \sqrt{1-(2x)^2} = \cos\theta$.



$$\begin{aligned}\int \sqrt{1-4x^2}\,dx &= \int \cos\theta\left(\tfrac{1}{2}\cos\theta\right)d\theta = \tfrac{1}{4}\int (1+\cos 2\theta)\,d\theta\\ &= \tfrac{1}{4}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{4}(\theta + \sin\theta\cos\theta) + C\\ &= \tfrac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\right] + C\end{aligned}$$

12. $\int_0^1 x\sqrt{x^2+4}\,dx = \int_4^5 \sqrt{u}\left(\frac{1}{2}\,du\right) \quad [u = x^2+4,\ du = 2x\,dx] \quad = \frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_4^5 = \frac{1}{3}\left(5\sqrt{5}-8\right)$

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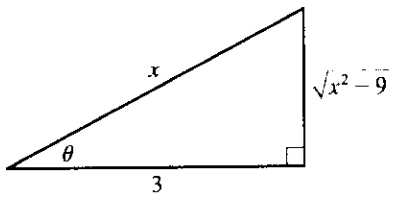
13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so

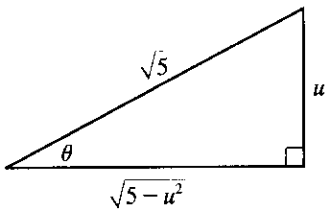


$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \tfrac{1}{3}\int \sin^2\theta\,d\theta = \tfrac{1}{3}\int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\frac{\sqrt{x^2-9}}{x}\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

14. Let $u = \sqrt{5}\sin\theta$, so $du = \sqrt{5}\cos\theta\,d\theta$. Then



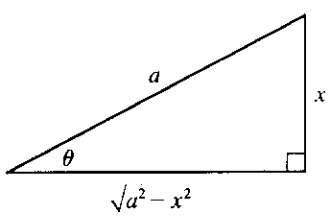
$$\int \frac{du}{u\sqrt{5-u^2}} = \int \frac{1}{\sqrt{5}\sin\theta\cdot\sqrt{5}\cos\theta}\sqrt{5}\cos\theta\,d\theta = \frac{1}{\sqrt{5}}\int \csc\theta\,d\theta$$

$$= \tfrac{1}{\sqrt{5}}\ln|\csc\theta - \cot\theta| + C \quad \text{[by Exercise 8.2.39]}$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u} - \frac{\sqrt{5-u^2}}{u}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}-\sqrt{5-u^2}}{u}\right| + C$$

15. Let $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = a\cos\theta\,d\theta$ and



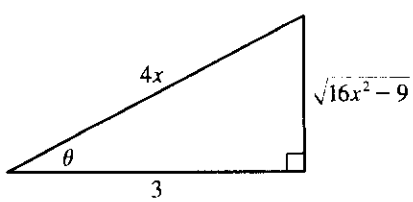
$$\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \int \frac{a^2\sin^2\theta\,a\cos\theta\,d\theta}{a^3\cos^3\theta} = \int \tan^2\theta\,d\theta$$

$$= \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C$$

16. Let $4x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = \frac{3}{4}\sec\theta\tan\theta\,d\theta$ and $\sqrt{16x^2-9} = 3\tan\theta$, so



$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{\frac{3}{4}\sec\theta\tan\theta d\theta}{\left(\frac{3}{4}\right)^2\sec^2\theta\,3\tan\theta}$$

$$= \frac{4}{9}\int \cos\theta\,d\theta = \frac{4}{9}\sin\theta + C = \frac{4}{9}\frac{\sqrt{16x^2-9}}{4x} + C = \frac{\sqrt{16x^2-9}}{9x} + C$$

17. Let $u = x^2 - 7$, so $du = 2x\,dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C.$

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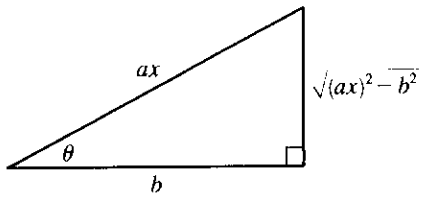
18. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \;\Rightarrow$

$(ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2(\sec^2\theta - 1) = b^2\tan^2\theta$. So

$\sqrt{(ax^2) - b^2} = b\tan\theta$, $dx = \dfrac{b}{a}\sec\theta\tan\theta\,d\theta$, and

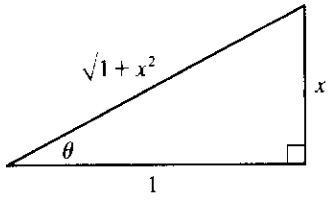


$$\int \frac{dx}{\left[(ax)^2 - b^2\right]^{3/2}} = \int \frac{\frac{b}{a}\sec\theta\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int \csc\theta\cot\theta\,d\theta$$

$$= -\frac{1}{ab^2}\csc\theta + C = -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2 - b^2}} + C = -\frac{x}{b^2\sqrt{(ax)^2 - b^2}} + C$$

19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$

and $\sqrt{1+x^2} = \sec\theta$, so



$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta$$

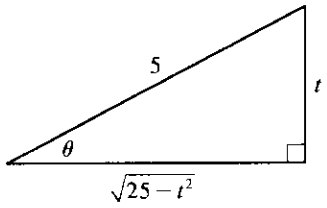
$$= \int (\csc\theta + \sec\theta\tan\theta)\,d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \quad \text{[by Exercise 8.2.39]}$$

$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C$$

20. Let $t = 5\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dt = 5\cos\theta\,d\theta$

and $\sqrt{25 - t^2} = 5\cos\theta$, so



$$\int \frac{t}{\sqrt{25-t^2}}\,dt = \int \frac{5\sin\theta}{5\cos\theta}\,5\cos\theta\,d\theta = 5\int \sin\theta\,d\theta$$

$$= -5\cos\theta + C = -5\cdot\frac{\sqrt{25-t^2}}{5} + C = -\sqrt{25-t^2} + C$$

Or: Let $u = 25 - t^2$, so $du = -2t\,dt$.

21. Let $u = 4 - 9x^2 \;\Rightarrow\; du = -18x\,dx$. Then $x^2 = \frac{1}{9}(4-u)$ and

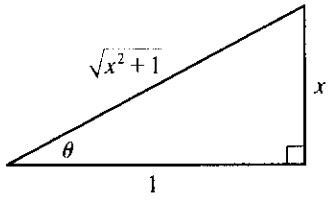
$$\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx = \int_4^0 \tfrac{1}{9}(4-u)u^{1/2}\left(-\tfrac{1}{18}\right)du = \tfrac{1}{162}\int_0^4 \left(4u^{1/2} - u^{3/2}\right)du$$

$$= \tfrac{1}{162}\left[\tfrac{8}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right]_0^4 = \tfrac{1}{162}\left[\tfrac{64}{3} - \tfrac{64}{5}\right] = \tfrac{64}{1215}$$

Or: Let $3x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

22. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$,

$\sqrt{x^2+1} = \sec\theta$ and $x = 0 \;\Rightarrow\; \theta = 0$, $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$, so



$$\int_0^1 \sqrt{x^2+1}\,dx = \int_0^{\pi/4} \sec\theta\sec^2\theta\,d\theta = \int_0^{\pi/4} \sec^3\theta\,d\theta$$

$$= \tfrac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} \quad \text{[by Example 8.2.8]}$$

$$= \tfrac{1}{2}\left[\sqrt{2}\cdot 1 + \ln\left(1+\sqrt{2}\right) - 0 - \ln(1+0)\right] = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

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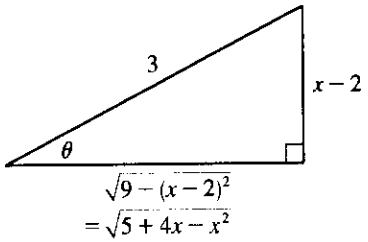
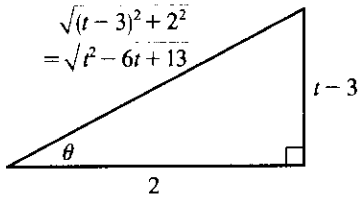
23. $5+4x-x^2 = -(x^2-4x+4)+9 = -(x-2)^2+9$. Let $x-2 = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $dx = 3\cos\theta\,d\theta$. Then



$$\begin{aligned}\int \sqrt{5+4x-x^2}\,dx &= \int \sqrt{9-(x-2)^2}\,dx = \int \sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta \\ &= \int \sqrt{9\cos^2\theta}\,3\cos\theta\,d\theta = \int 9\cos^2\theta\,d\theta \\ &= \tfrac{9}{2}\int(1+\cos 2\theta)\,d\theta = \tfrac{9}{2}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)+C \\ &= \tfrac{9}{2}\theta+\tfrac{9}{4}\sin 2\theta+C = \tfrac{9}{2}\theta+\tfrac{9}{4}(2\sin\theta\cos\theta)+C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3}+C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{1}{2}(x-2)\sqrt{5+4x-x^2}+C\end{aligned}$$

24. $t^2-6t+13 = (t^2-6t+9)+4 = (t-3)^2+2^2$. Let $t-3 = 2\tan\theta$, so $dt = 2\sec^2\theta\,d\theta$. Then



$$\int \frac{dt}{\sqrt{t^2-6t+13}} = \int \frac{1}{\sqrt{(2\tan\theta)^2+2^2}}\,2\sec^2\theta\,d\theta = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta$$

$$\begin{aligned}&= \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|+C_1 \quad \text{[by Formula 8.2.1]} \\ &= \ln\left|\frac{\sqrt{t^2-6t+13}}{2}+\frac{t-3}{2}\right|+C_1 \\ &= \ln\left|\sqrt{t^2-6t+13}+t-3\right|+C \qquad \text{where } C = C_1-\ln 2\end{aligned}$$

25. $9x^2+6x-8 = (3x+1)^2-9$, so let $u = 3x+1$, $du = 3dx$. Then $\displaystyle\int \frac{dx}{\sqrt{9x^2+6x-8}} = \int \frac{\frac{1}{3}\,du}{\sqrt{u^2-9}}$. Now let $u = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $du = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{u^2-9} = 3\tan\theta$, so

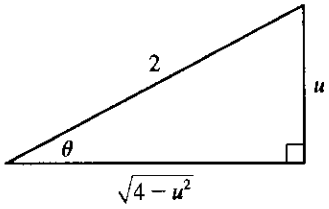
$$\int \frac{\frac{1}{3}\,du}{\sqrt{u^2-9}} = \int \frac{\sec\theta\tan\theta\,d\theta}{3\tan\theta} = \tfrac{1}{3}\int \sec\theta d\theta = \tfrac{1}{3}\ln|\sec\theta+\tan\theta|+C_1 = \tfrac{1}{3}\ln\left|\frac{u+\sqrt{u^2-9}}{3}\right|+C_1$$

$$= \tfrac{1}{3}\ln\left|u+\sqrt{u^2-9}\right|+C = \tfrac{1}{3}\ln\left|3x+1+\sqrt{9x^2+6x-8}\right|+C$$

26. $4x-x^2 = -(x^2-4x+4)+4 = 4-(x-2)^2$, so let $u = x-2$. Then $x = u+2$ and $dx = du$, so

$$\int \frac{x^2\,dx}{\sqrt{4x-x^2}} = \int \frac{(u+2)^2\,du}{\sqrt{4-u^2}} = \int \frac{(2\sin\theta+2)^2}{2\cos\theta}\,2\cos\theta\,d\theta \quad \text{[Put } u = 2\sin\theta\text{]}$$

$$\begin{aligned}&= 4\int(\sin^2\theta+2\sin\theta+1)\,d\theta \\ &= 2\int(1-\cos 2\theta)\,d\theta+8\int\sin\theta d\theta+4\int d\theta \\ &= 2\theta-\sin 2\theta-8\cos\theta+4\theta+C \\ &= 6\theta-8\cos\theta-2\sin\theta\cos\theta+C \\ &= 6\sin^{-1}\left(\tfrac{1}{2}u\right)-4\sqrt{4-u^2}-\tfrac{1}{2}u\sqrt{4-u^2}+C \\ &= 6\sin^{-1}\left(\frac{x-2}{2}\right)-4\sqrt{4x-x^2}-\left(\frac{x-2}{2}\right)\sqrt{4x-x^2}+C\end{aligned}$$



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27. $x^2+2x+2=(x+1)^2+1$. Let $u=x+1$, $du=dx$. Then

$$\int \frac{dx}{(x^2+2x+2)^2}=\int \frac{du}{(u^2+1)^2}=\int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} \qquad \left[\begin{array}{c}\text{where } u=\tan\theta,\ du=\sec^2\theta\, d\theta,\\ \text{and } u^2+1=\sec^2\theta\end{array}\right]$$

$$=\int \cos^2\theta\, d\theta=\tfrac{1}{2}\int(1+\cos 2\theta)\, d\theta=\tfrac{1}{2}(\theta+\sin\theta\cos\theta)+C$$

$$=\frac{1}{2}\left[\tan^{-1}u+\frac{u}{1+u^2}\right]+C=\frac{1}{2}\left[\tan^{-1}(x+1)+\frac{x+1}{x^2+2x+2}\right]+C$$

28. $5-4x-x^2=-\left(x^2+4x+4\right)+9=9-(x+2)^2$. Let $u=x+2 \;\Rightarrow\; du=dx$. Then

$$\int \frac{dx}{(5-4x-x^2)^{5/2}}=\int \frac{du}{(9-u^2)^{5/2}}=\int \frac{3\cos\theta\, d\theta}{(3\cos\theta)^5} \qquad \left[\begin{array}{c}\text{where } u=3\sin\theta,\ du=3\cos\theta\, d\theta,\\ \text{and } \sqrt{9-u^2}=3\cos\theta\end{array}\right]$$

$$=\tfrac{1}{81}\int \sec^4\theta\, d\theta=\tfrac{1}{81}\int\left(\tan^2\theta+1\right)\sec^2\theta\, d\theta=\tfrac{1}{81}\left[\tfrac{1}{3}\tan^3\theta+\tan\theta\right]+C$$

$$=\frac{1}{243}\left[\frac{u^3}{(9-u^2)^{3/2}}+\frac{3u}{\sqrt{9-u^2}}\right]+C=\frac{1}{243}\left[\frac{(x+2)^3}{(5-4x-x^2)^{3/2}}+\frac{3(x+2)}{\sqrt{5-4x-x^2}}\right]+C$$

29. Let $u=x^2$, $du=2x\, dx$. Then

$$\int x\sqrt{1-x^4}\, dx=\int \sqrt{1-u^2}\left(\tfrac{1}{2}\, du\right)=\tfrac{1}{2}\int \cos\theta\cdot\cos\theta\, d\theta \qquad \left[\begin{array}{c}\text{where } u=\sin\theta,\ du=\cos\theta\, d\theta,\\ \text{and } \sqrt{1-u^2}=\cos\theta\end{array}\right]$$

$$=\tfrac{1}{2}\int \tfrac{1}{2}(1+\cos 2\theta)d\theta=\tfrac{1}{4}\theta+\tfrac{1}{8}\sin 2\theta+C=\tfrac{1}{4}\theta+\tfrac{1}{4}\sin\theta\cos\theta+C$$

$$=\tfrac{1}{4}\sin^{-1}u+\tfrac{1}{4}u\sqrt{1-u^2}+C=\tfrac{1}{4}\sin^{-1}(x^2)+\tfrac{1}{4}x^2\sqrt{1-x^4}+C$$

30. Let $u=\sin t$, $du=\cos t\, dt$. Then

$$\int_0^{\pi/2}\frac{\cos t}{\sqrt{1+\sin^2 t}}\, dt=\int_0^1\frac{1}{\sqrt{1+u^2}}\, du=\int_0^{\pi/4}\frac{1}{\sec\theta}\sec^2\theta\, d\theta \qquad \left[\begin{array}{c}\text{where } u=\tan\theta,\ du=\sec^2\theta\, d\theta,\\ \text{and } \sqrt{1+u^2}=\sec\theta\end{array}\right]$$

$$=\textstyle\int_0^{\pi/4}\sec\theta\, d\theta=\left[\ln|\sec\theta+\tan\theta|\right]_0^{\pi/4} \qquad \text{[by (1) in Section 8.2]}$$

$$=\ln\left(\sqrt{2}+1\right)-\ln(1+0)=\ln\left(\sqrt{2}+1\right)$$

31. (a) Let $x=a\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $\sqrt{x^2+a^2}=a\sec\theta$ and

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\int\frac{a\sec^2\theta\, d\theta}{a\sec\theta}=\int\sec\theta\, d\theta=\ln|\sec\theta+\tan\theta|+C_1=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|+C_1$$

$$=\ln\left(x+\sqrt{x^2+a^2}\right)+C \quad \text{where } C=C_1-\ln|a|$$

(h) Let $x=a\sinh t$, so that $dx=a\cosh t\, dt$ and $\sqrt{x^2+a^2}=a\cosh t$. Then

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\int\frac{a\cosh t\, dt}{a\cosh t}=t+C=\sinh^{-1}\frac{x}{a}+C.$$

32. (a) Let $x=a\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then

$$I=\int\frac{x^2}{(x^2+a^2)^{3/2}}\, dx=\int\frac{a^2\tan^2\theta}{a^3\sec^3\theta}a\sec^2\theta\, d\theta=\int\frac{\tan^2\theta}{\sec\theta}\, d\theta=\int\frac{\sec^2\theta-1}{\sec\theta}\, d\theta$$

$$=\int(\sec\theta-\cos\theta)\, d\theta=\ln|\sec\theta+\tan\theta|-\sin\theta+C$$

$$=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|-\frac{x}{\sqrt{x^2+a^2}}+C=\ln\left(x+\sqrt{x^2+a^2}\right)-\frac{x}{\sqrt{x^2+a^2}}+C_1$$

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(b) Let $x = a\sinh t$. Then

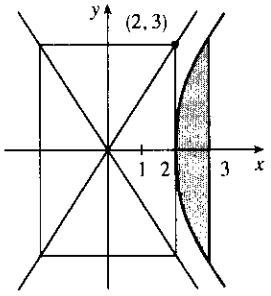
$$I = \int \frac{a^2\sinh^2 t}{a^3\cosh^3 t}\, a\cosh t\, dt = \int \tanh^2 t\, dt = \int \left(1 - \operatorname{sech}^2 t\right) dt = t - \tanh t + C$$
$$= \sinh^{-1}\frac{x}{a} - \frac{x}{\sqrt{a^2+x^2}} + C$$

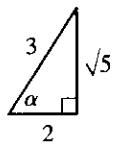
33. The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1, 7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\, dx = \frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot \sec\theta\tan\theta\, d\theta \quad \left[\begin{array}{c}\text{where } x = \sec\theta,\ dx = \sec\theta\tan\theta\, d\theta,\\ \sqrt{x^2-1} = \tan\theta, \text{ and } \alpha = \sec^{-1} 7\end{array}\right]$$
$$= \tfrac{1}{6}\int_0^\alpha \tan^2\theta\, d\theta = \tfrac{1}{6}\int_0^\alpha (\sec^2\theta - 1)\, d\theta$$
$$= \tfrac{1}{6}\Big[\tan\theta - \theta\Big]_0^\alpha = \tfrac{1}{6}(\tan\alpha - \alpha)$$
$$= \tfrac{1}{6}\left(\sqrt{48} - \sec^{-1} 7\right)$$

34. $9x^2 - 4y^2 = 36 \quad\Rightarrow\quad y = \pm\tfrac{3}{2}\sqrt{x^2-4} \quad\Rightarrow$

$$\text{area} = 2\int_2^3 \tfrac{3}{2}\sqrt{x^2-4}\, dx = 3\int_2^3 \sqrt{x^2-4}\, dx$$





$$= 3\int_0^\alpha 2\tan\theta\, 2\sec\theta\tan\theta\, d\theta \quad \left[\begin{array}{c}\text{where } x = 2\sec\theta,\\ dx = 2\sec\theta\tan\theta\, d\theta,\\ \alpha = \sec^{-1}\frac{3}{2}\end{array}\right]$$
$$= 12\int_0^\alpha \left(\sec^2\theta - 1\right)\sec\theta d\theta = 12\int_0^\alpha \left(\sec^3\theta - \sec\theta\right) d\theta$$
$$= 12\left[\tfrac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) - \ln|\sec\theta + \tan\theta|\right]_0^\alpha$$
$$= 6\Big[\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\Big]_0^\alpha$$
$$= 6\left[\tfrac{3\sqrt{5}}{4} - \ln\left(\tfrac{3}{2} + \tfrac{\sqrt{5}}{2}\right)\right] = \tfrac{9\sqrt{5}}{2} - 6\ln\left(\tfrac{3+\sqrt{5}}{2}\right)$$

35. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\, dx$. Let $x = r\cos u \quad\Rightarrow\quad dx = -r\sin u\, du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

$$\int \sqrt{r^2-x^2}\, dx = \int r\sin u\,(-r\sin u)\, du = -r^2\int \sin^2 u\, du = -\tfrac{1}{2}r^2(u - \sin u\cos u) + C$$
$$= -\tfrac{1}{2}r^2\cos^{-1}(x/r) + \tfrac{1}{2}x\sqrt{r^2-x^2} + C$$

so

$$\text{area of region } PQR = \tfrac{1}{2}\left[-r^2\cos^{-1}(x/r) + x\sqrt{r^2-x^2}\right]_{r\cos\theta}^{r}$$
$$= \tfrac{1}{2}\left[0 - \left(-r^2\theta + r\cos\theta\, r\sin\theta\right)\right]$$
$$= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\cos\theta$$

and thus, (area of sector POR) = (area of $\triangle POQ$) + (area of region PQR) = $\frac{1}{2}r^2\theta$.

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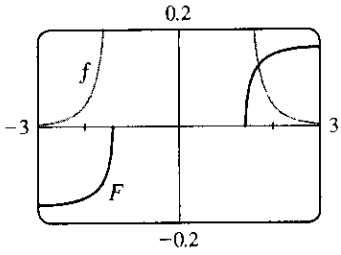
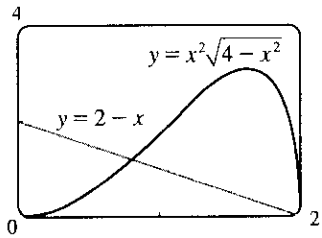
36. Let $x = \sqrt{2}\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, so $dx = \sqrt{2}\sec\theta\tan\theta d\theta$. Then

$$\begin{aligned}\int \frac{dx}{x^4\sqrt{x^2-2}} &= \int \frac{\sqrt{2}\sec\theta\tan\theta\,d\theta}{4\sec^4\theta\sqrt{2}\tan\theta} \\ &= \tfrac{1}{4}\int \cos^3\theta\,d\theta = \tfrac{1}{4}\int (1-\sin^2\theta)\cos\theta\,d\theta \\ &= \tfrac{1}{4}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right] + C \qquad [\text{substitute } u = \sin\theta] \\ &= \frac{1}{4}\left[\frac{\sqrt{x^2-2}}{x} - \frac{(x^2-2)^{3/2}}{3x^3}\right] + C\end{aligned}$$



From the graph, it appears that our answer is reasonable. [Notice that $f(x)$ is large when F increases rapidly and small when F levels out.]

37. From the graph, it appears that the curve $y = x^2\sqrt{4-x^2}$ and the line $y = 2 - x$ intersect at about $x = 0.81$ and $x = 2$, with $x^2\sqrt{4-x^2} > 2 - x$ on $(0.81, 2)$. So the area bounded by the curve and the line is $A \approx$ $\int_{0.81}^{2}\left[x^2\sqrt{4-x^2} - (2-x)\right]dx = \int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx - \left[2x - \frac{1}{2}x^2\right]_{0.81}^{2}$.
To evaluate the integral, we put $x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then



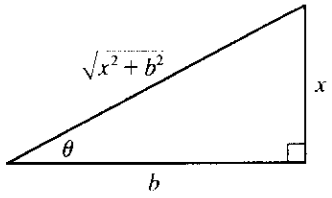
$dx = 2\cos\theta\,d\theta$, $x = 2 \;\Rightarrow\; \theta = \sin^{-1}1 = \frac{\pi}{2}$, and $x = 0.81 \;\Rightarrow\; \theta = \sin^{-1}0.405 \approx 0.417$. So

$$\begin{aligned}\int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx &\approx \int_{0.417}^{\pi/2} 4\sin^2\theta\,(2\cos\theta)(2\cos\theta\,d\theta) = 4\int_{0.417}^{\pi/2}\sin^2 2\theta\,d\theta = 4\int_{0.417}^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta \\ &= 2\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_{0.417}^{\pi/2} = 2\left[\left(\tfrac{\pi}{2}-0\right) - \left(0.417 - \tfrac{1}{4}(0.995)\right)\right] \approx 2.81\end{aligned}$$

Thus, $A \approx 2.81 - \left[\left(2\cdot 2 - \frac{1}{2}\cdot 2^2\right) - \left(2\cdot 0.81 - \frac{1}{2}\cdot 0.81^2\right)\right] \approx 2.10$.

38. Let $x = b\tan\theta$, so that $dx = b\sec^2\theta\,d\theta$ and $\sqrt{x^2+b^2} = b\sec\theta$.

$$\begin{aligned}E(P) &= \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0(x^2+b^2)^{3/2}}\,dx = \frac{\lambda b}{4\pi\varepsilon_0}\int_{\theta_1}^{\theta_2}\frac{1}{(b\sec\theta)^3}\,b\sec^2\theta\,d\theta \\ &= \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\frac{1}{\sec\theta}\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\left[\sin\theta\right]_{\theta_1}^{\theta_2} \\ &= \frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{x}{\sqrt{x^2+b^2}}\right]_{-a}^{L-a} = \frac{\lambda}{4\pi\varepsilon_0 b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\right)\end{aligned}$$



39. Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y-b)^2 = r^2$, where $b = \sqrt{R^2 - r^2}$ is the distance between the centers of the circles. The desired area is

$$\begin{aligned}A &= \int_{-r}^{r}\left[\left(b + \sqrt{r^2-x^2}\right) - \sqrt{R^2-x^2}\right]dx = 2\int_0^r\left(b + \sqrt{r^2-x^2} - \sqrt{R^2-x^2}\right)dx \\ &= 2\int_0^r b\,dx + 2\int_0^r\sqrt{r^2-x^2}\,dx - 2\int_0^r\sqrt{R^2-x^2}\,dx\end{aligned}$$

The first integral is just $2br = 2r\sqrt{R^2-r^2}$. To evaluate the other two integrals, note that

$$\begin{aligned}\int\sqrt{a^2-x^2}\,dx &= \int a^2\cos^2\theta\,d\theta \quad [x = a\sin\theta,\ dx = a\cos\theta\,d\theta] \quad = \tfrac{1}{2}a^2\int(1+\cos 2\theta)\,d\theta \\ &= \tfrac{1}{2}a^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}a^2(\theta + \sin\theta\cos\theta) + C \\ &= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a} + C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C\end{aligned}$$

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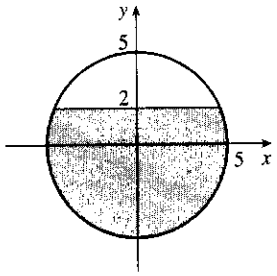
so the desired area is

$$A = 2r\sqrt{R^2-r^2} + \left[r^2\arcsin(x/r) + x\sqrt{r^2-x^2}\right]_0^r - \left[R^2\arcsin(x/R) + x\sqrt{R^2-x^2}\right]_0^r$$

$$= 2r\sqrt{R^2-r^2} + r^2\left(\tfrac{\pi}{2}\right) - \left[R^2\arcsin(r/R) + r\sqrt{R^2-r^2}\right] = r\sqrt{R^2-r^2} + \tfrac{\pi}{2}r^2 - R^2\arcsin(r/R)$$

40. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is



$$A = 2\int_{-5}^{2}\sqrt{25-y^2}\,dy$$

$$= \left[25\arcsin(y/5) + y\sqrt{25-y^2}\right]_{-5}^{2} \quad [\text{substitute } y = 5\sin\theta]$$

$$= 25\arcsin\tfrac{2}{5} + 2\sqrt{21} + \tfrac{25}{2}\pi \approx 58.72 \text{ ft}^2$$

so the fraction of the total capacity in use is $\dfrac{A}{\pi(5)^2} \approx \dfrac{58.72}{25\pi} \approx 0.748$ or 74.8%.

41. We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y-R)^2 \quad \Rightarrow \quad x = \pm\sqrt{r^2-(y-R)^2}$, so $g(y) = 2\sqrt{r^2-(y-R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2-(y-R)^2}\,dy = \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \quad [\text{where } u = y-R]$$

$$= 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du \quad \left[\begin{array}{c}\text{where } u = r\sin\theta,\ du = r\cos\theta\,d\theta \\ \text{in the second integral}\end{array}\right]$$

$$= 4\pi\left[-\tfrac{1}{3}\left(r^2-u^2\right)^{3/2}\right]_{-r}^{r} + 4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta = -\tfrac{4\pi}{3}(0-0) + 4\pi Rr^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 2\pi Rr^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta = 2\pi Rr^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 Rr^2$$

Another method: Use washers instead of shells, so $V = 8\pi R\int_0^r\sqrt{r^2-y^2}\,dy$ as in Exercise 6.2.61(a), but evaluate the integral using $y = r\sin\theta$.

8.4 Integration of Rational Functions by Partial Fractions

1. (a) $\dfrac{2x}{(x+3)(3x+1)} = \dfrac{A}{x+3} + \dfrac{B}{3x+1}$

(b) $\dfrac{1}{x^3+2x^2+x} = \dfrac{1}{x(x^2+2x+1)} = \dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

2. (a) $\dfrac{x-1}{x^3+x^2} = \dfrac{x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

(b) $\dfrac{x-1}{x^3+x} = \dfrac{x-1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

3. (a) $\dfrac{2}{x^2+3x-4} = \dfrac{2}{(x+4)(x-1)} = \dfrac{A}{x+4} + \dfrac{B}{x-1}$

(b) x^2+x+1 is irreducible, so $\dfrac{x^2}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$.

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4. (a) $\dfrac{x^3}{x^2+4x+3} = x-4+\dfrac{13x+12}{x^2+4x+3} = x-4+\dfrac{13x+12}{(x+1)(x+3)} = x-4+\dfrac{A}{x+1}+\dfrac{B}{x+3}$

(b) $\dfrac{2x+1}{(x+1)^3(x^2+4)^2} = \dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{(x+1)^3}+\dfrac{Dx+E}{x^2+4}+\dfrac{Fx+G}{(x^2+4)^2}$

5. (a) $\dfrac{x^4}{x^4-1} = \dfrac{(x^4-1)+1}{x^4-1} = 1+\dfrac{1}{x^4-1}$ [or use long division] $= 1+\dfrac{1}{(x^2-1)(x^2+1)}$

$= 1+\dfrac{1}{(x-1)(x+1)(x^2+1)} = 1+\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2+1}$

(b) $\dfrac{t^4+t^2+1}{(t^2+1)(t^2+4)^2} = \dfrac{At+B}{t^2+1}+\dfrac{Ct+D}{t^2+4}+\dfrac{Et+F}{(t^2+4)^2}$

6. (a) $\dfrac{x^4}{(x^3+x)(x^2-x+3)} = \dfrac{x^4}{x(x^2+1)(x^2-x+3)} = \dfrac{x^3}{(x^2+1)(x^2-x+3)} = \dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2-x+3}$

(b) $\dfrac{1}{x^6-x^3} = \dfrac{1}{x^3(x^3-1)} = \dfrac{1}{x^3(x-1)(x^2+x+1)} = \dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x-1}+\dfrac{Ex+F}{x^2+x+1}$

7. $\displaystyle\int \frac{x}{x-6}\,dx = \int \frac{(x-6)+6}{x-6}\,dx = \int \left(1+\frac{6}{x-6}\right)dx = x+6\ln|x-6|+C$

8. $\displaystyle\int \frac{r^2}{r+4}\,dr = \int \left(\frac{r^2-16}{r+4}+\frac{16}{r+4}\right)dr = \int \left(r-4+\frac{16}{r+4}\right)dr$ [or use long division]

$= \frac{1}{2}r^2-4r+16\ln|r+4|+C$

9. $\dfrac{x-9}{(x+5)(x-2)} = \dfrac{A}{x+5}+\dfrac{B}{x-2}$. Multiply both sides by $(x+5)(x-2)$ to get $x-9 = A(x-2)+B(x+5)$.
Substituting 2 for x gives $-7 = 7B \Leftrightarrow B = -1$. Substituting -5 for x gives $-14 = -7A \Leftrightarrow A = 2$. Thus,

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \left(\frac{2}{x+5}+\frac{-1}{x-2}\right)dx = 2\ln|x+5|-\ln|x-2|+C$$

10. $\dfrac{1}{(t+4)(t-1)} = \dfrac{A}{t+4}+\dfrac{B}{t-1} \Rightarrow 1 = A(t-1)+B(t+4)$.

$t=1 \Rightarrow 1=5B \Rightarrow B=\frac{1}{5}$. $t=-4 \Rightarrow 1=-5A \Rightarrow A=-\frac{1}{5}$. Thus,

$$\int \frac{1}{(t+4)(t-1)}\,dt = \int \left(\frac{-1/5}{t+4}+\frac{1/5}{t-1}\right)dt = -\tfrac{1}{5}\ln|t+4|+\tfrac{1}{5}\ln|t-1|+C \quad \text{or} \quad \frac{1}{5}\ln\left|\frac{t-1}{t+4}\right|+C$$

11. $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1}+\dfrac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get
$1 = A(x-1)+B(x+1)$. Substituting 1 for x gives $1 = 2B \Leftrightarrow B = \frac{1}{2}$.
Substituting -1 for x gives $1 = -2A \Leftrightarrow A = -\frac{1}{2}$. Thus,

$$\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3 \left(\frac{-1/2}{x+1}+\frac{1/2}{x-1}\right)dx = \left[-\tfrac{1}{2}\ln|x+1|+\tfrac{1}{2}\ln|x-1|\right]_2^3$$
$$= \left(-\tfrac{1}{2}\ln 4+\tfrac{1}{2}\ln 2\right)-\left(-\tfrac{1}{2}\ln 3+\tfrac{1}{2}\ln 1\right) = \tfrac{1}{2}(\ln 2+\ln 3-\ln 4) \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{3}{2}\right]$$

12. $\dfrac{x-1}{x^2+3x+2} = \dfrac{A}{x+1}+\dfrac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x-1 = A(x+2)+B(x+1)$.
Substituting -2 for x gives $-3 = -B \Leftrightarrow B = 3$. Substituting -1 for x gives $-2 = A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{-2}{x+1}+\frac{3}{x+2}\right)dx = \left[-2\ln|x+1|+3\ln|x+2|\right]_0^1$$
$$= (-2\ln 2+3\ln 3)-(-2\ln 1+3\ln 2) = 3\ln 3-5\ln 2 \quad \left[\text{or } \ln\tfrac{27}{32}\right]$$

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13. $\displaystyle\int \frac{ax}{x^2-bx}\,dx = \int \frac{ax}{x(x-b)}\,dx = \int \frac{a}{x-b}\,dx = a\ln|x-b|+C$

14. If $a \neq b$, $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{b-a}\left(\dfrac{1}{x+a} - \dfrac{1}{x+b}\right)$, so if $a \neq b$, then

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C$$

If $a = b$, then $\displaystyle\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C$.

15. $\dfrac{2x+3}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} \quad\Rightarrow\quad 2x+3 = A(x+1)+B$. Take $x=-1$ to get $B=1$, and equate coefficients of x to get $A=2$. Now

$$\int_0^1 \frac{2x+3}{(x+1)^2}dx = \int_0^1 \left[\frac{2}{x+1} + \frac{1}{(x+1)^2}\right]dx = \left[2\ln(x+1) - \frac{1}{x+1}\right]_0^1$$

$$= 2\ln 2 - \tfrac{1}{2} - (2\ln 1 - 1) = 2\ln 2 + \tfrac{1}{2}$$

16. $\dfrac{x^3-4x-10}{x^2-x-6} = x+1+\dfrac{3x-4}{(x-3)(x+2)}$. Write $\dfrac{3x-4}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$. Then $3x-4 = A(x+2)+B(x-3)$. Taking $x=3$ and $x=-2$, we get $5=5A \;\Leftrightarrow\; A=1$ and $-10=-5B \;\Leftrightarrow\; B=2$, so

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \int_0^1 \left(x+1+\frac{1}{x-3}+\frac{2}{x+2}\right)dx = \left[\frac{1}{2}x^2 + x + \ln|x-3| + 2\ln(x+2)\right]_0^1$$

$$= \left(\tfrac{1}{2}+1+\ln 2+2\ln 3\right) - (0+0+\ln 3+2\ln 2) = \tfrac{3}{2}+\ln 3-\ln 2 = \tfrac{3}{2}+\ln\tfrac{3}{2}$$

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3} \quad\Rightarrow\quad 4y^2-7y-12 = A(y+2)(y-3)+By(y-3)+Cy(y+2)$.

Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$. Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy = \int_1^2 \left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3}\right)dy = \left[2\ln|y| + \tfrac{9}{5}\ln|y+2| + \tfrac{1}{5}\ln|y-3|\right]_1^2$$

$$= 2\ln 2 + \tfrac{9}{5}\ln 4 + \tfrac{1}{5}\ln 1 - 2\ln 1 - \tfrac{9}{5}\ln 3 - \tfrac{1}{5}\ln 2$$

$$= 2\ln 2 + \tfrac{18}{5}\ln 2 - \tfrac{1}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2 - \ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}$$

18. $\dfrac{x^2+2x-1}{x^3-x} = \dfrac{x^2+2x-1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get $x^2+2x-1 = A(x+1)(x-1)+Bx(x-1)+Cx(x+1)$. Substituting 0 for x gives $-1=-A \;\Leftrightarrow\; A=1$. Substituting -1 for x gives $-2=2B \;\Leftrightarrow\; B=-1$. Substituting 1 for x gives $2=2C \;\Leftrightarrow\; C=1$. Thus,

$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int \left(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1}\right)dx = \ln|x| - \ln|x+1| + \ln|x-1| + C = \ln\left|\frac{x(x-1)}{x+1}\right| + C.$$

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19. $\dfrac{1}{(x+5)^2(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{(x+5)^2} + \dfrac{C}{x-1} \quad \Rightarrow \quad 1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$. Setting $x = -5$ gives $1 = -6B$, so $B = -\frac{1}{6}$. Setting $x = 1$ gives $1 = 36C$, so $C = \frac{1}{36}$. Setting $x = -2$ gives $1 = A(3)(-3) + B(-3) + C(3^2) = -9A - 3B + 9C = -9A + \frac{1}{2} + \frac{1}{4} = -9A + \frac{3}{4}$, so $9A = -\frac{1}{4}$ and $A = -\frac{1}{36}$. Now

$$\int \frac{1}{(x+5)^2(x-1)}\,dx = \int \left[\frac{-1/36}{x+5} - \frac{1/6}{(x+5)^2} + \frac{1/36}{x-1}\right] dx$$

$$= -\frac{1}{36}\ln|x+5| + \frac{1}{6(x+5)} + \frac{1}{36}\ln|x-1| + C$$

20. $\dfrac{x^2}{(x-3)(x+2)^2} = \dfrac{A}{x-3} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2} \quad \Rightarrow \quad x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$.

Setting $x = 3$ gives $A = \frac{9}{25}$. Take $x = -2$ to get $C = -\frac{4}{5}$, and equate the coefficients of x^2 to get $1 = A + B \quad \Rightarrow$ $B = \frac{16}{25}$. Then

$$\int \frac{x^2}{(x-3)(x+2)^2}dx = \int \left[\frac{9/25}{x-3} + \frac{16/25}{x+2} - \frac{4/5}{(x+2)^2}\right] dx$$

$$= \frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C$$

21. $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2 + 3x - 2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x = -2$ to get $C = 3$, and take $x = 0$ to get $B = -1$. Equating the coefficients of x^2 gives $5 = A + C \quad \Rightarrow \quad A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right) dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

22. $\dfrac{1}{s^2(s-1)^2} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2} \quad \Rightarrow \quad 1 = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2$.

Set $s = 0$, giving $B = 1$. Then set $s = 1$ to get $D = 1$. Equate the coefficients of s^3 to get $0 = A + C$ or $A = -C$, and finally set $s = 2$ to get $1 = 2A + 1 - 4A + 4$ or $A = 2$. Now

$$\int \frac{ds}{s^2(s-1)^2} = \int \left[\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}\right] ds = 2\ln|s| - \frac{1}{s} - 2\ln|s-1| - \frac{1}{s-1} + C.$$

23. $\dfrac{x^2}{(x+1)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3}$. Multiply by $(x+1)^3$ to get $x^2 = A(x+1)^2 + B(x+1) + C$. Setting $x = -1$ gives $C = 1$. Equating the coefficients of x^2 gives $A = 1$, and setting $x = 0$ gives $B = -2$. Now $\displaystyle\int \frac{x^2\,dx}{(x+1)^3} = \int \left[\frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}\right] dx = \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.$

24. $\dfrac{x}{x+1} = \dfrac{(x+1)-1}{x+1} = 1 - \dfrac{1}{x+1}$, so $\dfrac{x^3}{(x+1)^3} = \left[1 - \dfrac{1}{x+1}\right]^3 = 1 - \dfrac{3}{x+1} + \dfrac{3}{(x+1)^2} - \dfrac{1}{(x+1)^3}$. Thus,

$$\int \frac{x^3}{(x+1)^3}\,dx = \int \left[1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3}\right] dx = x - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C.$$

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25. $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)(x^2+9)$ to get

$10 = A(x^2+9) + (Bx+C)(x-1)$ $(*)$. Substituting 1 for x gives $10 = 10A \quad \Leftrightarrow \quad A = 1$. Substituting 0 for x gives $10 = 9A - C \quad \Rightarrow \quad C = 9(1) - 10 = -1$. The coefficients of the x^2-terms in $(*)$ must be equal, so $0 = A + B \quad \Rightarrow \quad B = -1$. Thus,

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int \left(\frac{1}{x-1} + \frac{-x-1}{x^2+9}\right) dx = \int \left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right) dx$$
$$= \ln|x-1| - \tfrac{1}{2}\ln(x^2+9) \ [\text{let } u = x^2+9] \ - \tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right) \ [\text{Formula 10}] + C$$

26. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get

$x^2 - x + 6 = A(x^2+3) + (Bx+C)x$. Substituting 0 for x gives $6 = 3A \quad \Leftrightarrow \quad A = 2$. The coefficients of the x^2-terms must be equal, so $1 = A + B \quad \Rightarrow \quad B = 1 - 2 = -1$. The coefficients of the x-terms must be equal, so $-1 = C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right) dx = \int \left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3}\right) dx$$
$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

27. $\dfrac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$x^3 + x^2 + 2x + 1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1) \quad \Leftrightarrow$

$x^3 + x^2 + 2x + 1 = (Ax^3 + Bx^2 + 2Ax + 2B) + (Cx^3 + Dx^2 + Cx + D) \quad \Leftrightarrow$

$x^3 + x^2 + 2x + 1 = (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$. Comparing coefficients gives us the following system of equations:

$$A + C = 1 \quad \textbf{(1)} \qquad B + D = 1 \quad \textbf{(2)}$$
$$2A + C = 2 \quad \textbf{(3)} \qquad 2B + D = 1 \quad \textbf{(4)}$$

Subtracting equation **(1)** from equation **(3)** gives us $A = 1$, so $C = 0$. Subtracting equation **(2)** from equation **(4)** gives us $B = 0$, so $D = 1$. Thus, $I = \displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx = \int \left(\frac{x}{x^2+1} + \frac{1}{x^2+2}\right) dx$. For $\displaystyle\int \frac{x}{x^2+1}\,dx$, let $u = x^2+1$ so $du = 2x\,dx$ and then $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$. For $\displaystyle\int \frac{1}{x^2+2}\,dx$, use Formula 10 with $a = \sqrt{2}$. So $\displaystyle\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+\left(\sqrt{2}\right)^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$.

Thus, $I = \dfrac{1}{2}\ln(x^2+1) + \dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}} + C$.

28. $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+1} \quad \Rightarrow$

$x^2 - 2x - 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$. Setting $x = 1$ gives $B = -1$.

Equating the coefficients of x^3 gives $A = -C$. Equating the constant terms gives $-1 = -A - 1 + D$, so $D = A$,

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and setting $x = 2$ gives $-1 = 5A - 5 - 2A + A$ or $A = 1$. We have

$$\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\,dx = \int \left[\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x-1}{x^2+1}\right] dx$$
$$= \ln|x-1| + \frac{1}{x-1} - \tfrac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$$

29. $$\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{(2x+2)\,dx}{x^2+2x+5} + \int \frac{3\,dx}{(x+1)^2+4}$$
$$= \frac{1}{2}\ln|x^2+2x+5| + 3\int \frac{2\,du}{4(u^2+1)} \qquad \left[\begin{array}{c}\text{where } x+1 = 2u,\\ \text{and } dx = 2\,du\end{array}\right]$$
$$= \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}u + C = \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

30. $$\frac{x^3-2x^2+x+1}{x^4+5x^2+4} = \frac{x^3-2x^2+x+1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \quad \Rightarrow$$

$x^3 - 2x^2 + x + 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$. Equating coefficients gives $A + C = 1$, $B + D = -2$, $4A + C = 1$, $4B + D = 1$ $\Rightarrow$ $A = 0$, $C = 1$, $B = 1$, $D = -3$. Now

$$\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx = \int \frac{dx}{x^2+1} + \int \frac{x-3}{x^2+4}\,dx = \tan^{-1}x + \tfrac{1}{2}\ln(x^2+4) - \tfrac{3}{2}\tan^{-1}(x/2) + C.$$

31. $$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \quad \Rightarrow \quad 1 = A(x^2+x+1) + (Bx+C)(x-1).$$

Take $x = 1$ to get $A = \frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0 = \frac{1}{3} + B$, $1 = \frac{1}{3} - C$, so $B = -\frac{1}{3}$, $C = -\frac{2}{3}$ $\Rightarrow$

$$\int \frac{1}{x^3-1}dx = \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}\,dx = \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx$$
$$= \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+1/2}{x^2+x+1}dx - \frac{1}{3}\int \frac{(3/2)\,dx}{(x+1/2)^2+3/4}$$
$$= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{2}\left(\tfrac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + K$$
$$= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x+1)\right) + K$$

32. $$\int_0^1 \frac{x}{x^2+4x+13}\,dx = \int_0^1 \frac{\frac{1}{2}(2x+4)}{x^2+4x+13}\,dx - 2\int_0^1 \frac{dx}{(x+2)^2+9}$$
$$= \frac{1}{2}\int_{13}^{18}\frac{dy}{y} - 2\int_{2/3}^{1}\frac{3\,du}{9u^2+9} \qquad \left[\begin{array}{c}\text{where } y = x^2+4x+13,\ dy = (2x+4)\,dx,\\ x+2 = 3u, \text{ and } dx = 3\,du\end{array}\right]$$
$$= \tfrac{1}{2}\left[\ln y\right]_{13}^{18} - \tfrac{2}{3}\left[\tan^{-1}u\right]_{2/3}^{1} = \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{2}{3}\left(\tfrac{\pi}{4} - \tan^{-1}\left(\tfrac{2}{3}\right)\right)$$
$$= \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{\pi}{6} + \tfrac{2}{3}\tan^{-1}\left(\tfrac{2}{3}\right)$$

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33. Let $u = x^3 + 3x^2 + 4$. Then $du = 3(x^2 + 2x)\,dx \;\Rightarrow$

$$\int_2^5 \frac{x^2+2x}{x^3+3x^2+4}\,dx = \frac{1}{3}\int_{24}^{204} \frac{du}{u} = \tfrac{1}{3}[\ln u]_{24}^{204} = \tfrac{1}{3}(\ln 204 - \ln 24) = \tfrac{1}{3}\ln\tfrac{204}{24} = \tfrac{1}{3}\ln\tfrac{17}{2}.$$

34. $\dfrac{x^3}{x^3+1} = \dfrac{(x^3+1)-1}{x^3+1} = 1 - \dfrac{1}{x^3+1} = 1 - \left(\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}\right) \;\Rightarrow$

$1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$. Equate the terms of degree 2, 1 and 0 to get $0 = A + B$, $0 = -A + B + C$, $1 = A + C$. Solve the three equations to get $A = \frac{1}{3}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$. So

$$\begin{aligned}\int \frac{x^3}{x^3+1}\,dx &= \int \left[1 - \frac{\frac{1}{3}}{x+1} + \frac{\frac{1}{3}x - \frac{2}{3}}{x^2-x+1}\right] dx \\ &= x - \tfrac{1}{3}\ln|x+1| + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= x - \tfrac{1}{3}\ln|x+1| + \tfrac{1}{6}\ln(x^2-x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x-1)\right) + K\end{aligned}$$

35. $\dfrac{1}{x^4-x^2} = \dfrac{1}{x^2(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{x+1}$. Multiply by $x^2(x-1)(x+1)$ to get

$1 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1)$. Setting $x = 1$ gives $C = \frac{1}{2}$, taking $x = -1$ gives $D = -\frac{1}{2}$. Equating the coefficients of x^3 gives $0 = A + C + D = A$. Finally, setting $x = 0$ yields $B = -1$. Now $\displaystyle\int \frac{dx}{x^4-x^2} = \int \left[\frac{-1}{x^2} + \frac{1/2}{x-1} - \frac{1/2}{x+1}\right] dx = \frac{1}{x} + \tfrac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$

36. Let $u = x^4 + 5x^2 + 4 \;\Rightarrow\; du = (4x^3 + 10x)\,dx = 2(2x^3 + 5x)\,dx$, so

$$\int_0^1 \frac{2x^3+5x}{x^4+5x^2+4}\,dx = \frac{1}{2}\int_4^{10} \frac{du}{u} = \tfrac{1}{2}\Big[\ln|u|\Big]_4^{10} = \tfrac{1}{2}(\ln 10 - \ln 4) = \tfrac{1}{2}\ln\tfrac{5}{2}.$$

37. $\displaystyle\int \frac{x-3}{(x^2+2x+4)^2}\,dx = \int \frac{x-3}{\left[(x+1)^2+3\right]^2}\,dx = \int \frac{u-4}{(u^2+3)^2}\,du$ [with $u = x+1$]

$$\begin{aligned}&= \int \frac{u\,du}{(u^2+3)^2} - 4\int \frac{du}{(u^2+3)^2} = \frac{1}{2}\int \frac{dv}{v^2} - 4\int \frac{\sqrt{3}\sec^2\theta\,d\theta}{9\sec^4\theta} \qquad \left[\begin{array}{c} v = u^2+3 \text{ in the first integral;} \\ u = \sqrt{3}\tan\theta \text{ in the second}\end{array}\right] \\ &= \frac{-1}{(2v)} - \frac{4\sqrt{3}}{9}\int \cos^2\theta\,d\theta = \frac{-1}{2(u^2+3)} - \frac{2\sqrt{3}}{9}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}\,(x+1)}{x^2+2x+4}\right] + C \\ &= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) - \frac{2(x+1)}{3(x^2+2x+4)} + C\end{aligned}$$

38. $\dfrac{x^4+1}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2} \;\Rightarrow\; x^4 + 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$

Setting $x = 0$ gives $A = 1$, and equating the coefficients of x^4 gives $1 = A + B$, so $B = 0$. Now

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$\dfrac{C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}=\dfrac{x^4+1}{x(x^2+1)^2}-\dfrac{1}{x}=\dfrac{1}{x}\left[\dfrac{x^4+1-(x^4+2x^2+1)}{(x^2+1)^2}\right]=\dfrac{-2x}{(x^2+1)^2}$, so we can take $C=0$, $D=-2$, and $E=0$. Hence, $\displaystyle\int\frac{x^4+1}{x(x^2+1)^2}\,dx=\int\left[\frac{1}{x}-\frac{2x}{(x^2+1)^2}\right]dx=\ln|x|+\frac{1}{x^2+1}+C$.

39. Let $u=\sqrt{x+1}$. Then $x=u^2-1$, $dx=2u\,du$ $\Rightarrow$

$$\int\frac{dx}{x\sqrt{x+1}}=\int\frac{2u\,du}{(u^2-1)\,u}=2\int\frac{du}{u^2-1}=\ln\left|\frac{u-1}{u+1}\right|+C=\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C.$$

40. Let $u=\sqrt{x+2}$. Then $x=u^2-2$, $dx=2u\,du$ $\Rightarrow$

$I=\displaystyle\int\frac{dx}{x-\sqrt{x+2}}=\int\frac{2u\,du}{u^2-2-u}=2\int\frac{u\,du}{u^2-u-2}$ and $\dfrac{u}{u^2-u-2}=\dfrac{A}{u-2}+\dfrac{B}{u+1}$ $\Rightarrow$

$u=A(u+1)+B(u-2)$. Substituting -1 for u gives $-1=-3B$ $\Leftrightarrow$ $B=\frac{1}{3}$ and substituting 2 for u gives $2=3A$ $\Leftrightarrow$ $A=\frac{2}{3}$. Thus,

$$\begin{aligned}I&=\frac{2}{3}\int\left[\frac{2}{u-2}+\frac{1}{u+1}\right]du=\tfrac{2}{3}(2\ln|u-2|+\ln|u+1|)+C\\&=\tfrac{2}{3}\left[2\ln\left|\sqrt{x+2}-2\right|+\ln\left(\sqrt{x+2}+1\right)\right]+C\end{aligned}$$

41. Let $u=\sqrt{x}$, so $u^2=x$ and $dx=2u\,du$. Thus,

$$\int_9^{16}\frac{\sqrt{x}}{x-4}\,dx=\int_3^4\frac{u}{u^2-4}2u\,du=2\int_3^4\frac{u^2}{u^2-4}\,du=2\int_3^4\left(1+\frac{4}{u^2-4}\right)du\qquad\text{[by long division]}$$

$$=2+8\int_3^4\frac{du}{(u+2)(u-2)}.\quad(*)$$

Multiply $\dfrac{1}{(u+2)(u-2)}=\dfrac{A}{u+2}+\dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1=A(u-2)+B(u+2)$. Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $B=\frac{1}{4}$ and $A=-\frac{1}{4}$, so

$\dfrac{1}{(u+2)(u-2)}=\dfrac{-1/4}{u+2}+\dfrac{1/4}{u-2}$ and $(*)$ is

$$\begin{aligned}2+8\int_3^4\left(\frac{-1/4}{u+2}+\frac{1/4}{u-2}\right)du&=2+8\left[-\tfrac{1}{4}\ln|u+2|+\tfrac{1}{4}\ln|u-2|\right]_3^4\\&=2+\left[2\ln|u-2|-2\ln|u+2|\right]_3^4=2+2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4\\&=2+2\left(\ln\tfrac{2}{6}-\ln\tfrac{1}{5}\right)=2+2\ln\tfrac{2/6}{1/5}\\&=2+2\ln\tfrac{5}{3}\ \text{ or }\ 2+\ln\left(\tfrac{5}{3}\right)^2=2+\ln\tfrac{25}{9}\end{aligned}$$

42. Let $u=\sqrt[3]{x}$. Then $x=u^3$, $dx=3u^2\,du$ $\Rightarrow$

$$\begin{aligned}\int_0^1\frac{1}{1+\sqrt[3]{x}}\,dx&=\int_0^1\frac{3u^2\,du}{1+u}=\int_0^1\left(3u-3+\frac{3}{1+u}\right)du=\left[\tfrac{3}{2}u^2-3u+3\ln(1+u)\right]_0^1\\&=3\left(\ln 2-\tfrac{1}{2}\right)\end{aligned}$$

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43. Let $u = \sqrt[3]{x^2+1}$. Then $x^2 = u^3 - 1$, $2x\,dx = 3u^2\,du$ $\Rightarrow$

$$\int \frac{x^3\,dx}{\sqrt[3]{x^2+1}} = \int \frac{(u^3-1)\frac{3}{2}u^2\,du}{u} = \tfrac{3}{2}\int (u^4 - u)\,du = \tfrac{3}{10}u^5 - \tfrac{3}{4}u^2 + C$$
$$= \tfrac{3}{10}(x^2+1)^{5/3} - \tfrac{3}{4}(x^2+1)^{2/3} + C$$

44. Let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u\,du$ $\Rightarrow$

$$\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u\cdot 2u\,du}{u^4+u^2} = 2\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1} = 2\left[\tan^{-1}u\right]_{1/\sqrt{3}}^{\sqrt{3}} = 2\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right) = \tfrac{\pi}{3}$$

45. If we were to substitute $u = \sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u = \sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u = \sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u = \sqrt[6]{x}$. Then $x = u^6$, so $dx = 6u^5\,du$ and $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$. Thus,

$$\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3 - u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$
$$= 6\int \left(u^2 + u + 1 + \frac{1}{u-1}\right)du \qquad \text{[by long division]}$$
$$= 6\left(\tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 + u + \ln|u-1|\right) + C = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x} - 1\right| + C$$

46. Let $u = \sqrt[12]{x}$. Then $x = u^{12}$, $dx = 12u^{11}\,du$ $\Rightarrow$

$$\int \frac{dx}{\sqrt[3]{x} + \sqrt[4]{x}} = \int \frac{12u^{11}du}{u^4+u^3} = 12\int \frac{u^8 du}{u+1} = 12\int \left(u^7 - u^6 + u^5 - u^4 + u^3 - u^2 + u - 1 + \frac{1}{u+1}\right)du$$
$$= \tfrac{3}{2}u^8 - \tfrac{12}{7}u^7 + 2u^6 - \tfrac{12}{5}u^5 + 3u^4 - 4u^3 + 6u^2 - 12u + 12\ln|u+1| + C$$
$$= \tfrac{3}{2}x^{2/3} - \tfrac{12}{7}x^{7/12} + 2\sqrt{x} - \tfrac{12}{5}x^{5/12} + 3\sqrt[3]{x} - 4\sqrt[4]{x} + 6\sqrt[6]{x} - 12\sqrt[12]{x} + 12\ln\left(\sqrt[12]{x} + 1\right) + C$$

47. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u}$ $\Rightarrow$

$$\int \frac{e^{2x}\,dx}{e^{2x} + 3e^x + 2} = \int \frac{u^2\,(du/u)}{u^2+3u+2} = \int \frac{u\,du}{(u+1)(u+2)} = \int \left[\frac{-1}{u+1} + \frac{2}{u+2}\right]du$$
$$= 2\ln|u+2| - \ln|u+1| + C = \ln\left[(e^x+2)^2/(e^x+1)\right] + C$$

48. Let $u = \sin x$. Then $du = \cos x\,dx$ $\Rightarrow$

$$\int \frac{\cos x\,dx}{\sin^2 x + \sin x} = \int \frac{du}{u^2+u} = \int \frac{du}{u(u+1)} = \int \left[\frac{1}{u} - \frac{1}{u+1}\right]du$$
$$= \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{\sin x}{1+\sin x}\right| + C$$

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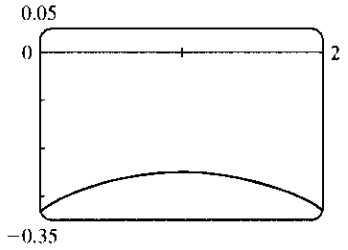
49. Let $u = \ln(x^2 - x + 2)$, $dv = dx$. Then $du = \dfrac{2x-1}{x^2-x+2}\,dx$, $v = x$, and (by integration by parts)

$$\begin{aligned}\int \ln(x^2-x+2)\,dx &= x\ln(x^2-x+2) - \int \frac{2x^2-x}{x^2-x+2}\,dx = x\ln(x^2-x+2) - \int \left(2 + \frac{x-4}{x^2-x+2}\right) dx \\ &= x\ln(x^2-x+2) - 2x - \int \frac{\frac{1}{2}(2x-1)}{x^2-x+2}\,dx + \frac{7}{2}\int \frac{dx}{(x-\frac{1}{2})^2 + \frac{7}{4}} \\ &= x\ln(x^2-x+2) - 2x - \frac{1}{2}\ln(x^2-x+2) + \frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)} \quad \left[\begin{array}{c}\text{where } x-\frac{1}{2} = \frac{\sqrt{7}}{2}u, \\ dx = \frac{\sqrt{7}}{2}\,du, \\ (x-\frac{1}{2})^2 + \frac{7}{4} = \frac{7}{4}(u^2+1)\end{array}\right. \\ &= (x-\tfrac{1}{2})\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}u + C \\ &= (x-\tfrac{1}{2})\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}} + C\end{aligned}$$

50. Let $u = \tan^{-1}x$, $dv = x\,dx \quad \Rightarrow \quad du = dx/(1+x^2)$, $v = \frac{1}{2}x^2$.

Then $\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \dfrac{1}{2}\int \dfrac{x^2}{1+x^2}\,dx$. To evaluate the last integral, use long division or observe that $\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \frac{(1+x^2)-1}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx = x - \tan^{-1}x + C_1$. So $\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\left(x - \tan^{-1}x + C_1\right) = \frac{1}{2}\left(x^2\tan^{-1}x + \tan^{-1}x - x\right) + C$.

51.



From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3, so we estimate the integral to be $-(2 \cdot 0.3) = -0.6$. Now

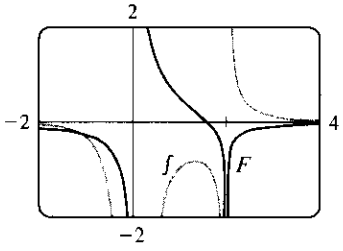
$$\frac{1}{x^2-2x-3} = \frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \quad \Leftrightarrow$$

$1 = (A+B)x + A - 3B$, so $A = -B$ and $A - 3B = 1 \quad \Leftrightarrow \quad A = \frac{1}{4}$ and $B = -\frac{1}{4}$, so the integral becomes

$$\begin{aligned}\int_0^2 \frac{dx}{x^2-2x-3} &= \frac{1}{4}\int_0^2 \frac{dx}{x-3} - \frac{1}{4}\int_0^2 \frac{dx}{x+1} = \tfrac{1}{4}\Big[\ln|x-3| - \ln|x+1|\Big]_0^2 \\ &= \frac{1}{4}\left[\ln\left|\frac{x-3}{x+1}\right|\right]_0^2 = \tfrac{1}{4}\left(\ln\tfrac{1}{3} - \ln 3\right) = -\tfrac{1}{2}\ln 3 \approx -0.55\end{aligned}$$

52. $\dfrac{1}{x^3-2x^2} = \dfrac{1}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2} \Rightarrow 1 = (A+C)x^2 + (B-2A)x - 2B$, so $A + C = B - 2A = 0$ and $-2B = 1 \quad \Rightarrow \quad B = -\frac{1}{2}$, $A = -\frac{1}{4}$, and $C = \frac{1}{4}$. So the general antiderivative of $\dfrac{1}{x^3-2x^2}$ is

$$\begin{aligned}\int \frac{dx}{x^3-2x^2} &= -\frac{1}{4}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x^2} + \frac{1}{4}\int \frac{dx}{x-2} \\ &= -\tfrac{1}{4}\ln|x| - \tfrac{1}{2}(-1/x) + \tfrac{1}{4}\ln|x-2| + C \\ &= \frac{1}{4}\ln\left|\frac{x-2}{x}\right| + \frac{1}{2x} + C\end{aligned}$$



We plot this function with $C = 0$ on the same screen as $y = \dfrac{1}{x^3-2x^2}$.

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(c) The CAS verifies our guesses. Now

$$\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} \quad \Rightarrow \quad 1 = A(x+b) + B(x+a)$$

Setting $x = -b$ gives $B = 1/(a-b)$ and setting $x = -a$ gives $A = 1/(b-a)$. So

$$\int \frac{1}{(x+a)(x+b)}\,dx = \int \left[\frac{1/(b-a)}{x+a} + \frac{1/(a-b)}{x+b}\right] dx = \frac{\ln|x+a|}{b-a} + \frac{\ln|x+b|}{a-b} + C$$

and our guess for $a \neq b$ is correct.

If $a = b$, then $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{(x+a)^2} = (x+a)^{-2}$. Letting $u = x + a \quad \Rightarrow \quad du = dx$, we have

$\displaystyle\int (x+a)^{-2}\,dx = \int u^{-2}\,du = -\frac{1}{u} + C = -\frac{1}{x+a} + C$, and our guess for $a = b$ is also correct.

2. (a) (i) $\displaystyle\int \sin x \cos 2x\,dx = \frac{\cos x}{2} - \frac{\cos 3x}{6}$

(ii) $\displaystyle\int \sin 3x \cos 7x\,dx = \frac{\cos 4x}{8} - \frac{\cos 10x}{20}$

(iii) $\displaystyle\int \sin 8x \cos 3x\,dx = -\frac{\cos 11x}{22} - \frac{\cos 5x}{10}$

(b) Looking at the sums and differences of a and b in part (a), we guess that

$$\int \sin ax \cos bx\,dx = \frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos((a+b)x)}{2(a+b)} + C$$

Note that $\cos((a-b)x) = \cos((b-a)x)$.

(c) The CAS verifies our guess. To integrate directly, we can use Formula 2(a) from Section 8.2.

$$\int \sin ax \cos bx\,dx = \int \left\{\tfrac{1}{2}[\sin(ax-bx) + \sin(ax+bx)]\right\} dx = \tfrac{1}{2}\int [\sin((a-b)x) + \sin((a+b)x)]\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos((a-b)x)}{b-a} - \frac{\cos((a+b)x)}{a+b}\right] + C = \frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos((a+b)x)}{2(a+b)} + C$$

Our formula is valid for $a \neq b$.

3. (a) (i) $\displaystyle\int \ln x\,dx = x\ln x - x$

(ii) $\displaystyle\int x \ln x\,dx = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2$

(iii) $\displaystyle\int x^2 \ln x\,dx = \tfrac{1}{3}x^3 \ln x - \tfrac{1}{9}x^3$

(iv) $\displaystyle\int x^3 \ln x\,dx = \tfrac{1}{4}x^4 \ln x - \tfrac{1}{16}x^4$

(v) $\displaystyle\int x^7 \ln x\,dx = \tfrac{1}{8}x^8 \ln x - \tfrac{1}{64}x^8$

(b) We guess that $\displaystyle\int x^n \ln x\,dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1}$.

(c) Let $u = \ln x$, $dv = x^n\,dx \quad \Rightarrow \quad du = \dfrac{dx}{x}$, $v = \dfrac{1}{n+1}x^{n+1}$.

Then
$$\int x^n \ln x\,dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^n\,dx$$
$$= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\cdot\frac{1}{n+1}x^{n+1},$$

which verifies our guess. We must have $n + 1 \neq 0 \quad \Leftrightarrow \quad n \neq -1$.

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4. (a) (i) $\int xe^x\,dx = e^x(x-1)$

(ii) $\int x^2e^x\,dx = e^x(x^2-2x+2)$

(iii) $\int x^3e^x\,dx = e^x(x^3-3x^2+6x-6)$

(iv) $\int x^4e^x\,dx = e^x(x^4-4x^3+12x^2-24x+24)$

(v) $\int x^5e^x\,dx = e^x(x^5-5x^4+20x^3-60x^2+120x-120)$

(b) Notice from part (a) that we can write

$$\int x^4e^x\,dx = e^x(x^4-4x^3+4\cdot3x^2-4\cdot3\cdot2x+4\cdot3\cdot2\cdot1)$$

and

$$\int x^5e^x\,dx = e^x(x^5-5x^4+5\cdot4x^3-5\cdot4\cdot3x^2+5\cdot4\cdot3\cdot2x-5\cdot4\cdot3\cdot2\cdot1)$$

So we guess that

$$\begin{aligned}\int x^6e^x\,dx &= e^x(x^6-6x^5+6\cdot5x^4-6\cdot5\cdot4x^3+6\cdot5\cdot4\cdot3x^2-6\cdot5\cdot4\cdot3\cdot2x+6\cdot5\cdot4\cdot3\cdot2\cdot1)\\ &= e^x(x^6-6x^5+30x^4-120x^3+360x^2-720x+720)\end{aligned}$$

The CAS verifies our guess.

(c) From the results in part (a), as well as our prediction in part (b), we speculate that

$$\begin{aligned}\int x^ne^x\,dx &= e^x\left[x^n-nx^{n-1}+n(n-1)x^{n-2}-n(n-1)(n-2)x^{n-3}+\cdots\pm n!x\mp n!\right]\\ &= e^x\sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!}x^i\end{aligned}$$

(We have reversed the order of the polynomial's terms.)

(d) Let S_n be the statement that $\int x^ne^x\,dx = e^x\sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!}x^i$.

S_1 is true by part (a)(i). Suppose S_k is true for some k, and consider S_{k+1}. Integrating by parts with $u=x^{k+1}$, $dv=e^x\,dx \;\Rightarrow\; du=(k+1)x^k\,dx$, $v=e^x$, we get

$$\begin{aligned}\int x^{k+1}e^x\,dx &= x^{k+1}e^x-(k+1)\int x^ke^x\,dx\\ &= x^{k+1}e^x-(k+1)\left[e^x\sum_{i=0}^{k}(-1)^{k-i}\frac{k!}{i!}x^i\right]\\ &= e^x\left[x^{k+1}-(k+1)\sum_{i=0}^{k}(-1)^{k-i}\frac{k!}{i!}x^i\right]\\ &= e^x\left[x^{k+1}+\sum_{i=0}^{k}(-1)^{k-i+1}\frac{(k+1)k!}{i!}x^i\right]\\ &= e^x\sum_{i=0}^{k+1}(-1)^{(k+1)-i}\frac{(k+1)!}{i!}x^i\end{aligned}$$

This verifies S_n for $n=k+1$. Thus, by mathematical induction, S_n is true for all n, where n is a positive integer.

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8.7 Approximate Integration

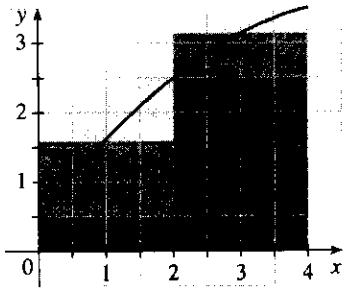
1. (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$

$$L_2 = \sum_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0)+f(2)] = 2(0.5+2.5) = 6$$

$$R_2 = \sum_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2)+f(4)] = 2(2.5+3.5) = 12$$

$$M_2 = \sum_{i=1}^{2} f(\overline{x}_i)\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1)+f(3)] \approx 2(1.6+3.2) = 9.6$$

(b)

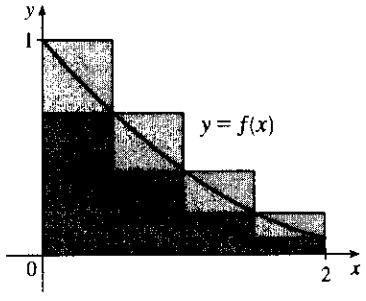


L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I. See the solution to Exercise 45 for a proof of the fact that if f is concave down on $[a,b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\,\Delta x\right)[f(x_0)+2f(x_1)+f(x_2)] = \frac{2}{2}[f(0)+2f(2)+f(4)] = 0.5+2(2.5)+3.5 = 9$.
This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 45 for a general proof of this conclusion.

(d) For any n, we will have $L_n < T_n < I < M_n < R_n$.

2.



The diagram shows that $L_4 > T_4 > \int_0^2 f(x)\,dx > R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x)\,dx$. In fact, for any function that is concave upward, it can be shown that

$L_n > T_n > \int_0^2 f(x)\,dx > M_n > R_n$.

(a) Since $0.9540 > 0.8675 > 0.8632 > 0.7811$, it follows that $L_n = 0.9540$, $T_n = 0.8675$, $M_n = 0.8632$, and $R_n = 0.7811$.

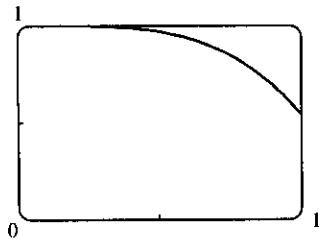
(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have

$0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

3. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

(a) $T_4 = \frac{1}{4\cdot 2}\left[f(0)+2f\left(\frac{1}{4}\right)+2f\left(\frac{2}{4}\right)+2f\left(\frac{3}{4}\right)+f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right] \approx 0.908907$



The graph shows that f is concave down on $[0,1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos(x^2)\,dx < 0.908907$.

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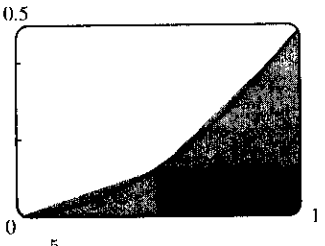
4.



(a) Since f is increasing on $[0, 1]$, L_2 will underestimate I (since the area of the darkest rectangle is less than the area under the curve), and R_2 will overestimate I. Since f is concave upward on $[0, 1]$, M_2 will underestimate I and T_2 will overestimate I (the area under the straight line segments is greater than the area under the curve).

(b) For any n, we will have $L_n < M_n < I < T_n < R_n$.

(c) $L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x = \frac{1}{5}[f(0.0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.1187$

$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 0.2146$

$M_5 = \sum_{i=1}^{5} f(\bar{x}_i)\,\Delta x = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.1622$

$T_5 = \left(\frac{1}{2}\,\Delta x\right)[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 0.1666$

From the graph, it appears that the Midpoint Rule gives the best approximation. (This is in fact the case, since $I \approx 0.16371405$.)

5. $f(x) = x^2 \sin x$, $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{8} = \dfrac{\pi}{8}$

(a) $M_8 = \frac{\pi}{8}\left[f\left(\frac{\pi}{16}\right) + f\left(\frac{3\pi}{16}\right) + f\left(\frac{5\pi}{16}\right) + \cdots + f\left(\frac{15\pi}{16}\right)\right] \approx 5.932957$

(b) $S_8 = \frac{\pi}{8\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 4f\left(\frac{3\pi}{8}\right) + 2f\left(\frac{4\pi}{8}\right) + 4f\left(\frac{5\pi}{8}\right) + 2f\left(\frac{6\pi}{8}\right) + 4f\left(\frac{7\pi}{8}\right) + f(\pi)\right]$
≈ 5.869247

Actual: $\int_0^\pi x^2 \sin x\,dx \overset{84}{=} \left[-x^2\cos x\right]_0^\pi + 2\int_0^\pi x\cos x\,dx \overset{83}{=} \left[-\pi^2(-1) - 0\right] + 2[\cos x + x\sin x]_0^\pi$
$= \pi^2 + 2[(-1+0) - (1+0)] = \pi^2 - 4 \approx 5.869604$

Errors: $E_M = \text{actual} - M_8 = \int_0^\pi x^2 \sin x\,dx - M_8 \approx -0.063353$

$E_S = \text{actual} - S_8 = \int_0^\pi x^2 \sin x\,dx - S_8 \approx 0.000357$

6. $f(x) = e^{-\sqrt{x}}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{6} = \dfrac{1}{6}$

(a) $M_6 = \frac{1}{6}\left[f\left(\frac{1}{12}\right) + f\left(\frac{3}{12}\right) + f\left(\frac{5}{12}\right) + f\left(\frac{7}{12}\right) + f\left(\frac{9}{12}\right) + f\left(\frac{11}{12}\right)\right] \approx 0.525100$

(b) $S_6 = \frac{1}{6\cdot 3}\left[f(0) + 4f\left(\frac{1}{6}\right) + 2f\left(\frac{2}{6}\right) + 4f\left(\frac{3}{6}\right) + 2f\left(\frac{4}{6}\right) + 4f\left(\frac{5}{6}\right) + f(1)\right] \approx 0.533979$

Actual: $\int_0^1 e^{-\sqrt{x}}\,dx = \int_0^{-1} e^u 2u\,du \qquad [u = -\sqrt{x},\ u^2 = x,\ 2u\,du = dx]$
$\overset{96}{=} 2[(u-1)e^u]_0^{-1} = 2\left[-2e^{-1} - \left(-1e^0\right)\right] = 2 - 4e^{-1} \approx 0.528482$

Errors: $E_M = \text{actual} - M_6 = \int_0^1 e^{-\sqrt{x}}\,dx - M_6 \approx 0.003382$

$E_S = \text{actual} - S_6 = \int_0^1 e^{-\sqrt{x}}\,dx - S_6 \approx -0.005497$

7. $f(x) = \sqrt[4]{1+x^2}$, $\Delta x = \dfrac{2-0}{8} = \dfrac{1}{4}$

(a) $T_8 = \frac{1}{4\cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + \cdots + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2)\right] \approx 2.413790$

(b) $M_8 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 2.411453$

(c) $S_8 = \frac{1}{4\cdot 3}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + 2f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right] \approx 2.412232$

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8. $f(x) = \sin(x^2)$, $\Delta x = \dfrac{\frac{1}{2} - 0}{4} = \dfrac{1}{8}$

(a) $T_4 = \frac{1}{8 \cdot 2}\left[f(0) + 2f\left(\frac{1}{8}\right) + 2f\left(\frac{2}{8}\right) + 2f\left(\frac{3}{8}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.042743$

(b) $M_4 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + f\left(\frac{7}{16}\right)\right] \approx 0.040850$

(c) $S_4 = \frac{1}{8 \cdot 3}\left[f(0) + 4f\left(\frac{1}{8}\right) + 2f\left(\frac{2}{8}\right) + 4f\left(\frac{3}{8}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.041478$

9. $f(x) = \dfrac{\ln x}{1 + x}$, $\Delta x = \dfrac{2 - 1}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(1) + 2f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 2f(1.9) + f(2)] \approx 0.146879$

(b) $M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + \cdots + f(1.85) + f(1.95)] \approx 0.147391$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) + 4f(1.7)$
$+ 2f(1.8) + 4f(1.9) + f(2)]$
≈ 0.147219

10. $f(t) = \dfrac{1}{1 + t^2 + t^4}$, $\Delta t = \dfrac{3 - 0}{6} = \dfrac{1}{2}$

(a) $T_6 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + f(3)\right] \approx 0.895122$

(b) $M_6 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right] \approx 0.895478$

(c) $S_6 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + f(3)\right] \approx 0.898014$

11. $f(t) = \sin(e^{t/2})$, $\Delta t = \dfrac{\frac{1}{2} - 0}{8} = \dfrac{1}{16}$

(a) $T_8 = \frac{1}{16 \cdot 2}\left[f(0) + 2f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 2f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451948$

(b) $M_8 = \frac{1}{16}\left[f\left(\frac{1}{32}\right) + f\left(\frac{3}{32}\right) + f\left(\frac{5}{32}\right) + \cdots + f\left(\frac{13}{32}\right) + f\left(\frac{15}{32}\right)\right] \approx 0.451991$

(c) $S_8 = \frac{1}{16 \cdot 3}\left[f(0) + 4f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 4f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451976$

12. $f(x) = \sqrt{1 + \sqrt{x}}$, $\Delta x = \dfrac{4 - 0}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + \cdots + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right] \approx 6.042985$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + \cdots + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx 6.084778$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx 6.061678$

13. $f(x) = e^{1/x}$, $\Delta x = \dfrac{2 - 1}{4} = \dfrac{1}{4}$

(a) $T_4 = \frac{1}{4 \cdot 2}[f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)] \approx 2.031893$

(b) $M_4 = \frac{1}{4}[f(1.125) + f(1.375) + f(1.625) + f(1.875)] \approx 2.014207$

(c) $S_4 = \frac{1}{4 \cdot 3}[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)] \approx 2.020651$

14. $f(x) = \sqrt{x} \sin x$, $\Delta x = \dfrac{4 - 0}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right)\right] + f(4)\right\} \approx 1.732865$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + \cdots + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx 1.787427$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx 1.772142$

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15. $f(x) = \dfrac{\cos x}{x}$, $\Delta x = \dfrac{5-1}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2\cdot 2}\left[f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + \cdots + 2f(4) + 2f\left(\frac{9}{2}\right) + f(5)\right] \approx -0.495333$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) + f\left(\frac{17}{4}\right) + f\left(\frac{19}{4}\right)\right] \approx -0.543321$

(c) $S_8 = \frac{1}{2\cdot 3}\left[f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + 2f(4) + 4f\left(\frac{9}{2}\right) + f(5)\right]$
≈ -0.526123

16. $f(x) = \ln(x^3 + 2)$, $\Delta x = \dfrac{6-4}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\left[f(4) + 2f(4.2) + 2f(4.4) + \cdots + 2f(5.6) + 2f(5.8) + f(6)\right] \approx 9.649753$

(b) $M_{10} = \frac{1}{5}\left[f(4.1) + f(4.3) + \cdots + f(5.7) + f(5.9)\right] \approx 9.650912$

(c) $S_{10} = \frac{1}{5\cdot 3}[f(4) + 4f(4.2) + 2f(4.4) + 4f(4.6) + 2f(4.8) + 4f(5)$
$+ 2f(5.2) + 4f(5.4) + 2f(5.6) + 4f(5.8) + f(6)]$
≈ 9.650526

17. $f(y) = \dfrac{1}{1+y^5}$, $\Delta y = \dfrac{3-0}{6} = \dfrac{1}{2}$

(a) $T_6 = \frac{1}{2\cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 2f\left(\frac{5}{2}\right) + f(3)\right] \approx 1.064275$

(b) $M_6 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right] \approx 1.067416$

(c) $S_6 = \frac{1}{2\cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 4f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 4f\left(\frac{5}{2}\right) + f(3)\right] \approx 1.074915$

18. $f(x) = \dfrac{e^x}{x}$, $\Delta x = \dfrac{4-2}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\{f(2) + 2[f(2.2) + f(2.4) + f(2.6) + \cdots + f(3.8)] + f(4)\} \approx 14.704592$

(b) $M_{10} = \frac{1}{5}[f(2.1) + f(2.3) + f(2.5) + f(2.7) + \cdots + f(3.7) + f(3.9)] \approx 14.662669$

(c) $S_{10} = \frac{1}{5\cdot 3}[f(2) + 4f(2.2) + 2f(2.4) + 4f(2.6) + \cdots + 2f(3.6) + 4f(3.8) + f(4)] \approx 14.676696$

19. $f(x) = e^{-x^2}$, $\Delta x = \dfrac{2-0}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\{f(0) + 2[f(0.2) + f(0.4) + \cdots + f(1.8)] + f(2)\} \approx 0.881839$
$M_{10} = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + \cdots + f(1.7) + f(1.9)] \approx 0.882202$

(b) $f(x) = e^{-x^2}$, $f'(x) = -2xe^{-x^2}$, $f''(x) = \left(4x^2 - 2\right)e^{-x^2}$, $f'''(x) = 4x\left(3 - 2x^2\right)e^{-x^2}$.

$f'''(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$. So to find the maximum value of $|f''(x)|$ on $[0, 2]$, we need only consider its values at $x = 0$, $x = 2$, and $x = \sqrt{\frac{3}{2}}$. $|f''(0)| = 2$, $|f''(2)| \approx 0.2564$ and $\left|f''\left(\sqrt{\frac{3}{2}}\right)\right| \approx 0.8925$. Thus, taking $K = 2$, $a = 0$, $b = 2$, and $n = 10$ in Theorem 3, we get $|E_T| \le 2 \cdot 2^3/\left(12 \cdot 10^2\right) = \frac{1}{75} = 0.01\overline{3}$, and $|E_M| \le |E_T|/2 \le 0.00\overline{6}$.

(c) Take $K = 2$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 10^{-5} \quad \Leftrightarrow \quad \dfrac{2(2-0)^3}{12n^2} \le 10^{-5} \quad \Leftrightarrow$ $\frac{3}{4}n^2 \ge 10^5 \quad \Leftrightarrow \quad n \ge 365.1\ldots \quad \Leftrightarrow \quad n \ge 366$. Take $n = 366$ for T_n. For E_M, again take $K = 2$ in Theorem 3 to get $|E_M| \le 10^{-5} \quad \Leftrightarrow \quad \frac{3}{2}n^2 \ge 10^5 \quad \Leftrightarrow \quad n \ge 258.2 \quad \Rightarrow \quad n \ge 259$. Take $n = 259$ for M_n.

20. (a) $T_8 = \frac{1}{8\cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] + f(1)\right\} \approx 0.902333$
$M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] = 0.905620$

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(b) $f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$, $f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$. For $0 \le x \le 1$, sin and cos are positive, so $|f''(x)| = 2\sin(x^2) + 4x^2\cos(x^2) \le 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6$ since $\sin(x^2) \le 1$ and $\cos(x^2) \le 1$ for all x, and $x^2 \le 1$ for $0 \le x \le 1$. So for $n = 8$, we take $K = 6$, $a = 0$, and $b = 1$ in Theorem 3, to get $|E_T| \le 6 \cdot 1^3/(12 \cdot 8^2) = \frac{1}{128} = 0.0078125$ and $|E_M| \le \frac{1}{256} = 0.00390625$. [A better estimate is obtained by noting from a graph of f'' that $|f''(x)| \le 4$ for $0 \le x \le 1$.]

(c) Using $K = 6$ as in part (b), we have $|E_T| \le 6 \cdot 1^3/(12n^2) = 1/(2n^2) \le 10^{-5} \Rightarrow 2n^2 \ge 10^5 \Rightarrow n \ge \sqrt{\frac{1}{2} \cdot 10^5}$ or $n \ge 224$. To guarantee that $|E_M| \le 0.00001$, we need $6 \cdot 1^3/(24n^2) \le 10^{-5} \Rightarrow 4n^2 \ge 10^5 \Rightarrow n \ge \sqrt{\frac{1}{4} \cdot 10^5}$ or $n \ge 159$.

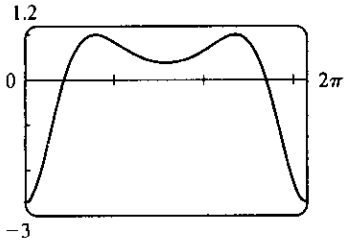
21. (a) $T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.71971349$
$S_{10} = \frac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + \cdots + 4f(0.9) + f(1)] \approx 1.71828278$
Since $I = \int_0^1 e^x dx = [e^x]_0^1 = e - 1 \approx 1.71828183$, $E_T = I - T_{10} \approx -0.00143166$ and $E_S = I - S_{10} \approx -0.00000095$.

(b) $f(x) = e^x \Rightarrow f''(x) = e^x \le e$ for $0 \le x \le 1$. Taking $K = e$, $a = 0$, $b = 1$, and $n = 10$ in Theorem 3, we get $|E_T| \le e(1)^3/(12 \cdot 10^2) \approx 0.002265 > 0.00143166$ [actual $|E_T|$ from (a)]. $f^{(4)}(x) = e^x < e$ for $0 \le x \le 1$. Using Theorem 4, we have $|E_S| \le e(1)^5/(180 \cdot 10^4) \approx 0.0000015 > 0.00000095$ [actual $|E_S|$ from (a)]. We see that the actual errors are about two-thirds the size of the error estimates.

(c) From part (b), we take $K = e$ to get $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.00001 \Rightarrow n^2 \ge \dfrac{e(1^3)}{12(0.00001)} \Rightarrow$ $n \ge 150.5$. Take $n = 151$ for T_n. Now $|E_M| \le \dfrac{K(b-a)^3}{24n^2} \le 0.00001 \Rightarrow n \ge 106.4$. Take $n = 107$ for M_n. Finally, $|E_S| \le \dfrac{K(b-a)^5}{180n^4} \le 0.00001 \Rightarrow n^4 \ge \dfrac{e(1^5)}{180(0.00001)} \Rightarrow n \ge 6.23$. Take $n = 8$ for S_n (since n has to be even for Simpson's Rule).

22. From Example 7(b), we take $K = 76e$ to get $|E_S| \le 76e(1)^5/(180n^4) \le 0.00001 \Rightarrow n^4 \ge 76e/[180(0.00001)] \Rightarrow n \ge 18.4$. Take $n = 20$ (since n must be even).

23. (a) Using a CAS, we differentiate $f(x) = e^{\cos x}$ twice, and find that $f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. From the graph, we see that the maximum value of $|f''(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f''(0) = -e$, we can use $K = e$ or $K = 2.8$.



(b) A CAS gives $M_{10} \approx 7.954926518$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = e$, we get $|E_M| \le \dfrac{e(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.280945995$. With $K = 2.8$, we get $|E_M| \le \dfrac{2.8(2\pi - 0)^3}{24 \cdot 10^2} = 0.289391916$.

(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9}, much less than the estimate in part (c).

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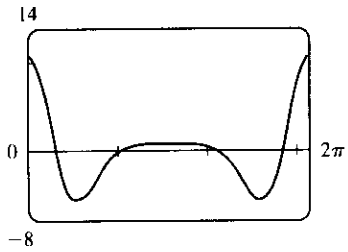
(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = e^{\cos x}\left(\sin^4 x - 6\sin^2 x\cos x + 3 - 7\sin^2 x + \cos x\right)$.

From the graph, we see that the maximum value of $\left|f^{(4)}(x)\right|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use $K = 4e$ or $K = 10.9$.



(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use `student[simpson]`.)

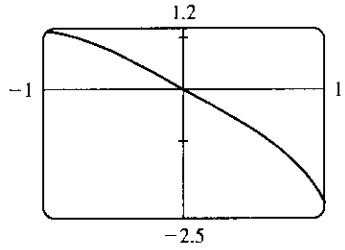
(h) Using Theorem 4 with $K = 4e$, we get $|E_S| \le \dfrac{4e(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$. With $K = 10.9$, we get $|E_S| \le \dfrac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059299814$.

(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{4e(2\pi)^5}{180 \cdot n^4} \le 0.0001 \quad \Rightarrow \quad \dfrac{4e(2\pi)^5}{180 \cdot 0.0001} \le n^4 \quad \Rightarrow$ $n^4 \ge 5{,}915{,}362 \quad \Leftrightarrow \quad n \ge 49.3$. So we must take $n \ge 50$ to ensure that $|I - S_n| \le 0.0001$. ($K = 10.9$ leads to the same value of n.)

24. (a) Using the CAS, we differentiate $f(x) = \sqrt{4 - x^3}$ twice, and find that $f''(x) = -\dfrac{9x^4}{4(4 - x^3)^{3/2}} - \dfrac{3x}{(4 - x^3)^{1/2}}$.

From the graph, we see that $|f''(x)| < 2.2$ on $[-1, 1]$.



(b) A CAS gives $M_{10} \approx 3.995804152$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = 2.2$, we get $|E_M| \le \dfrac{2.2\,[1 - (-1)]^3}{24 \cdot 10^2} \approx 0.00733$.

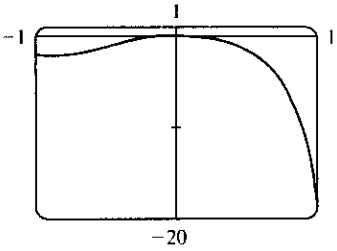
(d) A CAS gives $I \approx 3.995487677$.

(e) The actual error is about -0.0003165, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = \dfrac{9}{16}\dfrac{x^2\left(x^6 - 224x^3 - 1280\right)}{(4 - x^3)^{7/2}}$.

From the graph, we see that $\left|f^{(4)}(x)\right| < 18.1$ on $[-1, 1]$.



(g) A CAS gives $S_{10} \approx 3.995449790$. (In Maple, use `student[simpson]`.)

(h) Using Theorem 4 with $K = 18.1$, we get $|E_S| \le \dfrac{18.1\,[1 - (-1)]^5}{180 \cdot 10^4} \approx 0.000322$.

(i) The actual error is about $3.995487677 - 3.995449790 \approx 0.0000379$. This is quite a bit smaller than the estimate in part (h).

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(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{18.1(2)^5}{180 \cdot n^4} \le 0.0001 \quad \Rightarrow \quad \dfrac{18.1(2)^5}{180 \cdot 0.0001} \le n^4 \quad \Rightarrow$

$n^4 \ge 32{,}178 \quad \Rightarrow n \ge 13.4$. So we must take $n \ge 14$ to ensure that $|I - S_n| \le 0.0001$.

25. $I = \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = 0.25$. $f(x) = x^3$.

$n = 4$: $L_4 = \frac{1}{4}\left[0^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{2}{4}\right)^3 + \left(\frac{3}{4}\right)^3\right] = 0.140625$

$R_4 = \frac{1}{4}\left[\left(\frac{1}{4}\right)^3 + \left(\frac{2}{4}\right)^3 + \left(\frac{3}{4}\right)^3 + 1^3\right] = 0.390625$

$T_4 = \frac{1}{4\cdot 2}\left[0^3 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 1^3\right] = 0.265625,$

$M_4 = \frac{1}{4}\left[\left(\frac{1}{8}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{5}{8}\right)^3 + \left(\frac{7}{8}\right)^3\right] = 0.2421875,$

$E_L = I - L_4 = \frac{1}{4} - 0.140625 = 0.109375$, $E_R = \frac{1}{4} - 0.390625 = -0.140625,$

$E_T = \frac{1}{4} - 0.265625 = -0.015625$, $E_M = \frac{1}{4} - 0.2421875 = 0.0078125$

$n = 8$: $L_8 = \frac{1}{8}\left[f(0) + f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] \approx 0.191406$

$R_8 = \frac{1}{8}\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right) + f(1)\right] \approx 0.316406$

$T_8 = \frac{1}{8\cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] + f(1)\right\} \approx 0.253906$

$M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + \cdots + f\left(\frac{13}{16}\right) + f\left(\frac{15}{16}\right)\right] = 0.248047$

$E_L \approx \frac{1}{4} - 0.191406 \approx 0.058594$, $E_R \approx \frac{1}{4} - 0.316406 \approx -0.066406,$

$E_T \approx \frac{1}{4} - 0.253906 \approx -0.003906$, $E_M \approx \frac{1}{4} - 0.248047 \approx 0.001953.$

$n = 16$: $L_{16} = \frac{1}{16}\left[f(0) + f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] \approx 0.219727$

$R_{16} = \frac{1}{16}\left[f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right) + f(1)\right] \approx 0.282227$

$T_{16} = \frac{1}{16\cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] + f(1)\right\} \approx 0.250977$

$M_{16} = \frac{1}{16}\left[f\left(\frac{1}{32}\right) + f\left(\frac{3}{32}\right) + \cdots + f\left(\frac{31}{32}\right)\right] \approx 0.249512$

$E_L \approx \frac{1}{4} - 0.219727 \approx 0.030273$, $E_R \approx \frac{1}{4} - 0.282227 \approx -0.032227,$

$E_T \approx \frac{1}{4} - 0.250977 \approx -0.000977$, $E_M \approx \frac{1}{4} - 0.249512 \approx 0.000488.$

n	L_n	R_n	T_n	M_n
4	0.140625	0.390625	0.265625	0.242188
8	0.191406	0.316406	0.253906	0.248047
16	0.219727	0.282227	0.250977	0.249512

n	E_L	E_R	E_T	E_M
4	0.109375	−0.140625	−0.015625	0.007813
8	0.058594	−0.066406	−0.003906	0.001953
16	0.030273	−0.032227	−0.000977	0.000488

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

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26. $\int_0^2 e^x\,dx = [e^x]_0^2 = e^2 - 1 \approx 6.389056$. $f(x) = e^x$

$n = 4$: $\Delta x = (2-0)/4 = \frac{1}{2}$

$L_4 = \frac{1}{2}\left[e^0 + e^{1/2} + e^1 + e^{3/2}\right] \approx 4.924346$

$R_4 = \frac{1}{2}\left[e^{1/2} + e^1 + e^{3/2} + e^2\right] \approx 8.118874$

$T_4 = \frac{1}{2\cdot 2}\left[e^0 + 2e^{1/2} + 2e^1 + 2e^{3/2} + e^2\right] \approx 6.521610$

$M_4 = \frac{1}{2}\left[e^{1/4} + e^{3/4} + e^{5/4} + e^{7/4}\right] \approx 6.322986.$

$E_L \approx 6.389056 - 4.924346 \approx 1.464710$, $E_R \approx 6.389056 - 8.118874 = -1.729818$,

$E_T \approx 6.389056 - 6.521610 \approx -0.132554$, $E_M \approx 6.389056 - 6.322986 = 0.0660706$.

$n = 8$: $\Delta x = (2-0)/8 = \frac{1}{4}$

$L_8 = \frac{1}{4}\left[e^0 + e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4}\right] \approx 5.623666$

$R_8 = \frac{1}{4}\left[e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4} + e^2\right] \approx 7.220930$

$T_8 = \frac{1}{4\cdot 2}\left[e^0 + 2e^{1/4} + 2e^{1/2} + 2e^{3/4} + 2e^1 + 2e^{5/4} + 2e^{3/2} + 2e^{7/4} + e^2\right] \approx 6.422298$

$M_8 = \frac{1}{4}\left[e^{1/8} + e^{3/8} + e^{5/8} + e^{7/8} + e^{9/8} + e^{11/8} + e^{13/8} + e^{15/8}\right] \approx 6.372448$

$E_L \approx 6.389056 - 5.623666 \approx 0.765390$, $E_R \approx 6.389056 - 7.220930 \approx -0.831874$,

$E_T \approx 6.389056 - 6.422298 \approx -0.033242$, $E_M \approx 6.389056 - 6.372448 \approx 0.016608$.

$n = 16$: $\Delta x = (2-0)/16 = \frac{1}{8}$

$L_{16} = \frac{1}{8}\left[f(0) + f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{14}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 5.998057$

$R_{16} = \frac{1}{8}\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{15}{8}\right) + f(2)\right] \approx 6.796689$

$T_{16} = \frac{1}{8\cdot 2}\left(f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{15}{8}\right)\right] + f(2)\right) \approx 6.397373$

$M_{16} = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + \cdots + f\left(\frac{29}{16}\right) + f\left(\frac{31}{16}\right)\right] \approx 6.384899$

$E_L \approx 6.389056 - 5.998057 \approx 0.390999$, $E_R \approx 6.389056 - 6.796689 \approx -0.407633$,

$E_T \approx 6.389056 - 6.397373 \approx -0.008317$, $E_M \approx 6.389056 - 6.384899 \approx 0.004158$.

n	E_L	E_R	E_T	E_M
4	1.464710	−1.729818	−0.132554	0.066071
8	0.765390	−0.831874	−0.033242	0.016608
16	0.390999	−0.407633	−0.008317	0.004158

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by a factor of about 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

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27. $\int_1^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3} \approx 4.666667$

$n = 6$: $\Delta x = (4-1)/6 = \frac{1}{2}$

$T_6 = \frac{1}{2\cdot 2}\left[\sqrt{1} + 2\sqrt{1.5} + 2\sqrt{2} + 2\sqrt{2.5} + 2\sqrt{3} + 2\sqrt{3.5} + \sqrt{4}\right] \approx 4.661488$

$M_6 = \frac{1}{2}\left[\sqrt{1.25} + \sqrt{1.75} + \sqrt{2.25} + \sqrt{2.75} + \sqrt{3.25} + \sqrt{3.75}\right] \approx 4.669245$

$S_6 = \frac{1}{2\cdot 3}\left[\sqrt{1} + 4\sqrt{1.5} + 2\sqrt{2} + 4\sqrt{2.5} + 2\sqrt{3} + 4\sqrt{3.5} + \sqrt{4}\right] \approx 4.666563$

$E_T \approx \frac{14}{3} - 4.661488 \approx 0.005178$, $E_M \approx \frac{14}{3} - 4.669245 \approx -0.002578$,

$E_S \approx \frac{14}{3} - 4.666563 \approx 0.000104$.

$n = 12$: $\Delta x = (4-1)/12 = \frac{1}{4}$

$T_{12} = \frac{1}{4\cdot 2}(f(1) + 2[f(1.25) + f(1.5) + \cdots + f(3.5) + f(3.75)] + f(4)) \approx 4.665367$

$M_{12} = \frac{1}{4}[f(1.125) + f(1.375) + f(1.625) + \cdots + f(3.875)] \approx 4.667316$

$S_{12} = \frac{1}{4\cdot 3}[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + \cdots + 4f(3.75) + f(4)] \approx 4.666659$

$E_T \approx \frac{14}{3} - 4.665367 \approx 0.001300$, $E_M \approx \frac{14}{3} - 4.667316 \approx -0.000649$,

$E_S \approx \frac{14}{3} - 4.666659 \approx 0.000007$.

Note: These errors were computed more precisely and then rounded to six places. That is, they were not computed by comparing the rounded values of T_n, M_n, and S_n with the rounded value of the actual integral.

n	T_n	M_n	S_n
6	4.661488	4.669245	4.666563
12	4.665367	4.667316	4.666659

n	E_T	E_M	E_S
6	0.005178	−0.002578	0.000104
12	0.001300	−0.000649	0.000007

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

28. $I = \int_{-1}^2 xe^x dx = [xe^x - e^x]_{-1}^2 = e^2 + 2/e \approx 8.124815$. $f(x) = xe^x$.

$n = 6$: $\Delta x = [2-(-1)]/6 = \frac{1}{2}$

$T_6 = \frac{1}{2\cdot 2}\{f(-1) + 2[f(-0.5) + f(0) + \cdots + f(1.5)] + f(2)\} \approx 8.583514$

$M_6 = \frac{1}{2}[f(-0.75) + f(-0.25) + \cdots + f(1.75)] \approx 7.896632$

$S_6 = \frac{1}{2\cdot 3}[f(-1) + 4f(-0.5) + 2f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)] \approx 8.136885$

$E_T \approx I - 8.583514 \approx -0.458699$, $E_M \approx I - 7.896632 \approx 0.228183$,

$E_S \approx I - 8.136885 \approx -0.012070$.

$n = 12$: $\Delta x = [2-(-1)]/12 = \frac{1}{4}$

$T_{12} = \frac{1}{4\cdot 2}\{f(-1) + 2[f(-0.75) + f(-0.5) + \cdots + f(1.75)] + f(2)\} \approx 8.240073$

$M_{12} = \frac{1}{4}\left[f\left(-\frac{7}{8}\right) + f\left(-\frac{5}{8}\right) + \cdots + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 8.067259$

$S_{12} = \frac{1}{4\cdot 3}[f(-1) + 4f(-0.75) + 2f(-0.5) + \cdots + 2f(1.5) + 4f(1.75) + f(2)] \approx 8.125593$

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$E_T \approx I - 8.240073 \approx -0.115258$, $E_M \approx I - 8.067259 \approx 0.057556$,

$E_S \approx I - 8.125593 \approx -0.000778$

n	T_n	M_n	S_n
6	8.583514	7.896632	8.136885
12	8.240073	8.067259	8.125593

n	E_T	E_M	E_S
6	−0.458699	0.228183	−0.012070
12	−0.115258	0.057556	−0.000778

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

29. $\Delta x = (4-0)/4 = 1$

(a) $T_4 = \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)] \approx \frac{1}{2}[0 + 2(3) + 2(5) + 2(3) + 1] = 11.5$

(b) $M_4 = 1 \cdot [f(0.5) + f(1.5) + f(2.5) + f(3.5)] \approx 1 + 4.5 + 4.5 + 2 = 12$

(c) $S_4 = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx \frac{1}{3}[0 + 4(3) + 2(5) + 4(3) + 1] = 11.\overline{6}$

30. If $x =$ distance from left end of pool and $w = w(x) =$ width at x, then Simpson's Rule with $n = 8$ and $\Delta x = 2$ gives Area $= \int_0^{16} w\,dx \approx \frac{2}{3}[0 + 4(6.2) + 2(7.2) + 4(6.8) + 2(5.6) + 4(5.0) + 2(4.8) + 4(4.8) + 0] \approx 84$ m^2.

31. (a) We are given the function values at the endpoints of 8 intervals of length 0.4, so we'll use the Midpoint Rule with $n = 8/2 = 4$ and $\Delta x = (3.2 - 0)/4 = 0.8$.

$$\begin{aligned}\int_0^{3.2} f(x)\,dx \approx M_4 &= 0.8[f(0.4) + f(1.2) + f(2.0) + f(2.8)] \\ &= 0.8[6.5 + 6.4 + 7.6 + 8.8] \\ &= 0.8(29.3) = 23.44\end{aligned}$$

(b) $-4 \le f''(x) \le 1 \quad \Rightarrow \quad |f''(x)| \le 4$, so use $K = 4$, $a = 0$, $b = 3.2$, and $n = 4$ in Theorem 3. So

$$|E_M| \le \frac{4(3.2-0)^3}{24(4)^2} = \frac{128}{375} = 0.341\overline{3}.$$

32. We use Simpson's Rule with $n = 10$ and $\Delta x = \frac{1}{2}$:

$$\begin{aligned}\text{distance} = \int_0^5 v(t)\,dt \approx S_{10} &= \tfrac{1}{2\cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 4f(4.5) + f(5)] \\ &= \tfrac{1}{6}[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) \\ &\qquad + 2(10.51) + 4(10.67) + 2(10.76) + 4(10.81) + 10.81] \\ &= \tfrac{1}{6}(268.41) = 44.735 \text{ m}\end{aligned}$$

33. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = (6-0)/6 = 1$ to estimate this integral:

$$\begin{aligned}\int_0^6 a(t)\,dt \approx S_6 &= \tfrac{1}{3}[a(0) + 4a(1) + 2a(2) + 4a(3) + 2a(4) + 4a(5) + a(6)] \\ &\approx \tfrac{1}{3}[0 + 4(0.5) + 2(4.1) + 4(9.8) + 2(12.9) + 4(9.5) + 0] = \tfrac{1}{3}(113.2) = 37.7\overline{3} \text{ ft/s}\end{aligned}$$

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34. By the Net Change Theorem, the total amount of water that leaked out during the first six hours is equal to $\int_0^6 r(t)\,dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-0}{6} = 1$ to estimate this integral:

$$\begin{aligned}\int_0^6 r(t)\,dt \approx S_6 &= \tfrac{1}{3}[r(0) + 4r(1) + 2r(2) + 4r(3) + 2r(4) + 4r(5) + r(6)] \\ &\approx \tfrac{1}{3}[4 + 4(3) + 2(2.4) + 4(1.9) + 2(1.4) + 4(1.1) + 1] \\ &= \tfrac{1}{3}(36.6) = 12.2 \text{ liters}\end{aligned}$$

The function values were obtained from a high-resolution graph.

35. By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t)\,dt$. We use Simpson's Rule with $n = 12$ and $\Delta t = (6-0)/12 = \frac{1}{2}$ to estimate this integral:

$$\begin{aligned}\int_0^6 P(t)\,dt \approx S_{12} &= \tfrac{1/2}{3}[P(0) + 4P(0.5) + 2P(1) + 4P(1.5) + 2P(2) + 4P(2.5) \\ &\quad + 2P(3) + 4P(3.5) + 2P(4) + 4P(4.5) + 2P(5) + 4P(5.5) + P(6)] \\ &= \tfrac{1}{6}[1814 + 4(1735) + 2(1686) + 4(1646) + 2(1637) + 4(1609) + 2(1604) \\ &\quad + 4(1611) + 2(1621) + 4(1666) + 2(1745) + 4(1886) + 2052] \\ &= \tfrac{1}{6}(61{,}064) = 10{,}177.\overline{3} \text{ megawatt-hours.}\end{aligned}$$

36. By the Net Change Theorem, the total amount of data transmitted is equal to $\int_0^8 D(t)\,dt \times 3600$ [since $D(t)$ is measured in megabits per second and t is in hours]. We use Simpson's Rule with $n = 8$ and $\Delta t = (8-0)/8 = 1$ to estimate this integral:

$$\begin{aligned}\int_0^8 D(t)\,dt \approx S_8 &= \tfrac{1}{3}[D(0) + 4D(1) + 2D(2) + 4D(3) + 2D(4) + 4D(5) + 2D(6) + 4D(7) + D(8)] \\ &\approx \tfrac{1}{3}[0.35 + 4(0.32) + 2(0.41) + 4(0.50) + 2(0.51) + 4(0.56) + 2(0.56) + 4(0.83) + 0.88] \\ &= \tfrac{1}{3}(13.03) = 4.34\overline{3}\end{aligned}$$

Now multiply by 3600 to obtain 15,636 megabits.

37. Let $y = f(x)$ denote the curve. Using cylindrical shells, $V = \int_2^{10} 2\pi x f(x)\,dx = 2\pi \int_2^{10} x f(x)\,dx = 2\pi I$. Now use Simpson's Rule to approximate I:

$$\begin{aligned}I \approx S_8 &= \tfrac{10-2}{3(8)}[2f(2) + 4\cdot 3f(3) + 2\cdot 4f(4) + 4\cdot 5f(5) + 2\cdot 6f(6) \\ &\quad + 4\cdot 7f(7) + 2\cdot 8f(8) + 4\cdot 9f(9) + 10f(10)] \\ &\approx \tfrac{1}{3}[2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0)] \\ &= \tfrac{1}{3}(395.2)\end{aligned}$$

Thus, $V \approx 2\pi \cdot \frac{1}{3}(395.2) \approx 827.7$ or 828 cubic units.

38. Work $= \displaystyle\int_0^{18} f(x)\,dx \approx S_6 = \frac{18-0}{6\cdot 3}[f(0) + 4f(3) + 2f(6) + 4f(9) + 2f(12) + 4f(15) + f(18)]$

$= 1\cdot[9.8 + 4(9.1) + 2(8.5) + 4(8.0) + 2(7.7) + 4(7.5) + 7.4] = 148$ joules

39. Volume $= \pi \int_0^2 \left(\sqrt[3]{1+x^3}\right)^2 dx = \pi \int_0^2 \left(1+x^3\right)^{2/3} dx$. $V \approx \pi \cdot S_{10}$ where $f(x) = \left(1+x^3\right)^{2/3}$ and $\Delta x = (2-0)/10 = \frac{1}{5}$. Therefore,

$$\begin{aligned}V \approx \pi \cdot S_{10} = \pi \tfrac{1}{5\cdot 3}[&f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8) + 4f(1) \\ &+ 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)] \approx 12.325078\end{aligned}$$

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40. Using Simpson's Rule with $n = 10$, $\Delta x = \frac{\pi/2}{10}$, $L = 1$, $\theta_0 = \frac{42\pi}{180}$ radians, $g = 9.8\ \text{m/s}^2$, $k^2 = \sin^2\left(\frac{1}{2}\theta_0\right)$, and $f(x) = 1/\sqrt{1 - k^2 \sin^2 x}$, we get

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \approx 4\sqrt{\frac{L}{g}}\, S_{10}$$

$$= 4\sqrt{\tfrac{1}{9.8}}\left(\tfrac{\pi/2}{10 \cdot 3}\right)\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 2.07665$$

41. $I(\theta) = \dfrac{N^2 \sin^2 k}{k^2}$, where $k = \dfrac{\pi N d \sin\theta}{\lambda}$, $N = 10{,}000$, $d = 10^{-4}$, and $\lambda = 632.8 \times 10^{-9}$. So $I(\theta) = \dfrac{(10^4)^2 \sin^2 k}{k^2}$, where $k = \dfrac{\pi(10^4)(10^{-4})\sin\theta}{632.8 \times 10^{-9}}$. Now $n = 10$ and $\Delta\theta = \dfrac{10^{-6} - (-10^{-6})}{10} = 2 \times 10^{-7}$, so $M_{10} = 2 \times 10^{-7}[I(-0.0000009) + I(-0.0000007) + \cdots + I(0.0000009)] \approx 59.4$.

42. $f(x) = \cos(\pi x)$, $\Delta x = \frac{20-0}{10} = 2 \Rightarrow$

$$T_{10} = \tfrac{2}{2}\{f(0) + 2[f(2) + f(4) + \cdots + f(18)] + f(20)\}$$

$$= 1[\cos 0 + 2(\cos 2\pi + \cos 4\pi + \cdots + \cos 18\pi) + \cos 20\pi]$$

$$= 1 + 2(1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) + 1 = 20$$

The actual value is $\int_0^{20} \cos(\pi x)\, dx = \frac{1}{\pi}[\sin \pi x]_0^{20} = \frac{1}{\pi}(\sin 20\pi - \sin 0) = 0$. The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value is $f(2n) = \cos(2n\pi) = 1$.

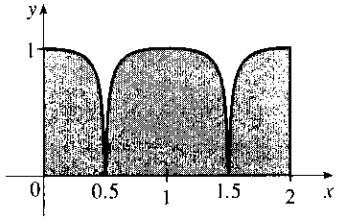
43. Consider the function f whose graph is shown. The area $\int_0^2 f(x)\, dx$ is close to 2. The Trapezoidal Rule gives

$T_2 = \frac{2-0}{2\cdot 2}[f(0) + 2f(1) + f(2)] = \frac{1}{2}[1 + 2 \cdot 1 + 1] = 2$.

The Midpoint Rule gives

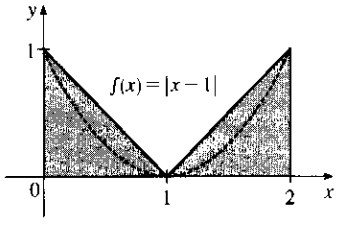
$M_2 = \frac{2-0}{2}[f(0.5) + f(1.5)] = 1[0 + 0] = 0$,

so the Trapezoidal Rule is more accurate.



44. Consider the function $f(x) = |x - 1|$, $0 \le x \le 2$. The area $\int_0^2 f(x)\, dx$ is exactly 1. So is the right endpoint approximation: $R_2 = f(1)\,\Delta x + f(2)\,\Delta x = 0 \cdot 1 + 1 \cdot 1 = 1$. But Simpson's Rule approximates f with the parabola $y = (x - 1)^2$, shown dashed, and

$$S_2 = \frac{\Delta x}{3}[f(0) + 4f(1) + f(2)] = \frac{1}{3}[1 + 4 \cdot 0 + 1] = \frac{2}{3}.$$



45. Since the Trapezoidal and Midpoint approximations on the interval $[a, b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we can focus our attention on one such interval. The condition $f''(x) < 0$ for $a \le x \le b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x)\, dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n < \int_a^b f(x)\, dx < M_n$. In general, the condition $f'' < 0$ implies that the graph of f on $[a, b]$ lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$. Thus, $\int_a^b f(x)\, dx > T_n$. Since M_n is the area under a tangent to the graph, and since $f'' < 0$ implies that the tangent lies above the graph, we also have $M_n > \int_a^b f(x)\, dx$. Thus, $T_n < \int_a^b f(x)\, dx < M_n$.

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46. Let f be a polynomial of degree ≤ 3; say $f(x) = Ax^3 + Bx^2 + Cx + D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n = 2$), because for a larger even number of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then Simpson's approximation is

$$\begin{aligned}\int_{-h}^{h} f(x)\,dx &\approx \tfrac{1}{3}h[f(-h) + 4f(0) + f(h)] \\ &= \tfrac{1}{3}h\left[\left(-Ah^3 + Bh^2 - Ch + D\right) + 4D + \left(Ah^3 + Bh^2 + Ch + D\right)\right] \\ &= \tfrac{1}{3}h\left[2Bh^2 + 6D\right] = \tfrac{2}{3}Bh^3 + 2Dh\end{aligned}$$

The exact value of the integral is

$$\begin{aligned}\int_{-h}^{h}\left(Ax^3 + Bx^2 + Cx + D\right)dx &= 2\int_0^h \left(Bx^2 + D\right)dx \quad \text{[by Theorem 5.5.6(a) and (b)]} \\ &= 2\left[\tfrac{1}{3}Bx^3 + Dx\right]_0^h = \tfrac{2}{3}Bh^3 + 2Dh\end{aligned}$$

Thus, Simpson's Rule is exact.

47. $T_n = \frac{1}{2}\,\Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$ and $M_n = \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_{n-1}) + f(\overline{x}_n)]$, where $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Now

$$\begin{aligned}T_{2n} = \tfrac{1}{2}\left(\tfrac{1}{2}\Delta x\right)[f(x_0) &+ 2f(\overline{x}_1) + 2f(x_1) + 2f(\overline{x}_2) + 2f(x_2) + \cdots \\ &+ 2f(\overline{x}_{n-1}) + 2f(x_{n-1}) + 2f(\overline{x}_n) + f(x_n)]\end{aligned}$$

so
$$\begin{aligned}\tfrac{1}{2}(T_n + M_n) &= \tfrac{1}{2}T_n + \tfrac{1}{2}M_n \\ &= \tfrac{1}{4}\,\Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \\ &\quad + \tfrac{1}{4}\Delta x\,[2f(\overline{x}_1) + 2f(\overline{x}_2) + \cdots + 2f(\overline{x}_{n-1}) + 2f(\overline{x}_n)] \\ &= T_{2n}\end{aligned}$$

48. $T_n = \dfrac{\Delta x}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$ and $M_n = \Delta x \sum_{i=1}^{n} f\left(x_i - \dfrac{\Delta x}{2}\right)$, so

$$\tfrac{1}{3}T_n + \tfrac{2}{3}M_n = \tfrac{1}{3}(T_n + 2M_n) = \frac{\Delta x}{3\cdot 2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f\left(x_i - \frac{\Delta x}{2}\right)\right]$$

where $\Delta x = \dfrac{b-a}{n}$. Let $\delta x = \dfrac{b-a}{2n}$. Then $\Delta x = 2\delta x$, so

$$\begin{aligned}\tfrac{1}{3}T_n + \tfrac{2}{3}M_n &= \frac{\delta x}{3}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f(x_i - \delta x)\right] \\ &= \tfrac{1}{3}\delta x[f(x_0) + 4f(x_1 - \delta x) + 2f(x_1) + 4f(x_2 - \delta x) \\ &\quad + 2f(x_2) + \cdots + 2f(x_{n-1}) + 4f(x_n - \delta x) + f(x_n)]\end{aligned}$$

Since $x_0, x_1 - \delta x, x_1, x_2 - \delta x, x_2, \ldots, x_{n-1}, x_n - \delta x, x_n$ are the subinterval endpoints for S_{2n}, and since $\delta x = \dfrac{b-a}{2n}$ is the width of the subintervals for S_{2n}, the last expression for $\frac{1}{3}T_n + \frac{2}{3}M_n$ is the usual expression for S_{2n}. Therefore, $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

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8.8 Improper Integrals

1. (a) Since $\int_1^\infty x^4 e^{-x^4}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(b) Since $y = \sec x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi/2} \sec x\,dx$ is a Type II improper integral.

(c) Since $y = \dfrac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x = 2$, $\displaystyle\int_0^2 \frac{x}{x^2-5x+6}\,dx$ is a Type II improper integral.

(d) Since $\displaystyle\int_{-\infty}^0 \frac{1}{x^2+5}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

2. (a) Since $y = 1/(2x-1)$ is defined and continuous on $[1,2]$, the integral is proper.

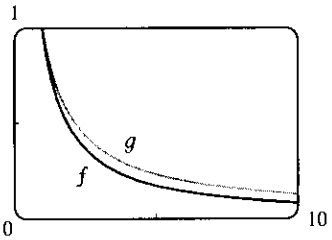
(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

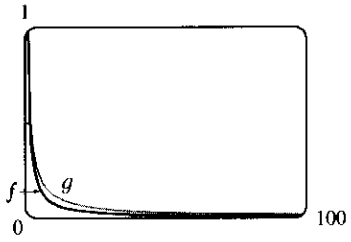
(c) Since $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)dx$ is a Type II improper integral.

3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is $A(t) = \int_1^t x^{-3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2}t^{-2} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 1/(2t^2)$. So the area for $1 \le x \le 10$ is $A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \le x \le 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for $1 \le x \le 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \ge 1$ is $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

4. (a)





(b) The area under the graph of f from $x = 1$ to $x = t$ is

$$F(t) = \int_1^t f(x)\,dx = \int_1^t x^{-1.1}dx = \left[-\tfrac{1}{0.1}x^{-0.1}\right]_1^t$$
$$= -10\left(t^{-0.1} - 1\right) = 10\left(1 - t^{-0.1}\right)$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)dx = \int_1^t x^{-0.9}dx = \left[\tfrac{1}{0.1}x^{0.1}\right]_1^t$$
$$= 10\left(t^{0.1} - 1\right)$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} 10\left(1 - t^{-0.1}\right) = 10$.

The total area under the graph of g does not exist, since $\lim_{t\to\infty} G(t) = \lim_{t\to\infty} 10\left(t^{0.1} - 1\right) = \infty$.

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5. $I = \int_1^\infty \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(3x+1)^2}\,dx$. Now

$$\int \frac{1}{(3x+1)^2}\,dx = \frac{1}{3}\int \frac{1}{u^2}\,du \quad [u = 3x+1,\ du = 3\,dx]$$

$$= -\frac{1}{3u} + C = -\frac{1}{3(3x+1)} + C,$$

so $I = \lim_{t\to\infty}\left[-\frac{1}{3(3x+1)}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{3(3t+1)} + \frac{1}{12}\right] = 0 + \frac{1}{12} = \frac{1}{12}$. Convergent

6. $\int_{-\infty}^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\left[\frac{1}{2}\ln|2x-5|\right]_t^0 = \lim_{t\to\infty}\left[\frac{1}{2}\ln 5 - \frac{1}{2}\ln|2t-5|\right] = -\infty$.
Divergent

7. $\int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\int_t^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\left[-2\sqrt{2-w}\right]_t^{-1}$ $[u = 2-w,\ du = -dw]$

$= \lim_{t\to-\infty}\left[-2\sqrt{3} + 2\sqrt{2-t}\right] = \infty$. Divergent

8. $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2} + \frac{1}{2}\right)$

$= \frac{1}{2}\left(0 + \frac{1}{2}\right) = \frac{1}{4}$. Convergent

9. $\int_4^\infty e^{-y/2}\,dy = \lim_{t\to\infty}\int_4^t e^{-y/2}\,dy = \lim_{t\to\infty}\left[-2e^{-y/2}\right]_4^t = \lim_{t\to\infty}\left(-2e^{-t/2} + 2e^{-2}\right) = 0 + 2e^{-2} = 2e^{-2}$.
Convergent

10. $\int_{-\infty}^{-1} e^{-2t}\,dt = \lim_{x\to-\infty}\int_x^{-1} e^{-2t}\,dt = \lim_{x\to-\infty}\left[-\frac{1}{2}e^{-2t}\right]_x^{-1} = \lim_{x\to-\infty}\left[-\frac{1}{2}e^2 + \frac{1}{2}e^{-2x}\right] = \infty$. Divergent

11. $\int_{-\infty}^\infty \frac{x\,dx}{1+x^2} = \int_{-\infty}^0 \frac{x\,dx}{1+x^2} + \int_0^\infty \frac{x\,dx}{1+x^2}$ and

$\int_{-\infty}^0 \frac{x\,dx}{1+x^2} = \lim_{t\to-\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_t^0 = \lim_{t\to-\infty}\left[0 - \frac{1}{2}\ln\left(1+t^2\right)\right] = -\infty$. Divergent

12. $I = \int_{-\infty}^\infty \left(2-v^4\right)dv = I_1 + I_2 = \int_{-\infty}^0 \left(2-v^4\right)dv + \int_0^\infty \left(2-v^4\right)dv$, but

$I_1 = \lim_{t\to-\infty}\left[2v - \frac{1}{5}v^5\right]_t^0 = \lim_{t\to-\infty}\left(-2t + \frac{1}{5}t^5\right) = -\infty$. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2. Divergent

13. $\int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx$.

$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left(1 - e^{-t^2}\right) = -\frac{1}{2}\cdot 1 = -\frac{1}{2}$, and

$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2} - 1\right) = -\frac{1}{2}\cdot(-1) = \frac{1}{2}$.

Therefore, $\int_{-\infty}^\infty xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$. Convergent

14. $\int_{-\infty}^\infty x^2e^{-x^3}\,dx = \int_{-\infty}^0 x^2e^{-x^3}\,dx + \int_0^\infty x^2e^{-x^3}\,dx$, and

$\int_{-\infty}^0 x^2e^{-x^3}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{3}e^{-x^3}\right]_t^0 = -\frac{1}{3} + \frac{1}{3}\left(\lim_{t\to-\infty} e^{-t^3}\right) = \infty$. Divergent

15. $\int_{2\pi}^\infty \sin\theta\,d\theta = \lim_{t\to\infty}\int_{2\pi}^t \sin\theta\,d\theta = \lim_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^t = \lim_{t\to\infty}(-\cos t + 1)$. This limit does not exist, so the integral is divergent. Divergent

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16. $\int_0^\infty \cos^2\alpha\,d\alpha = \lim_{t\to\infty}\int_0^t \tfrac{1}{2}(1+\cos 2\alpha)\,d\alpha = \lim_{t\to\infty}\left[\tfrac{1}{2}\alpha + \tfrac{1}{4}\sin 2\alpha\right]_0^t = \lim_{t\to\infty}\left[\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right] = \infty$ since $\left|\tfrac{1}{4}\sin 2t\right| \le \tfrac{1}{4}$ for all t, but $\tfrac{1}{2}t \to \infty$ as $t \to \infty$. Divergent

17. $\displaystyle\int_1^\infty \frac{x+1}{x^2+2x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(2x+2)}{x^2+2x}\,dx = \tfrac{1}{2}\lim_{t\to\infty}\left[\ln(x^2+2x)\right]_1^t = \tfrac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+2t) - \ln 3\right]$

$= \infty$. Divergent

18. $\displaystyle\int_0^\infty \frac{dz}{z^2+3z+2} = \lim_{t\to\infty}\int_0^t \left[\frac{1}{z+1} - \frac{1}{z+2}\right]dz = \lim_{t\to\infty}\left[\ln\left(\frac{z+1}{z+2}\right)\right]_0^t$

$\displaystyle= \lim_{t\to\infty}\left[\ln\left(\frac{t+1}{t+2}\right) - \ln\left(\frac{1}{2}\right)\right] = \ln 1 + \ln 2 = \ln 2$. Convergent

19. $\displaystyle\int_0^\infty se^{-5s}\,ds = \lim_{t\to\infty}\int_0^t se^{-5s}\,ds = \lim_{t\to\infty}\left[-\frac{1}{5}se^{-5s} - \frac{1}{25}e^{-5s}\right]_0^t$ [by integration by parts with $u = s$]

$= \lim_{t\to\infty}\left(-\tfrac{1}{5}te^{-5t} - \tfrac{1}{25}e^{-5t} + \tfrac{1}{25}\right) = 0 - 0 + \tfrac{1}{25}$ [by l'Hospital's Rule]

$= \tfrac{1}{25}$. Convergent

20. $\displaystyle\int_{-\infty}^6 re^{r/3}\,dr = \lim_{t\to-\infty}\int_t^6 re^{r/3}\,dr = \lim_{t\to-\infty}\left[3re^{r/3} - 9e^{r/3}\right]_t^6$ [by integration by parts with $u = r$]

$= \lim_{t\to-\infty}(18e^2 - 9e^2 - 3te^{t/3} + 9e^{t/3}) = 9e^2 - 0 + 0$ [by l'Hospital's Rule]

$= 9e^2$. Convergent

21. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$ (by substitution with $u = \ln x$, $du = dx/x$) $\displaystyle= \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty$. Divergent

22. $\int_{-\infty}^\infty e^{-|x|}\,dx = \int_{-\infty}^0 e^x\,dx + \int_0^\infty e^{-x}\,dx$, $\int_{-\infty}^0 e^x\,dx = \lim_{t\to-\infty}\left[e^x\right]_t^0 = \lim_{t\to-\infty}\left(1 - e^t\right) = 1$, and $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left(1 - e^{-t}\right) = 1$. Therefore, $\int_{-\infty}^\infty e^{-|x|}\,dx = 1 + 1 = 2$. Convergent

23. $\displaystyle\int_{-\infty}^\infty \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx + \int_0^\infty \frac{x^2}{9+x^6}\,dx = 2\int_0^\infty \frac{x^2}{9+x^6}\,dx$ [since the integrand is even].

Now $\displaystyle\int \frac{x^2\,dx}{9+x^6}$ $\left[\begin{matrix}u = x^3\\ du = 3x^2dx\end{matrix}\right]$ $\displaystyle= \int \frac{\frac{1}{3}\,du}{9+u^2}$ $\left[\begin{matrix}u = 3v\\ du = 3\,dv\end{matrix}\right]$ $\displaystyle= \int \frac{\frac{1}{3}(3\,dv)}{9+9v^2} = \frac{1}{9}\int \frac{dv}{1+v^2}$

$\displaystyle= \frac{1}{9}\tan^{-1}v + C = \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) + C$,

so $\displaystyle 2\int_0^\infty \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\int_0^t \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^t$

$\displaystyle= 2\lim_{t\to\infty}\frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) = \frac{2}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9}$. Convergent

24. Integrate by parts with $u = \ln x$, $dv = dx/x^3$ $\Rightarrow$ $du = dx/x$, $v = -1/(2x^2)$.

$\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\left(\left[-\frac{1}{2x^2}\ln x\right]_1^t + \frac{1}{2}\int_1^t \frac{1}{x^3}\,dx\right)$

$\displaystyle= \lim_{t\to\infty}\left(-\frac{1}{2}\frac{\ln t}{t^2} + 0 - \frac{1}{4t^2} + \frac{1}{4}\right) = \frac{1}{4}$

since $\displaystyle\lim_{t\to\infty}\frac{\ln t}{t^2} \overset{H}{=} \lim_{t\to\infty}\frac{1/t}{2t} = \lim_{t\to\infty}\frac{1}{2t^2} = 0$. Convergent

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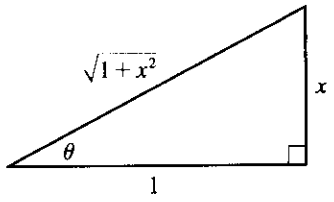
25. Integrate by parts with $u = \ln x$, $dv = dx/x^2 \;\Rightarrow\; du = dx/x$, $v = -1/x$.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{\ln t}{t} - \frac{1}{t} + 0 + 1\right)$$
$$= -0 - 0 + 0 + 1 = 1$$

since $\lim_{t\to\infty}\dfrac{\ln t}{t} \overset{\text{H}}{=} \lim_{t\to\infty}\dfrac{1/t}{1} = 0$. Convergent

26. $\displaystyle\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x\arctan x}{(1+x^2)^2}\,dx$. Let $u = \arctan x$, $dv = \dfrac{x\,dx}{(1+x^2)^2}$. Then $du = \dfrac{dx}{1+x^2}$, $v = \dfrac{1}{2}\displaystyle\int \frac{2x\,dx}{(1+x^2)^2} = \frac{-1/2}{1+x^2}$, and

$$\int \frac{x\arctan x}{(1+x^2)^2}\,dx = -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \frac{dx}{(1+x^2)^2} \qquad \left[\begin{matrix} x = \tan\theta, \\ dx = \sec^2\theta\,d\theta \end{matrix}\right]$$
$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2}$$
$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \cos^2\theta\,d\theta$$
$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{\theta}{4} + \frac{\sin\theta\cos\theta}{4} + C$$
$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{4}\arctan x + \frac{1}{4}\frac{x}{1+x^2} + C$$



It follows that

$$\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{4}\arctan x + \frac{1}{4}\frac{x}{1+x^2}\right]_0^t$$
$$= \lim_{t\to\infty}\left(-\frac{1}{2}\frac{\arctan t}{1+t^2} + \frac{1}{4}\arctan t + \frac{1}{4}\frac{t}{1+t^2}\right) = 0 + \frac{1}{4}\cdot\frac{\pi}{2} + 0 = \frac{\pi}{8}.$$

Convergent.

27. There is an infinite discontinuity at the left endpoint of $[0, 3]$.

$$\int_0^3 \frac{dx}{\sqrt{x}} = \lim_{t\to0^+}\int_t^3 \frac{dx}{\sqrt{x}} = \lim_{t\to0^+}\left[2\sqrt{x}\right]_t^3 = \lim_{t\to0^+}\left(2\sqrt{3} - 2\sqrt{t}\right) = 2\sqrt{3}.$$ Convergent

28. There is an infinite discontinuity at the left endpoint of $[0, 3]$.

$$\int_0^3 \frac{dx}{x\sqrt{x}} = \lim_{t\to0^+}\int_t^3 \frac{dx}{x^{3/2}} = \lim_{t\to0^+}\left[\frac{-2}{\sqrt{x}}\right]_t^3 = \frac{-2}{\sqrt{3}} + \lim_{t\to0^+}\frac{2}{\sqrt{t}} = \infty.$$ Divergent

29. There is an infinite discontinuity at the right endpoint of $[-1, 0]$.

$$\int_{-1}^0 \frac{dx}{x^2} = \lim_{t\to0^-}\int_{-1}^t \frac{dx}{x^2} = \lim_{t\to0^-}\left[\frac{-1}{x}\right]_{-1}^t = \lim_{t\to0^-}\left[-\frac{1}{t} + \frac{1}{-1}\right] = \infty.$$ Divergent

30. $\displaystyle\int_1^9 \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to9^-}\int_1^t \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to9^-}\left[\tfrac{3}{2}(x-9)^{2/3}\right]_1^t = \lim_{t\to9^-}\left[\tfrac{3}{2}(t-9)^{2/3} - \tfrac{3}{2}(4)\right] = 0 - 6 = -6.$
Convergent

31. $\displaystyle\int_{-2}^3 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}$, but $\displaystyle\int_{-2}^0 \frac{dx}{x^4} = \lim_{t\to0^-}\left[-\frac{x^{-3}}{3}\right]_{-2}^t = \lim_{t\to0^-}\left[-\frac{1}{3t^3} - \frac{1}{24}\right] = \infty.$ Divergent

32. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to1^-}\left[\sin^{-1}x\right]_0^t = \lim_{t\to1^-}\sin^{-1}t = \frac{\pi}{2}.$ Convergent

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7. Let $u = \ln t$, $du = dt/t$. Then $\displaystyle\int \frac{\sin(\ln t)}{t}\,dt = \int \sin u\,du = -\cos u + C = -\cos(\ln t) + C$.

8. Let $x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sec\theta} = \int \frac{\sec\theta\,d\theta}{\tan^2\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = \int \frac{du}{u^2} \quad [\text{put } u = \sin\theta]$$

$$= -\frac{1}{u} + C = -\frac{1}{\sin\theta} + C = -\frac{\sqrt{1+x^2}}{x} + C$$

9. $\displaystyle\int_1^4 x^{3/2}\ln x\,dx \quad \begin{bmatrix} u = \ln x, & dv = x^{3/2}\,dx, \\ du = dx/x & v = \frac{2}{5}x^{5/2} \end{bmatrix} = \frac{2}{5}\left[x^{5/2}\ln x\right]_1^4 - \frac{2}{5}\int_1^4 x^{3/2}\,dx$

$$= \tfrac{2}{5}(32\ln 4 - \ln 1) - \tfrac{2}{5}\left[\tfrac{2}{5}x^{5/2}\right]_1^4$$

$$= \tfrac{2}{5}(64\ln 2) - \tfrac{4}{25}(32-1)$$

$$= \tfrac{128}{5}\ln 2 - \tfrac{124}{25} \quad \left(\text{or } \tfrac{64}{5}\ln 4 - \tfrac{124}{25}\right)$$

10. Let $u = \arctan x$, $du = dx/(1+x^2)$. Then

$$\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2}\,dx = \int_0^{\pi/4} \sqrt{u}\,du = \frac{2}{3}\left[u^{3/2}\right]_0^{\pi/4} = \frac{2}{3}\left[\frac{\pi^{3/2}}{4^{3/2}} - 0\right] = \frac{2}{3}\cdot\frac{1}{8}\pi^{3/2} = \frac{1}{12}\pi^{3/2}.$$

11. Let $x = \sec\theta$. Then

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3} \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int_0^{\pi/3} \tan^2\theta\,d\theta = \int_0^{\pi/3} \left(\sec^2\theta - 1\right)d\theta$$

$$= \left[\tan\theta - \theta\right]_0^{\pi/3} = \sqrt{3} - \tfrac{\pi}{3}$$

12. $\displaystyle\int_{-1}^1 \frac{\sin x}{1+x^2}\,dx = 0$ by Theorem 5.5.6(b), since $f(x) = \dfrac{\sin x}{1+x^2}$ is an odd function.

13. $\displaystyle\int \frac{dx}{x^3+x} = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \ln|x| - \tfrac{1}{2}\ln\left(x^2+1\right) + C$

14. $\displaystyle\int \frac{x^2+2}{x+2}\,dx = \int \left(x - 2 + \frac{6}{x+2}\right)dx = \tfrac{1}{2}x^2 - 2x + 6\ln|x+2| + C$

15. $\displaystyle\int \sin^2\theta\cos^5\theta\,d\theta = \int \sin^2\theta\left(\cos^2\theta\right)^2\cos\theta\,d\theta = \int \sin^2\theta\left(1-\sin^2\theta\right)^2\cos\theta\,d\theta$

$$= \int u^2\left(1-u^2\right)^2\,du \quad [u = \sin\theta,\ du = \cos\theta\,d\theta] \quad = \int u^2\left(1 - 2u^2 + u^4\right)du$$

$$= \int \left(u^2 - 2u^4 + u^6\right)du = \tfrac{1}{3}u^3 - \tfrac{2}{5}u^5 + \tfrac{1}{7}u^7 + C = \tfrac{1}{3}\sin^3\theta - \tfrac{2}{5}\sin^5\theta + \tfrac{1}{7}\sin^7\theta + C$$

16. $\displaystyle\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta = \int \frac{(\tan^2\theta+1)^2\sec^2\theta}{\tan^2\theta}\,d\theta \qquad \begin{bmatrix} u = \tan\theta, \\ du = \sec^2\theta\,d\theta \end{bmatrix}$

$$= \int \frac{(u^2+1)^2}{u^2}\,du = \int \frac{u^4+2u^2+1}{u^2}\,du = \int \left(u^2 + 2 + \frac{1}{u^2}\right)du$$

$$= \frac{u^3}{3} + 2u - \frac{1}{u} + C = \frac{1}{3}\tan^3\theta + 2\tan\theta - \cot\theta + C$$

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17. Integrate by parts with $u = x$, $dv = \sec x \tan x\,dx \;\Rightarrow\; du = dx$, $v = \sec x$:

$$\int x \sec x \tan x\,dx = x \sec x - \int \sec x\,dx \overset{14}{=} x \sec x - \ln|\sec x + \tan x| + C.$$

18. $\dfrac{x^2+8x-3}{x^3+3x^2} = \dfrac{x^2+8x-3}{x^2(x+3)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+3} \;\Rightarrow\; x^2+8x-3 = Ax(x+3) + B(x+3) + Cx^2.$

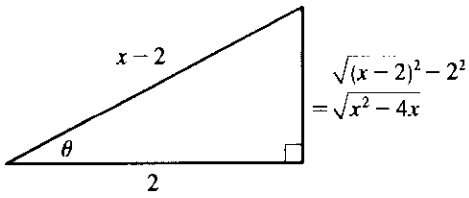
Taking $x = 0$, we get $-3 = 3B$, so $B = -1$. Taking $x = -3$, we get $-18 = 9C$, so $C = -2$. Taking $x = 1$, we get $6 = 4A + 4B + C = 4A - 4 - 2$, so $4A = 12$ and $A = 3$. Now

$$\int \frac{x^2+8x-3}{x^3+3x^2}\,dx = \int \left(\frac{3}{x} - \frac{1}{x^2} - \frac{2}{x+3}\right) dx = 3\ln|x| + \frac{1}{x} - 2\ln|x+3| + C$$

19.
$$\int \frac{x+1}{9x^2+6x+5}\,dx = \int \frac{x+1}{(9x^2+6x+1)+4}\,dx = \int \frac{x+1}{(3x+1)^2+4}\,dx \quad \begin{bmatrix} u = 3x+1, \\ du = 3\,dx \end{bmatrix}$$
$$= \int \frac{\left[\frac{1}{3}(u-1)\right]+1}{u^2+4}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{3}\int \frac{(u-1)+3}{u^2+4}\,du$$
$$= \frac{1}{9}\int \frac{u}{u^2+4}\,du + \frac{1}{9}\int \frac{2}{u^2+2^2}\,du = \frac{1}{9}\cdot\frac{1}{2}\ln(u^2+4) + \frac{2}{9}\cdot\frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right) + C$$
$$= \tfrac{1}{18}\ln(9x^2+6x+5) + \tfrac{1}{9}\tan^{-1}\left[\tfrac{1}{2}(3x+1)\right] + C$$

20. $\displaystyle\int \frac{dt}{\sin^2 t + \cos 2t} = \int \frac{dt}{\sin^2 t + (\cos^2 t - \sin^2 t)} = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\,dt = \tan t + C$

21.



$$\int \frac{dx}{\sqrt{x^2-4x}} = \int \frac{dx}{\sqrt{(x^2-4x+4)-4}} = \int \frac{dx}{\sqrt{(x-2)^2-2^2}}$$
$$= \int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta} \quad \begin{bmatrix} x-2 = 2\sec\theta, \\ dx = 2\sec\theta\tan\theta\,d\theta \end{bmatrix} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1$$
$$= \ln\left|\frac{x-2}{2} + \frac{\sqrt{x^2-4x}}{2}\right| + C_1 = \ln\left|x-2+\sqrt{x^2-4x}\right| + C, \text{ where } C = C_1 - \ln 2$$

22. Let $u = x + 1$. Then

$$\int \frac{x^3}{(x+1)^{10}}\,dx = \int \frac{(u-1)^3}{u^{10}}\,dx = \int \frac{u^3-3u^2+3u-1}{u^{10}}\,du$$
$$= \int \left(u^{-7} - 3u^{-8} + 3u^{-9} - u^{-10}\right) du = -\tfrac{1}{6}u^{-6} + \tfrac{3}{7}u^{-7} - \tfrac{3}{8}u^{-8} + \tfrac{1}{9}u^{-9} + C$$
$$= \frac{-1}{6(x+1)^6} + \frac{3}{7(x+1)^7} - \frac{3}{8(x+1)^8} + \frac{1}{9(x+1)^9} + C$$

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23. Let $u = \cot 4x$. Then $du = -4\csc^2 4x\,dx \;\Rightarrow$

$$\int \csc^4 4x\,dx = \int \left(\cot^2 4x + 1\right)\csc^2 4x\,dx = \int \left(u^2+1\right)\left(-\tfrac{1}{4}\,du\right)$$

$$= -\tfrac{1}{4}\left(\tfrac{1}{3}u^3 + u\right) + C = -\tfrac{1}{12}\left(\cot^3 4x + 3\cot 4x\right) + C$$

24. Let $u = \cos x$, $dv = e^x\,dx \;\Rightarrow\; du = -\sin x\,dx$, $v = e^x$: $(*)\; I = \int e^x \cos x\,dx = e^x\cos x + \int e^x \sin x\,dx$.

To integrate $\int e^x \sin x\,dx$, let $U = \sin x$, $dV = e^x\,dx \;\Rightarrow\; dU = \cos x\,dx$, $V = e^x$. Then

$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - I$. By substitution in $(*)$, $I = e^x\cos x + e^x \sin x - I \;\Rightarrow$

$2I = e^x(\cos x + \sin x) \;\Rightarrow\; I = \tfrac{1}{2}e^x(\cos x + \sin x) + C$.

25. $\dfrac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2} \;\Rightarrow$

$3x^3 - x^2 + 6x - 4 = (Ax+B)\left(x^2+2\right) + (Cx+D)\left(x^2+1\right)$. Equating the coefficients gives $A + C = 3$, $B + D = -1$, $2A + C = 6$, and $2B + D = -4 \;\Rightarrow\; A = 3$, $C = 0$, $B = -3$, and $D = 2$. Now

$$\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\,dx = 3\int \frac{x-1}{x^2+1}\,dx + 2\int \frac{dx}{x^2+2}$$

$$= \tfrac{3}{2}\ln\left(x^2+1\right) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}x\right) + C$$

26. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u}$, so

$$\int \frac{dx}{1+e^x} = \int \frac{du/u}{1+u} = \int \left[\frac{1}{u} - \frac{1}{u+1}\right] du = \ln u - \ln(u+1) + C = \ln e^x - \ln(1+e^x) + C$$

$$= x - \ln(1+e^x) + C$$

27. $\int_0^{\pi/2} \cos^3 x \sin 2x\,dx = \int_0^{\pi/2} \cos^3 x\,(2\sin x\cos x)\,dx = \int_0^{\pi/2} 2\cos^4 x \sin x\,dx = \left[-\tfrac{2}{5}\cos^5 x\right]_0^{\pi/2} = \tfrac{2}{5}$

28. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2 du \;\Rightarrow$

$$\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx = \int \frac{u+1}{u-1}\,3u^2 du = 3\int \left(u^2 + 2u + 2 + \frac{2}{u-1}\right) du$$

$$= u^3 + 3u^2 + 6u + 6\ln|u-1| + C = x + 3x^{2/3} + 6\sqrt[3]{x} + 6\ln\left|\sqrt[3]{x} - 1\right| + C$$

29. The product of an odd function and an even function is an odd function, so $f(x) = x^5 \sec x$ is an odd function. By Theorem 5.5.6(b), $\int_{-1}^{1} x^5 \sec x\,dx = 0$.

30. Let $u = e^{-x}$, $du = -e^{-x}\,dx$. Then

$$\int \frac{dx}{e^x\sqrt{1-e^{-2x}}} = \int \frac{e^{-x}\,dx}{\sqrt{1-(e^{-x})^2}} = \int \frac{-du}{\sqrt{1-u^2}} = -\sin^{-1}u + C = -\sin^{-1}(e^{-x}) + C.$$

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31. Let $u = \sqrt{e^x - 1}$. Then $u^2 = e^x - 1$ and $2u\,du = e^x\,dx$. Also, $e^x + 8 = u^2 + 9$. Thus,

$$\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,dx = \int_0^3 \frac{u\cdot 2u\,du}{u^2+9} = 2\int_0^3 \frac{u^2}{u^2+9}\,du = 2\int_0^3 \left(1 - \frac{9}{u^2+9}\right)du$$

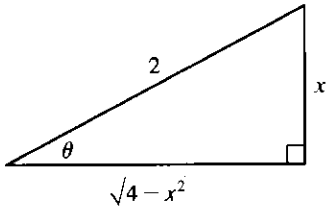
$$= 2\left[u - \frac{9}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]_0^3 = 2\left[(3 - 3\tan^{-1}1) - 0\right] = 2\left(3 - 3\cdot\frac{\pi}{4}\right) = 6 - \frac{3\pi}{2}$$

32. $\displaystyle\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\,dx = \int_0^{\pi/4} x\tan x\sec^2 x\,dx \quad \begin{bmatrix} u = x, & dv = \tan x\sec^2 x\,dx, \\ du = dx & v = \frac{1}{2}\tan^2 x \end{bmatrix}$

$$= \left[\frac{x}{2}\tan^2 x\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\tan^2 x\,dx = \frac{\pi}{8}\cdot 1^2 - 0 - \frac{1}{2}\int_0^{\pi/4}(\sec^2 x - 1)\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[\tan x - x\right]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$

33. Let $x = 2\sin\theta \;\Rightarrow\; (4 - x^2)^{3/2} = (2\cos\theta)^3$, $dx = 2\cos\theta\,d\theta$, so



$$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}\,2\cos\theta\,d\theta$$

$$= \int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

34. Integrate by parts twice, first with $u = (\arcsin x)^2$, $dv = dx$:

$$I = \int (\arcsin x)^2\,dx = x(\arcsin x)^2 - \int 2x\arcsin x\left(\frac{dx}{\sqrt{1-x^2}}\right)$$

Now let $U = \arcsin x$, $dV = \dfrac{x}{\sqrt{1-x^2}}\,dx \;\Rightarrow\; dU = \dfrac{1}{\sqrt{1-x^2}}\,dx$, $V = -\sqrt{1-x^2}$. So

$$I = x(\arcsin x)^2 - 2\left[\arcsin x\left(-\sqrt{1-x^2}\right) + \int dx\right] = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$$

35. $\displaystyle\int \frac{1}{\sqrt{x}+x^{3/2}}\,dx = \int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = \int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}} \quad \begin{bmatrix} u = 1+\sqrt{x}, \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \quad = \int \frac{2\,du}{\sqrt{u}} = \int 2u^{-1/2}\,du$

$$= 4\sqrt{u} + C = 4\sqrt{1+\sqrt{x}} + C$$

36. $\displaystyle\int \frac{1-\tan\theta}{1+\tan\theta}\,d\theta = \int \frac{\frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}\,d\theta = \int \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\,d\theta = \ln|\cos\theta + \sin\theta| + C$

37. $\displaystyle\int (\cos x + \sin x)^2\cos 2x\,dx = \int \left(\cos^2 x + 2\sin x\cos x + \sin^2 x\right)\cos 2x\,dx$

$$= \int (1 + \sin 2x)\cos 2x\,dx = \int \cos 2x\,dx + \tfrac{1}{2}\int \sin 4x\,dx = \tfrac{1}{2}\sin 2x - \tfrac{1}{8}\cos 4x + C$$

Or: $\displaystyle\int (\cos x + \sin x)^2\cos 2x\,dx = \int (\cos x + \sin x)^2(\cos^2 x - \sin^2 x)\,dx$

$$= \int (\cos x + \sin x)^3(\cos x - \sin x)\,dx = \tfrac{1}{4}(\cos x + \sin x)^4 + C_1$$

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38. Let $u = (\tan^{-1} x)^2$, $dv = x\,dx \;\Rightarrow\; du = 2(\tan^{-1} x)/(1+x^2)\,dx$, $v = \frac{1}{2}x^2$. Then

$$I = \int x(\tan^{-1} x)^2\,dx = \tfrac{1}{2}x^2(\tan^{-1} x)^2 - \int \frac{x^2 \tan^{-1} x}{1+x^2}\,dx$$

Now let $w = \tan^{-1} x$, $dw = 1/(1+x^2)\,dx$, and $x^2 = \tan^2 w$. So

$$\begin{aligned} I &= \tfrac{1}{2}x^2(\tan^{-1} x)^2 - \int w\tan^2 w\,dw = \tfrac{1}{2}x^2(\tan^{-1} x)^2 - \int w\sec^2 w\,dw + \int w\,dw \\ &= \tfrac{1}{2}x^2(\tan^{-1} x)^2 - \left(x\tan^{-1} x - \ln\sqrt{x^2+1}\right) + \tfrac{1}{2}(\tan^{-1} x)^2 \quad [\text{parts with } u = w,\ dv = \sec^2 w\,dw] \\ &= \tfrac{1}{2}(x^2+1)(\tan^{-1} x)^2 - x\tan^{-1} x + \ln\sqrt{x^2+1} + C \\ &\qquad \text{or } \tfrac{1}{2}(x^2+1)(\tan^{-1} x)^2 - x\tan^{-1} x + \tfrac{1}{2}\ln(x^2+1) + C \end{aligned}$$

39. We'll integrate $I = \displaystyle\int \frac{xe^{2x}}{(1+2x)^2}\,dx$ by parts with $u = xe^{2x}$ and $dv = \dfrac{dx}{(1+2x)^2}$. Then

$du = (x\cdot 2e^{2x} + e^{2x}\cdot 1)\,dx$ and $v = -\dfrac{1}{2}\cdot\dfrac{1}{1+2x}$, so

$$\begin{aligned} I &= -\frac{1}{2}\cdot\frac{xe^{2x}}{1+2x} - \int\left[-\frac{1}{2}\cdot\frac{e^{2x}(2x+1)}{1+2x}\right]dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2}\cdot\frac{1}{2}e^{2x} + C \\ &= e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right) + C. \end{aligned}$$

Thus, $\displaystyle\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,dx = \left[e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right)\right]_0^{1/2} = e\left(\frac{1}{4} - \frac{1}{8}\right) - 1\left(\frac{1}{4} - 0\right) = \frac{1}{8}e - \frac{1}{4}.$

40. $$\begin{aligned} \int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta &= \int_{\pi/4}^{\pi/3} \frac{\sqrt{\frac{\sin\theta}{\cos\theta}}}{2\sin\theta\cos\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2}(\sin\theta)^{-1/2}(\cos\theta)^{-3/2}d\theta \\ &= \int_{\pi/4}^{\pi/3} \frac{1}{2}\left(\frac{\sin\theta}{\cos\theta}\right)^{-1/2}(\cos\theta)^{-2}d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta\,d\theta \\ &= \left[\sqrt{\tan\theta}\right]_{\pi/4}^{\pi/3} = \sqrt{\sqrt{3}} - \sqrt{1} = \sqrt[4]{3} - 1 \end{aligned}$$

41. $$\begin{aligned} \int_1^\infty \frac{1}{(2x+1)^3}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \tfrac{1}{2}(2x+1)^{-3}\,2\,dx \\ &= \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t = -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2} - \frac{1}{9}\right] = -\frac{1}{4}\left(0 - \frac{1}{9}\right) = \frac{1}{36} \end{aligned}$$

42. $\dfrac{t^2+1}{t^2-1} = \dfrac{(t^2-1)+2}{t^2-1} = 1 + \dfrac{2}{(t+1)(t-1)}$. Now $\dfrac{2}{(t+1)(t-1)} = \dfrac{A}{t+1} + \dfrac{B}{t-1} \;\Rightarrow$

$2 = A(t-1) + B(t+1)$. Letting $t = 1 \;\Rightarrow\; B = 1$ and letting $t = -1 \;\Rightarrow\; A = -1$. So

$$\begin{aligned} \int_0^1 \frac{t^2+1}{t^2-1}\,dt &= \lim_{b\to 1^-}\int_0^b \left(1 + \frac{-1}{t+1} + \frac{1}{t-1}\right)dt = \lim_{b\to 1^-}\left[t - \ln|t+1| + \ln|t-1|\right]_0^b \\ &= \lim_{b\to 1^-}\left[t + \ln\left|\frac{t-1}{t+1}\right|\right]_0^b = \lim_{b\to 1^-}\left[\left(b + \ln\left|\frac{b-1}{b+1}\right|\right) - (0+0)\right] = -\infty. \quad \text{Divergent} \end{aligned}$$

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43. $\displaystyle\int \frac{dx}{x\ln x} \quad \begin{bmatrix} u=\ln x, \\ du=\dfrac{dx}{x} \end{bmatrix} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$, so

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{t\to\infty}\int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty}\Big[\ln|\ln x|\Big]_2^t = \lim_{t\to\infty}[\ln(\ln t) - \ln(\ln 2)] = \infty,$$ so the integral is divergent.

44. Let $u = \sqrt{y-2}$. Then $y = u^2+2$ and $dy = 2u\,du$, so

$$\int \frac{y\,dy}{\sqrt{y-2}} = \int \frac{(u^2+2)2u\,du}{u} = 2\int (u^2+2)\,du = 2\big[\tfrac{1}{3}u^3 + 2u\big] + C$$

Thus, $\displaystyle\int_2^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+}\int_t^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+}\Big[\tfrac{2}{3}(y-2)^{3/2} + 4\sqrt{y-2}\Big]_t^6$

$$= \lim_{t\to 2^+}\Big[\tfrac{16}{3} + 8 - \tfrac{2}{3}(t-2)^{3/2} - 4\sqrt{t-2}\Big] = \tfrac{40}{3}.$$

45. $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^4 \frac{\ln x}{\sqrt{x}}\,dx \overset{*}{=} \lim_{t\to 0^+}\big[2\sqrt{x}\ln x - 4\sqrt{x}\big]_t^4$

$$= \lim_{t\to 0^+}\big[(2\cdot 2\ln 4 - 4\cdot 2) - \big(2\sqrt{t}\ln t - 4\sqrt{t}\big)\big] \overset{**}{=} (4\ln 4 - 8) - (0-0) = 4\ln 4 - 8$$

(*) Let $u = \ln x$, $dv = \dfrac{1}{\sqrt{x}}\,dx \;\Rightarrow\; du = \dfrac{1}{x}\,dx$, $v = 2\sqrt{x}$. Then

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 2\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

(**) $\displaystyle\lim_{t\to 0^+}\big(2\sqrt{t}\ln t\big) = \lim_{t\to 0^+}\frac{2\ln t}{t^{-1/2}} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{2/t}{-\frac{1}{2}t^{-3/2}} = \lim_{t\to 0^+}\big(-4\sqrt{t}\big) = 0$

46. Note that $f(x) = 1/(2-3x)$ has an infinite discontinuity at $x = \frac{2}{3}$. Now

$$\int_0^{2/3} \frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-}\int_0^t \frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-}\Big[-\tfrac{1}{3}\ln|2-3x|\Big]_0^t$$

$$= -\tfrac{1}{3}\lim_{t\to(2/3)^-}\Big[\ln|2-3t| - \ln 2\Big] = \infty$$

Since $\displaystyle\int_0^{2/3} \frac{1}{2-3x}\,dx$ diverges, so does $\displaystyle\int_0^1 \frac{1}{2-3x}\,dx$.

47. $\displaystyle\int_0^3 \frac{dx}{x^2-x-2} = \int_0^3 \frac{dx}{(x+1)(x-2)} = \int_0^2 \frac{dx}{(x+1)(x-2)} + \int_2^3 \frac{dx}{(x+1)(x-2)}$, and

$$\int_2^3 \frac{dx}{x^2-x-2} = \lim_{t\to 2^+}\int_t^3 \left[\frac{-1/3}{x+1} + \frac{1/3}{x-2}\right]dx = \lim_{t\to 2^+}\left[\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|\right]_t^3$$

$$= \lim_{t\to 2^+}\left[\tfrac{1}{3}\ln\tfrac{1}{4} - \frac{1}{3}\ln\left|\frac{t-2}{t+1}\right|\right] = \infty$$

so $\displaystyle\int_0^3 \frac{dx}{x^2-x-2}$ diverges.

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18. $\displaystyle\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x^4}}\,dx = \int_{-1}^{1} \left(x^{-1/3} + x^{-4/3}\right) dx = \int_{-1}^{0} \left(x^{-1/3} + x^{-4/3}\right) dx + \int_{0}^{1} \left(x^{-1/3} + x^{-4/3}\right) dx$. But

$$\int_0^1 \left(x^{-1/3} + x^{-4/3}\right) dx = \lim_{t\to 0^+} \int_t^1 \left(x^{-1/3} + x^{-4/3}\right) dx = \lim_{t\to 0^+} \left[\tfrac{3}{2}x^{2/3} - 3x^{-1/3}\right]_t^1$$

$$= \lim_{t\to 0^+} \left[\tfrac{3}{2} - 3 - \tfrac{3}{2}t^{2/3} + 3t^{-1/3}\right] = \infty. \quad \text{Divergent}$$

19. Let $u = 2x + 1$. Then

$$\int_{-\infty}^{\infty} \frac{dx}{4x^2+4x+5} = \int_{-\infty}^{\infty} \frac{\frac{1}{2}\,du}{u^2+4} = \frac{1}{2}\int_{-\infty}^{0} \frac{du}{u^2+4} + \frac{1}{2}\int_{0}^{\infty} \frac{du}{u^2+4}$$

$$= \tfrac{1}{2} \lim_{t\to-\infty} \left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_t^0 + \tfrac{1}{2} \lim_{t\to\infty} \left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_0^t$$

$$= \tfrac{1}{4}\left[0 - \left(-\tfrac{\pi}{2}\right)\right] + \tfrac{1}{4}\left[\tfrac{\pi}{2} - 0\right] = \tfrac{\pi}{4}$$

50. $\displaystyle\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\tan^{-1}x}{x^2}\,dx$. Integrate by parts:

$$\int \frac{\tan^{-1}x}{x^2}\,dx = \frac{-\tan^{-1}x}{x} + \int \frac{1}{x}\frac{dx}{1+x^2} = \frac{-\tan^{-1}x}{x} + \int \left[\frac{1}{x} - \frac{x}{x^2+1}\right] dx$$

$$= \frac{-\tan^{-1}x}{x} + \ln|x| - \tfrac{1}{2}\ln\left(x^2+1\right) + C = \frac{-\tan^{-1}x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1} + C$$

Thus,

$$\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{\tan^{-1}x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1}\right]_1^t = \lim_{t\to\infty} \left[-\frac{\tan^{-1}t}{t} + \frac{1}{2}\ln\frac{t^2}{t^2+1} + \tfrac{\pi}{4} - \tfrac{1}{2}\ln\tfrac{1}{2}\right]$$

$$= 0 + \tfrac{1}{2}\ln 1 + \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2 = \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2$$

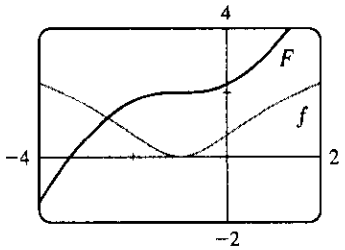
51. We first make the substitution $t = x + 1$, so $\ln\left(x^2+2x+2\right) = \ln\left[(x+1)^2+1\right] = \ln\left(t^2+1\right)$. Then we use parts with $u = \ln\left(t^2+1\right)$, $dv = dt$:

$$\int \ln\left(t^2+1\right) dt = t\,\ln\left(t^2+1\right) - \int \frac{t(2t)\,dt}{t^2+1} = t\,\ln\left(t^2+1\right) - 2\int \frac{t^2\,dt}{t^2+1}$$

$$= t\,\ln\left(t^2+1\right) - 2\int \left(1 - \frac{1}{t^2+1}\right) dt = t\,\ln\left(t^2+1\right) - 2t + 2\arctan t + C$$

$$= (x+1)\,\ln\left(x^2+2x+2\right) - 2x + 2\arctan(x+1) + K, \text{ where } K = C - 2$$

[Alternatively, we could have integrated by parts immediately with $u = \ln\left(x^2+2x+2\right)$.] Notice from the graph that $f = 0$ where F has a horizontal tangent. Also, F is always increasing, and $f \geq 0$.



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52. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x\,dx = \frac{1}{2}\,du$, so

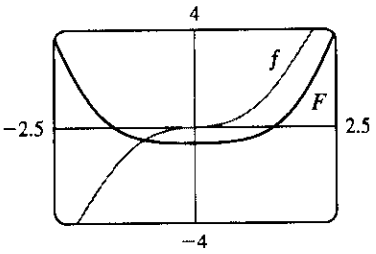
$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C$$

$$= \tfrac{1}{3}\left(x^2+1\right)^{3/2} - \left(x^2+1\right)^{1/2} + C$$

$$= \tfrac{1}{3}\left(x^2+1\right)^{1/2}\left[\left(x^2+1\right) - 3\right] + C$$

$$= \tfrac{1}{3}\sqrt{x^2+1}\,\left(x^2-2\right) + C.$$

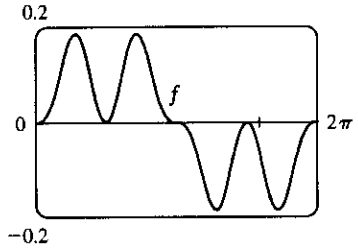


53. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x\,\sin^3 x\,dx$ is equal to 0.

To evaluate the integral, we write the integral as

$I = \int_0^{2\pi} \cos^2 x\,(1 - \cos^2 x)\sin x\,dx$ and let $u = \cos x \;\Rightarrow$

$du = -\sin x\,dx$. Thus, $I = \int_1^1 u^2(1-u^2)(-du) = 0$.



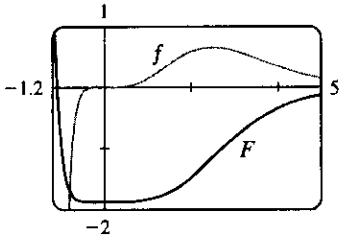
54. (a) To evaluate $\int x^5 e^{-2x}\,dx$ by hand, we would integrate by parts repeatedly, always taking $dv = e^{-2x}$ and starting with $u = x^5$. Each time we would reduce the degree of the x-factor by 1.

(b) To evaluate the integral using tables, we would use Formula 97 (which is proved using integration by parts) until the exponent of x was reduced to 1, and then we would use Formula 96.

(c) $\int x^5 e^{-2x}\,dx$

$$= -\tfrac{1}{8}e^{-2x}\left(4x^5 + 10x^4 + 20x^3 + 30x^2 + 30x + 15\right) + C$$

(d)



55. $u = e^x \;\Rightarrow\; du = e^x\,dx$, so

$$\int e^x\sqrt{1-e^{2x}}\,dx = \int \sqrt{1-u^2}\,du \overset{30}{=} \tfrac{1}{2}u\sqrt{1-u^2} + \tfrac{1}{2}\sin^{-1}u + C = \tfrac{1}{2}\left[e^x\sqrt{1-e^{2x}} + \sin^{-1}(e^x)\right] + C$$

56. $\int \csc^5 t\,dt \overset{78}{=} -\tfrac{1}{4}\cot t\,\csc^3 t + \tfrac{3}{4}\int \csc^3 t\,dt$

$$\overset{72}{=} -\tfrac{1}{4}\cot t\,\csc^3 t + \tfrac{3}{4}\left[-\tfrac{1}{2}\csc t\,\cot t + \tfrac{1}{2}\ln|\csc t - \cot t|\right] + C$$

$$= -\tfrac{1}{4}\cot t\,\csc^3 t - \tfrac{3}{8}\csc t\,\cot t + \tfrac{3}{8}\ln|\csc t - \cot t| + C$$

57. $\int \sqrt{x^2+x+1}\,dx = \int \sqrt{x^2+x+\tfrac{1}{4}+\tfrac{3}{4}}\,dx = \int \sqrt{\left(x+\tfrac{1}{2}\right)^2 + \tfrac{3}{4}}\,dx$

$$= \int \sqrt{u^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2}\,du \quad [u = x + \tfrac{1}{2},\, du = dx]$$

$$\overset{21}{=} \tfrac{1}{2}u\sqrt{u^2+\tfrac{3}{4}} + \tfrac{3}{8}\ln\left(u + \sqrt{u^2+\tfrac{3}{4}}\right) + C$$

$$= \frac{2x+1}{4}\sqrt{x^2+x+1} + \tfrac{3}{8}\ln\left(x + \tfrac{1}{2} + \sqrt{x^2+x+1}\right) + C$$

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58. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cot x\,dx}{\sqrt{1+2\sin x}} = \int \frac{du}{u\sqrt{1+2u}} \overset{\substack{57 \text{ with} \\ a=1,\, b=2}}{=} \ln\left|\frac{\sqrt{1+2u}-1}{\sqrt{1+2u}+1}\right| + C = \ln\left|\frac{\sqrt{1+2\sin x}-1}{\sqrt{1+2\sin x}+1}\right| + C$$

59. (a) $\dfrac{d}{du}\left[-\dfrac{1}{u}\sqrt{a^2-u^2} - \sin^{-1}\left(\dfrac{u}{a}\right) + C\right] = \dfrac{1}{u^2}\sqrt{a^2-u^2} + \dfrac{1}{\sqrt{a^2-u^2}} - \dfrac{1}{\sqrt{1-u^2/a^2}}\cdot\dfrac{1}{a}$

$$= (a^2-u^2)^{-1/2}\left[\frac{1}{u^2}(a^2-u^2) + 1 - 1\right] = \frac{\sqrt{a^2-u^2}}{u^2}$$

(b) Let $u = a\sin\theta \;\Rightarrow\; du = a\cos\theta\,d\theta$, $a^2-u^2 = a^2(1-\sin^2\theta) = a^2\cos^2\theta$.

$$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \int \frac{a^2\cos^2\theta}{a^2\sin^2\theta}\,d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta = \int (\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\left(\frac{u}{a}\right) + C$$

60. Work backward, and use integration by parts with $U = u^{-(n-1)}$ and $dV = (a+bu)^{-1/2}\,du \;\Rightarrow$ $dU = \dfrac{-(n-1)\,du}{u^n}$ and $V = \dfrac{2}{b}\sqrt{a+bu}$, to get

$$\int \frac{du}{u^{n-1}\sqrt{a+bu}} = \int U\,dV = UV - \int V\,dU = \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b}\int \frac{\sqrt{a+bu}}{u^n}\,du$$

$$= \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b}\int \frac{a+bu}{u^n\sqrt{a+bu}}\,du$$

$$= \frac{2\sqrt{a+bu}}{bu^{n-1}} + 2(n-1)\int \frac{du}{u^{n-1}\sqrt{a+bu}} + \frac{2a(n-1)}{b}\int \frac{du}{u^n\sqrt{a+bu}}$$

Rearranging the equation gives $\dfrac{2a(n-1)}{b}\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{2\sqrt{a+bu}}{bu^{n-1}} - (2n-3)\int \frac{du}{u^{n-1}\sqrt{a+bu}} \;\Rightarrow$

$$\int \frac{du}{u^n\sqrt{a+bu}} = \frac{-\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

61. For $n \geq 0$, $\int_0^\infty x^n\,dx = \lim_{t\to\infty}\left[x^{n+1}/(n+1)\right]_0^t = \infty$. For $n < 0$, $\int_0^\infty x^n\,dx = \int_0^1 x^n\,dx + \int_1^\infty x^n\,dx$. Both integrals are improper. By (8.8.2), the second integral diverges if $-1 \leq n < 0$. By Exercise 8.8.57, the first integral diverges if $n \leq -1$. Thus, $\int_0^\infty x^n\,dx$ is divergent for all values of n.

62. $I = \displaystyle\int_0^\infty e^{ax}\cos x\,dx = \lim_{t\to\infty}\int_0^t e^{ax}\cos x\,dx \overset{\substack{99 \text{ with} \\ b=1}}{=} \lim_{t\to\infty}\left[\frac{e^{ax}}{a^2+1}(a\cos x + \sin x)\right]_0^t$

$$= \lim_{t\to\infty}\left[\frac{e^{at}}{a^2+1}(a\cos t + \sin t) - \frac{1}{a^2+1}(a)\right] = \frac{1}{a^2+1}\lim_{t\to\infty}\left[e^{at}(a\cos t + \sin t) - a\right].$$

For $a \geq 0$, the limit does not exist due to oscillation. For $a < 0$, $\lim_{t\to\infty}\left[e^{at}(a\cos t + \sin t)\right] = 0$ by the Squeeze Theorem, because $\left|e^{at}(a\cos t + \sin t)\right| \leq e^{at}(|a|+1)$, so $I = \dfrac{1}{a^2+1}(-a) = -\dfrac{a}{a^2+1}$.

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63. $f(x) = \sqrt{1+x^4}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{10} = \dfrac{1}{10}$.

(a) $T_{10} = \frac{1}{10\cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.090608$

(b) $M_{10} = \frac{1}{10}\left[f\left(\frac{1}{20}\right) + f\left(\frac{3}{20}\right) + f\left(\frac{5}{20}\right) + \cdots + f\left(\frac{19}{20}\right)\right] \approx 1.088840$

(c) $S_{10} = \frac{1}{10\cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + \cdots + 4f(0.9) + f(1)] \approx 1.089429$

f is concave upward, so the Trapezoidal Rule gives us an overestimate, the Midpoint Rule gives an underestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

64. $f(x) = \sqrt{\sin x}$, $\Delta x = \dfrac{\frac{\pi}{2} - 0}{10} = \dfrac{\pi}{20}$.

(a) $T_{10} = \frac{\pi}{20\cdot 2}\left\{f(0) + 2\left[f\left(\frac{\pi}{20}\right) + f\left(\frac{2\pi}{20}\right) + \cdots + f\left(\frac{9\pi}{20}\right)\right] + f\left(\frac{\pi}{2}\right)\right\} \approx 1.185197$

(b) $M_{10} = \frac{\pi}{20}\left[f\left(\frac{\pi}{40}\right) + f\left(\frac{3\pi}{40}\right) + f\left(\frac{5\pi}{40}\right) + \cdots + f\left(\frac{17\pi}{40}\right) + f\left(\frac{19\pi}{40}\right)\right] \approx 1.201932$

(c) $S_{10} = \frac{\pi}{20\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{20}\right) + 2f\left(\frac{2\pi}{20}\right) + \cdots + 4f\left(\frac{9\pi}{20}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 1.193089$

f is concave downward, so the Trapezoidal Rule gives us an underestimate, the Midpoint Rule gives an overestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

65. $f(x) = (1+x^4)^{1/2}$, $f'(x) = \frac{1}{2}(1+x^4)^{-1/2}(4x^3) = 2x^3(1+x^4)^{-1/2}$, $f''(x) = (2x^6 + 6x^2)(1+x^4)^{-3/2}$.

A graph of f'' on $[0, 1]$ shows that it has its maximum at $x = 1$, so $|f''(x)| \le f''(1) = \sqrt{8}$ on $[0, 1]$. By taking $K = \sqrt{8}$, we find that the error in Exercise 63(a) is bounded by $\dfrac{K(b-a)^3}{12n^2} = \dfrac{\sqrt{8}}{1200} \approx 0.0024$, and in (b) by about $\frac{1}{2}(0.0024) = 0.0012$.

Note: Another way to estimate K is to let $x = 1$ in the factor $2x^6 + 6x^2$ (maximizing the numerator) and let $x = 0$ in the factor $(1+x^4)^{-3/2}$ (minimizing the denominator). Doing so gives us $K = 8$ and errors of $0.00\overline{6}$ and $0.00\overline{3}$.

Using $K = 8$ for the Trapezoidal Rule, we have $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.00001 \quad \Leftrightarrow \quad \dfrac{8(1-0)^3}{12n^2} \le \dfrac{1}{100{,}000}$

$\Leftrightarrow \quad n^2 \ge \dfrac{800{,}000}{12} \quad \Leftrightarrow \quad n \gtrsim 258.2$, so we should take $n = 259$.

For the Midpoint Rule, $|E_M| \le \dfrac{K(b-a)^3}{24n^2} \le 0.00001 \quad \Leftrightarrow \quad n^2 \ge \dfrac{800{,}000}{24} \quad \Leftrightarrow \quad n \gtrsim 182.6$, so we should take $n = 183$.

66. $\displaystyle\int_1^4 \frac{e^x}{x}\,dx \approx S_6 = \frac{(4-1)/6}{3}[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 17.739438$

67. $\Delta t = \left(\frac{10}{60} - 0\right)/10 = \frac{1}{60}$.

Distance traveled $= \int_0^{10} v\,dt \approx S_{10}$

$$= \tfrac{1}{60\cdot 3}[40 + 4(42) + 2(45) + 4(49) + 2(52) + 4(54) + 2(56) + 4(57) + 2(57) + 4(55) + 56]$$

$$= \tfrac{1}{180}(1544) = 8.5\overline{7} \text{ mi}$$

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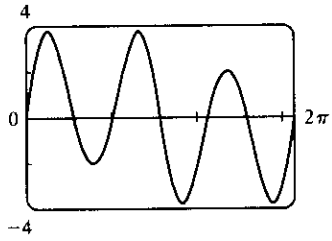
58. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$:

$$\begin{aligned}\text{Increase in bee population} &= \int_0^{24} r(t)\,dt \approx S_6 \\ &= \tfrac{4}{3}[r(0) + 4r(4) + 2r(8) + 4r(12) + 2r(16) + 4r(20) + r(24)] \\ &= \tfrac{4}{3}[0 + 4(300) + 2(3000) + 4(11{,}000) + 2(4000) + 4(400) + 0] \\ &= \tfrac{4}{3}(60{,}800) \approx 81{,}067 \text{ bees}\end{aligned}$$

59. (a) $f(x) = \sin(\sin x)$. A CAS gives

$$\begin{aligned}f^{(4)}(x) = \sin(\sin x)\left[\cos^4 x + 7\cos^2 x - 3\right] \\ + \cos(\sin x)\left[6\cos^2 x \sin x + \sin x\right]\end{aligned}$$



From the graph, we see that $\left|f^{(4)}(x)\right| < 3.8$ for $x \in [0, \pi]$.

(b) We use Simpson's Rule with $f(x) = \sin(\sin x)$ and $\Delta x = \frac{\pi}{10}$:

$$\int_0^{\pi} f(x)\,dx \approx \tfrac{\pi}{10\cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{10}\right) + 2f\left(\tfrac{2\pi}{10}\right) + \cdots + 4f\left(\tfrac{9\pi}{10}\right) + f(\pi)\right] \approx 1.786721$$

From part (a), we know that $\left|f^{(4)}(x)\right| < 3.8$ on $[0, \pi]$, so we use Theorem 8.7.4 with $K = 3.8$, and estimate the error as $|E_S| \le \dfrac{3.8(\pi - 0)^5}{180(10)^4} \approx 0.000646$.

(c) If we want the error to be less than 0.00001, we must have $|E_S| \le \frac{3.8\pi^5}{180n^4} \le 0.00001$, so $n^4 \ge \frac{3.8\pi^5}{180(0.00001)} \approx 646{,}041.6 \quad \Rightarrow \quad n \ge 28.35$. Since n must be even for Simpson's Rule, we must have $n \ge 30$ to ensure the desired accuracy.

70. With an x-axis in the normal position, at $x = 7$ we have $C = 2\pi r = 45 \quad \Rightarrow \quad r(7) = \frac{2\pi}{45}$. Using Simpson's Rule with $n = 4$ and $\Delta x = 7$, we have

$$V = \int_0^{28} \pi[r(x)]^2\,dx \approx S_4 = \tfrac{7}{3}\left[0 + 4\pi\left(\tfrac{45}{2\pi}\right)^2 + 2\pi\left(\tfrac{53}{2\pi}\right)^2 + 4\pi\left(\tfrac{45}{2\pi}\right)^2 + 0\right] = \tfrac{7}{3}\left(\tfrac{21{,}818}{4\pi}\right) \approx 4051 \text{ cm}^3$$

71. $\dfrac{x^3}{x^5+2} \le \dfrac{x^3}{x^5} = \dfrac{1}{x^2}$ for x in $[1, \infty)$. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent by (8.8.2) with $p = 2 > 1$. Therefore, $\displaystyle\int_1^{\infty} \frac{x^3}{x^5+2}\,dx$ is convergent by the Comparison Theorem.

72. The line $y = 3$ intersects the hyperbola $y^2 - x^2 = 1$ at two points on its upper branch, namely $\left(-2\sqrt{2}, 3\right)$ and $\left(2\sqrt{2}, 3\right)$. The desired area is

$$\begin{aligned}A &= \textstyle\int_{-2\sqrt{2}}^{2\sqrt{2}} \left(3 - \sqrt{x^2+1}\right) dx = 2\int_0^{2\sqrt{2}} \left(3 - \sqrt{x^2+1}\right) dx \\ &\overset{21}{=} 2\left[3x - \tfrac{1}{2}x\sqrt{x^2+1} - \tfrac{1}{2}\ln\left(x + \sqrt{x^2+1}\right)\right]_0^{2\sqrt{2}} \\ &= \left[6x - x\sqrt{x^2+1} - \ln\left(x + \sqrt{x^2+1}\right)\right]_0^{2\sqrt{2}} = 12\sqrt{2} - 2\sqrt{2}\cdot 3 - \ln\left(2\sqrt{2}+3\right) \\ &= 6\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)\end{aligned}$$

Another method: $A = 2\int_1^3 \sqrt{y^2-1}\,dy$ and use Formula 39.

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73. For x in $[0, \frac{\pi}{2}]$, $0 \le \cos^2 x \le \cos x$. For x in $[\frac{\pi}{2}, \pi]$, $\cos x \le 0 \le \cos^2 x$. Thus,

$$\begin{aligned}\text{area} &= \int_0^{\pi/2} (\cos x - \cos^2 x)\,dx + \int_{\pi/2}^{\pi} (\cos^2 x - \cos x)\,dx \\ &= \left[\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} + \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - \sin x\right]_{\pi/2}^{\pi} \\ &= \left[\left(1 - \tfrac{\pi}{4}\right) - 0\right] + \left[\tfrac{\pi}{2} - \left(\tfrac{\pi}{4} - 1\right)\right] = 2\end{aligned}$$

74. The curves $y = \dfrac{1}{2 \pm \sqrt{x}}$ are defined for $x \ge 0$. For $x > 0$, $\dfrac{1}{2-\sqrt{x}} > \dfrac{1}{2+\sqrt{x}}$. Thus, the required area is

$$\begin{aligned}\int_0^1 \left(\frac{1}{2-\sqrt{x}} - \frac{1}{2+\sqrt{x}}\right) dx &= \int_0^1 \left(\frac{1}{2-u} - \frac{1}{2+u}\right) 2u\,du \quad [\text{put } u = \sqrt{x}\,] \\ &= 2\int_0^1 \left(-\frac{u}{u-2} - \frac{u}{u+2}\right) du = 2\int_0^1 \left(-1 - \frac{2}{u-2} - 1 + \frac{2}{u+2}\right) du \\ &= 2\left[2\ln\left|\frac{u+2}{u-2}\right| - 2u\right]_0^1 = 4\ln 3 - 4\end{aligned}$$

75. Using the formula for disks, the volume is

$$\begin{aligned}V &= \int_0^{\pi/2} \pi\,[f(x)]^2\,dx = \pi\int_0^{\pi/2} \left(\cos^2 x\right)^2\,dx = \pi\int_0^{\pi/2} \left[\tfrac{1}{2}(1+\cos 2x)\right]^2\,dx \\ &= \tfrac{\pi}{4}\int_0^{\pi/2} (1 + \cos^2 2x + 2\cos 2x)\,dx = \tfrac{\pi}{4}\int_0^{\pi/2} \left[1 + \tfrac{1}{2}(1+\cos 4x) + 2\cos 2x\right] dx \\ &= \tfrac{\pi}{4}\left[\tfrac{3}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\sin 4x\right) + 2\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} = \tfrac{\pi}{4}\left[\left(\tfrac{3\pi}{4} + \tfrac{1}{8}\cdot 0 + 0\right) - 0\right] = \tfrac{3\pi^2}{16}\end{aligned}$$

76. Using the formula for cylindrical shells, the volume is

$$\begin{aligned}V &= \int_0^{\pi/2} 2\pi x f(x)\,dx = 2\pi\int_0^{\pi/2} x\cos^2 x\,dx \\ &= 2\pi\int_0^{\pi/2} x\left[\tfrac{1}{2}(1+\cos 2x)\right] dx = 2\left(\tfrac{1}{2}\right)\pi\int_0^{\pi/2} (x + x\cos 2x)\,dx \\ &= \pi\left(\left[\tfrac{1}{2}x^2\right]_0^{\pi/2} + \left[x\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} - \int_0^{\pi/2} \tfrac{1}{2}\sin 2x\,dx\right) \quad [\text{parts with } u = x,\ dv = \cos 2x\,dx] \\ &= \pi\left[\tfrac{1}{2}\left(\tfrac{\pi}{2}\right)^2 + 0 - \tfrac{1}{2}\left[-\tfrac{1}{2}\cos 2x\right]_0^{\pi/2}\right] = \tfrac{\pi^3}{8} + \tfrac{\pi}{4}(-1-1) = \tfrac{1}{8}\left(\pi^3 - 4\pi\right)\end{aligned}$$

77. By the Fundamental Theorem of Calculus,

$$\int_0^\infty f'(x)\,dx = \lim_{t\to\infty}\int_0^t f'(x)\,dx = \lim_{t\to\infty}[f(t) - f(0)] = \lim_{t\to\infty} f(t) - f(0) = 0 - f(0) = -f(0).$$

78. (a) $\left(\tan^{-1} x\right)_{\text{ave}} = \lim\limits_{t\to\infty} \dfrac{1}{t-0}\displaystyle\int_0^t \tan^{-1} x\,dx \overset{89}{=} \lim\limits_{t\to\infty}\left\{\dfrac{1}{t}\left[x\tan^{-1} x - \tfrac{1}{2}\ln\left(1+x^2\right)\right]_0^t\right\}$

$$\begin{aligned}&= \lim_{t\to\infty}\left[\frac{1}{t}\left(t\tan^{-1} t - \tfrac{1}{2}\ln\left(1+t^2\right)\right)\right] = \lim_{t\to\infty}\left[\tan^{-1} t - \frac{\ln\left(1+t^2\right)}{2t}\right] \\ &\overset{\text{H}}{=} \frac{\pi}{2} - \lim_{t\to\infty}\frac{2t/\left(1+t^2\right)}{2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

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(b) $f(x) \geq 0$ and $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow$ $\lim_{t\to\infty} \int_a^t f(x)\,dx = \infty$.

$$f_{\text{ave}} = \lim_{t\to\infty} \frac{\int_a^t f(x)\,dx}{t-a}\,dx \overset{\text{H}}{=} \lim_{t\to\infty} \frac{f(t)}{1} \quad \text{[by FTC1]} \quad = \lim_{x\to\infty} f(x), \text{ if this limit exists.}$$

(c) Suppose $\int_a^\infty f(x)\,dx$ converges; that is, $\lim_{t\to\infty} \int_a^t f(x)\,dx = L < \infty$. Then

$$f_{\text{ave}} = \lim_{t\to\infty} \left[\frac{1}{t-a}\int_a^t f(x)\,dx\right] = \lim_{t\to\infty} \frac{1}{t-a} \cdot \lim_{t\to\infty} \int_a^t f(x)\,dx = 0 \cdot L = 0$$

(d) $(\sin x)_{\text{ave}} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \sin x\,dx = \lim_{t\to\infty} \left(\frac{1}{t}\left[-\cos x\right]_0^t\right) = \lim_{t\to\infty} \left(-\frac{\cos t}{t} + \frac{1}{t}\right) = \lim_{t\to\infty} \frac{1-\cos t}{t} = 0$

9. Let $u = 1/x \quad \Rightarrow \quad x = 1/u \quad \Rightarrow \quad dx = -\left(1/u^2\right) du$.

$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = \int_\infty^0 \frac{\ln(1/u)}{1+1/u^2}\left(-\frac{du}{u^2}\right) = \int_\infty^0 \frac{-\ln u}{u^2+1}\,(-du) = \int_\infty^0 \frac{\ln u}{1+u^2}\,du = -\int_0^\infty \frac{\ln u}{1+u^2}\,du$$

Therefore, $\int_0^\infty \frac{\ln x}{1+x^2}\,dx = -\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$.

10. If the distance between P and the point charge is d, then the potential V at P is

$$V = W = \int_\infty^d F\,dr = \int_\infty^d \frac{q}{4\pi\varepsilon_0 r^2}\,dr = \lim_{t\to\infty} \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_t^d = \frac{q}{4\pi\varepsilon_0} \lim_{t\to\infty} \left(-\frac{1}{d} + \frac{1}{t}\right) = -\frac{q}{4\pi\varepsilon_0 d}$$

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(d) $P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds \quad \Rightarrow \quad P(4a_0) = \int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds$. Using $(*)$ from part (a) (with $b = -2/a_0$),

$$P(4a_0) = \frac{4}{a_0^3}\left[\frac{e^{-2s/a_0}}{-8/a_0^3}\left(\frac{4}{a_0^2}s^2 + \frac{4}{a_0}s + 2\right)\right]_0^{4a_0} = \frac{4}{a_0^3}\left(\frac{a_0^3}{-8}\right)\left[e^{-8}(64 + 16 + 2) - 1(2)\right]$$

$$= -\tfrac{1}{2}\left(82e^{-8} - 2\right) = 1 - 41e^{-8} \approx 0.986$$

(e) $\mu = \int_{-\infty}^{\infty} rp(r)\, dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^3 e^{-2r/a_0}\, dr$. Integrating by parts three times or using a CAS, we find that $\int x^3 e^{bx}\, dx = \frac{e^{bx}}{b^4}\left(b^3x^3 - 3b^2x^2 + 6bx - 6\right)$. So with $b = -\frac{2}{a_0}$, we use l'Hospital's Rule, and get

$$\mu = \frac{4}{a_0^3}\left[-\frac{a_0^4}{16}(-6)\right] = \tfrac{3}{2}a_0.$$

9 Review

CONCEPT CHECK

1. (a) The length of a curve is defined to be the limit of the lengths of the inscribed polygons, as described near Figure 3 in Section 9.1.

(b) See Equation 9.1.2.

(c) See Equation 9.1.4.

2. (a) $S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$

(b) If $x = g(y)$, $c \le y \le d$, then $S = \int_c^d 2\pi y\sqrt{1 + [g'(y)]^2}\, dy$.

(c) $S = \int_a^b 2\pi x\sqrt{1 + [f'(x)]^2}\, dx$ or $S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy$

3. Let $c(x)$ be the cross-sectional length of the wall (measured parallel to the surface of the fluid) at depth x. Then the hydrostatic force against the wall is given by $F = \int_a^b \delta x c(x)\, dx$, where a and b are the lower and upper limits for x at points of the wall and δ is the weight density of the fluid.

4. (a) The center of mass is the point at which the plate balances horizontally.

(b) See Equations 9.3.8.

5. If a plane region $\mathcal{R}$ that lies entirely on one side of a line ℓ in its plane is rotated about ℓ, then the volume of the resulting solid is the product of the area of $\mathcal{R}$ and the distance traveled by the centroid of $\mathcal{R}$.

6. See Figure 3 in Section 9.4, and the discussion which precedes it.

7. (a) See the definition in the first paragraph of the subsection *Cardiac Output* in Section 9.4.

(b) See the discussion in the second paragraph of the subsection *Cardiac Output* in Section 9.4.

8. A probability density function f is a function on the domain of a continuous random variable X such that $\int_a^b f(x)\, dx$ measures the probability that X lies between a and b. Such a function f has nonnegative values and satisfies the relation $\int_D f(x)\, dx = 1$, where D is the domain of the corresponding random variable X. If $D = \mathbb{R}$, or if we define $f(x) = 0$ for real numbers $x \notin D$, then $\int_{-\infty}^{\infty} f(x)\, dx = 1$. (Of course, to work with f in this way, we must assume that the integrals of f exist.)

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9. (a) $\int_0^{100} f(x)\,dx$ represents the probability that the weight of a randomly chosen female college student is less than 100 pounds.

(b) $\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{\infty} xf(x)\,dx$

(c) The median of f is the number m such that $\int_m^{\infty} f(x)\,dx = \frac{1}{2}$.

10. See the discussion near Equation 3 in Section 9.5.

EXERCISES

1. $y = \frac{1}{6}(x^2+4)^{3/2} \Rightarrow dy/dx = \frac{1}{4}(x^2+4)^{1/2}(2x) \Rightarrow$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left[\frac{1}{2}x(x^2+4)^{1/2}\right]^2 = 1 + \frac{1}{4}x^2(x^2+4) = \frac{1}{4}x^4 + x^2 + 1 = \left(\frac{1}{2}x^2+1\right)^2.$

Thus, $L = \int_0^3 \sqrt{\left(\frac{1}{2}x^2+1\right)^2}\,dx = \int_0^3 \left(\frac{1}{2}x^2+1\right)dx = \left[\frac{1}{6}x^3 + x\right]_0^3 = \frac{15}{2}$.

2. $y = 2\ln(\sin\frac{1}{2}x) \Rightarrow \frac{dy}{dx} = 2\cdot\frac{1}{\sin(\frac{1}{2}x)}\cdot\cos(\frac{1}{2}x)\cdot\frac{1}{2} = \cot(\frac{1}{2}x) \Rightarrow$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cot^2(\frac{1}{2}x) = \csc^2(\frac{1}{2}x)$. Thus,

$$L = \int_{\pi/3}^{\pi} \sqrt{\csc^2(\tfrac{1}{2}x)}\,dx = \int_{\pi/3}^{\pi} \left|\csc(\tfrac{1}{2}x)\right|dx = \int_{\pi/3}^{\pi} \csc(\tfrac{1}{2}x)\,dx = \int_{\pi/6}^{\pi/2} \csc u\,(2\,du) \quad \begin{bmatrix} u = \frac{1}{2}x, \\ du = \frac{1}{2}\,dx \end{bmatrix}$$

$$= 2\left[\ln|\csc u - \cot u|\right]_{\pi/6}^{\pi/2} = 2\left[\ln\left|\csc\tfrac{\pi}{2} - \cot\tfrac{\pi}{2}\right| - \ln\left|\csc\tfrac{\pi}{6} - \cot\tfrac{\pi}{6}\right|\right]$$

$$= 2\left[\ln|1-0| - \ln\left|2-\sqrt{3}\,\right|\right] = -2\ln\left(2-\sqrt{3}\right) \approx 2.63$$

3. (a) $y = \frac{x^4}{16} + \frac{1}{2x^2} = \frac{1}{16}x^4 + \frac{1}{2}x^{-2} \Rightarrow \frac{dy}{dx} = \frac{1}{4}x^3 - x^{-3} \Rightarrow$

$1 + (dy/dx)^2 = 1 + \left(\frac{1}{4}x^3 - x^{-3}\right)^2 = 1 + \frac{1}{16}x^6 - \frac{1}{2} + x^{-6} = \frac{1}{16}x^6 + \frac{1}{2} + x^{-6} = \left(\frac{1}{4}x^3 + x^{-3}\right)^2.$

Thus, $L = \int_1^2 \left(\frac{1}{4}x^3 + x^{-3}\right)dx = \left[\frac{1}{16}x^4 - \frac{1}{2}x^{-2}\right]_1^2 = \left(1 - \frac{1}{8}\right) - \left(\frac{1}{16} - \frac{1}{2}\right) = \frac{21}{16}$.

(b) $S = \int_1^2 2\pi x\left(\frac{1}{4}x^3 + x^{-3}\right)dx = 2\pi\int_1^2 \left(\frac{1}{4}x^4 + x^{-2}\right)dx = 2\pi\left[\frac{1}{20}x^5 - \frac{1}{x}\right]_1^2$

$= 2\pi\left[\left(\frac{32}{20} - \frac{1}{2}\right) - \left(\frac{1}{20} - 1\right)\right] = 2\pi\left(\frac{8}{5} - \frac{1}{2} - \frac{1}{20} + 1\right) = 2\pi\left(\frac{41}{20}\right) = \frac{41}{10}\pi$

4. (a) $y = x^2 \Rightarrow 1 + (y')^2 = 1 + 4x^2 \Rightarrow$

$$S = \int_0^1 2\pi x\sqrt{1+4x^2}\,dx = \int_1^5 \tfrac{\pi}{4}\sqrt{u}\,du \;\; [u = 1+4x^2] \;\; = \tfrac{\pi}{6}\left[u^{3/2}\right]_1^5 = \tfrac{\pi}{6}\left(5^{3/2} - 1\right)$$

(b) $y = x^2 \Rightarrow 1 + (y')^2 = 1 + 4x^2$. So

$$S = 2\pi\int_0^1 x^2\sqrt{1+4x^2}\,dx = 2\pi\int_0^2 \tfrac{1}{4}u^2\sqrt{1+u^2}\,\tfrac{1}{2}\,du \quad [u = 2x] \quad = \tfrac{\pi}{4}\int_0^2 u^2\sqrt{1+u^2}\,du$$

$$= \tfrac{\pi}{4}\left[\tfrac{1}{8}u(1+2u^2)\sqrt{1+u^2} - \tfrac{1}{8}\ln\left|u + \sqrt{1+u^2}\,\right|\right]_0^2 \qquad [u = \tan\theta \text{ or use Formula 22}]$$

$$= \tfrac{\pi}{4}\left[\tfrac{1}{4}(9)\sqrt{5} - \tfrac{1}{8}\ln\left(2+\sqrt{5}\right) - 0\right] = \tfrac{\pi}{32}\left[18\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right]$$

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5. $y = e^{-x^2} \quad \Rightarrow \quad dy/dx = -2xe^{-x^2} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + 4x^2e^{-2x^2}$.

Let $f(x) = \sqrt{1 + 4x^2e^{-2x^2}}$. Then

$$L = \int_0^3 f(x)\,dx \approx S_6 = \frac{(3-0)/6}{3}[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)]$$
$$\approx 3.292287$$

6. $S = \int_0^3 2\pi y\,ds = \int_0^3 2\pi e^{-x^2}\sqrt{1 + 4x^2e^{-2x^2}}\,dx$. Let $g(x) = 2\pi e^{-x^2}\sqrt{1 + 4x^2e^{-2x^2}}$. Then

$$S = \int_0^3 g(x)\,dx \approx S_6 = \frac{(3-0)/6}{3}[g(0) + 4g(0.5) + 2g(1) + 4g(1.5) + 2g(2) + 4g(2.5) + g(3)]$$
$$\approx 6.648327$$

7. $y = \int_1^x \sqrt{\sqrt{t} - 1}\,dt \quad \Rightarrow \quad dy/dx = \sqrt{\sqrt{x} - 1} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + (\sqrt{x} - 1) = \sqrt{x}$.

Thus, $L = \int_1^{16} \sqrt{\sqrt{x}}\,dx = \int_1^{16} x^{1/4}\,dx = \frac{4}{5}\left[x^{5/4}\right]_1^{16} = \frac{4}{5}(32 - 1) = \frac{124}{5}$.

8. $S = \displaystyle\int_1^{16} 2\pi x\,ds = 2\pi\int_1^{16} x \cdot x^{1/4}\,dx = 2\pi\int_1^{16} x^{5/4}\,dx = 2\pi \cdot \frac{4}{9}\left[x^{9/4}\right]_1^{16} = \frac{8\pi}{9}(512 - 1) = \frac{4088\pi}{9}$

9. As in Example 1 of Section 9.3, $\dfrac{a}{2-x} = \dfrac{1}{2} \quad \Rightarrow \quad 2a = 2 - x$ and

$w = 2(1.5 + a) = 3 + 2a = 3 + 2 - x = 5 - x$. Thus,

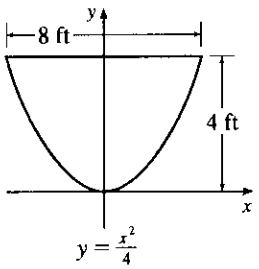
$F = \int_0^2 \rho g x(5 - x)\,dx = \rho g\left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^2 = \rho g\left(10 - \frac{8}{3}\right) = \frac{22}{3}\delta \ \ [\rho g = \delta] \ \approx \frac{22}{3} \cdot 62.5 \approx 458$ lb.

10. $F = \int_0^4 \delta(4 - y)2\left(2\sqrt{y}\right)dy = 4\delta\int_0^4 \left(4y^{1/2} - y^{3/2}\right)dy$

$= 4\delta\left[\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^4 = 4\delta\left(\frac{64}{3} - \frac{64}{5}\right)$

$= 256\delta\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{512}{15}\delta$

≈ 2133.3 lb $\quad [\delta \approx 62.5$ lb/ft$^3]$



11. $A = \int_{-2}^1 \left[(4 - x^2) - (x + 2)\right]dx = \int_{-2}^1 (2 - x - x^2)\,dx = \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^1$

$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = \frac{9}{2} \quad \Rightarrow$

$$\overline{x} = A^{-1}\int_{-2}^1 x(2 - x - x^2)dx = \frac{2}{9}\int_{-2}^1 (2x - x^2 - x^3)\,dx = \frac{2}{9}\left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{-2}^1$$
$$= \frac{2}{9}\left[\left(1 - \frac{1}{3} - \frac{1}{4}\right) - \left(4 + \frac{8}{3} - 4\right)\right] = -\frac{1}{2}$$

and $$\overline{y} = A^{-1}\int_{-2}^1 \frac{1}{2}\left[(4 - x^2)^2 - (x + 2)^2\right]dx = \frac{1}{9}\int_{-2}^1 (x^4 - 9x^2 - 4x + 12)\,dx$$
$$= \frac{1}{9}\left[\frac{1}{5}x^5 - 3x^3 - 2x^2 + 12x\right]_{-2}^1 = \frac{1}{9}\left[\left(\frac{1}{5} - 3 - 2 + 12\right) - \left(-\frac{32}{5} + 24 - 8 - 24\right)\right] = \frac{12}{5}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(-\frac{1}{2}, \frac{12}{5}\right)$.

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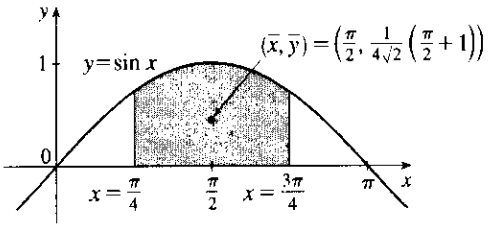
12. From the symmetry of the region, $\overline{x} = \frac{\pi}{2}$. $A = \int_{\pi/4}^{3\pi/4} \sin x\,dx = \left[-\cos x\right]_{\pi/4}^{3\pi/4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

$$\begin{aligned}\overline{y} &= \frac{1}{A}\int_{\pi/4}^{3\pi/4} \frac{1}{2}\sin^2 x\,dx = \frac{1}{A}\int_{\pi/4}^{3\pi/4} \frac{1}{4}(1-\cos 2x)\,dx \\ &= \frac{1}{4\sqrt{2}}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/4}^{3\pi/4} \\ &= \frac{1}{4\sqrt{2}}\left[\frac{3\pi}{4} - \frac{1}{2}(-1) - \frac{\pi}{4} + \frac{1}{2}\cdot 1\right] \\ &= \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2}+1\right)\end{aligned}$$



Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{\pi}{2}, \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2}+1\right)\right) \approx (1.57, 0.45)$.

13. An equation of the line passing through $(0,0)$ and $(3,2)$ is $y = \frac{2}{3}x$. $A = \frac{1}{2}\cdot 3\cdot 2 = 3$. Therefore, using Equations 9.3.8, $\overline{x} = \frac{1}{3}\int_0^3 x\left(\frac{2}{3}x\right)dx = \frac{2}{27}\left[x^3\right]_0^3 = 2$, and $\overline{y} = \frac{1}{3}\int_0^3 \frac{1}{2}\left(\frac{2}{3}x\right)^2 dx = \frac{2}{81}\left[x^3\right]_0^3 = \frac{2}{3}$.

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(2, \frac{2}{3}\right)$.

14. Suppose first that the large rectangle were complete, so that its mass would be $6\cdot 3 = 18$. Its centroid would be $\left(1, \frac{3}{2}\right)$. The mass removed from this object to create the one being studied is 3. The centroid of the cut-out piece is $\left(\frac{3}{2}, \frac{3}{2}\right)$. Therefore, for the actual lamina, whose mass is 15, $\overline{x} = \frac{18}{15}(1) - \frac{3}{15}\left(\frac{3}{2}\right) = \frac{9}{10}$, and $\overline{y} = \frac{3}{2}$, since the lamina is symmetric about the line $y = \frac{3}{2}$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{9}{10}, \frac{3}{2}\right)$.

15. The centroid of this circle, $(1,0)$, travels a distance $2\pi(1)$ when the lamina is rotated about the y-axis. The area of the circle is $\pi(1)^2$. So by the Theorem of Pappus, $V = A(2\pi\overline{x}) = \pi(1)^2 2\pi(1) = 2\pi^2$.

16. The semicircular region has an area of $\frac{1}{2}\pi r^2$, and sweeps out a sphere of radius r when rotated about the x-axis. $\overline{x} = 0$ because of symmetry about the line $x = 0$. And by the Theorem of Pappus, $V = A(2\pi\overline{y}) \quad \Rightarrow$ $\frac{4}{3}\pi r^3 = \frac{1}{2}\pi r^2(2\pi\overline{y}) \quad \Rightarrow \quad \overline{y} = \frac{4}{3\pi}r$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{4}{3\pi}r\right)$.

17. $x = 100 \quad \Rightarrow \quad P = 2000 - 0.1(100) - 0.01(100)^2 = 1890$

$$\begin{aligned}\text{Consumer surplus} &= \int_0^{100}[p(x) - P]\,dx = \int_0^{100}\left(2000 - 0.1x - 0.01x^2 - 1890\right)dx \\ &= \left[110x - 0.05x^2 - \tfrac{0.01}{3}x^3\right]_0^{100} = 11{,}000 - 500 - \tfrac{10{,}000}{3} \approx \$7166.67\end{aligned}$$

18. $\int_0^{24} c(t)\,dt \approx S_{12} = \frac{24-0}{12\cdot 3}[1(0) + 4(1.9) + 2(3.3) + 4(5.1) + 2(7.6) + 4(7.1) + 2(5.8)$
$+ 4(4.7) + 2(3.3) + 4(2.1) + 2(1.1) + 4(0.5) + 1(0)]$
$= \frac{2}{3}(127.8) = 85.2 \text{ mg}\cdot\text{s/L}$

Therefore, $F \approx A/85.2 = 6/85.2 \approx 0.0704$ L/s or 4.225 L/min.

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19. $f(x) = \begin{cases} \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) & \text{if } 0 \le x \le 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$

(a) $f(x) \ge 0$ for all real numbers x and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} \tfrac{\pi}{20}\sin\left(\tfrac{\pi}{10}x\right)dx = \tfrac{\pi}{20}\cdot\tfrac{10}{\pi}\left[-\cos\left(\tfrac{\pi}{10}x\right)\right]_0^{10}$$

$$= \tfrac{1}{2}(-\cos\pi + \cos 0) = \tfrac{1}{2}(1+1) = 1$$

Therefore, f is a probability density function.

(b) $P(X < 4) = \int_{-\infty}^{4} f(x)\,dx = \int_0^4 \tfrac{\pi}{20}\sin\left(\tfrac{\pi}{10}x\right)dx = \tfrac{1}{2}\left[-\cos\left(\tfrac{\pi}{10}x\right)\right]_0^4 = \tfrac{1}{2}\left(-\cos\tfrac{2\pi}{5} + \cos 0\right)$

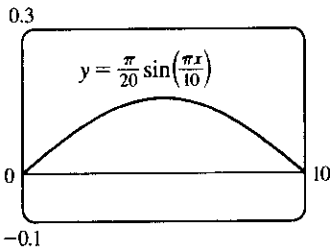
$$\approx \tfrac{1}{2}(-0.309017 + 1) \approx 0.3455$$

(c) $\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{10} \tfrac{\pi}{20}x\sin\left(\tfrac{\pi}{10}x\right)dx$

$$= \int_0^{\pi} \tfrac{\pi}{20}\cdot\tfrac{10}{\pi}u(\sin u)\left(\tfrac{10}{\pi}\right)du \quad [u = \tfrac{\pi}{10}x,\ du = \tfrac{\pi}{10}\,dx]$$

$$= \tfrac{5}{\pi}\int_0^{\pi} u\sin u\,du \overset{82}{=} \tfrac{5}{\pi}\left[\sin u - u\cos u\right]_0^{\pi} = \tfrac{5}{\pi}[0 - \pi(-1)] = 5$$

This answer is expected because the graph of f is symmetric about the line $x = 5$.



20. $P(250 \le X \le 280) = \displaystyle\int_{250}^{280} \frac{1}{15\sqrt{2\pi}}\exp\left(\frac{-(x-268)^2}{2\cdot 15^2}\right)dx \approx 0.673$. Thus, the percentage of pregnancies that last between 250 and 280 days is about 67.3%.

21. (a) The probability density function is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \ge 0 \end{cases}$

$$P(0 \le X \le 3) = \int_0^3 \tfrac{1}{8}e^{-t/8}\,dt = \left[-e^{-t/8}\right]_0^3 = -e^{-3/8} + 1 \approx 0.3127$$

(b) $P(X > 10) = \int_{10}^{\infty} \tfrac{1}{8}e^{-t/8}\,dt = \lim_{x\to\infty}\left[-e^{-t/8}\right]_{10}^{x} = \lim_{x\to\infty}\left(-e^{-x/8} + e^{-10/8}\right) = 0 + e^{-5/4} \approx 0.2865$

(c) We need to find m such that $P(X \ge m) = \frac{1}{2} \Rightarrow \int_m^{\infty} \frac{1}{8}e^{-t/8}\,dt = \frac{1}{2} \Rightarrow \lim_{x\to\infty}\left[-e^{-t/8}\right]_m^x = \frac{1}{2} \Rightarrow$

$\lim_{x\to\infty}\left(-e^{-x/8} + e^{-m/8}\right) = \frac{1}{2} \Rightarrow e^{-m/8} = \frac{1}{2} \Rightarrow -m/8 = \ln\frac{1}{2} \Rightarrow$

$m = -8\ln\frac{1}{2} = 8\ln 2 \approx 5.55$ minutes.

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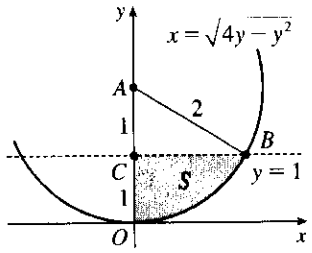
PROBLEMS PLUS

1. $x^2 + y^2 \le 4y \iff x^2 + (y-2)^2 \le 4$, so S is part of a circle, as shown in the diagram. The area of S is



$$\int_0^1 \sqrt{4y-y^2}\,dy \overset{113}{=} \left[\tfrac{y-2}{2}\sqrt{4y-y^2} + 2\cos^{-1}\left(\tfrac{2-y}{2}\right)\right]_0^1 \quad [a=2]$$
$$= -\tfrac{1}{2}\sqrt{3} + 2\cos^{-1}\left(\tfrac{1}{2}\right) - 2\cos^{-1}1$$
$$= -\tfrac{\sqrt{3}}{2} + 2\left(\tfrac{\pi}{3}\right) - 2(0) = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$$

Another method (without calculus): Note that $\theta = \angle CAB = \frac{\pi}{3}$, so the area is

$$(\text{area of sector } OAB) - (\text{area of } \triangle ABC) = \tfrac{1}{2}(2^2)\tfrac{\pi}{3} - \tfrac{1}{2}(1)\sqrt{3} = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$$

2. $y = \pm\sqrt{x^3 - x^4} \quad \Rightarrow$ The loop of the curve is symmetric about $y = 0$, and therefore $\overline{y} = 0$. At each point x where $0 \le x \le 1$, the lamina has a vertical length of $\sqrt{x^3-x^4} - \left(-\sqrt{x^3-x^4}\right) = 2\sqrt{x^3-x^4}$. Therefore, $\overline{x} = \dfrac{\int_0^1 x \cdot 2\sqrt{x^3-x^4}\,dx}{\int_0^1 2\sqrt{x^3-x^4}\,dx} = \dfrac{\int_0^1 x\sqrt{x^3-x^4}\,dx}{\int_0^1 \sqrt{x^3-x^4}\,dx}$. We evaluate the integrals separately:

$$\int_0^1 x\sqrt{x^3-x^4}\,dx = \int_0^1 x^{5/2}\sqrt{1-x}\,dx$$
$$= \int_0^{\pi/2} 2\sin^6\theta\cos\theta\sqrt{1-\sin^2\theta}\,d\theta \quad \left[\begin{matrix}\sin\theta = \sqrt{x},\ \cos\theta\,d\theta = dx/(2\sqrt{x}\,),\\ 2\sin\theta\cos\theta\,d\theta = dx\end{matrix}\right]$$
$$= \int_0^{\pi/2} 2\sin^6\theta\cos^2\theta\,d\theta = \int_0^{\pi/2} 2\left[\tfrac{1}{2}(1-\cos 2\theta)\right]^3 \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$
$$= \int_0^{\pi/2} \tfrac{1}{8}(1 - 2\cos 2\theta + 2\cos^3 2\theta - \cos^4 2\theta)\,d\theta$$
$$= \int_0^{\pi/2} \tfrac{1}{8}\left[1 - 2\cos 2\theta + 2\cos 2\theta(1-\sin^2 2\theta) - \tfrac{1}{4}(1+\cos 4\theta)^2\right] d\theta$$
$$= \tfrac{1}{8}\left[\theta - \tfrac{1}{3}\sin^3 2\theta\right]_0^{\pi/2} - \tfrac{1}{32}\int_0^{\pi/2}\left(1 + 2\cos 4\theta + \cos^2 4\theta\right) d\theta$$
$$= \tfrac{\pi}{16} - \tfrac{1}{32}\left[\theta + \tfrac{1}{2}\sin 4\theta\right]_0^{\pi/2} - \tfrac{1}{64}\int_0^{\pi/2}(1+\cos 8\theta)\,d\theta$$
$$= \tfrac{3\pi}{64} - \tfrac{1}{64}\left[\theta + \tfrac{1}{8}\sin 8\theta\right]_0^{\pi/2} = \tfrac{5\pi}{128}$$

$$\int_0^1 \sqrt{x^3-x^4}\,dx = \int_0^1 x^{3/2}\sqrt{1-x}\,dx = \int_0^{\pi/2} 2\sin^4\theta\cos\theta\sqrt{1-\sin^2\theta}\,d\theta \qquad [\,\sin\theta = \sqrt{x}\,]$$
$$= \int_0^{\pi/2} 2\sin^4\theta\cos^2\theta\,d\theta = \int_0^{\pi/2} 2\cdot\tfrac{1}{4}(1-\cos 2\theta)^2\cdot\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$
$$= \int_0^{\pi/2} \tfrac{1}{4}(1 - \cos 2\theta - \cos^2 2\theta + \cos^3 2\theta)\,d\theta$$
$$= \int_0^{\pi/2} \tfrac{1}{4}\left[1 - \cos 2\theta - \tfrac{1}{2}(1+\cos 4\theta) + \cos 2\theta(1-\sin^2 2\theta)\right] d\theta$$
$$= \tfrac{1}{4}\left[\tfrac{\theta}{2} - \tfrac{1}{8}\sin 4\theta - \tfrac{1}{6}\sin^3 2\theta\right]_0^{\pi/2} = \tfrac{\pi}{16}$$

Therefore, $\overline{x} = \dfrac{5\pi/128}{\pi/16} = \dfrac{5}{8}$, and $(\overline{x}, \overline{y}) = \left(\tfrac{5}{8}, 0\right)$.

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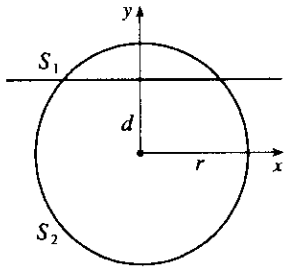
3. (a) The two spherical zones, whose surface areas we will call S_1 and S_2, are generated by rotation about the y-axis of circular arcs, as indicated in the figure. The arcs are the upper and lower portions of the circle $x^2 + y^2 = r^2$ that are obtained when the circle is cut with the line $y = d$. The portion of the upper arc in the first quadrant is sufficient to generate the upper spherical zone. That portion of the arc can be described by the relation $x = \sqrt{r^2 - y^2}$ for $d \le y \le r$. Thus, $dx/dy = -y/\sqrt{r^2 - y^2}$ and



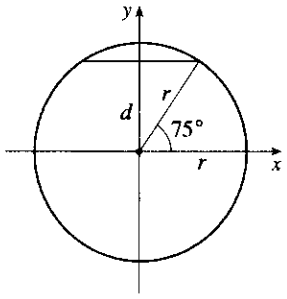
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{1 + \frac{y^2}{r^2 - y^2}}\,dy = \sqrt{\frac{r^2}{r^2 - y^2}}\,dy = \frac{r\,dy}{\sqrt{r^2 - y^2}}$$

From Formula 9.2.8 we have

$$S_1 = \int_d^r 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_d^r 2\pi \sqrt{r^2 - y^2}\,\frac{r\,dy}{\sqrt{r^2 - y^2}} = \int_d^r 2\pi r\,dy = 2\pi r(r - d)$$

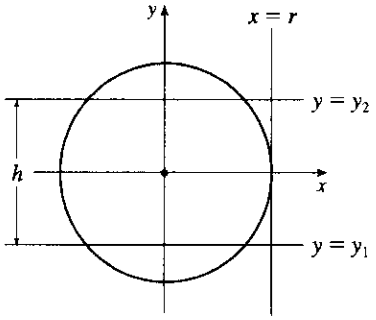
Similarly, we can compute $S_2 = \int_{-r}^{d} 2\pi x \sqrt{1 + (dx/dy)^2}\,dy = \int_{-r}^{d} 2\pi r\,dy = 2\pi r(r + d)$. Note that $S_1 + S_2 = 4\pi r^2$, the surface area of the entire sphere.

(b) $r = 3960$ mi and $d = r\,(\sin 75^\circ) \approx 3825$ mi, so the surface area of the Arctic Ocean is about $2\pi r(r - d) \approx 2\pi(3960)(135) \approx 3.36 \times 10^6$ mi^2.



(c) The area on the sphere lies between planes $y = y_1$ and $y = y_2$, where $y_2 - y_1 = h$. Thus, we compute the surface area on the sphere to be $S = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_{y_1}^{y_2} 2\pi r\,dy = 2\pi r(y_2 - y_1) = 2\pi r h$.

This equals the lateral area of a cylinder of radius r and height h, since such a cylinder is obtained by rotating the line $x = r$ about the y-axis, so the surface area of the cylinder between the planes $y = y_1$ and $y = y_2$ is



$$A = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_{y_1}^{y_2} 2\pi r \sqrt{1 + 0^2}\,dy$$

$$= 2\pi r y\Big|_{y=y_1}^{y_2} = 2\pi r(y_2 - y_1) = 2\pi r h$$

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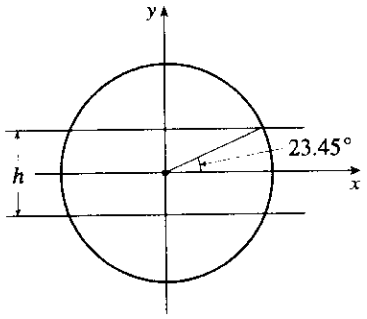
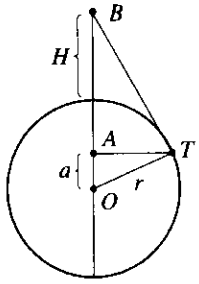
(d) $h = 2r \sin 23.45° \approx 3152$ mi, so the surface area of the Torrid Zone is $2\pi rh \approx 2\pi(3960)(3152) \approx 7.84 \times 10^7$ mi^2.



4. (a) Since the right triangles OAT and OTB are similar, we have $\frac{r+H}{r} = \frac{r}{a} \quad \Rightarrow \quad a = \frac{r^2}{r+H}$. The surface area visible from B is $S = \int_a^r 2\pi x \sqrt{1 + (dx/dy)^2}\, dy$. From $x^2 + y^2 = r^2$, we get

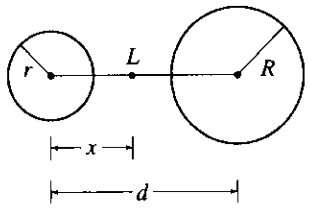


$$\frac{d}{dy}(x^2 + y^2) = \frac{d}{dy}(r^2) \quad \Rightarrow \quad 2x\frac{dx}{dy} + 2y = 0 \quad \Rightarrow$$

$\frac{dx}{dy} = -\frac{y}{x}$ and $1 + \left(\frac{dx}{dy}\right)^2 = \frac{x^2 + y^2}{x^2} = \frac{r^2}{x^2}$. Thus,

$$S = \int_a^r 2\pi x \cdot \frac{r}{x}\, dy = 2\pi r(r - a) = 2\pi r\left(r - \frac{r^2}{r+H}\right) = 2\pi r^2\left(1 - \frac{r}{r+H}\right) = 2\pi r^2 \cdot \frac{H}{r+H} = \frac{2\pi r^2 H}{r+H}.$$

(b) Assume $R \geq r$. If a light is placed at point L, at a distance x from the center of the sphere of radius r, then from part (a) we find that the total illuminated area A on the two spheres is [with $r + H = x$ and $r + H = d - x$].



$A(x) = \frac{2\pi r^2(x-r)}{x} + \frac{2\pi R^2(d-x-R)}{d-x}$ $[r \leq x \leq d - R]$. $\frac{A(x)}{2\pi} = r^2\left(1 - \frac{r}{x}\right) + R^2\left(1 - \frac{R}{d-x}\right)$, so

$$A'(x) = 0 \quad \Leftrightarrow \quad 0 = r^2 \cdot \frac{r}{x^2} + R^2 \cdot \frac{-R}{(d-x)^2} \quad \Leftrightarrow \quad \frac{r^3}{x^2} = \frac{R^3}{(d-x)^2} \quad \Leftrightarrow \quad \frac{(d-x)^2}{x^2} = \frac{R^3}{r^3} \quad \Leftrightarrow$$

$$\left(\frac{d}{x} - 1\right)^2 = \left(\frac{R}{r}\right)^3 \quad \Rightarrow \quad \frac{d}{x} - 1 = \left(\frac{R}{r}\right)^{3/2} \quad \Leftrightarrow \quad \frac{d}{x} = 1 + \left(\frac{R}{r}\right)^{3/2} \quad \Leftrightarrow \quad x = x^* = \frac{d}{1 + (R/r)^{3/2}}.$$

Now $A'(x) = 2\pi\left(\frac{r^3}{x^2} - \frac{R^3}{(d-x)^2}\right) \quad \Rightarrow \quad A''(x) = 2\pi\left(-\frac{2r^3}{x^3} - \frac{2R^3}{(d-x)^3}\right)$ and $A''(x^*) < 0$, so we have a local maximum at $x = x^*$.

However, x^* may not be an allowable value of x—we must show that x^* is between r and $d - R$.

(1) $x^* \geq r \quad \Leftrightarrow \quad \frac{d}{1 + (R/r)^{3/2}} \geq r \quad \Leftrightarrow \quad d \geq r + R\sqrt{R/r}$

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(2) $x^* \le d - R \quad \Leftrightarrow \quad \dfrac{d}{1+(R/r)^{3/2}} \le d - R \quad \Leftrightarrow \quad d \le d - R + d\left(\dfrac{R}{r}\right)^{3/2} - R\left(\dfrac{R}{r}\right)^{3/2} \quad \Leftrightarrow$

$R + R\left(\dfrac{R}{r}\right)^{3/2} \le d\left(\dfrac{R}{r}\right)^{3/2} \quad \Leftrightarrow \quad d \ge \dfrac{R}{(R/r)^{3/2}} + R = R + r\sqrt{r/R}$, but

$R + r\sqrt{r/R} \le R + r$, and since $d > r + R$ [given], we conclude that $x^* \le d - R$.

Thus, from (1) and (2), x^* is not an allowable value of x if $d < r + R\sqrt{R/r}$.

So A may have a maximum at $x = r$, x^*, or $d - R$.

$$A(r) = \frac{2\pi R^2(d - r - R)}{d - r} \qquad \text{and} \qquad A(d - R) = \frac{2\pi r^2(d - r - R)}{d - R}$$

$A(r) > A(d-R) \quad \Leftrightarrow \quad \dfrac{R^2}{d-r} > \dfrac{r^2}{d-R} \quad \Leftrightarrow \quad R^2(d-R) > r^2(d-r) \quad \Leftrightarrow \quad R^2d - R^3 > r^2d - r^3 \quad \Leftrightarrow$

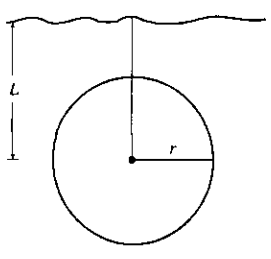
$R^2d - r^2d > R^3 - r^3 \quad \Leftrightarrow \quad d(R-r)(R+r) > (R-r)(R^2 + Rr + r^2) \quad \Leftrightarrow$

$d > (R^2 + Rr + r^2)/(R+r) \quad \Leftrightarrow \quad d > [(R+r)^2 - Rr]/(R+r) \quad \Leftrightarrow \quad d > R + r - Rr/(R+r)$. Now $R + r - Rr/(R+r) < R + r$, and we know that $d > R + r$, so we conclude that $A(r) > A(d-R)$.

In conclusion, A has an absolute maximum at $x = x^*$ provided $d \ge r + R\sqrt{R/r}$; otherwise, A has its maximum at $x = r$.

5. (a) Choose a vertical x-axis pointing downward with its origin at the surface. In order to calculate the pressure at depth z, consider n subintervals of the interval $[0, z]$ by points x_i and choose a point $x_i^* \in [x_{i-1}, x_i]$ for each i. The thin layer of water lying between depth x_{i-1} and depth x_i has a density of approximately $\rho(x_i^*)$, so the weight of a piece of that layer with unit cross-sectional area is $\rho(x_i^*)g\,\Delta x$. The total weight of a column of water extending from the surface to depth z (with unit cross-sectional area) would be approximately $\sum_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. The estimate becomes exact if we take the limit as $n \to \infty$; weight (or force) per unit area at depth z is $W = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. In other words, $P(z) = \int_0^z \rho(x)g\,dx$. More generally, if we make no assumptions about the location of the origin, then $P(z) = P_0 + \int_0^z \rho(x)g\,dx$, where P_0 is the pressure at $x = 0$. Differentiating, we get $dP/dz = \rho(z)g$.

(b)



$$\begin{aligned} F &= \int_{-r}^{r} P(L+x)\cdot 2\sqrt{r^2-x^2}\,dx \\ &= \int_{-r}^{r}\left(P_0 + \int_0^{L+x} \rho_0 e^{z/H} g\,dz\right)\cdot 2\sqrt{r^2-x^2}\,dx \\ &= P_0\int_{-r}^{r} 2\sqrt{r^2-x^2}\,dx + \rho_0 gH\int_{-r}^{r}\left(e^{(L+x)/H} - 1\right)\cdot 2\sqrt{r^2-x^2}\,dx \\ &= (P_0 - \rho_0 gH)\int_{-r}^{r} 2\sqrt{r^2-x^2}\,dx + \rho_0 gH\int_{-r}^{r} e^{(L+x)/H}\cdot 2\sqrt{r^2-x^2}\,dx \\ &= (P_0 - \rho_0 gH)(\pi r^2) + \rho_0 gHe^{L/H}\int_{-r}^{r} e^{x/H}\cdot 2\sqrt{r^2-x^2}\,dx \end{aligned}$$

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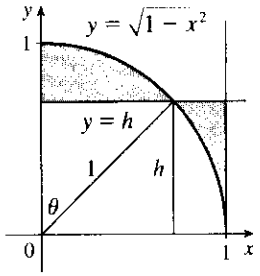
6. The problem can be reduced to finding the line which minimizes the shaded area in the diagram. The equation of the circle in the first quadrant is $y = \sqrt{1 - x^2}$, so if the equation of the line is $y = h$, then the circle and the line intersect where $h = \sqrt{1 - x^2} \quad \Rightarrow \quad x = \sqrt{1 - h^2}$. So the shaded area is



$$
\begin{aligned}
A &= \int_0^{\sqrt{1-h^2}} \left(\sqrt{1-x^2} - h\right) dx + \int_{\sqrt{1-h^2}}^{1} \left(h - \sqrt{1-x^2}\right) dx \\
&\overset{\star}{=} \left[-hx\right]_0^{\sqrt{1-h^2}} + \left[hx\right]_{\sqrt{1-h^2}}^{1} + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx \\
&= -h\sqrt{1-h^2} + h - h\sqrt{1-h^2} + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx \\
&= h\left(1 - 2\sqrt{1-h^2}\right) + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx
\end{aligned}
$$

Note that at $(\star)$, we reversed the limits of integration and changed the sign in the last integral. We are interested in the minimum of

$A(h) = h\left(1 - 2\sqrt{1-h^2}\right) + \int_0^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1-x^2}\, dx$, so we find dA/dh using FTC1 and the Chain Rule:

$$
\begin{aligned}
\frac{dA}{dh} &= h\left(-2\frac{-h}{\sqrt{1-h^2}}\right) + \left(1 - 2\sqrt{1-h^2}\right) + 2\left[\sqrt{1 - \left(\sqrt{1-h^2}\right)^2}\right] \frac{d}{dh}\left(\sqrt{1-h^2}\right) \\
&= \frac{1}{\sqrt{1-h^2}}\left[2h^2 + \sqrt{1-h^2} - 2\left(1-h^2\right)\right] + 2h\,\frac{-h}{\sqrt{1-h^2}} \\
&= \frac{1}{\sqrt{1-h^2}}\left[\sqrt{1-h^2} - 2\left(1-h^2\right)\right]
\end{aligned}
$$

This is 0 when $\sqrt{1-h^2} - 2\left(1-h^2\right) = 0 \quad \Leftrightarrow \quad u - 2u^2 = 0$ (where $u = \sqrt{1-h^2}$) $\quad \Leftrightarrow \quad u = 0$ or $\frac{1}{2} \quad \Leftrightarrow$ $h = 1$ or $\frac{\sqrt{3}}{2}$. By the First Derivative Test, $h = \frac{\sqrt{3}}{2}$ represents a minimum for $A(h)$, since $A'(h) = 1 - \dfrac{2}{\sqrt{1-h^2}}$ goes from negative to positive at $h = \frac{\sqrt{3}}{2}$.

Another method: Use FTC2 to evaluate all of the integrals before differentiating.

Note: Another strategy is to use the angle θ as the variable (see diagram above) and show that $A = \theta + \cos\theta - \frac{\pi}{4} - \frac{1}{2}\sin 2\theta$, which is minimized when $\theta = \frac{\pi}{6}$.

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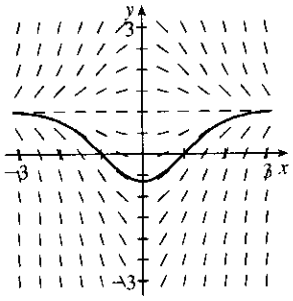
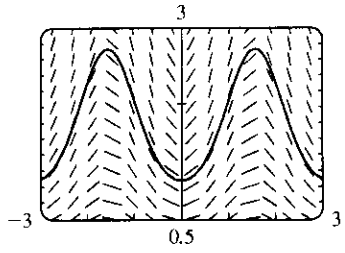
14.

x	y	$y' = x - xy$
± 2	0	± 2
± 2	3	∓ 4
± 2	-1	± 4

Note that $y' = x(1 - y) = 0$ for any point on $x = 0$ or on $y = 1$. The slopes are positive when the factors x and $1 - y$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(1, 0)$.

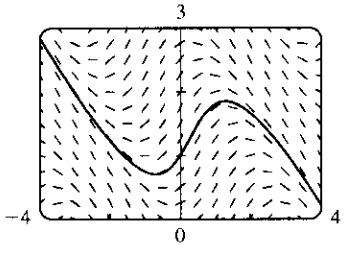


15. In Maple, we can use either `directionfield` (in Maple's share library) or `plots[fieldplot]` to plot the direction field. To plot the solution, we can either use the initial-value option in `directionfield`, or actually solve the equation. In Mathematica, we use `PlotVectorField` for the direction field, and the `Plot[Evaluate[...]]` construction to plot the solution, which is $y = e^{(1-\cos 2x)/2}$. In Derive, use `Direction_Field` (in utility file `ODE_APPR`) to plot the direction field. Then use `DSOLVE1(-y*SIN(2*x),1,x,y,0,1)` (in utility file `ODE1`) to solve the equation. Simplify each result.

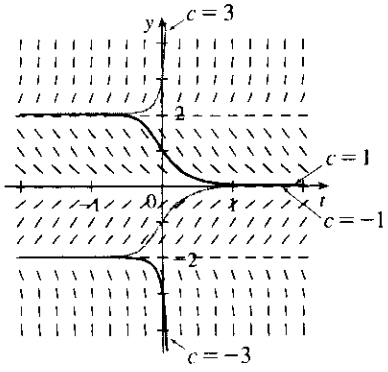


16. See Exercise 15 for specific CAS directions. The exact solution is

$$y = -x - 2\arctan\frac{2 + x - \dfrac{2}{1 + \tan(1/2)}}{x - \dfrac{2}{1 + \tan(1/2)}}$$



17.



$L = \lim_{t \to \infty} y(t)$ exists for $-2 \le c \le 2$; $L = \pm 2$ for $c = \pm 2$ and $L = 0$ for $-2 < c < 2$. For other values of c, L does not exist.

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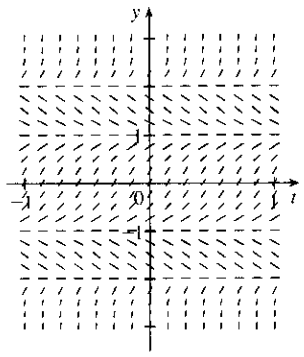
18.



Note that when $f(y) = 0$ on the graph in the text, we have $y' = f(y) = 0$; so we get horizontal segments at $y = \pm 1, \pm 2$. We get segments with negative slopes only for $1 < |y| < 2$. All other segments have positive slope. For the limiting behavior of solutions:

- If $y(0) > 2$, then $\lim_{t\to\infty} y = \infty$ and $\lim_{t\to-\infty} y = 2$.
- If $1 < y(0) < 2$, then $\lim_{t\to\infty} y = 1$ and $\lim_{t\to-\infty} y = 2$.
- If $-1 < y(0) < 1$, then $\lim_{t\to\infty} y = 1$ and $\lim_{t\to-\infty} y = -1$.
- If $-2 < y(0) < -1$, then $\lim_{t\to\infty} y = -2$ and $\lim_{t\to-\infty} y = -1$.
- If $y < -2$, then $\lim_{t\to\infty} y = -2$ and $\lim_{t\to-\infty} y = -\infty$.

19. (a) $y' = F(x, y) = y$ and $y(0) = 1 \quad \Rightarrow \quad x_0 = 0, y_0 = 1$.

(i) $h = 0.4$ and $y_1 = y_0 + hF(x_0, y_0) \quad \Rightarrow \quad y_1 = 1 + 0.4 \cdot 1 = 1.4$. $x_1 = x_0 + h = 0 + 0.4 = 0.4$, so $y_1 = y\,(0.4) = 1.4$.

(ii) $h = 0.2 \quad \Rightarrow \quad x_1 = 0.2$ and $x_2 = 0.4$, so we need to find y_2.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2$,

$y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44$.

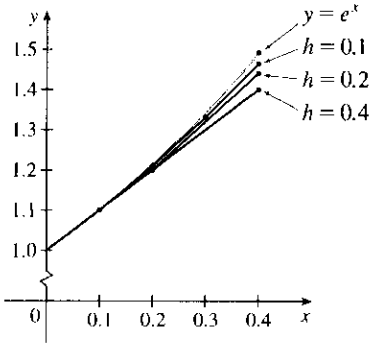
(iii) $h = 0.1 \quad \Rightarrow \quad x_4 = 0.4$, so we need to find y_4. $y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1$,

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21$,

$y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331$,

$y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641$.

(b)



We see that the estimates are underestimates since they are all below the graph of $y = e^x$.

(c) (i) For $h = 0.4$: (exact value) − (approximate value) $= e^{0.4} - 1.4 \approx 0.0918$

(ii) For $h = 0.2$: (exact value) − (approximate value) $= e^{0.4} - 1.44 \approx 0.0518$

(iii) For $h = 0.1$: (exact value) − (approximate value) $= e^{0.4} - 1.4641 \approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

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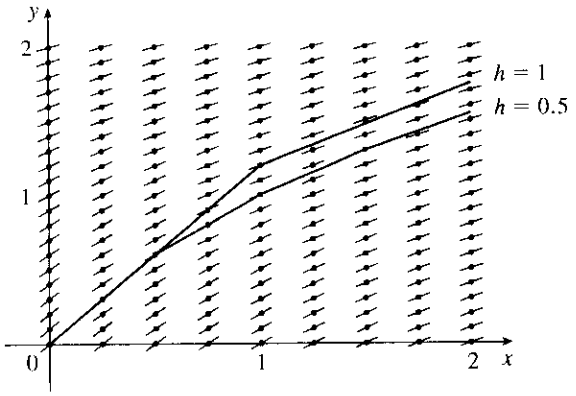
20.



As x increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

21. $h = 0.5$, $x_0 = 1$, $y_0 = 0$, and $F(x, y) = y - 2x$.

Note that $x_1 = x_0 + h = 1 + 0.5 = 1.5$, $x_2 = 2$, and $x_3 = 2.5$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.5F(1, 0) = 0.5[0 - 2(1)] = -1.$

$y_2 = y_1 + hF(x_1, y_1) = -1 + 0.5F(1.5, -1) = -1 + 0.5[-1 - 2(1.5)] = -3.$

$y_3 = y_2 + hF(x_2, y_2) = -3 + 0.5F(2, -3) = -3 + 0.5[-3 - 2(2)] = -6.5.$

$y_4 = y_3 + hF(x_3, y_3) = -6.5 + 0.5F(2.5, -6.5) = -6.5 + 0.5[-6.5 - 2(2.5)] = -12.25.$

22. $h = 0.2$, $x_0 = 0$, $y_0 = 0$, and $F(x, y) = 1 - xy$.

Note that $x_1 = x_0 + h = 0 + 0.2 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, and $x_4 = 0.8$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.2F(0, 0) = 0.2[1 - (0)(0)] = 0.2.$

$y_2 = y_1 + hF(x_1, y_1) = 0.2 + 0.2F(0.2, 0.2) = 0.2 + 0.2[1 - (0.2)(0.2)] = 0.392.$

$y_3 = y_2 + hF(x_2, y_2) = 0.392 + 0.2F(0.4, 0.392) = 0.392 + 0.2[1 - (0.4)(0.392)] = 0.56064.$

$y_4 = y_3 + hF(x_3, y_3) = 0.56064 + 0.2[1 - (0.6)(0.56064)] = 0.6933632.$

$y_5 = y_4 + hF(x_4, y_4) = 0.6933632 + 0.2[1 - (0.8)(0.6933632)] = 0.782425088.$

Thus, $y(1) \approx 0.7824$.

23. $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = y + xy$.

Note that $x_1 = x_0 + h = 0 + 0.1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, and $x_4 = 0.4$.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1F(0, 1) = 1 + 0.1[1 + (0)(1)] = 1.1.$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1F(0.1, 1.1) = 1.1 + 0.1[1.1 + (0.1)(1.1)] = 1.221.$

$y_3 = y_2 + hF(x_2, y_2) = 1.221 + 0.1F(0.2, 1.221) = 1.221 + 0.1[1.221 + (0.2)(1.221)] = 1.36752.$

$y_4 = y_3 + hF(x_3, y_3) = 1.36752 + 0.1F(0.3, 1.36752) = 1.36752 + 0.1[1.36752 + (0.3)(1.36752)]$
$= 1.5452976.$

$y_5 = y_4 + hF(x_4, y_4) = 1.5452976 + 0.1F(0.4, 1.5452976)$
$= 1.5452976 + 0.1[1.5452976 + (0.4)(1.5452976)] = 1.761639264.$

Thus, $y(0.5) \approx 1.7616$.

24. (a) $h = 0.2$, $x_0 = 1$, $y_0 = 0$, and $F(x, y) = x - xy$.

We need to find y_2, because $x_1 = 1.2$ and $x_2 = 1.4$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.2F(1, 0) = 0.2[1 - (1)(0)] = 0.2.$

$y_2 = y_1 + hF(x_1, y_1) = 0.2 + 0.2F(1.2, 0.2) = 0.2 + 0.2[1.2 - (1.2)(0.2)] = 0.392 \approx y(1.4).$

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(b) Now $h = 0.1$, so we need to find y_4.

$y_1 = 0 + 0.1[1 - (1)(0)] = 0.1,$

$y_2 = 0.1 + 0.1[1.1 - (1.1)(0.1)] = 0.199,$

$y_3 = 0.199 + 0.1[1.2 - (1.2)(0.199)] = 0.29512,$ and

$y_4 = 0.29512 + 0.1[1.3 - (1.3)(0.29512)] = 0.3867544 \approx y(1.4).$

25. (a) $dy/dx + 3x^2y = 6x^2 \quad \Rightarrow \quad y' = 6x^2 - 3x^2y$. Store this expression in Y_1 and use the following simple program to evaluate $y(1)$ for each part, using $H = h = 1$ and $N = 1$ for part (i), $H = 0.1$ and $N = 10$ for part (ii), and so forth.

$h \rightarrow$ H: 0 $\rightarrow$ X: 3 $\rightarrow$ Y:
For(I, 1, N): $Y + H \times Y_1 \rightarrow$ Y: $X + H \rightarrow$ X:
End(loop):
Display Y. [To see all iterations, include this statement in the loop.]

(i) $H = 1, N = 1 \quad \Rightarrow \quad y(1) = 3$

(ii) $H = 0.1, N = 10 \quad \Rightarrow \quad y(1) \approx 2.3928$

(iii) $H = 0.01, N = 100 \quad \Rightarrow \quad y(1) \approx 2.3701$

(iv) $H = 0.001, N = 1000 \quad \Rightarrow \quad y(1) \approx 2.3681$

(b) $y = 2 + e^{-x^3} \quad \Rightarrow \quad y' = -3x^2e^{-x^3}$

$$\text{LHS} = y' + 3x^2y = -3x^2e^{-x^3} + 3x^2\left(2 + e^{-x^3}\right) = -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 = \text{RHS}$$

$y(0) = 2 + e^{-0} = 2 + 1 = 3$

(c) The exact value of $y(1)$ is $2 + e^{-1^3} = 2 + e^{-1}$.

(i) For $h = 1$: (exact value) − (approximate value) $= 2 + e^{-1} - 3 \approx -0.6321$

(ii) For $h = 0.1$: (exact value) − (approximate value) $= 2 + e^{-1} - 2.3928 \approx -0.0249$

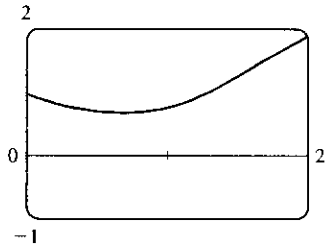
(iii) For $h = 0.01$: (exact value) − (approximate value) $= 2 + e^{-1} - 2.3701 \approx -0.0022$

(iv) For $h = 0.001$: (exact value) − (approximate value) $= 2 + e^{-1} - 2.3681 \approx -0.0002$

In (ii)–(iv), it seems that when the step size is divided by 10, the error estimate is also divided by 10 (approximately).

26. (a) We use the program from the solution to Exercise 25 with $Y_1 = x^3 - y^3$, $H = 0.01$, and $N = \frac{2-0}{0.01} = 200$. With $(x_0, y_0) = (0, 1)$, we get $y(2) \approx 1.9000$.

(b)



Notice from the graph that $y(2) \approx 1.9$, which serves as a check on our calculation in part (a).

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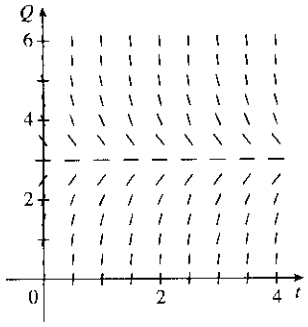
27. (a) $R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t)$ becomes

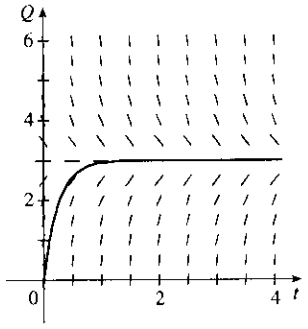
$5Q' + \frac{1}{0.05}Q = 60$ or $Q' + 4Q = 12$.



(b) From the graph, it appears that the limiting value of the charge Q is about 3.

(c) If $Q' = 0$, then $4Q = 12 \Rightarrow Q = 3$ is an equilibrium solution.

(d)



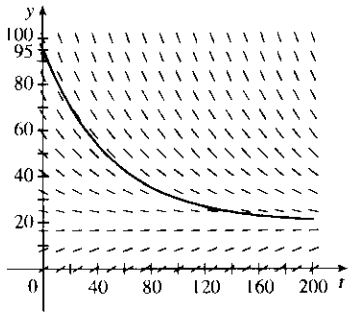
(e) $Q' + 4Q = 12 \Rightarrow Q' = 12 - 4Q$. Now $Q(0) = 0$, so $t_0 = 0$ and $Q_0 = 0$.

$$Q_1 = Q_0 + hF(t_0, Q_0) = 0 + 0.1(12 - 4 \cdot 0) = 1.2$$
$$Q_2 = Q_1 + hF(t_1, Q_1) = 1.2 + 0.1(12 - 4 \cdot 1.2) = 1.92$$
$$Q_3 = Q_2 + hF(t_2, Q_2) = 1.92 + 0.1(12 - 4 \cdot 1.92) = 2.352$$
$$Q_4 = Q_3 + hF(t_3, Q_3) = 2.352 + 0.1(12 - 4 \cdot 2.352) = 2.6112$$
$$Q_5 = Q_4 + hF(t_4, Q_4) = 2.6112 + 0.1(12 - 4 \cdot 2.6112) = 2.76672$$

Thus, $Q_5 = Q(0.5) \approx 2.77$ C.

28. (a) From Exercise 10.1.14, we have $dy/dt = k(y - R)$. We are given that $R = 20°\text{C}$ and $dy/dt = -1°\text{C/min}$ when $y = 70°\text{C}$. Thus, $-1 = k(70 - 20) \Rightarrow k = -\frac{1}{50}$ and the differential equation becomes $dy/dt = -\frac{1}{50}(y - 20)$.

(b)



The limiting value of the temperature is 20°C; that is, the temperature of the room.

(c) From part (a), $dy/dt = -\frac{1}{50}(y - 20)$. With $t_0 = 0$, $y_0 = 95$, and $h = 2$ min, we get

$$y_1 = y_0 + hF(t_0, y_0) = 95 + 2\left[-\tfrac{1}{50}(95 - 20)\right] = 92$$
$$y_2 = y_1 + hF(t_1, y_1) = 92 + 2\left[-\tfrac{1}{50}(92 - 20)\right] = 89.12$$
$$y_3 = y_2 + hF(t_2, y_2) = 89.12 + 2\left[-\tfrac{1}{50}(89.12 - 20)\right] = 86.3552$$
$$y_4 = y_3 + hF(t_3, y_3) = 86.3552 + 2\left[-\tfrac{1}{50}(86.3552 - 20)\right] = 83.700992$$
$$y_5 = y_4 + hF(t_4, y_4) = 83.700992 + 2\left[-\tfrac{1}{50}(83.700992 - 20)\right] = 81.15295232$$

Thus, $y(10) \approx 81.15°\text{C}$.

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10.3 Separable Equations

1. $\dfrac{dy}{dx} = \dfrac{y}{x} \;\Rightarrow\; \dfrac{dy}{y} = \dfrac{dx}{x}\;\; [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int \frac{dx}{x} \;\Rightarrow\; \ln|y| = \ln|x| + C \;\Rightarrow$

$|y| = e^{\ln|x|+C} = e^{\ln|x|}e^C = e^C\,|x| \;\Rightarrow\; y = Kx$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

2. $\dfrac{dy}{dx} = \dfrac{e^{2x}}{4y^3} \;\Rightarrow\; 4y^3\,dy = e^{2x}\,dx \;\Rightarrow\; \displaystyle\int 4y^3\,dy = \int e^{2x}\,dx \;\Rightarrow\; y^4 = \tfrac{1}{2}e^{2x} + C \;\Rightarrow$

$y = \pm\sqrt[4]{\tfrac{1}{2}e^{2x} + C}$

3. $(x^2+1)y' = xy \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{xy}{x^2+1} \;\Rightarrow\; \dfrac{dy}{y} = \dfrac{x\,dx}{x^2+1}\;\; [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int \frac{x\,dx}{x^2+1} \;\Rightarrow$

$\ln|y| = \tfrac{1}{2}\ln(x^2+1) + C\;\; [u = x^2+1,\, du = 2x\,dx] \;= \ln(x^2+1)^{1/2} + \ln c^C = \ln\left(e^C\sqrt{x^2+1}\right) \;\Rightarrow$

$|y| = e^C\sqrt{x^2+1} \;\Rightarrow\; y = K\sqrt{x^2+1}$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

4. $y' = y^2\sin x \;\Rightarrow\; \dfrac{dy}{dx} = y^2\sin x \;\Rightarrow\; \dfrac{dy}{y^2} = \sin x\,dx\;\; [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y^2} = \int \sin x\,dx \;\Rightarrow$

$-\dfrac{1}{y} = -\cos x + C \;\Rightarrow\; \dfrac{1}{y} = \cos x - C \;\Rightarrow\; y = \dfrac{1}{\cos x + K}$, where $K = -C$. $y = 0$ is also a solution.

5. $(1+\tan y)\,y' = x^2+1 \;\Rightarrow\; (1+\tan y)\dfrac{dy}{dx} = x^2+1 \;\Rightarrow\; \left(1 + \dfrac{\sin y}{\cos y}\right)dy = (x^2+1)\,dx \;\Rightarrow$

$\displaystyle\int \left(1 - \frac{-\sin y}{\cos y}\right)dy = \int (x^2+1)\,dx \;\Rightarrow\; y - \ln|\cos y| = \tfrac{1}{3}x^3 + x + C$. Note: The left side is equivalent to $y + \ln|\sec y|$.

6. $\dfrac{du}{dr} = \dfrac{1+\sqrt{r}}{1+\sqrt{u}} \;\Rightarrow\; (1+\sqrt{u}\,)\,du = (1+\sqrt{r}\,)\,dr \;\Rightarrow\; \displaystyle\int (1+u^{1/2})\,du = \int (1+r^{1/2})\,dr \;\Rightarrow$

$u + \tfrac{2}{3}u^{3/2} = r + \tfrac{2}{3}r^{3/2} + C$

7. $\dfrac{dy}{dt} = \dfrac{te^t}{y\sqrt{1+y^2}} \;\Rightarrow\; y\sqrt{1+y^2}\,dy = te^t\,dt \;\Rightarrow\; \displaystyle\int y\sqrt{1+y^2}\,dy = \int te^t\,dt \;\Rightarrow$

$\tfrac{1}{3}\left(1+y^2\right)^{3/2} = te^t - e^t + C$ [where the first integral is evaluated by substitution and the second by parts] $\Rightarrow$

$1 + y^2 = [3(te^t - e^t + C)]^{2/3} \;\Rightarrow\; y = \pm\sqrt{[3(te^t - e^t + C)]^{2/3} - 1}$

8. $y' = \dfrac{xy}{2\ln y} \;\Rightarrow\; \dfrac{2\ln y}{y}\,dy = x\,dx \;\Rightarrow\; \displaystyle\int \frac{2\ln y}{y}\,dy = \int x\,dx \;\Rightarrow\; (\ln y)^2 = \frac{x^2}{2} + C \;\Rightarrow$

$\ln y = \pm\sqrt{x^2/2 + C} \;\Rightarrow\; y = e^{\pm\sqrt{x^2/2 + C}}$

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9. $\dfrac{du}{dt} = 2 + 2u + t + tu \;\Rightarrow\; \dfrac{du}{dt} = (1+u)(2+t) \;\Rightarrow\; \displaystyle\int \frac{du}{1+u} = \int (2+t)\,dt \;\; [u \neq -1] \;\Rightarrow$

$\ln|1+u| = \frac{1}{2}t^2 + 2t + C \;\Rightarrow\; |1+u| = e^{t^2/2 + 2t + C} = Ke^{t^2/2+2t}$, where $K = e^C \;\Rightarrow$

$1 + u = \pm Ke^{t^2/2+2t} \;\Rightarrow\; u = -1 \pm Ke^{t^2/2+2t}$ where $K > 0$. $u = -1$ is also a solution, so

$u = -1 + Ae^{t^2/2+2t}$, where A is an arbitrary constant.

10. $\dfrac{dz}{dt} + e^{t+z} = 0 \;\Rightarrow\; \dfrac{dz}{dt} = -e^t e^z \;\Rightarrow\; \displaystyle\int e^{-z}\,dz = -\int e^t\,dt \;\Rightarrow\; -e^{-z} = -e^t + C \;\Rightarrow\; e^{-z} = e^t - C$

$\Rightarrow\; \dfrac{1}{e^z} = e^t - C \;\Rightarrow\; e^z = \dfrac{1}{e^t - C} \;\Rightarrow\; z = \ln\left(\dfrac{1}{e^t - C}\right) \;\Rightarrow\; z = -\ln\left(e^t - C\right)$

11. $\dfrac{dy}{dx} = y^2 + 1$, $y(1) = 0$. $\displaystyle\int \frac{dy}{y^2+1} = \int dx \;\Rightarrow\; \tan^{-1} y = x + C$. $y = 0$ when $x = 1$, so

$1 + C = \tan^{-1} 0 = 0 \;\Rightarrow\; C = -1$. Thus, $\tan^{-1} y = x - 1$ and $y = \tan(x-1)$.

12. $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2}$, $y(0) = 1$. $\left(1+y^2\right)dy = y\cos x\,dx \;\Rightarrow\; \dfrac{1+y^2}{y}dy = \cos x\,dx \;\Rightarrow$

$\displaystyle\int \left(\tfrac{1}{y} + y\right)dy = \int \cos x\,dx \;\Rightarrow\; \ln|y| + \tfrac{1}{2}y^2 = \sin x + C$. $y(0) = 1 \;\Rightarrow\; \ln 1 + \tfrac{1}{2} = \sin 0 + C \;\Rightarrow$

$C = \frac{1}{2}$, so $\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}$. We cannot solve explicitly for y.

13. $x\cos x = (2y + e^{3y})\,y' \;\Rightarrow\; x\cos x\,dx = (2y + e^{3y})\,dy \;\Rightarrow\; \displaystyle\int (2y + e^{3y})\,dy = \int x\cos x\,dx \;\Rightarrow$

$y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x + C$ [where the second integral is evaluated using integration by parts]. Now

$y(0) = 0 \;\Rightarrow\; 0 + \frac{1}{3} = 0 + 1 + C \;\Rightarrow\; C = -\frac{2}{3}$. Thus, a solution is $y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x - \frac{2}{3}$.

We cannot solve explicitly for y.

14. $\dfrac{dP}{dt} = \sqrt{Pt} \;\Rightarrow\; dP/\sqrt{P} = \sqrt{t}\,dt \;\Rightarrow\; \displaystyle\int P^{-1/2}\,dP = \int t^{1/2}\,dt \;\Rightarrow\; 2P^{1/2} = \frac{2}{3}t^{3/2} + C$.

$P(1) = 2 \;\Rightarrow\; 2\sqrt{2} = \frac{2}{3} + C \;\Rightarrow\; C = 2\sqrt{2} - \frac{2}{3}$, so $2P^{1/2} = \frac{2}{3}t^{3/2} + 2\sqrt{2} - \frac{2}{3} \;\Rightarrow$

$\sqrt{P} = \frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3} \;\Rightarrow\; P = \left(\frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3}\right)^2$.

15. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}$, $u(0) = -5$. $\displaystyle\int 2u\,du = \int \left(2t + \sec^2 t\right)dt \;\Rightarrow\; u^2 = t^2 + \tan t + C$, where

$[u(0)]^2 = 0^2 + \tan 0 + C \;\Rightarrow\; C = (-5)^2 = 25$. Therefore, $u^2 = t^2 + \tan t + 25$, so $u = \pm\sqrt{t^2 + \tan t + 25}$.

Since $u(0) = -5$, we must have $u = -\sqrt{t^2 + \tan t + 25}$.

16. $\dfrac{dy}{dt} = te^y$, $y(1) = 0$. $\displaystyle\int e^{-y}\,dy = \int t\,dt \;\Rightarrow\; -e^{-y} = \frac{1}{2}t^2 + C$. Since $y(1) = 0$, $-e^0 = \frac{1}{2}\cdot 1^2 + C$. Therefore,

$C = -1 - \frac{1}{2} = -\frac{3}{2}$ and $-e^{-y} = \frac{1}{2}t^2 - \frac{3}{2}$. So $e^{-y} = \frac{3}{2} - \frac{1}{2}t^2 = \dfrac{3 - t^2}{2} \;\Rightarrow\; e^y = \dfrac{2}{3 - t^2} \;\Rightarrow$

$y = \ln 2 - \ln\left(3 - t^2\right)$ for $|t| < \sqrt{3}$.

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17. $y' \tan x = a + y,\ 0 < x < \pi/2 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{a+y}{\tan x} \quad \Rightarrow \quad \dfrac{dy}{a+y} = \cot x\, dx \quad [a + y \neq 0] \quad \Rightarrow$

$\displaystyle\int \frac{dy}{a+y} = \int \frac{\cos x}{\sin x}\, dx \quad \Rightarrow \quad \ln|a+y| = \ln|\sin x| + C \quad \Rightarrow$

$|a+y| = e^{\ln|\sin x| + C} = e^{\ln|\sin x|} \cdot e^C = e^C\,|\sin x| \quad \Rightarrow \quad a + y = K \sin x$, where $K = \pm e^C$. (In our derivation, K was nonzero, but we can restore the excluded case $y = -a$ by allowing K to be zero.) $y(\pi/3) = a \quad \Rightarrow$

$a + a = K \sin\left(\dfrac{\pi}{3}\right) \quad \Rightarrow \quad 2a = K\,\dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad K = \dfrac{4a}{\sqrt{3}}$. Thus, $a + y = \dfrac{4a}{\sqrt{3}} \sin x$ and so $y = \dfrac{4a}{\sqrt{3}} \sin x - a$.

18. $xy' + y = y^2 \quad \Rightarrow \quad x\,\dfrac{dy}{dx} = y^2 - y \quad \Rightarrow \quad x\,dy = (y^2 - y)\,dx \quad \Rightarrow \quad \dfrac{dy}{y^2 - y} = \dfrac{dx}{x} \quad \Rightarrow$

$\displaystyle\int \frac{dy}{y(y-1)} = \int \frac{dx}{x} \quad [y \neq 0, 1] \quad \Rightarrow \quad \int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy = \int \frac{dx}{x} \quad \Rightarrow \quad \ln|y-1| - \ln|y| = \ln|x| + C$

$\Rightarrow \quad \ln\left|\dfrac{y-1}{y}\right| = \ln\left(e^C\,|x|\right) \quad \Rightarrow \quad \left|\dfrac{y-1}{y}\right| = e^C\,|x| \quad \Rightarrow \quad \dfrac{y-1}{y} = Kx$, where $K = \pm e^C \quad \Rightarrow$

$1 - \dfrac{1}{y} = Kx \quad \Rightarrow \quad \dfrac{1}{y} = 1 - Kx \quad \Rightarrow \quad y = \dfrac{1}{1 - Kx}$. [The excluded cases, $y = 0$ and $y = 1$, are ruled out by the initial condition $y(1) = -1$.] Now $y(1) = -1 \quad \Rightarrow \quad -1 = \dfrac{1}{1-K} \quad \Rightarrow \quad 1 - K = -1 \quad \Rightarrow \quad K = 2$,

so $y = \dfrac{1}{1 - 2x}$.

19. $\dfrac{dy}{dx} = 4x^3 y,\ y(0) = 7.\ \dfrac{dy}{y} = 4x^3\,dx$ [if $y \neq 0$] $\Rightarrow \quad \displaystyle\int \frac{dy}{y} = \int 4x^3\,dx \quad \Rightarrow \quad \ln|y| = x^4 + C \quad \Rightarrow$

$e^{\ln|y|} = e^{x^4 + C} \quad \Rightarrow \quad |y| = e^{x^4} e^C \quad \Rightarrow \quad y = Ae^{x^4};\ y(0) = 7 \quad \Rightarrow \quad A = 7 \quad \Rightarrow \quad y = 7e^{x^4}$.

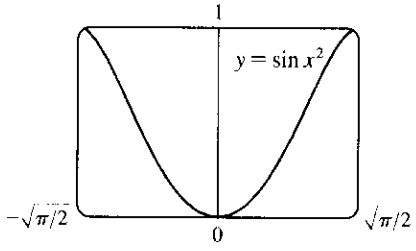
20. $\dfrac{dy}{dx} = \dfrac{y^2}{x^3}$, $y(1) = 1$. $\displaystyle\int \frac{dy}{y^2} = \int \frac{dx}{x^3} \quad \Rightarrow \quad -\frac{1}{y} = -\frac{1}{2x^2} + C$. $y(1) = 1 \quad \Rightarrow \quad -1 = -\frac{1}{2} + C \quad \Rightarrow$

$C = -\frac{1}{2}$. So $\dfrac{1}{y} = \dfrac{1}{2x^2} + \dfrac{1}{2} = \dfrac{2 + 2x^2}{2 \cdot 2x^2} \quad \Rightarrow \quad y = \dfrac{2x^2}{x^2 + 1}$.

21. (a) $y' = 2x\sqrt{1 - y^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = 2x\sqrt{1 - y^2} \quad \Rightarrow \quad \dfrac{dy}{\sqrt{1 - y^2}} = 2x\,dx \quad \Rightarrow \quad \displaystyle\int \frac{dy}{\sqrt{1 - y^2}} = \int 2x\,dx \quad \Rightarrow$

$\sin^{-1} y = x^2 + C$ for $-\frac{\pi}{2} \leq x^2 + C \leq \frac{\pi}{2}$.

(b) $y(0) = 0 \quad \Rightarrow \quad \sin^{-1} 0 = 0^2 + C \quad \Rightarrow \quad C = 0$, so $\sin^{-1} y = x^2$ and $y = \sin\left(x^2\right)$ for $-\sqrt{\pi/2} \leq x \leq \sqrt{\pi/2}$.



(c) For $\sqrt{1 - y^2}$ to be a real number, we must have $-1 \leq y \leq 1$; that is, $-1 \leq y(0) \leq 1$. Thus, the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 2$ does *not* have a solution.

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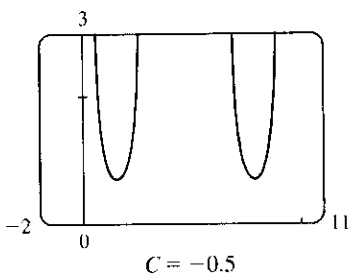
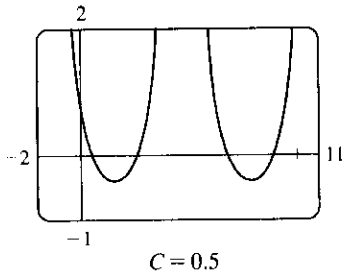
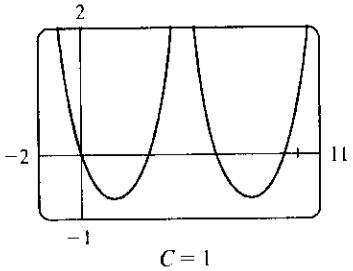
22. $e^{-y}y' + \cos x = 0 \quad \Leftrightarrow \quad \int e^{-y}\,dy = -\int \cos x\,dx \quad \Leftrightarrow \quad -e^{-y} = -\sin x + C_1 \quad \Leftrightarrow \quad y = -\ln(\sin x + C).$
The solution is periodic, with period 2π. Note that for $C > 1$, the domain of the solution is $\mathbb{R}$, but for $-1 < C \le 1$ it is only defined on the intervals where $\sin x + C > 0$, and it is meaningless for $C \le -1$, since then $\sin x + C \le 0$, and the logarithm is undefined.



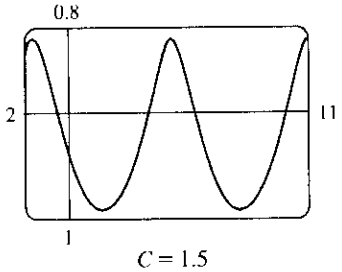
$C = -0.5$



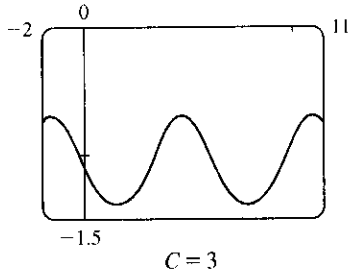
$C = 0.5$



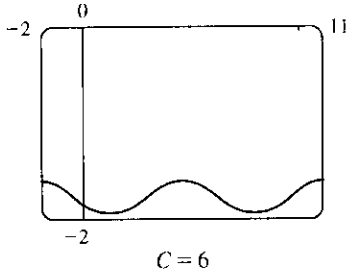
$C = 1$



$C = 1.5$



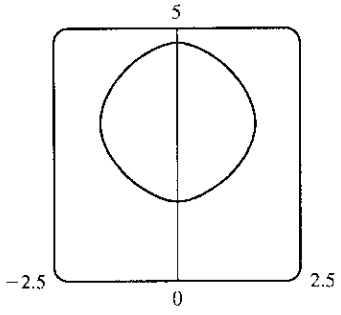
$C = 3$



$C = 6$

For $-1 < C < 1$, the solution curve consists of concave-up pieces separated by intervals on which the solution is not defined (where $\sin x + C \le 0$). For $C = 1$, the solution curve consists of concave-up pieces separated by vertical asymptotes at the points where $\sin x + C = 0 \quad \Leftrightarrow \quad \sin x = -1$. For $C > 1$, the curve is continuous, and as C increases, the graph moves downward, and the amplitude of the oscillations decreases.

23. $\dfrac{dy}{dx} = \dfrac{\sin x}{\sin y}$, $y(0) = \frac{\pi}{2}$. So $\int \sin y\,dy = \int \sin x\,dx \quad \Leftrightarrow$
$-\cos y = -\cos x + C \quad \Leftrightarrow \quad \cos y = \cos x - C$. From the initial condition, we need $\cos\frac{\pi}{2} = \cos 0 - C \quad \Rightarrow \quad 0 = 1 - C \quad \Rightarrow \quad C = 1$, so the solution is $\cos y = \cos x - 1$. Note that we cannot take $\cos^{-1}$ of both sides, since that would unnecessarily restrict the solution to the case where $-1 \le \cos x - 1 \quad \Leftrightarrow \quad 0 \le \cos x$, as $\cos^{-1}$ is defined only on $[-1, 1]$. Instead we plot the graph using Maple's `plots[implicitplot]` or Mathematica's `Plot[Evaluate[···]]`.



24. $\dfrac{dy}{dx} = \dfrac{x\sqrt{x^2+1}}{ye^y} \quad \Leftrightarrow \quad \int ye^y\,dy = \int x\sqrt{x^2+1}\,dx$. We use parts on the LHS with $u = y$, $dv = e^y\,dy$, and on the RHS we use the substitution $z = x^2 + 1$, so $dz = 2x\,dx$. The equation becomes $ye^y - \int e^y\,dy = \frac{1}{2}\int \sqrt{z}\,dz$
$\Leftrightarrow \quad e^y(y-1) = \frac{1}{3}(x^2+1)^{3/2} + C$, so we see that the curves are symmetric about the y-axis. Every point (x, y) in the plane lies on one of the curves, namely the one for which $C = (y-1)e^y - \frac{1}{3}(x^2+1)^{3/2}$. For example,

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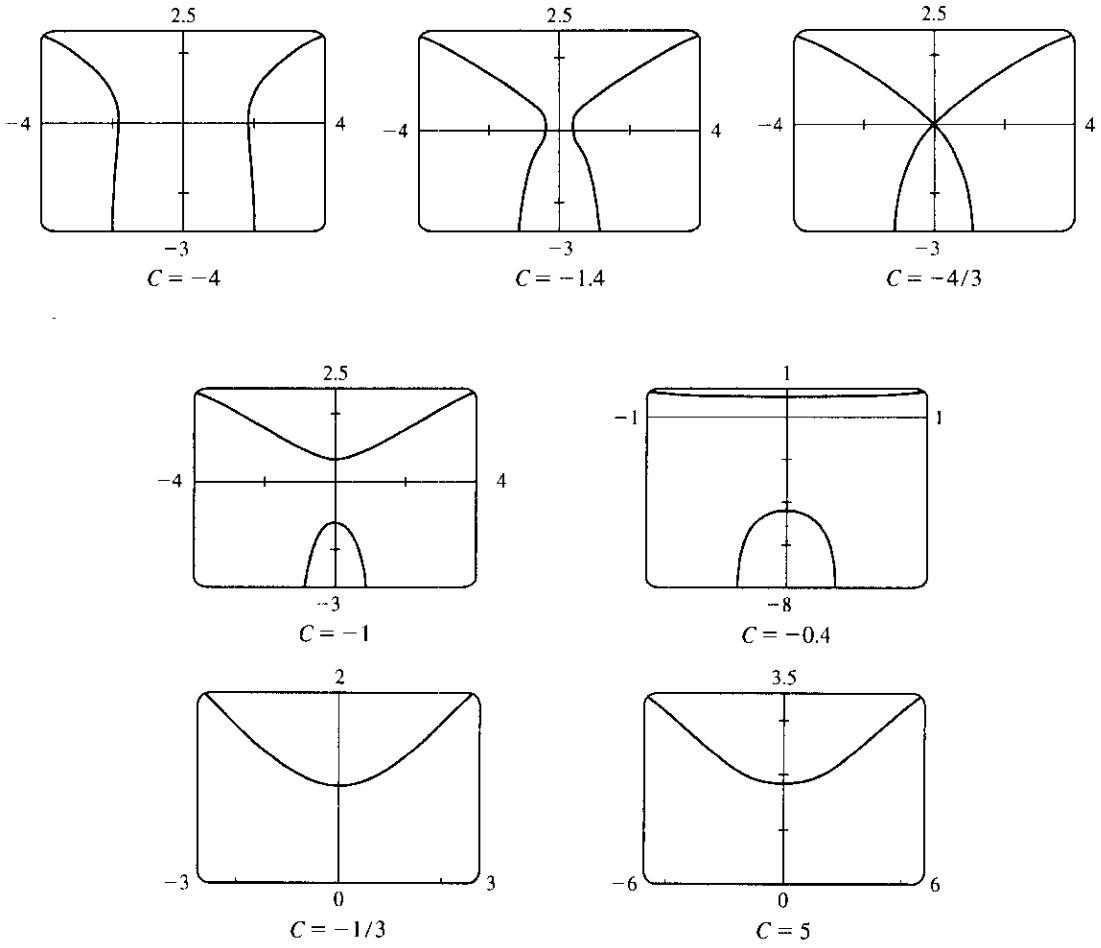
along the y-axis, $C = (y-1)e^y - \frac{1}{3}$, so the origin lies on the curve with $C = -\frac{4}{3}$. We use Maple's `plots[implicitplot]` command or `Plot[Evaluate[···]]` in Mathematica to plot the solution curves for various values of C.

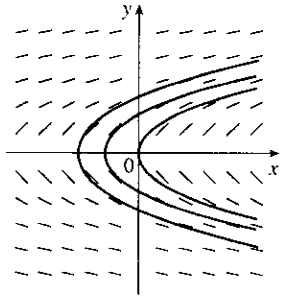


It seems that the transitional values of C are $-\frac{4}{3}$ and $-\frac{1}{3}$. For $C < -\frac{4}{3}$, the graph consists of left and right branches. At $C = -\frac{4}{3}$, the two branches become connected at the origin, and as C increases, the graph splits into top and bottom hranches. At $C = -\frac{1}{3}$, the bottom half disappears. As C increases further, the graph moves upward, but doesn't change shape much.

25. (a)

x	y	$y' = 1/y$
0	0.5	2
0	−0.5	−2
0	1	1
0	−1	−1
0	2	0.5

x	y	$y' = 1/y$
0	−2	−0.5
0	4	0.25
0	3	$0.\overline{3}$
0	0.25	4
0	$0.\overline{3}$	3



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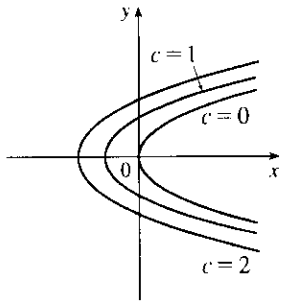
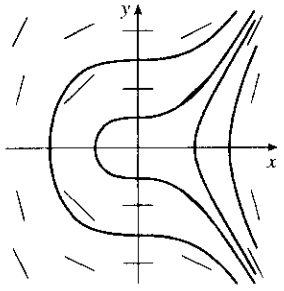
(b) $y' = 1/y \quad \Rightarrow \quad dy/dx = 1/y \quad \Rightarrow$
$y\,dy = dx \quad \Rightarrow \quad \int y\,dy = \int dx \quad \Rightarrow$
$\frac{1}{2}y^2 = x + c \quad \Rightarrow \quad y^2 = 2(x + c)$
or $y = \pm\sqrt{2(x + c)}$.

(c)



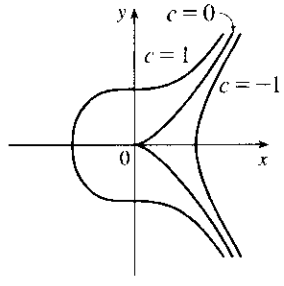
26. (a)

x	y	$y' = x^2/y$
1	1	1
−1	1	1
−1	−1	−1
1	−1	−1
1	2	0.5
2	1	4
2	2	2
1	0.5	2
0.5	1	0.25
2	0.5	8

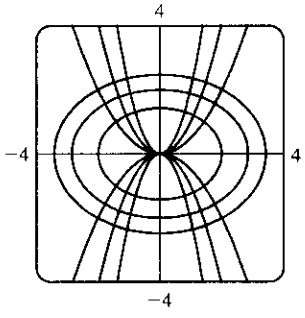


(b) $y' = x^2/y \quad \Rightarrow \quad y\,dy = x^2\,dx$,
so $\frac{1}{2}y^2 = \frac{1}{3}x^3 + c_1$, or
$y = \pm\left(\frac{2}{3}x^3 + c\right)^{1/2}$.

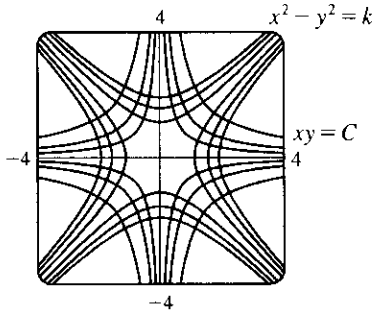
(c)



27. The curves $y = kx^2$ form a family of parabolas with axis the y-axis. Differentiating gives $y' = 2kx$, but $k = y/x^2$, so $y' = 2y/x$. Thus, the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = 2y/x$, so the orthogonal trajectories must satisfy $y' = -x/(2y)$ $\Leftrightarrow \quad 2y\,dy = -x\,dx \quad \Leftrightarrow \quad y^2 = -x^2/2 + C_1 \quad \Leftrightarrow \quad x^2 + 2y^2 = C$. This is a family of ellipses.



28. The curves $x^2 - y^2 = k$ form a family of hyperbolas. Differentiating gives $2x - 2y\,(dy/dx) = 0$ or $y' = x/y$, the slope of the tangent line at (x, y) on one of the hyperbolas. Thus, the orthogonal trajectories must satisfy $y' = -y/x \quad \Leftrightarrow \quad dy/y = -dx/x \quad \Leftrightarrow \quad \ln|y| = -\ln|x| + C_1 \quad \Leftrightarrow$ $\ln|x| + \ln|y| = C_1 \quad \Leftrightarrow \quad \ln|xy| = C_1 \quad \Leftrightarrow$ $|xy| = e^{C_1} \quad \Leftrightarrow \quad xy = C$. This is a family of hyperbolas.



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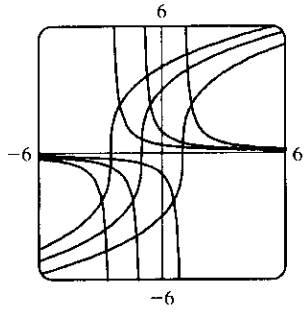
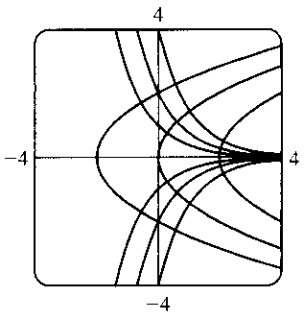
29. Differentiating $y = (x+k)^{-1}$ gives $y' = -\dfrac{1}{(x+k)^2}$, but $k = \dfrac{1}{y} - x$, so $y' = -\dfrac{1}{(1/y)^2} = -y^2$. Thus, the orthogonal trajectories must satisfy $y' = -\dfrac{1}{-y^2} = \dfrac{1}{y^2} \quad \Leftrightarrow \quad y^2\,dy = dx \quad \Leftrightarrow \quad \dfrac{y^3}{3} = x + C$ or $y = [3(x+C)]^{1/3}$



30. Differentiating $y = ke^{-x}$ gives $y' = -ke^{-x}$, but $k = ye^{x}$, so $y' = -y$. Thus, the orthogonal trajectories must satisfy $y' = -1/(-y) = 1/y \quad \Leftrightarrow$ $y\,dy = dx \quad \Leftrightarrow \quad \frac{1}{2}y^2 = x + C \quad \Leftrightarrow \quad y = \pm\,[2(C+x)]^{1/2}$. This is a family of parabolas with axis the x-axis.



31. From Exercise 10.2.27, $\dfrac{dQ}{dt} = 12 - 4Q \quad \Leftrightarrow \quad \displaystyle\int \frac{dQ}{12-4Q} = \int dt \quad \Leftrightarrow \quad -\tfrac{1}{4}\ln|12-4Q| = t + C \quad \Leftrightarrow$ $\ln|12-4Q| = -4t - 4C \quad \Leftrightarrow \quad |12-4Q| = e^{-4t-4C} \quad \Leftrightarrow \quad 12 - 4Q = Ke^{-4t}\ [K = \pm e^{-4C}] \quad \Leftrightarrow$ $4Q = 12 - Ke^{-4t} \quad \Leftrightarrow \quad Q = 3 - Ae^{-4t}\ [A = K/4]$. $Q(0) = 0 \quad \Leftrightarrow \quad 0 = 3 - A \quad \Leftrightarrow \quad A = 3 \quad \Leftrightarrow$ $Q(t) = 3 - 3e^{-4t}$. As $t \to \infty$, $Q(t) \to 3 - 0 = 3$ (the limiting value).

32. From Exercise 10.2.28, $\dfrac{dy}{dt} = -\frac{1}{50}(y - 20) \quad \Leftrightarrow \quad \displaystyle\int \frac{dy}{y-20} = \int \left(-\tfrac{1}{50}\right) dt \quad \Leftrightarrow \quad \ln|y-20| = -\tfrac{1}{50}t + C \quad \Leftrightarrow$ $y - 20 = Ke^{-t/50} \quad \Leftrightarrow \quad y(t) = Ke^{-t/50} + 20$. $y(0) = 95 \quad \Leftrightarrow \quad 95 = K + 20 \quad \Leftrightarrow \quad K = 75 \quad \Leftrightarrow$ $y(t) = 75e^{-t/50} + 20$.

33. $\dfrac{dP}{dt} = k(M - P) \quad \Leftrightarrow \quad \displaystyle\int \frac{dP}{P-M} = \int (-k)\,dt \quad \Leftrightarrow \quad \ln|P-M| = -kt + C \quad \Leftrightarrow \quad |P-M| = e^{-kt+C}$ $\Leftrightarrow \quad P - M = Ae^{-kt}\ [A = \pm e^{C}] \quad \Leftrightarrow \quad P = M + Ae^{-kt}$. If we assume that performance is at level 0 when $t = 0$, then $P(0) = 0 \quad \Leftrightarrow \quad 0 = M + A \quad \Leftrightarrow \quad A = -M \quad \Leftrightarrow \quad P(t) = M - Me^{-kt}$.
$\lim\limits_{t\to\infty} P(t) = M - M \cdot 0 = M$.

34. (a) $\dfrac{dx}{dt} = k\,(a-x)\,(b-x)$, $a \neq b$. Using partial fractions, $\dfrac{1}{(a-x)(b-x)} = \dfrac{1/(b-a)}{a-x} - \dfrac{1/(b-a)}{b-x}$, so $\displaystyle\int \frac{dx}{(a-x)(b-x)} = \int k\,dt \quad \Rightarrow \quad \frac{1}{b-a}(-\ln|a-x| + \ln|b-x|) = kt + C \quad \Rightarrow$ $\ln\left|\dfrac{b-x}{a-x}\right| = (b-a)\,(kt+C)$. The concentrations $[\mathrm{A}] = a - x$ and $[\mathrm{B}] = b - x$ cannot he negative, so $\dfrac{b-x}{a-x} \geq 0$ and $\left|\dfrac{b-x}{a-x}\right| = \dfrac{b-x}{a-x}$. We now have $\ln\left(\dfrac{b-x}{a-x}\right) = (b-a)(kt+C)$. Since $x(0) = 0$, we get

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$\ln\left(\frac{b}{a}\right) = (b-a)C$. Hence, $\ln\left(\frac{b-x}{a-x}\right) = (b-a)kt + \ln\left(\frac{b}{a}\right) \quad \Rightarrow \quad \frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)kt} \quad \Rightarrow$

$x = \dfrac{b\left[e^{(b-a)kt}-1\right]}{be^{(b-a)kt}/a - 1} = \dfrac{ab\left[e^{(b-a)kt}-1\right]}{be^{(b-a)kt}-a}$ moles/L.

(b) If $b = a$, then $\dfrac{dx}{dt} = k(a-x)^2$, so $\displaystyle\int \frac{dx}{(a-x)^2} = \int k\,dt$ and $\dfrac{1}{a-x} = kt + C$. Since $x(0) = 0$, we get

$C = \dfrac{1}{a}$. Thus, $a - x = \dfrac{1}{kt + 1/a}$ and $x = a - \dfrac{a}{akt+1} = \dfrac{a^2kt}{akt+1}\dfrac{\text{moles}}{\text{L}}$.

Suppose $x = [\text{C}] = a/2$ when $t = 20$. Then $x(20) = a/2 \quad \Rightarrow \quad \dfrac{a}{2} = \dfrac{20a^2k}{20ak+1} \quad \Rightarrow \quad 40a^2k = 20a^2k + a$

$\Rightarrow \quad 20a^2k = a \quad \Rightarrow \quad k = \dfrac{1}{20a}$, so $x = \dfrac{a^2t/(20a)}{1 + at/(20a)} = \dfrac{at/20}{1+t/20} = \dfrac{at}{t+20}\dfrac{\text{moles}}{\text{L}}$.

35. (a) If $a = b$, then $\dfrac{dx}{dt} = k(a-x)(b-x)^{1/2}$ becomes $\dfrac{dx}{dt} = k(a-x)^{3/2} \quad \Rightarrow \quad (a-x)^{-3/2}dx = k\,dt \quad \Rightarrow$

$\displaystyle\int (a-x)^{-3/2}dx = \int k\,dt \quad \Rightarrow \quad 2(a-x)^{-1/2} = kt + C$ [by substitution] $\Rightarrow \quad \dfrac{2}{kt+C} = \sqrt{a-x} \quad \Rightarrow$

$\left(\dfrac{2}{kt+C}\right)^2 = a - x \quad \Rightarrow \quad x(t) = a - \dfrac{4}{(kt+C)^2}$. The initial concentration of HBr is 0, so $x(0) = 0 \quad \Rightarrow$

$0 = a - \dfrac{4}{C^2} \quad \Rightarrow \quad \dfrac{4}{C^2} = a \quad \Rightarrow \quad C^2 = \dfrac{4}{a} \quad \Rightarrow \quad C = 2/\sqrt{a}$ (C is positive since

$kt + C = 2(a-x)^{-1/2} > 0$). Thus, $x(t) = a - \dfrac{4}{\left(kt + 2/\sqrt{a}\,\right)^2}$.

(b) $\dfrac{dx}{dt} = k(a-x)(b-x)^{1/2} \quad \Rightarrow \quad \dfrac{dx}{(a-x)\sqrt{b-x}} = k\,dt \quad \Rightarrow \quad \displaystyle\int \frac{dx}{(a-x)\sqrt{b-x}} = \int k\,dt$ $(\star)$. From the

hint, $u = \sqrt{b-x} \quad \Rightarrow \quad u^2 = b - x \quad \Rightarrow \quad 2u\,du = -dx$, so $\displaystyle\int \frac{dx}{(a-x)\sqrt{b-x}} = \int \frac{-2u\,du}{[a-(b-u^2)]u} =$

$\displaystyle -2\int \frac{du}{a-b+u^2} = -2\int \frac{du}{\left(\sqrt{a-b}\right)^2 + u^2} \overset{17}{=} -2\left(\frac{1}{\sqrt{a-b}}\tan^{-1}\frac{u}{\sqrt{a-b}}\right)$. So $(\star)$ becomes

$\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}} = kt + C$. Now $x(0) = 0 \quad \Rightarrow \quad C = \dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}}$

and we have $\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}} = kt - \dfrac{2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}} \quad \Rightarrow$

$\dfrac{2}{\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}} - \tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right) = kt \quad \Rightarrow$

$t(x) = \dfrac{2}{k\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}} - \tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right)$.

36. If $S = \dfrac{dT}{dr}$, then $\dfrac{dS}{dr} = \dfrac{d^2T}{dr^2}$. The differential equation $\dfrac{d^2T}{dr^2} + \dfrac{2}{r}\dfrac{dT}{dr} = 0$ can be written as $\dfrac{dS}{dr} + \dfrac{2}{r}S = 0$. Thus,

$\dfrac{dS}{dr} = \dfrac{-2S}{r} \quad \Rightarrow \quad \dfrac{dS}{S} = -\dfrac{2}{r}\,dr \quad \Rightarrow \quad \displaystyle\int \frac{1}{S}\,dS = \int -\frac{2}{r}\,dr \quad \Rightarrow \quad \ln|S| = -2\ln|r| + C$. Assuming

$S = dT/dr > 0$ and $r > 0$, we have $S = e^{-2\ln r + C} = e^{\ln r^{-2}}e^C = r^{-2}k \;\; [k = e^C] \quad \Rightarrow \quad S = \dfrac{1}{r^2}\,k \quad \Rightarrow$

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$\dfrac{dT}{dr} = \dfrac{1}{r^2}\,k \quad \Rightarrow \quad dT = \dfrac{1}{r^2}\,k\,dr \quad \Rightarrow \quad \displaystyle\int dT = \int \frac{1}{r^2}\,k\,dr \quad \Rightarrow \quad T(r) = -\frac{k}{r} + A.$

$T(1) = 15 \quad \Rightarrow \quad 15 = -k + A$ **(1)** and $T(2) = 25 \quad \Rightarrow \quad 25 = -\frac{1}{2}k + A$ **(2)**.

Now solve for k and A: -2**(2)** + **(1)** $\Rightarrow \quad -35 = -A$, so $A = 35$ and $k = 20$, and $T(r) = -20/r + 35$.

37. (a) $\dfrac{dC}{dt} = r - kC \quad \Rightarrow \quad \dfrac{dC}{dt} = -(kC - r) \quad \Rightarrow \quad \displaystyle\int \frac{dC}{kC - r} = \int -dt \quad \Rightarrow \quad (1/k)\ln|kC - r| = -t + M_1$

$\Rightarrow \quad \ln|kC - r| = -kt + M_2 \quad \Rightarrow \quad |kC - r| = e^{-kt+M_2} \quad \Rightarrow \quad kC - r = M_3 e^{-kt} \quad \Rightarrow$

$kC = M_3 e^{-kt} + r \quad \Rightarrow \quad C(t) = M_4 e^{-kt} + r/k.\ C(0) = C_0 \quad \Rightarrow \quad C_0 = M_4 + r/k \quad \Rightarrow$

$M_4 = C_0 - r/k \quad \Rightarrow \quad C(t) = (C_0 - r/k)e^{-kt} + r/k.$

(b) If $C_0 < r/k$, then $C_0 - r/k < 0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim\limits_{t\to\infty} C(t) = r/k$.

As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of r/k increases.

38. (a) Use 1 billion dollars as the x-unit and 1 day as the t-unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t, the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]\,/10$, and the amount of old currency being returned to the banks each day is $\dfrac{10 - x(t)}{10}0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so

$\dfrac{dx}{dt} = \dfrac{10 - x}{10} \cdot 0.05 = 0.005(10 - x)$ billion dollars per day.

(b) $\dfrac{dx}{10 - x} = 0.005\,dt \quad \Rightarrow \quad \dfrac{-dx}{10 - x} = -0.005\,dt \quad \Rightarrow \quad \ln(10 - x) = -0.005t + c \quad \Rightarrow$

$10 - x = Ce^{-0.005t}$, where $C = e^c \quad \Rightarrow \quad x(t) = 10 - Ce^{-0.005t}$. From $x(0) = 0$, we get $C = 10$, so $x(t) = 10\big(1 - e^{-0.005t}\big)$.

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.

$9 = 10\big(1 - e^{-0.005t}\big) \quad \Rightarrow \quad 0.9 = 1 - e^{-0.005t} \quad \Rightarrow \quad e^{-0.005t} = 0.1 \quad \Rightarrow \quad -0.005t = -\ln 10 \quad \Rightarrow$

$t = 200 \ln 10 \approx 460.517$ days ≈ 1.26 years.

39. (a) Let $y(t)$ be the amount of salt (in kg) after t minutes. Then $y(0) = 15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time t (in minutes) is $y(t)/1000$ kg/L and

$\dfrac{dy}{dt} = -\left[\dfrac{y(t)}{1000}\ \dfrac{\text{kg}}{\text{L}}\right]\left(10\ \dfrac{\text{L}}{\text{min}}\right) = -\dfrac{y(t)}{100}\ \dfrac{\text{kg}}{\text{min}}. \displaystyle\int \frac{dy}{y} = -\frac{1}{100}\int dt \quad \Rightarrow \quad \ln y = -\frac{t}{100} + C$, and

$y(0) = 15 \quad \Rightarrow \quad \ln 15 = C$, so $\ln y = \ln 15 - \dfrac{t}{100}$. It follows that $\ln\left(\dfrac{y}{15}\right) = -\dfrac{t}{100}$ and $\dfrac{y}{15} = e^{-t/100}$, so $y = 15e^{-t/100}$ kg.

(b) After 20 minutes, $y = 15e^{-20/100} = 15e^{-0.2} \approx 12.3$ kg.

40. (a) If $y(t)$ is the amount of salt (in kg) after t minutes, then $y(0) = 0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\frac{dy}{dt} = \left(0.05\ \frac{\text{kg}}{\text{L}}\right)\left(5\ \frac{\text{L}}{\text{min}}\right) + \left(0.04\ \frac{\text{kg}}{\text{L}}\right)\left(10\ \frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{1000}\ \frac{\text{kg}}{\text{L}}\right)\left(15\ \frac{\text{L}}{\text{min}}\right)$$

$$= 0.25 + 0.40 - 0.015y = 0.65 - 0.015y = \frac{130 - 3y}{200}\ \frac{\text{kg}}{\text{min}}$$

so $\displaystyle\int \frac{dy}{130 - 3y} = \int \frac{dt}{200}$ and $-\frac{1}{3}\ln|130 - 3y| = \frac{1}{200}t + C$; since $y(0) = 0$, we have $-\frac{1}{3}\ln 130 = C$,

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so $-\frac{1}{3}\ln|130-3y| = \frac{1}{200}t - \frac{1}{3}\ln 130 \Rightarrow \ln|130-3y| = -\frac{3}{200}t + \ln 130 = \ln\left(130e^{-3t/200}\right)$, and $|130-3y| = 130e^{-3t/200}$. Since y is continuous, $y(0)=0$, and the right-hand side is never zero, we deduce that $130-3y$ is always positive. Thus, $130-3y = 130e^{-3t/200}$ and $y = \frac{130}{3}\left(1-e^{-3t/200}\right)$ kg.

(b) After one hour, $y = \frac{130}{3}\left(1-e^{-3\cdot 60/200}\right) = \frac{130}{3}\left(1-e^{-0.9}\right) \approx 25.7$ kg.

Note: As $t \to \infty$, $y(t) \to \frac{130}{3} = 43\frac{1}{3}$ kg.

41. Assume that the raindrop begins at rest, so that $v(0)=0$. $dm/dt = km$ and $(mv)' = gm \Rightarrow$ $mv' + vm' = gm \Rightarrow mv' + v(km) = gm \Rightarrow v' + vk = g \Rightarrow dv/dt = g - kv \Rightarrow$ $\int \frac{dv}{g-kv} = \int dt \Rightarrow -(1/k)\ln|g-kv| = t + C \Rightarrow \ln|g-kv| = -kt - kC \Rightarrow g - kv = Ae^{-kt}$. $v(0)=0 \Rightarrow A = g$. So $kv = g - ge^{-kt} \Rightarrow v = (g/k)\left(1-e^{-kt}\right)$. Since $k>0$, as $t\to\infty$, $e^{-kt}\to 0$ and therefore, $\lim_{t\to\infty} v(t) = g/k$.

42. (a) $m\frac{dv}{dt} = -kv \Rightarrow \frac{dv}{v} = -\frac{k}{m}\,dt \Rightarrow \ln|v| = -\frac{k}{m}t + C$. Since $v(0)=v_0$, $\ln|v_0| = C$. Therefore, $\ln\left|\frac{v}{v_0}\right| = -\frac{k}{m}t \Rightarrow \left|\frac{v}{v_0}\right| = e^{-kt/m} \Rightarrow v(t) = \pm v_0 e^{-kt/m}$. The sign is $+$ when $t=0$, and we assume v is continuous, so that the sign is $+$ for all t. Thus, $v(t) = v_0e^{-kt/m}$. $ds/dt = v_0e^{-kt/m} \Rightarrow$ $s(t) = -\frac{mv_0}{k}e^{-kt/m} + C'$. From $s(0)=s_0$, we get $s_0 = -\frac{mv_0}{k} + C'$, so $C' = s_0 + \frac{mv_0}{k}$ and $s(t) = s_0 + \frac{mv_0}{k}\left(1-e^{-kt/m}\right)$. The distance traveled from time 0 to time t is $s(t)-s_0$, so the total distance traveled is $\lim_{t\to\infty}[s(t)-s_0] = \frac{mv_0}{k}$.

Note: In finding the limit, we use the fact that $k>0$ to conclude that $\lim_{t\to\infty} e^{-kt/m} = 0$.

(b) $m\frac{dv}{dt} = -kv^2 \Rightarrow \frac{dv}{v^2} = -\frac{k}{m}\,dt \Rightarrow \frac{-1}{v} = -\frac{kt}{m} + C \Rightarrow \frac{1}{v} = \frac{kt}{m} - C$. Since $v(0)=v_0$, $C = -\frac{1}{v_0}$ and $\frac{1}{v} = \frac{kt}{m} + \frac{1}{v_0}$. Therefore, $v(t) = \frac{1}{kt/m + 1/v_0} = \frac{mv_0}{kv_0t+m}$. $\frac{ds}{dt} = \frac{mv_0}{kv_0t+m} \Rightarrow$

$s(t) = \frac{m}{k}\int \frac{kv_0\,dt}{kv_0t+m} = \frac{m}{k}\ln|kv_0t+m| + C'$. Since $s(0)=s_0$, we get $s_0 = \frac{m}{k}\ln m + C' \Rightarrow$

$C' = s_0 - \frac{m}{k}\ln m \Rightarrow s(t) = s_0 + \frac{m}{k}(\ln|kv_0t+m| - \ln m) = s_0 + \frac{m}{k}\ln\left|\frac{kv_0t+m}{m}\right|$. We can rewrite the formulas for $v(t)$ and $s(t)$ as $v(t) = \frac{v_0}{1+(kv_0/m)t}$ and $s(t) = s_0 + \frac{m}{k}\ln\left|1+\frac{kv_0}{m}t\right|$.

Remarks: This model of horizontal motion through a resistive medium was designed to handle the case in which $v_0>0$. Then the term $-kv^2$ representing the resisting force causes the object to decelerate. The absolute value in the expression for $s(t)$ is unnecessary (since k, v_0, and m are all positive), and $\lim_{t\to\infty} s(t) = \infty$. In other words, the object travels infinitely far. However, $\lim_{t\to\infty} v(t) = 0$. When $v_0<0$, the term $-kv^2$ increases the magnitude of the object's negative velocity. According to the formula for $s(t)$, the position of the object

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approaches $-\infty$ as t approaches $m/k(-v_0)$: $\lim_{t\to -m/(kv_0)} s(t) = -\infty$. Again the object travels infinitely far, but this time the feat is accomplished in a finite amount of time. Notice also that $\lim_{t\to -m/(kv_0)} v(t) = -\infty$ when $v_0 < 0$, showing that the speed of the object increases without limit.

43. (a) The rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M - A(t)$; that is, the rate is proportional to the product of those two quantities. So for some constant k, $dA/dt = k\sqrt{A}\,(M - A)$. We are interested in the maximum of the function dA/dt (when the tissue grows the fastest), so we differentiate, using the Chain Rule and then substituting for dA/dt from the differential equation:

$$\frac{d}{dt}\left(\frac{dA}{dt}\right) = k\left[\sqrt{A}\,(-1)\frac{dA}{dt} + (M-A)\cdot \tfrac{1}{2}A^{-1/2}\frac{dA}{dt}\right] = \tfrac{1}{2}kA^{-1/2}\frac{dA}{dt}\left[-2A + (M-A)\right]$$

$$= \tfrac{1}{2}kA^{-1/2}\left[k\sqrt{A}(M-A)\right][M-3A] = \tfrac{1}{2}k^2(M-A)(M-3A)$$

This is 0 when $M - A = 0$ [this situation never actually occurs, since the graph of $A(t)$ is asymptotic to the line $y = M$, as in the logistic model] and when $M - 3A = 0 \;\Leftrightarrow\; A(t) = M/3$. This represents a maximum by the First Derivative Test, since $\dfrac{d}{dt}\left(\dfrac{dA}{dt}\right)$ goes from positive to negative when $A(t) = M/3$.

(b) From the CAS, we get $A(t) = M\left(\dfrac{Ce^{\sqrt{M}kt} - 1}{Ce^{\sqrt{M}kt} + 1}\right)^2$. To get C in terms of the initial area A_0 and the maximum area M, we substitute $t = 0$ and $A = A_0 = A(0)$: $A_0 = M\left(\dfrac{C-1}{C+1}\right)^2 \;\Leftrightarrow\; (C+1)\sqrt{A_0} = (C-1)\sqrt{M}$

$\Leftrightarrow\; C\sqrt{A_0} + \sqrt{A_0} = C\sqrt{M} - \sqrt{M} \;\Leftrightarrow\; \sqrt{M} + \sqrt{A_0} = C\sqrt{M} - C\sqrt{A_0} \;\Leftrightarrow$

$\sqrt{M} + \sqrt{A_0} = C\left(\sqrt{M} - \sqrt{A_0}\right) \;\Leftrightarrow\; C = \dfrac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}}$. (Notice that if $A_0 = 0$, then $C = 1$.)

44. (a) According to the hint we use the Chain Rule: $m\dfrac{dv}{dt} = m\,\dfrac{dv}{dx}\cdot\dfrac{dx}{dt} = mv\,\dfrac{dv}{dx} = -\dfrac{mgR^2}{(x+R)^2} \;\Rightarrow$

$\displaystyle\int v\,dv = \int \frac{-gR^2\,dx}{(x+R)^2} \;\Rightarrow\; \frac{v^2}{2} = \frac{gR^2}{x+R} + C$. When $x = 0$, $v = v_0$, so $\dfrac{v_0^2}{2} = \dfrac{gR^2}{0+R} + C \;\Rightarrow$

$C = \frac{1}{2}v_0^2 - gR \;\Rightarrow\; \frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \dfrac{gR^2}{x+R} - gR$. Now at the top of its flight, the rocket's velocity will be 0, and its height will be $x = h$. Solving for v_0: $-\frac{1}{2}v_0^2 = \dfrac{gR^2}{h+R} - gR \;\Rightarrow$

$$\frac{v_0^2}{2} = g\left[-\frac{R^2}{R+h} + \frac{R(R+h)}{R+h}\right] = \frac{gRh}{R+h} \;\Rightarrow\; v_0 = \sqrt{\frac{2gRh}{R+h}}.$$

(b) $v_e = \lim_{h\to\infty} v_0 = \lim_{h\to\infty}\sqrt{\dfrac{2gRh}{R+h}} = \lim_{h\to\infty}\sqrt{\dfrac{2gR}{(R/h)+1}} = \sqrt{2gR}$

(c) $v_e = \sqrt{2\cdot 32\ \text{ft/s}^2\cdot 3960\ \text{mi}\cdot 5280\ \text{ft/mi}} \approx 36{,}581\ \text{ft/s} \approx 6.93\ \text{mi/s}$

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APPLIED PROJECT How Fast Does a Tank Drain?

1. (a) $V = \pi r^2 h \quad \Rightarrow \quad \dfrac{dV}{dt} = \pi r^2 \dfrac{dh}{dt}$ [implicit differentiation] $\Rightarrow$

$$\frac{dh}{dt} = \frac{1}{\pi r^2}\frac{dV}{dt} = \frac{1}{\pi r^2}\left(-a\sqrt{2gh}\right) = \frac{1}{\pi 2^2}\left[-\pi\left(\tfrac{1}{12}\right)^2\sqrt{2\cdot 32}\sqrt{h}\right] = -\tfrac{1}{72}\sqrt{h}$$

(b) $\dfrac{dh}{dt} = -\tfrac{1}{72}\sqrt{h} \quad \Rightarrow \quad h^{-1/2}\,dh = -\tfrac{1}{72}\,dt \quad \Rightarrow \quad 2\sqrt{h} = -\tfrac{1}{72}t + C.$

$h(0) = 6 \quad \Rightarrow \quad 2\sqrt{6} = 0 + C \quad \Rightarrow \quad C = 2\sqrt{6} \quad \Rightarrow \quad h(t) = \left(-\tfrac{1}{144}t + \sqrt{6}\right)^2.$

(c) We want to find t when $h = 0$, so we set $h = 0 = \left(-\tfrac{1}{144}t + \sqrt{6}\right)^2 \quad \Rightarrow \quad t = 144\sqrt{6} \approx 5$ min 53 s.

2. (a) $\dfrac{dh}{dt} = k\sqrt{h} \quad \Rightarrow \quad h^{-1/2}\,dh = k\,dt \quad [h \neq 0] \quad \Rightarrow \quad 2\sqrt{h} = kt + C$

$\Rightarrow \quad h(t) = \tfrac{1}{4}(kt + C)^2$. Since $h(0) = 10$ cm, the relation $2\sqrt{h(t)} = kt + C$ gives us $2\sqrt{10} = C$. Also, $h(68) = 3$ cm, so $2\sqrt{3} = 68k + 2\sqrt{10}$ and $k = -\dfrac{\sqrt{10} - \sqrt{3}}{34}$. Thus,

$$h(t) = \frac{1}{4}\left(2\sqrt{10} - \frac{\sqrt{10} - \sqrt{3}}{34}t\right)^2 \approx 10 - 0.133t + 0.00044t^2.$$

Here is a table of values of $h(t)$ correct to one decimal place.

t (in s)	$h(t)$ (in cm)
10	8.7
20	7.5
30	6.4
40	5.4
50	4.5
60	3.6

(b) The answers to this part are to be obtained experimentally. See the article by Tom Farmer and Fred Gass, *Physical Demonstrations in the Calculus Classroom*, College Mathematics Journal 1992, pp. 146–148.

3. $V(t) = \pi r^2 h(t) = 100\pi h(t) \quad \Rightarrow \quad \dfrac{dV}{dh} = 100\pi$ and $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} = 100\pi\dfrac{dh}{dt}$.

Diameter $= 2.5$ inches $\quad \Rightarrow \quad$ radius $= 1.25$ inches $= \tfrac{5}{4}\cdot\tfrac{1}{12}$ foot $= \tfrac{5}{48}$ foot. Thus,

$$\frac{dV}{dt} = -a\sqrt{2gh} \quad \Rightarrow \quad 100\pi\frac{dh}{dt} = -\pi\left(\tfrac{5}{48}\right)^2\sqrt{2\cdot 32h} = -\frac{25\pi}{288}\sqrt{h} \quad \Rightarrow \quad \frac{dh}{dt} = -\frac{\sqrt{h}}{1152} \quad \Rightarrow$$

$$\int h^{-1/2}\,dh = \int -\tfrac{1}{1152}\,dt \quad \Rightarrow \quad 2\sqrt{h} = -\tfrac{1}{1152}t + C \quad \Rightarrow \quad \sqrt{h} = -\tfrac{1}{2304}t + k \quad \Rightarrow \quad h(t) = \left(-\tfrac{1}{2304}t + k\right)^2.$$

The water pressure after t seconds is $62.5h(t)$ lb/ft^2, so the condition that the pressure be at least 2160 lb/ft^2 for 10 minutes (600 seconds) is the condition $62.5 \cdot h(600) \geq 2160$; that is, $\left(k - \tfrac{600}{2304}\right)^2 \geq \tfrac{2160}{62.5} \quad \Rightarrow$

$\left|k - \tfrac{25}{96}\right| \geq \sqrt{34.56} \quad \Rightarrow \quad k \geq \tfrac{25}{96} + \sqrt{34.56}$. Now $h(0) = k^2$, so the height of the tank should be at least $\left(\tfrac{25}{96} + \sqrt{34.56}\right)^2 \approx 37.69$ ft.

4. (a) If the radius of the circular cross-section at height h is r, then the Pythagorean Theorem gives $r^2 = 2^2 - (2 - h)^2$ since the radius of the tank is 2 m. So $A(h) = \pi r^2 = \pi\left[4 - (2 - h)^2\right] = \pi(4h - h^2)$.

Thus, $A(h)\dfrac{dh}{dt} = -a\sqrt{2gh} \quad \Rightarrow \quad \pi(4h - h^2)\dfrac{dh}{dt} = -\pi(0.01)^2\sqrt{2\cdot 10h} \quad \Rightarrow$

$(4h - h^2)\dfrac{dh}{dt} = -0.0001\sqrt{20h}.$

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(b) From part (a) we have $\left(4h^{1/2} - h^{3/2}\right)dh = \left(-0.0001\sqrt{20}\right)dt \Rightarrow$

$\frac{8}{3}h^{3/2} - \frac{2}{5}h^{5/2} = \left(-0.0001\sqrt{20}\right)t + C$. $h(0) = 2 \Rightarrow \frac{8}{3}(2)^{3/2} - \frac{2}{5}(2)^{5/2} = C \Rightarrow$

$C = \left(\frac{16}{3} - \frac{8}{5}\right)\sqrt{2} = \frac{56}{15}\sqrt{2}$. To find out how long it will take to drain all the water we evaluate t when $h = 0$:

$0 = \left(-0.0001\sqrt{20}\right)t + C \Rightarrow$

$$t = \frac{C}{0.0001\sqrt{20}} = \frac{56\sqrt{2}/15}{0.0001\sqrt{20}} = \frac{11{,}200\sqrt{10}}{3} \approx 11{,}806 \text{ s} \approx 3 \text{ h } 17 \text{ min.}$$

APPLIED PROJECT Which is Faster, Going Up or Coming Down?

1. $mv' = -pv - mg \Rightarrow m\frac{dv}{dt} = -(pv + mg) \Rightarrow \int \frac{dv}{pv + mg} = \int -\frac{1}{m}\,dt \Rightarrow$

$\frac{1}{p}\ln(pv + mg) = -\frac{1}{m}t + C$ $[pv + mg > 0]$. At $t = 0$, $v = v_0$, so $C = \frac{1}{p}\ln(pv_0 + mg)$. Thus,

$\frac{1}{p}\ln(pv + mg) = -\frac{1}{m}t + \frac{1}{p}\ln(pv_0 + mg) \Rightarrow \ln(pv + mg) = -\frac{p}{m}t + \ln(pv_0 + mg) \Rightarrow$

$pv + mg = e^{-pt/m}(pv_0 + mg) \Rightarrow pv = (pv_0 + mg)e^{-pt/m} - mg \Rightarrow$

$v(t) = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}$.

2. $y(t) = \int v(t)\,dt = \int \left[\left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}\right]dt = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m}\left(-\frac{m}{p}\right) - \frac{mg}{p}t + C$

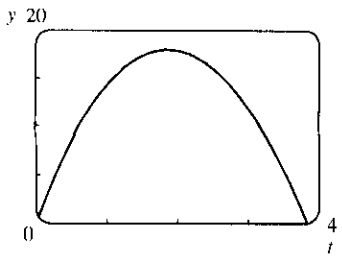
At $t = 0$, $y = 0$, so $C = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}$. Thus,

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p} - \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}e^{-pt/m} - \frac{mgt}{p} = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{-pt/m}\right) - \frac{mgt}{p}$$

3. $v(t) = 0 \Rightarrow \frac{mg}{p} = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} \Rightarrow e^{pt/m} = \frac{pv_0}{mg} + 1 \Rightarrow \frac{pt}{m} = \ln\left(\frac{pv_0}{mg} + 1\right) \Rightarrow$

$t_1 = \frac{m}{p}\ln\left(\frac{mg + pv_0}{mg}\right)$. With $m = 1$, $v_0 = 20$, $p = \frac{1}{10}$, and $g = 9.8$, we have $t_1 = 10\ln\left(\frac{11.8}{9.8}\right) \approx 1.86$ s.

4.



The figure shows the graph of $y = 1180\left(1 - e^{-0.1t}\right) - 98t$. The zeros are at $t = 0$ and $t_2 \approx 3.84$. Thus, $t_1 - 0 \approx 1.86$ and $t_2 - t_1 \approx 1.98$. So the time it takes to come down is about 0.12 s longer than the time it takes to go up; hence, going up is faster.

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5. $y(2t_1) = \left(v_0 + \dfrac{mg}{p}\right)\dfrac{m}{p}\left(1 - e^{-2pt_1/m}\right) - \dfrac{mg}{p}\cdot 2t_1$

$= \left(\dfrac{pv_0 + mg}{p}\right)\dfrac{m}{p}\left[1 - \left(e^{pt_1/m}\right)^{-2}\right] - \dfrac{mg}{p}\cdot 2\dfrac{m}{p}\ln\left(\dfrac{pv_0 + mg}{mg}\right)$

Substituting $x = e^{pt_1/m} = \dfrac{pv_0}{mg} + 1 = \dfrac{pv_0 + mg}{mg}$ (from Problem 3), we get

$y(2t_1) = \left(x\cdot\dfrac{mg}{p}\right)\dfrac{m}{p}\left(1 - x^{-2}\right) - \dfrac{m^2g}{p^2}\cdot 2\ln x = \dfrac{m^2g}{p^2}\left(x - \dfrac{1}{x} - 2\ln x\right)$. Now $p > 0, m > 0, t_1 > 0 \Rightarrow$

$x = e^{pt_1/m} > e^0 = 1$. $f(x) = x - \dfrac{1}{x} - 2\ln x \Rightarrow f'(x) = 1 + \dfrac{1}{x^2} - \dfrac{2}{x} = \dfrac{x^2 - 2x + 1}{x^2} = \dfrac{(x-1)^2}{x^2} > 0$ for

$x > 1 \Rightarrow f(x)$ is increasing for $x > 1$. Since $f(1) = 0$, it follows that $f(x) > 0$ for every $x > 1$. Therefore,

$y(2t_1) = \dfrac{m^2g}{p^2}f(x)$ is positive, which means that the ball has not yet reached the ground at time $2t_1$. This tells us that the time spent going up is always less than the time spent coming down, so *ascent is faster*.

10.4 Exponential Growth and Decay

1. The relative growth rate is $\dfrac{1}{P}\dfrac{dP}{dt} = 0.7944$, so $\dfrac{dP}{dt} = 0.7944P$ and, by Theorem 2, $P(t) = P(0)e^{0.7944t} = 2e^{0.7944t}$. Thus, $P(6) = 2e^{0.7944(6)} \approx 234.99$ or about 235 members.

2. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 60e^{kt}$. In 20 minutes ($\frac{1}{3}$ hour), there are 120 cells, so $P\left(\frac{1}{3}\right) = 60e^{k/3} = 120 \Rightarrow e^{k/3} = 2 \Rightarrow k/3 = \ln 2 \Rightarrow k = 3\ln 2 = \ln\left(2^3\right) = \ln 8$.

(b) $P(t) = 60e^{(\ln 8)t} = 60\cdot 8^t$

(c) $P(8) = 60\cdot 8^8 = 60\cdot 2^{24} = 1{,}006{,}632{,}960$

(d) $dP/dt = kP \Rightarrow P'(8) = kP(8) = (\ln 8)P(8) \approx 2.093$ billion cells/h

(e) $P(t) = 20{,}000 \Rightarrow 60\cdot 8^t = 20{,}000 \Rightarrow 8^t = 1000/3 \Rightarrow t\ln 8 = \ln(1000/3) \Rightarrow$ $t = \dfrac{\ln(1000/3)}{\ln 8} \approx 2.79$ h

3. (a) By Theorem 2, $y(t) = y(0)e^{kt} = 500e^{kt}$. Now $y(3) = 500e^{k(3)} = 8000 \Rightarrow e^{3k} = \frac{8000}{500} \Rightarrow$ $3k = \ln 16 \Rightarrow k = (\ln 16)/3$. So $y(t) = 500e^{(\ln 16)t/3} = 500\cdot 16^{t/3}$

(b) $y(4) = 500\cdot 16^{4/3} \approx 20{,}159$

(c) $dy/dt = ky \Rightarrow y'(4) = ky(4) = \frac{1}{3}\ln 16\left(500\cdot 16^{4/3}\right)$ [from part (a)] $\approx 18{,}631$ cells/h

(d) $y(t) = 500\cdot 16^{t/3} = 30{,}000 \Rightarrow 16^{t/3} = 60 \Rightarrow \frac{1}{3}t\ln 16 = \ln 60 \Rightarrow t = 3(\ln 60)/(\ln 16) \approx 4.4$ h

4. (a) $y(t) = y(0)e^{kt} \Rightarrow y(2) = y(0)e^{2k} = 600$, $y(8) = y(0)e^{8k} = 75{,}000$. Dividing these equations, we get $e^{8k}/e^{2k} = 75{,}000/600 \Rightarrow e^{6k} = 125 \Rightarrow 6k = \ln 125 = \ln 5^3 = 3\ln 5 \Rightarrow k = \frac{3}{6}\ln 5 = \frac{1}{2}\ln 5$. Thus, $y(0) = 600/e^{2k} = 600/e^{\ln 5} = \frac{600}{5} = 120$.

(b) $y(t) = y(0)e^{kt} = 120e^{(\ln 5)t/2}$ or $y = 120\cdot 5^{t/2}$

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(c) $y(5) = 120 \cdot 5^{5/2} = 120 \cdot 25\sqrt{5} = 3000\sqrt{5} \approx 6708$ bacteria.

(d) $y(t) = 120 \cdot 5^{t/2} \quad \Rightarrow \quad y'(t) = 120 \cdot 5^{t/2} \cdot \ln 5 \cdot \frac{1}{2} = 60 \cdot \ln 5 \cdot 5^{t/2}$.
$y'(5) = 60 \cdot \ln 5 \cdot 5^{5/2} = 60 \cdot \ln 5 \cdot 25\sqrt{5} \approx 5398$ bacteria/hour.

(e) $y(t) = 200{,}000 \quad \Leftrightarrow \quad 120e^{(\ln 5)t/2} = 200{,}000 \quad \Leftrightarrow \quad e^{(\ln 5)t/2} = \frac{5000}{3} \quad \Leftrightarrow \quad (\ln 5)t/2 = \ln\frac{5000}{3} \quad \Leftrightarrow$
$t = \left(2\ln\frac{5000}{3}\right)/\ln 5 \approx 9.2$ h.

5. (a) Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t - 1750$ for t in Theorem 2, so the exponential model gives $P(t) = P(1750)e^{k(t-1750)}$. Then $P(1800) = 980 = 790e^{k(1800-1750)} \quad \Rightarrow \quad \frac{980}{790} = e^{k(50)} \quad \Rightarrow \quad \ln\frac{980}{790} = 50k \quad \Rightarrow$ $k = \frac{1}{50}\ln\frac{980}{790} \approx 0.0043104$. So with this model, we have $P(1900) = 790e^{k(1900-1750)} \approx 1508$ million, and $P(1950) = 790e^{k(1950-1750)} \approx 1871$ million. Both of these estimates are much too low.

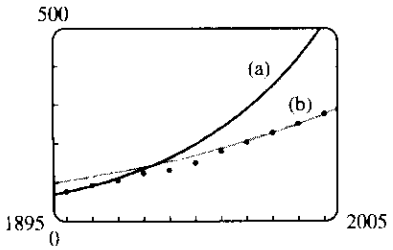
(b) In this case, the exponential model gives $P(t) = P(1850)e^{k(t-1850)} \quad \Rightarrow$ $P(1900) = 1650 = 1260e^{k(1900-1850)} \quad \Rightarrow \quad \ln\frac{1650}{1260} = k(50) \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{1650}{1260} \approx 0.005393$. So with this model, we estimate $P(1950) = 1260e^{k(1950-1850)} \approx 2161$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

(c) The exponential model gives $P(t) = P(1900)e^{k(t-1900)} \quad \Rightarrow \quad P(1950) = 2560 = 1650e^{k(1950-1900)} \quad \Rightarrow$ $\ln\frac{2560}{1650} = k(50) \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{2560}{1650} \approx 0.008785$. With this model, we estimate $P(2000) = 1650e^{k(2000-1900)} \approx 3972$ million. This is much too low. The discrepancy is explained by the fact that tbe world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in tbe first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

6. (a) Let $P(t)$ be the population (in millions) in the year t. Since the initial time is the year 1900, we substitute $t - 1900$ for t in Theorem 2, and find that the exponential model gives $P(t) = P(1900)e^{k(t-1900)} \quad \Rightarrow$ $P(1910) = 92 = 76e^{k(1910-1900)} \quad \Rightarrow \quad k = \frac{1}{10}\ln\frac{92}{76} \approx 0.0191$. With this model, we estimate $P(2000) = 76e^{k(2000-1900)} \approx 514$ million. This estimate is much too higb. The discrepancy is explained by the fact that, between the years 1900 and 1910, an enormous number of immigrants (compared to the total population) came to the United States. Since that time, immigration (as a proportion of total population) has been much lower. Also, the birth rate in the United States has declined since the turn of the century. So our calculation of the constant k was based partly on factors which no longer exist.

(b) Substituting $t - 1980$ for t in Theorem 2, we find that the exponential model gives $P(t) = P(1980)e^{k(t-1980)}$ $\Rightarrow \quad P(1990) = 250 = 227e^{k(1990-1980)} \quad \Rightarrow \quad k = \frac{1}{10}\ln\frac{250}{227} \approx 0.00965$. With this model, we estimate $P(2000) = 227e^{k(2000-1980)} \approx 275.3$ million. This is quite accurate. The further estimates are $P(2010) = 227e^{30k} \approx 303$ million and $P(2020) = 227e^{40k} \approx 334$ million.

(c)



The model in part (a) is quite inaccurate after 1910 (off by 5 million in 1920 and 12 million in 1930). The model in part (b) is more accurate (which is not surprising, since it is based on more recent information).

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7. (a) If $y = [N_2O_5]$ then by Theorem 2, $\dfrac{dy}{dt} = -0.0005y \;\Rightarrow\; y(t) = y(0)e^{-0.0005t} = Ce^{-0.0005t}$.

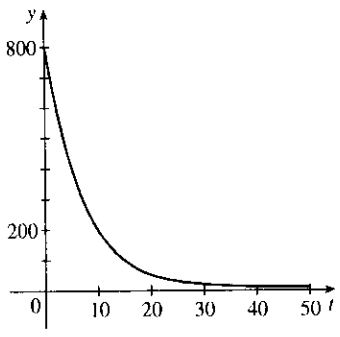
(b) $y(t) = Ce^{-0.0005t} = 0.9C \;\Rightarrow\; e^{-0.0005t} = 0.9 \;\Rightarrow\; -0.0005t = \ln 0.9 \;\Rightarrow$
$t = -2000 \ln 0.9 \approx 211$ s

8. (a) The mass remaining after t days is
$y(t) = y(0)e^{kt} = 800e^{kt}$. Since the half-life is 5.0 days,
$y(5) = 800e^{5k} = 400 \;\Rightarrow\; e^{5k} = \frac{1}{2} \;\Rightarrow$
$5k = \ln \frac{1}{2} \;\Rightarrow\; k = -(\ln 2)/5$, so
$y(t) = 800e^{-(\ln 2)t/5} = 800 \cdot 2^{-t/5}$.

(b) $y(30) = 800 \cdot 2^{-30/5} = 12.5$ mg

(c) $800e^{-(\ln 2)t/5} = 1 \;\Leftrightarrow\; -(\ln 2)\,\frac{t}{5} = \ln \frac{1}{800} = -\ln 800$
$\Leftrightarrow\; t = 5\,\frac{\ln 800}{\ln 2} \approx 48$ days

(d)



9. (a) If $y(t)$ is the mass (in mg) remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$. $y(30) = 100e^{30k} = \frac{1}{2}(100)$
$\Rightarrow\; e^{30k} = \frac{1}{2} \;\Rightarrow\; k = -(\ln 2)/30 \;\Rightarrow\; y(t) = 100e^{-(\ln 2)t/30} = 100 \cdot 2^{-t/30}$

(b) $y(100) = 100 \cdot 2^{-100/30} \approx 9.92$ mg

(c) $100e^{-(\ln 2)t/30} = 1 \;\Rightarrow\; -(\ln 2)t/30 = \ln \frac{1}{100} \;\Rightarrow\; t = -30\,\frac{\ln 0.01}{\ln 2} \approx 199.3$ years

10. (a) If $y(t)$ is the mass after t days and $y(0) = A$, then $y(t) = Ae^{kt}$. $y(3) = Ae^{3k} = 0.58A \;\Rightarrow$
$e^{3k} = 0.58 \;\Rightarrow\; 3k = \ln 0.58 \;\Rightarrow\; k = \frac{1}{3}\ln 0.58$. Then $Ae^{(\ln 0.58)t/3} = \frac{1}{2}A \;\Leftrightarrow$
$\ln e^{(\ln 0.58)t/3} = \ln \frac{1}{2} \;\Leftrightarrow\; \dfrac{(\ln 0.58)t}{3} = \ln \frac{1}{2}$, so the half-life is $t = -\dfrac{3\ln 2}{\ln 0.58} \approx 3.82$ days.

(b) $Ae^{(\ln 0.58)t/3} = 0.10A \;\Leftrightarrow\; \dfrac{(\ln 0.58)t}{3} = \ln \frac{1}{10} \;\Leftrightarrow\; t = -\dfrac{3\ln 10}{\ln 0.58} \approx 12.68$ days

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t) = y(0)e^{-kt}$ and k is determined by using the half-life:
$y(5730) = \frac{1}{2}y(0) \;\Rightarrow\; y(0)e^{-k(5730)} = \frac{1}{2}y(0) \;\Rightarrow\; e^{-5730k} = \frac{1}{2} \;\Rightarrow$
$-5730k = \ln \frac{1}{2} \;\Rightarrow\; k = -\dfrac{\ln \frac{1}{2}}{5730} = \dfrac{\ln 2}{5730}$. If 74% of the ^{14}C remains, then we know that $y(t) = 0.74y(0)$
$\Rightarrow\; 0.74 = e^{-t(\ln 2)/5730} \;\Rightarrow\; \ln 0.74 = -\dfrac{t\ln 2}{5730} \;\Rightarrow\; t = -\dfrac{5730(\ln 0.74)}{\ln 2} \approx 2489 \approx 2500$ years.

12. From the information given, we know that $\dfrac{dy}{dx} = 2y \;\Rightarrow\; y = Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0, 5)$: $5 = Ce^{2(0)} \;\Rightarrow\; C = 5$. Thus, the equation of the curve is $y = 5e^{2x}$.

13. (a) Using Newton's Law of Cooling, $\dfrac{dT}{dt} = k(T - T_s)$, we have $\dfrac{dT}{dt} = k(T - 75)$.
Now let $y = T - 75$, so $y(0) = T(0) - 75 = 185 - 75 = 110$, so y is a solution of the initial-value problem $dy/dt = ky$ with $y(0) = 110$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 110e^{kt}$.
$y(30) = 110e^{30k} = 150 - 75 \;\Rightarrow\; e^{30k} = \frac{75}{110} = \frac{15}{22} \;\Rightarrow\; k = \frac{1}{30}\ln\frac{15}{22}$,
so $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}$ and $y(45) = 110e^{\frac{45}{30}\ln\left(\frac{15}{22}\right)} \approx 62°$F. Thus, $T(45) \approx 62 + 75 = 137°$F.

(b) $T(t) = 100 \;\Rightarrow\; y(t) = 25$. $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = 25 \;\Rightarrow\; e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = \frac{25}{110} \;\Rightarrow$
$\frac{1}{30}t\ln\frac{15}{22} = \ln\frac{25}{110} \;\Rightarrow\; t = \dfrac{30\ln\frac{25}{110}}{\ln\frac{15}{22}} \approx 116$ min.

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14. (a) Let $T(t)$ = temperature after t minutes. Newton's Law of Cooling implies that $\dfrac{dT}{dt} = k(T-5)$. Let $y(t) = T(t) - 5$. Then $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt} = 15e^{kt} \Rightarrow T(t) = 5 + 15e^{kt} \Rightarrow T(1) = 5 + 15e^{k} = 12 \Rightarrow e^{k} = \frac{7}{15} \Rightarrow k = \ln\frac{7}{15}$, so $T(t) = 5 + 15e^{\ln(7/15)t}$ and $T(2) = 5 + 15e^{2\ln(7/15)} \approx 8.3^\circ$C.

(b) $5 + 15e^{\ln(7/15)t} = 6$ when $e^{\ln(7/15)t} = \frac{1}{15} \Rightarrow \ln\left(\frac{7}{15}\right)t = \ln\frac{1}{15} \Rightarrow t = \dfrac{\ln\frac{1}{15}}{\ln\frac{7}{15}} \approx 3.6$ min.

15. $\dfrac{dT}{dt} = k(T-20)$. Letting $y = T - 20$, we get $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$.

$y(0) = T(0) - 20 = 5 - 20 = -15$, so $y(25) = y(0)e^{25k} = -15e^{25k}$, and

$y(25) = T(25) - 20 = 10 - 20 = -10$, so $-15e^{25k} = -10 \Rightarrow e^{25k} = \frac{2}{3}$. Thus, $25k = \ln\left(\frac{2}{3}\right)$ and

$k = \frac{1}{25}\ln\left(\frac{2}{3}\right)$, so $y(t) = y(0)e^{kt} = -15e^{(1/25)\ln(2/3)t}$. More simply, $e^{25k} = \frac{2}{3} \Rightarrow e^{k} = \left(\frac{2}{3}\right)^{1/25} \Rightarrow$

$e^{kt} = \left(\frac{2}{3}\right)^{t/25} \Rightarrow y(t) = -15 \cdot \left(\frac{2}{3}\right)^{t/25}$.

(a) $T(50) = 20 + y(50) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{50/25} = 20 - 15 \cdot \left(\frac{2}{3}\right)^{2} = 20 - \frac{20}{3} = 13.\overline{3}\,^\circ$C

(b) $15 = T(t) = 20 + y(t) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{t/25} \Rightarrow 15 \cdot \left(\frac{2}{3}\right)^{t/25} = 5 \Rightarrow \left(\frac{2}{3}\right)^{t/25} = \frac{1}{3} \Rightarrow$
$(t/25)\ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{3}\right) \Rightarrow t = 25\ln\left(\frac{1}{3}\right)/\ln\left(\frac{2}{3}\right) \approx 67.74$ min.

16. $\dfrac{dT}{dt} = k(T-20)$. Let $y = T - 20$. Then $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 95 - 20 = 75$, so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\dfrac{dT}{dt} = -1^\circ$C/min. Equivalently, $\dfrac{dy}{dt} = -1$ when $y(t) = 50$. Thus, $-1 = \dfrac{dy}{dt} = ky(t) = 50k$ and $50 = y(t) = 75e^{kt}$. The first relation implies $k = -1/50$, so the second relation says $50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \Rightarrow -t/50 = \ln\left(\frac{2}{3}\right) \Rightarrow t = -50\ln\left(\frac{2}{3}\right) \approx 20.27$ min.

17. (a) Let $P(h)$ be the pressure at altitude h. Then $dP/dh = kP \Rightarrow P(h) = P(0)e^{kh} = 101.3e^{kh}$.
$P(1000) = 101.3e^{1000k} = 87.14 \Rightarrow 1000k = \ln\left(\frac{87.14}{101.3}\right) \Rightarrow$
$k = \frac{1}{1000}\ln\left(\frac{87.14}{101.3}\right) \Rightarrow P(h) = 101.3\,e^{\frac{1}{1000}h\ln\left(\frac{87.14}{101.3}\right)}$, so $P(3000) = 101.3e^{3\ln\left(\frac{87.14}{101.3}\right)} \approx 64.5$ kPa.

(b) $P(6187) = 101.3\,e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

18. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 500$, $r = 0.14$, and $t = 2$, we have:

(i) Annually: $n = 1$; $A = 500\left(1 + \frac{0.14}{1}\right)^{1\cdot 2} = \$649.80$

(ii) Quarterly: $n = 4$; $A = 500\left(1 + \frac{0.14}{4}\right)^{4\cdot 2} = \$658.40$

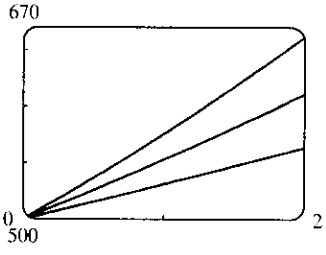
(iii) Monthly: $n = 12$; $A = 500\left(1 + \frac{0.14}{12}\right)^{12\cdot 2} = \$660.49$

(iv) Daily: $n = 365$; $A = 500\left(1 + \frac{0.14}{365}\right)^{365\cdot 2} = \$661.53$

(v) Hourly: $n = 365 \cdot 24$; $A = 500\left(1 + \frac{0.14}{365\cdot 24}\right)^{365\cdot 24\cdot 2} = \$661.56$

(vi) Continuously: $A = 500e^{(0.14)2} = \$661.56$

(b)



$A_{0.14}(2) = \$661.56$,
$A_{0.10}(2) = \$610.70$, and
$A_{0.06}(2) = \$563.75$.

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19. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 3000$, $r = 0.05$, and $t = 5$, we have:

(i) Annually: $n = 1$; $A = 3000\left(1 + \frac{0.05}{1}\right)^{1\cdot 5} = \$3828.84$

(ii) Semiannually: $n = 2$; $A = 3000\left(1 + \frac{0.05}{2}\right)^{2\cdot 5} = \$3840.25$

(iii) Monthly: $n = 12$; $A = 3000\left(1 + \frac{0.05}{12}\right)^{12\cdot 5} = \$3850.08$

(iv) Weekly: $n = 52$; $A = 3000\left(1 + \frac{0.05}{52}\right)^{52\cdot 5} = \$3851.61$

(v) Daily: $n = 365$; $A = 3000\left(1 + \frac{0.05}{365}\right)^{365\cdot 5} = \$3852.01$

(vi) Continuously: $A = 3000e^{(0.05)5} = \$3852.08$

(b) $dA/dt = 0.05A$ and $A(0) = 3000$.

20. (a) $A_0e^{0.06t} = 2A_0 \;\Leftrightarrow\; e^{0.06t} = 2 \;\Leftrightarrow\; 0.06t = \ln 2 \;\Leftrightarrow\; t = \frac{50}{3}\ln 2 \approx 11.55$, so the investment will double in about 11.55 years.

(b) The annual interest rate in $A = A_0(1 + r)^t$ is r. From part (a), we have $A = A_0e^{0.06t}$. These amounts must be equal, so $(1 + r)^t = e^{0.06t} \;\Rightarrow\; 1 + r = e^{0.06} \;\Rightarrow\; r = e^{0.06} - 1 \approx 0.0618 = 6.18\%$, which is the equivalent annual interest rate.

21. (a) $\dfrac{dP}{dt} = kP - m = k\left(P - \dfrac{m}{k}\right)$. Let $y = P - \dfrac{m}{k}$, so $\dfrac{dy}{dt} = \dfrac{dP}{dt}$ and the differential equation becomes $\dfrac{dy}{dt} = ky$. The solution is $y = y_0e^{kt} \;\Rightarrow\; P - \dfrac{m}{k} = \left(P_0 - \dfrac{m}{k}\right)e^{kt} \;\Rightarrow\; P(t) = \dfrac{m}{k} + \left(P_0 - \dfrac{m}{k}\right)e^{kt}$.

(b) Since $k > 0$, there will be an exponential expansion $\;\Leftrightarrow\; P_0 - \dfrac{m}{k} > 0 \;\Leftrightarrow\; m < kP_0$.

(c) The population will be constant if $P_0 - \dfrac{m}{k} = 0 \;\Leftrightarrow\; m = kP_0$. It will decline if $P_0 - \dfrac{m}{k} < 0 \;\Leftrightarrow\; m > kP_0$.

(d) $P_0 = 8{,}000{,}000$, $k = \alpha - \beta = 0.016$, $m = 210{,}000 \;\Rightarrow\; m > kP_0\ (= 128{,}000)$, so by part (c), the population was declining.

22. (a) $\dfrac{dy}{dt} = ky^{1+c} \;\Rightarrow\; y^{-1-c}\,dy = k\,dt \;\Rightarrow\; \dfrac{y^{-c}}{-c} = kt + C$. Since $y(0) = y_0$, we have $C = \dfrac{y_0^{-c}}{-c}$. Thus, $\dfrac{y^{-c}}{-c} = kt + \dfrac{y_0^{-c}}{-c}$, or $y^{-c} = y_0^{-c} - ckt$. So $y^c = \dfrac{1}{y_0^{-c} - ckt} = \dfrac{y_0^c}{1 - cy_0^ckt}$ and $y(t) = \dfrac{y_0}{(1 - cy_0^ckt)^{1/c}}$.

(b) $y(t) \to \infty$ as $1 - cy_0^ckt \to 0$, that is, as $t \to \dfrac{1}{cy_0^ck}$. Define $T = \dfrac{1}{cy_0^ck}$. Then $\lim\limits_{t\to T^-} y(t) = \infty$.

(c) According to the data given, we have $c = 0.01$, $y(0) = 2$, and $y(3) = 16$, where the time t is given in months. Thus, $y_0 = 2$ and $16 = y(3) = \dfrac{y_0}{(1 - cy_0^ck\cdot 3)^{1/c}}$. Since $T = \dfrac{1}{cy_0^ck}$, we will solve for cy_0^ck.

$16 = \dfrac{2}{(1 - 3cy_0^ck)^{100}} \;\Rightarrow\; 1 - 3cy_0^ck = \left(\frac{1}{8}\right)^{0.01} = 8^{-0.01} \;\Rightarrow\; cy_0^ck = \frac{1}{3}\left(1 - 8^{-0.01}\right)$. Thus, doomsday occurs when $t = T = \dfrac{1}{cy_0^ck} = \dfrac{3}{1 - 8^{-0.01}} \approx 145.77$ months or 12.15 years.

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APPLIED PROJECT Calculus and Baseball

1. (a) $F = ma = m\,\dfrac{dv}{dt}$, so by the Substitution Rule we have

$$\int_{t_0}^{t_1} F(t)\,dt = \int_{t_0}^{t_1} m\left(\frac{dv}{dt}\right)dt = m\int_{v_0}^{v_1} dv = [mv]_{v_0}^{v_1} = mv_1 - mv_0 = p(t_1) - p(t_0)$$

(b) (i) We have $v_1 = 110$ mi/h $= \frac{110(5280)}{3600}$ ft/s $= 161.\overline{3}$ ft/s, $v_0 = -90$ mi/h $= -132$ ft/s, and the mass of the baseball is $m = \dfrac{w}{g} = \frac{5/16}{32} = \frac{5}{512}$. So the change in momentum is

$p(t_1) - p(t_0) = mv_1 - mv_0 = \frac{5}{512}\left[161.\overline{3} - (-132)\right] \approx 2.86$ slug-ft/s.

(ii) From part (a) and part (b)(i), we have $\int_0^{0.001} F(t)dt = p(0.001) - p(0) \approx 2.86$, so the average force over the interval $[0, 0.001]$ is $\frac{1}{0.001}\int_0^{0.001} F(t)\,dt \approx \frac{1}{0.001}(2.86) = 2860$ lb.

2. (a) $W = \displaystyle\int_{s_0}^{s_1} F(s)\,ds$, where $F(s) = m\,\dfrac{dv}{dt} = m\,\dfrac{dv}{ds}\dfrac{ds}{dt} = mv\,\dfrac{dv}{ds}$ and so, by the Substitution Rule,

$$W = \int_{s_0}^{s_1} F(s)\,ds = \int_{s_0}^{s_1} mv\,\frac{dv}{ds}\,ds = \int_{v(s_0)}^{v(s_1)} mv\,dv = \left[\tfrac{1}{2}mv^2\right]_{v_0}^{v_1} = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2$$

(b) From part (b)(i), 90 mi/h $= 132$ ft/s. Assume $v_0 = v(s_0) = 0$ and $v_1 = v(s_1) = 132$ ft/s (note that s_1 is the point of release of the baseball). $m = \frac{5}{512}$, so the work done is

$$W = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}\cdot\tfrac{5}{512}\cdot(132)^2 \approx 85 \text{ ft-lb}$$

3. (a) Here we have a differential equation of the form $dv/dt = kv$, so by Theorem 10.4.2, the solution is $v(t) = v(0)e^{kt}$. In this case $k = -\frac{1}{10}$ and $v(0) = 100$ ft/s, so $v(t) = 100e^{-t/10}$. We are interested in the time t that the ball takes to travel 280 ft, so we find the distance function

$$s(t) = \int_0^t v(x)\,dx = \int_0^t 100e^{-x/10}\,dx = 100\Big[-10e^{-x/10}\Big]_0^t = -1000\Big(e^{-t/10} - 1\Big)$$

$$= 1000\Big(1 - e^{-t/10}\Big)$$

Now we set $s(t) = 280$ and solve for t: $280 = 1000\Big(1 - e^{-t/10}\Big) \;\Rightarrow\; 1 - e^{-t/10} = \frac{7}{25} \;\Rightarrow$

$-\frac{1}{10}t = \ln\left(1 - \frac{7}{25}\right) \;\Rightarrow\; t \approx 3.285$ seconds.

(b) Let x be the distance of the shortstop from home plate. We calculate the time for the ball to reach home plate as a function of x, then differentiate with respect to x to find the value of x which corresponds to the minimum time. The total time that it takes the ball to reach home is the sum of the times of the two throws, plus the relay time ($\frac{1}{2}$ s). The distance from the fielder to the shortstop is $280 - x$, so to find the time t_1 taken by the first throw, we solve the equation $s_1(t_1) = 280 - x \;\Leftrightarrow\; 1 - e^{-t_1/10} = \dfrac{280 - x}{1000} \;\Leftrightarrow\; t_1 = -10\ln\dfrac{720 + x}{1000}$.

We find the time t_2 taken by the second throw if the shortstop throws with velocity w, since we see that this velocity varies in the rest of the problem. We use $v = we^{-t/10}$ and isolate t_2 in the equation

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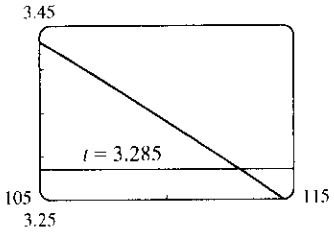
$s(t_2) = 10w\left(1 - e^{-t_2/10}\right) = x \quad \Leftrightarrow \quad e^{-t_2/10} = 1 - \dfrac{x}{10w} \quad \Leftrightarrow \quad t_2 = -10 \ln \dfrac{10w - x}{10w}$, so the total time is $t_w(x) = \dfrac{1}{2} - 10\left[\ln \dfrac{720 + x}{1000} + \ln \dfrac{10w - x}{10w}\right]$. To find the minimum, we differentiate:

$\dfrac{dt_w}{dx} = -10\left[\dfrac{1}{720 + x} - \dfrac{1}{10w - x}\right]$, which changes from negative to positive when $720 + x = 10w - x \quad \Leftrightarrow$ $x = 5w - 360$. By the First Derivative Test, t_w has a minimum at this distance from the shortstop to home plate. So if the shortstop throws at $w = 105$ ft/s from a point $x = 5(105) - 360 = 165$ ft from home plate, the minimum time is $t_{105}(165) = \frac{1}{2} - 10\left(\ln \frac{720 + 165}{1000} + \ln \frac{1050 - 165}{1050}\right) \approx 3.431$ seconds. This is longer than the time taken in part (a), so in this case the manager should encourage a direct throw.

If $w = 115$ ft/s, then $x = 215$ ft from home, and the minimum time is $t_{115}(215) = \frac{1}{2} - 10\left(\ln \frac{720 + 215}{1000} + \ln \frac{1150 - 215}{1150}\right) \approx 3.242$ seconds. This is less than the time taken in part (a), so in this case, the manager should encourage a relayed throw.

(c) In general, the minimum time is

$$t_w(5w - 360) = \frac{1}{2} - 10\left[\ln \frac{360 + 5w}{1000} + \ln \frac{360 + 5w}{10w}\right]$$
$$= \frac{1}{2} - 10 \ln \frac{(w + 72)^2}{400w}$$



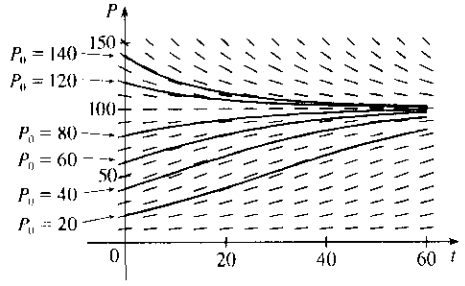
We want to find out when this is about 3.285 seconds, the same time as the direct throw. From the graph, we estimate that this is the case for $w \approx 112.8$ ft/s. So if the shortstop can throw the ball with this velocity, then a relayed throw takes the same time as a direct throw.

10.5 The Logistic Equation

1. (a) $dP/dt = 0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100)$. Comparing to Equation 1, $dP/dt = kP(1 - P/K)$, we see that the carrying capacity is $K = 100$ and the value of k is 0.05.

(b) The slopes close to 0 occur where P is near 0 or 100. The largest slopes appear to be on the line $P = 50$. The solutions are increasing for $0 < P_0 < 100$ and decreasing for $P_0 > 100$.

(c)



All of the solutions approach $P = 100$ as t increases. As in part (b), the solutions differ since for $0 < P_0 < 100$ they are increasing, and for $P_0 > 100$ they are decreasing. Also, some have an IP and some don't. It appears that the solutions which have $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$.

(d) The equilibrium solutions are $P = 0$ (trivial solution) and $P = 100$. The increasing solutions move away from $P = 0$ and all nonzero solutions approach $P = 100$ as $t \to \infty$.

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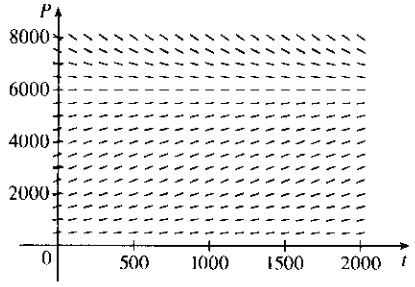
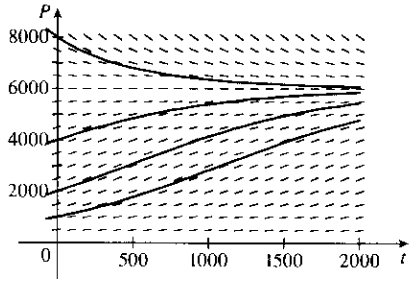
2. (a) $K = 6000$ and $k = 0.0015 \quad \Rightarrow \quad dP/dt = 0.0015P(1 - P/6000)$.

(b)



All of the solution curves approach 6000 as $t \to \infty$.

(c)

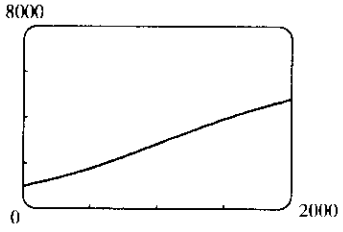


The curves with $P_0 = 1000$ and $P_0 = 2000$ appear to be concave upward at first and then concave downward. The curve with $P_0 = 4000$ appears to be concave downward everywbere. The curve with $P_0 = 8000$ appears to be concave upward everywhere. The inflection points are where the population grows the fastest.

(d) See the solution to Exercise 10.2.25 for a possible program to calculate $P(50)$. [In this case, we use X = 0, H = 1, N = 50, $Y_1 = 0.0015y(1 - y/6000)$, and Y = 1000.] We find that $P(50) \approx 1064$.

(e) Using Equation 4 with $K = 6000$, $k = 0.0015$, and $P_0 = 1000$, we have $P(t) = \dfrac{K}{1 + Ae^{-kt}} = \dfrac{6000}{1 + Ae^{-0.0015t}}$, where $A = \dfrac{K - P_0}{P_0} = \dfrac{6000 - 1000}{1000} = 5$. Thus, $P(50) = \dfrac{6000}{1 + 5e^{-0.0015(50)}} \approx 1064.1$, which is extremely close to the estimate obtained in part (d).

(f)



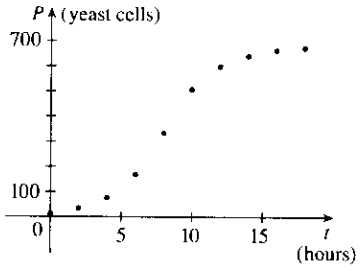
The curves are very similar.

3. (a) $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{K}\right) \quad \Rightarrow \quad y(t) = \dfrac{K}{1 + Ae^{-kt}}$ with $A = \dfrac{K - y(0)}{y(0)}$. With $K = 8 \times 10^7$, $k = 0.71$, and $y(0) = 2 \times 10^7$, we get the model $y(t) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}}$, so $y(1) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.23 \times 10^7$ kg.

(b) $y(t) = 4 \times 10^7 \quad \Rightarrow \quad \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \quad \Rightarrow \quad 2 = 1 + 3e^{-0.71t} \quad \Rightarrow \quad e^{-0.71t} = \frac{1}{3} \quad \Rightarrow$

$-0.71t = \ln \frac{1}{3} \quad \Rightarrow \quad t = \dfrac{\ln 3}{0.71} \approx 1.55$ years

4. (a)



From the graph, we estimate the carrying capacity K for the yeast population to be 680.

(b) An estimate of the initial relative growth rate is

$$\frac{1}{P_0}\frac{dP}{dt} = \frac{1}{18} \cdot \frac{39 - 18}{2 - 0} = \frac{7}{12} = 0.58\overline{3}.$$

(c) An exponential model is $P(t) = 18e^{7t/12}$. A logistic model is $P(t) = \dfrac{680}{1 + Ae^{-7t/12}}$, where $A = \frac{680 - 18}{18} = \frac{331}{9}$.

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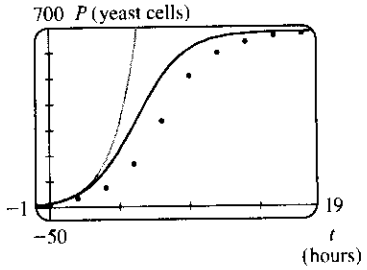
(d)

Time in Hours	Observed Values	Exponential Model	Logistic Model
0	18	18	18
2	39	58	55
4	80	186	149
6	171	596	322
8	336	1914	505
10	509	6147	614
12	597	19,739	658
14	640	63,389	673
16	664	203,558	678
18	672	653,679	679



The exponential model is a poor fit for anything beyond the first two observed values.
The logistic model varies more for the middle values than it does for the values at either end, but provides a good general fit, as shown in the figure.

(e) $P(7) = \dfrac{680}{1 + \frac{331}{9}e^{-7(7/12)}} \approx 420$ yeast cells

5. (a) We will assume that the difference in the birth and death rates is 20 million/year. Let $t = 0$ correspond to the year 1990 and use a unit of 1 billion for all calculations. $k \approx \dfrac{1}{P}\dfrac{dP}{dt} = \dfrac{1}{5.3}(0.02) = \dfrac{1}{265}$, so

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) = \frac{1}{265}P\left(1 - \frac{P}{100}\right), \ P \text{ in billions}$$

(b) $A = \dfrac{K - P_0}{P_0} = \dfrac{100 - 5.3}{5.3} = \dfrac{947}{53} \approx 17.8679$. $P(t) = \dfrac{K}{1 + Ae^{-kt}} = \dfrac{100}{1 + \frac{947}{53}e^{-(1/265)t}}$, so $P(10) \approx 5.49$ billion.

(c) $P(110) \approx 7.81$, and $P(510) \approx 27.72$. The predictions are 7.81 billion in the year 2100 and 27.72 billion in 2500.

(d) If $K = 50$, then $P(t) = \dfrac{50}{1 + \frac{447}{53}e^{-(1/265)t}}$. So $P(10) \approx 5.48$, $P(110) \approx 7.61$, and $P(510) \approx 22.41$. The predictions become 5.48 billion in the year 2000, 7.61 billion in 2100, and 22.41 billion in the year 2500.

6. (a) If we assume that the carrying capacity for the world population is 100 billion, it would seem reasonable that the carrying capacity for the U.S. is 3–5 billion by using current populations and simple proportions. We will use $K = 4$ billion or 4000 million. With $t = 0$ corresponding to 1980, we have

$$P(t) = \frac{4000}{1 + \left(\frac{4000 - 250}{250}\right)e^{-kt}} = \frac{4000}{1 + 15e^{-kt}}$$

(b) $P(10) = 275 \ \Rightarrow \ \dfrac{4000}{1 + 15e^{-10k}} = 275 \ \Rightarrow \ 1 + 15e^{-10k} = \frac{4000}{275} \ \Rightarrow \ e^{-10k} = \dfrac{\frac{160}{11} - 1}{15} \ \Rightarrow$
$-10k = \ln\frac{149}{165} \ \Rightarrow \ k = -\frac{1}{10}\ln\frac{149}{165} \approx 0.01019992$.

(c) $2100 - 1990 = 110$ and $P(110) \approx 680$ million.
$2200 - 1990 = 210$ and $P(210) \approx 1449$ million, or about 1.4 billion.

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(d) $P(t) = 300 \quad \Rightarrow \quad \dfrac{4000}{1+15e^{-kt}} = 300 \quad \Rightarrow \quad 1+15e^{-kt} = \frac{40}{3} \quad \Rightarrow \quad e^{-kt} = \frac{37}{3}\cdot\frac{1}{15} \quad \Rightarrow \quad -kt = \ln\frac{37}{45}$

$\Rightarrow \quad t = 10\,\dfrac{\ln\frac{37}{45}}{\ln\frac{149}{165}} \approx 19.19 \approx 19$. So we predict that the U.S. population will exceed 300 million in the year $1990 + 19 = 2009$.

7. (a) Our assumption is that $\dfrac{dy}{dt} = ky(1-y)$, where y is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (1), $\dfrac{dP}{dt} = kP\left(1-\dfrac{P}{K}\right)$, we substitute $y = \dfrac{P}{K}$, $P = Ky$, and $\dfrac{dP}{dt} = K\dfrac{dy}{dt}$, to obtain $K\,\dfrac{dy}{dt} = k(Ky)(1-y) \quad \Leftrightarrow \quad \dfrac{dy}{dt} = ky(1-y)$, our equation in part (a). Now the solution to (1) is $P(t) = \dfrac{K}{1+Ae^{-kt}}$, where $A = \dfrac{K-P_0}{P_0}$. We use the same substitution to obtain $Ky = \dfrac{K}{1+\dfrac{K-Ky_0}{Ky_0}e^{-kt}}$

$\Rightarrow \quad y = \dfrac{y_0}{y_0 + (1-y_0)e^{-kt}}$.

Alternatively, we could use the same steps as outlined in "The Analytic Solution," following Example 2.

(c) Let t be the number of hours since 8 A.M. Then $y_0 = y(0) = \frac{80}{1000} = 0.08$ and $y(4) = \frac{1}{2}$, so $\dfrac{1}{2} = y(4) = \dfrac{0.08}{0.08+0.92e^{-4k}}$. Thus, $0.08 + 0.92e^{-4k} = 0.16$, $e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23}$, and $e^{-k} = \left(\frac{2}{23}\right)^{1/4}$, so $y = \dfrac{0.08}{0.08+0.92(2/23)^{t/4}} = \dfrac{2}{2+23(2/23)^{t/4}}$. Solving this equation for t, we get

$2y + 23y\left(\dfrac{2}{23}\right)^{t/4} = 2 \quad \Rightarrow \quad \left(\dfrac{2}{23}\right)^{t/4} = \dfrac{2-2y}{23y} \quad \Rightarrow \quad \left(\dfrac{2}{23}\right)^{t/4} = \dfrac{2}{23}\cdot\dfrac{1-y}{y} \quad \Rightarrow$

$\left(\dfrac{2}{23}\right)^{t/4\ \ 1} = \dfrac{1-y}{y}$. It follows that $\dfrac{t}{4} - 1 = \dfrac{\ln[(1-y)/y]}{\ln\frac{2}{23}}$, so $t = 4\left[1 + \dfrac{\ln((1-y)/y)}{\ln\frac{2}{23}}\right]$.

When $y = 0.9$, $\dfrac{1-y}{y} = \frac{1}{9}$, so $t = 4\left(1 - \dfrac{\ln 9}{\ln\frac{2}{23}}\right) \approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 P.M.

8. (a) $P(0) = P_0 = 400$, $P(1) = 1200$ and $K = 10{,}000$. From the solution to the logistic differential equation $P(t) = \dfrac{P_0K}{P_0 + (K-P_0)e^{-kt}}$, we get $P = \dfrac{400\,(10{,}000)}{400 + (9600)e^{-kt}} = \dfrac{10{,}000}{1+24e^{-kt}}$. $P(1) = 1200 \quad \Rightarrow$

$1 + 24e^{-k} = \frac{100}{12} \quad \Rightarrow \quad e^k = \frac{288}{88} \quad \Rightarrow \quad k = \ln\frac{36}{11}$. So $P = \dfrac{10{,}000}{1+24e^{-t\ln(36/11)}} = \dfrac{10{,}000}{1+24\cdot(11/36)^t}$.

(b) $5000 = \dfrac{10{,}000}{1+24(11/36)^t} \quad \Rightarrow \quad 24\left(\frac{11}{36}\right)^t = 1 \quad \Rightarrow \quad t\ln\frac{11}{36} = \ln\frac{1}{24} \quad \Rightarrow \quad t \approx 2.68$ years.

9. (a) $\dfrac{dP}{dt} = kP\left(1-\dfrac{P}{K}\right) \quad \Rightarrow$

$$\frac{d^2P}{dt^2} = k\left[P\left(-\frac{1}{K}\frac{dP}{dt}\right) + \left(1-\frac{P}{K}\right)\frac{dP}{dt}\right] = k\,\frac{dP}{dt}\left(-\frac{P}{K} + 1 - \frac{P}{K}\right)$$

$$= k\left[kP\left(1-\frac{P}{K}\right)\right]\left(1-\frac{2P}{K}\right) = k^2P\left(1-\frac{P}{K}\right)\left(1-\frac{2P}{K}\right)$$

(b) P grows fastest when P' has a maximum, that is, when $P'' = 0$. From part (a), $P'' = 0 \quad \Leftrightarrow \quad P = 0$, $P = K$, or $P = K/2$. Since $0 < P < K$, we see that $P'' = 0 \quad \Leftrightarrow \quad P = K/2$.

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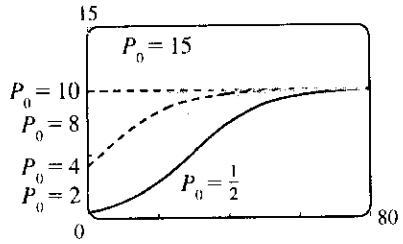
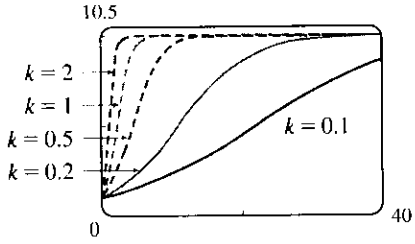
10.



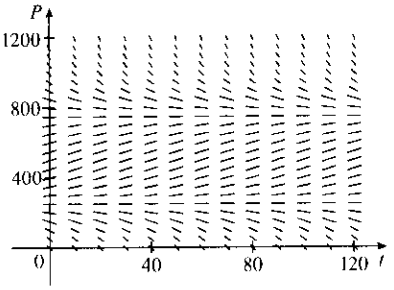
First we keep k constant (at 0.1, say) and change P_0 in the function $P = \dfrac{10P_0}{P_0 + (10 - P_0)e^{-0.1t}}$. (Notice that P_0 is the P-intercept.) If $P_0 = 0$, the function is 0 everywhere. For $0 < P_0 < 5$, the curve has an inflection point, which moves to the right as P_0 decreases. If $5 < P_0 < 10$, the graph is concave down everywhere. (We are considering only $t \geq 0$.) If $P_0 = 10$, the function is the constant function $P = 10$, and if $P_0 > 10$, the function decreases. For all $P_0 \neq 0$, $\lim_{t \to \infty} P = 10$.



Now we instead keep P_0 constant (at $P_0 = 1$) and change k in the function $P = \dfrac{10}{1 + 9e^{-kt}}$. It seems that as k increases, the graph approaches the line $P = 10$ more and more quickly. (Note that the only difference in the shape of the curves is in the horizontal scaling; if we choose suitable x-scales, the graphs all look the same.)

11. (a) The term -15 represents a harvesting of fish at a constant rate—in this case, 15 fish/week. This is the rate at which fish are caught.

(b)

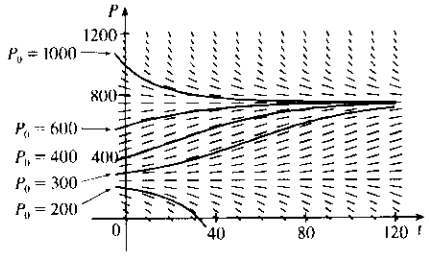


(c) From the graph in part (b), it appears that $P(t) = 250$ and $P(t) = 750$ are the equilibrium solutions. We confirm this analytically by solving the equation $dP/dt = 0$ as follows: $0.08P(1 - P/1000) - 15 = 0 \quad \Rightarrow$

$0.08P - 0.00008P^2 - 15 = 0 \quad \Rightarrow \quad -0.00008\left(P^2 - 1000P + 187{,}500\right) = 0 \quad \Rightarrow$

$(P - 250)(P - 750) = 0 \quad \Rightarrow \quad P = 250$ or 750.

(d)



For $0 < P_0 < 250$, $P(t)$ decreases to 0. For $P_0 = 250$, $P(t)$ remains constant. For $250 < P_0 < 750$, $P(t)$ increases and approaches 750. For $P_0 = 750$, $P(t)$ remains constant. For $P_0 > 750$, $P(t)$ decreases and approaches 750.

(e) $\dfrac{dP}{dt} = 0.08P\left(1 - \dfrac{P}{1000}\right) - 15 \quad \Leftrightarrow \quad -\dfrac{100{,}000}{8} \cdot \dfrac{dP}{dt} = \left(0.08P - 0.00008P^2 - 15\right) \cdot \left(-\dfrac{100{,}000}{8}\right) \quad \Leftrightarrow$

$-12{,}500\,\dfrac{dP}{dt} = P^2 - 1000P + 187{,}500 \quad \Leftrightarrow \quad \dfrac{dP}{(P - 250)(P - 750)} = -\dfrac{1}{12{,}500}\,dt \quad \Leftrightarrow$

$\displaystyle\int \left(\frac{-1/500}{P - 250} + \frac{1/500}{P - 750}\right) dP = -\frac{1}{12{,}500}\,dt \quad \Leftrightarrow \quad \int \left(\frac{1}{P - 250} - \frac{1}{P - 750}\right) dP = \tfrac{1}{25}\,dt \quad \Leftrightarrow$

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$\ln|P-250| - \ln|P-750| = \frac{1}{25}t + C \quad \Leftrightarrow \quad \ln\left|\dfrac{P-250}{P-750}\right| = \frac{1}{25}t + C \quad \Leftrightarrow$

$\left|\dfrac{P-250}{P-750}\right| = e^{t/25+C} = ke^{t/25} \quad \Leftrightarrow \quad \dfrac{P-250}{P-750} = ke^{t/25} \quad \Leftrightarrow \quad P-250 = Pke^{t/25} - 750ke^{t/25} \quad \Leftrightarrow$

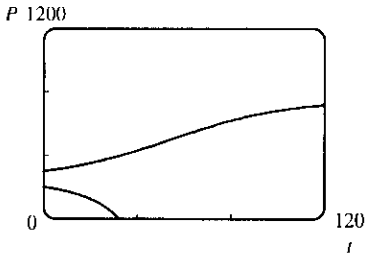
$P - Pke^{t/25} = 250 - 750ke^{t/25} \quad \Leftrightarrow \quad P(t) = \dfrac{250 - 750ke^{t/25}}{1 - ke^{t/25}}$. If $t = 0$ and $P = 200$, then

$200 = \dfrac{250 - 750k}{1-k} \quad \Leftrightarrow \quad 200 - 200k = 250 - 750k \quad \Leftrightarrow$

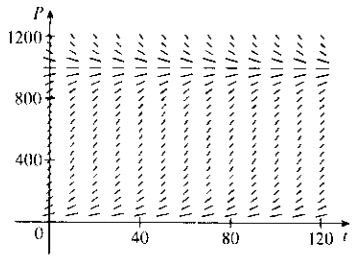
$550k = 50 \quad \Leftrightarrow \quad k = \frac{1}{11}$. Similarly, if $t = 0$ and $P = 300$, then

$k = -\frac{1}{9}$. Simplifying P with these two values of k gives us

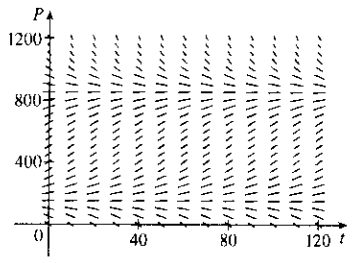
$P(t) = \dfrac{250\left(3e^{t/25} - 11\right)}{e^{t/25} - 11}$ and $P(t) = \dfrac{750\left(e^{t/25} + 3\right)}{e^{t/25} + 9}$.



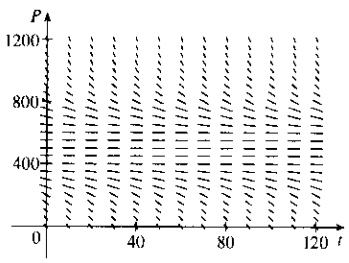
12. (a)



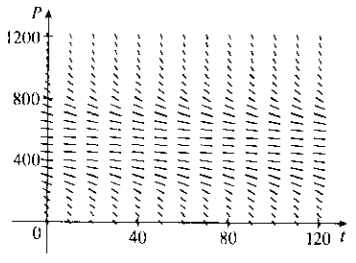
$c = 0$



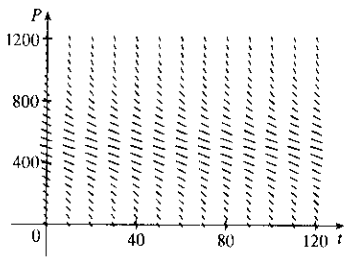
$c = 10$



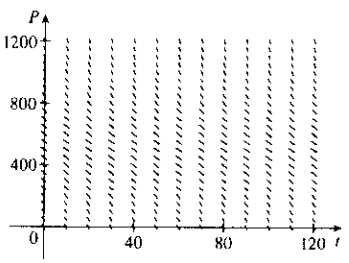
$c = 20$



$c = 21$



$c = 25$



$c = 30$

(b) For $0 \le c \le 20$, there is at least one equilibrium solution. For $c > 20$, the population always dies out.

(c) $\dfrac{dP}{dt} = 0.08P - 0.00008P^2 - c$. $\dfrac{dP}{dt} = 0 \quad \Leftrightarrow \quad P = \dfrac{-0.08 \pm \sqrt{(0.08)^2 - 4(-0.00008)(-c)}}{2(-0.00008)}$, which has at least one solution when the discriminant is nonnegative $\Rightarrow \quad 0.0064 - 0.00032c \ge 0 \quad \Leftrightarrow \quad c \le 20$. For $0 \le c \le 20$, there is at least one value of P such that $dP/dt = 0$ and hence, at least one equilibrium solution. For $c > 20$, $dP/dt < 0$ and the population always dies out.

(d) The weekly catch should be less than 20 fish per week.

13. (a) $\dfrac{dP}{dt} = (kP)\left(1 - \dfrac{P}{K}\right)\left(1 - \dfrac{m}{P}\right)$. If $m < P < K$, then $dP/dt = (+)(+)(+) = + \quad \Rightarrow \quad P$ is increasing. If $0 < P < m$, then $dP/dt = (+)(+)(-) = - \quad \Rightarrow \quad P$ is decreasing.

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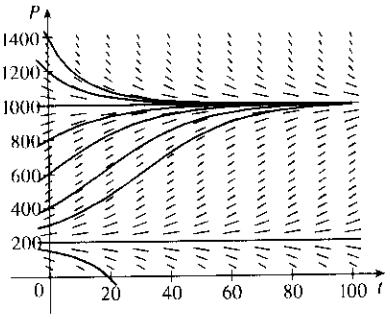
(b)



$k = 0.08$, $K = 1000$, and $m = 200 \Rightarrow$

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)\left(1 - \frac{200}{P}\right)$$

For $0 < P_0 < 200$, the population dies out. For $P_0 = 200$, the population is steady. For $200 < P_0 < 1000$, the population increases and approaches 1000. For $P_0 > 1000$, the population decreases and approaches 1000.

The equilibrium solutions are $P(t) = 200$ and $P(t) = 1000$.

(c) $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right)\left(1 - \dfrac{m}{P}\right) = kP\left(\dfrac{K-P}{K}\right)\left(\dfrac{P-m}{P}\right) = \dfrac{k}{K}(K-P)(P-m) \Leftrightarrow$

$$\int \frac{dP}{(K-P)(P-m)} = \int \frac{k}{K}\,dt.$$

By partial fractions, $\dfrac{1}{(K-P)(P-m)} = \dfrac{A}{K-P} + \dfrac{B}{P-m}$, so $A(P-m) + B(K-P) = 1$.

If $P = m$, $B = \dfrac{1}{K-m}$; if $P = K$, $A = \dfrac{1}{K-m}$, so $\dfrac{1}{K-m}\displaystyle\int\left(\frac{1}{K-P} + \frac{1}{P-m}\right)dP = \int \frac{k}{K}\,dt \Rightarrow$

$$\frac{1}{K-m}\left(-\ln|K-P| + \ln|P-m|\right) = \frac{k}{K}t + M \Rightarrow \frac{1}{K-m}\ln\left|\frac{P-m}{K-P}\right| = \frac{k}{K}t + M \Rightarrow$$

$$\ln\left|\frac{P-m}{K-P}\right| = (K-m)\frac{k}{K}t + M_1 \Leftrightarrow \frac{P-m}{K-P} = De^{(K-m)(k/K)t} \quad [D = \pm e^{M_1}].$$

Let $t = 0$: $\dfrac{P_0 - m}{K - P_0} = D$. So $\dfrac{P-m}{K-P} = \dfrac{P_0 - m}{K - P_0}e^{(K-m)(k/K)t}$. Solving for P, we get

$$P(t) = \frac{m(K-P_0) + K(P_0 - m)e^{(K-m)(k/K)t}}{K - P_0 + (P_0 - m)e^{(K-m)(k/K)t}}.$$

(d) If $P_0 < m$, then $P_0 - m < 0$. Let $N(t)$ be the numerator of the expression for $P(t)$ in part (c). Then

$N(0) = P_0(K-m) > 0$, and $P_0 - m < 0 \Leftrightarrow \lim\limits_{t\to\infty} K(P_0 - m)\,e^{(K-m)(k/K)t} = -\infty \Rightarrow$

$\lim\limits_{t\to\infty} N(t) = -\infty$. Since N is continuous, there is a number t such that $N(t) = 0$ and thus $P(t) = 0$. So the species will become extinct.

14. (a) $\dfrac{dP}{dt} = c\ln\left(\dfrac{K}{P}\right)P \Rightarrow \displaystyle\int \frac{dP}{P\ln(K/P)} = \int c\,dt$. Let $u = \ln\left(\dfrac{K}{P}\right) = \ln K - \ln P \Rightarrow du = -\dfrac{dP}{P}$

$$\Rightarrow \int -\frac{du}{u} = ct + D \Rightarrow \ln|u| = -ct - D \Rightarrow |u| = e^{-(ct+D)} \Rightarrow |\ln(K/P)| = e^{-(ct+D)} \Rightarrow$$

$\ln(K/P) = \pm e^{-(ct+D)}$. Letting $t = 0$, we get $\ln(K/P_0) = \pm e^{-D}$, so

$$\ln(K/P) = \pm e^{-ct-D} = \pm e^{-ct}e^{-D} = \ln(K/P_0)\,e^{-ct} \Rightarrow K/P = e^{\ln(K/P_0)e^{-ct}} \Rightarrow$$

$P(t) = Ke^{-\ln(K/P_0)e^{-ct}}$, $c \neq 0$.

(b) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} Ke^{-\ln(K/P_0)e^{-ct}} = Ke^{-\ln(K/P_0)\cdot 0} = Ke^0 = K$

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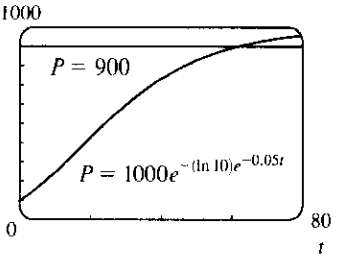
(c) 

The graphs look very similar. For the Gompertz function, $P(40) \approx 732$, nearly the same as the logistic function. The Gompertz function reaches $P = 900$ at $t \approx 61.7$ and its value at $t = 80$ is about 959, so it doesn't increase quite as fast as the logistic curve.

(d) $\dfrac{dP}{dt} = c \ln\left(\dfrac{K}{P}\right) P = cP(\ln K - \ln P) \quad \Rightarrow$

$$\frac{d^2P}{dt^2} = c\left[P\left(-\frac{1}{P}\frac{dP}{dt}\right) + (\ln K - \ln P)\frac{dP}{dt}\right] = c\frac{dP}{dt}\left[-1 + \ln\left(\frac{K}{P}\right)\right]$$

$$= c\left[c \ln(K/P)\, P\right]\left[\ln(K/P) - 1\right] = c^2 P \ln(K/P)\left[\ln(K/P) - 1\right]$$

Since $0 < P < K$, $P'' = 0 \quad \Leftrightarrow \quad \ln(K/P) = 1 \quad \Leftrightarrow \quad K/P = e \quad \Leftrightarrow \quad P = K/e$. $P'' > 0$ for $0 < P < K/e$ and $P'' < 0$ for $K/e < P < K$, so P' is a maximum (and P grows fastest) when $P = K/e$.

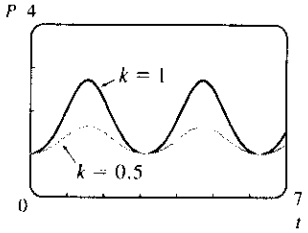
Note: If $P > K$, then $\ln(K/P) < 0$, so $P''(t) > 0$.

15. (a) $dP/dt = kP\cos(rt - \phi) \quad \Rightarrow \quad (dP)/P = k\cos(rt - \phi)\, dt \quad \Rightarrow \quad \int (dP)/P = k \int \cos(rt - \phi)\, dt \quad \Rightarrow$ $\ln P = (k/r)\sin(rt - \phi) + C$. (Since this is a growth model, $P > 0$ and we can write $\ln P$ instead of $\ln|P|$.) Since $P(0) = P_0$, we obtain $\ln P_0 = (k/r)\sin(-\phi) + C = -(k/r)\sin\phi + C \quad \Rightarrow$ $C = \ln P_0 + (k/r)\sin\phi$. Thus, $\ln P = (k/r)\sin(rt - \phi) + \ln P_0 + (k/r)\sin\phi$, which we can rewrite as $\ln(P/P_0) = (k/r)[\sin(rt - \phi) + \sin\phi]$ or, after exponentiation, $P(t) = P_0 e^{(k/r)[\sin(rt-\phi)+\sin\phi]}$.

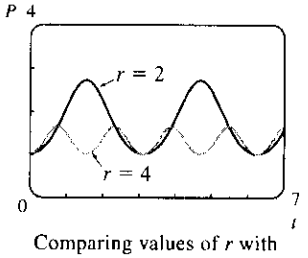
(b) As k increases, the amplitude increases, but the minimum value stays the same.

As r increases, the amplitude and the period decrease.

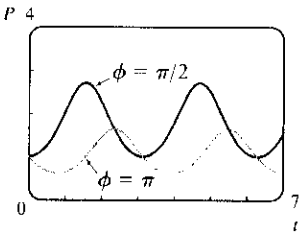
A change in ϕ produces slight adjustments in the phase shift and amplitude.



Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$



Comparing values of r with $P_0 = 1$, $k = 1$, and $\phi = \pi/2$



Comparing values of ϕ with $P_0 = 1$, $k = 1$, and $r = 2$

$P(t)$ oscillates between $P_0 e^{(k/r)(1+\sin\phi)}$ and $P_0 e^{(k/r)(-1+\sin\phi)}$ (the extreme values are attained when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$), so $\lim_{t\to\infty} P(t)$ does not exist.

16. (a) $dP/dt = kP\cos^2(rt - \phi) \quad \Rightarrow \quad (dP)/P = k\cos^2(rt - \phi)\, dt \quad \Rightarrow \quad \int (dP)/P = k \int \cos^2(rt - \phi)\, dt \quad \Rightarrow$ $\ln P = k \displaystyle\int \frac{1 + \cos(2(rt - \phi))}{2}\, dt = \frac{k}{2}t + \frac{k}{4r}\sin(2(rt - \phi)) + C$. From $P(0) = P_0$, we get $\ln P_0 = \dfrac{k}{4r}\sin(-2\phi) + C = C - \dfrac{k}{4r}\sin 2\phi$, so $C = \ln P_0 + \dfrac{k}{4r}\sin 2\phi$ and

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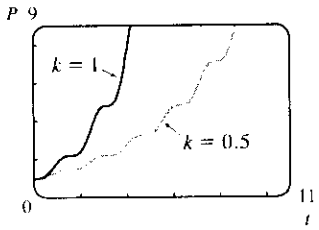
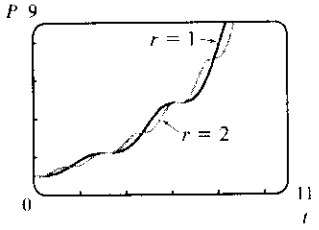
$\ln P = \frac{k}{2}t + \frac{k}{4r}\sin(2(rt-\phi)) + \ln P_0 + \frac{k}{4r}\sin 2\phi$. Simplifying, we get $\ln\frac{P}{P_0} = \frac{k}{2}t + \frac{k}{4r}[\sin(2(rt-\phi)) + \sin 2\phi] = f(t)$, or $P(t) = P_0 e^{f(t)}$.

(b) An increase in k stretches the graph of P vertically while maintaining $P(0) = P_0$.



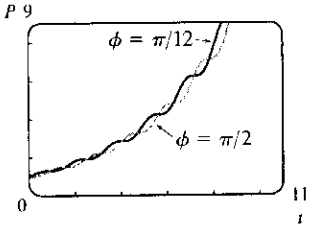
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

An increase in r compresses the graph of P horizontally—similar to changing the period in Exercise 15.



Comparing values of r with $P_0 = 1$, $k = 0.5$, and $\phi = \pi/2$

As in Exercise 15, a change in ϕ only makes slight adjustments in the growth of P, as shown in the figure.



Comparing values of ϕ with $P_0 = 1$, $k = 0.5$, and $r = 2$

$f'(t) = k/2 + [k/(4r)][2r\cos(2(rt-\phi))] = (k/2)[1+\cos(2(rt-\phi))] \geq 0$. Since $P(t) = P_0 e^{f(t)}$, we have $P'(t) = P_0 f'(t) e^{f(t)} \geq 0$, with equality only when $\cos(2(rt-\phi)) = -1$; that is, when $rt-\phi$ is an odd multiple of $\frac{\pi}{2}$. Therefore, $P(t)$ is an increasing function on $(0,\infty)$. P can also be written as $P(t) = P_0 e^{kt/2} e^{(k/4r)[\sin(2(rt-\phi))+\sin 2\phi]}$. The second exponential oscillates between $e^{(k/4r)(1+\sin 2\phi)}$ and $e^{(k/4r)(-1+\sin 2\phi)}$, while the first one, $e^{kt/2}$, grows without bound. So $\lim_{t\to\infty} P(t) = \infty$.

17. By Equation (4), $P(t) = \dfrac{K}{1+Ae^{-kt}}$. By comparison, if $c = (\ln A)/k$ and $u = \frac{1}{2}k(t-c)$, then

$$1+\tanh u = 1 + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{e^u + e^{-u}}{e^u + e^{-u}} + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{2e^u}{e^u + e^{-u}} \cdot \frac{e^{-u}}{e^{-u}} = \frac{2}{1+e^{-2u}}$$

and $e^{-2u} = e^{-k(t-c)} = e^{kc}e^{-kt} = e^{\ln A}e^{-kt} = Ae^{-kt}$, so

$$\tfrac{1}{2}K\big[1+\tanh\big(\tfrac{1}{2}k(t-c)\big)\big] = \frac{K}{2}[1+\tanh u] = \frac{K}{2}\cdot\frac{2}{1+e^{-2u}} = \frac{K}{1+e^{-2u}} = \frac{K}{1+Ae^{-kt}} = P(t).$$

10.6 Linear Equations

1. $y' + e^x y = x^2 y^2$ is not linear since it cannot be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$.

2. $y + \sin x = x^3 y' \;\Rightarrow\; x^3 y' - y = \sin x \;\Rightarrow\; y' + \left(-\dfrac{1}{x^3}\right) y = \dfrac{\sin x}{x^3}$. This equation is in the standard linear form (1), so it is linear.

3. $xy' + \ln x - x^2 y = 0 \;\Rightarrow\; xy' - x^2 y = -\ln x \;\Rightarrow\; y' + (-x)\,y = -\dfrac{\ln x}{x}$, which is in the standard linear form (1), so this equation is linear.

4. $y' + \cos y = \tan x$ is not linear since it cannot be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$. [$\cos y$ is not of the form $P(x)y$.]

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5. Comparing the given equation, $y' + 2y = 2e^x$, with the general form, $y' + P(x)y = Q(x)$, we see that $P(x) = 2$ and the integrating factor is $I(x) = e^{\int P(x)dx} = e^{\int 2\,dx} = e^{2x}$. Multiplying the differential equation by $I(x)$ gives $e^{2x}y' + 2e^{2x}y = 2e^{3x} \;\Rightarrow\; \left(e^{2x}y\right)' = 2e^{3x} \;\Rightarrow\; e^{2x}y = \int 2e^{3x}\,dx \;\Rightarrow\; e^{2x}y = \frac{2}{3}e^{3x} + C \;\Rightarrow\; y = \frac{2}{3}e^{x} + Ce^{-2x}$.

6. $y' = x + 5y \;\Rightarrow\; y' - 5y = x$. $I(x) = e^{\int P(x)dx} = e^{\int(-5)dx} = e^{-5x}$. Multiplying the differential equation by $I(x)$ gives $e^{-5x}y' - 5e^{-5x}y = xe^{-5x} \;\Rightarrow\; \left(e^{-5x}y\right)' = xe^{-5x} \;\Rightarrow\; e^{-5x}y = \int xe^{-5x}\,dx = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$ [by parts] $\Rightarrow\; y = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}$

7. $xy' - 2y = x^2$ [divide by x] $\Rightarrow\; y' + \left(-\frac{2}{x}\right)y = x$ $(*)$.

$I(x) = e^{\int P(x)\,dx} = e^{\int(-2/x)\,dx} = e^{-2\ln|x|} = e^{\ln|x|^{-2}} = e^{\ln(1/x^2)} = 1/x^2$. Multiplying the differential equation $(*)$ by $I(x)$ gives $\frac{1}{x^2}y' - \frac{2}{x^3}y = \frac{1}{x} \;\Rightarrow\; \left(\frac{1}{x^2}y\right)' = \frac{1}{x} \;\Rightarrow\; \frac{1}{x^2}y = \ln|x| + C \;\Rightarrow\; y = x^2(\ln|x| + C) = x^2\ln|x| + Cx^2$.

8. $x^2y' + 2xy = \cos^2 x \;\Rightarrow\; y' + \frac{2}{x}y = \frac{\cos^2 x}{x^2}$. $I(x) = e^{\int P(x)\,dx} = e^{\int 2/x\,dx} = e^{2\ln|x|} = e^{\ln(x^2)} = x^2$. Multiplying by $I(x)$ gives us our original equation back. You may have noticed this immediately, since $P(x)$ is the derivative of the coefficient of y'. We rewrite it as $(x^2y)' = \cos^2 x$. Thus,

$x^2y = \int \cos^2 x\,dx = \int \frac{1}{2}(1 + \cos 2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \;\Rightarrow\; y = \frac{1}{2x} + \frac{1}{4x^2}\sin 2x + \frac{C}{x^2}$ or

$y = \frac{1}{2x} + \frac{1}{2x^2}\sin x\cos x + \frac{C}{x^2}$.

9. Since $P(x)$ is the derivative of the coefficient of y' [$P(x) = 1$ and the coefficient is x], we can write the differential equation $xy' + y = \sqrt{x}$ in the easily integrable form $(xy)' = \sqrt{x} \;\Rightarrow\; xy = \frac{2}{3}x^{3/2} + C \;\Rightarrow\; y = \frac{2}{3}\sqrt{x} + C/x$.

10. $y' - y = 1/x$ $[x \neq 0]$, so $I(x) = e^{\int(-1)dx} = e^{-x}$. Multiplying the differential equation by $I(x)$ gives $e^{-x}y' - e^{-x}y = e^{-x}/x \;\Rightarrow\; \left(e^{-x}y\right)' = e^{-x}/x \;\Rightarrow\; y = e^{x}\left[\int \left(e^{-x}/x\right)dx + C\right]$.

11. $I(x) = e^{\int 2x\,dx} = e^{x^2}$. Multiplying the differential equation $y' + 2xy = x^2$ by $I(x)$ gives $e^{x^2}y' + 2xe^{x^2}y = x^2e^{x^2} \;\Rightarrow\; \left(e^{x^2}y\right)' = x^2e^{x^2}$. Thus

$y = e^{-x^2}\left[\int x^2e^{x^2}dx + C\right] = e^{-x^2}\left[\frac{1}{2}xe^{x^2} - \int \frac{1}{2}e^{x^2}\,dx + C\right] = \frac{1}{2}x + Ce^{-x^2} - e^{-x^2}\int \frac{1}{2}e^{x^2}\,dx$.

12. $I(x) = e^{\int -\tan x\,dx} = e^{\ln|\cos x|} = \cos x$ (since $-\frac{\pi}{2} < x < \frac{\pi}{2}$). Multiplying the differential equation by $I(x)$ gives $y'\cos x - y\tan x\cos x = x\cos x\sin 2x \;\Rightarrow\; (y\cos x)' = x\cos x\sin 2x$. So

$$y = \frac{1}{\cos x}\left[\int x\cos x\sin 2x\,dx + C\right] = \frac{1}{\cos x}\left[\int 2x\cos^2 x\sin x\,dx + C\right]$$

$$= \frac{1}{\cos x}\left[\frac{-2x\cos^3 x}{3} + \frac{2}{3}\left(\sin x - \frac{\sin^3 x}{3}\right) + C\right] = \frac{-2x\cos^2 x}{3} + \frac{C}{\cos x} + 2\tan x\,\frac{3 - \sin^2 x}{9}$$

13. $(1 + t)\,\frac{du}{dt} + u = 1 + t,\ t > 0$ [divide by $1 + t$] $\Rightarrow\; \frac{du}{dt} + \frac{1}{1+t}u = 1$ $(*)$, which has the form $u' + P(t)\,u = Q(t)$. The integrating factor is $I(t) = e^{\int P(t)\,dt} = e^{\int[1/(1+t)]\,dt} = e^{\ln(1+t)} = 1 + t$.

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Multiplying $(*)$ by $I(t)$ gives us our original equation back. We rewrite it as $[(1+t)\,u]' = 1+t$. Thus,

$$(1+t)\,u = \int (1+t)\,dt = t + \tfrac{1}{2}t^2 + C \quad \Rightarrow \quad u = \frac{t+\frac{1}{2}t^2+C}{1+t} \text{ or } u = \frac{t^2+2t+2C}{2\,(t+1)}.$$

14. $t \ln t \dfrac{dr}{dt} + r = te^t \quad \Rightarrow \quad \dfrac{dr}{dt} + \dfrac{1}{t \ln t}\, r = \dfrac{e^t}{\ln t}$. $I(t) = e^{\int dt/(t \ln t)} = e^{\ln(\ln t)} = \ln t$. Multiplying by $\ln t$ gives

$$\ln t \frac{dr}{dt} + \frac{1}{t} r = e^t \quad \Rightarrow \quad [(\ln t) r]' = e^t \quad \Rightarrow \quad (\ln t) r = e^t + C \quad \Rightarrow \quad r = \frac{e^t + C}{\ln t}.$$

15. $y' = x + y \quad \Rightarrow \quad y' + (-1)y = x$. $I(x) = e^{\int (-1)\,dx} = e^{-x}$. Multiplying by e^{-x} gives $e^{-x}y' - e^{-x}y = xe^{-x}$

$\Rightarrow \quad (e^{-x}y)' = xe^{-x} \quad \Rightarrow \quad e^{-x}y = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$ [integration by parts with $u = x$,

$dv = e^{-x}\,dx$] $\Rightarrow \quad y = -x - 1 + Ce^x$. $y(0) = 2 \quad \Rightarrow \quad -1 + C = 2 \quad \Rightarrow \quad C = 3$, so $y = -x - 1 + 3e^x$.

16. $t\dfrac{dy}{dt} + 2y = t^3, t > 0, y(1) = 0$. Divide by t to get $\dfrac{dy}{dt} + \dfrac{2}{t}y = t^2$, which is linear.

$I(t) = e^{\int (2/t)dt} = e^{2\ln t} = t^2$. Multiplying by t^2 gives $t^2\dfrac{dy}{dt} + 2ty = t^4 \quad \Rightarrow \quad (t^2 y)' = t^4 \quad \Rightarrow$

$t^2 y = \frac{1}{5}t^5 + C \quad \Rightarrow \quad y = \dfrac{t^3}{5} + \dfrac{C}{t^2}$. Thus, $0 = y(1) = \frac{1}{5} + C \quad \Rightarrow \quad C = -\frac{1}{5}$, so $y = \dfrac{t^3}{5} - \dfrac{1}{5t^2}$.

17. $\dfrac{dv}{dt} - 2tv = 3t^2 e^{t^2}, v(0) = 5$. $I(t) = e^{\int (-2t)dt} = e^{-t^2}$. Multiply the differential equation by $I(t)$ to get

$$e^{-t^2}\frac{dv}{dt} - 2te^{-t^2}v = 3t^2 \quad \Rightarrow \quad \left(e^{-t^2}v\right)' = 3t^2 \quad \Rightarrow \quad e^{-t^2}v = \int 3t^2\,dt = t^3 + C \quad \Rightarrow \quad v = t^3 e^{t^2} + Ce^{t^2}.$$

$5 = v(0) = 0 \cdot 1 + C \cdot 1 = C$, so $v = t^3 e^{t^2} + 5e^{t^2}$.

18. $2xy' + y = 6x,\ x > 0 \quad \Rightarrow \quad y' + \dfrac{1}{2x}\,y = 3$. $I(x) = e^{\int 1/(2x)\,dx} = e^{(1/2)\ln x} = e^{\ln x^{1/2}} = \sqrt{x}$. Multiplying

by $\sqrt{x}$ gives $\sqrt{x}\,y' + \dfrac{1}{2\sqrt{x}}\,y = 3\sqrt{x} \quad \Rightarrow \quad (\sqrt{x}\,y)' = 3\sqrt{x} \quad \Rightarrow \quad \sqrt{x}\,y = \int 3\sqrt{x}\,dx = 2x^{3/2} + C \quad \Rightarrow$

$y = 2x + \dfrac{C}{\sqrt{x}}$. $y(4) = 20 \quad \Rightarrow \quad 8 + \dfrac{C}{2} = 20 \quad \Rightarrow \quad C = 24$, so $y = 2x + \dfrac{24}{\sqrt{x}}$.

19. $xy' = y + x^2 \sin x \quad \Rightarrow \quad y' - \dfrac{1}{x}\,y = x \sin x$. $I(x) = e^{\int (-1/x)\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = \dfrac{1}{x}$.

Multiplying by $\dfrac{1}{x}$ gives $\dfrac{1}{x}\,y' - \dfrac{1}{x^2}\,y = \sin x \quad \Rightarrow \quad \left(\dfrac{1}{x}\,y\right)' = \sin x \quad \Rightarrow \quad \dfrac{1}{x}\,y = -\cos x + C \quad \Rightarrow$

$y = -x\cos x + Cx$. $y(\pi) = 0 \quad \Rightarrow \quad -\pi \cdot (-1) + C\pi = 0 \quad \Rightarrow \quad C = -1$, so $y = -x\cos x - x$.

20. $x\dfrac{dy}{dx} - \dfrac{y}{x+1} = x \quad \Rightarrow \quad y' - \dfrac{y}{x\,(x+1)} = 1\ (x > 0)$, so $I(x) = e^{-\int 1/[x(x+1)]dx} = e^{-(\ln|x| - \ln|x+1|)} = \dfrac{x+1}{x}$.

Multiplying the differential equation by $I(x)$ gives $\dfrac{x+1}{x}y' - \dfrac{y}{x\,(x+1)}\dfrac{x+1}{x} = \dfrac{x+1}{x} \quad \Rightarrow$

$\left(\dfrac{x+1}{x}y\right)' = \dfrac{x+1}{x}$. Then $y = \dfrac{x}{x+1}\left[\displaystyle\int \left(1 + \frac{1}{x}\right)dx + C\right] = \dfrac{x}{x+1}\,(x + \ln x + C)$. But

$0 = y(1) = \frac{1}{2}\,[1 + C]$ so $C = -1$ and the solution to the initial-value problem is $y = \dfrac{x}{x+1}\,(x - 1 + \ln x)$.

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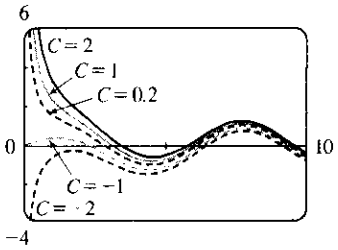
21. $y' + \dfrac{1}{x}y = \cos x \ \ (x \neq 0)$, so $I(x) = e^{\int (1/x)dx} = e^{\ln|x|} = x$ (for $x > 0$). Multiplying the differential equation by $I(x)$ gives $xy' + y = x\cos x \ \Rightarrow \ (xy)' = x\cos x$. Thus,

$$y = \frac{1}{x}\left[\int x\cos x\,dx + C\right] = \frac{1}{x}\left[x\sin x + \cos x + C\right]$$

$$= \sin x + \frac{\cos x}{x} + \frac{C}{x}$$

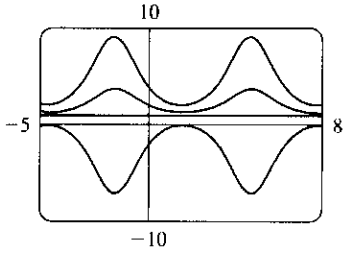


The solutions are asymptotic to the y-axis (except for $C = -1$). In fact, for $C > -1$, $y \to \infty$ as $x \to 0^+$, whereas for $C < -1$, $y \to -\infty$ as $x \to 0^+$. As x gets larger, the solutions approximate $y = \sin x$ more closely. The graphs for larger C lie above those for smaller C. The distance between the graphs lessens as x increases.

22. $I(x) = e^{\int \cos x\,dx} = e^{\sin x}$. Multiplying the differential equation by $I(x)$ gives $e^{\sin x}y' + \cos x \cdot e^{\sin x}y = \cos x \cdot e^{\sin x} \ \Rightarrow$

$\left(e^{\sin x}y\right)' = \cos x \cdot e^{\sin x} \ \Rightarrow$

$y = e^{-\sin x}\left[\int \cos x \cdot e^{\sin x}\,dx + C\right] = 1 + Ce^{-\sin x}$. The graphs for $C = -3, 0, 1$, and 3 are shown. As the values of C get further from zero the graph is stretched away from the line $y = 1$, which is the value for $C = 0$. The graphs are all periodic in x, with a period of 2π.



23. Setting $u = y^{1-n}$, $\dfrac{du}{dx} = (1-n)\,y^{-n}\dfrac{dy}{dx}$ or $\dfrac{dy}{dx} = \dfrac{y^n}{1-n}\dfrac{du}{dx} = \dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx}$. Then the Bernoulli differential equation becomes $\dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx} + P(x)u^{1/(1-n)} = Q(x)u^{n/(1-n)}$ or $\dfrac{du}{dx} + (1-n)P(x)u = Q(x)(1-n)$.

24. Here $y' + \dfrac{y}{x} = -y^2$, so $n = 2$, $P(x) = \dfrac{1}{x}$ and $Q(x) = -1$. Setting $u = y^{-1}$, u satisfies $u' - \dfrac{1}{x}u = 1$. Then $I(x) = e^{\int(-1/x)dx} = \dfrac{1}{x}$ (for $x > 0$) and $u = x\left(\displaystyle\int \frac{1}{x}\,dx + C\right) = x(\ln|x| + C)$. Thus, $y = \dfrac{1}{x(C + \ln|x|)}$.

25. $y' + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$. Here $n = 3$, $P(x) = \dfrac{2}{x}$, $Q(x) = \dfrac{1}{x^2}$ and setting $u = y^{-2}$, u satisfies $u' - \dfrac{4u}{x} = -\dfrac{2}{x^2}$.

Then $I(x) = e^{\int(-4/x)dx} = x^{-4}$ and $u = x^4\left(\displaystyle\int -\frac{2}{x^6}\,dx + C\right) = x^4\left(\frac{2}{5x^5} + C\right) = Cx^4 + \dfrac{2}{5x}$.

Thus, $y = \pm\left(Cx^4 + \dfrac{2}{5x}\right)^{-1/2}$.

26. Here $n = 3$, $P(x) = 1$, $Q(x) = x$ and setting $u = y^{-2}$, u satisfies $u' - 2u = -2x$. Then $I(x) = e^{\int(-2)dx} = e^{-2x}$ and $u = e^{2x}\left[\int -2xe^{-2x}\,dx + C\right] = e^{2x}\left(xe^{-2x} + \frac{1}{2}e^{-2x} + C\right) = x + \frac{1}{2} + Ce^{2x}$.

So $y^{-2} = x + \frac{1}{2} + Ce^{2x} \ \Rightarrow \ y = \pm\left[x + \frac{1}{2} + Ce^{2x}\right]^{-1/2}$.

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27. (a) $2\dfrac{dI}{dt} + 10I = 40$ or $\dfrac{dI}{dt} + 5I = 20$. Then the integrating factor is $e^{\int 5\,dt} = e^{5t}$. Multiplying the differential equation by the integrating factor gives $e^{5t}\dfrac{dI}{dt} + 5Ie^{5t} = 20e^{5t} \;\Rightarrow\; \left(e^{5t}I\right)' = 20e^{5t} \;\Rightarrow$

$I(t) = e^{-5t}\left[\int 20e^{5t}\,dt + C\right] = 4 + Ce^{-5t}$. But $0 = I(0) = 4 + C$, so $I(t) = 4 - 4e^{-5t}$.

(b) $I(0.1) = 4 - 4e^{-0.5} \approx 1.57$ A

28. (a) $\dfrac{dI}{dt} + 20I = 40\sin 60t$, so the integrating factor is e^{20t}. Multiplying the differential equation by the integrating factor gives $e^{20t}\dfrac{dI}{dt} + 20Ie^{20t} = 40e^{20t}\sin 60t \;\Rightarrow\; \left(e^{20t}I\right)' = 40e^{20t}\sin 60t \;\Rightarrow$

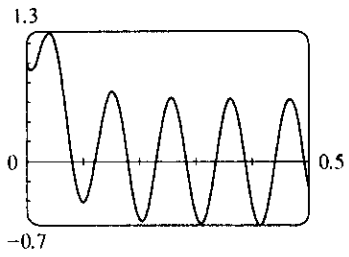
$$I(t) = e^{-20t}\left[\int 40e^{20t}\sin 60t\,dt + C\right]$$
$$= e^{-20t}\left[40e^{20t}\left(\tfrac{1}{4000}\right)(20\sin 60t - 60\cos 60t)\right] + Ce^{-20t}$$
$$= \frac{\sin 60t - 3\cos 60t}{5} + Ce^{-20t}$$

But $1 = I(0) = -\frac{3}{5} + C$, so

$$I(t) = \frac{\sin 60t - 3\cos 60t + 8e^{-20t}}{5}.$$

(b) $I(0.1) = \dfrac{\sin 6 - 3\cos 6 + 8e^{-2}}{5} \approx -0.42$ A

(c)



29. $5\dfrac{dQ}{dt} + 20Q = 60$ with $Q(0) = 0$ C. Then the integrating factor is $e^{\int 4\,dt} = e^{4t}$, and multiplying the differential equation by the integrating factor gives $e^{4t}\dfrac{dQ}{dt} + 4e^{4t}Q = 12e^{4t} \;\Rightarrow\; \left(e^{4t}Q\right)' = 12e^{4t} \;\Rightarrow$

$Q(t) = e^{-4t}\left[\int 12e^{4t}\,dt + C\right] = 3 + Ce^{-4t}$. But $0 = Q(0) = 3 + C$ so $Q(t) = 3\left(1 - e^{-4t}\right)$ is the charge at time t and $I = dQ/dt = 12e^{-4t}$ is the current at time t.

30. $2\dfrac{dQ}{dt} + 100Q = 10\sin 60t$ or $\dfrac{dQ}{dt} + 50Q = 5\sin 60t$. Then the integrating factor is $e^{\int 50\,dt} = e^{50t}$, and multiplying the differential equation by the integrating factor gives $e^{50t}\dfrac{dQ}{dt} + 50e^{50t}Q = 5e^{50t}\sin 60t \;\Rightarrow$

$\left(e^{50t}Q\right)' = 5e^{50t}\sin 60t \;\Rightarrow$

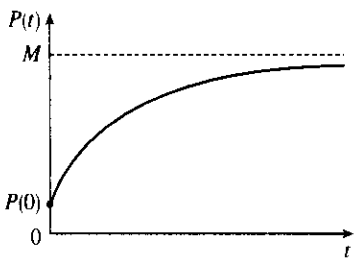
$$Q(t) = e^{-50t}\left[\int 5e^{50t}\sin 60t\,dt + C\right] = e^{-50t}\left[5e^{50t}\left(\tfrac{1}{6100}\right)(50\sin 60t - 60\cos 60t)\right] + Ce^{-50t}$$
$$= \tfrac{1}{122}(5\sin 60t - 6\cos 60t) + Ce^{-50t}$$

But $0 = Q(0) = -\frac{6}{122} + C$ so $C = \frac{3}{61}$ and $Q(t) = \dfrac{5\sin 60t - 6\cos 60t}{122} + \dfrac{3e^{-50t}}{61}$ is the charge at time t, while the current is $I(t) = \dfrac{dQ}{dt} = \dfrac{150\cos 60t + 180\sin 60t - 150e^{-50t}}{61}$.

31. $\dfrac{dP}{dt} + kP = kM$, so $I(t) = e^{\int k\,dt} = e^{kt}$. Multiplying the differential equation by $I(t)$ gives $e^{kt}\dfrac{dP}{dt} + kPe^{kt} = kMe^{kt} \;\Rightarrow$

$\left(e^{kt}P\right)' = kMe^{kt} \;\Rightarrow$

$P(t) = e^{-kt}\left(\int kMe^{kt}dt + C\right) = M + Ce^{-kt}$, $k > 0$. Furthermore, it is reasonable to assume that $0 \le P(0) \le M$, so $-M \le C \le 0$.



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32. Since $P(0) = 0$, we have $P(t) = M(1 - e^{-kt})$. If $P_1(t)$ is Jim's learning curve, then $P_1(1) = 25$ and $P_1(2) = 45$. Hence, $25 = M_1(1 - e^{-k})$ and $45 = M_1(1 - e^{-2k})$, so $1 - 25/M_1 = e^{-k}$ or $k = -\ln\left(1 - \dfrac{25}{M_1}\right) = \ln\left(\dfrac{M_1}{M_1 - 25}\right)$. But $45 = M_1(1 - e^{-2k})$ so $45 = M_1\left[1 - \left(\dfrac{M_1 - 25}{M_1}\right)^2\right]$ or $45 = \dfrac{50M_1 - 625}{M_1}$. Thus, $M_1 = 125$ is the maximum number of units per hour Jim is capable of processing.

Similarly, if $P_2(t)$ is Mark's learning curve, then $P_2(1) = 35$ and $P_2(2) = 50$. So $k = \ln\left(\dfrac{M_2}{M_2 - 35}\right)$ and $50 = M_2\left[1 - \left(\dfrac{M_2 - 35}{M_2}\right)^2\right]$ or $M_2 = 61.25$. Hence the maximum number of units per hour for Mark is approximately 61. Another approach would be to use the midpoints of the intervals so that $P_1(0.5) = 25$ and $P_1(1.5) = 45$. Doing so gives us $M_1 \approx 52.6$ and $M_2 \approx 51.8$.

33. $y(0) = 0$ kg. Salt is added at a rate of $\left(0.4 \dfrac{\text{kg}}{\text{L}}\right)\left(5 \dfrac{\text{L}}{\text{min}}\right) = 2 \dfrac{\text{kg}}{\text{min}}$. Since solution is drained from the tank at a rate of 3 L/min, but salt solution is added at a rate of 5 L/min, the tank, which starts out with 100 L of water, contains $(100 + 2t)$ L of liquid after t min. Thus, the salt concentration at time t is $\dfrac{y(t)}{100 + 2t} \dfrac{\text{kg}}{\text{L}}$. Salt therefore leaves the tank at a rate of $\left(\dfrac{y(t)}{100 + 2t} \dfrac{\text{kg}}{\text{L}}\right)\left(3 \dfrac{\text{L}}{\text{min}}\right) = \dfrac{3y}{100 + 2t} \dfrac{\text{kg}}{\text{min}}$. Combining the rates at which salt enters and leaves the tank, we get $\dfrac{dy}{dt} = 2 - \dfrac{3y}{100 + 2t}$. Rewriting this equation as $\dfrac{dy}{dt} + \left(\dfrac{3}{100 + 2t}\right) y = 2$, we see that it is linear. $I(t) = \exp\left(\displaystyle\int \frac{3\,dt}{100 + 2t}\right) = \exp\left(\frac{3}{2}\ln(100 + 2t)\right) = (100 + 2t)^{3/2}$. Multiplying the differential equation by $I(t)$ gives $(100 + 2t)^{3/2}\dfrac{dy}{dt} + 3(100 + 2t)^{1/2}y = 2(100 + 2t)^{3/2} \Rightarrow$

$\left[(100 + 2t)^{3/2}y\right]' = 2(100 + 2t)^{3/2} \Rightarrow (100 + 2t)^{3/2}y = \frac{2}{5}(100 + 2t)^{5/2} + C \Rightarrow$

$y = \frac{2}{5}(100 + 2t) + C(100 + 2t)^{-3/2}$. Now $0 = y(0) = \frac{2}{5}(100) + C \cdot 100^{-3/2} = 40 + \frac{1}{1000}C \Rightarrow$

$C = -40{,}000$, so $y = \left[\frac{2}{5}(100 + 2t) - 40{,}000(100 + 2t)^{-3/2}\right]$ kg. From this solution (no pun intended), we calculate the salt concentration at time t to be $C(t) = \dfrac{y(t)}{100 + 2t} = \left[\dfrac{-40{,}000}{(100 + 2t)^{5/2}} + \dfrac{2}{5}\right] \dfrac{\text{kg}}{\text{L}}$. In particular, $C(20) = \dfrac{-40{,}000}{140^{5/2}} + \dfrac{2}{5} \approx 0.2275 \dfrac{\text{kg}}{\text{L}}$ and $y(20) = \frac{2}{5}(140) - 40{,}000(140)^{-3/2} \approx 31.85$ kg.

34. Let $y(t)$ denote the amount of chlorine in the tank at time t (in seconds). $y(0) = (0.05 \text{ g/L})\,(400 \text{ L}) = 20$ g. The amount of liquid in the tank at time t is $(400 - 6t)$ L since 4 L of water enters the tank each second and 10 L of liquid leaves the tank each second. Thus, the concentration of chlorine at time t is $\dfrac{y(t)}{400 - 6t} \dfrac{\text{g}}{\text{L}}$. Chlorine doesn't enter the tank, but it leaves at a rate of $\left[\dfrac{y(t)}{400 - 6t} \dfrac{\text{g}}{\text{L}}\right]\left[10 \dfrac{\text{L}}{\text{s}}\right] = \dfrac{10\,y(t)}{400 - 6t} \dfrac{\text{g}}{\text{s}} = \dfrac{5\,y(t)}{200 - 3t} \dfrac{\text{g}}{\text{s}}$. Therefore,

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$\frac{dy}{dt} = -\frac{5y}{200-3t} \Rightarrow \int \frac{dy}{y} = \int \frac{-5\,dt}{200-3t} \Rightarrow \ln y = \frac{5}{3}\ln(200-3t) + C \Rightarrow$

$y = \exp\left(\frac{5}{3}\ln(200-3t) + C\right) = e^C(200-3t)^{5/3}$. Now $20 = y(0) = e^C \cdot 200^{5/3} \Rightarrow e^C = \frac{20}{200^{5/3}}$, so

$y(t) = 20\frac{(200-3t)^{5/3}}{200^{5/3}} = 20(1-0.015t)^{5/3}$ g for $0 \le t \le 66\frac{2}{3}$ s, at which time the tank is empty.

35. (a) $\frac{dv}{dt} + \frac{c}{m}v = g$ and $I(t) = e^{\int (c/m)dt} = e^{(c/m)t}$, and multiplying the differential equation by $I(t)$ gives $e^{(c/m)t}\frac{dv}{dt} + \frac{vce^{(c/m)t}}{m} = ge^{(c/m)t} \Rightarrow \left[e^{(c/m)t}v\right]' = ge^{(c/m)t}$. Hence, $v(t) = e^{-(c/m)t}\left[\int ge^{(c/m)t}\,dt + K\right] = mg/c + Ke^{-(c/m)t}$. But the object is dropped from rest, so $v(0) = 0$ and $K = -mg/c$. Thus, the velocity at time t is $v(t) = (mg/c)\left[1 - e^{-(c/m)t}\right]$.

(b) $\lim_{t\to\infty} v(t) = mg/c$

(c) $s(t) = \int v(t)\,dt = (mg/c)\left[t + (m/c)e^{-(c/m)t}\right] + c_1$ where $c_1 = s(0) - m^2g/c^2$. $s(0)$ is the initial position, so $s(0) = 0$ and $s(t) = (mg/c)\left[t + (m/c)e^{-(c/m)t}\right] - m^2g/c^2$.

36. $v = (mg/c)(1 - e^{-ct/m}) \Rightarrow \frac{dv}{dm} = \frac{mg}{c}\left(0 - e^{-ct/m}\cdot\frac{ct}{m^2}\right) + \frac{g}{c}\left(1 - e^{-ct/m}\right)\cdot 1 =$

$-\frac{gt}{m}e^{-ct/m} + \frac{g}{c} - \frac{g}{c}e^{-ct/m} = \frac{g}{c}\left(1 - e^{-ct/m} - \frac{ct}{m}e^{-ct/m}\right) \Rightarrow$

$\frac{c}{g}\frac{dv}{dm} = 1 - \left(1 + \frac{ct}{m}\right)e^{-ct/m} = 1 - \frac{1 + ct/m}{e^{ct/m}} = 1 - \frac{1+Q}{e^Q}$, where $Q = \frac{ct}{m} \ge 0$. Since $e^Q > 1 + Q$ for all $Q > 0$, it follows that $dv/dm > 0$ for $t > 0$. In other words, for all $t > 0$, v increases as m increases.

10.7 Predator-Prey Systems

1. (a) $dx/dt = -0.05x + 0.0001xy$. If $y = 0$, we have $dx/dt = -0.05x$, which indicates that in the absence of y, x declines at a rate proportional to itself. So x represents the predator population and y represents the prey population. The growth of the prey population, $0.1y$ (from $dy/dt = 0.1y - 0.005xy$), is restricted only by encounters with predators (the term $-0.005xy$). The predator population increases only through the term $0.0001xy$; that is, by encounters with the prey and not through additional food sources.

(b) $dy/dt = -0.015y + 0.00008xy$. If $x = 0$, we have $dy/dt = -0.015y$, which indicates that in the absence of x, y would decline at a rate proportional to itself. So y represents the predator population and x represents the prey population. The growth of the prey population, $0.2x$ (from $dx/dt = 0.2x - 0.0002x^2 - 0.006xy = 0.2x(1 - 0.001x) - 0.006xy$), is restricted by a carrying capacity of 1000 [from the term $1 - 0.001x = 1 - x/1000$] and by encounters with predators (the term $-0.006xy$). The predator population increases only through the term $0.00008xy$; that is, by encounters with the prey and not through additional food sources.

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2. (a) $dx/dt = 0.12x - 0.0006x^2 + 0.00001xy$. $dy/dt = 0.08y + 0.00004xy$.

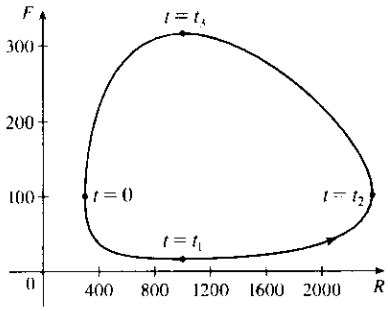
The xy terms represent encounters between the two species x and y. An increase in y makes dx/dt (the growth rate of x) larger due to the positive term $0.00001xy$. An increase in x makes dy/dt (the growth rate of y) larger due to the positive term $0.00004xy$. Hence, the system describes a cooperation model.

(b) $dx/dt = 0.15x - 0.0002x^2 - 0.0006xy = 0.15x(1 - x/750) - 0.0006xy$.

$dy/dt = 0.2y - 0.00008y^2 - 0.0002xy = 0.2y(1 - y/2500) - 0.0002xy$.

The system shows that x and y have carrying capacities of 750 and 2500. An increase in x reduces the growth rate of y due to the negative term $-0.0002xy$. An increase in y reduces the growth rate of x due to the negative term $-0.0006xy$. Hence, the system describes a competition model.

3. (a) At $t = 0$, there are about 300 rabbits and 100 foxes. At $t = t_1$, the number of foxes reaches a minimum of about 20 while the number of rabbits is about 1000. At $t = t_2$, the number of rabbits reaches a maximum of about 2400, while the number of foxes rebounds to 100. At $t = t_3$, the number of rabbits decreases to about 1000 and the number of foxes reaches a maximum of about 315. As t increases, the number of foxes decreases greatly to 100, and the number of rabbits decreases to 300 (the initial populations), and the cycle starts again.



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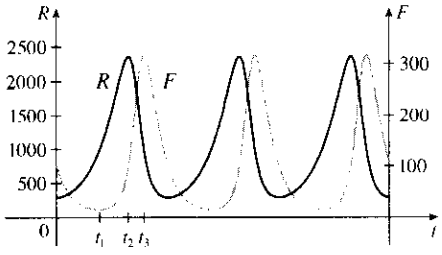
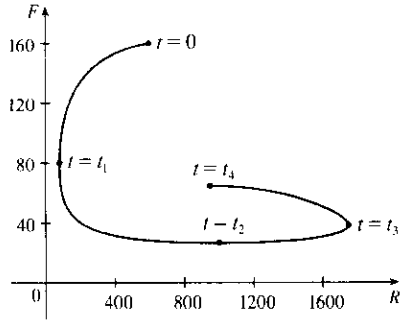


4. (a) At $t = 0$, there are about 600 rabbits and 160 foxes. At $t = t_1$, the number of rabbits reaches a minimum of about 80 and the number of foxes is also 80. At $t = t_2$, the number of foxes reaches a minimum of about 25 while the number of rabbits rebounds to 1000. At $t = t_3$, the number of foxes has increased to 40 and the rabbit population has reached a maximum of about 1750. The curve ends at $t = t_4$, where the number of foxes has increased to 65 and the number of rabbits has decreased to about 950.



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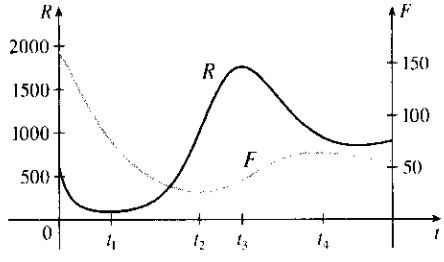

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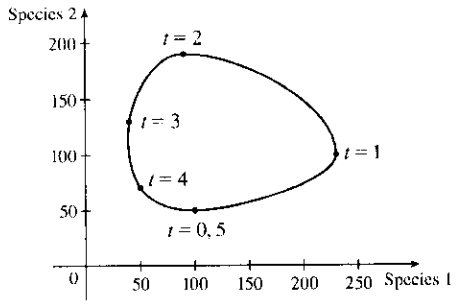

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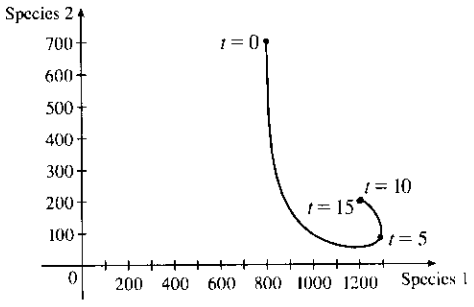


7. $\dfrac{dW}{dR} = \dfrac{-0.02W + 0.00002RW}{0.08R - 0.001RW} \quad \Leftrightarrow \quad (0.08 - 0.001W)R\,dW = (-0.02 + 0.00002R)W\,dR \quad \Leftrightarrow$

$\dfrac{0.08 - 0.001W}{W}\,dW = \dfrac{-0.02 + 0.00002R}{R}\,dR \quad \Leftrightarrow \quad \displaystyle\int \left(\frac{0.08}{W} - 0.001\right) dW = \int \left(-\frac{0.02}{R} + 0.00002\right) dR$

$\Leftrightarrow \quad 0.08 \ln|W| - 0.001W = -0.02 \ln|R| + 0.00002R + K \quad \Leftrightarrow$

$0.08 \ln W + 0.02 \ln R = 0.001W + 0.00002R + K \quad \Leftrightarrow \quad \ln\left(W^{0.08}R^{0.02}\right) = 0.00002R + 0.001W + K \quad \Leftrightarrow$

$W^{0.08}R^{0.02} = e^{0.00002R + 0.001W + K} \quad \Leftrightarrow \quad R^{0.02}W^{0.08} = Ce^{0.00002R}e^{0.001W} \quad \Leftrightarrow \quad \dfrac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C.$

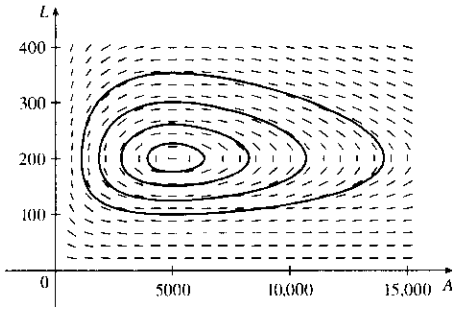
In general, if $\dfrac{dy}{dx} = \dfrac{-ry + bxy}{kx - axy}$, then $C = \dfrac{x^r y^k}{e^{bx}e^{ay}}$.

8. (a) A and L are constant $\Rightarrow A' = 0$ and $L' = 0 \Rightarrow \begin{cases} 0 = 2A - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{cases} \Rightarrow \begin{cases} 0 = A(2 - 0.01L) \\ 0 = L(-0.5 + 0.0001A) \end{cases}$

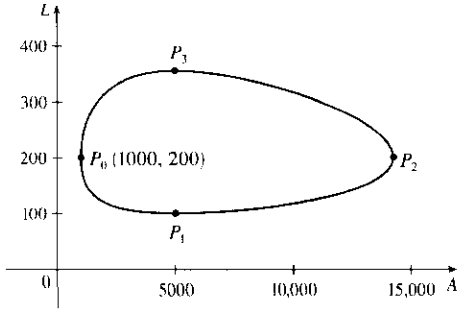
So either $A = L = 0$ or $L = \frac{2}{0.01} = 200$ and $A = \frac{0.5}{0.0001} = 5000$. The trivial solution $A = L = 0$ just says that if there aren't any aphids or ladybugs, then the populations will not change. The non-trivial solution, $L = 200$ and $A = 5000$, indicates the population sizes needed so that there are no changes in either the number of aphids or the number of ladybugs.

(b) $\dfrac{dL}{dA} = \dfrac{dL/dt}{dA/dt} = \dfrac{-0.5L + 0.0001AL}{2A - 0.01AL}$

(c) The solution curves (phase trajectories) are all closed curves that have the equilibrium point $(5000, 200)$ inside them.



(d)



At $P_0(1000, 200)$, $dA/dt = 0$ and $dL/dt = -80 < 0$, so the number of ladybugs is decreasing and hence, we are proceeding in a counterclockwise direction. At P_0, there aren't enough aphids to support the ladybug population, so the number of ladybugs decreases and the number of aphids begins to increase. The ladybug population reaches a minimum at $P_1(5000, 100)$ while the aphid population increases in a dramatic way, reaching its maximum at $P_2(14{,}250, 200)$. Meanwhile, the ladybug population is increasing from P_1 to $P_3(5000, 355)$, and as we pass through P_2, the increasing number of ladybugs starts to deplete the aphid population. At P_3 the ladybugs reach a maximum

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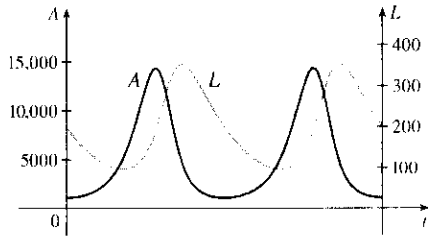
population, and start to decrease due to the reduced aphid population. Both populations then decrease until P_0, where the cycle starts over again.

(e) Both graphs have the same period and the graph of L peaks about a quarter of a cycle after the graph of A.



9. (a) Letting $W = 0$ gives us $dR/dt = 0.08R(1 - 0.0002R)$. $dR/dt = 0 \quad \Leftrightarrow \quad R = 0$ or 5000. Since $dR/dt > 0$ for $0 < R < 5000$, we would expect the rabbit population to *increase* to 5000 for these values of R. Since $dR/dt < 0$ for $R > 5000$, we would expect the rabbit population to *decrease* to 5000 for these values of R. Hence, in the absence of wolves, we would expect the rabbit population to stabilize at 5000.

(b) R and W are constant $\Rightarrow \quad R' = 0$ and $W' = 0 \quad \Rightarrow$

$$\left.\begin{aligned} 0 &= 0.08R(1 - 0.0002R) - 0.001RW \\ 0 &= -0.02W + 0.00002RW \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} 0 &= R[0.08(1 - 0.0002R) - 0.001W] \\ 0 &= W(-0.02 + 0.00002R) \end{aligned}$$

The second equation is true if $W = 0$ or $R = \frac{0.02}{0.00002} = 1000$. If $W = 0$ in the first equation, then either $R = 0$ or $R = \frac{1}{0.0002} = 5000$ [as in part (a)]. If $R = 1000$, then $0 = 1000[0.08(1 - 0.0002 \cdot 1000) - 0.001W] \quad \Leftrightarrow$ $0 = 80(1 - 0.2) - W \quad \Leftrightarrow \quad W = 64$.

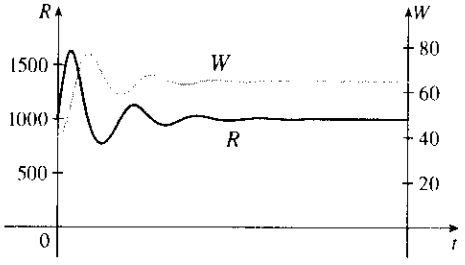
Case (i): $W = 0$, $R = 0$: both populations are zero

Case (ii): $W = 0$, $R = 5000$: see part (a)

Case (iii): $R = 1000$, $W = 64$: the predator/prey interaction balances and the populations are stable.

(c) The populations of wolves and rabbits fluctuate around 64 and 1000, respectively, and eventually stabilize at those values.

(d)



10. (a) If $L = 0$, $dA/dt = 2A\,(1 - 0.0001A)$, so $dA/dt = 0 \quad \Leftrightarrow \quad A = 0$ or $A = \frac{1}{0.0001} = 10{,}000$. Since $dA/dt > 0$ for $0 < A < 10{,}000$, we expect the aphid population to *increase* to 10,000 for these values of A. Since $dA/dt < 0$ for $A > 10{,}000$, we expect the aphid population to *decrease* to 10,000 for these values of A. Hence, in the absence of ladybugs we expect the aphid population to stabilize at 10,000.

(b) A and L are constant $\Rightarrow \quad A' = 0$ and $L' = 0 \quad \Rightarrow$

$$\left.\begin{aligned} 0 &= 2A(1 - 0.0001A) - 0.01AL \\ 0 &= -0.5L + 0.0001AL \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} 0 &= A[2(1 - 0.0001A) - 0.01L] \\ 0 &= L(-0.5 + 0.0001A) \end{aligned}$$

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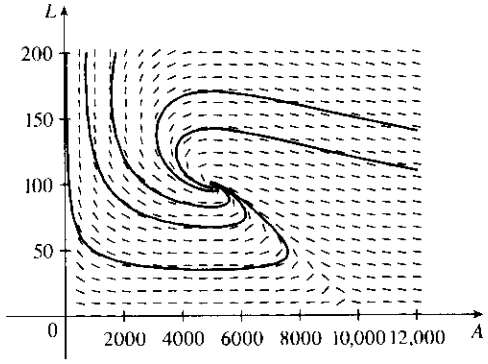
The second equation is true if $L = 0$ or $A = \frac{0.5}{0.0001} = 5000$. If $L = 0$ in the first equation, then either $A = 0$ or $A = \frac{1}{0.0001} = 10{,}000$. If $A = 5000$, then $0 = 5000[2(1 - 0.0001 \cdot 5000) - 0.01L] \Leftrightarrow$ $0 = 10{,}000(1 - 0.5) - 50L \Leftrightarrow 50L = 5000 \Leftrightarrow L = 100$. The equilibrium solutions are:

(i) $L = 0, A = 0$ (ii) $L = 0, A = 10{,}000$ (iii) $A = 5000, L = 100$

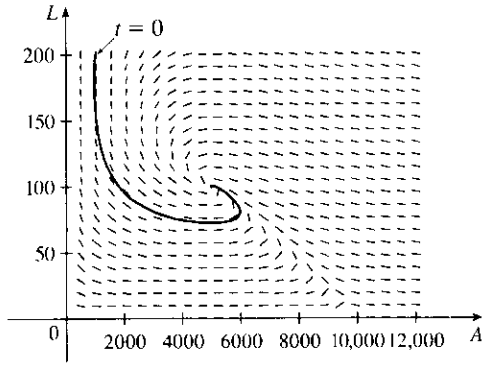
(c) $\dfrac{dL}{dA} = \dfrac{dL/dt}{dA/dt} = \dfrac{-0.5L + 0.0001AL}{2A(1 - 0.0001A) - 0.01AL}$

(d)



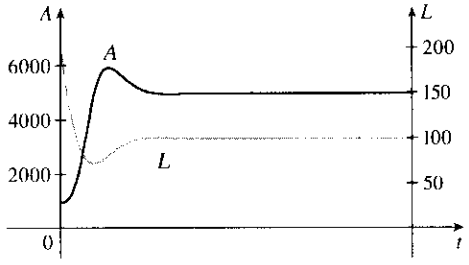
All of the phase trajectories spiral tightly around the equilibrium solution $(5000, 100)$.

(e)



At $t = 0$, the ladybug population decreases rapidly and the aphid population decreases slightly before beginning to increase. As the aphid population continues to increase, the ladybug population reaches a minimum at about $(5000, 75)$. The ladybug population starts to increase and quickly stabilizes at 100, while the aphid population stabilizes at 5000.

(f)



The graph of A peaks just after the graph of L has a minimum.

10 Review

CONCEPT CHECK

1. (a) A differential equation is an equation that contains an unknown function and one or more of its derivatives.

   (b) The order of a differential equation is the order of the highest derivative that occurs in the equation.

   (c) An initial condition is a condition of the form $y(t_0) = y_0$.

2. $y' = x^2 + y^2 \geq 0$ for all x and y. $y' = 0$ only at the origin, so there is a horizontal tangent at $(0, 0)$, but nowhere else. The graph of the solution is increasing on every interval.

3. See the paragraph preceding Example 1 in Section 10.2.

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4. See the paragraph after Figure 14 in Section 10.2.

5. A separable equation is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y, that is, $dy/dx = g(x)f(y)$. We can solve the equation by integrating both sides of the equation $dy/f(y) = g(x)dx$ and solving for y.

6. A first-order linear differential equation is a differential equation that can be put in the form $\dfrac{dy}{dx} + P(x)\,y = Q(x)$, where P and Q are continuous functions on a given interval. To solve such an equation, multiply it by the integrating factor $I(x) = e^{\int P(x)dx}$ to put it in the form $[I(x)\,y]' = I(x)\,Q(x)$ and then integrate both sides to get $I(x)\,y = \int I(x)\,Q(x)\,dx$, that is, $e^{\int P(x)\,dx}y = \int e^{\int P(x)\,dx}Q(x)\,dx$. Solving for y gives us $y = e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx}Q(x)\,dx$.

7. (a) $\dfrac{dy}{dt} = ky$; the relative growth rate, $\dfrac{1}{y}\dfrac{dy}{dt}$, is constant.

   (b) The equation in part (a) is an appropriate model for population growth, assuming that there is enough room and nutrition to support the growth.

   (c) If $y(0) = y_0$, then the solution is $y(t) = y_0e^{kt}$.

8. (a) $dP/dt = kP(1 - P/K)$, where K is the carrying capacity.

   (b) The equation in part (a) is an appropriate model for population growth, assuming that the population grows at a rate proportional to the size of the population in the beginning, but eventually levels off and approaches its carrying capacity because of limited resources.

9. (a) $dF/dt = kF - aFS$ and $dS/dt = -rS + bFS$.

   (b) In the absence of sharks, an ample food supply would support exponential growth of the fish population, that is, $dF/dt = kF$, where k is a positive constant. In the absence of fish, we assume that the shark population would decline at a rate proportional to itself, that is, $dS/dt = -rS$, where r is a positive constant.

TRUE-FALSE QUIZ

1. True. Since $y^4 \geq 0$, $y' = -1 - y^4 < 0$ and the solutions are decreasing functions.

2. True. $y = \dfrac{\ln x}{x} \quad \Rightarrow \quad y' = \dfrac{1 - \ln x}{x^2}$.

   LHS $= x^2y' + xy = x^2 \cdot \dfrac{1 - \ln x}{x^2} + x \cdot \dfrac{\ln x}{x} = (1 - \ln x) + \ln x = 1 =$ RHS, so $y = \dfrac{\ln x}{x}$ is a solution of $x^2y' + xy = 1$.

3. False. $x + y$ cannot be written in the form $g(x)f(y)$.

4. True. $y' = 3y - 2x + 6xy - 1 = 6xy - 2x + 3y - 1 = 2x(3y - 1) + 1(3y - 1) = (2x + 1)(3y - 1)$, so y' can be written in the form $g(x)f(y)$, and hence, is separable.

5. True. $e^xy' = y \quad \Rightarrow \quad y' = e^{-x}y \quad \Rightarrow \quad y' + (-e^{-x})y = 0$, which is of the form $y' + P(x)\,y = Q(x)$, so the equation is linear.

6. False. $y' + xy = e^y$ cannot be put in the form $y' + P(x)\,y = Q(x)$, so it is not linear.

7. True. By comparing $\dfrac{dy}{dt} = 2y\left(1 - \dfrac{y}{5}\right)$ with the logistic differential equation (10.5.1), we see that the carrying capacity is 5; that is, $\lim\limits_{t\to\infty} y = 5$.

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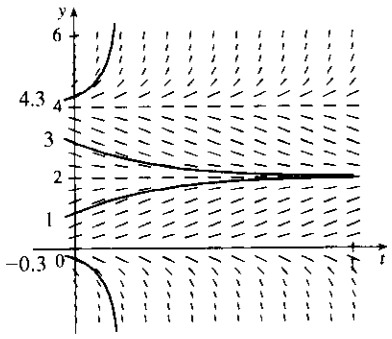
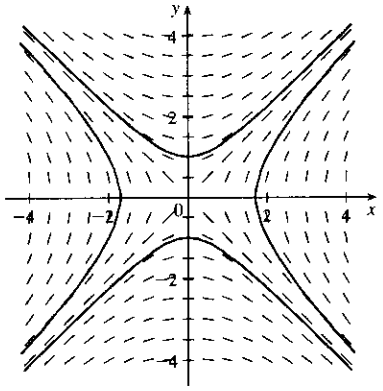
EXERCISES

1. (a)



(b) $\lim_{t\to\infty} y(t)$ appears to be finite for $0 \le c \le 4$. In fact $\lim_{t\to\infty} y(t) = 4$ for $c = 4$, $\lim_{t\to\infty} y(t) = 2$ for $0 < c < 4$, and $\lim_{t\to\infty} y(t) = 0$ for $c = 0$. The equilibrium solutions are $y(t) = 0$, $y(t) = 2$, and $y(t) = 4$.

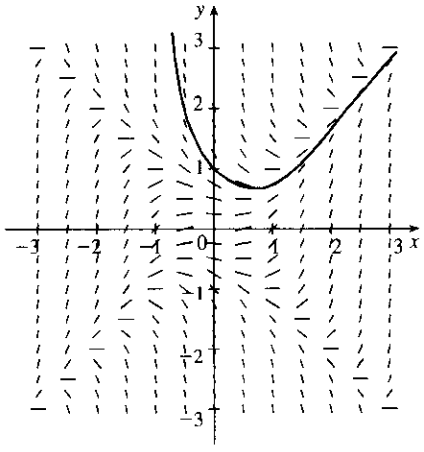
2. (a)



We sketch the direction field and four solution curves, as shown. Note that the slope $y' = x/y$ is not defined on the line $y = 0$.

(b) $y' = x/y \quad \Leftrightarrow \quad y\,dy = x\,dx \quad \Leftrightarrow y^2 = x^2 + C$. For $C = 0$, this is the pair of lines $y = \pm x$. For $C \neq 0$, it is the hyperbola $x^2 - y^2 = -C$.

3. (a)



We estimate that when $x = 0.3$, $y = 0.8$, so $y(0.3) \approx 0.8$.

(b) $h = 0.1$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = x^2 - y^2$. So $y_n = y_{n-1} + 0.1\left(x_{n-1}^2 - y_{n-1}^2\right)$. Thus,
$y_1 = 1 + 0.1\left(0^2 - 1^2\right) = 0.9$,
$y_2 = 0.9 + 0.1\left(0.1^2 - 0.9^2\right) = 0.82$,
$y_3 = 0.82 + 0.1\left(0.2^2 - 0.82^2\right) = 0.75676$. This is close to our graphical estimate of $y(0.3) \approx 0.8$.

(c) The centers of the horizontal line segments of the direction field are located on the lines $y = x$ and $y = -x$. When a solution curve crosses one of these lines, it has a local maximum or minimum.

4. (a) $h = 0.2$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = 2xy^2$. We need y_2.
$y_1 = 1 + 0.2\left(2 \cdot 0 \cdot 1^2\right) = 1$, $y_2 = 1 + 0.2\left(2 \cdot 0.2 \cdot 1^2\right) = 1.08 \approx y(0.4)$.

(b) $h = 0.1$ now, so $y_1 = 1 + 0.1\left(2 \cdot 0 \cdot 1^2\right) = 1$, $y_2 = 1 + 0.1\left(2 \cdot 0.1 \cdot 1^2\right) = 1.02$,
$y_3 = 1.02 + 0.1\left(2 \cdot 0.2 \cdot 1.02^2\right) \approx 1.06162$, $y_4 = 1.06162 + 0.1\left(2 \cdot 0.3 \cdot 1.06162^2\right) \approx 1.1292 \approx y(0.4)$.

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(c) The equation is separable, so we write $\frac{dy}{y^2} = 2x\,dx \;\Rightarrow\; \int \frac{dy}{y^2} = \int 2x\,dx \;\Leftrightarrow\; -\frac{1}{y} = x^2 + C$, but $y(0) = 1$, so $C = -1$ and $y(x) = \dfrac{1}{1-x^2} \;\Leftrightarrow\; y(0.4) = \dfrac{1}{1-0.16} \approx 1.1905$. From this we see that the approximation was greatly improved by increasing the number of steps, but the approximations were still far off.

5. $y' = xe^{-\sin x} - y\cos x \;\Rightarrow\; y' + (\cos x)\,y = xe^{-\sin x}$ $(*)$. This is a linear equation and the integrating factor is $I(x) = e^{\int \cos x\,dx} = e^{\sin x}$. Multiplying $(*)$ by $e^{\sin x}$ gives $e^{\sin x}\,y' + e^{\sin x}(\cos x)\,y = x \;\Rightarrow\; (e^{\sin x}\,y)' = x$ $\Rightarrow\; e^{\sin x}\,y = \frac{1}{2}x^2 + C \;\Rightarrow\; y = \left(\frac{1}{2}x^2 + C\right)e^{-\sin x}$.

6. $\dfrac{dx}{dt} = 1 - t + x - tx = 1(1-t) + x(1-t) = (1+x)(1-t) \;\Rightarrow\; \dfrac{dx}{1+x} = (1-t)\,dt \;\Rightarrow$ $\displaystyle\int \frac{dx}{1+x} = \int (1-t)\,dt \;\Rightarrow\; \ln|1+x| = t - \tfrac{1}{2}t^2 + C \;\Rightarrow\; |1+x| = e^{t - t^2/2 + C} \;\Rightarrow$ $1 + x = \pm e^{t-t^2/2}\cdot e^C \;\Rightarrow\; x = -1 + Ke^{t-t^2/2}$, where K is any nonzero constant.

7. $(3y^2 + 2y)\,y' = x\cos x \;\Rightarrow\; (3y^2 + 2y)\,dy = (x\cos x)\,dx \;\Rightarrow\; \int (3y^2 + 2y)\,dy = \int (x\cos x)\,dx \;\Rightarrow$ $y^3 + y^2 = \cos x + x\sin x + C$. For the last step, use integration by parts or Formula 83 in the Table of Integrals.

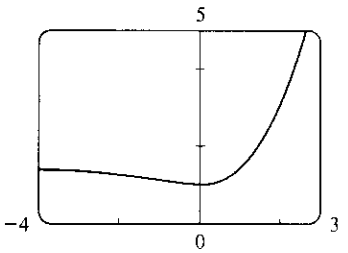
8. $x^2y' - y = 2x^3e^{-1/x} \;\Rightarrow\; y' - \dfrac{1}{x^2}\,y = 2xe^{-1/x}$ $(*)$. This is a linear equation and the integrating factor is $I(x) = e^{\int(-1/x^2)\,dx} = e^{1/x}$. Multiplying $(*)$ by $e^{1/x}$ gives $e^{1/x}\,y' - e^{1/x}\cdot\dfrac{1}{x^2}\,y = 2x \;\Rightarrow\; (e^{1/x}\,y)' = 2x \;\Rightarrow$ $e^{1/x}\,y = x^2 + C \;\Rightarrow\; y = e^{-1/x}(x^2 + C)$.

9. $xyy' = \ln x \;\Rightarrow\; y\,dy = \dfrac{\ln x}{x}\,dx \;\Rightarrow\; \displaystyle\int y\,dy = \int \frac{\ln x}{x}\,dx$ (Make the substitution $u = \ln x$; then $du = dx/x$.) So $\int y\,dy = \int u\,du \;\Rightarrow\; \frac{1}{2}y^2 = \frac{1}{2}u^2 + C \;\Rightarrow\; \frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + C$. $y(1) = 2 \;\Rightarrow$ $\frac{1}{2}2^2 = \frac{1}{2}(\ln 1)^2 + C = C \;\Leftrightarrow\; C = 2$. Therefore, $\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + 2$, or $y = \sqrt{(\ln x)^2 + 4}$. The negative square root is inadmissible, since $y(1) > 0$.

10. $1 + x = 2xyy' \;\Rightarrow\; y' = \dfrac{1+x}{2xy} \;\Leftrightarrow\; y\,dy = \dfrac{1+x}{2x}\,dx \;\Rightarrow\; \dfrac{y^2}{2} = \dfrac{\ln|x|}{2} + \dfrac{x}{2} + c_1$. But $x > 0$, so $y^2 = \ln x + x + c \;\Leftrightarrow\; y(x) = \pm\sqrt{c + x + \ln x}$. But $-2 = y(1)$ so choose the negative square root and $-2 = -\sqrt{c+1}$ so $c = 3$. Thus, the solution is $y(x) = -\sqrt{3 + x + \ln x}$.

11. Since the equation is linear, let $I(x) = e^{\int dx} = e^x$. Then multiplying by $I(x)$ gives $e^xy' + e^xy = \sqrt{x} \;\Rightarrow$ $(e^xy)' = \sqrt{x} \;\Rightarrow\; y(x) = e^{-x}\left(\int \sqrt{x}\,dx + c\right) = e^{-x}\left(\frac{2}{3}x^{3/2} + c\right)$. But $3 = y(0) = c$, so the solution to the initial-value problem is $y(x) = e^{-x}\left(\frac{2}{3}x^{3/2} + 3\right)$.

12. $2yy' = xe^x \;\Rightarrow\; \int 2y\,dy = \int xe^x\,dx \;\Rightarrow$ $y^2 = xe^x - \int e^x\,dx$ (by parts) $= (x-1)\,e^x + C$. We substitute the initial condition: $1^2 = (0-1)\,e^0 + C \;\Rightarrow\; C = 2$. So the solution is $y = \sqrt{(x-1)\,e^x + 2}$. The negative square root is inadmissible due to the initial condition.



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13. The curves $kx^2 + y^2 = 1$ form a family of ellipses for $k > 0$, a family of hyperbolas for $k < 0$, and two parallel lines $y = \pm 1$ for $k = 0$. Solving $kx^2 + y^2 = 1$ for k gives $k = \dfrac{1 - y^2}{x^2}$. Differentiating gives $2kx + 2yy' = 0 \quad \Leftrightarrow$ $y' = -\dfrac{kx}{y} = -\left(1 - y^2\right)\dfrac{x}{yx^2} = \dfrac{y^2 - 1}{xy}$. Thus, for $k \neq 0$ the orthogonal trajectories must satisfy $y' = -\dfrac{xy}{y^2 - 1}$ $\Rightarrow \quad \dfrac{y^2 - 1}{y}\,dy = -x\,dx \quad \Rightarrow \quad \dfrac{y^2}{2} - \ln|y| = \dfrac{-x^2}{2} + K \quad \Rightarrow \quad y^2 - 2\ln|y| + x^2 = C$. For $k = 0$, the orthogonal trajectories are given by $x = C_1$ for C_1 an arbitrary constant.

14. Differentiating both sides of $y = \dfrac{k}{1 + x^2}$ gives $y' = -\dfrac{2kx}{(1 + x^2)^2} = -2xy\,\dfrac{1 + x^2}{(1 + x^2)^2} = -\dfrac{2xy}{1 + x^2}$. Thus, for $k \neq 0$ the orthogonal trajectories must satisfy $y' = \dfrac{1 + x^2}{2xy} \quad \Rightarrow \quad 2y\,dy = \left(\dfrac{1}{x} + x\right)dx \quad \Rightarrow$

$y^2 = \dfrac{x^2}{2} + \ln|x| + C$. For $k = 0$, the orthogonal trajectories are given by $x = C_2$ for C_2 an arbitrary constant.

15. (a) $y(t) = y(0)e^{kt} = 1000e^{kt} \quad \Rightarrow \quad y(2) = 1000e^{2k} = 9000 \quad \Rightarrow \quad e^{2k} = 9 \quad \Rightarrow \quad 2k = \ln 9 \quad \Rightarrow$ $k = \frac{1}{2}\ln 9 = \ln 3 \quad \Rightarrow \quad y(t) = 1000e^{(\ln 3)t} = 1000 \cdot 3^t$

(b) $y(3) = 1000 \cdot 3^3 = 27{,}000$

(c) $y'(t) = 1000 \cdot 3^t \cdot \ln 3$, so $y'(3) = 27{,}000 \ln 3 \approx 29{,}663$ bacteria per hour

(d) $1000 \cdot 3^t = 2 \cdot 1000 \quad \Rightarrow \quad 3^t = 2 \quad \Rightarrow \quad t \ln 3 = \ln 2 \quad \Rightarrow \quad t = (\ln 2)/\ln 3 \approx 0.63$ h

16. (a) If $y(t)$ is the mass remaining after t years, then $y(t) = y(0)e^{kt} = 18e^{kt}$. $y(25) = 18e^{25k} = \frac{1}{2} \cdot 18 \quad \Rightarrow$ $e^{25k} = \frac{1}{2} \quad \Rightarrow \quad 25k = -\ln 2 \quad \Rightarrow \quad k = -\frac{1}{25}\ln 2 \quad \Rightarrow \quad y(t) = 18\,e^{-(\ln 2)t/25} = 18 \cdot 2^{-t/25}$.

(b) $18 \cdot 2^{-t/25} = 2 \quad \Rightarrow \quad 2^{-t/25} = \frac{1}{9} \quad \Rightarrow \quad -\frac{1}{25}t \ln 2 = -\ln 9 \quad \Rightarrow \quad t = 25\,\frac{\ln 9}{\ln 2} \approx 79$ years

17. (a) $C'(t) = -kC(t) \quad \Rightarrow \quad C(t) = C(0)e^{-kt}$ by Theorem 10.4.2. But $C(0) = C_0$, so $C(t) = C_0 e^{-kt}$.

(b) $C(30) = \frac{1}{2}C_0$ since the concentration is reduced by half. Thus, $\frac{1}{2}C_0 = C_0 e^{-30k} \quad \Rightarrow \ln\frac{1}{2} = -30k \quad \Rightarrow$ $k = -\frac{1}{30}\ln\frac{1}{2} = \frac{1}{30}\ln 2$. Since 10% of the original concentration remains if 90% is eliminated, we want the value of t such that $C(t) = \frac{1}{10}C_0$. Therefore, $\frac{1}{10}C_0 = C_0 e^{-t(\ln 2)/30} \quad \Rightarrow \quad \ln 0.1 = -t(\ln 2)/30 \quad \Rightarrow$ $t = -\frac{30}{\ln 2}\ln 0.1 \approx 100$ h.

18. (a) Let $t = 0$ correspond to 1990 so that $P(t) = 5.28e^{kt}$ is a starting point for the model. When $t = 10$, $P = 6.07$. So $6.07 = 5.28e^{10k} \quad \Rightarrow \quad 10k = \ln\frac{6.07}{5.28} \quad \Rightarrow \quad k = \frac{1}{10}\ln\frac{6.07}{5.28} \approx 0.01394$. For the year 2020, $t = 30$, and $P(30) = 5.28e^{30k} \approx 8.02$ billion.

(b) $P = 10 \quad \Rightarrow \quad 5.28e^{kt} = 10 \quad \Rightarrow \quad \frac{10}{5.28} = e^{kt} \quad \Rightarrow \quad kt = \ln\frac{10}{5.28} \quad \Rightarrow \quad t = 10\dfrac{\ln\frac{10}{5.28}}{\ln\frac{6.07}{5.28}} \approx 45.8$ years; that is, in $1990 + 45 = 2035$.

(c) $P(t) = \dfrac{K}{1 + Ae^{-kt}} = \dfrac{100}{1 + Ae^{-kt}}$, where $A = \dfrac{100 - 5.28}{5.28} \approx 17.94$. Using $k = \dfrac{1}{10}\ln\dfrac{6.07}{5.28}$ from part (a), a model is $P(t) \approx \dfrac{100}{1 + 17.94e^{-0.01394t}}$ and $P(30) \approx 7.81$ billion, slightly lower than our estimate of 8.02 billion in part (a).

(d) $P = 10 \quad \Rightarrow \quad 1 + Ae^{-kt} = \frac{100}{10} \quad \Rightarrow \quad Ae^{-kt} = 9 \quad \Rightarrow \quad e^{-kt} = 9/A \quad \Rightarrow \quad -kt = \ln(9/A) \quad \Rightarrow$ $t = -\dfrac{1}{k}\ln\dfrac{9}{A} \approx 49.47$ years (that is, in 2039), which is later than the prediction of 2035 in part (b).

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19. (a) $\dfrac{dL}{dt} \propto L_\infty - L \;\Rightarrow\; \dfrac{dL}{dt} = k(L_\infty - L) \;\Rightarrow\; \displaystyle\int \frac{dL}{L_\infty - L} = \int k\,dt \;\Rightarrow\; -\ln|L_\infty - L| = kt + C \;\Rightarrow$
$\ln|L_\infty - L| = -kt - C \;\Rightarrow\; |L_\infty - L| = e^{-kt-C} \;\Rightarrow\; L_\infty - L = Ae^{-kt} \;\Rightarrow\; L = L_\infty - Ae^{-kt}.$
At $t = 0$, $L = L(0) = L_\infty - A \;\Rightarrow\; A = L_\infty - L(0) \;\Rightarrow\; L(t) = L_\infty - [L_\infty - L(0)]\,e^{-kt}.$

(b) $L_\infty = 53$ cm, $L(0) = 10$ cm, and $k = 0.2 \;\Rightarrow\; L(t) = 53 - (53 - 10)e^{-0.2t} = 53 - 43e^{-0.2t}.$

20. Denote the amount of salt in the tank (in kg) by y. $y(0) = 0$ since initially there is only water in the tank. The rate at which y increases is equal to the rate at which salt flows into the tank minus the rate at which it flows out. That rate is $\dfrac{dy}{dt} = 0.1\dfrac{\text{kg}}{\text{L}} \times 10\dfrac{\text{L}}{\text{min}} - \dfrac{y}{100}\dfrac{\text{kg}}{\text{L}} \times 10\dfrac{\text{L}}{\text{min}} = 1 - \dfrac{y}{10}\dfrac{\text{kg}}{\text{min}} \;\Rightarrow\; \displaystyle\int \frac{dy}{10 - y} = \int \frac{1}{10}\,dt \;\Rightarrow$

$-\ln|10 - y| = \frac{1}{10}t + C \;\Rightarrow\; 10 - y = Ae^{-t/10}.\; y(0) = 0 \;\Rightarrow\; 10 = A \;\Rightarrow\; y = 10\left(1 - e^{-t/10}\right).$

At $t = 6$ minutes, $y = 10\left(1 - e^{-6/10}\right) \approx 4.512$ kg.

21. Let P be the population and I be the number of infected people. The rate of spread dI/dt is jointly proportional to I and to $P - I$, so for some constant k, $dI/dt = kI(P - I) \;\Rightarrow\; I = \dfrac{I_0 P}{I_0 + (P - I_0)e^{-kPt}}$ (from the discussion of logistic growth in Section 10.5).

Now, measuring t in days, we substitute $t = 7$, $P = 5000$, $I_0 = 160$ and $I(7) = 1200$ to find k:

$1200 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)e^{-5000 \cdot 7 \cdot k}} \;\Leftrightarrow\; k \approx 0.00006448$. So, putting $I = 5000 \times 80\% = 4000$, we solve for t: $4000 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)e^{-0.00006448 \cdot 5000 \cdot t}} \;\Leftrightarrow\; 160 + 4840e^{-0.3224t} = 200 \;\Leftrightarrow$

$-0.3224t = \ln\frac{40}{4840} \;\Leftrightarrow\; t \approx 14.9$. So it takes about 15 days for 80% of the population to be infected.

22. $\dfrac{1}{R}\dfrac{dR}{dt} = \dfrac{k}{S}\dfrac{dS}{dt} \;\Rightarrow\; \dfrac{d}{dt}(\ln R) = \dfrac{d}{dt}(k \ln S) \;\Rightarrow\; \ln R = k \ln S + C \;\Rightarrow$

$R = e^{k \ln S + C} = e^C\left(e^{\ln S}\right)^k \;\Rightarrow\; R = AS^k$, where $A = e^C$ is a positive constant.

23. $\dfrac{dh}{dt} = -\dfrac{R}{V}\left(\dfrac{h}{k + h}\right) \;\Rightarrow\; \displaystyle\int \frac{k + h}{h}\,dh = \int\left(-\frac{R}{V}\right)dt \;\Rightarrow\; \int\left(1 + \frac{k}{h}\right)dh = -\frac{R}{V}\int 1\,dt \;\Rightarrow$

$h + k \ln h = -\dfrac{R}{V}t + C$. This equation gives a relationship between h and t, but it is not possible to isolate h and express it in terms of t.

24. $dx/dt = 0.4x - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$

(a) The xy terms represent encounters between the birds and the insects. Since the y-population increases from these terms and the x-population decreases, we expect y to represent the birds and x the insects.

(b) x and y are constant $\;\Rightarrow\; x' = 0$ and $y' = 0 \;\Rightarrow$

$$\left\{\begin{matrix} 0 = 0.4x - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{matrix}\right\} \;\Rightarrow\; \left\{\begin{matrix} 0 = 0.4x(1 - 0.005y) \\ 0 = -0.2y(1 - 0.00004x) \end{matrix}\right. \;\Rightarrow\; y = 0 \text{ and } x = 0 \text{ (zero populations)}$$

or $y = \frac{1}{0.005} = 200$ and $x = \frac{1}{0.00004} = 25{,}000$. The non-trivial solution represents the population sizes needed so that there are no changes in either the number of birds or the number of insects.

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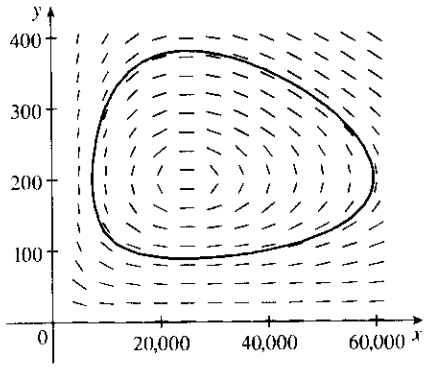
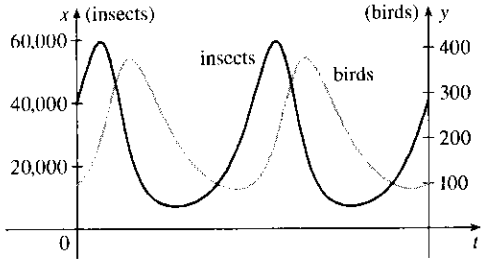
(c) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-0.2y + 0.000008xy}{0.4x - 0.002xy}$

(d)



At $(x, y) = (40{,}000, 100)$, $dx/dt = 8000 > 0$, so as t increases we are proceeding in a counterclockwise direction. The populations increase to approximately $(59{,}646, 200)$, at which point the insect population starts to decrease. The birds attain a maximum population of about 380 when the insect population is 25,000. The populations decrease to about $(7370, 200)$, at which point the insect population starts to increase. The birds attain a minimum population of about 88 when the insect population is 25,000, and then the cycle repeats.

(e)



Both graphs have the same period and the bird population peaks about a quarter-cycle after the insect population.

25. (a) $dx/dt = 0.4x(1 - 0.000005x) - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$. If $y = 0$, then $dx/dt = 0.4x(1 - 0.000005x)$, so $dx/dt = 0 \;\Leftrightarrow\; x = 0$ or $x = 200{,}000$, which shows that the insect population increases logistically with a carrying capacity of 200,000. Since $dx/dt > 0$ for $0 < x < 200{,}000$ and $dx/dt < 0$ for $x > 200{,}000$, we expect the insect population to stabilize at 200,000.

(b) x and y are constant $\Rightarrow$ $x' = 0$ and $y' = 0$ $\Rightarrow$

$$\left.\begin{aligned} 0 &= 0.4x(1 - 0.000005x) - 0.002xy \\ 0 &= -0.2y + 0.000008xy \end{aligned}\right\} \;\Rightarrow\; \begin{cases} 0 = 0.4x[(1 - 0.000005x) - 0.005y] \\ 0 = y(-0.2 + 0.000008x) \end{cases}$$

The second equation is true if $y = 0$ or $x = \frac{0.2}{0.000008} = 25{,}000$. If $y = 0$ in the first equation, then either $x = 0$ or $x = \frac{1}{0.000005} = 200{,}000$. If $x = 25{,}000$, then $0 = 0.4(25{,}000)[(1 - 0.000005 \cdot 25{,}000) - 0.005y] \;\Rightarrow$ $0 = 10{,}000[(1 - 0.125) - 0.005y] \;\Rightarrow\; 0 = 8750 - 50y \;\Rightarrow\; y = 175$.

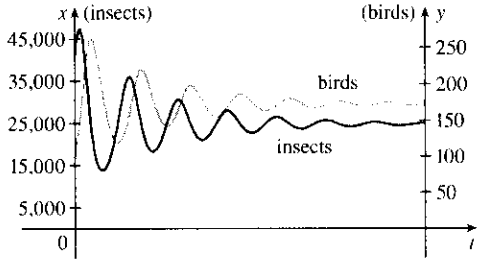
Case (i): $y = 0$, $x = 0$: Zero populations

Case (ii): $y = 0$, $x = 200{,}000$: In the absence of birds, the insect population is always 200,000.

Case (iii): $x = 25{,}000$, $y = 175$: The predator/prey interaction balances and the populations are stable.

(c) The populations of the birds and insects fluctuate around 175 and 25,000, respectively, and eventually stabilize at those values.

(d)



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26. First note that, in this question, "weighs" is used in the informal sense, so what we really require is Barbara's mass m in kg as a function of t. Barbara's net intake of calories per day at time t (measured in days) is $c(t) = 1600 - 850 - 15m(t) = 750 - 15m(t)$, where $m(t)$ is her mass at time t. We are given that $m(0) = 60$ kg and $\dfrac{dm}{dt} = \dfrac{c(t)}{10{,}000}$, so $\dfrac{dm}{dt} = \dfrac{750 - 15m}{10{,}000} = \dfrac{150 - 3m}{2000} = \dfrac{-3(m-50)}{2000}$ with $m(0) = 60$. From $\displaystyle\int \frac{dm}{m-50} = \int \frac{-3\,dt}{2000}$, we get $\ln|m-50| = -\frac{3}{2000}t + C$. Since $m(0) = 60$, $C = \ln 10$. Now $\ln\dfrac{|m-50|}{10} = -\dfrac{3t}{2000}$, so $|m-50| = 10e^{-3t/2000}$. The quantity $m - 50$ is continuous, initially positive, and the right-hand side is never zero. Thus, $m - 50$ is positive for all t, and $m(t) = 50 + 10e^{-3t/2000}$ kg. As $t \to \infty$, $m(t) \to 50$ kg. Thus, Barbara's mass gradually settles down to 50 kg.

27. (a) $\dfrac{d^2y}{dx^2} = k\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$. Setting $z = \dfrac{dy}{dx}$, we get $\dfrac{dz}{dx} = k\sqrt{1+z^2} \quad \Rightarrow \quad \dfrac{dz}{\sqrt{1+z^2}} = k\,dx$. Using Formula 25 gives $\ln\left(z + \sqrt{1+z^2}\right) = kx + c \quad \Rightarrow \quad z + \sqrt{1+z^2} = Ce^{kx}$ (where $C = e^c$) $\Rightarrow$ $\sqrt{1+z^2} = Ce^{kx} - z \quad \Rightarrow \quad 1 + z^2 = C^2e^{2kx} - 2Ce^{kx}z + z^2 \quad \Rightarrow \quad 2Ce^{kx}z = C^2e^{2kx} - 1 \quad \Rightarrow$ $z = \dfrac{C}{2}e^{kx} - \dfrac{1}{2C}e^{-kx}$. Now $\dfrac{dy}{dx} = \dfrac{C}{2}e^{kx} - \dfrac{1}{2C}e^{-kx} \quad \Rightarrow \quad y = \dfrac{C}{2k}e^{kx} + \dfrac{1}{2Ck}e^{-kx} + C'$. From the diagram in the text, we see that $y(0) = a$ and $y(\pm b) = h$. $a = y(0) = \dfrac{C}{2k} + \dfrac{1}{2Ck} + C' \quad \Rightarrow$ $C' = a - \dfrac{C}{2k} - \dfrac{1}{2Ck} \quad \Rightarrow \quad y = \dfrac{C}{2k}\left(e^{kx} - 1\right) + \dfrac{1}{2Ck}\left(e^{-kx} - 1\right) + a$. From $h = y(\pm b)$, we find $h = \dfrac{C}{2k}\left(e^{kb} - 1\right) + \dfrac{1}{2Ck}\left(e^{-kb} - 1\right) + a$ and $h = \dfrac{C}{2k}\left(e^{-kb} - 1\right) + \dfrac{1}{2Ck}\left(e^{kb} - 1\right) + a$. Subtracting the second equation from the first, we get $0 = \dfrac{C}{k}\dfrac{e^{kb} - e^{-kb}}{2} - \dfrac{1}{Ck}\dfrac{e^{kb} - e^{-kb}}{2} = \dfrac{1}{k}\left(C - \dfrac{1}{C}\right)\sinh kb$. Now $k > 0$ and $b > 0$, so $\sinh kb > 0$ and $C = \pm 1$. If $C = 1$, then $y = \dfrac{1}{2k}\left(e^{kx} - 1\right) + \dfrac{1}{2k}\left(e^{-kx} - 1\right) + a = \dfrac{1}{k}\dfrac{e^{kx} + e^{-kx}}{2} - \dfrac{1}{k} + a = a + \dfrac{1}{k}(\cosh kx - 1)$. If $C = -1$, then $y = -\dfrac{1}{2k}\left(e^{kx} - 1\right) - \dfrac{1}{2k}\left(e^{-kx} - 1\right) + a = \dfrac{-1}{k}\dfrac{e^{kx} + e^{-kx}}{2} + \dfrac{1}{k} + a = a - \dfrac{1}{k}(\cosh kx - 1)$. Since $k > 0$, $\cosh kx \geq 1$, and $y \geq a$, we conclude that $C = 1$ and $y = a + \dfrac{1}{k}(\cosh kx - 1)$, where $h = y(b) = a + \dfrac{1}{k}(\cosh kb - 1)$. Since $\cosh(kb) = \cosh(-kb)$, there is no further information to extract from the condition that $y(b) = y(-b)$. However, we could replace a with the expression $h - \dfrac{1}{k}(\cosh kb - 1)$, obtaining $y = h + \dfrac{1}{k}(\cosh kx - \cosh kb)$. It would be better still to keep a in the expression for y, and use the expression for h to solve for k in terms of a, b, and h. That would enable us to express y in terms of x and the given parameters a, b, and h. Sadly, it is not possible to solve for k in closed form. That would bave to be done by numerical methods when specific parameter values are given.

(b) The length of the cable is

$$L = \int_{-b}^{b} \sqrt{1 + (dy/dx)^2}\,dx = \int_{-b}^{b} \sqrt{1 + \sinh^2 kx}\,dx = \int_{-b}^{b} \cosh kx\,dx = 2\int_0^b \cosh kx\,dx$$
$$= 2\Big[(1/k)\sinh kx\Big]_0^b = (2/k)\sinh kb$$

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PROBLEMS PLUS

1. We use the Fundamental Theorem of Calculus to differentiate the given equation:

$[f(x)]^2 = 100 + \int_0^x \left\{[f(t)]^2 + [f'(t)]^2\right\} dt \quad \Rightarrow \quad 2f(x)f'(x) = [f(x)]^2 + [f'(x)]^2 \quad \Rightarrow$

$[f(x)]^2 + [f'(x)]^2 - 2f(x)f'(x) = 0 \quad \Rightarrow \quad [f(x) - f'(x)]^2 = 0 \quad \Leftrightarrow \quad f(x) = f'(x)$. We can solve this as a separable equation, or else use Theorem 10.4.2 with $k = 1$, which says that the solutions are $f(x) = Ce^x$. Now $[f(0)]^2 = 100$, so $f(0) = C = \pm 10$, and hence $f(x) = \pm 10e^x$ are the only functions satisfying the given equation.

2. $(fg)' = f'g'$, where $f(x) = e^{x^2} \quad \Rightarrow \quad \left(e^{x^2}g\right)' = 2xe^{x^2}g'$. Since the student's mistake did not affect the answer,

$\left(e^{x^2}g\right)' = e^{x^2}g' + 2xe^{x^2}g = 2xe^{x^2}g'$. So $(2x-1)g' = 2xg$, or $\dfrac{g'}{g} = \dfrac{2x}{2x-1} = 1 + \dfrac{1}{2x-1} \quad \Rightarrow$

$\ln|g(x)| = x + \frac{1}{2}\ln(2x-1) + C \quad \Rightarrow \quad g(x) = Ae^x\sqrt{2x-1}$.

3. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)\,[f(h) - 1]}{h}$ [since $f(x+h) = f(x)f(h)$]

$= f(x) \lim\limits_{h\to 0} \dfrac{f(h) - 1}{h} = f(x) \lim\limits_{h\to 0} \dfrac{f(h) - f(0)}{h - 0} = f(x)f'(0) = f(x)$

Therefore, $f'(x) = f(x)$ for all x and from Theorem 10.4.2 we get $f(x) = Ae^x$. Now $f(0) = 1 \quad \Rightarrow \quad A = 1 \quad \Rightarrow$ $f(x) = e^x$.

4. $\left(\int f(x)\,dx\right)\left(\int \dfrac{dx}{f(x)}\right) = -1 \Rightarrow \int \dfrac{dx}{f(x)} = \dfrac{-1}{\int f(x)\,dx} \Rightarrow \dfrac{1}{f(x)} = \dfrac{f(x)}{\left[\int f(x)\,dx\right]^2}$ [after differentiating] $\Rightarrow$

$\int f(x)\,dx = \pm f(x)$ [after taking square roots] $\Rightarrow \quad f(x) = \pm f'(x)$ [after differentiating again] $\Rightarrow$

$y = Ae^x$ or $y = Ae^{-x}$ by Theorem 10.4.2. Therefore, $f(x) = Ae^x$ or $f(x) = Ae^{-x}$, for all nonzero constants A, are the functions satisfying the original equation.

5. Let $y(t)$ denote the temperature of the peach pie t minutes after 5:00 P.M. and R the temperature of the room.

Newton's Law of Cooling gives us $dy/dt = k(y - R)$. Solving for y we get $\dfrac{dy}{y-R} = k\,dt \quad \Rightarrow$

$\ln|y - R| = kt + C \quad \Rightarrow \quad |y - R| = e^{kt+C} \quad \Rightarrow \quad y - R = \pm e^{kt}\cdot e^C \quad \Rightarrow \quad y = Me^{kt} + R$, where M is a nonzero constant. We are given temperatures at three times.

$$\begin{aligned} y(0) &= 100 \quad \Rightarrow \quad 100 = M + R \quad \Rightarrow \quad R = 100 - M \\ y(10) &= 80 \quad \Rightarrow \quad 80 = Me^{10k} + R \quad \textbf{(1)} \\ y(20) &= 65 \quad \Rightarrow \quad 65 = Me^{20k} + R \quad \textbf{(2)} \end{aligned}$$

Substituting $100 - M$ for R in **(1)** and **(2)** gives us

$$-20 = Me^{10k} - M \ \textbf{(3)} \quad \text{and} \quad -35 = Me^{20k} - M \ \textbf{(4)}$$

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Dividing (3) by (4) gives us $\dfrac{-20}{-35} = \dfrac{M(e^{10k}-1)}{M(e^{20k}-1)} \Rightarrow \dfrac{4}{7} = \dfrac{e^{10k}-1}{e^{20k}-1} \Rightarrow 4e^{20k} - 4 = 7e^{10k} - 7 \Rightarrow$ $4e^{20k} - 7e^{10k} + 3 = 0$. This is a quadratic equation in e^{10k}. $(4e^{10k} - 3)(e^{10k} - 1) = 0 \Rightarrow e^{10k} = \frac{3}{4}$ or 1 $\Rightarrow 10k = \ln\frac{3}{4}$ or $\ln 1 \Rightarrow k = \frac{1}{10}\ln\frac{3}{4}$ since k is a nonzero constant of proportionality. Substituting $\frac{3}{4}$ for e^{10k} in (3) gives us $-20 = M \cdot \frac{3}{4} - M \Rightarrow -20 = -\frac{1}{4}M \Rightarrow M = 80$. Now $R = 100 - M$ so $R = 20\,^\circ\text{C}$.

6. Let b be the number of hours before noon that it began to snow, t the time measured in hours after noon, and $x = x(t) =$ distance traveled by the plow at time t. Then $dx/dt =$ speed of plow. Since the snow falls steadily, the height at time t is $h(t) = k(t+b)$, where k is a constant. We are given that the rate of removal is constant, say R (in m^3/h). If the width of the path is w, then $R =$ height $\times$ width $\times$ speed $= h(t) \times w \times \dfrac{dx}{dt} = k(t+b)w\dfrac{dx}{dt}$.

Thus, $\dfrac{dx}{dt} = \dfrac{C}{t+b}$, where $C = \dfrac{R}{kw}$ is a constant. This is a separable equation. $\displaystyle\int dx = C\int \frac{dt}{t+b} \Rightarrow$ $x(t) = C\ln(t+b) + K$.

Put $t = 0$: $0 = C\ln b + K \Rightarrow K = -C\ln b$, so $x(t) = C\ln(t+b) - C\ln b = C\ln(1 + t/b)$.

Put $t = 1$: $6000 = C\ln(1 + 1/b)$ [$x = 6$ km].

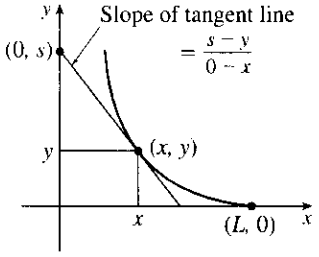
Put $t = 2$: $9000 = C\ln(1 + 2/b)$ [$x = (6+3)$ km].

Solve for b: $\dfrac{\ln(1+1/b)}{6000} = \dfrac{\ln(1+2/b)}{9000} \Rightarrow 3\ln\left(1+\frac{1}{b}\right) = 2\ln\left(1+\frac{2}{b}\right) \Rightarrow \left(1+\frac{1}{b}\right)^3 = \left(1+\frac{2}{b}\right)^2$

$\Rightarrow 1 + \dfrac{3}{b} + \dfrac{3}{b^2} + \dfrac{1}{b^3} = 1 + \dfrac{4}{b} + \dfrac{4}{b^2} \Rightarrow \dfrac{1}{b} + \dfrac{1}{b^2} - \dfrac{1}{b^3} = 0 \Rightarrow b^2 + b - 1 = 0 \Rightarrow b = \frac{-1\pm\sqrt{5}}{2}$.

But $b > 0$, so $b = \frac{-1+\sqrt{5}}{2} \approx 0.618$ h ≈ 37 min. The snow began to fall $\frac{\sqrt{5}-1}{2}$ hours before noon; that is, at about 11:23 A.M.

7. (a) While running from $(L, 0)$ to (x, y), the dog travels a distance $s = \displaystyle\int_x^L \sqrt{1 + (dy/dx)^2}\,dx = -\int_L^x \sqrt{1 + (dy/dx)^2}\,dx$, so $\dfrac{ds}{dx} = -\sqrt{1 + (dy/dx)^2}$. The dog and rabbit run at the same speed, so the rabbit's position when the dog has traveled a distance s is $(0, s)$. Since the dog runs straight for the rabbit, $\dfrac{dy}{dx} = \dfrac{s-y}{0-x}$ (see the figure).



Thus, $s = y - x\dfrac{dy}{dx} \Rightarrow \dfrac{ds}{dx} = \dfrac{dy}{dx} - \left(x\dfrac{d^2y}{dx^2} + 1\dfrac{dy}{dx}\right) = -x\dfrac{d^2y}{dx^2}$. Equating the two expressions for $\dfrac{ds}{dx}$ gives us $x\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$, as claimed.

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(b) Letting $z = \dfrac{dy}{dx}$, we obtain the differential equation $x\,\dfrac{dz}{dx} = \sqrt{1+z^2}$, or $\dfrac{dz}{\sqrt{1+z^2}} = \dfrac{dx}{x}$. Integrating:

$\ln x = \displaystyle\int \frac{dz}{\sqrt{1+z^2}} \overset{25}{=} \ln\left(z + \sqrt{1+z^2}\right) + C$. When $x = L$, $z = dy/dx = 0$, so $\ln L = \ln 1 + C$.

Therefore, $C = \ln L$, so $\ln x = \ln\left(\sqrt{1+z^2} + z\right) + \ln L = \ln\left[L\left(\sqrt{1+z^2} + z\right)\right] \;\Rightarrow$

$x = L\left(\sqrt{1+z^2} + z\right) \;\Rightarrow\; \sqrt{1+z^2} = \dfrac{x}{L} - z \;\Rightarrow\; 1 + z^2 = \left(\dfrac{x}{L}\right)^2 - \dfrac{2xz}{L} + z^2 \;\Rightarrow$

$\left(\dfrac{x}{L}\right)^2 - 2z\left(\dfrac{x}{L}\right) - 1 = 0 \;\Rightarrow\; z = \dfrac{(x/L)^2 - 1}{2(x/L)} = \dfrac{x^2 - L^2}{2Lx} = \dfrac{x}{2L} - \dfrac{L}{2}\dfrac{1}{x}$ [for $x > 0$]. Since $z = \dfrac{dy}{dx}$,

$y = \dfrac{x^2}{4L} - \dfrac{L}{2}\ln x + C_1$. Since $y = 0$ when $x = L$, $0 = \dfrac{L}{4} - \dfrac{L}{2}\ln L + C_1 \;\Rightarrow\; C_1 = \dfrac{L}{2}\ln L - \dfrac{L}{4}$. Thus,

$y = \dfrac{x^2}{4L} - \dfrac{L}{2}\ln x + \dfrac{L}{2}\ln L - \dfrac{L}{4} = \dfrac{x^2 - L^2}{4L} - \dfrac{L}{2}\ln\left(\dfrac{x}{L}\right)$.

(c) As $x \to 0^+$, $y \to \infty$, so the dog never catches the rabbit.

8. (a) If the dog runs twice as fast as the rabbit, then the rabbit's position when the dog has traveled a distance s is $(0, s/2)$. Since the dog runs straight toward the rabbit, the tangent line to the dog's path has slope $\dfrac{dy}{dx} = \dfrac{s/2 - y}{0 - x}$. Thus, $s = 2y - 2x\,\dfrac{dy}{dx} \;\Rightarrow\; \dfrac{ds}{dx} = 2\,\dfrac{dy}{dx} - \left(2x\,\dfrac{d^2y}{dx^2} + 2\,\dfrac{dy}{dx}\right) = -2x\,\dfrac{d^2y}{dx^2}$.

From Problem 7(a), $\dfrac{ds}{dx} = -\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$, so $2x\,\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$.

Letting $z = \dfrac{dy}{dx}$, we obtain the differential equation $2x\,\dfrac{dz}{dx} = \sqrt{1+z^2}$, or $\dfrac{2\,dz}{\sqrt{1+z^2}} = \dfrac{dx}{x}$.

Integrating, we get $\ln x = \displaystyle\int \frac{2\,dz}{\sqrt{1+z^2}} = 2\ln\left(\sqrt{1+z^2} + z\right) + C$. [See Problem 7(b).]

When $x = L$, $z = dy/dx = 0$, so $\ln L = 2\ln 1 + C = C$. Thus,

$\ln x = 2\ln\left(\sqrt{1+z^2} + z\right) + \ln L = \ln\left(L\left(\sqrt{1+z^2} + z\right)^2\right) \;\Rightarrow\; x = L\left(\sqrt{1+z^2} + z\right)^2 \;\Rightarrow$

$\sqrt{1+z^2} = \sqrt{\dfrac{x}{L}} - z \;\Rightarrow\; 1 + z^2 = \dfrac{x}{L} - 2\sqrt{\dfrac{x}{L}}\,z + z^2 \;\Rightarrow\; 2\sqrt{\dfrac{x}{L}}\,z = \dfrac{x}{L} - 1 \;\Rightarrow$

$\dfrac{dy}{dx} = z = \dfrac{1}{2}\sqrt{\dfrac{x}{L}} - \dfrac{1}{2\sqrt{x/L}} = \dfrac{1}{2\sqrt{L}}x^{1/2} - \dfrac{\sqrt{L}}{2}x^{-1/2} \;\Rightarrow\; y = \dfrac{1}{3\sqrt{L}}x^{3/2} - \sqrt{L}\,x^{1/2} + C_1$. When

$x = L$, $y = 0$, so $0 = \dfrac{1}{3\sqrt{L}}L^{3/2} - \sqrt{L}\,L^{1/2} + C_1 = \dfrac{L}{3} - L + C_1 = C_1 - \frac{2}{3}L$. Therefore, $C_1 = \frac{2}{3}L$ and

$y = \dfrac{x^{3/2}}{3\sqrt{L}} - \sqrt{L}\,x^{1/2} + \frac{2}{3}L$. As $x \to 0$, $y \to \frac{2}{3}L$, so the dog catches the rabbit when the rabbit is at $\left(0, \frac{2}{3}L\right)$.

(At that point, the dog has traveled a distance of $\frac{4}{3}L$, twice as far as the rabbit has run.)

(b) As in the solutions to part (a) and Problem 7, we get $z = \dfrac{dy}{dx} = \dfrac{x^2}{2L^2} - \dfrac{L^2}{2x^2}$ and hence $y = \dfrac{x^3}{6L^2} + \dfrac{L^2}{2x} - \frac{2}{3}L$.

We want to minimize the distance D from the dog at (x, y) to the rabbit at $(0, 2s)$. Now $s = \frac{1}{2}y - \frac{1}{2}x\,\dfrac{dy}{dx} \;\Rightarrow$

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$2s = y - xz \quad \Rightarrow \quad y - 2s = xz = x\left(\dfrac{x^2}{2L^2} - \dfrac{L^2}{2x^2}\right) = \dfrac{x^3}{2L^2} - \dfrac{L^2}{2x}$, so

$$D = \sqrt{(x-0)^2 + (y-2s)^2} = \sqrt{x^2 + \left(\frac{x^3}{2L^2} - \frac{L^2}{2x}\right)^2}$$

$$= \sqrt{\frac{x^6}{4L^4} + \frac{x^2}{2} + \frac{L^4}{4x^2}} = \sqrt{\left(\frac{x^3}{2L^2} + \frac{L^2}{2x}\right)^2} = \frac{x^3}{2L^2} + \frac{L^2}{2x}$$

$D' = 0 \quad \Leftrightarrow \quad \dfrac{3x^2}{2L^2} - \dfrac{L^2}{2x^2} = 0 \quad \Leftrightarrow \quad \dfrac{3x^2}{2L^2} = \dfrac{L^2}{2x^2} \quad \Leftrightarrow \quad x^4 = \dfrac{L^4}{3} \quad \Leftrightarrow \quad x = \dfrac{L}{\sqrt[4]{3}}, x > 0, L > 0.$

Since $D''(x) = \dfrac{3x}{L^2} + \dfrac{L^2}{x^3} > 0$ for all $x > 0$, we know that

$D\left(\dfrac{L}{\sqrt[4]{3}}\right) = \dfrac{\left(L \cdot 3^{-1/4}\right)^3}{2L^2} + \dfrac{L^2}{2(L \cdot 3^{-1/4})} = \dfrac{2L}{3^{3/4}}$ is the minimum value of D, that is, the closest the dog gets to the rabbit. The positions at this distance are

Dog: $(x, y) = \left(\dfrac{L}{\sqrt[4]{3}}, \left(\dfrac{5}{3^{7/4}} - \dfrac{2}{3}\right)L\right) = \left(\dfrac{L}{\sqrt[4]{3}}, \dfrac{5\sqrt[4]{3} - 6}{9}L\right)$

Rabbit: $(0, 2s) = \left(0, \dfrac{8\sqrt[4]{3}L}{9} - \dfrac{2L}{3}\right) = \left(0, \dfrac{8\sqrt[4]{3} - 6}{9}L\right)$

9. (a) We are given that $V = \frac{1}{3}\pi r^2 h$, $dV/dt = 60{,}000\pi$ ft^3/h, and $r = 1.5h = \frac{3}{2}h$. So $V = \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h = \frac{3}{4}\pi h^3$ $\Rightarrow$ $\dfrac{dV}{dt} = \frac{3}{4}\pi \cdot 3h^2 \dfrac{dh}{dt} = \frac{9}{4}\pi h^2 \dfrac{dh}{dt}$. Therefore, $\dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2} = \dfrac{240{,}000\pi}{9\pi h^2} = \dfrac{80{,}000}{3h^2}$ $(\star)$ $\Rightarrow$ $\int 3h^2\, dh = \int 80{,}000\, dt \quad \Rightarrow \quad h^3 = 80{,}000t + C$. When $t = 0$, $h = 60$. Thus, $C = 60^3 = 216{,}000$, so $h^3 = 80{,}000t + 216{,}000$. Let $h = 100$. Then $100^3 = 1{,}000{,}000 = 80{,}000t + 216{,}000$ $\Rightarrow$ $80{,}000t = 784{,}000 \quad \Rightarrow \quad t = 9.8$, so the time required is 9.8 hours.

(b) The floor area of the silo is $F = \pi \cdot 200^2 = 40{,}000\pi$ ft^2, and the area of the base of the pile is $A = \pi r^2 = \pi\left(\frac{3}{2}h\right)^2 = \frac{9\pi}{4}h^2$. So the area of the floor which is not covered when $h = 60$ is $F - A = 40{,}000\pi - 8100\pi = 31{,}900\pi \approx 100{,}217$ ft^2. Now $A = \frac{9\pi}{4}h^2 \quad \Rightarrow \quad dA/dt = \frac{9\pi}{4} \cdot 2h\,(dh/dt)$, and from $(\star)$ in part (a) we know that when $h = 60$, $dh/dt = \frac{80{,}000}{3(60)^2} = \frac{200}{27}$ ft/h. Therefore, $dA/dt = \frac{9\pi}{4}(2)(60)\left(\frac{200}{27}\right) = 2000\pi \approx 6283$ ft^2/h.

(c) At $h = 90$ ft, $dV/dt = 60{,}000\pi - 20{,}000\pi = 40{,}000\pi$ ft^3/h. From $(\star)$ in part (a), $\dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2} = \dfrac{4(40{,}000\pi)}{9\pi h^2} = \dfrac{160{,}000}{9h^2} \quad \Rightarrow \quad \int 9h^2\, dh = \int 160{,}000\, dt \quad \Rightarrow \quad 3h^3 = 160{,}000t + C$. When $t = 0$, $h = 90$; therefore, $C = 3 \cdot 729{,}000 = 2{,}187{,}000$. So $3h^3 = 160{,}000t + 2{,}187{,}000$. At the top, $h = 100 \quad \Rightarrow \quad 3(100)^3 = 160{,}000t + 2{,}187{,}000 \quad \Rightarrow \quad t = \frac{813{,}000}{160{,}000} \approx 5.1$. The pile reaches the top after about 5.1 h.

10. Let $P(a, b)$ be any first-quadrant point on the curve $y = f(x)$. The tangent line at P has equation $y - b = f'(a)(x - a)$, or equivalently, $y = mx + b - ma$, where $m = f'(a)$. If $Q(0, c)$ is the y-intercept, then $c = b - am$. If $R(k, 0)$ is the x-intercept, then $k = \dfrac{am - b}{m} = a - \dfrac{b}{m}$. Since the tangent line is bisected at P, we know that $|PQ| = |PR|$; that is,

$$\sqrt{(a-0)^2 + [b - (b - am)]^2} = \sqrt{[a - (a - b/m)]^2 + (b-0)^2}$$

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Squaring and simplifying gives us $a^2 + a^2m^2 = b^2/m^2 + b^2 \Rightarrow a^2m^2 + a^2m^4 = b^2 + b^2m^2 \Rightarrow a^2m^4 + (a^2 - b^2)m^2 - b^2 = 0 \Rightarrow (a^2m^2 - b^2)(m^2 + 1) = 0 \Rightarrow m^2 = b^2/a^2$. Since m is the slope of the line from a positive y-intercept to a positive x-intercept, m must be negative. Since a and b are positive, we have $m = -b/a$, so we will solve the equivalent differential equation $\dfrac{dy}{dx} = -\dfrac{y}{x} \Rightarrow \dfrac{dy}{y} = -\dfrac{dx}{x} \Rightarrow$

$\displaystyle\int \frac{dy}{y} = -\int \frac{dx}{x} \Rightarrow \ln y = -\ln x + C \ \ [x, y > 0] \Rightarrow y = e^{-\ln x + C} = e^{\ln x^{-1}} \cdot e^C = x^{-1} \cdot A \Rightarrow$ $y = A/x$. Since the point $(3, 2)$ is on the curve, $3 = A/2 \Rightarrow A = 6$ and the curve is $y = 6/x$ with $x > 0$.

11. Let $P(a, b)$ be any point on the curve. If m is the slope of the tangent line at P, then $m = y'(a)$, and an equation of the normal line at P is $y - b = -\dfrac{1}{m}(x - a)$, or equivalently, $y = -\dfrac{1}{m}x + b + \dfrac{a}{m}$.

The y-intercept is always 6, so $b + \dfrac{a}{m} = 6 \Rightarrow \dfrac{a}{m} = 6 - b \Rightarrow m = \dfrac{a}{6 - b}$.

We will solve the equivalent differential equation $\dfrac{dy}{dx} = \dfrac{x}{6 - y} \Rightarrow (6 - y)\,dy = x\,dx \Rightarrow$

$\displaystyle\int (6 - y)\,dy = \int x\,dx \Rightarrow 6y - \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C \Rightarrow 12y - y^2 = x^2 + K$. Since $(3, 2)$ is on the curve, $12(2) - 2^2 = 3^2 + K \Rightarrow K = 11$. So the curve is given by $12y - y^2 = x^2 + 11 \Rightarrow$ $x^2 + y^2 - 12y + 36 = -11 + 36 \Rightarrow x^2 + (y - 6)^2 = 25$, a circle with center $(0, 6)$ and radius 5.

12. Suppose C is a curve with the required property and let $P = (x_0, y_0)$ be a point on C. The equation of the normal line to C at P is $y - y_0 = -\dfrac{1}{y_0'}(x - x_0)$, where y_0' is the value of $\dfrac{dy}{dx}$ at $x = x_0$. This equation makes sense only if $y_0' \neq 0$. If $y_0' = 0$, then the normal line at P is $x = x_0$, which does not intersect the y-axis at all unless $x_0 = 0$.

So let's assume that $y_0' \neq 0$. Then the normal line to C at P intersects the x-axis at $(x_0 + y_0y_0', 0)$, and it intersects the y-axis at $(0, y_0 + x_0/y_0')$. The condition on C implies that

$$\left[\text{distance from } P(x_0, y_0) \text{ to } (0, y_0 + x_0/y_0')\right] = \left[\text{distance from } (0, y_0 + x_0/y_0') \text{ to } (x_0 + y_0y_0', 0)\right]$$

$$\sqrt{(0 - x_0)^2 + (y_0 + x_0/y_0' - y_0)^2} = \sqrt{(x_0 + y_0y_0' - 0)^2 + [0 - (y_0 + x_0/y_0')]^2}$$

Squaring both sides, we get $x_0^2 + x_0^2/(y_0')^2 = (x_0 + y_0y_0')^2 + (y_0 + x_0/y_0')^2$ or

$x_0^2 + \dfrac{x_0^2}{(y_0')^2} = x_0^2 + 2x_0y_0y_0' + y_0^2(y_0')^2 + y_0^2 + 2\dfrac{x_0y_0}{y_0'} + \dfrac{x_0^2}{(y_0')^2}$. Subtracting $x_0^2 + \dfrac{x_0^2}{(y_0')^2}$ from both sides and multiplying by y_0', we get

$$0 = y_0^2y_0' + y_0^2(y_0')^3 + 2x_0y_0\left[1 + (y_0')^2\right] = y_0\left\{y_0y_0' + y_0(y_0')^3 + 2x_0\left[1 + (y_0')^2\right]\right\}$$

$$= y_0\left\{y_0y_0'\left[1 + (y_0')^2\right] + 2x_0\left[1 + (y_0')^2\right]\right\} = y_0(y_0y_0' + 2x_0)\left[1 + (y_0')^2\right]$$

Since $1 + (y_0')^2 \geq 1 > 0$, we conclude that $y_0(y_0y_0' + 2x_0) = 0$. Now P is an arbitrary point on C for which $y_0' \neq 0$. Thus, we have shown that $y(yy' + 2x) = 0$ for points (x, y) along C where $y' \neq 0$. One solution of this equation is $y = 0$, but that curve (the x-axis) doesn't satisfy the condition required of C, since its normal lines at points for $x \neq 0$ don't intersect the y-axis. Thus, we can focus our attention on points of C where $y \neq 0$, and conclude that $yy' + 2x = 0$ at points of C where $y \neq 0$ and $y' \neq 0$. Integrating both sides of $yy' + 2x = 0$, we get $\frac{1}{2}y^2 + x^2 = c$. Clearly $c > 0$ (since $y \neq 0$), so we can write $c = a^2$, where $a = \sqrt{c} > 0$. Thus, $\frac{1}{2}y^2 + x^2 = a^2$ and

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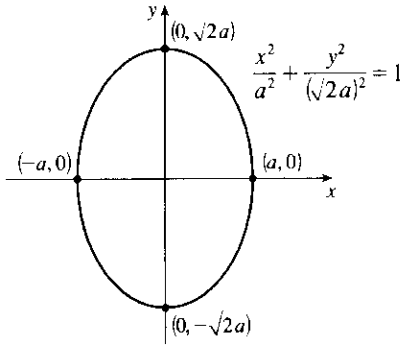
$x^2/a^2 + y^2/(\sqrt{2}a)^2 = 1$. This shows that C is (part of) the ellipse centered at $(0, 0)$ with semimajor axis $\sqrt{2}\,a$ in the y-direction and semiminor axis a in the x-direction. The points of C where $y = 0$ or $y' = 0$ are the vertices $\left(0, \pm\sqrt{2}\,a\right)$ and $(\pm a, 0)$. At these points, the condition on C is satisfied in a degenerate way. [When $P = (\pm a, 0)$, the normal line at P *is* the x-axis, so *all* the points of the normal line can be viewed as points of intersection with the x-axis. The intersection with the y-axis at $(0, 0)$ is midway between $(a, 0)$ and $(-a, 0)$; one of these points is P, and the other can be regarded as an intersection of the normal line with the x-axis. Similarly, when $P = \left(0, \pm\sqrt{2}\,a\right)$, the normal line is the y-axis, and the point $\left(0, \pm\sqrt{2}\,a/2\right)$, which can be regarded as an intersection of the normal line with the y-axis, is midway between P and $(0, 0)$, the intersection with the x-axis.]



Conversely, if C is part of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{2a^2} = 1$ for some $a > 0$, then the normal line at a point (x_0, y_0) of C (other than the four vertices) has equation $y - y_0 = \dfrac{y_0}{2x_0}(x - x_0)$. Its intersections with the coordinate axes are $\left(0, \dfrac{y_0}{2}\right)$ and $(-x_0, 0)$. $\left[\text{distance from } (x_0, y_0) \text{ to } \left(0, \dfrac{y_0}{2}\right)\right]^2 = x_0^2 + \dfrac{y_0^2}{4}$ and $\left[\text{distance from } \left(0, \dfrac{y_0}{2}\right) \text{ to } (-x_0, 0)\right]^2 = x_0^2 + \dfrac{y_0^2}{4}$, so the required condition is met at points other than the four vertices. As we have explained, if we are willing to interpret the condition broadly, then it can be viewed as holding even at the four vertices.

Another method: Let $P(x_0, y_0)$ be a point on the curve. Since the midpoint of the line segment determined by the normal line from (x_0, y_0) to its intersection with the x-axis has x-coordinate 0, the x-coordinate of the point of intersection with the x-axis must be $-x_0$. Hence, the normal line has slope $\dfrac{y_0 - 0}{x_0 - (-x_0)} = \dfrac{y_0}{2x_0}$. So the tangent line has slope $-\dfrac{2x_0}{y_0}$. This gives the differential equation $y' = -\dfrac{2x}{y} \;\Rightarrow\; y\,dy = -2x\,dx \;\Rightarrow$ $\int y\,dy = \int(-2x)\,dx \;\Rightarrow\; \frac{1}{2}y^2 = -x^2 + C \;\Rightarrow\; x^2 + \frac{1}{2}y^2 = C \;\; (C > 0)$.

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Chapter 11

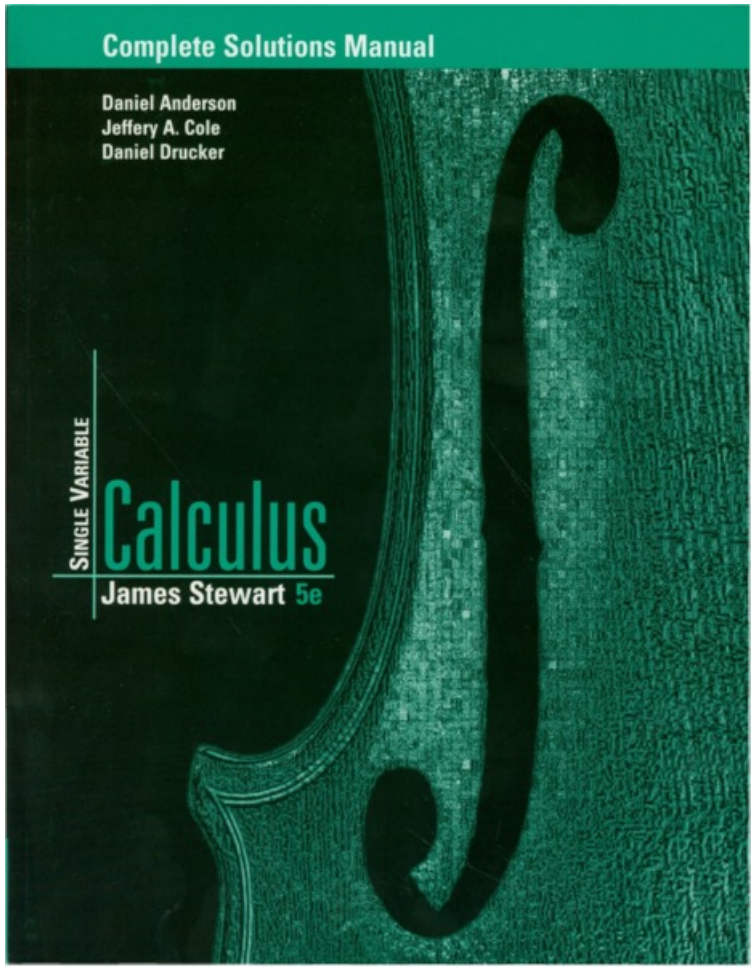
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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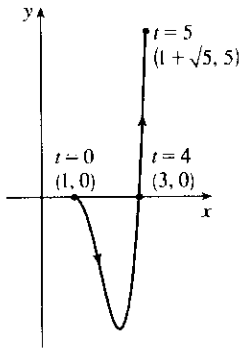
11 □ PARAMETRIC EQUATIONS AND POLAR COORDINATES

11.1 Curves Defined by Parametric Equations

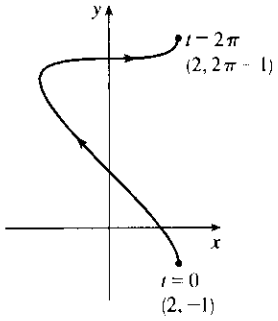
1. $x = 1 + \sqrt{t}, \quad y = t^2 - 4t, \quad 0 \le t \le 5$

t	0	1	2	3	4	5
x	1	2	$1+\sqrt{2}$	$1+\sqrt{3}$	3	$1+\sqrt{5}$
			2.41	2.73		3.24
y	0	−3	−4	−3	0	5



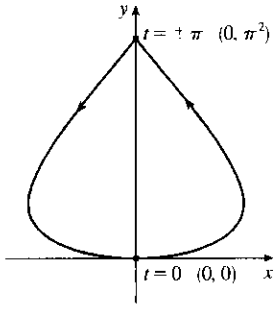
2. $x = 2\cos t, \quad y = t - \cos t, \quad 0 \le t \le 2\pi$

t	0	$\pi/2$	π	$3\pi/2$	2π
x	2	0	−2	0	2
y	−1	$\pi/2$	$\pi+1$	$3\pi/2$	$2\pi-1$
		1.57	4.14	4.71	5.28



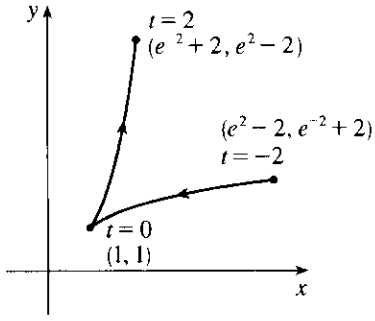
3. $x = 5\sin t, \quad y = t^2, \quad -\pi \le t \le \pi$

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	0	−5	0	5	0
y	π^2	$\pi^2/4$	0	$\pi^2/4$	π^2
	9.87	2.47		2.47	9.87



4. $x = e^{-t} + t, \quad y = e^t - t, \quad -2 \le t \le 2$

t	−2	−1	0	1	2
x	e^2-2	$e-1$	1	$e^{-1}+1$	$e^{-2}+2$
	5.39	1.72		1.37	2.14
y	$e^{-2}+2$	$e^{-1}+1$	1	$e-1$	e^2-2
	2.14	1.37		1.72	5.39



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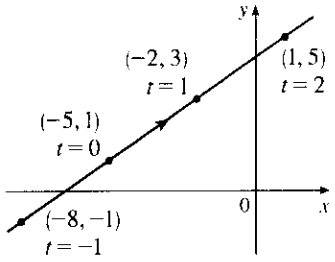
5. $x = 3t - 5, \quad y = 2t + 1$

(a)

t	-2	-1	0	1	2	3	4
x	-11	-8	-5	-2	1	4	7
y	-3	-1	1	3	5	7	9

(b) $x = 3t - 5 \Rightarrow 3t = x + 5 \Rightarrow t = \frac{1}{3}(x + 5) \Rightarrow$
$y = 2 \cdot \frac{1}{3}(x + 5) + 1$, so $y = \frac{2}{3}x + \frac{13}{3}$.

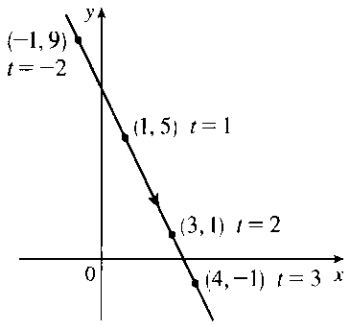


6. $x = 1 + t, \quad y = 5 - 2t, \quad -2 \le t \le 3$

(a)

t	-2	-1	0	1	2	3
x	-1	0	1	2	3	4
y	9	7	5	3	1	-1

(b) $x = 1 + t \Rightarrow t = x - 1 \Rightarrow y = 5 - 2(x - 1)$,
so $y = -2x + 7, \ -1 \le x \le 4$.

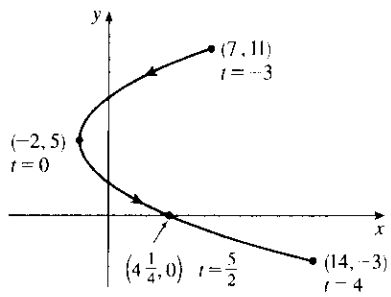


7. $x = t^2 - 2, \quad y = 5 - 2t, \quad -3 \le t \le 4$

(a)

t	-3	-2	-1	0	1	2	3	4
x	7	2	-1	-2	-1	2	7	14
y	11	9	7	5	3	1	-1	-3

(b) $y = 5 - 2t \Rightarrow 2t = 5 - y \Rightarrow t = \frac{1}{2}(5 - y) \Rightarrow$
$x = \left[\frac{1}{2}(5 - y)\right]^2 - 2$, so $x = \frac{1}{4}(5 - y)^2 - 2$,
$-3 \le y \le 11$.

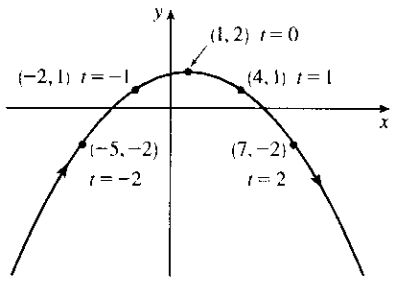


8. $x = 1 + 3t, \quad y = 2 - t^2$

(a)

t	-3	-2	-1	0	1	2	3
x	-8	-5	-2	1	4	7	10
y	-7	-2	1	2	1	-2	-7

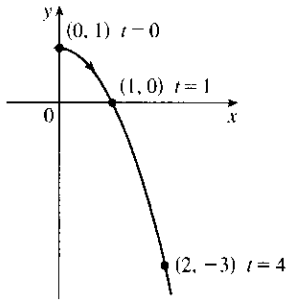
(b) $x = 1 + 3t \Rightarrow t = \frac{1}{3}(x - 1) \Rightarrow$
$y = 2 - \left[\frac{1}{3}(x - 1)\right]^2$, so $y = -\frac{1}{9}(x - 1)^2 + 2$.



9. (a) $x = \sqrt{t}, \ y = 1 - t$

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3

(b) $x = \sqrt{t} \Rightarrow t = x^2 \Rightarrow y = 1 - t = 1 - x^2$.
Since $t \ge 0$, $x \ge 0$.



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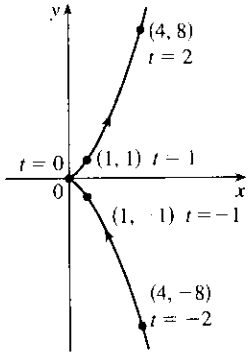
10. (a) $x = t^2$, $y = t^3$

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8

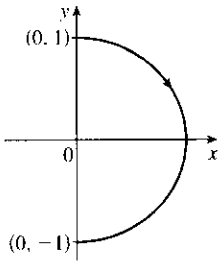


(b) $y = t^3 \Rightarrow t = \sqrt[3]{y} \Rightarrow x = t^2 = \left(\sqrt[3]{y}\right)^2 = y^{2/3}$.
$t \in \mathbb{R}$, $y \in \mathbb{R}$, $x \geq 0$.

11. (a) $x = \sin\theta$, $y = \cos\theta$, $0 \leq \theta \leq \pi$.

$x^2 + y^2 = \sin^2\theta + \cos^2\theta = 1$. Since $0 \leq \theta \leq \pi$, we have $\sin\theta \geq 0$, so $x \geq 0$.

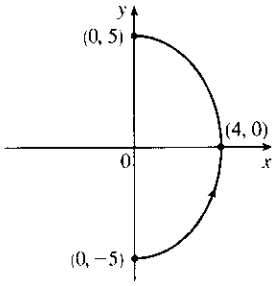
(b)



12. (a) $x = 4\cos\theta$, $y = 5\sin\theta$, $-\pi/2 \leq \theta \leq \pi/2$.

$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = \cos^2\theta + \sin^2\theta = 1$, which is an ellipse with x-intercepts $(\pm 4, 0)$ and y-intercepts $(0, \pm 5)$. We obtain the portion of the ellipse with $x \geq 0$ since $4\cos\theta \geq 0$ for $-\pi/2 \leq \theta \leq \pi/2$.

(b)

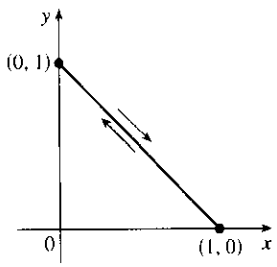


13. (a) $x = \sin^2\theta$, $y = \cos^2\theta$.

$x + y = \sin^2\theta + \cos^2\theta = 1$, $0 \leq x \leq 1$.

Note that the curve is at $(0, 1)$ whenever $\theta = \pi n$ and is at $(1, 0)$ whenever $\theta = \frac{\pi}{2}n$ for every integer n.

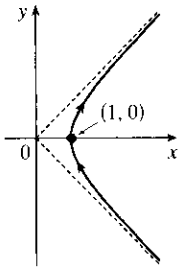
(b)



14. (a) $x = \sec\theta$, $y = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$x^2 - y^2 = \sec^2\theta - \tan^2\theta = 1$, $x \geq 1$, or $x = \sqrt{y^2 + 1}$.

(b)



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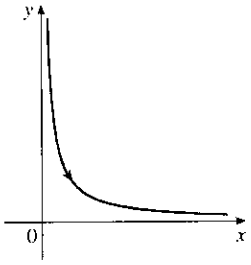
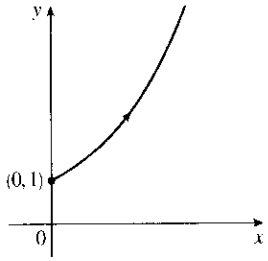
15. (a) $x = e^t$, $y = e^{-t}$.
$y = 1/e^t = 1/x$, $x > 0$

(b)



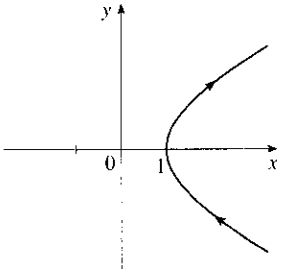
16. (a) $x = \ln t$, $y = \sqrt{t}$, $t \geq 1$.
$x = \ln t \quad \Rightarrow \quad t = e^x \quad \Rightarrow$
$y = \sqrt{t} = e^{x/2}$, $x \geq 0$.

(b)



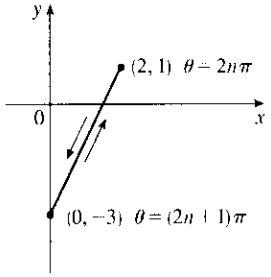
17. (a) $x = \cosh\, t$, $y = \sinh\, t$,
$x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$, $x \geq 1$

(b)



18. (a) $x = 1 + \cos\theta \quad \Rightarrow \quad \cos\theta = x - 1$.
$y = 2\cos\theta - 1 = 2(x-1) - 1 = 2x - 3$,
$0 \leq x \leq 2$.

(b)



19. $x^2 + y^2 = \cos^2 \pi t + \sin^2 \pi t = 1$, $1 \leq t \leq 2$, so the particle moves counterclockwise along the circle $x^2 + y^2 = 1$ from $(-1, 0)$ to $(1, 0)$, along the lower half of the circle.

20. $(x-2)^2 + (y-3)^2 = \cos^2 t + \sin^2 t = 1$, so the motion takes place on a unit circle centered at $(2, 3)$. As t goes from 0 to 2π, the particle makes one complete counterclockwise rotation around the circle, starting and ending at $(3, 3)$.

21. $\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{3}y\right)^2 = \sin^2 t + \cos^2 t = 1$, so the particle moves once clockwise along the ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$, starting and ending at $(0, 3)$.

22. $x = \cos^2 t = y^2$, so the particle moves along the parabola $x = y^2$. As t goes from 0 to 4π, the particle moves from $(1, 1)$ down to $(1, -1)$ (at $t = \pi$), back up to $(1, 1)$ again (at $t = 2\pi$), and then repeats this entire cycle between $t = 2\pi$ and $t = 4\pi$.

23. We must have $1 \leq x \leq 4$ and $2 \leq y \leq 3$. So the graph of the curve must be contained in the rectangle $[1, 4]$ by $[2, 3]$.

24. (a) From the first graph, we have $1 \leq x \leq 2$. From the second graph, we have $-1 \leq y \leq 1$. The only choice that satisfies either of those conditions is III.

(b) From the first graph, the values of x cycle through the values from -2 to 2 four times. From the second graph, the values of y cycle through the values from -2 to 2 six times. Choice I satisfies these conditions.

(c) From the first graph, the values of x cycle through the values from -2 to 2 three times. From the second graph, we have $0 \leq y \leq 2$. Choice IV satisfies these conditions.

(d) From the first graph, the values of x cycle through the values from -2 to 2 two times. From the second graph, the values of y do the same thing. Choice II satisfies these conditions.

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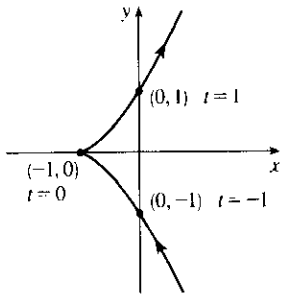
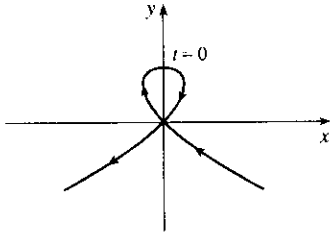
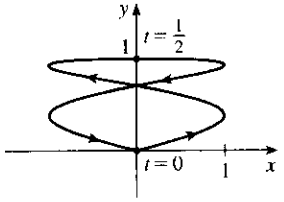
25. When $t = -1$, $(x, y) = (0, -1)$. As t increases to 0, x decreases to -1 and y increases to 0. As t increases from 0 to 1, x increases to 0 and y increases to 1. As t increases beyond 1, both x and y increase. For $t < -1$, x is positive and decreasing and y is negative and increasing. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



26. For $t < -1$, x is positive and decreasing, while y is negative and increasing (these points are in Quadrant IV). When $t = -1$, $(x, y) = (0, 0)$ and, as t increases from -1 to 0, x becomes negative and y increases from 0 to 1. At $t = 0$, $(x, y) = (0, 1)$ and, as t increases from 0 to 1, y decreases from 1 to 0 and x is positive. At $t = 1$, $(x, y) = (0, 0)$ again, so the loop is completed. For $t > 1$, x and y both become large negative. This enables us to draw a rough sketch. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



27. When $t = 0$ we see that $x = 0$ and $y = 0$, so the curve starts at the origin. As t increases from 0 to $\frac{1}{2}$, the graphs show that y increases from 0 to 1 while x increases from 0 to 1, decreases to 0 and to -1, then increases back to 0, so we arrive at the point $(0, 1)$. Similarly, as t increases from $\frac{1}{2}$ to 1, y decreases from 1 to 0 while x repeats its pattern, and we arrive back at the origin. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



28. (a) Note that as $t \to -\infty$, we have $x \to -\infty$ and $y \to \infty$, whereas when $t \to \infty$, both x and $y \to \infty$. This description fits only IV. [But also note that $x(t)$ increases, then decreases, then increases again.]

(b) Note that as $t \to \pm\infty$, $y \to -\infty$. This is only the case with VI.

(c) If $t = 0$, then $(x, y) = (\sin 0, \sin 0) = (0, 0)$. Also, $|x| = |\sin 3t| \le 1$ for all t, and $|y| = |\sin 4t| \le 1$ for all t. The only graph which includes the point $(0, 0)$ and which has $|x| \le 1$ and $|y| \le 1$, is V.

(d) Note that as $t \to -\infty$, both x and $y \to -\infty$, and as $t \to \infty$, both x and $y \to \infty$. This description fits only III. (Also note that, since $\sin 2t$ and $\sin 3t$ lie between -1 and 1, the curve never strays very far from the line $y = x$.)

(e) Note that both $x(t)$ and $y(t)$ are periodic with period 2π and satisfy $|x| \le 1$ and $|y| \le 1$. Now the only y-intercepts occur when $x = \sin(t + \sin t) = 0 \quad \Leftrightarrow \quad t = 0$ or π. So there should be two y-intercepts: $y(0) = \cos 1 \approx 0.54$ and $y(\pi) = \cos(\pi - 1) \approx -0.54$. Similarly, there should be two x-intercepts: $x\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 1\right) \approx 0.54$ and $x\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2} - 1\right) \approx -0.54$. The only curve with these x- and y-intercepts is I.

(f) Note that $x(t)$ is periodic with period 2π, so the only y-intercepts occur when $x = \cos t = 0 \quad \Leftrightarrow$ $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Also, the graph is symmetric about the x-axis, since $y(-t) = \sin(-t + \sin 5(-t)) = \sin(-t - \sin 5t) = -\sin(t + \sin 5t) = -y(t)$, and $x(-t) = \cos(-t) = \cos t = x(t)$. The only graph which has only two y-intercepts, and is symmetric about the x-axis, is II.

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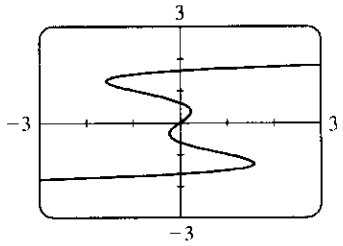
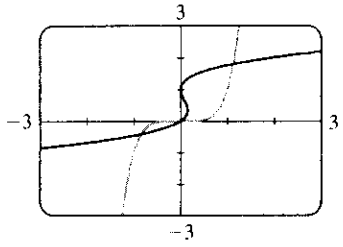
29. As in Example 5, we let $y = t$ and $x = t - 3t^3 + t^5$ and use a t-interval of $[-2\pi, 2\pi]$.



30. We use $x_1 = t$, $y_1 = t^5$ and $x_2 = t\,(t-1)^2$, $y_2 = t$ with $-2\pi \le t \le 2\pi$. There are 3 points of intersection; $(0, 0)$ is fairly obvious. The point in quadrant III is approximately $(-0.8, -0.4)$ and the point in quadrant I is approximately $(1.1, 1.8)$.

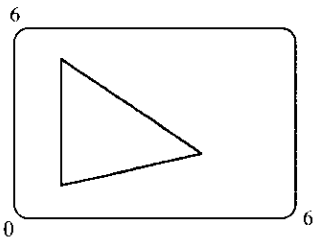


31. (a) $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \le t \le 1$. Clearly the curve passes through $P_1(x_1, y_1)$ when $t = 0$ and through $P_2(x_2, y_2)$ when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2. For every value of t, x and y satisfy the relation $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}\,(x - x_1)$, which is the equation of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Finally, any point (x, y) on that line satisfies $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$; if we call that common value t, then the given parametric equations yield the point (x, y); and any (x, y) on the line between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$.

(b) $x = -2 + [3 - (-2)]t = -2 + 5t$ and $y = 7 + (-1 - 7)t = 7 - 8t$ for $0 \le t \le 1$.

32. For the side of the triangle from A to B, use $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$. Hence, the equations are $x = x_1 + (x_2 - x_1)\,t = 1 + (4 - 1)\,t = 1 + 3t$, $y = y_1 + (y_2 - y_1)\,t = 1 + (2 - 1)\,t = 1 + t$. Graphing $x = 1 + 3t$ and $y = 1 + t$ with $0 \le t \le 1$ gives us the side of the triangle from A to B. Similarly, for the side BC we use $x = 4 - 3t$ and $y = 2 + 3t$, and for the side AC we use $x = 1$ and $y = 1 + 4t$.



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33. The circle $x^2 + y^2 = 4$ can be represented parametrically by $x = 2\cos t$, $y = 2\sin t$; $0 \le t \le 2\pi$. The circle $x^2 + (y-1)^2 = 4$ can be represented by $x = 2\cos t$, $y = 1 + 2\sin t$; $0 \le t \le 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.

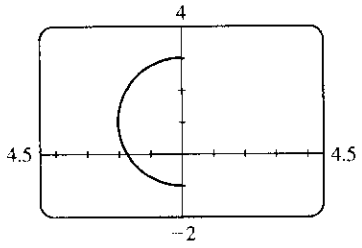
(a) To get a clockwise orientation, we could change the equations to $x = 2\cos t$, $y = 1 - 2\sin t$, $0 \le t \le 2\pi$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x = 2\cos t$, $y = 1 + 2\sin t$ with the domain expanded to $0 \le t \le 6\pi$.

(c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2\cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use $x = 2\cos t$, $y = 1 + 2\sin t$; $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

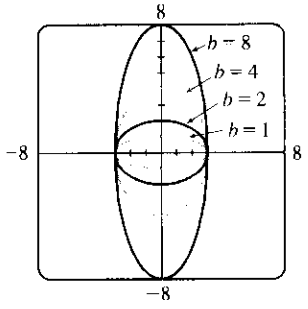
Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x = -2\sin t$, $y = 1 + 2\cos t$, $0 \le t \le \pi$.

34.



35. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a\sin t$ and $y = b\cos t$ with $0 \le t \le 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

(b) The equations are $x = 3\sin t$ and $y = b\cos t$ for $b \in \{1, 2, 4, 8\}$.



(c) As b increases, the ellipse stretches vertically.

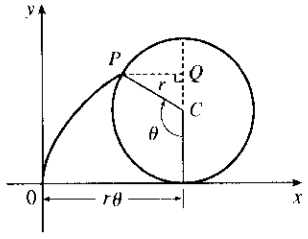
36. The possible parametrizations of the curve $y = x^3$ include

(1) $x = t$, $y = t^3$, $t \in \mathbb{R}$

(2) $x = -t$, $y = -t^3$, $t \in \mathbb{R}$

(3) $x = t + 1$, $y = (t+1)^3$, $t \in \mathbb{R}$

37. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 6, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$ [since $\cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha = -\cos\alpha$], so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\theta))$ [since $\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha = \sin\alpha$]. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



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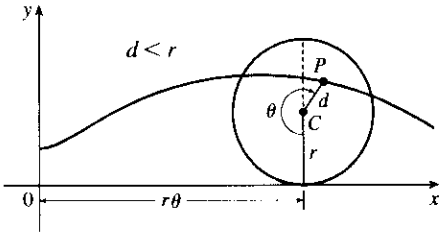
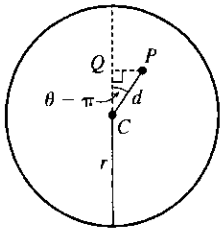
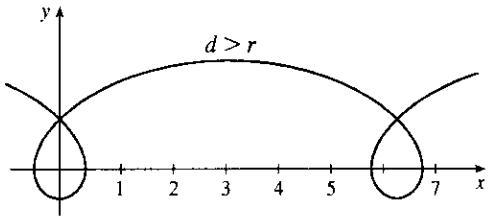
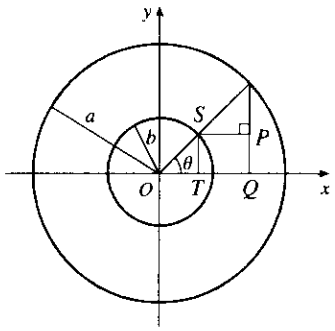
38. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d < r$. As in Example 6, C has coordinates $(r\theta, r)$. Now Q (in the second diagram) has coordinates $(r\theta, r + d\cos(\theta - \pi)) = (r\theta, r - d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta + d\sin(\theta - \pi), r - d\cos\theta)$. That is, P has coordinates (x, y), where $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$. When $d = r$, these equations agree with those of the cycloid.





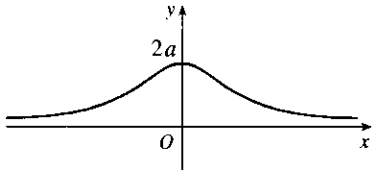


39. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a\cos\theta$ and $y = |ST| = b\sin\theta$. Thus, the parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$. To eliminate θ we rearrange: $\sin\theta = y/b \;\Rightarrow\; \sin^2\theta = (y/b)^2$ and $\cos\theta = x/a \;\Rightarrow\; \cos^2\theta = (x/a)^2$. Adding the two equations: $\sin^2\theta + \cos^2\theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



40. A has coordinates $(a\cos\theta, a\sin\theta)$. Since OA is perpendicular to AB, ΔOAB is a right triangle and B has coordinates $(a\sec\theta, 0)$. It follows that P has coordinates $(a\sec\theta, b\sin\theta)$. Thus, the parametric equations are $x = a\sec\theta$, $y = b\sin\theta$.

41. $C = (2a\cot\theta, 2a)$, so the x-coordinate of P is $x = 2a\cot\theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a\sin\theta$ and $A = ((2a\sin\theta)\cos\theta, (2a\sin\theta)\sin\theta)$. Thus, the y-coordinate of P is $y = 2a\sin^2\theta$.



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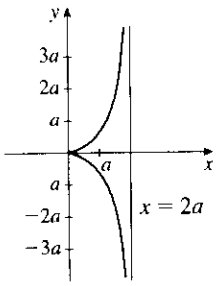
42. Let θ be the angle of inclination of segment OP. Then $|OB| = \dfrac{2a}{\cos\theta}$. Let $C = (2a, 0)$. Then by use of right triangle OAC we see that $|OA| = 2a\cos\theta$. Now

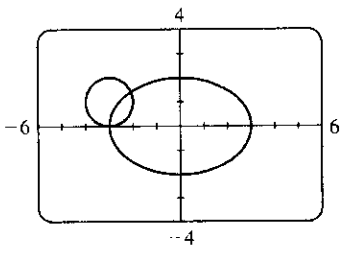


$$|OP| = |AB| = |OB| - |OA| = 2a\left(\frac{1}{\cos\theta} - \cos\theta\right)$$

$$= 2a\frac{1-\cos^2\theta}{\cos\theta} = 2a\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\tan\theta$$

So P has coordinates $x = 2a\sin\theta\tan\theta\cdot\cos\theta = 2a\sin^2\theta$ and $y = 2a\sin\theta\tan\theta\cdot\sin\theta = 2a\sin^2\theta\tan\theta$.

43. (a)



There are 2 points of intersection: $(-3, 0)$ and approximately $(-2.1, 1.4)$.

(b) A collision point occurs when $x_1 = x_2$ and $y_1 = y_2$ for the same t. So solve the equations:

$$3\sin t = -3 + \cos t \quad \textbf{(1)}$$
$$2\cos t = 1 + \sin t \quad \textbf{(2)}$$

From **(2)**, $\sin t = 2\cos t - 1$. Substituting into **(1)**, we get $3(2\cos t - 1) = -3 + \cos t \;\Rightarrow$ $5\cos t = 0$ $(*)$ $\Rightarrow$ $\cos t = 0$ $\Rightarrow$ $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. We check that $t = \frac{3\pi}{2}$ satisfies **(1)** and **(2)** but $t = \frac{\pi}{2}$ does not. So the only collision point occurs when $t = \frac{3\pi}{2}$, and this gives the point $(-3, 0)$. [We could check our work by graphing x_1 and x_2 together as functions of t and, on another plot, y_1 and y_2 as functions of t. If we do so, we see that the only value of t for which *both* pairs of graphs intersect is $t = \frac{3\pi}{2}$.]

(c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but there are no collision points, since $(*)$ in part (b) becomes $5\cos t = 6 \;\Rightarrow\; \cos t = \frac{6}{5} > 1$.

44. (a) If $\alpha = 30^\circ$ and $v_0 = 500$ m/s, then the equations become $x = (500\cos 30^\circ)t = 250\sqrt{3}t$ and $y = (500\sin 30^\circ)t - \frac{1}{2}(9.8)t^2 = 250t - 4.9t^2$. $y = 0$ when $t = 0$ (when the gun is fired) and again when $t = \frac{250}{4.9} \approx 51$ s. Then $x = \left(250\sqrt{3}\,\right)\left(\frac{250}{4.9}\right) \approx 22{,}092$ m, so the bullet hits the ground about 22 km from the gun.

The formula for y is quadratic in t. To find the maximum y-value, we will complete the square:

$$y = -4.9\left(t^2 - \tfrac{250}{4.9}t\right) = -4.9\left[t^2 - \tfrac{250}{4.9}t + \left(\tfrac{125}{4.9}\right)^2\right] + \tfrac{125^2}{4.9} = -4.9\left(t - \tfrac{125}{4.9}\right)^2 + \tfrac{125^2}{4.9} \le \tfrac{125^2}{4.9}$$

with equality when $t = \frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9} \approx 3189$ m.

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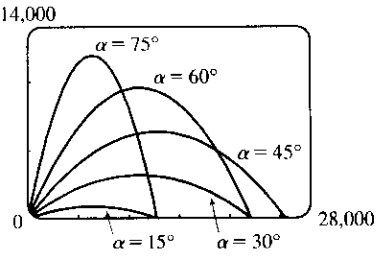
(b)



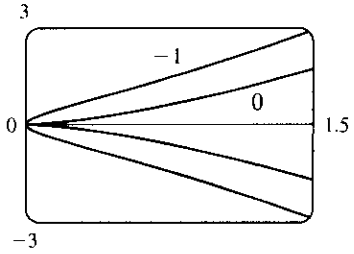
As α ($0° < \alpha < 90°$) increases up to $45°$, the projectile attains a greater height and a greater range. As α increases past $45°$, the projectile attains a greater height, but its range decreases.

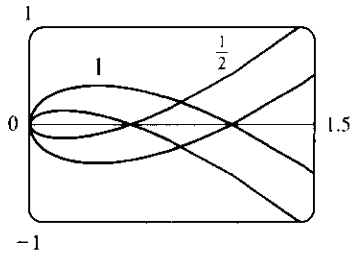
(c) $x = (v_0 \cos\alpha)t \quad \Rightarrow \quad t = \dfrac{x}{v_0 \cos\alpha}$.

$$y = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2 \quad \Rightarrow \quad y = (v_0 \sin\alpha)\frac{x}{v_0 \cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0 \cos\alpha}\right)^2 = (\tan\alpha)x - \left(\frac{g}{2v_0^2 \cos^2\alpha}\right)x^2,$$

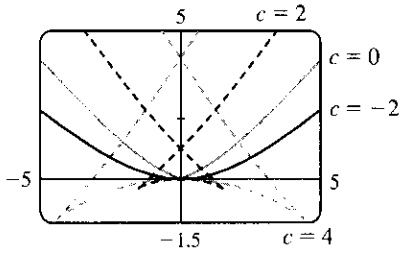
which is the equation of a parabola (quadratic in x).

45. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the x-axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.





46. $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the y-axis. When $c < 0$, the graph resembles that of a polynomial of even degree, but when $c = 0$ there is a corner at the origin, and when $c > 0$, the graph crosses itself at the origin, and has two cusps below the x-axis. The size of the "swallowtail" increases as c increases.



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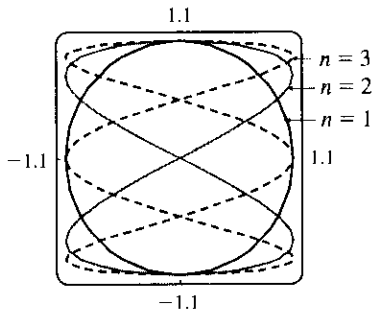
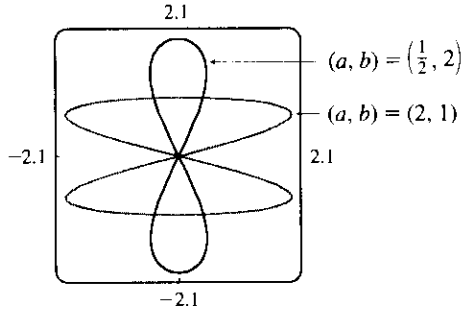
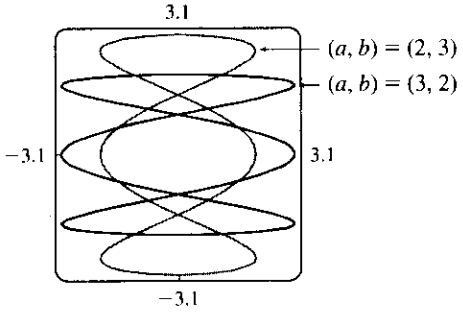
47. Note that all the Lissajous figures are symmetric about the x-axis. The parameters a and b simply stretch the graph in the x- and y-directions respectively. For $a = b = n = 1$ the graph is simply a circle with radius 1. For $n = 2$ the graph crosses itself at the origin and there are loops above and below the x-axis. In general, the figures have $n - 1$ points of intersection, all of which are on the y-axis, and a total of n closed loops.



$a = b = 1$

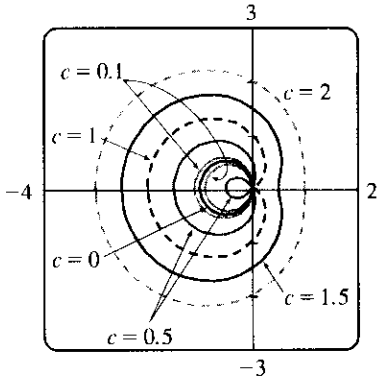


$n = 2$



$n = 3$

48. We use $-\pi \le t \le \pi$ in the viewing rectangle $[-4, 2] \times [-3, 3]$. We first observe that for $c = 0$, we obtain a circle with center $\left(-\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$. As the value of c increases, there is a larger outer loop and a smaller inner loop until $c = 1$, when we obtain a curve with a dent (called a **cardioid**). As c increases, we get a curve with a dimple (called a **limaçon**) until $c = 2$. For $c > 2$, we have convex limaçons. For negative values of c, we obtain the same graphs as for positive c, but with different values of t corresponding to the points on the curve.



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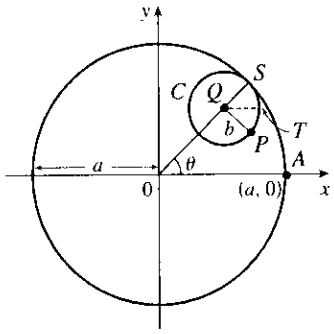
LABORATORY PROJECT Running Circles Around Circles

1. The center Q of the smaller circle has coordinates $((a-b)\cos\theta, (a-b)\sin\theta)$. Arc PS on circle C has length $a\theta$ since it is equal in length to arc AS (the smaller circle rolls without slipping against the larger.) Thus, $\angle PQS = \dfrac{a}{b}\theta$ and $\angle PQT = \dfrac{a}{b}\theta - \theta$, so P has coordinates

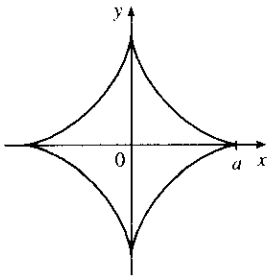


$$x = (a-b)\cos\theta + b\cos(\angle PQT) = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

and

$$y = (a-b)\sin\theta - b\sin(\angle PQT) = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$$

2. With $b = 1$ and a a positive integer greater than 2, we obtain a hypocycloid of a cusps. Shown in the figure is the graph for $a = 4$. Let $a = 4$ and $b = 1$. Using the sum identities to expand $\cos 3\theta$ and $\sin 3\theta$, we obtain

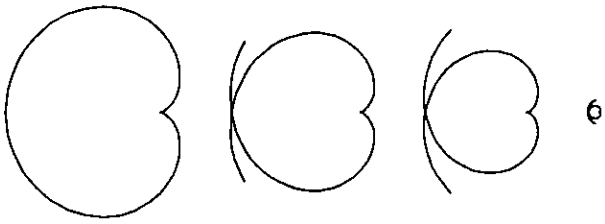


$$x = 3\cos\theta + \cos 3\theta = 3\cos\theta + \left(4\cos^3\theta - 3\cos\theta\right) = 4\cos^3\theta$$

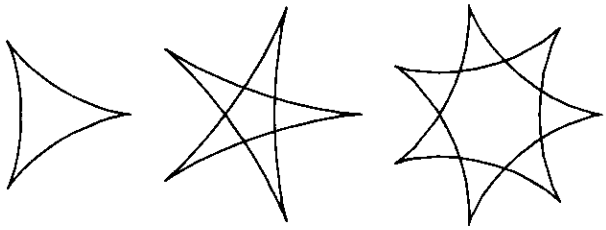
and

$$y = 3\sin\theta - \sin 3\theta = 3\sin\theta - \left(3\sin\theta - 4\sin^3\theta\right) = 4\sin^3\theta$$

3. The following graphs are obtained with $b = 1$ and $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{10}$ with $-2\pi \le \theta \le 2\pi$. We conclude that as the denominator d increases, the graph gets smaller, but maintains the basic shape shown.



Letting $d = 2$ and $n = 3, 5$, and 7 with $-2\pi \le \theta \le 2\pi$ gives us the following:



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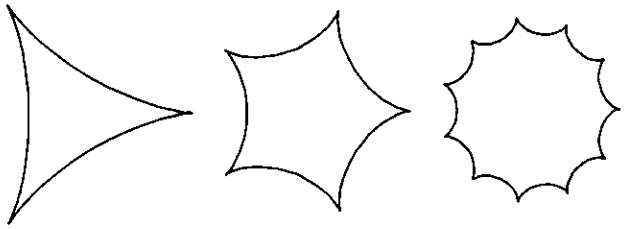
So if d is held constant and n varies, we get a graph with n cusps (assuming n/d is in lowest form).

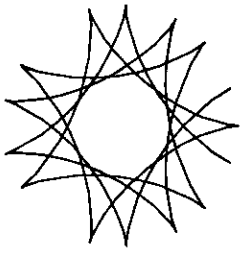
When $n = d + 1$, we obtain a hypocycloid of n cusps. As n increases, we must expand the range of θ in order to get a closed curve. The following graphs have $a = \frac{3}{2}$, $\frac{5}{4}$, and $\frac{11}{10}$.



4. If $b = 1$, the equations for the hypocycloid are

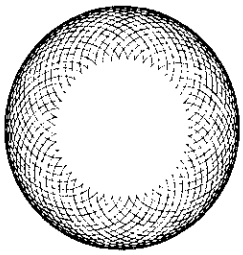
$$x = (a-1)\cos\theta + \cos((a-1)\theta) \qquad y = (a-1)\sin\theta - \sin((a-1)\theta)$$

which is a hypocycloid of a cusps (from Problem 2). In general, if $a > 1$, we get a figure with cusps on the "outside ring" and if $a < 1$, the cusps are on the "inside ring". In any case, as the values of θ get larger, we get a figure that looks more and more like a washer. If we were to graph the hypocycloid for all values of θ, every point on the washer would eventually be arbitrarily close to a point on the curve.



$a = \sqrt{2}$

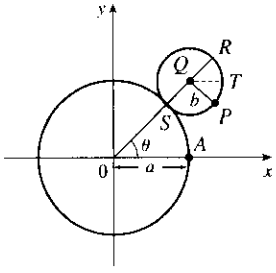
$-10\pi \le \theta \le 10\pi$



$a = e - 2$

$0 \le \theta \le 446$

5. The center Q of the smaller circle has coordinates $((a+b)\cos\theta, (a+b)\sin\theta)$. Arc PS has length $a\theta$ (as in Problem 1), so that $\angle PQS = \dfrac{a\theta}{b}$, $\angle PQR = \pi - \dfrac{a\theta}{b}$, and $\angle PQT = \pi - \dfrac{a\theta}{b} - \theta = \pi - \left(\dfrac{a+b}{b}\right)\theta$ since $\angle RQT = \theta$.



Thus, the coordinates of P are

$$x = (a+b)\cos\theta + b\cos\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

and

$$y = (a+b)\sin\theta - b\sin\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right).$$

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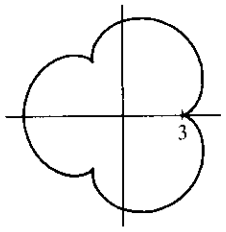
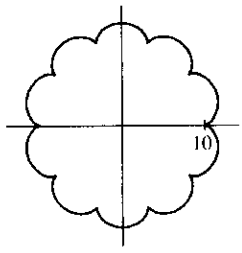
6. Let $b = 1$ and the equations become

$$x = (a+1)\cos\theta - \cos((a+1)\theta) \qquad y = (a+1)\sin\theta - \sin((a+1)\theta)$$

If $a = 1$, we have a cardioid. If a is a positive integer greater than 1, we get the graph of an "a-leafed clover", with cusps that are a units from the origin. (Some of the pairs of figures are not to scale.)

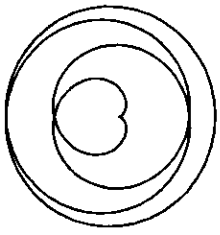


$a = 3, -2\pi \le \theta \le 2\pi$

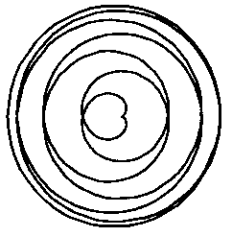


$a = 10, -2\pi \le \theta \le 2\pi$

If $a = n/d$ with $n = 1$, we obtain a figure that does not increase in size and requires $-d\pi \le \theta \le d\pi$ to be a closed curve traced exactly once.

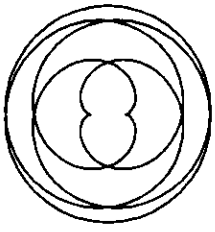


$a = \frac{1}{4}, -4\pi \le \theta \le 4\pi$

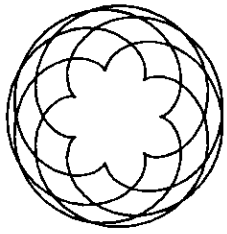


$a = \frac{1}{7}, -7\pi \le \theta \le 7\pi$

Next, we keep d constant and let n vary. As n increases, so does the size of the figure. There is an n-pointed star in the middle.

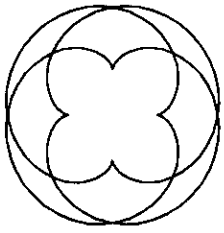


$a = \frac{2}{5}, -5\pi \le \theta \le 5\pi$

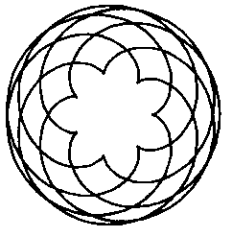


$a = \frac{7}{5}, -5\pi \le \theta \le 5\pi$

Now if $n = d + 1$ we obtain figures similar to the previous ones, but the size of the figure does not increase.

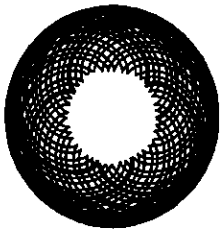


$a = \frac{4}{3}, -3\pi \le \theta \le 3\pi$

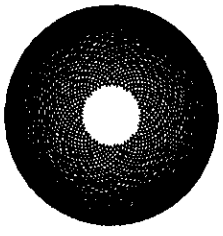


$a = \frac{7}{6}, -6\pi \le \theta \le 6\pi$

If a is irrational, we get washers that increase in size as a increases.



$a = \sqrt{2}, 0 \le \theta \le 200$



$a = e - 2, 0 \le \theta \le 446$

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11.2 Calculus with Parametric Curves

1. $x = t - t^3,\ y = 2 - 5t \quad \Rightarrow \quad \dfrac{dy}{dt} = -5,\ \dfrac{dx}{dt} = 1 - 3t^2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-5}{1 - 3t^2}$ or $\dfrac{5}{3t^2 - 1}$.

2. $x = te^t,\ y = t + e^t \quad \Rightarrow \quad \dfrac{dy}{dt} = 1 + e^t,\ \dfrac{dx}{dt} = te^t + e^t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + e^t}{te^t + e^t}$.

3. $x = t^4 + 1,\ y = t^3 + t;\ t = -1$. $\dfrac{dy}{dt} = 3t^2 + 1,\ \dfrac{dx}{dt} = 4t^3$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 + 1}{4t^3}$.

When $t = -1$, $(x, y) = (2, -2)$ and $dy/dx = \frac{4}{-4} = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is $y - (-2) = (-1)(x - 2)$, or $y = -x$.

4. $x = 2t^2 + 1,\ y = \frac{1}{3}t^3 - t;\ t = 3$. $\dfrac{dy}{dt} = t^2 - 1,\ \dfrac{dx}{dt} = 4t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t^2 - 1}{4t}$. When $t = 3$, $(x, y) = (19, 6)$ and $dy/dx = \frac{8}{12} = \frac{2}{3}$, so an equation of the tangent line is $y - 6 = \frac{2}{3}(x - 19)$, or $y = \frac{2}{3}x - \frac{20}{3}$.

5. $x = e^{\sqrt{t}},\ y = t - \ln t^2;\ t = 1$. $\dfrac{dy}{dt} = 1 - \dfrac{2t}{t^2} = 1 - \dfrac{2}{t},\ \dfrac{dx}{dt} = \dfrac{e^{\sqrt{t}}}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - 2/t}{e^{\sqrt{t}}/(2\sqrt{t})} \cdot \dfrac{2t}{2t} = \dfrac{2t - 4}{\sqrt{t}e^{\sqrt{t}}}$. When $t = 1$, $(x, y) = (e, 1)$ and $\dfrac{dy}{dx} = -\dfrac{2}{e}$, so an equation of the tangent line is $y - 1 = -\frac{2}{e}(x - e)$, or $y = -\frac{2}{e}x + 3$.

6. $x = \cos\theta + \sin 2\theta,\ y = \sin\theta + \cos 2\theta;\ \theta = 0$. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\cos\theta - 2\sin 2\theta}{-\sin\theta + 2\cos 2\theta}$. When $\theta = 0$, $(x, y) = (1, 1)$ and $dy/dx = \frac{1}{2}$, so an equation of the tangent to the curve is $y - 1 = \frac{1}{2}(x - 1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

7. (a) $x = e^t,\ y = (t - 1)^2;\ (1, 1)$. $\dfrac{dy}{dt} = 2(t - 1),\ \dfrac{dx}{dt} = e^t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2(t - 1)}{e^t}$.

At $(1, 1)$, $t = 0$ and $\dfrac{dy}{dx} = -2$, so an equation of the tangent is $y - 1 = -2(x - 1)$, or $y = -2x + 3$.

(b) $x = e^t \quad \Rightarrow \quad t = \ln x$, so $y = (t - 1)^2 = (\ln x - 1)^2$ and $\dfrac{dy}{dx} = 2(\ln x - 1)\left(\dfrac{1}{x}\right)$. When $x = 1$, $\dfrac{dy}{dx} = 2(-1)(1) = -2$, so an equation of the tangent is $y = -2x + 3$, as in part (a).

8. (a) $x = \tan\theta,\ y = \sec\theta;\ \left(1, \sqrt{2}\right)$. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\sec\theta\tan\theta}{\sec^2\theta} = \dfrac{\tan\theta}{\sec\theta} = \sin\theta$.

When $(x, y) = \left(1, \sqrt{2}\right)$, $\theta = \frac{\pi}{4}$ (or $\frac{\pi}{4} + 2\pi n$ for some integer n), so $dy/dx = \sin\frac{\pi}{4} = \sqrt{2}/2$. Thus, an equation of the tangent to the curve is $y - \sqrt{2} = \left(\sqrt{2}/2\right)(x - 1)$, or $y = \left(\sqrt{2}/2\right)x + \left(\sqrt{2}/2\right)$.

(b) $\tan^2\theta + 1 = \sec^2\theta \quad \Rightarrow \quad x^2 + 1 = y^2$, so $\dfrac{d}{dx}(x^2 + 1) = \dfrac{d}{dx}(y^2) \quad \Rightarrow \quad 2x = 2y\dfrac{dy}{dx}$.

When $(x, y) = \left(1, \sqrt{2}\right)$, $\dfrac{dy}{dx} = \dfrac{x}{y} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$, so an equation of the tangent is $y - \sqrt{2} = \left(\sqrt{2}/2\right)(x - 1)$, as in part (a).

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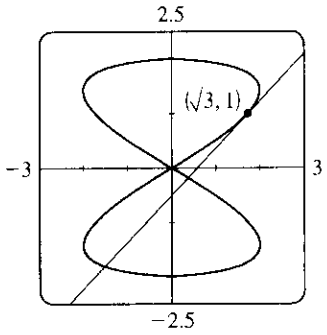
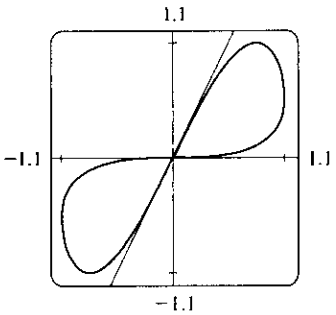
9. $x = 2\sin 2t,\ y = 2\sin t;\ (\sqrt{3}, 1)$.



$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2\cos t}{2 \cdot 2\cos 2t} = \dfrac{\cos t}{2\cos 2t}$. The point $(\sqrt{3}, 1)$ corresponds to $t = \frac{\pi}{6}$, so the slope of the tangent at that point is $\dfrac{\cos\frac{\pi}{6}}{2\cos\frac{\pi}{3}} = \dfrac{\frac{\sqrt{3}}{2}}{2 \cdot \frac{1}{2}} = \dfrac{\sqrt{3}}{2}$. An equation of the tangent is therefore $(y - 1) = \frac{\sqrt{3}}{2}(x - \sqrt{3})$, or $y = \frac{\sqrt{3}}{2}x - \frac{1}{2}$.

10. $x = \sin t,\ y = \sin(t + \sin t);\ (0, 0)$.



$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos(t + \sin t)(1 + \cos t)}{\cos t} = (\sec t + 1)\cos(t + \sin t)$

Note that there are two tangents at the point $(0, 0)$, since both $t = 0$ and $t = \pi$ correspond to the origin. The tangent corresponding to $t = 0$ has slope $(\sec 0 + 1)\cos(0 + \sin 0) = 2\cos 0 = 2$, and its equation is $y = 2x$. The tangent corresponding to $t = \pi$ has slope $(\sec \pi + 1)\cos(\pi + \sin \pi) = 0$, so it is the x-axis.

11. $x = 4 + t^2,\ \ y = t^2 + t^3 \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + 3t^2}{2t} = 1 + \dfrac{3}{2}t \ \Rightarrow$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dx}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{d(dy/dx)/dt}{dx/dt} = \dfrac{(d/dt)(1 + \frac{3}{2}t)}{2t} = \dfrac{3/2}{2t} = \dfrac{3}{4t}$. The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t > 0$.

12. $x = t^3 - 12t,\ y = t^2 - 1 \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{3t^2 - 12} \ \Rightarrow$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{\frac{(3t^2 - 12)\cdot 2 - 2t(6t)}{(3t^2 - 12)^2}}{3t^2 - 12} = \dfrac{-6t^2 - 24}{(3t^2 - 12)^3} = \dfrac{-6(t^2 + 4)}{3^3(t^2 - 4)^3} = \dfrac{-2(t^2 + 4)}{9(t^2 - 4)^3}$. Thus, the curve is CU when $t^2 - 4 < 0 \ \Rightarrow\ |t| < 2 \ \Rightarrow\ -2 < t < 2$.

13. $x = t - e^t,\ y = t + e^{-t} \ \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - e^{-t}}{1 - e^t} = \dfrac{1 - \frac{1}{e^t}}{1 - e^t} = \dfrac{\frac{e^t - 1}{e^t}}{1 - e^t} = -e^{-t} \ \Rightarrow\ \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{\frac{d}{dt}(-e^{-t})}{dx/dt} = \dfrac{e^{-t}}{1 - e^t}$.

The curve is CU when $e^t < 1$ [since $e^{-t} > 0$] $\ \Rightarrow\ t < 0$.

14. $x = t + \ln t,\ y = t - \ln t \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - 1/t}{1 + 1/t} = \dfrac{t - 1}{t + 1} = 1 - \dfrac{2}{t + 1} \ \Rightarrow$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{\frac{d}{dt}\left(1 - \frac{2}{t + 1}\right)}{1 + 1/t} = \dfrac{2/(t + 1)^2}{(t + 1)/t} = \dfrac{2t}{(t + 1)^3}$, so the curve is CU for all t in its domain, that is, $t > 0$ [$t < -1$ not in domain].

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15. $x = 2\sin t,\ y = 3\cos t,\ 0 < t < 2\pi$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3\sin t}{2\cos t} = -\frac{3}{2}\tan t, \text{ so } \frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{3}{2}\sec^2 t}{2\cos t} = -\frac{3}{4}\sec^3 t.$$

The curve is CU when $\sec^3 t < 0 \ \Rightarrow\ \sec t < 0 \ \Rightarrow\ \cos t < 0 \ \Rightarrow\ \frac{\pi}{2} < t < \frac{3\pi}{2}$.

16. $x = \cos 2t,\ y = \cos t,\ 0 < t < \pi$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-\sin t}{-2\sin 2t} = \dfrac{\sin t}{2 \cdot 2\sin t\cos t} = \dfrac{1}{4\cos t} = \dfrac{1}{4}\sec t$, so

$\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \dfrac{\frac{1}{4}\sec t\tan t}{-4\sin t\cos t} = -\dfrac{1}{16}\sec^3 t$. The curve is CU when $\sec^3 t < 0 \ \Rightarrow\ \sec t < 0 \ \Rightarrow\ \cos t < 0 \ \Rightarrow\ \frac{\pi}{2} < t < \pi$.

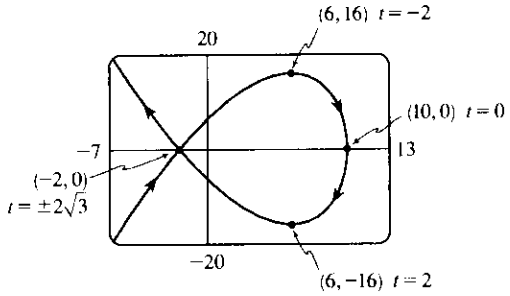
17. $x = 10 - t^2,\ y = t^3 - 12t$.

$dy/dt = 3t^2 - 12 = 3(t+2)(t-2)$, so

$dy/dt = 0 \ \Leftrightarrow\ t = \pm 2 \ \Leftrightarrow$

$(x, y) = (6, \mp 16)$. $dx/dt = -2t$, so $dx/dt = 0$

$\Leftrightarrow\ t = 0 \ \Leftrightarrow\ (x, y) = (10, 0)$. The curve has horizontal tangents at $(6, \pm 16)$ and a vertical tangent at $(10, 0)$.



18. $x = 2t^3 + 3t^2 - 12t,\ y = 2t^3 + 3t^2 + 1$.

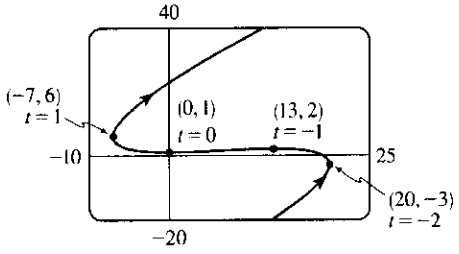
$dy/dt = 6t^2 + 6t = 6t(t+1)$, so $dy/dt = 0 \ \Leftrightarrow$

$t = 0$ or $-1 \ \Leftrightarrow\ (x, y) = (0, 1)$ or $(13, 2)$.

$dx/dt = 6t^2 + 6t - 12 = 6(t+2)(t-1)$, so

$dx/dt = 0 \ \Leftrightarrow\ t = -2$ or $1 \ \Leftrightarrow$

$(x, y) = (20, -3)$ or $(-7, 6)$. The curve has horizontal tangents at $(0, 1)$ and $(13, 2)$, and vertical tangents at $(20, -3)$ and $(-7, 6)$.



19. $x = 2\cos\theta,\ y = \sin 2\theta$.

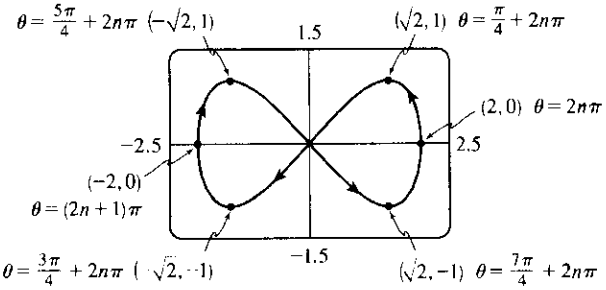
$dy/d\theta = 2\cos 2\theta$, so $dy/d\theta = 0 \ \Leftrightarrow$

$2\theta = \frac{\pi}{2} + n\pi$ (n an integer) $\Leftrightarrow\ \theta = \frac{\pi}{4} + \frac{\pi}{2}n$

$\Leftrightarrow\ (x, y) = \left(\pm\sqrt{2}, \pm 1\right)$. Also,

$dx/d\theta = -2\sin\theta$, so $dx/d\theta = 0 \ \Leftrightarrow\ \theta = n\pi$

$\Leftrightarrow\ (x, y) = (\pm 2, 0)$. The curve has horizontal tangents at $\left(\pm\sqrt{2}, \pm 1\right)$ (four points), and vertical tangents at $(\pm 2, 0)$.



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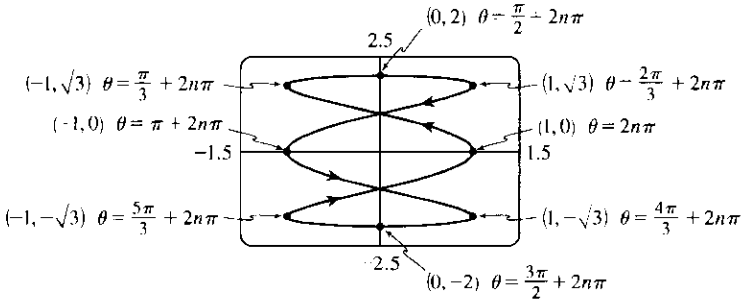
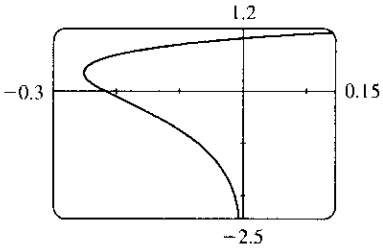
20. $x = \cos 3\theta,\ y = 2\sin\theta$. $dy/d\theta = 2\cos\theta$, so $dy/d\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} + n\pi$ (n an integer) $\Leftrightarrow$ $(x, y) = (0, \pm 2)$. Also, $dx/d\theta = -3\sin 3\theta$, so $dx/d\theta = 0 \Leftrightarrow 3\theta = n\pi \Leftrightarrow \theta = \frac{\pi}{3}n \Leftrightarrow$ $(x, y) = (\pm 1, 0)$ or $(\pm 1, \pm\sqrt{3})$. The curve has horizontal tangents at $(0, \pm 2)$, and vertical tangents at $(\pm 1, 0)$, $(\pm 1, -\sqrt{3})$ and $(\pm 1, \sqrt{3})$.

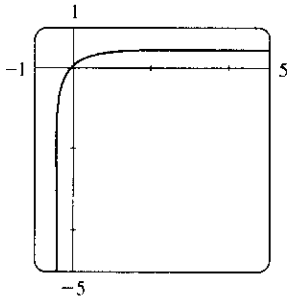


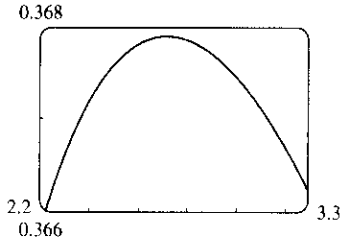
21. From the graph, it appears that the leftmost point on the curve $x = t^4 - t^2$, $y = t + \ln t$ is about $(-0.25, 0.36)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 4t^3 - 2t \Leftrightarrow 2t(2t^2 - 1) = 0 \Leftrightarrow$ $2t(\sqrt{2}\,t + 1)(\sqrt{2}\,t - 1) = 0 \Leftrightarrow t = 0$ or $\pm\frac{1}{\sqrt{2}}$. The negative and 0 roots are inadmissible since $y(t)$ is only defined for $t > 0$, so the leftmost point must be



$$\left(x\left(\tfrac{1}{\sqrt{2}}\right), y\left(\tfrac{1}{\sqrt{2}}\right)\right) = \left(\left(\tfrac{1}{\sqrt{2}}\right)^4 - \left(\tfrac{1}{\sqrt{2}}\right)^2, \tfrac{1}{\sqrt{2}} + \ln\tfrac{1}{\sqrt{2}}\right) = \left(-\tfrac{1}{4}, \tfrac{1}{\sqrt{2}} - \tfrac{1}{2}\ln 2\right)$$

22. The curve is symmetric about the line $y = -x$ since replacing t with $-t$ has the effect of replacing (x, y) with $(-y, -x)$, so if we can find the highest point (x_h, y_h), then the leftmost point is $(x_l, y_l) = (-y_h, -x_h)$. After carefully zooming in, we estimate that the highest point on the curve $x = te^t$, $y = te^{-t}$ is about $(2.7, 0.37)$.





To find the exact coordinates of the highest point, we find the value of t for which the curve has a horizontal tangent, that is, $dy/dt = 0 \Leftrightarrow t(-e^{-t}) + e^{-t} = 0 \Leftrightarrow (1 - t)e^{-t} = 0 \Leftrightarrow t = 1$. This corresponds to the point $(x(1), y(1)) = (e, 1/e)$. To find the leftmost point, we find the value of t for which $0 = dx/dt = te^t + e^t \Leftrightarrow$ $(1 + t)e^t = 0 \Leftrightarrow t = -1$. This corresponds to the point $(x(-1), y(-1)) = (-1/e, -e)$. As $t \to -\infty$, $x(t) = te^t \to 0^-$ by l'Hospital's Rule and $y(t) = te^{-t} \to -\infty$, so the y-axis is an asymptote. As $t \to \infty$, $x(t) \to \infty$ and $y(t) \to 0^+$, so the x-axis is the other asymptote. The asymptotes can also be determined from the graph, if we use a larger t-interval.

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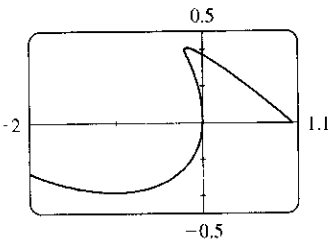
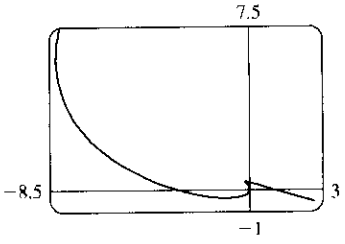
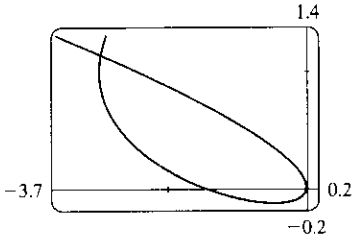
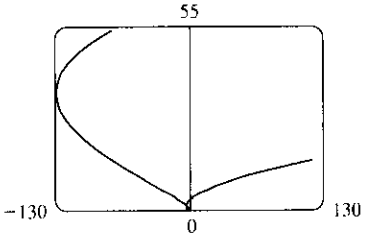
23. We graph the curve $x = t^4 - 2t^3 - 2t^2$, $y = t^3 - t$ in the viewing rectangle $[-2, 1.1]$ by $[-0.5, 0.5]$. This rectangle corresponds approximately to $t \in [-1, 0.8]$. We estimate that the curve has horizontal tangents at about $(-1, -0.4)$ and $(-0.17, 0.39)$ and vertical tangents at about $(0, 0)$ and $(-0.19, 0.37)$. We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 1}{4t^3 - 6t^2 - 4t}$. The horizontal tangents occur when $dy/dt = 3t^2 - 1 = 0 \quad \Leftrightarrow \quad t = \pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when $dx/dt = 2t(2t^2 - 3t - 2) = 0$ $\Leftrightarrow \quad 2t(2t + 1)(t - 2) = 0 \quad \Leftrightarrow \quad t = 0, -\frac{1}{2}$ or 2. It seems that we have missed one vertical tangent, and indeed if we plot the curve on the t-interval $[-1.2, 2.2]$ we see that there is another vertical tangent at $(-8, 6)$.





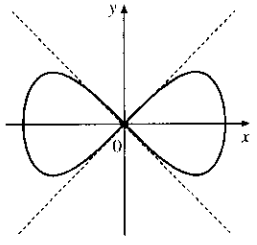
24. We graph the curve $x = t^4 + 4t^3 - 8t^2$, $y = 2t^2 - t$ in the viewing rectangle $[-3.7, 0.2]$ by $[-0.2, 1.4]$. It appears that there is a horizontal tangent at about $(-0.4, -0.1)$, and vertical tangents at about $(-3, 1)$ and $(0, 0)$.





We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t - 1}{4t^3 + 12t^2 - 16t}$, so there is a horizontal tangent where $dy/dt = 4t - 1 = 0 \quad \Leftrightarrow$ $t = \frac{1}{4}$. This point (the lowest point) is shown in the first graph. There are vertical tangents where $dx/dt = 4t^3 + 12t^2 - 16t = 0 \quad \Leftrightarrow \quad 4t(t^2 + 3t - 4) = 0 \quad \Leftrightarrow \quad 4t(t + 4)(t - 1) = 0$. We have missed one vertical tangent corresponding to $t = -4$, and if we plot the graph for $t \in [-5, 3]$, we see that the curve has another vertical tangent line at approximately $(-128, 36)$.

25. $x = \cos t$, $y = \sin t \cos t$. $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = -\sin^2 t + \cos^2 t = \cos 2t$. $(x, y) = (0, 0) \quad \Leftrightarrow \quad \cos t = 0 \quad \Leftrightarrow$ t is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = -1$, so $\frac{dy}{dx} = 1$. When $t = \frac{3\pi}{2}$, $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -1$. So $\frac{dy}{dx} = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



26. $x = 1 - 2\cos^2 t = -\cos 2t$, $y = (\tan t)(1 - 2\cos^2 t) = -(\tan t)\cos 2t$. To find a point where the curve crosses itself, we look for two values of t that give the same point (x, y). Call these values t_1 and t_2. Then $\cos^2 t_1 = \cos^2 t_2$ (from the equation for x) and either $\tan t_1 = \tan t_2$ or $\cos^2 t_1 = \cos^2 t_2 = \frac{1}{2}$ (from the equation for y). We can satisfy $\cos^2 t_1 = \cos^2 t_2$ and $\tan t_1 = \tan t_2$ by choosing t_1 arbitrarily and taking $t_2 = t_1 + \pi$, so evidently the whole curve is retraced every time t traverses an interval of length π. Thus, we can restrict our attention to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $t_2 = -t_1$, then $\cos^2 t_2 = \cos^2 t_1$, but $\tan t_2 = -\tan t_1$. This suggests that

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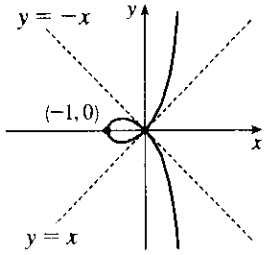
we try to satisfy the condition $\cos^2 t_1 = \cos^2 t_2 = \frac{1}{2}$. Taking $t_1 = \frac{\pi}{4}$ and $t_2 = -\frac{\pi}{4}$ gives $(x, y) = (0, 0)$ for both values of t. $\frac{dx}{dt} = 2\sin 2t$,and $\frac{dy}{dt} = 2\sin 2t \tan t - \cos 2t \sec^2 t$. When $t = \frac{\pi}{4}$, $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 2$, so $\frac{dy}{dx} = 1$. When $t = -\frac{\pi}{4}$, $\frac{dx}{dt} = -2$ and $\frac{dy}{dt} = 2$, so $\frac{dy}{dx} = -1$. Thus, the equations of the two tangents at $(0, 0)$ are $y = x$ and $y = -x$.



27. (a) $x = r\theta - d\sin\theta$, $y = r - d\cos\theta$; $\frac{dx}{d\theta} = r - d\cos\theta$, $\frac{dy}{d\theta} = d\sin\theta$. So $\frac{dy}{dx} = \frac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

28. $x = a\cos^3\theta$, $y = a\sin^3\theta$.

(a) $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$, so $\frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$.

(b) The tangent is horizontal $\Leftrightarrow$ $dy/dx = 0$ $\Leftrightarrow$ $\tan\theta = 0$ $\Leftrightarrow$ $\theta = n\pi$ $\Leftrightarrow$ $(x, y) = (\pm a, 0)$. The tangent is vertical $\Leftrightarrow$ $\cos\theta = 0$ $\Leftrightarrow$ θ is an odd multiple of $\frac{\pi}{2}$ $\Leftrightarrow$ $(x, y) = (0, \pm a)$.

(c) $dy/dx = \pm 1$ $\Leftrightarrow$ $\tan\theta = \pm 1$ $\Leftrightarrow$ θ is an odd multiple of $\frac{\pi}{4}$ $\Leftrightarrow$ $(x, y) = \left(\pm\frac{\sqrt{2}}{4}a, \pm\frac{\sqrt{2}}{4}a\right)$ (All sign choices are valid.)

29. The line with parametric equations $x = -7t$, $y = 12t - 5$ is $y = 12\left(-\frac{1}{7}x\right) - 5$, which has slope $-\frac{12}{7}$. The curve $x = t^3 + 4t$, $y = 6t^2$ has slope $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12t}{3t^2 + 4}$. This equals $-\frac{12}{7}$ $\Leftrightarrow$ $3t^2 + 4 = -7t$ $\Leftrightarrow$ $(3t + 4)(t + 1) = 0$ $\Leftrightarrow$ $t = -1$ or $t = -\frac{4}{3}$ $\Leftrightarrow$ $(x, y) = (-5, 6)$ or $\left(-\frac{208}{27}, \frac{32}{3}\right)$.

30. $x = 3t^2 + 1$, $y = 2t^3 + 1$, $\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6t^2$, so $\frac{dy}{dx} = \frac{6t^2}{6t} = t$ (even where $t = 0$).
So at the point corresponding to parameter value t, an equation of the tangent line is $y - (2t^3 + 1) = t\left[x - (3t^2 + 1)\right]$. If this line is to pass through $(4, 3)$, we must have $3 - (2t^3 + 1) = t\left[4 - (3t^2 + 1)\right]$ $\Leftrightarrow$ $2t^3 - 2 = 3t^3 - 3t$ $\Leftrightarrow$ $t^3 - 3t + 2 = 0$ $\Leftrightarrow$ $(t - 1)^2(t + 2) = 0$ $\Leftrightarrow$ $t = 1$ or -2. Hence, the desired equations are $y - 3 = x - 4$, or $y = x - 1$, tangent to the curve at $(4, 3)$, and $y - (-15) = -2(x - 13)$, or $y = -2x + 11$, tangent to the curve at $(13, -15)$.

31. By symmetry of the ellipse about the x- and y-axes,

$$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 b\sin\theta\,(-a\sin\theta)\,d\theta = 4ab\int_0^{\pi/2}\sin^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1 - \cos 2\theta)\,d\theta$$

$$= 2ab\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left(\tfrac{\pi}{2}\right) = \pi ab$$

32. $t + 1/t = 2.5$ $\Leftrightarrow$ $t = \frac{1}{2}$ or 2, and for $\frac{1}{2} < t < 2$, we have $t + 1/t < 2.5$. $x = -\frac{3}{2}$ when $t = \frac{1}{2}$ and $x = \frac{3}{2}$ when $t = 2$.

$$A = \int_{-3/2}^{3/2}(2.5 - y)\,dx = \int_{1/2}^{2}\left(\tfrac{5}{2} - t - 1/t\right)(1 + 1/t^2)\,dt \quad [x = t - 1/t,\ dx = (1 + 1/t^2)\,dt]$$

$$= \int_{1/2}^{2}\left(-t + \tfrac{5}{2} - 2t^{-1} + \tfrac{5}{2}t^{-2} - t^{-3}\right)dt = \left[\frac{-t^2}{2} + \frac{5t}{2} - 2\ln|t| - \frac{5}{2t} + \frac{1}{2t^2}\right]_{1/2}^{2}$$

$$= \left(-2 + 5 - 2\ln 2 - \tfrac{5}{4} + \tfrac{1}{8}\right) - \left(-\tfrac{1}{8} + \tfrac{5}{4} + 2\ln 2 - 5 + 2\right) = \tfrac{15}{4} - 4\ln 2$$

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33. $A = \int_0^1 (y-1)\,dx = \int_{\pi/2}^0 (e^t - 1)(-\sin t)\,dt = \int_0^{\pi/2} (e^t \sin t - \sin t)\,dt \overset{98}{=} \left[\frac{1}{2}e^t(\sin t - \cos t) + \cos t\right]_0^{\pi/2}$
$= \frac{1}{2}(e^{\pi/2} - 1)$

34. By symmetry, $A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 a\sin^3\theta(-3a\cos^2\theta\sin\theta)\,d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$. Now

$$\int \sin^4\theta\cos^2\theta\,d\theta = \int \sin^2\theta\left(\tfrac{1}{4}\sin^2 2\theta\right)d\theta = \tfrac{1}{8}\int(1-\cos 2\theta)\sin^2 2\theta\,d\theta$$
$$= \tfrac{1}{8}\int\left[\tfrac{1}{2}(1-\cos 4\theta) - \sin^2 2\theta\cos 2\theta\right]d\theta = \tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C$$

so $\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{32}$. Thus, $A = 12a^2\left(\frac{\pi}{32}\right) = \frac{3}{8}\pi a^2$.

35. $A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi}(r - d\cos\theta)(r - d\cos\theta)\,d\theta = \int_0^{2\pi}(r^2 - 2dr\cos\theta + d^2\cos^2\theta)\,d\theta$
$= \left[r^2\theta - 2dr\sin\theta + \tfrac{1}{2}d^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = 2\pi r^2 + \pi d^2$

36. (a) By symmetry, the area of $\mathcal{R}$ is twice the area inside $\mathcal{R}$ above the x-axis. The top half of the loop is described by $x = t^2$, $y = t^3 - 3t$, $-\sqrt{3} \le t \le 0$, so, using the Substitution Rule with $y = t^3 - 3t$ and $dx = 2t\,dt$, we find that

$$\text{area} = 2\int_0^3 y\,dx = 2\int_0^{-\sqrt{3}}(t^3 - 3t)2t\,dt = 2\int_0^{-\sqrt{3}}(2t^4 - 6t^2)\,dt = 2\left[\tfrac{2}{5}t^5 - 2t^3\right]_0^{-\sqrt{3}}$$
$$= 2\left[\tfrac{2}{5}(-3^{1/2})^5 - 2(-3^{1/2})^3\right] = 2\left[\tfrac{2}{5}\left(-9\sqrt{3}\right) - 2\left(-3\sqrt{3}\right)\right] = \tfrac{24}{5}\sqrt{3}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

$$\text{volume} = \pi\int_0^3 y^2\,dx = \pi\int_0^{-\sqrt{3}}(t^3 - 3t)^2 2t\,dt = 2\pi\int_0^{-\sqrt{3}}(t^6 - 6t^4 + 9t^2)t\,dt$$
$$= 2\pi\left[\tfrac{1}{8}t^8 - t^6 + \tfrac{9}{4}t^4\right]_0^{-\sqrt{3}} = 2\pi\left[\tfrac{1}{8}(-3^{1/2})^8 - (-3^{1/2})^6 + \tfrac{9}{4}(-3^{1/2})^4\right]$$
$$= 2\pi\left[\tfrac{81}{8} - 27 + \tfrac{81}{4}\right] = \tfrac{27}{4}\pi$$

(c) By symmetry, the y-coordinate of the centroid is 0. To find the x-coordinate, we note that it is the same as the x-coordinate of the centroid of the top half of $\mathcal{R}$, the area of which is $\frac{1}{2}\cdot\frac{24}{5}\sqrt{3} = \frac{12}{5}\sqrt{3}$. So, using Formula 9.3.8 with $A = \frac{12}{5}\sqrt{3}$, we get

$$\overline{x} = \tfrac{5}{12\sqrt{3}}\int_0^3 xy\,dx = \tfrac{5}{12\sqrt{3}}\int_0^{-\sqrt{3}} t^2(t^3 - 3t)2t\,dt = \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}t^7 - \tfrac{3}{5}t^5\right]_0^{-\sqrt{3}}$$
$$= \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}(-3^{1/2})^7 - \tfrac{3}{5}(-3^{1/2})^5\right] = \tfrac{5}{6\sqrt{3}}\left[-\tfrac{27}{7}\sqrt{3} + \tfrac{27}{5}\sqrt{3}\right] = \tfrac{9}{7}$$

So the coordinates of the centroid of $\mathcal{R}$ are $(x, y) = \left(\frac{9}{7}, 0\right)$.

37. $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \le t \le 2$. $dx/dt = 1 - 2t$ and $dy/dt = 2t^{1/2}$, so $(dx/dt)^2 + (dy/dt)^2 = (1-2t)^2 + (2t^{1/2})^2 = 1 - 4t + 4t^2 + 4t = 1 + 4t^2$. Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^2 \sqrt{1 + 4t^2}\,dt$.

38. $x = 1 + e^t$, $y = t^2$, $-3 \le t \le 3$. $dx/dt = e^t$ and $dy/dt = 2t$, so $(dx/dt)^2 + (dy/dt)^2 = e^{2t} + 4t^2$. Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_{-3}^3 \sqrt{e^{2t} + 4t^2}\,dt$.

39. $x = t + \cos t$, $y = t - \sin t$, $0 \le t \le 2\pi$. $dx/dt = 1 - \sin t$ and $dy/dt = 1 - \cos t$, so
$(dx/dt)^2 + (dy/dt)^2 = (1 - \sin t)^2 + (1 - \cos t)^2 = (1 - 2\sin t + \sin^2 t) + (1 - 2\cos t + \cos^2 t)$
$= 3 - 2\sin t - 2\cos t$
Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{2\pi}\sqrt{3 - 2\sin t - 2\cos t}\,dt$.

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40. $x = \ln t,\ y = \sqrt{t+1},\ 1 \le t \le 5.\ \dfrac{dx}{dt} = \dfrac{1}{t}$ and $\dfrac{dy}{dt} = \dfrac{1}{2\sqrt{t+1}}$, so

$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \dfrac{1}{t^2} + \dfrac{1}{4(t+1)} = \dfrac{t^2+4t+4}{4t^2(t+1)}$. Thus,

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^5 \sqrt{\frac{t^2+4t+4}{4t^2(t+1)}}\,dt = \int_1^5 \sqrt{\frac{(t+2)^2}{(2t)^2(t+1)}}\,dt = \int_1^5 \frac{t+2}{2t\sqrt{t+1}}\,dt.$$

41. $x = 1+3t^2,\ y = 4+2t^3,\ 0 \le t \le 1.\ dx/dt = 6t$ and $dy/dt = 6t^2$, so $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$.

Thus, $L = \int_0^1 \sqrt{36t^2+36t^4}\,dt = \int_0^1 6t\sqrt{1+t^2}\,dt = 6\int_1^2 \sqrt{u}\,\left(\frac{1}{2}du\right)$ $[u = 1+t^2,\ du = 2t\,dt]$

$$= 3\left[\tfrac{2}{3}u^{3/2}\right]_1^2 = 2(2^{3/2}-1) = 2\left(2\sqrt{2}-1\right)$$

42. $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta),\ 0 \le \theta \le \pi.$

$$(dx/d\theta)^2 + (dy/d\theta)^2 = a^2\left[(-\sin\theta + \theta\cos\theta + \sin\theta)^2 + (\cos\theta + \theta\sin\theta - \cos\theta)^2\right]$$
$$= a^2\theta^2\left(\cos^2\theta + \sin^2\theta\right) = (a\theta)^2$$

Thus, $L = \int_0^\pi a\theta\,d\theta = a\left[\frac{1}{2}\theta^2\right]_0^\pi = \frac{1}{2}\pi^2 a$.

43. $x = \dfrac{t}{1+t},\ y = \ln(1+t),\ 0 \le t \le 2.\ \dfrac{dx}{dt} = \dfrac{(1+t)\cdot 1 - t\cdot 1}{(1+t)^2} = \dfrac{1}{(1+t)^2}$ and $\dfrac{dy}{dt} = \dfrac{1}{1+t}$, so

$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \dfrac{1}{(1+t)^4} + \dfrac{1}{(1+t)^2} = \dfrac{1}{(1+t)^4}\left[1+(1+t)^2\right] = \dfrac{t^2+2t+2}{(1+t)^4}$. Thus,

$$L = \int_0^2 \frac{\sqrt{t^2+2t+2}}{(1+t)^2}\,dt = \int_1^3 \frac{\sqrt{u^2+1}}{u^2}\,du\ \ [u = t+1,\ du = dt] \overset{24}{=} \left[-\frac{\sqrt{u^2+1}}{u} + \ln\left(u+\sqrt{u^2+1}\right)\right]_1^3$$
$$= -\tfrac{\sqrt{10}}{3} + \ln\left(3+\sqrt{10}\right) + \sqrt{2} - \ln\left(1+\sqrt{2}\right)$$

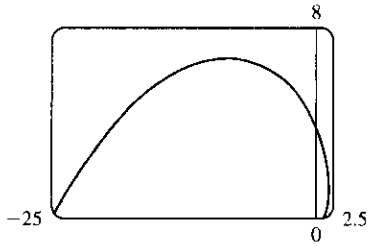
44. $x = e^t + e^{-t},\ y = 5-2t,\ 0 \le t \le 3.\ \ dx/dt = e^t - e^{-t}$ and $dy/dt = -2$, so

$(dx/dt)^2 + (dy/dt)^2 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2$. Thus,

$L = \int_0^3 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^3 = e^3 - e^{-3} - (1-1) = e^3 - e^{-3}$.

45. $x = e^t\cos t,\ y = e^t\sin t,\ 0 \le t \le \pi.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left[e^t(\cos t - \sin t)\right]^2 + \left[e^t(\sin t + \cos t)\right]^2$$
$$= \left(e^t\right)^2\left(\cos^2 t - 2\cos t\,\sin t + \sin^2 t\right)$$
$$+ \left(e^t\right)^2\left(\sin^2 t + 2\sin t\,\cos t + \cos^2 t\right)$$
$$= e^{2t}\left(2\cos^2 t + 2\sin^2 t\right) = 2e^{2t}$$



Thus, $L = \int_0^\pi \sqrt{2e^{2t}}\,dt = \int_0^\pi \sqrt{2}\,e^t\,dt = \sqrt{2}\left[e^t\right]_0^\pi = \sqrt{2}\,(e^\pi - 1)$.

46. $x = \cos t + \ln(\tan\frac{1}{2}t),\quad y = \sin t,\quad \pi/4 \le t \le 3\pi/4.$

$$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2}\sec^2(t/2)}{\tan(t/2)} = -\sin t + \frac{1}{2\sin(t/2)\cos(t/2)} = -\sin t + \frac{1}{\sin t}$$

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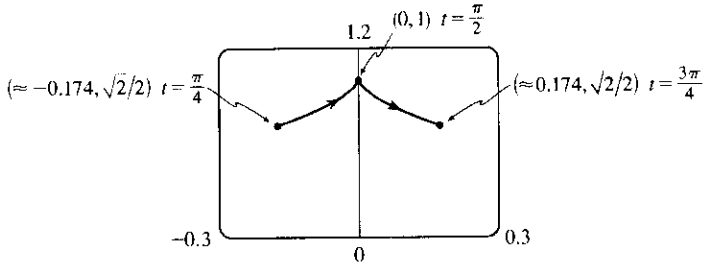
and $\dfrac{dy}{dt} = \cos t$, so $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \sin^2 t - 2 + \dfrac{1}{\sin^2 t} + \cos^2 t = 1 - 2 + \csc^2 t = \cot^2 t$. Thus,

$$L = \int_{\pi/4}^{3\pi/4} \left|\cot t\right| \, dt = 2\int_{\pi/4}^{\pi/2} \cot t \, dt = 2\left[\ln|\sin t|\right]_{\pi/4}^{\pi/2} = 2\left(\ln 1 - \ln\frac{1}{\sqrt{2}}\right)$$

$$= 2\left(0 + \ln\sqrt{2}\right) = 2\left(\tfrac{1}{2}\ln 2\right) = \ln 2.$$



47. $x = e^t - t$, $y = 4e^{t/2}$, $-8 \le t \le 3$.

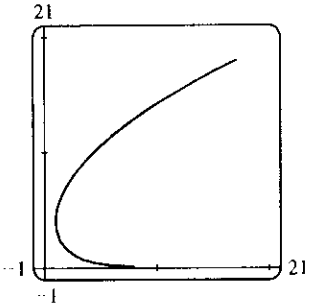
$$(dx/dt)^2 + (dy/dt)^2 = (e^t - 1)^2 + \left(2e^{t/2}\right)^2 = e^{2t} - 2e^t + 1 + 4e^t$$

$$= e^{2t} + 2e^t + 1 = (e^t + 1)^2$$

Thus,

$$L = \int_{-8}^{3} \sqrt{(e^t + 1)^2}\, dt = \int_{-8}^{3} (e^t + 1)\, dt = \Big[e^t + t\Big]_{-8}^{3}$$

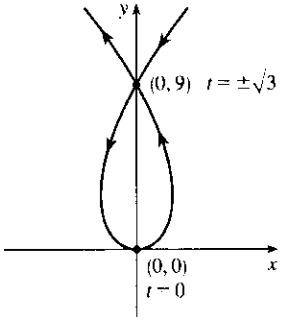
$$= (e^3 + 3) - (e^{-8} - 8) = e^3 - e^{-8} + 11.$$



48. $x = 3t - t^3$, $y = 3t^2$. $dx/dt = 3 - 3t^2$ and $dy/dt = 6t$, so $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = (3 + 3t^2)^2$ and the length of the loop is given by

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2)\, dt = 2\int_{0}^{\sqrt{3}} (3 + 3t^2)\, dt = 2\left[3t + t^3\right]_0^{\sqrt{3}}$$

$$= 2\left(3\sqrt{3} + 3\sqrt{3}\right) = 12\sqrt{3}.$$



49. $x = t - e^t$, $y = t + e^t$, $-6 \le t \le 6$.
$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 - e^t\right)^2 + \left(1 + e^t\right)^2 = \left(1 - 2e^t + e^{2t}\right) + \left(1 + 2e^t + e^{2t}\right) = 2 + 2e^{2t}$, so
$L = \int_{-6}^{6} \sqrt{2 + 2e^{2t}}\, dt$. Set $f(t) = \sqrt{2 + 2e^{2t}}$. Then by Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-(-6)}{6} = 2$, we get $L \approx \frac{2}{3}[f(-6) + 4f(-4) + 2f(-2) + 4f(0) + 2f(2) + 4f(4) + f(6)] \approx 612.3053$.

50. $x = 2a\cot\theta \;\Rightarrow\; dx/dt = -2a\csc^2\theta$ and $y = 2a\sin^2\theta \;\Rightarrow\; dy/dt = 4a\sin\theta\cos\theta = 2a\sin 2\theta$.
So $L = \int_{\pi/4}^{\pi/2} \sqrt{4a^2\csc^4\theta + 4a^2\sin^2 2\theta}\, d\theta = 2a\int_{\pi/4}^{\pi/2} \sqrt{\csc^4\theta + \sin^2 2\theta}\, d\theta$. Using Simpson's Rule with $n = 4$, $\Delta\theta = \frac{\pi/2 - \pi/4}{4} = \frac{\pi}{16}$, and $f(\theta) = \sqrt{\csc^4\theta + \sin^2 2\theta}$, we get
$L \approx 2a \cdot S_4 = (2a)\frac{\pi}{16\cdot 3}\left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 2.2605a$.

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51. $x = \sin^2 t,\ y = \cos^2 t,\ 0 \le t \le 3\pi$.

$(dx/dt)^2 + (dy/dt)^2 = (2\sin t\cos t)^2 + (-2\cos t\sin t)^2 = 8\sin^2 t\cos^2 t = 2\sin^2 2t \ \Rightarrow$

$$\text{Distance} = \int_0^{3\pi} \sqrt{2}\,|\sin 2t|\,dt = 6\sqrt{2}\int_0^{\pi/2} \sin 2t\,dt \ \text{[by symmetry]} \ = -3\sqrt{2}\left[\cos 2t\right]_0^{\pi/2}$$

$$= -3\sqrt{2}\,(-1-1) = 6\sqrt{2}$$

The full curve is traversed as t goes from 0 to $\frac{\pi}{2}$, because the curve is the segment of $x + y = 1$ that lies in the first quadrant (since $x,\ y \ge 0$), and this segment is completely traversed as t goes from 0 to $\frac{\pi}{2}$.

Thus, $L = \int_0^{\pi/2} \sin 2t\,dt = \sqrt{2}$, as above.

52. $x = \cos^2 t,\ y = \cos t,\ 0 \le t \le 4\pi$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-2\cos t\sin t)^2 + (-\sin t)^2 = \sin^2 t(4\cos^2 t + 1)$

$$\text{Distance} = \int_0^{4\pi} |\sin t|\sqrt{4\cos^2 t + 1}\,dt = 4\int_0^{\pi} \sin t\sqrt{4\cos^2 t + 1}\,dt$$

$$= -4\int_1^{-1} \sqrt{4u^2+1}\,du \ \ [u = \cos t,\ du = -\sin t\,dt] = 4\int_{-1}^{1} \sqrt{4u^2+1}\,du = 8\int_0^1 \sqrt{4u^2+1}\,du$$

$$= 8\int_0^{\tan^{-1}2} \sec\theta\cdot\tfrac{1}{2}\sec^2\theta\,d\theta \qquad [2u = \tan\theta,\ 2\,du = \sec^2\theta\,d\theta]$$

$$= 4\int_0^{\tan^{-1}2} \sec^3\theta\,d\theta \overset{71}{=} \left[2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta|\right]_0^{\tan^{-1}2} = 4\sqrt{5} + 2\ln\left(\sqrt{5}+2\right)$$

Thus, $L = \int_0^{\pi} |\sin t|\sqrt{4\cos^2 t + 1}\,dt = \sqrt{5} + \frac{1}{2}\ln\left(\sqrt{5}+2\right)$.

53. $x = a\sin\theta,\ y = b\cos\theta,\ 0 \le \theta \le 2\pi$.

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (a\cos\theta)^2 + (-b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2\left(1-\sin^2\theta\right) + b^2\sin^2\theta$$

$$= a^2 - \left(a^2 - b^2\right)\sin^2\theta = a^2 - c^2\sin^2\theta = a^2\left(1 - \frac{c^2}{a^2}\sin^2\theta\right) = a^2\left(1 - e^2\sin^2\theta\right)$$

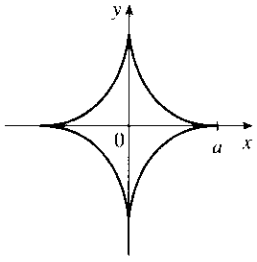
So $L = 4\int_0^{\pi/2} \sqrt{a^2\left(1 - e^2\sin^2\theta\right)}\,d\theta$ [by symmetry] $= 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\,d\theta$.

54. $x = a\cos^3\theta,\ y = a\sin^3\theta$.

$$(dx/d\theta)^2 + (dy/d\theta)^2 = \left(-3a\cos^2\theta\,\sin\theta\right)^2 + \left(3a\sin^2\theta\,\cos\theta\right)^2$$

$$= 9a^2\cos^4\theta\,\sin^2\theta + 9a^2\sin^4\theta\,\cos^2\theta$$

$$= 9a^2\sin^2\theta\,\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right) = 9a^2\sin^2\theta\,\cos^2\theta.$$

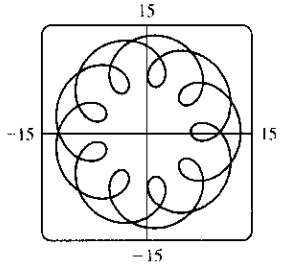


The graph has four-fold symmetry and the curve in the first quadrant corresponds to $0 \le \theta \le \pi/2$. Thus,

$$L = 4\int_0^{\pi/2} 3a\sin\theta\,\cos\theta\,d\theta \qquad [\text{since } a > 0 \text{ and } \sin\theta \text{ and } \cos\theta \text{ are positive for } 0 \le \theta \le \pi/2]$$

$$= 12a\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = 12a\left(\tfrac{1}{2} - 0\right) = 6a.$$

55. (a) Notice that $0 \le t \le 2\pi$ does not give the complete curve because $x(0) \ne x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2} = \frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π.



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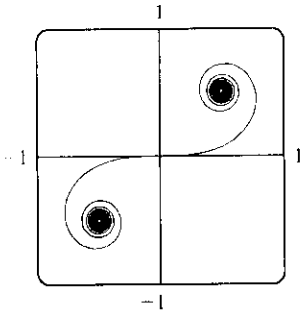
(b) We use the CAS to find the derivatives dx/dt and dy/dt, and then use Formula 1 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ as $88E\left(2\sqrt{2}\,i\right)$, where $E(x)$ is the elliptic integral $\displaystyle\int_0^1 \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}\, dt$ and i is the imaginary number $\sqrt{-1}$. Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03. Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to $11\int_0^{4\pi} \sqrt{-4\cos t\,\cos\left(\frac{11t}{2}\right) - 4\sin t\,\sin\left(\frac{11t}{2}\right) + 5}\, dt$.

56. (a) It appears that as $t \to \infty$, $(x, y) \to \left(\frac{1}{2}, \frac{1}{2}\right)$, and as $t \to -\infty$, $(x, y) \to \left(-\frac{1}{2}, -\frac{1}{2}\right)$.



(b) By the Fundamental Theorem of Calculus, $dx/dt = \cos\left(\frac{\pi}{2}t^2\right)$ and $dy/dt = \sin\left(\frac{\pi}{2}t^2\right)$, so by Formula 6, the length of the curve from the origin to the point with parameter value t is

$$L = \int_0^t \sqrt{(dx/du)^2 + (dy/du)^2}\, du = \int_0^t \sqrt{\cos^2\left(\tfrac{\pi}{2}u^2\right) + \sin^2\left(\tfrac{\pi}{2}u^2\right)}\, du$$
$$= \int_0^t 1\, du = t \quad [\text{or } -t \text{ if } t < 0]$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

57. $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \le t \le 2$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1-2t)^2 + (2t^{1/2})^2 = 1 - 4t + 4t^2 + 4t = 1 + 4t^2$, so $S = \int_1^2 2\pi y\, ds = \int_1^2 2\pi \cdot \frac{4}{3}t^{3/2}\sqrt{1+4t^2}\, dt = \int_1^2 \frac{8\pi}{3}t^{3/2}\sqrt{1+4t^2}\, dt$.

58. $x = \sin^2 t$, $y = \sin 3t$, $0 \le t \le \frac{\pi}{3}$. $dx/dt = 2\sin t\cos t = \sin 2t$ and $dy/dt = 3\cos 3t$, so $(dx/dt)^2 + (dy/dt)^2 = \sin^2 2t + 9\cos^2 3t$ and $S = \int 2\pi y\, ds = \int_0^{\pi/3} 2\pi \sin 3t\sqrt{\sin^2 2t + 9\cos^2 3t}\, dt$.

59. $x = t^3$, $y = t^2$, $0 \le t \le 1$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3t^2)^2 + (2t)^2 = 9t^4 + 4t^2$.

$$S = \int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi t^2\sqrt{9t^4+4t^2}\, dt = 2\pi\int_0^1 t^2\sqrt{t^2(9t^2+4)}\, dt$$
$$= 2\pi\int_4^{13} \left(\frac{u-4}{9}\right)\sqrt{u}\left(\tfrac{1}{18}du\right) \quad \begin{bmatrix} u = 9t^2+4,\ t^2 = (u-4)/9 \\ du = 18t\, dt,\ \text{so } t\, dt = \frac{1}{18}\, du \end{bmatrix} = \frac{2\pi}{9\cdot 18}\int_4^{13} (u^{3/2} - 4u^{1/2})\, du$$
$$= \tfrac{\pi}{81}\left[\tfrac{2}{5}u^{5/2} - \tfrac{8}{3}u^{3/2}\right]_4^{13} = \tfrac{\pi}{81}\cdot\tfrac{2}{15}\left[3u^{5/2} - 20u^{3/2}\right]_4^{13}$$
$$= \tfrac{2\pi}{1215}\left[\left(3\cdot 13^2\sqrt{13} - 20\cdot 13\sqrt{13}\right) - (3\cdot 32 - 20\cdot 8)\right]$$
$$= \tfrac{2\pi}{1215}\left(247\sqrt{13} + 64\right)$$

60. $x = 3t - t^3$, $y = 3t^2$, $0 \le t \le 1$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3-3t^2)^2 + (6t)^2 = 9(1+2t^2+t^4) = \left[3(1+t^2)\right]^2$.

$S = \int_0^1 2\pi\cdot 3t^2\cdot 3(1+t^2)\, dt = 18\pi\int_0^1 (t^2+t^4)\, dt = 18\pi\left[\frac{1}{3}t^3 + \frac{1}{5}t^5\right]_0^1 = \frac{48}{5}\pi$

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61. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le \frac{\pi}{2}$.

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2 = 9a^2\sin^2\theta\cos^2\theta.$$

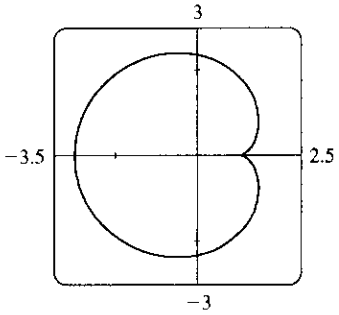
$$S = \int_0^{\pi/2} 2\pi\cdot a\sin^3\theta\cdot 3a\sin\theta\cos\theta\,d\theta = 6\pi a^2\int_0^{\pi/2}\sin^4\theta\cos\theta\,d\theta = \tfrac{6}{5}\pi a^2\left[\sin^5\theta\right]_0^{\pi/2} = \tfrac{6}{5}\pi a^2$$

62. $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (-2\sin\theta + 2\sin 2\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2$

$$= 4\left[\left(\sin^2\theta - 2\sin\theta\sin 2\theta + \sin^2 2\theta\right) + \left(\cos^2\theta - 2\cos\theta\cos 2\theta + \cos^2 2\theta\right)\right]$$

$$= 4[1 + 1 - 2(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta)] = 8\left[1 - \cos(2\theta - \theta)\right] = 8(1 - \cos\theta)$$

We plot the graph with parameter interval $[0, 2\pi]$, and see that we should only integrate between 0 and π. (If the interval $[0, 2\pi]$ were taken, the surface of revolution would be generated twice.) Also note that $y = 2\sin\theta - \sin 2\theta = 2\sin\theta(1 - \cos\theta)$. So



$$S = \int_0^{\pi} 2\pi\cdot 2\sin\theta(1-\cos\theta)\,2\sqrt{2}\sqrt{1-\cos\theta}\,d\theta$$

$$= 8\sqrt{2}\pi\int_0^{\pi}(1-\cos\theta)^{3/2}\sin\theta\,d\theta$$

$$= 8\sqrt{2}\pi\int_0^{2}\sqrt{u^3}\,du \quad [\text{where } u = 1-\cos\theta,\ du = \sin\theta\,d\theta]$$

$$= 8\sqrt{2}\pi\left[\left(\tfrac{2}{5}\right)u^{5/2}\right]_0^2 = \tfrac{16}{5}\sqrt{2}\pi(2^{5/2}) = \tfrac{128}{5}\pi$$

63. $x = t + t^3,\ y = t - \dfrac{1}{t^2},\ 1 \le t \le 2.\ \dfrac{dx}{dt} = 1 + 3t^2$ and $\dfrac{dy}{dt} = 1 + \dfrac{2}{t^3}$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1+3t^2)^2 + \left(1+\frac{2}{t^3}\right)^2 \text{ and}$$

$$S = \int 2\pi y\,ds = \int_1^2 2\pi\left(t - \frac{1}{t^2}\right)\sqrt{(1+3t^2)^2 + \left(1+\frac{2}{t^3}\right)^2}\,dt \approx 59.101.$$

64. $S = \int_{\pi/4}^{\pi/2} 2\pi\cdot 2a\sin^2\theta\sqrt{\csc^4\theta + \sin^2 2\theta}\,dt = 4\pi a\int_{\pi/4}^{\pi/2}\sin^2\theta\sqrt{\csc^4\theta + \sin^2 2\theta}\,d\theta.$

Using Simpson's Rule with $n = 4$, $\Delta\theta = \frac{\pi/2 - \pi/4}{4} = \frac{\pi}{16}$, and $f(\theta) = \sin^2\theta\sqrt{\csc^4\theta + \sin^2 2\theta}$, we get

$$S \approx (4\pi a)\,\tfrac{\pi}{16\cdot 3}\left[f\left(\tfrac{\pi}{4}\right) + 4f\left(\tfrac{5\pi}{16}\right) + 2f\left(\tfrac{3\pi}{8}\right) + 4f\left(\tfrac{7\pi}{16}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 11.0893a.$$

65. $x = 3t^2,\ y = 2t^3,\ 0 \le t \le 5 \ \Rightarrow\ \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = (6t)^2 + (6t^2)^2 = 36t^2(1+t^2) \ \Rightarrow$

$$S = \int_0^5 2\pi x\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^5 2\pi(3t^2)6t\sqrt{1+t^2}\,dt = 18\pi\int_0^5 t^2\sqrt{1+t^2}\,2t\,dt$$

$$= 18\pi\int_1^{26}(u-1)\sqrt{u}\,du \ [\text{where } u = 1+t^2,\ du = 2t\,dt] = 18\pi\int_1^{26}\left(u^{3/2} - u^{1/2}\right)du$$

$$= 18\pi\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^{26} = 18\pi\left[\left(\tfrac{2}{5}\cdot 676\sqrt{26} - \tfrac{2}{3}\cdot 26\sqrt{26}\right) - \left(\tfrac{2}{5} - \tfrac{2}{3}\right)\right]$$

$$= \tfrac{24}{5}\pi\left(949\sqrt{26} + 1\right)$$

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66. $x = e^t - t,\ y = 4e^{t/2},\ 0 \le t \le 1.$ $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - 1)^2 + \left(2e^{t/2}\right)^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2.$

$$S = \int_0^1 2\pi(e^t - t)\sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\,dt = \int_0^1 2\pi(e^t - t)(e^t + 1)\,dt$$
$$= 2\pi\left[\tfrac{1}{2}e^{2t} + e^t - (t-1)e^t - \tfrac{1}{2}t^2\right]_0^1 = \pi(e^2 + 2e - 6)$$

67. If f' is continuous and $f'(t) \neq 0$ for $a \le t \le b$, then either $f'(t) > 0$ for all t in $[a, b]$ or $f'(t) < 0$ for all t in $[a, b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a, b]$. It follows that f has an inverse. Set $F = g \circ f^{-1}$, that is, define F by $F(x) = g(f^{-1}(x))$. Then $x = f(t) \Rightarrow f^{-1}(x) = t$, so $y = g(t) = g(f^{-1}(x)) = F(x)$.

68. By Formula 9.2.5 with $y = F(x)$, $S = \int_a^b 2\pi F(x)\sqrt{1 + [F'(x)]^2}\,dx$. But by Formula 11.2.2, $1 + [F'(x)]^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy/dt}{dx/dt}\right)^2 = \frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}$. Using the Suhstitution Rule with $x = x(t)$, where $a = x(\alpha)$ and $b = x(\beta)$, we have (since $dx = \frac{dx}{dt}\,dt$)

$$S = \int_\alpha^\beta 2\pi F(x(t))\sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\,\frac{dx}{dt}\,dt = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

which is Formula 11.2.7.

69. (a) $\phi = \tan^{-1}\left(\frac{dy}{dx}\right) \Rightarrow \frac{d\phi}{dt} = \frac{d}{dt}\tan^{-1}\left(\frac{dy}{dx}\right) = \frac{1}{1 + (dy/dx)^2}\left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]$. But $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot y}{\dot x} \Rightarrow$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dot y}{\dot x}\right) = \frac{\ddot y\dot x - \ddot x\dot y}{\dot x^2} \Rightarrow \frac{d\phi}{dt} = \frac{1}{1 + (\dot y/\dot x)^2}\left(\frac{\ddot y\dot x - \ddot x\dot y}{\dot x^2}\right) = \frac{\dot x\ddot y - \ddot x\dot y}{\dot x^2 + \dot y^2}$.

Using the Chain Rule, and the fact that $s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \Rightarrow$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = (\dot x^2 + \dot y^2)^{1/2}$, we have that

$\frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt} = \left(\frac{\dot x\ddot y - \ddot x\dot y}{\dot x^2 + \dot y^2}\right)\frac{1}{(\dot x^2 + \dot y^2)^{1/2}} = \frac{\dot x\ddot y - \ddot x\dot y}{(\dot x^2 + \dot y^2)^{3/2}}$. So

$\kappa = \left|\frac{d\phi}{ds}\right| = \left|\frac{\dot x\ddot y - \ddot x\dot y}{(\dot x^2 + \dot y^2)^{3/2}}\right| = \frac{|\dot x\ddot y - \ddot x\dot y|}{(\dot x^2 + \dot y^2)^{3/2}}$.

(b) $x = x$ and $y = f(x) \Rightarrow \dot x = 1, \ddot x = 0$ and $\dot y = \frac{dy}{dx}, \ddot y = \frac{d^2y}{dx^2}$.

So $\kappa = \frac{|1 \cdot (d^2y/dx^2) - 0 \cdot (dy/dx)|}{\left[1 + (dy/dx)^2\right]^{3/2}} = \frac{|d^2y/dx^2|}{\left[1 + (dy/dx)^2\right]^{3/2}}$.

70. (a) $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow \frac{d^2y}{dx^2} = 2$. So $\kappa = \frac{|d^2y/dx^2|}{\left[1 + (dy/dx)^2\right]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$, and at $(1, 1)$, $\kappa = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}}$.

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(b) $\kappa' = \dfrac{d\kappa}{dx} = -3(1+4x^2)^{-5/2}(8x) = 0 \quad \Leftrightarrow \quad x = 0 \quad \Rightarrow \quad y = 0$. This is a maximum since $\kappa' > 0$ for $x < 0$ and $\kappa' < 0$ for $x > 0$. So the parabola $y = x^2$ has maximum curvature at the origin.

71. $x = \theta - \sin\theta \quad \Rightarrow \quad \dot{x} = 1 - \cos\theta \quad \Rightarrow \quad \ddot{x} = \sin\theta$, and $y = 1 - \cos\theta \quad \Rightarrow \quad \dot{y} = \sin\theta \quad \Rightarrow \quad \ddot{y} = \cos\theta$.

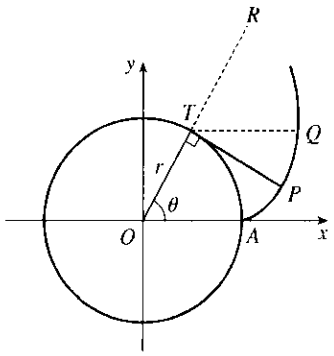
Therefore, $\kappa = \dfrac{|\cos\theta - \cos^2\theta - \sin^2\theta|}{\left[(1-\cos\theta)^2 + \sin^2\theta\right]^{3/2}} = \dfrac{\left|\cos\theta - \left(\cos^2\theta + \sin^2\theta\right)\right|}{\left(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right)^{3/2}} = \dfrac{|\cos\theta - 1|}{(2-2\cos\theta)^{3/2}}$. The top of the arch is characterized by a horizontal tangent, and from Example 2(b) in Section 11.2, the tangent is horizontal when $\theta = (2n-1)\pi$, so take $n = 1$ and substitute $\theta = \pi$ into the expression for κ:

$\kappa = \dfrac{|\cos\pi - 1|}{(2 - 2\cos\pi)^{3/2}} = \dfrac{|-1-1|}{[2-2(-1)]^{3/2}} = \dfrac{1}{4}$.

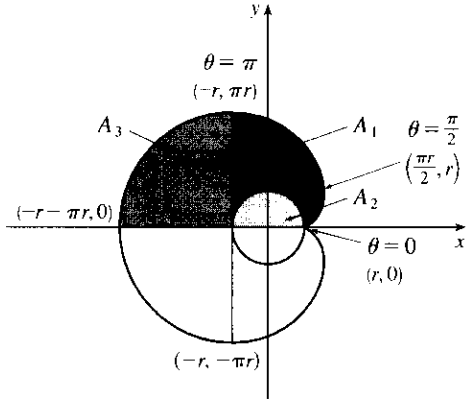
72. (a) Every straight line has parametrizations of the form $x = a + vt$, $y = b + wt$, where a, b are arbitrary and v, $w \neq 0$. For example, a straight line passing through distinct points (a, b) and (c, d) can be described as the parametrized curve $x = a + (c-a)t$, $y = b + (d-b)t$. Starting with $x = a + vt$, $y = b + wt$, we compute $\dot{x} = v$, $\dot{y} = w$, $\ddot{x} = \ddot{y} = 0$, and $\kappa = \dfrac{|v \cdot 0 - w \cdot 0|}{(v^2 + w^2)^{3/2}} = 0$.

(b) Parametric equations for a circle of radius r are $x = r\cos\theta$ and $y = r\sin\theta$. We can take the center to be the origin. So $\dot{x} = -r\sin\theta \quad \Rightarrow \quad \ddot{x} = -r\cos\theta$ and $\dot{y} = r\cos\theta \quad \Rightarrow \quad \ddot{y} = -r\sin\theta$. Therefore, $\kappa = \dfrac{|r^2\sin^2\theta + r^2\cos^2\theta|}{\left(r^2\sin^2\theta + r^2\cos^2\theta\right)^{3/2}} = \dfrac{r^2}{r^3} = \dfrac{1}{r}$. And so for any θ (and thus any point), $\kappa = \dfrac{1}{r}$.

73. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA, TP has length $r\theta$. Also $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$, so P has coordinates $x = r\cos\theta + r\theta\cos\left(\frac{1}{2}\pi - \theta\right) = r(\cos\theta + \theta\sin\theta)$, $y = r\sin\theta - r\theta\sin\left(\frac{1}{2}\pi - \theta\right) = r(\sin\theta - \theta\cos\theta)$.



74. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 73 corresponding to the range $0 \le \theta \le \pi$, arriving at the point $(-r, \pi r)$ when $\theta = \pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr, arriving at $(-r, -\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the x-axis of the initial involute path. (This corresponds to the range $-\pi \le \theta \le 0$.) Referring to the figure, we see that the total grazing area is $2(A_1 + A_3)$. A_3 is one-quarter of the area of a circle of radius πr, so $A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2$. We will compute $A_1 + A_2$ and then subtract $A_2 = \frac{1}{2}\pi r^2$ to obtain A_1.



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To find $A_1 + A_2$, first note that the rightmost point of the involute is $\left(\frac{\pi}{2}r, r\right)$. [To see this, note that $dx/d\theta = 0$ when $\theta = 0$ or $\frac{\pi}{2}$. $\theta = 0$ corresponds to the cusp at $(r, 0)$ and $\theta = \frac{\pi}{2}$ corresponds to $\left(\frac{\pi}{2}r, r\right)$.] The leftmost point of the involute is $(-r, \pi r)$. Thus, $A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y\,dx - \int_{\theta=0}^{\pi/2} y\,dx = \int_{\theta=\pi}^{0} y\,dx$. Now $y\,dx = r(\sin\theta - \theta\cos\theta)r\theta\cos\theta\,d\theta = r^2(\theta\sin\theta\cos\theta - \theta^2\cos^2\theta)d\theta$. Integrate: $(1/r^2)\int y\,dx = -\theta\cos^2\theta - \frac{1}{2}(\theta^2 - 1)\sin\theta\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + C$. This enables us to compute

$$A_1 + A_2 = r^2\left[-\theta\cos^2\theta - \tfrac{1}{2}(\theta^2 - 1)\sin\theta\cos\theta - \tfrac{1}{6}\theta^3 + \tfrac{1}{2}\theta\right]_{\pi}^{0} = r^2\left[0 - \left(-\pi - \tfrac{\pi^3}{6} + \tfrac{\pi}{2}\right)\right]$$

$$= r^2\left(\tfrac{\pi}{2} + \tfrac{\pi^3}{6}\right)$$

Therefore, $A_1 = (A_1 + A_2) - A_2 = \frac{1}{6}\pi^3 r^2$, so the grazing area is $2(A_1 + A_3) = 2\left(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2\right) = \frac{5}{6}\pi^3 r^2$.

LABORATORY PROJECT Bezier Curves

1. The parametric equations for a cubic Bézier curve are

$$x = x_0(1-t)^3 + 3x_1t(1-t)^2 + 3x_2t^2(1-t) + x_3t^3$$

$$y = y_0(1-t)^3 + 3y_1t(1-t)^2 + 3y_2t^2(1-t) + y_3t^3$$

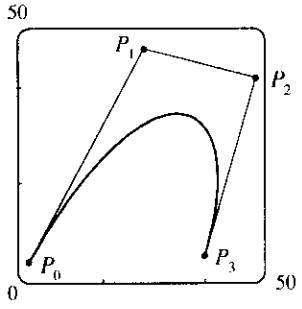
where $0 \le t \le 1$. We are given the points $P_0(x_0, y_0) = (4, 1)$, $P_1(x_1, y_1) = (28, 48)$, $P_2(x_2, y_2) = (50, 42)$, and $P_3(x_3, y_3) = (40, 5)$. The curve is then given by

$$x(t) = 4(1-t)^3 + 3\cdot 28t(1-t)^2 + 3\cdot 50t^2(1-t) + 40t^3$$

$$y(t) = 1(1-t)^3 + 3\cdot 48t(1-t)^2 + 3\cdot 42t^2(1-t) + 5t^3$$

where $0 \le t \le 1$. The line segments are of the form $x = x_0 + (x_1 - x_0)t$, $y = y_0 + (y_1 - y_0)t$:

$$\begin{array}{lll} P_0P_1 & x = 4 + 24t, & y = 1 + 47t \\ P_1P_2 & x = 28 + 22t, & y = 48 - 6t \\ P_2P_3 & x = 50 - 10t, & y = 42 - 37t \end{array}$$



2. It suffices to show that the slope of the tangent at P_0 is the same as that of line segment P_0P_1, namely $\dfrac{y_1 - y_0}{x_1 - x_0}$. We calculate the slope of the tangent to the Bézier curve:

$$\frac{dy/dt}{dx/dt} = \frac{-3y_0(1-t)^2 + 3y_1\left[-2t(1-t) + (1-t)^2\right] + 3y_2\left[-t^2 + (2t)(1-t)\right] + 3y_3t^2}{-3x_0^2(1-t) + 3x_1\left[-2t(1-t) + (1-t)^2\right] + 3x_2\left[-t^2 + (2t)(1-t)\right] + 3x_3t^2}$$

At point P_0, $t = 0$, so the slope of the tangent is $\dfrac{-3y_0 + 3y_1}{-3x_0 + 3x_1} = \dfrac{y_1 - y_0}{x_1 - x_0}$. So the tangent to the curve at P_0 passes through P_1. Similarly, the slope of the tangent at point P_3 (where $t = 1$) is $\dfrac{-3y_2 + 3y_3}{-3x_2 + 3x_3} = \dfrac{y_3 - y_2}{x_3 - x_2}$, which is also the slope of line P_2P_3.

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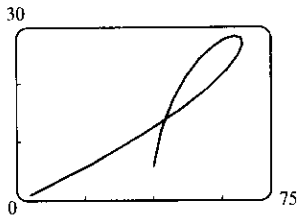
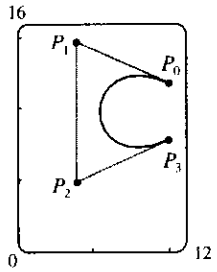
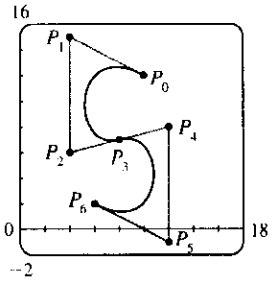
3. It seems that if P_1 were to the right of P_2, a loop would appear. We try setting $P_1 = (110, 30)$, and the resulting curve does indeed have a loop.



4. Based on the behavior of the Bézier curve in Problems 1–3, we suspect that the four control points should be in an exaggerated C shape. We try $P_0(10, 12)$, $P_1(4, 15)$, $P_2(4, 5)$, and $P_3(10, 8)$, and these produce a decent C. If you are using a CAS, it may be necessary to instruct it to make the x- and y-scales the same so as not to distort the figure (this is called a "constrained projection" in Maple.)



5. We use the same P_0 and P_1 as in Problem 4, and use part of our C as the top of an S. To prevent the center line from slanting up too much, we move P_2 up to $(4, 6)$ and P_3 down and to the left, to $(8, 7)$. In order to have a smooth joint between the top and bottom halves of the S (and a symmetric S), we determine points P_4, P_5, and P_6 by rotating points P_2, P_1, and P_0 about the center of the letter (point P_3). The points are therefore $P_4(12, 8)$, $P_5(12, -1)$, and $P_6(6, 2)$.



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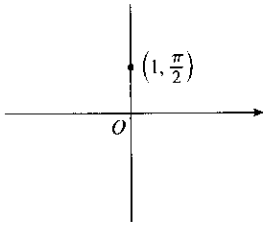
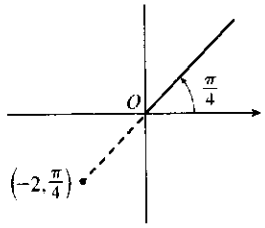
11.3 Polar Coordinates

1. (a) By adding 2π to $\frac{\pi}{2}$, we obtain the point $\left(1, \frac{5\pi}{2}\right)$. The direction opposite $\frac{\pi}{2}$ is $\frac{3\pi}{2}$, so $\left(-1, \frac{3\pi}{2}\right)$ is a point that satisfies the $r < 0$ requirement.

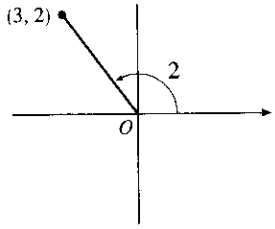


(b) $\left(-2, \frac{\pi}{4}\right)$



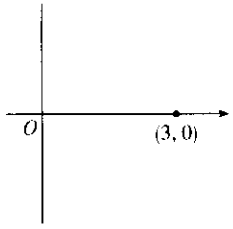
$\left(2, \frac{5\pi}{4}\right), \left(-2, \frac{9\pi}{4}\right)$

(c) $(3, 2)$



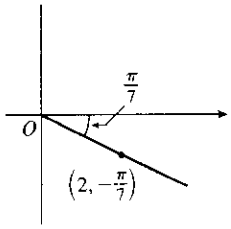
$(3, 2 + 2\pi), (-3, 2 + \pi)$

2. (a) $(3, 0)$



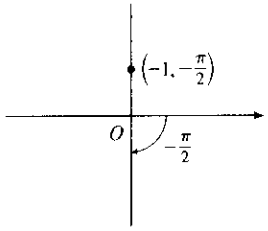
$(3, 2\pi), (-3, \pi)$

(b) $\left(2, -\frac{\pi}{7}\right)$



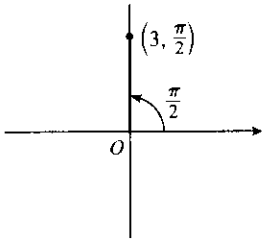
$\left(2, \frac{13\pi}{7}\right), \left(-2, \frac{6\pi}{7}\right)$

(c) $\left(-1, -\frac{\pi}{2}\right)$



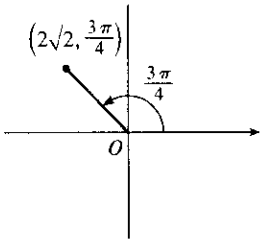
$\left(1, \frac{\pi}{2}\right), \left(-1, \frac{3\pi}{2}\right)$

3. (a)



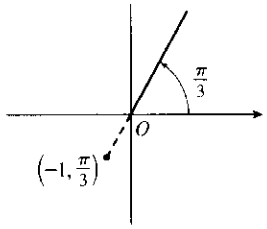
$x = 3\cos\frac{\pi}{2} = 3(0) = 0$ and $y = 3\sin\frac{\pi}{2} = 3(1) = 3$ give us the Cartesian coordinates $(0, 3)$.

(b)



$x = 2\sqrt{2}\cos\frac{3\pi}{4}$
$= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -2$ and
$y = 2\sqrt{2}\sin\frac{3\pi}{4} = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2$
give us $(-2, 2)$.

(c)



$x = -1\cos\frac{\pi}{3} = -\frac{1}{2}$ and $y = -1\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$ give us $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

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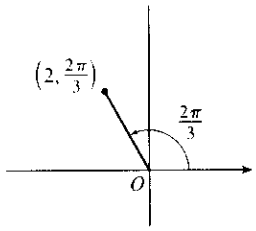
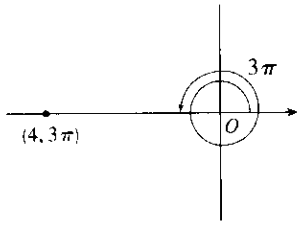
4. (a)



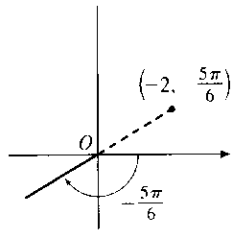
$x = 2\cos\frac{2\pi}{3} = -1$ and $y = 2\sin\frac{2\pi}{3} = \sqrt{3}$ give us $(-1, \sqrt{3})$.

(b)



$x = 4\cos 3\pi = -4$ and $y = 4\sin 3\pi = 0$ give us $(-4, 0)$.

(c)



$x = -2\cos\left(-\frac{5\pi}{6}\right) = \sqrt{3}$ and $y = -2\sin\left(-\frac{5\pi}{6}\right) = 1$ give us $(\sqrt{3}, 1)$.

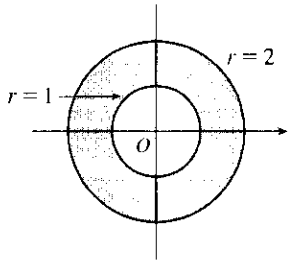
5. (a) $x = 1$ and $y = 1 \quad \Rightarrow \quad r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$. Since $(1, 1)$ is in the first quadrant, the polar coordinates are (i) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ and (ii) $\left(-\sqrt{2}, \frac{5\pi}{4}\right)$.

(b) $x = 2\sqrt{3}$ and $y = -2 \quad \Rightarrow \quad r = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$ and $\theta = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$. Since $(2\sqrt{3}, -2)$ is in the fourth quadrant and $0 \le \theta \le 2\pi$, the polar coordinates are (i) $\left(4, \frac{11\pi}{6}\right)$ and (ii) $\left(-4, \frac{5\pi}{6}\right)$.

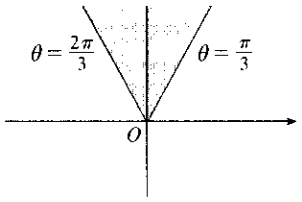
6. (a) $(x, y) = \left(-1, -\sqrt{3}\right)$, $r = \sqrt{1 + 3} = 2$, $\tan\theta = y/x = \sqrt{3}$ and (x, y) is in the third quadrant, so $\theta = \frac{4\pi}{3}$. The polar coordinates are (i) $\left(2, \frac{4\pi}{3}\right)$ and (ii) $\left(-2, \frac{\pi}{3}\right)$.

(b) $(x, y) = (-2, 3)$, $r = \sqrt{4 + 9} = \sqrt{13}$, $\tan\theta = y/x = -\frac{3}{2}$ and (x, y) is in the second quadrant, so $\theta = \tan^{-1}\left(-\frac{3}{2}\right) + \pi$. The polar coordinates are (i) $\left(\sqrt{13}, \theta\right)$ and (ii) $\left(-\sqrt{13}, \theta + \pi\right)$.

7. The curves $r = 1$ and $r = 2$ represent circles with center O and radii 1 and 2. The region in the plane satisfying $1 \le r \le 2$ consists of both circles and the shaded region between them in the figure.



8. $r \ge 0, \quad \pi/3 \le \theta \le 2\pi/3$



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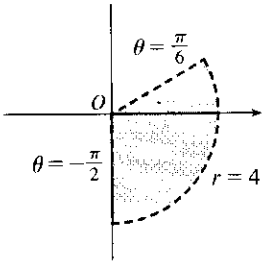
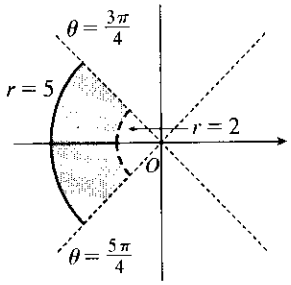
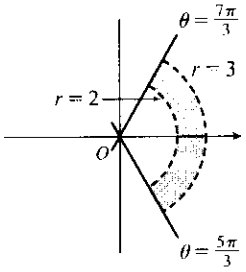
9. The region satisfying $0 \le r < 4$ and $-\pi/2 \le \theta < \pi/6$ does not include the circle $r = 4$ nor the line $\theta = \frac{\pi}{6}$.



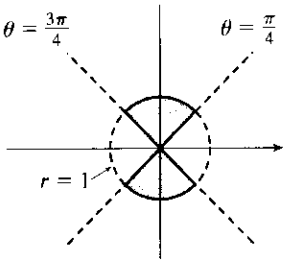
10. $2 < r \le 5, \quad 3\pi/4 < \theta < 5\pi/4$



11. $2 < r < 3, \quad \frac{5\pi}{3} \le \theta \le \frac{7\pi}{3}$



12. $-1 \le r \le 1, \quad \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$



13. $(r, \theta) = \left(1, \frac{\pi}{6}\right) \quad \Rightarrow \quad x = 1\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $y = 1\sin\frac{\pi}{6} = \frac{1}{2}$.
$(r, \theta) = \left(3, \frac{3\pi}{4}\right) \quad \Rightarrow \quad x = 3\cos\frac{3\pi}{4} = -\frac{3\sqrt{2}}{2}$ and $y = 3\sin\frac{3\pi}{4} = \frac{3\sqrt{2}}{2}$. The distance between them is

$$\sqrt{\left[\frac{\sqrt{3}}{2} - \left(-\frac{3\sqrt{2}}{2}\right)\right]^2 + \left(\frac{1}{2} - \frac{3\sqrt{2}}{2}\right)^2} = \sqrt{\tfrac{1}{4}\left(\sqrt{3} + 3\sqrt{2}\right)^2 + \tfrac{1}{4}\left(1 - 3\sqrt{2}\right)^2}$$

$$= \sqrt{\tfrac{1}{4}}\sqrt{\left(3 + 6\sqrt{6} + 18\right) + \left(1 - 6\sqrt{2} + 18\right)} = \tfrac{1}{2}\sqrt{40 + 6\sqrt{6} - 6\sqrt{2}}$$

14. The points (r_1, θ_1) and (r_2, θ_2) in Cartesian coordinates are $(r_1\cos\theta_1, r_1\sin\theta_1)$ and $(r_2\cos\theta_2, r_2\sin\theta_2)$, respectively. The *square* of the distance between them is

$$\begin{aligned}
&(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2 \\
&\quad = \left(r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1\right) + \left(r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1\right) \\
&\quad = r_1^2\left(\sin^2\theta_1 + \cos^2\theta_1\right) + r_2^2\left(\sin^2\theta_2 + \cos^2\theta_2\right) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\
&\quad = r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2,
\end{aligned}$$

so the distance between them is $\sqrt{r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2}$.

15. $r = 2 \quad \Leftrightarrow \quad \sqrt{x^2 + y^2} = 2 \quad \Leftrightarrow \quad x^2 + y^2 = 4$, a circle of radius 2 centered at the origin.

16. $r\cos\theta = 1 \quad \Leftrightarrow \quad x = 1$, a vertical line.

17. $r = 3\sin\theta \quad \Rightarrow \quad r^2 = 3r\sin\theta \quad \Leftrightarrow \quad x^2 + y^2 = 3y \quad \Leftrightarrow \quad x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$, a circle of radius $\frac{3}{2}$ centered at $\left(0, \frac{3}{2}\right)$. The first two equations are actually equivalent since $r^2 = 3r\sin\theta \quad \Rightarrow \quad r(r - 3\sin\theta) = 0 \quad \Rightarrow \quad r = 0$ or $r = 3\sin\theta$. But $r = 3\sin\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the single equation $r = 3\sin\theta$ is equivalent to the compound condition ($r = 0$ or $r = 3\sin\theta$).

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18. $r = 2\sin\theta + 2\cos\theta \;\Rightarrow\; r^2 = 2r\sin\theta + 2r\cos\theta \;\Leftrightarrow\; x^2 + y^2 = 2y + 2x \;\Leftrightarrow\; (x^2 - 2x + 1) + (y^2 - 2y + 1) = 2 \;\Leftrightarrow\; (x-1)^2 + (y-1)^2 = 2$. The first implication is reversible since $r^2 = 2r\sin\theta + 2r\cos\theta \;\Rightarrow\; r = 0$ or $r = 2\sin\theta + 2\cos\theta$, but the curve $r = 2\sin\theta + 2\cos\theta$ passes through the pole ($r = 0$) when $\theta = -\frac{\pi}{4}$, so $r = 2\sin\theta + 2\cos\theta$ includes the single point of $r = 0$. The curve is a circle of radius $\sqrt{2}$, centered at $(1, 1)$.

19. $r = \csc\theta \;\Leftrightarrow\; r = \dfrac{1}{\sin\theta} \;\Leftrightarrow\; r\sin\theta = 1 \;\Leftrightarrow\; y = 1$, a horizontal line 1 unit above the x-axis.

20. $r = \tan\theta\sec\theta = \dfrac{\sin\theta}{\cos^2\theta} \;\Rightarrow\; r\cos^2\theta = \sin\theta \;\Leftrightarrow\; (r\cos\theta)^2 = r\sin\theta \;\Leftrightarrow\; x^2 = y$, a parabola with vertex at the origin opening upward. The first implication is reversible since $\cos\theta = 0$ would imply $\sin\theta = r\cos^2\theta = 0$, contradicting the fact that $\cos^2\theta + \sin^2\theta = 1$.

21. $x = 3 \;\Leftrightarrow\; r\cos\theta = 3 \;\Leftrightarrow\; r = 3/\cos\theta \;\Leftrightarrow\; r = 3\sec\theta$.

22. $x^2 + y^2 = 9 \;\Leftrightarrow\; r^2 = 9 \;\Leftrightarrow\; r = 3$. [$r = -3$ gives the same curve.]

23. $x = -y^2 \;\Leftrightarrow\; r\cos\theta = -r^2\sin^2\theta \;\Leftrightarrow\; \cos\theta = -r\sin^2\theta \;\Leftrightarrow\; r = -\dfrac{\cos\theta}{\sin^2\theta} = -\cot\theta\csc\theta$.

24. $x + y = 9 \;\Leftrightarrow\; r\cos\theta + r\sin\theta = 9 \;\Leftrightarrow\; r = 9/(\cos\theta + \sin\theta)$.

25. $x^2 + y^2 = 2cx \;\Leftrightarrow\; r^2 = 2cr\cos\theta \;\Leftrightarrow\; r^2 - 2cr\cos\theta = 0 \;\Leftrightarrow\; r(r - 2c\cos\theta) = 0 \;\Leftrightarrow\; r = 0$ or $r = 2c\cos\theta$. $r = 0$ is included in $r = 2c\cos\theta$ when $\theta = \frac{\pi}{2} + n\pi$, so the curve is represented by the single equation $r = 2c\cos\theta$.

26. $x^2 - y^2 = 1 \;\Leftrightarrow\; (r\cos\theta)^2 - (r\sin\theta)^2 = 1 \;\Leftrightarrow\; r^2(\cos^2\theta - \sin^2\theta) = 1 \;\Leftrightarrow\; r^2\cos 2\theta = 1 \;\Rightarrow\; r^2 = \sec 2\theta$

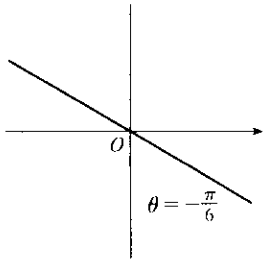
27. (a) The description leads immediately to the polar equation $\theta = \frac{\pi}{6}$, and the Cartesian equation $\tan\theta = y/x \;\Rightarrow\; y = (\tan\frac{\pi}{6})x = \frac{1}{\sqrt{3}}x$ is slightly more difficult to derive.

(b) The easier description here is the Cartesian equation $x = 3$.

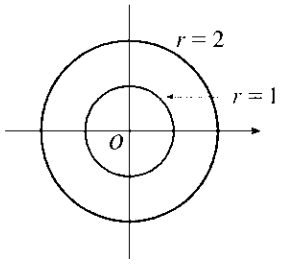
28. (a) Because its center is not at the origin, it is more easily described by its Cartesian equation, $(x-2)^2 + (y-3)^2 = 5^2$.

(b) This circle is more easily given in polar coordinates: $r = 4$. The Cartesian equation is also simple: $x^2 + y^2 = 16$.

29. $\theta = -\pi/6$



30. $r^2 - 3r + 2 = 0 \;\Leftrightarrow\; (r-1)(r-2) = 0 \;\Leftrightarrow\; r = 1$ or $r = 2$



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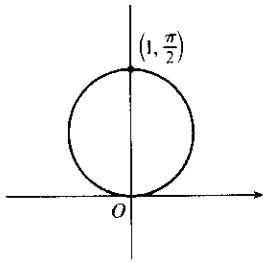
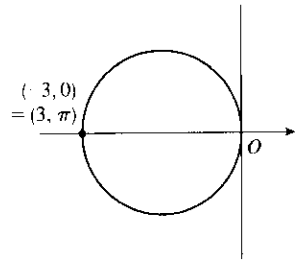
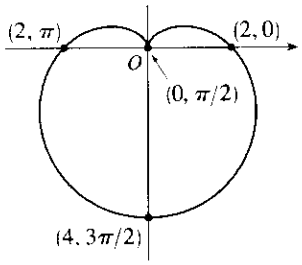
31. $r = \sin\theta \quad \Leftrightarrow \quad r^2 = r\sin\theta \quad \Leftrightarrow \quad x^2 + y^2 = y$
$\Leftrightarrow \quad x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$. The reasoning here is the same as in Exercise 17. This is a circle of radius $\frac{1}{2}$ centered at $\left(0, \frac{1}{2}\right)$.



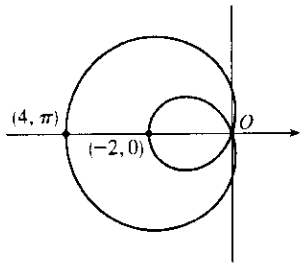
32. $r = -3\cos\theta \quad \Leftrightarrow \quad r^2 = -3r\cos\theta \quad \Leftrightarrow$
$x^2 + y^2 = -3x \quad \Leftrightarrow \quad \left(x + \frac{3}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$.
This curve is a circle of radius $\frac{3}{2}$ centered at $\left(-\frac{3}{2}, 0\right)$.



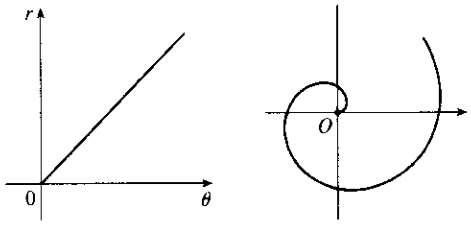
33. $r = 2(1 - \sin\theta)$. This curve is a cardioid.



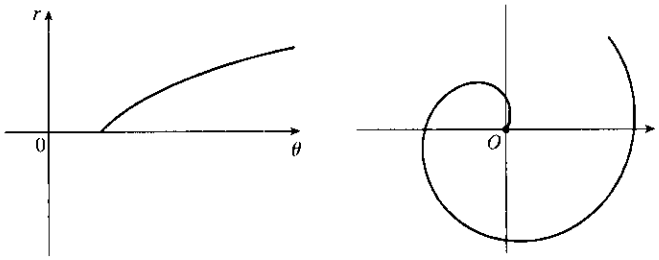
34. $r = 1 - 3\cos\theta$. This is a limaçon.



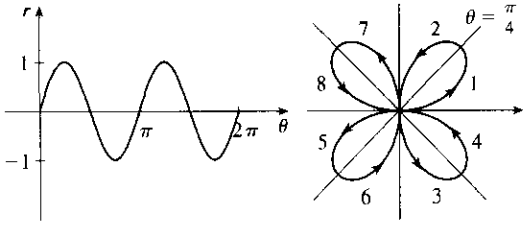
35. $r = \theta, \ \theta \geq 0$



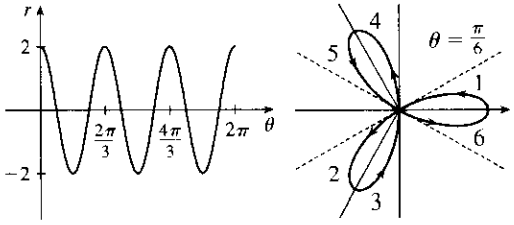
36. $r = \ln\theta, \ \theta \geq 1$



37. $r = \sin 2\theta$



38. $r = 2\cos 3\theta$



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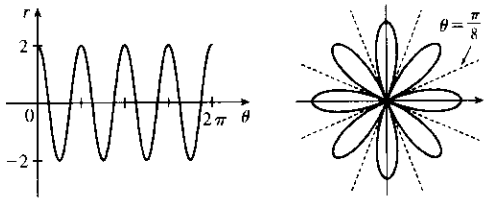
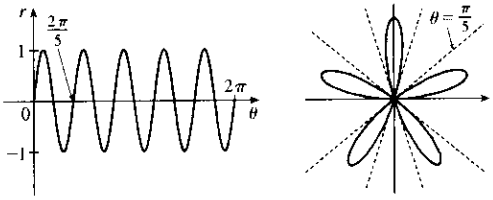
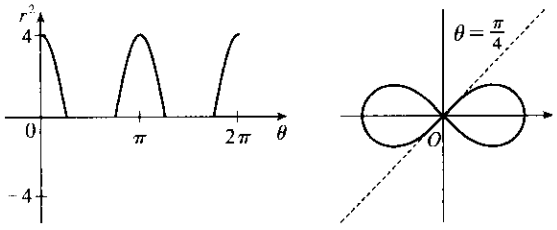
39. $r = 2\cos 4\theta$



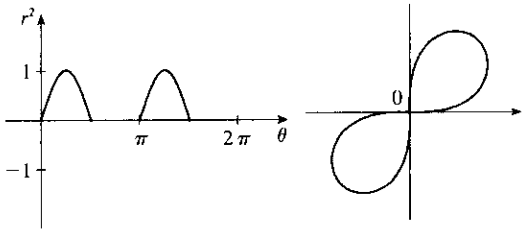
40. $r = \sin 5\theta$



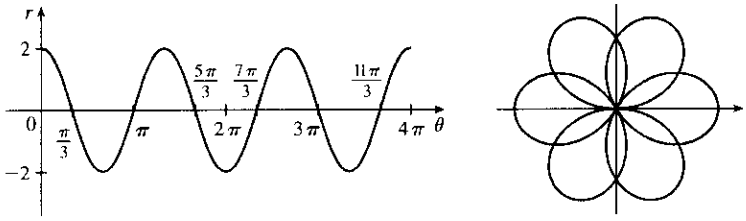
41. $r^2 = 4\cos 2\theta$



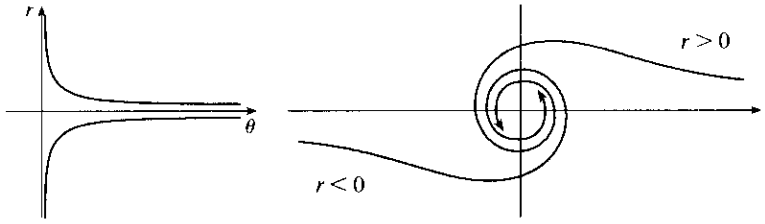
42. $r^2 = \sin 2\theta$



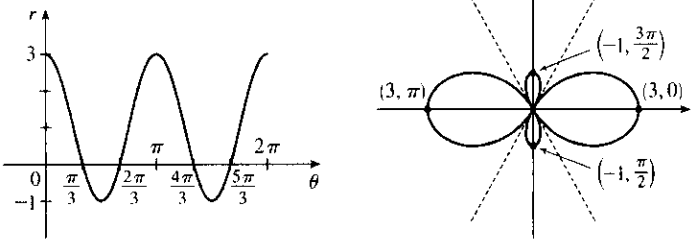
43. $r = 2\cos\left(\frac{3}{2}\theta\right)$



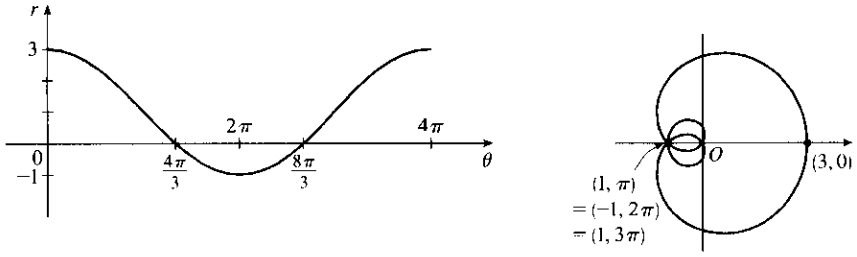
44. $r^2\theta = 1 \quad \Leftrightarrow \quad r = \pm 1/\sqrt{\theta}$ for $\theta > 0$



45. $r = 1 + 2\cos 2\theta$



46. $r = 1 + 2\cos(\theta/2)$



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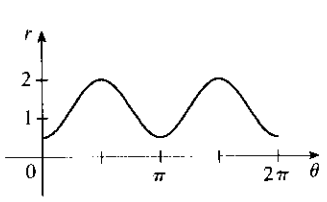
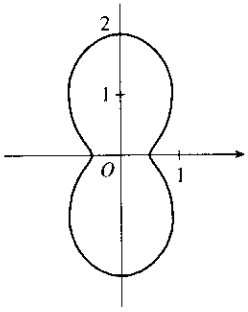
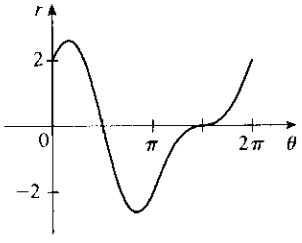
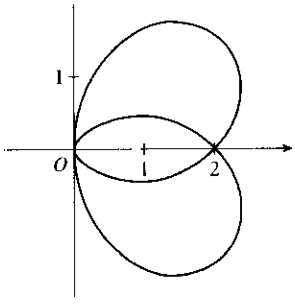
47. For $\theta = 0$, π, and 2π, r has its minimum value of about 0.5. For $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, r attains its maximum value of 2. We see that the graph has a similar shape for $0 \le \theta \le \pi$ and $\pi \le \theta \le 2\pi$.



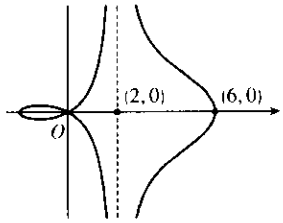


48.

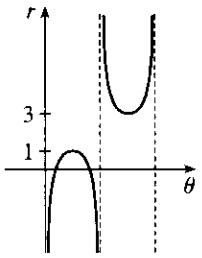


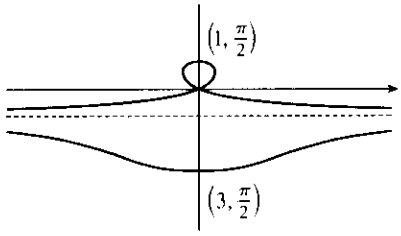


49. $x = r\cos\theta = (4 + 2\sec\theta)\cos\theta = 4\cos\theta + 2$. Now, $r \to \infty \Rightarrow$ $(4 + 2\sec\theta) \to \infty \Rightarrow \theta \to \left(\frac{\pi}{2}\right)^-$ or $\theta \to \left(\frac{3\pi}{2}\right)^+$ (since we need only consider $0 \le \theta < 2\pi$), so $\lim_{r\to\infty} x = \lim_{\theta\to\pi/2^-} (4\cos\theta + 2) = 2$. Also, $r \to -\infty$ $\Rightarrow$ $(4 + 2\sec\theta) \to -\infty \Rightarrow \theta \to \left(\frac{\pi}{2}\right)^+$ or $\theta \to \left(\frac{3\pi}{2}\right)^-$, so $\lim_{r\to-\infty} x = \lim_{\theta\to\pi/2^+} (4\cos\theta + 2) = 2$. Therefore, $\lim_{r\to\pm\infty} x = 2 \Rightarrow x = 2$ is a vertical asymptote.



50. $y = r\sin\theta = 2\sin\theta - \csc\theta\sin\theta = 2\sin\theta - 1$. $r \to \infty \Rightarrow (2 - \csc\theta) \to \infty \Rightarrow$ $\csc\theta \to -\infty \Rightarrow \theta \to \pi^+$ (since we need only consider $0 \le \theta < 2\pi$) and so $\lim_{r\to\infty} y = \lim_{\theta\to\pi^+} 2\sin\theta - 1 = -1$. Also $r \to -\infty \Rightarrow (2 - \csc\theta) \to -\infty \Rightarrow$ $\csc\theta \to \infty \Rightarrow$ $\theta \to \pi^-$ and so $\lim_{r\to-\infty} x = \lim_{\theta\to\pi^-} 2\sin\theta - 1 = -1$. Therefore $\lim_{r\to\pm\infty} y = -1 \Rightarrow y = -1$ is a horizontal asymptote.

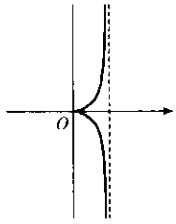




51. To show that $x = 1$ is an asymptote we must prove $\lim_{r\to\pm\infty} x = 1$.

$x = r\cos\theta = (\sin\theta\tan\theta)\cos\theta = \sin^2\theta$. Now, $r \to \infty \Rightarrow \sin\theta\tan\theta \to \infty$ $\Rightarrow \theta \to \left(\frac{\pi}{2}\right)^-$, so $\lim_{r\to\infty} x = \lim_{\theta\to\pi/2^-} \sin^2\theta = 1$. Also, $r \to -\infty \Rightarrow$ $\sin\theta\tan\theta \to -\infty \Rightarrow \theta \to \left(\frac{\pi}{2}\right)^+$, so $\lim_{r\to-\infty} x = \lim_{\theta\to\pi/2^+} \sin^2\theta = 1$.

Therefore, $\lim_{r\to\pm\infty} x = 1 \Rightarrow x = 1$ is a vertical asymptote. Also notice that $x = \sin^2\theta \ge 0$ for all θ, and



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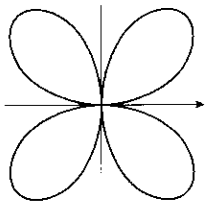
$x = \sin^2\theta \le 1$ for all θ. And $x \ne 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \le x < 1$.

52. The equation is $(x^2+y^2)^3 = 4x^2y^2$, but using polar coordinates we know that $x^2+y^2 = r^2$ and $x = r\cos\theta$ and $y = r\sin\theta$. Substituting into the given equation: $r^6 = 4r^2\cos^2\theta\, r^2\sin^2\theta \;\Rightarrow\; r^2 = 4\cos^2\theta\sin^2\theta \;\Rightarrow$ $r = \pm 2\cos\theta\sin\theta = \pm\sin 2\theta$. $r = \pm\sin 2\theta$ is sketched at right.



53. (a) We see that the curve crosses itself at the origin, where $r = 0$ (in fact the inner loop corresponds to negative r-values), so we solve the equation of the limaçon for $r = 0 \;\Leftrightarrow\; c\sin\theta = -1 \;\Leftrightarrow\; \sin\theta = -1/c$. Now if $|c| < 1$, then this equation has no solution and hence there is no inner loop. But if $c < -1$, then on the interval $(0, 2\pi)$ the equation has the two solutions $\theta = \sin^{-1}(-1/c)$ and $\theta = \pi - \sin^{-1}(-1/c)$, and if $c > 1$, the solutions are $\theta = \pi + \sin^{-1}(1/c)$ and $\theta = 2\pi - \sin^{-1}(1/c)$. In each case, $r < 0$ for θ between the two solutions, indicating a loop.

(b) For $0 < c < 1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta = \frac{3\pi}{2}$. So we determine for what c-values $\dfrac{d^2y}{d\theta^2}$ is negative at $\theta = \frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum: $y = r\sin\theta = \sin\theta + c\sin^2\theta \;\Rightarrow\; \dfrac{dy}{d\theta} = \cos\theta + 2c\sin\theta\cos\theta = \cos\theta + c\sin 2\theta \;\Rightarrow$ $\dfrac{d^2y}{d\theta^2} = -\sin\theta + 2c\cos 2\theta$. At $\theta = \frac{3\pi}{2}$, this is equal to $-(-1) + 2c(-1) = 1 - 2c$, which is negative only for $c > \frac{1}{2}$. A similar argument shows that for $-1 < c < 0$, y only has a local minimum at $\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.

54. (a) $r = \sin(\theta/2)$. This equation must correspond to one of II, III or VI, since these are the only graphs which are bounded. In fact it must be VI, since this is the only graph which is completed after a rotation of exactly 4π.

(b) $r = \sin(\theta/4)$. This equation must correspond to III, since this is the only graph which is completed after a rotation of exactly 8π.

(c) $r = \sec(3\theta)$. This must correspond to IV, since the graph is unbounded at $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{2\pi}{3}$, and so on.

(d) $r = \theta\sin\theta$. This must correspond to V. Note that $r = 0$ whenever θ is a multiple of π. This graph is unbounded, and each time θ moves through an interval of 2π, the same basic shape is repeated (because of the periodic $\sin\theta$ factor) but it gets larger each time (since θ increases each time we go around.)

(e) $r = 1 + 4\cos 5\theta$. This corresponds to II, since it is bounded, has fivefold rotational symmetry, and takes only one takes only one rotation through 2π to be complete.

(f) $r = 1/\sqrt{\theta}$. This corresponds to I, since it is unbounded at $\theta = 0$, and r decreases as θ increases; in fact $r \to 0$ as $\theta \to \infty$.

55. $r = 2\sin\theta \;\Rightarrow\; x = r\cos\theta = 2\sin\theta\cos\theta = \sin 2\theta,\ y = r\sin\theta = 2\sin^2\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cdot 2\sin\theta\cos\theta}{\cos 2\theta\cdot 2} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

When $\theta = \dfrac{\pi}{6}$, $\dfrac{dy}{dx} = \tan\left(2\cdot\dfrac{\pi}{6}\right) = \tan\dfrac{\pi}{3} = \sqrt{3}$.

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56. $r = 2 - \sin\theta \;\Rightarrow\; x = r\cos\theta = (2-\sin\theta)\cos\theta,\ y = r\sin\theta = (2-\sin\theta)\sin\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2-\sin\theta)\cos\theta + \sin\theta(-\cos\theta)}{(2-\sin\theta)(-\sin\theta) + \cos\theta(-\cos\theta)} = \frac{2\cos\theta - 2\sin\theta\cos\theta}{-2\sin\theta + \sin^2\theta - \cos^2\theta} = \frac{2\cos\theta - \sin 2\theta}{-2\sin\theta - \cos 2\theta}$$

$$\text{When } \theta = \frac{\pi}{3},\ \frac{dy}{dx} = \frac{2(1/2) - (\sqrt{3}/2)}{-2(\sqrt{3}/2) - (-1/2)} = \frac{1 - \sqrt{3}/2}{-\sqrt{3} + 1/2}\cdot\frac{2}{2} = \frac{2-\sqrt{3}}{1-2\sqrt{3}}.$$

57. $r = 1/\theta \;\Rightarrow\; x = r\cos\theta = (\cos\theta)/\theta,\ y = r\sin\theta = (\sin\theta)/\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-1/\theta^2) + (1/\theta)\cos\theta}{\cos\theta(-1/\theta^2) - (1/\theta)\sin\theta}\cdot\frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

$$\text{When } \theta = \pi,\ \frac{dy}{dx} = \frac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \frac{-\pi}{1} = -\pi.$$

58. $r = \ln\theta \;\Rightarrow\; x = r\cos\theta = \ln\theta\cos\theta,\ y = r\sin\theta = \ln\theta\,\sin\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(1/\theta) + \ln\theta\,\cos\theta}{\cos\theta(1/\theta) - \ln\theta\,\sin\theta}\cdot\frac{\theta}{\theta} = \frac{\sin\theta + \theta\ln\theta\cos\theta}{\cos\theta - \theta\ln\theta\sin\theta}$$

$$\text{When } \theta = e,\ \frac{dy}{dx} = \frac{\sin e + e\ln e\cos e}{\cos e - e\ln e\sin e} = \frac{\sin e + e\cos e}{\cos e - e\sin e}.$$

59. $r = 1 + \cos\theta \;\Rightarrow\; x = r\cos\theta = \cos\theta + \cos^2\theta,\ y = r\sin\theta = \sin\theta + \sin\theta\,\cos\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\,\sin\theta} = \frac{\cos\theta + \cos 2\theta}{-\sin\theta - \sin 2\theta}$$

$$\text{When } \theta = \frac{\pi}{6},\ \frac{dy}{dx} = \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}}{-\frac{1}{2} - \frac{\sqrt{3}}{2}} = \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}}{-\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)} = -1.$$

60. $r = \sin 3\theta \;\Rightarrow\; x = r\cos\theta = \sin 3\theta\cos\theta,\ y = r\sin\theta = \sin 3\theta\sin\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{3\cos 3\theta\cos\theta - \sin 3\theta\sin\theta}$$

$$\text{When } \theta = \frac{\pi}{6},\ \frac{dy}{dx} = \frac{3(0)(1/2) + 1(\sqrt{3}/2)}{3(0)(\sqrt{3}/2) - 1(1/2)} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}.$$

61. $r = 3\cos\theta \;\Rightarrow\; x = r\cos\theta = 3\cos\theta\,\cos\theta,\ y = r\sin\theta = 3\cos\theta\,\sin\theta \;\Rightarrow$
$dy/d\theta = -3\sin^2\theta + 3\cos^2\theta = 3\cos 2\theta = 0 \;\Rightarrow\; 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \;\Leftrightarrow\; \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. So the tangent is horizontal at $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)$ $\left[\text{same as } \left(\frac{3}{\sqrt{2}}, -\frac{\pi}{4}\right)\right]$. $dx/d\theta = -6\sin\theta\cos\theta = -3\sin 2\theta = 0 \;\Rightarrow$ $2\theta = 0$ or $\pi \;\Leftrightarrow\; \theta = 0$ or $\frac{\pi}{2}$. So the tangent is vertical at $(3,0)$ and $\left(0, \frac{\pi}{2}\right)$.

62. $y = r\sin\theta = \cos\theta\sin\theta + \sin^2\theta = \frac{1}{2}\sin 2\theta + \sin^2\theta \;\Rightarrow\; dy/d\theta = \cos 2\theta + \sin 2\theta = 0 \;\Rightarrow\; \tan 2\theta = -1$
$\Rightarrow\; 2\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4} \;\Leftrightarrow\; \theta = \frac{3\pi}{8}$ or $\frac{7\pi}{8} \;\Rightarrow$ horizontal tangents at $\left(\cos\frac{3\pi}{8} + \sin\frac{3\pi}{8}, \frac{3\pi}{8}\right)$ and $\left(\cos\frac{7\pi}{8} + \sin\frac{7\pi}{8}, \frac{7\pi}{8}\right)$. $x = r\cos\theta = \cos^2\theta + \cos\theta\sin\theta \;\Rightarrow\; dx/d\theta = -\sin 2\theta + \cos 2\theta = 0 \;\Rightarrow$ $\tan 2\theta = 1 \;\Rightarrow\; 2\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4} \;\Leftrightarrow\; \theta = \frac{\pi}{8}$ or $\frac{5\pi}{8} \;\Rightarrow$ vertical tangents at $\left(\cos\frac{\pi}{8} + \sin\frac{\pi}{8}, \frac{\pi}{8}\right)$ and $\left(\cos\frac{5\pi}{8} + \sin\frac{5\pi}{8}, \frac{5\pi}{8}\right)$.

Note: These expressions can be simplified using trigonometric identities. For example, $\cos\frac{\pi}{8} + \sin\frac{\pi}{8} = \frac{1}{2}\sqrt{4 + 2\sqrt{2}}$.

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63. $r = 1 + \cos\theta \;\Rightarrow\; x = r\cos\theta = \cos\theta(1+\cos\theta),\ y = r\sin\theta = \sin\theta(1+\cos\theta) \;\Rightarrow$
$dy/d\theta = (1+\cos\theta)\cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0 \;\Rightarrow\; \cos\theta = \frac{1}{2}$ or $-1 \;\Rightarrow\; \theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3} \;\Rightarrow$ horizontal tangent at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $(0,\pi)$ [the pole], and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.
$dx/d\theta = -(1+\cos\theta)\sin\theta - \cos\theta\sin\theta = -\sin\theta(1+2\cos\theta) = 0 \;\Rightarrow\; \sin\theta = 0$ or $\cos\theta = -\frac{1}{2} \;\Rightarrow$
$\theta = 0, \pi, \frac{2\pi}{3}$, or $\frac{4\pi}{3} \;\Rightarrow$ vertical tangent at $(2,0)$, $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$, and $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$. Note that the tangent is horizontal, not vertical when $\theta = \pi$, since $\lim_{\theta\to\pi} \dfrac{dy/d\theta}{dx/d\theta} = 0$.

64. $\dfrac{dy}{d\theta} = e^\theta\sin\theta + e^\theta\cos\theta = e^\theta(\sin\theta + \cos\theta) = 0 \;\Rightarrow\; \sin\theta = -\cos\theta \;\Rightarrow\; \tan\theta = -1 \;\Rightarrow$
$\theta = -\frac{1}{4}\pi + n\pi$ (n any integer) $\Rightarrow$ horizontal tangents at $\left(e^{\pi(n-1/4)}, \pi\left(n - \frac{1}{4}\right)\right)$.
$\dfrac{dx}{d\theta} = e^\theta\cos\theta - e^\theta\sin\theta = e^\theta(\cos\theta - \sin\theta) = 0 \;\Rightarrow\; \sin\theta = \cos\theta \;\Rightarrow\; \tan\theta = 1 \;\Rightarrow$
$\theta = \frac{1}{4}\pi + n\pi$ (n any integer) $\Rightarrow$ vertical tangents at $\left(e^{\pi(n+1/4)}, \pi\left(n + \frac{1}{4}\right)\right)$.

65. $r = \cos 2\theta \;\Rightarrow\; x = r\cos\theta = \cos 2\theta\cos\theta,\ y = r\sin\theta = \cos 2\theta\sin\theta \;\Rightarrow$

$$dy/d\theta = -2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta = -4\sin^2\theta\cos\theta + \left(\cos^3\theta - \sin^2\theta\cos\theta\right)$$
$$= \cos\theta\left(\cos^2\theta - 5\sin^2\theta\right) = \cos\theta\left(1 - 6\sin^2\theta\right) = 0 \;\Rightarrow$$

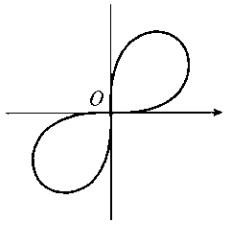
$\cos\theta = 0$ or $\sin\theta = \pm\frac{1}{\sqrt{6}} \;\Rightarrow\; \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \alpha, \pi-\alpha, \pi+\alpha$, or $2\pi - \alpha$ (where $\alpha = \sin^{-1}\frac{1}{\sqrt{6}}$).
So the tangent is horizontal at $\left(-1, \frac{\pi}{2}\right)$, $\left(-1, \frac{3\pi}{2}\right)$, $\left(\frac{2}{3}, \alpha\right)$, $\left(\frac{2}{3}, \pi-\alpha\right)$, $\left(\frac{2}{3}, \pi+\alpha\right)$, and $\left(\frac{2}{3}, 2\pi-\alpha\right)$.

$$dx/d\theta = -2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta = -4\sin\theta\cos^2\theta - (2\cos^2\theta - 1)\sin\theta$$
$$= \sin\theta(1 - 6\cos^2\theta) = 0 \;\Rightarrow$$

$\sin\theta = 0$ or $\cos\theta = \pm\frac{1}{\sqrt{6}} \;\Rightarrow\; \theta = 0, \pi, \beta, \pi-\beta, \pi+\beta$, or $2\pi - \beta$ (where $\beta = \cos^{-1}\frac{1}{\sqrt{6}}$).
So the tangent is vertical at $(1,0)$, $(1,\pi)$, $\left(-\frac{2}{3}, \beta\right)$, $\left(-\frac{2}{3}, \pi-\beta\right)$, $\left(-\frac{2}{3}, \pi+\beta\right)$, and $\left(-\frac{2}{3}, 2\pi-\beta\right)$.

66. By differentiating implicitly, $r^2 = \sin 2\theta \;\Rightarrow\; 2r\,(dr/d\theta) = 2\cos 2\theta \;\Rightarrow$
$dr/d\theta = (1/r)\cos 2\theta$, so

$$\frac{dy}{d\theta} = \frac{1}{r}\cos 2\theta\,\sin\theta + r\cos\theta = \frac{1}{r}\left(\cos 2\theta\,\sin\theta + r^2\cos\theta\right)$$
$$= \frac{1}{r}(\cos 2\theta\,\sin\theta + \sin 2\theta\,\cos\theta) = \frac{1}{r}\sin 3\theta$$



This is 0 when $\sin 3\theta = 0 \;\Rightarrow\; \theta = 0, \frac{\pi}{3}$ or $\frac{4\pi}{3}$ (restricting θ to the domain of the lemniscate), so there are horizontal tangents at $\left(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{3}\right)$, $\left(\sqrt[4]{\frac{3}{4}}, \frac{4\pi}{3}\right)$ and $(0,0)$. Similarly, $dx/d\theta = (1/r)\cos 3\theta = 0$ when $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$, so there are vertical tangents at $\left(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{6}\right)$ and $\left(\sqrt[4]{\frac{3}{4}}, \frac{7\pi}{6}\right)$ [and $(0,0)$].

67. $r = a\sin\theta + b\cos\theta \;\Rightarrow\; r^2 = ar\sin\theta + br\cos\theta \;\Rightarrow\; x^2 + y^2 = ay + bx \;\Rightarrow$
$x^2 - bx + \left(\frac{1}{2}b\right)^2 + y^2 - ay + \left(\frac{1}{2}a\right)^2 = \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}a\right)^2 \;\Rightarrow\; \left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}\left(a^2 + b^2\right)$, and this is a circle with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2+b^2}$.

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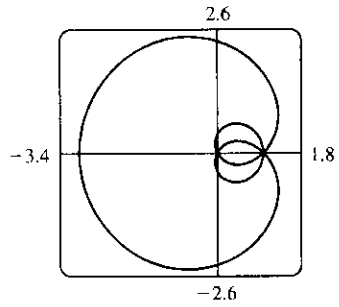
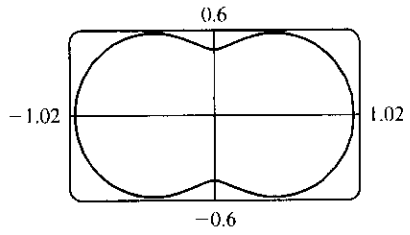
68. These curves are circles which intersect at the origin and at $\left(\frac{1}{\sqrt{2}}a, \frac{\pi}{4}\right)$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle ($r = a\sin\theta$), $dy/d\theta = a\cos\theta\sin\theta + a\sin\theta\cos\theta = a\sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $dx/d\theta = a\cos^2\theta - a\sin^2\theta = a\cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle ($r = a\cos\theta$), $dy/d\theta = a\cos 2\theta = 0$ and $dx/d\theta = -a\sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

Note for Exercises 69–74: Maple is able to plot polar curves using the `polarplot` command, or using the `coords=polar` option in a regular `plot` command. In Mathematica, use `PolarPlot`. In Derive, change to `Polar` under `Options State`. If your graphing device cannot plot polar equations, you must convert to parametric equations. For example, in Exercise 69, $x = r\cos\theta = [1 + 2\sin(\theta/2)]\cos\theta$, $y = r\sin\theta = [1 + 2\sin(\theta/2)]\sin\theta$.

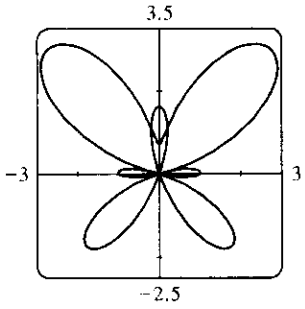
69. $r = 1 + 2\sin(\theta/2)$. The parameter interval is $[0, 4\pi]$.



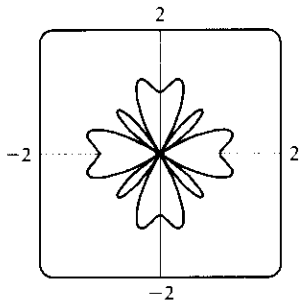
70. $r = \sqrt{1 - 0.8\sin^2\theta}$. The parameter interval is $[0, 2\pi]$.



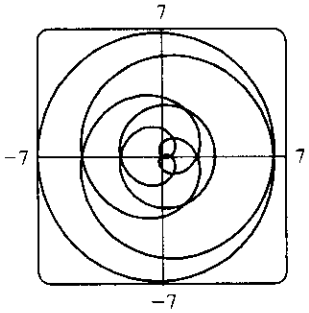
71. $r = e^{\sin\theta} - 2\cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



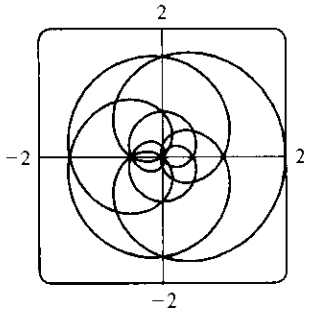
72. $r = \sin^2(4\theta) + \cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



73. $r = 2 - 5\sin(\theta/6)$. The parameter interval is $[-6\pi, 6\pi]$.



74. $r = \cos(\theta/2) + \cos(\theta/3)$. The parameter interval is $[-6\pi, 6\pi]$.



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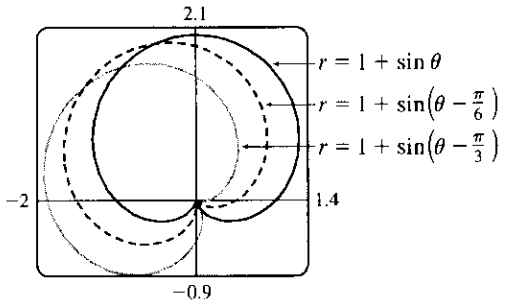
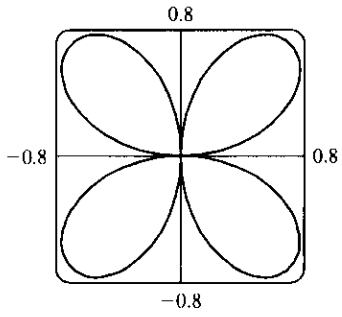
75.



It appears that the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{6}\right)$ is the same shape as the graph of $r = 1 + \sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r = f(\theta - \alpha)$ is the same shape as that of $r = f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point (r_0, θ_0) on the curve $r = f(\theta)$, the point $(r_0, \theta_0 + \alpha)$ is on the curve $r = f(\theta - \alpha)$, since $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$.

76.



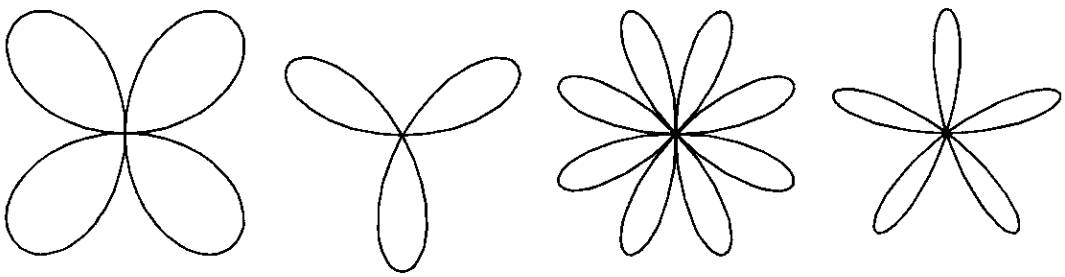
From the graph, the highest points seem to have $y \approx 0.77$. To find the exact value, we solve $dy/d\theta = 0$. $y = r\sin\theta = \sin\theta\sin 2\theta \quad \Rightarrow$

$$\begin{aligned} dy/d\theta &= 2\sin\theta\,\cos 2\theta + \cos\theta\,\sin 2\theta \\ &= 2\sin\theta\left(2\cos^2\theta - 1\right) + \cos\theta(2\sin\theta\,\cos\theta) \\ &= 2\sin\theta\left(3\cos^2\theta - 1\right) \end{aligned}$$

In the first quadrant, this is 0 when $\cos\theta = \frac{1}{\sqrt{3}} \quad \Leftrightarrow \quad \sin\theta = \sqrt{\frac{2}{3}} \quad \Leftrightarrow$ $y = 2\sin^2\theta\cos\theta = 2 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4\sqrt{3}}{9} \approx 0.77$.

77. (a) $r = \sin n\theta$. From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n.
This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

$$r(\theta + \pi) = \sin\left[n(\theta + \pi)\right] = \sin n\theta\,\cos n\pi + \cos n\theta\,\sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even} \\ -\sin n\theta & \text{if } n \text{ is odd} \end{cases}$$



$n = 2$ $n = 3$ $n = 4$ $n = 5$

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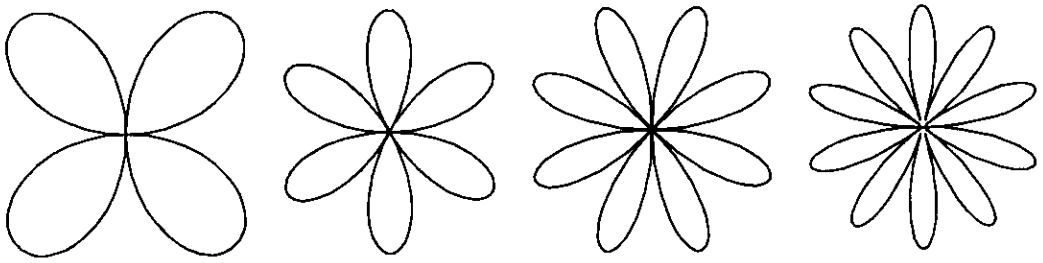
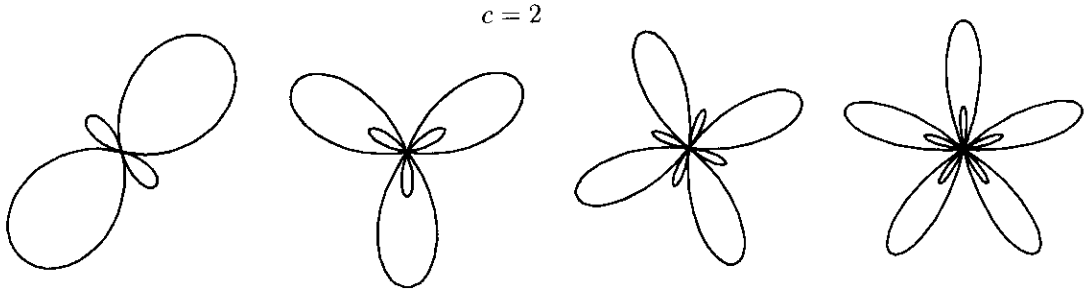
(b) The graph of $r = |\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r(\theta + \pi) = r(\theta)$.



$n = 2$ $n = 3$ $n = 4$ $n = 5$

78. $r = 1 + c \sin n\theta$. We vary n while keeping c constant at 2. As n changes, the curves change in the same way as those in Exercise 77: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, they are inside.

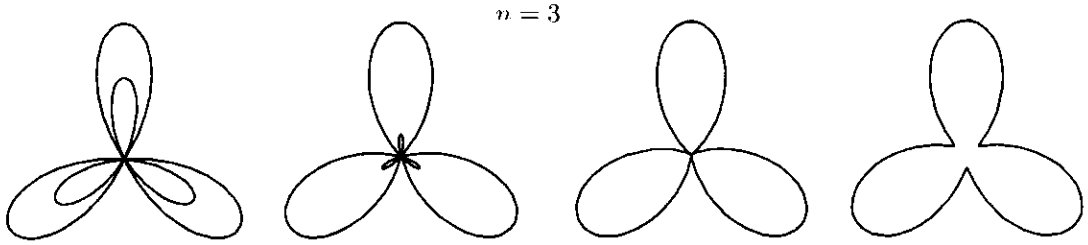
$c = 2$



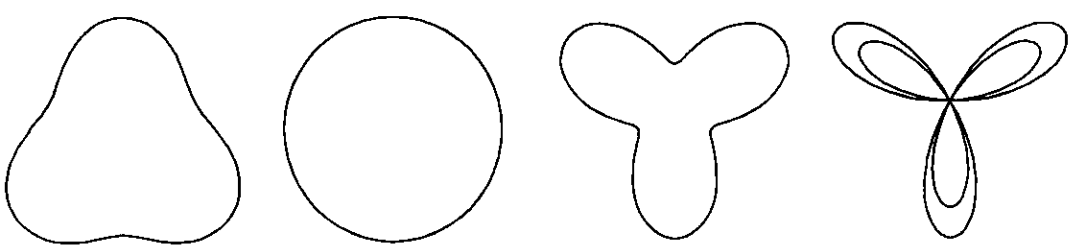
$n = 2$ $n = 3$ $n = 4$ $n = 5$

Now we vary c while keeping $n = 3$. As c increases toward 0, the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c = -1$, the small loops disappear entirely, and for $-1 < c < 1$, the graph is a simple, closed curve (at $c = 0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1 + (-c) \sin n\theta = 1 + c \sin n(\theta + \pi)$, so the graph for $c = c_0$ is the same as that for $c = -c_0$, with a rotation through π. As $c \to \infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2. Maple's `animate` command (or Mathematica's `Animate`) is very useful for seeing the changes that occur as c varies.

$n = 3$



$c = -4$ $c = -1.4$ $c = -1$ $c = -0.8$



$c = -0.2$ $c = 0$ $c = 0.5$ $c = 8$

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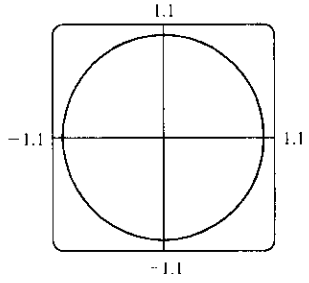
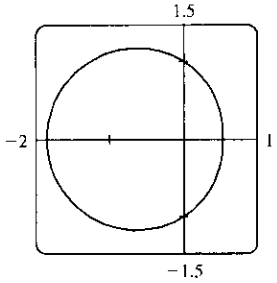
79. $r = \dfrac{1 - a\cos\theta}{1 + a\cos\theta}$. We start with $a = 0$, since in this case the curve is simply the circle $r = 1$.

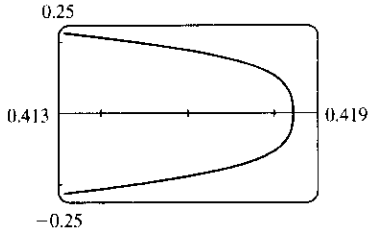
As a increases, the graph moves to the left, and its right side becomes flattened. As a increases through about 0.4, the right side seems to grow a dimple, which upon closer investigation (with narrower θ-ranges) seems to appear at $a \approx 0.42$ (the actual value is $\sqrt{2} - 1$). As $a \to 1$, this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at $a = 1$ the denominator vanishes at $\theta = \pi$, and the dimple becomes an actual cusp. For $a > 1$ we must choose our parameter interval carefully, since $r \to \infty$ as $1 + a\cos\theta \to 0 \quad \Leftrightarrow$ $\theta \to \pm\cos^{-1}(-1/a)$. As a increases from 1, the curve splits into two parts. The left part has a loop, which grows larger as a increases, and the right part grows broader vertically, and its left tip develops a dimple when $a \approx 2.42$ (actually, $\sqrt{2} + 1$). As a increases, the dimple grows more and more pronounced. If $a < 0$, we get the same graph as we do for the corresponding positive a-value, but with a rotation through π about the pole, as happened when c was replaced with $-c$ in Exercise 78.



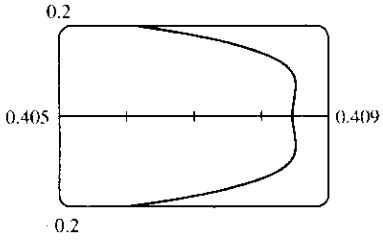
$a = 0$



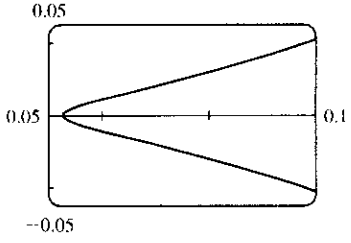
$a = 0.3$



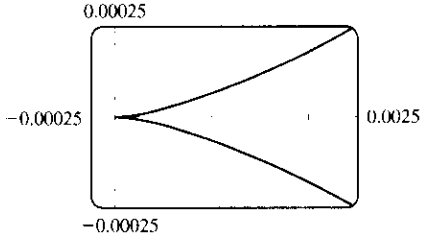
$a = 0.41, |\theta| \le 0.5$



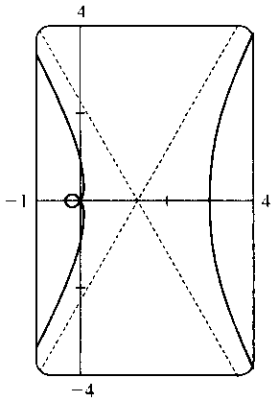
$a = 0.42, |\theta| \le 0.5$



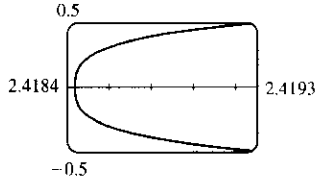
$a = 0.9, |\theta| \le 0.5$



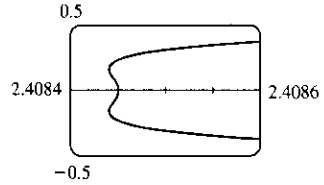
$a = 1, |\theta| \le 0.1$



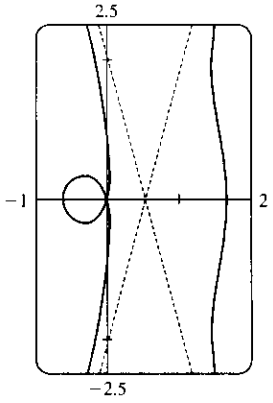
$a = 2$



$a = 2.41, |\theta - \pi| \le 0.2$



$a = 2.42, |\theta - \pi| \le 0.2$



$a = 4$

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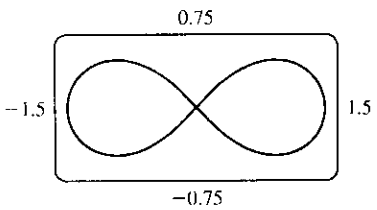
80. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for r in terms of θ, a, and c. We note that the given equation is a quadratic in r^2, so we use the quadratic formula and find that

$$r^2 = \frac{2c^2 \cos 2\theta \pm \sqrt{4c^4 \cos^2 2\theta - 4\left(c^4 - a^4\right)}}{2}$$

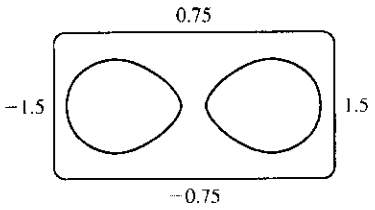
$$= c^2 \cos 2\theta \pm \sqrt{a^4 - c^4 \sin^2 2\theta}$$

so $r = \pm\sqrt{c^2 \cos 2\theta \pm \sqrt{a^4 - c^4 \sin^2 2\theta}}$. So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all four functions have period π.

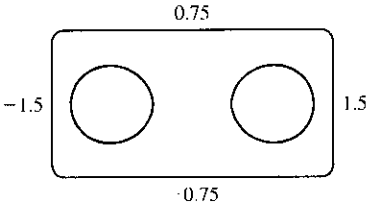
We start with the case $a = c = 1$, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decreases further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1, the two lobes of the curve join together, and as a increases further they continue to merge, until at $a \approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a \to \infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a, while the size of the graph varies as c and a jointly increase.



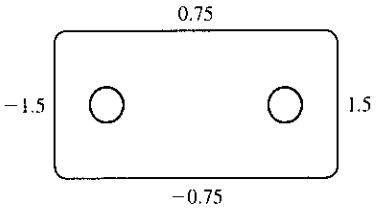
$(a, c) = (1, 1)$



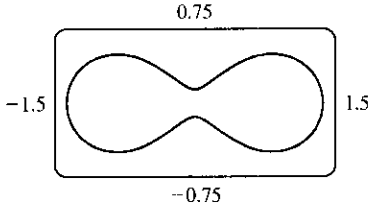
$(a, c) = (0.99, 1)$



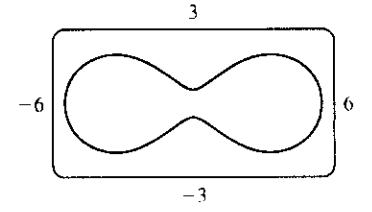
$(a, c) = (0.9, 1)$



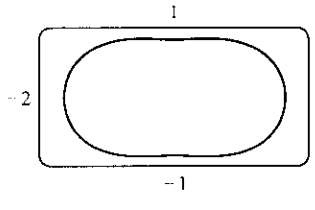
$(a, c) = (0.6, 1)$



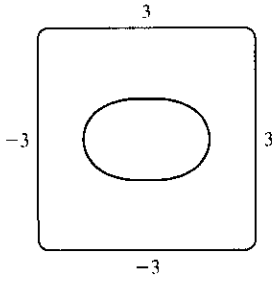
$(a, c) = (1.01, 1)$



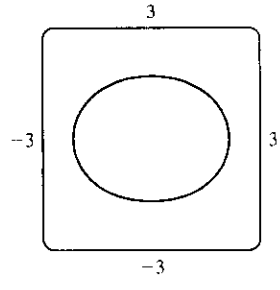
$(a, c) = (4.04, 4)$



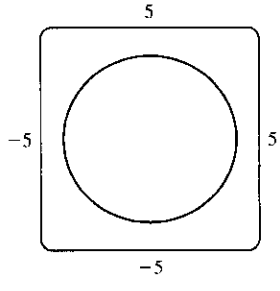
$(a, c) = (1.3, 1)$



$(a, c) = (1.5, 1)$



$(a, c) = (2, 1)$



$(a, c) = (4, 1)$

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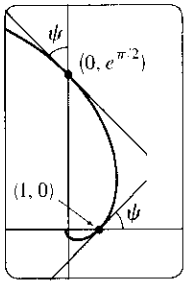
81.
$$\tan\psi = \tan(\phi-\theta) = \frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta} = \frac{\dfrac{dy}{dx}-\tan\theta}{1+\dfrac{dy}{dx}\tan\theta} = \frac{\dfrac{dy/d\theta}{dx/d\theta}-\tan\theta}{1+\dfrac{dy/d\theta}{dx/d\theta}\tan\theta}$$
$$= \frac{\dfrac{dy}{d\theta}-\dfrac{dx}{d\theta}\tan\theta}{\dfrac{dx}{d\theta}+\dfrac{dy}{d\theta}\tan\theta} = \frac{\left(\dfrac{dr}{d\theta}\sin\theta+r\cos\theta\right)-\tan\theta\left(\dfrac{dr}{d\theta}\cos\theta-r\sin\theta\right)}{\left(\dfrac{dr}{d\theta}\cos\theta-r\sin\theta\right)+\tan\theta\left(\dfrac{dr}{d\theta}\sin\theta+r\cos\theta\right)}$$
$$= \frac{r\cos\theta+r\cdot\dfrac{\sin^2\theta}{\cos\theta}}{\dfrac{dr}{d\theta}\cos\theta+\dfrac{dr}{d\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}} = \frac{r\cos^2\theta+r\sin^2\theta}{\dfrac{dr}{d\theta}\cos^2\theta+\dfrac{dr}{d\theta}\sin^2\theta} = \frac{r}{dr/d\theta}$$

82. (a) $r=e^\theta \Rightarrow dr/d\theta = e^\theta$, so by Exercise 81, $\tan\psi = r/e^\theta = 1 \Rightarrow \psi = \arctan 1 = \frac{\pi}{4}$.

(b) The Cartesian equation of the tangent line at $(1,0)$ is $y=x-1$, and that of the tangent line at $\left(0,e^{\pi/2}\right)$ is $y=e^{\pi/2}-x$.

(c) Let a be the tangent of the angle between the tangent and radial lines, that is, $a=\tan\psi$. Then, by Exercise 81,
$a=\dfrac{r}{dr/d\theta} \Rightarrow \dfrac{dr}{d\theta}=\dfrac{1}{a}r \Rightarrow r=Ce^{\theta/a}$ (by Theorem 10.4.2).



11.4 Areas and Lengths in Polar Coordinates

1. $r=\sqrt{\theta}$, $0\le\theta\le\frac{\pi}{4}$. $A=\int_0^{\pi/4}\frac{1}{2}r^2\,d\theta=\int_0^{\pi/4}\frac{1}{2}\left(\sqrt{\theta}\right)^2d\theta=\int_0^{\pi/4}\frac{1}{2}\theta\,d\theta=\left[\frac{1}{4}\theta^2\right]_0^{\pi/4}=\frac{1}{64}\pi^2$

2. $r=e^{\theta/2}$, $\pi\le\theta\le2\pi$. $A=\int_\pi^{2\pi}\frac{1}{2}\left(e^{\theta/2}\right)^2d\theta=\int_\pi^{2\pi}\frac{1}{2}e^\theta\,d\theta=\frac{1}{2}\left[e^\theta\right]_\pi^{2\pi}=\frac{1}{2}\left(e^{2\pi}-e^\pi\right)$

3. $r=\sin\theta$, $\frac{\pi}{3}\le\theta\le\frac{2\pi}{3}$.

$A=\int_{\pi/3}^{2\pi/3}\frac{1}{2}\sin^2\theta\,d\theta=\frac{1}{4}\int_{\pi/3}^{2\pi/3}(1-\cos2\theta)\,d\theta=\frac{1}{4}\left[\theta-\frac{1}{2}\sin2\theta\right]_{\pi/3}^{2\pi/3}$

$=\frac{1}{4}\left[\frac{2\pi}{3}-\frac{1}{2}\sin\frac{4\pi}{3}-\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}\right]=\frac{1}{4}\left[\frac{2\pi}{3}-\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)-\frac{\pi}{3}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right]=\frac{1}{4}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)=\frac{\pi}{12}+\frac{\sqrt{3}}{8}$

4. $r=\sqrt{\sin\theta}$, $0\le\theta\le\pi$. $A=\int_0^\pi\frac{1}{2}\left(\sqrt{\sin\theta}\right)^2d\theta=\int_0^\pi\frac{1}{2}\sin\theta\,d\theta=\left[-\frac{1}{2}\cos\theta\right]_0^\pi=\frac{1}{2}+\frac{1}{2}=1$

5. $r=\theta$, $0\le\theta\le\pi$. $A=\int_0^\pi\frac{1}{2}\theta^2d\theta=\left[\frac{1}{6}\theta^3\right]_0^\pi=\frac{1}{6}\pi^3$

6. $r=1+\sin\theta$, $\frac{\pi}{2}\le\theta\le\pi$.

$A=\int_{\pi/2}^{\pi}\frac{1}{2}(1+\sin\theta)^2\,d\theta=\frac{1}{2}\int_{\pi/2}^{\pi}\left(1+2\sin\theta+\sin^2\theta\right)d\theta=\frac{1}{2}\int_{\pi/2}^{\pi}\left[1+2\sin\theta+\frac{1}{2}(1-\cos2\theta)\right]d\theta$

$=\frac{1}{2}\left[\theta-2\cos\theta+\frac{1}{2}\theta-\frac{1}{4}\sin2\theta\right]_{\pi/2}^{\pi}=\frac{1}{2}\left[\pi+2+\frac{\pi}{2}-0-\left(\frac{\pi}{2}-0+\frac{\pi}{4}-0\right)\right]=\frac{1}{2}\left(\frac{3\pi}{4}+2\right)=\frac{3\pi}{8}+1$

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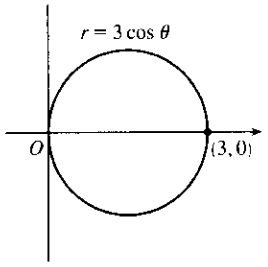
7. $r = 4 + 3\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}(4 + 3\sin\theta)^2 d\theta = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} \left(16 + 24\sin\theta + 9\sin^2\theta\right) d\theta \\ &= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} \left(16 + 9\sin^2\theta\right) d\theta \quad \text{[by Theorem 5.5.6(b)]} \\ &= \tfrac{1}{2}\cdot 2\int_0^{\pi/2} \left[16 + 9\cdot\tfrac{1}{2}(1 - \cos 2\theta)\right] d\theta \quad \text{[by Theorem 5.5.6(a)]} \\ &= \int_0^{\pi/2} \left(\tfrac{41}{2} - \tfrac{9}{2}\cos 2\theta\right) d\theta = \left[\tfrac{41}{2}\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/2} = \left(\tfrac{41\pi}{4} - 0\right) - (0 - 0) = \tfrac{41\pi}{4} \end{aligned}$$

8. $r = \sin 4\theta,\ 0 \le \theta \le \frac{\pi}{4}$. $A = \int_0^{\pi/4} \frac{1}{2}\sin^2 4\theta\, d\theta = \int_0^{\pi/4} \frac{1}{4}(1 - \cos 8\theta)\, d\theta = \left[\frac{1}{4}\theta - \frac{1}{32}\sin 8\theta\right]_0^{\pi/4} = \frac{\pi}{16}$

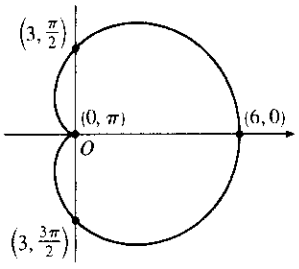
9. The area above the polar axis is bounded by $r = 3\cos\theta$ for $\theta = 0$ to $\theta = \pi/2$ (*not* π). By symmetry,



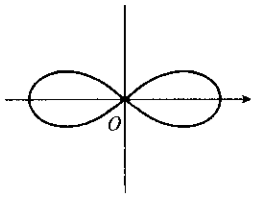
$$\begin{aligned} A &= 2\int_0^{\pi/2} \tfrac{1}{2}r^2\, d\theta = \int_0^{\pi/2} (3\cos\theta)^2 d\theta \\ &= 3^2\int_0^{\pi/2} \cos^2\theta\, d\theta = 9\int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta \\ &= \tfrac{9}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{9}{2}\left[\left(\tfrac{\pi}{2} + 0\right) - (0 + 0)\right] = \tfrac{9\pi}{4}. \end{aligned}$$

Also, note that this is a circle with radius $\frac{3}{2}$, so its area is $\pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$.

10. $$\begin{aligned} A &= \int_0^{2\pi} \tfrac{1}{2}r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}\left[3(1 + \cos\theta)\right]^2 d\theta \\ &= \tfrac{9}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta \\ &= \tfrac{9}{2}\int_0^{2\pi} \left[1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\ &= \tfrac{9}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{27}{2}\pi \end{aligned}$$

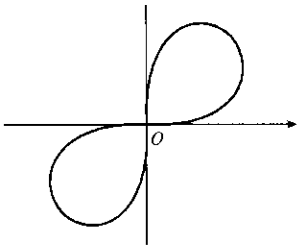


11. The curve $r^2 = 4\cos 2\theta$ goes through the pole when $\theta = \pi/4$, so we'll find the area for $0 \le \theta \le \pi/4$ and multiply it by 4.



$$\begin{aligned} A &= 4\int_0^{\pi/4} \tfrac{1}{2}r^2\, d\theta = 2\int_0^{\pi/4} (4\cos 2\theta)\, d\theta \\ &= 8\int_0^{\pi/4} \cos 2\theta\, d\theta = 4\left[\sin 2\theta\right]_0^{\pi/4} = 4(1 - 0) = 4 \end{aligned}$$

12. The curve $r^2 = \sin 2\theta$ goes through the pole when $\theta = \pi/2$, so we'll find the area for $0 \le \theta \le \pi/2$ and multiply it by 2.



$$\begin{aligned} A &= 2\int_0^{\pi/2} \tfrac{1}{2}r^2\, d\theta = \int_0^{\pi/2} \sin 2\theta\, d\theta = -\tfrac{1}{2}\left[\cos 2\theta\right]_0^{\pi/2} \\ &= -\tfrac{1}{2}(-1 - 1) = -\tfrac{1}{2}(-2) = 1 \end{aligned}$$

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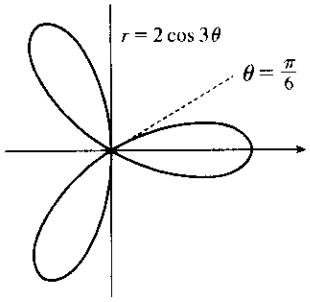
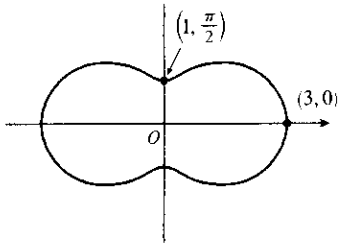
13. One-sixth of the area lies above the polar axis and is bounded by the curve $r = 2\cos 3\theta$ for $\theta = 0$ to $\theta = \pi/6$.



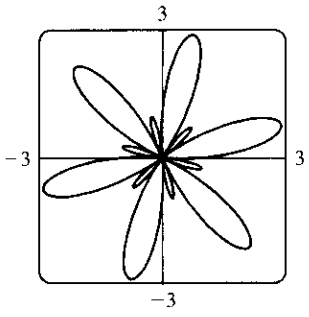
$$A = 6\int_0^{\pi/6} \tfrac{1}{2}(2\cos 3\theta)^2\,d\theta = 12\int_0^{\pi/6} \cos^2 3\theta\,d\theta$$
$$= \tfrac{12}{2}\int_0^{\pi/6}(1+\cos 6\theta)\,d\theta$$
$$= 6\left[\theta + \tfrac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 6\left(\tfrac{\pi}{6}\right) = \pi$$

14.



$$A = \int_0^{2\pi} \tfrac{1}{2}(2+\cos 2\theta)^2\,d\theta = \tfrac{1}{2}\int_0^{2\pi}(4+4\cos 2\theta+\cos^2 2\theta)\,d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi}\left(4+4\cos 2\theta+\tfrac{1}{2}+\tfrac{1}{2}\cos 4\theta\right)\,d\theta$$
$$= \tfrac{1}{2}\left[\tfrac{9}{2}\theta + 2\sin 2\theta + \tfrac{1}{8}\sin 4\theta\right]_0^{2\pi}$$
$$= \tfrac{1}{2}(9\pi) = \tfrac{9\pi}{2}$$

15.

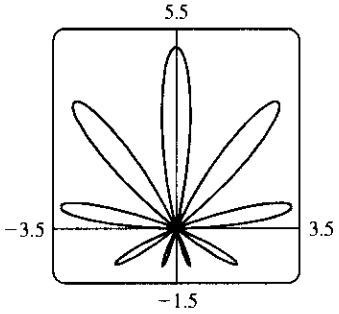


$$A = \int_0^{2\pi} \tfrac{1}{2}(1+2\sin 6\theta)^2\,d\theta = \tfrac{1}{2}\int_0^{2\pi}(1+4\sin 6\theta+4\sin^2 6\theta)\,d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi}\left[1+4\sin 6\theta+4\cdot\tfrac{1}{2}(1-\cos 12\theta)\right]\,d\theta$$
$$= \tfrac{1}{2}\int_0^{2\pi}(3+4\sin 6\theta-2\cos 12\theta)\,d\theta$$
$$= \tfrac{1}{2}\left[3\theta - \tfrac{2}{3}\cos 6\theta - \tfrac{1}{6}\sin 12\theta\right]_0^{2\pi}$$
$$= \tfrac{1}{2}\left[\left(6\pi - \tfrac{2}{3} - 0\right) - \left(0 - \tfrac{2}{3} - 0\right)\right] = 3\pi.$$

16.
$$A = \int_0^{\pi} \tfrac{1}{2}(2\sin\theta+3\sin 9\theta)^2\,d\theta = 2\int_0^{\pi/2} \tfrac{1}{2}(2\sin\theta+3\sin 9\theta)^2\,d\theta$$
$$= \int_0^{\pi/2}(4\sin^2\theta+12\sin\theta\sin 9\theta+9\sin^2 9\theta)\,d\theta$$
$$= \int_0^{\pi/2}\left[2(1-\cos 2\theta)+12\cdot\tfrac{1}{2}(\cos(\theta-9\theta)-\cos(\theta+9\theta))+\tfrac{9}{2}(1-\cos 18\theta)\right]\,d\theta$$

[integration by parts could be used for $\int \sin\theta\sin 9\theta\,d\theta$]

$$= \int_0^{\pi/2}(2-2\cos 2\theta+6\cos 8\theta-6\cos 10\theta+\tfrac{9}{2}-\tfrac{9}{2}\cos 18\theta)\,d\theta$$
$$= \left[\tfrac{13}{2}\theta - \sin 2\theta + \tfrac{3}{4}\sin 8\theta - \tfrac{3}{5}\sin 10\theta - \tfrac{1}{4}\sin 18\theta\right]_0^{\pi/2} = \tfrac{13}{4}\pi$$



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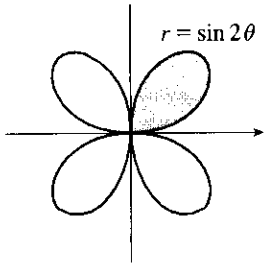
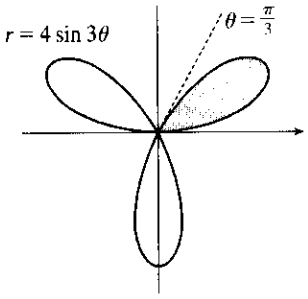
17. The shaded loop is traced out from $\theta = 0$ to $\theta = \pi/2$.

$$A = \int_0^{\pi/2} \tfrac{1}{2} r^2 \, d\theta = \tfrac{1}{2} \int_0^{\pi/2} \sin^2 2\theta \, d\theta$$
$$= \tfrac{1}{2} \int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 4\theta) \, d\theta$$
$$= \tfrac{1}{4} \left[\theta - \tfrac{1}{4} \sin 4\theta\right]_0^{\pi/2} = \tfrac{1}{4}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{8}$$



18. $A = \int_0^{\pi/3} \tfrac{1}{2} (4 \sin 3\theta)^2 \, d\theta = 8 \int_0^{\pi/3} \sin^2 3\theta \, d\theta$

$$= 4 \int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta$$
$$= 4\left[\theta - \tfrac{1}{6} \sin 6\theta\right]_0^{\pi/3} = \tfrac{4\pi}{3}$$

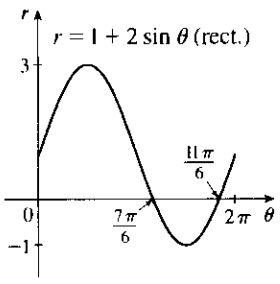


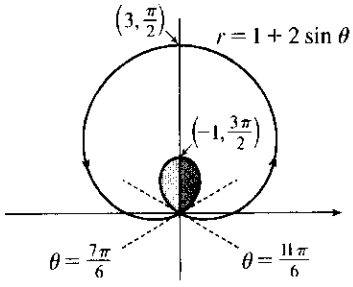
19. $r = 0 \;\Rightarrow\; 3\cos 5\theta = 0 \;\Rightarrow\; 5\theta = \tfrac{\pi}{2} \;\Rightarrow\; \theta = \tfrac{\pi}{10}$.

$$A = \int_{-\pi/10}^{\pi/10} \tfrac{1}{2}(3\cos 5\theta)^2 \, d\theta = \int_0^{\pi/10} 9\cos^2 5\theta \, d\theta = \tfrac{9}{2} \int_0^{\pi/10} (1 + \cos 10\theta) \, d\theta = \tfrac{9}{2}\left[\theta + \tfrac{1}{10} \sin 10\theta\right]_0^{\pi/10} = \tfrac{9\pi}{20}$$

20. $A = 2\int_0^{\pi/8} \tfrac{1}{2}(2\cos 4\theta)^2 \, d\theta = 2\int_0^{\pi/8} (1 + \cos 8\theta) \, d\theta = 2\left[\theta + \tfrac{1}{8} \sin 8\theta\right]_0^{\pi/8} = \tfrac{\pi}{4}$

21.





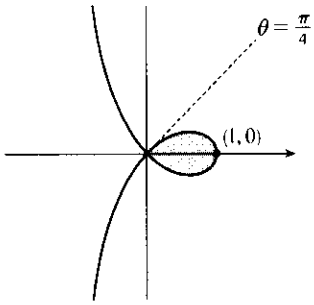
This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

$$A = 2\int_{7\pi/6}^{3\pi/2} \tfrac{1}{2}(1 + 2\sin\theta)^2 \, d\theta = \int_{7\pi/6}^{3\pi/2} (1 + 4\sin\theta + 4\sin^2\theta) \, d\theta$$
$$= \int_{7\pi/6}^{3\pi/2} \left[1 + 4\sin\theta + 4 \cdot \tfrac{1}{2}(1 - \cos 2\theta)\right] d\theta = \left[\theta - 4\cos\theta + 2\theta - \sin 2\theta\right]_{7\pi/6}^{3\pi/2}$$
$$= \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2}$$

22. To determine when the strophoid $r = 2\cos\theta - \sec\theta$ passes through the pole, we solve $r = 0 \;\Rightarrow\; 2\cos\theta - \dfrac{1}{\cos\theta} = 0 \;\Rightarrow$

$2\cos^2\theta - 1 = 0 \;\Rightarrow\; \cos^2\theta = \dfrac{1}{2} \;\Rightarrow\; \cos\theta = \pm\dfrac{1}{\sqrt{2}} \;\Rightarrow$

$\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ for $0 \le \theta \le \pi$ with $\theta \ne \frac{\pi}{2}$.



$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(2\cos\theta - \sec\theta)^2 \, d\theta = \int_0^{\pi/4} (4\cos^2\theta - 4 + \sec^2\theta) \, d\theta$$
$$= \int_0^{\pi/4} \left[4 \cdot \tfrac{1}{2}(1 + \cos 2\theta) - 4 + \sec^2\theta\right] d\theta = \int_0^{\pi/4} (-2 + 2\cos 2\theta + \sec^2\theta) \, d\theta$$
$$= \left[-2\theta + \sin 2\theta + \tan\theta\right]_0^{\pi/4} = \left(-\tfrac{\pi}{2} + 1 + 1\right) - 0 = 2 - \tfrac{\pi}{2}$$

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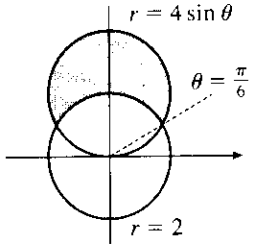
23. $4\sin\theta = 2 \;\Leftrightarrow\; \sin\theta = \frac{1}{2} \;\Leftrightarrow\; \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ (for $0 \le \theta \le 2\pi$). We'll subtract the unshaded area from the shaded area for $\pi/6 \le \theta \le \pi/2$ and double that value.



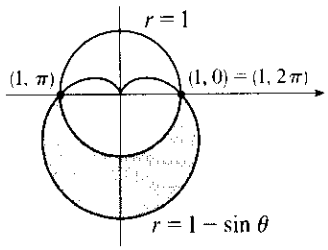
$$A = 2\int_{\pi/6}^{\pi/2} \tfrac{1}{2}(4\sin\theta)^2\,d\theta - 2\int_{\pi/6}^{\pi/2} \tfrac{1}{2}(2)^2\,d\theta = 2\int_{\pi/6}^{\pi/2} \tfrac{1}{2}\left[(4\sin\theta)^2 - 2^2\right]d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left(16\sin^2\theta - 4\right)d\theta = \int_{\pi/6}^{\pi/2} [8(1-\cos 2\theta) - 4]\,d\theta$$

$$= \int_{\pi/6}^{\pi/2} (4 - 8\cos 2\theta)\,d\theta = [4\theta - 4\sin 2\theta]_{\pi/6}^{\pi/2}$$

$$= (2\pi - 0) - \left(\tfrac{2\pi}{3} - 4\cdot\tfrac{\sqrt{3}}{2}\right) = \tfrac{4}{3}\pi + 2\sqrt{3}$$

24. $1 - \sin\theta = 1 \;\Rightarrow\; \sin\theta = 0 \;\Rightarrow\; \theta = 0$ or $\pi \;\Rightarrow$

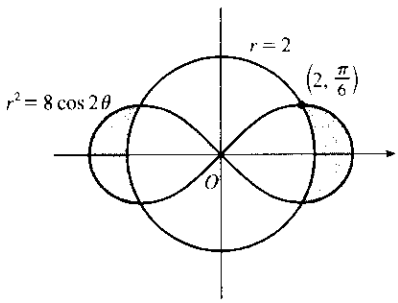


$$A = \int_{\pi}^{2\pi} \tfrac{1}{2}\left[(1-\sin\theta)^2 - 1\right]d\theta = \tfrac{1}{2}\int_{\pi}^{2\pi} \left(\sin^2\theta - 2\sin\theta\right)d\theta$$

$$= \tfrac{1}{4}\int_{\pi}^{2\pi} (1 - \cos 2\theta - 4\sin\theta)\,d\theta = \tfrac{1}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta + 4\cos\theta\right]_{\pi}^{2\pi}$$

$$= \tfrac{1}{4}\pi + 2$$

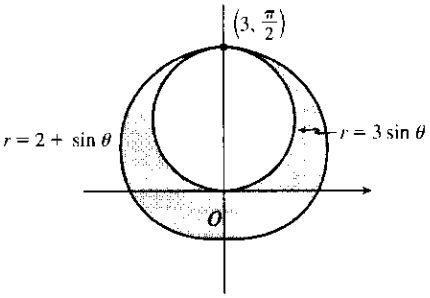
25. To find the area inside the leminiscate $r^2 = 8\cos 2\theta$ and outside the circle $r = 2$, we first note that the two curves intersect when $r^2 = 8\cos 2\theta$ and $r = 2$; i.e., when $\cos 2\theta = \frac{1}{2}$. For $-\pi < \theta \le \pi$, $\cos 2\theta = \frac{1}{2} \;\Leftrightarrow\; 2\theta = \pm\pi/3$ or $\pm 5\pi/3 \;\Leftrightarrow\; \theta = \pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,



$$A = 4\int_0^{\pi/6} \left[\tfrac{1}{2}(8\cos 2\theta) - \tfrac{1}{2}(2)^2\right]d\theta = 8\int_0^{\pi/6} (2\cos 2\theta - 1)\,d\theta$$

$$= 8\left[\sin 2\theta - \theta\right]_0^{\pi/6} = 8\left(\sqrt{3}/2 - \pi/6\right) = 4\sqrt{3} - 4\pi/3$$

26. To find the shaded area A, we'll find the area A_1 inside the curve $r = 2 + \sin\theta$ and subtract $\pi\left(\frac{3}{2}\right)^2$ since $r = 3\sin\theta$ is a circle with radius $\frac{3}{2}$.



$$A_1 = \int_0^{2\pi} \tfrac{1}{2}(2+\sin\theta)^2\,d\theta = \tfrac{1}{2}\int_0^{2\pi} (4 + 4\sin\theta + \sin^2\theta)\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \left[4 + 4\sin\theta + \tfrac{1}{2}\cdot(1-\cos 2\theta)\right]d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \left(\tfrac{9}{2} + 4\sin\theta - \tfrac{1}{2}\cos 2\theta\right)d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{9}{2}\theta - 4\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{1}{2}\left[(9\pi - 4) - (-4)\right] = \tfrac{9\pi}{2}$$

So $A = A_1 - \frac{9\pi}{4} = \frac{9\pi}{2} - \frac{9\pi}{4} = \frac{9\pi}{4}$.

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47. The parabola opens upward with vertex $(0, 4)$ [midway between the focus $(0, 6)$ and the directrix $y = 2$] and $p = 2$, so its equation is $(x - 0)^2 = 4 \cdot 2(y - 4) \quad \Leftrightarrow \quad x^2 = 8(y - 4)$.

48. Center is $(0, 0)$, and $c = 5$, $a = 2 \Rightarrow b = \sqrt{21}$; foci on y-axis $\quad \Rightarrow \quad$ equation of the hyperbola is $\dfrac{y^2}{4} - \dfrac{x^2}{21} = 1$.

49. The hyperbola has center $(0, 0)$ and foci on the x-axis. $c = 3$ and $b/a = \frac{1}{2}$ (from the asymptotes)

$\Rightarrow \quad 9 = c^2 = a^2 + b^2 = (2b)^2 + b^2 = 5b^2 \quad \Rightarrow \quad b = \frac{3}{\sqrt{5}} \quad \Rightarrow \quad a = \frac{6}{\sqrt{5}} \quad \Rightarrow$ an equation of the hyperbola is

$\dfrac{x^2}{36/5} - \dfrac{y^2}{9/5} = 1 \quad \Leftrightarrow \quad 5x^2 - 20y^2 = 36.$

50. Center is $(3, 0)$, and $a = \frac{8}{2} = 4$, $c = 2 \quad \Leftrightarrow \quad b = \sqrt{4^2 - 2^2} = \sqrt{12} \quad \Rightarrow$ an equation of the ellipse is

$\dfrac{(x-3)^2}{12} + \dfrac{y^2}{16} = 1.$

51. $x^2 = -(y - 100)$ has its vertex at $(0, 100)$, so one of the vertices of the ellipse is $(0, 100)$. Another form of the equation of a parabola is $x^2 = 4p(y - 100)$ so $4p(y - 100) = -(y - 100) \quad \Rightarrow \quad 4p = -1 \quad \Rightarrow \quad p = -\frac{1}{4}$. Therefore the shared focus is found at $\left(0, \frac{399}{4}\right)$ so $2c = \frac{399}{4} - 0 \quad \Rightarrow \quad c = \frac{399}{8}$ and the center of the ellipse is $\left(0, \frac{399}{8}\right)$. So $a = 100 - \frac{399}{8} = \frac{401}{8}$ and $b^2 = a^2 - c^2 = \frac{401^2 - 399^2}{8^2} = 25$. So the equation of the ellipse is

$\dfrac{x^2}{b^2} + \dfrac{\left(y - \frac{399}{8}\right)^2}{a^2} = 1 \quad \Rightarrow \quad \dfrac{x^2}{25} + \dfrac{\left(y - \frac{399}{8}\right)^2}{\left(\frac{401}{8}\right)^2} = 1$ or $\dfrac{x^2}{25} + \dfrac{(8y - 399)^2}{160{,}801} = 1.$

52. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\dfrac{x}{y}$. Therefore $\dfrac{dy}{dx} = m \quad \Leftrightarrow \quad y = -\dfrac{b^2}{a^2}\dfrac{x}{m}$.

Combining this condition with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we find that $x = \pm\dfrac{a^2 m}{\sqrt{a^2m^2 + b^2}}$. In other words, the two points on the ellipse where the tangent has slope m are $\left(\pm\dfrac{a^2m}{\sqrt{a^2m^2 + b^2}}, \mp\dfrac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$. The tangent lines at these points have the equations $y \pm \dfrac{b^2}{\sqrt{a^2m^2 + b^2}} = m\left(x \mp \dfrac{a^2m}{\sqrt{a^2m^2 + b^2}}\right)$ or

$y = mx \mp \dfrac{a^2m^2}{\sqrt{a^2m^2 + b^2}} \mp \dfrac{b^2}{\sqrt{a^2m^2 + b^2}} = mx \mp \sqrt{a^2m^2 + b^2}.$

53. Directrix $x = 4 \quad \Rightarrow \quad d = 4$, so $e = \frac{1}{3} \quad \Rightarrow \quad r = \dfrac{ed}{1 + e\cos\theta} = \dfrac{4}{3 + \cos\theta}$.

54. See the end of the proof of Theorem 11.6.1. If $e > 1$, then $1 - e^2 < 0$ and Equations 11.6.4 become $a^2 = \dfrac{e^2d^2}{(e^2 - 1)^2}$ and $b^2 = \dfrac{e^2d^2}{e^2 - 1}$, so $\dfrac{b^2}{a^2} = e^2 - 1$. The asymptotes $y = \pm\dfrac{b}{a}x$ have slopes $\pm\dfrac{b}{a} = \pm\sqrt{e^2 - 1}$, so the angles they make with the polar axis are $\pm\tan^{-1}\left[\sqrt{e^2 - 1}\right] = \cos^{-1}(\pm 1/e)$.

55. In polar coordinates, an equation for the circle is $r = 2a\sin\theta$. Thus, the coordinates of Q are $x = r\cos\theta = 2a\sin\theta\cos\theta$ and $y = r\sin\theta = 2a\sin^2\theta$. The coordinates of R are $x = 2a\cot\theta$ and $y = 2a$. Since P is the midpoint of QR, we use the midpoint formula to get $x = a(\sin\theta\cos\theta + \cot\theta)$ and $y = a(1 + \sin^2\theta)$.

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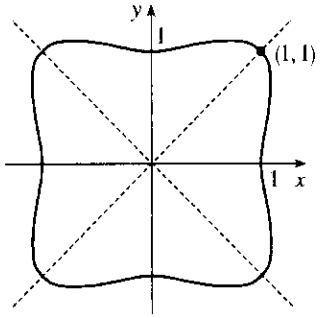
PROBLEMS PLUS

1. $x = \int_1^t \frac{\cos u}{u}\,du$, $y = \int_1^t \frac{\sin u}{u}\,du$, so by FTC1, we have $\frac{dx}{dt} = \frac{\cos t}{t}$ and $\frac{dy}{dt} = \frac{\sin t}{t}$. Vertical tangent lines occur when $\frac{dx}{dt} = 0 \quad \Leftrightarrow \quad \cos t = 0$. The parameter value corresponding to $(x, y) = (0, 0)$ is $t = 1$, so the nearest vertical tangent occurs when $t = \frac{\pi}{2}$. Therefore, the arc length between these points is

$$L = \int_1^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^{\pi/2} \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}}\,dt = \int_1^{\pi/2} \frac{dt}{t} = [\ln t]_1^{\pi/2} = \ln \tfrac{\pi}{2}$$

2. (a) The curve $x^4 + y^4 = x^2 + y^2$ is symmetric about both axes and about the line $y = x$ (since interchanging x and y does not change the equation) so we need only consider $y \geq x \geq 0$ to begin with. Implicit differentiation gives $4x^3 + 4y^3y' = 2x + 2yy' \quad \Rightarrow \quad y' = \frac{x\left(1 - 2x^2\right)}{y\left(2y^2 - 1\right)} \quad \Rightarrow \quad y' = 0$ when $x = 0$ and when $x = \pm\frac{1}{\sqrt{2}}$. If $x = 0$, then $y^4 = y^2 \quad \Rightarrow \quad y^2\left(y^2 - 1\right) = 0 \quad \Rightarrow \quad y = 0$ or ± 1. The point $(0, 0)$ can't be a highest or lowest point because it is isolated. [If $-1 < x < 1$ and $-1 < y < 1$, then $x^4 < x^2$ and $y^4 < y^2 \quad \Rightarrow$ $x^4 + y^4 < x^2 + y^2$, except for $(0, 0)$.] If $x = \frac{1}{\sqrt{2}}$, then $x^2 = \frac{1}{2}$, $x^4 = \frac{1}{4}$, so $\frac{1}{4} + y^4 = \frac{1}{2} + y^2 \quad \Rightarrow$ $4y^4 - 4y^2 - 1 = 0 \quad \Rightarrow \quad y^2 = \frac{4 \pm \sqrt{16+16}}{8} = \frac{1 \pm \sqrt{2}}{2}$. But $y^2 > 0$, so $y^2 = \frac{1+\sqrt{2}}{2} \quad \Rightarrow$ $y = \pm\sqrt{\frac{1}{2}\left(1 + \sqrt{2}\right)}$. Near the point $(0, 1)$, the denominator of y' is positive and the numerator changes from negative to positive as x increases through 0, so $(0, 1)$ is a local minimum point. At $\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$, y' changes from positive to negative, so that point gives a maximum. By symmetry, the highest points on the curve are $\left(\pm\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$ and the lowest points are $\left(\pm\frac{1}{\sqrt{2}}, -\sqrt{\frac{1+\sqrt{2}}{2}}\right)$.

(b) We use the information from part (a), together with symmetry with respect to the axes and the lines $y = \pm x$, to sketch the curve.



(c) In polar coordinates, $x^4 + y^4 = x^2 + y^2$ becomes $r^4\cos^4\theta + r^4\sin^4\theta = r^2$ or $r^2 = 1/\left(\cos^4\theta + \sin^4\theta\right)$. By the symmetry shown in part (b), the area enclosed by the curve is

$$A = 8\int_0^{\pi/4} \frac{1}{2}r^2\,d\theta = 4\int_0^{\pi/4} \frac{d\theta}{\cos^4\theta + \sin^4\theta} \overset{\text{CAS}}{=} \sqrt{2}\pi.$$

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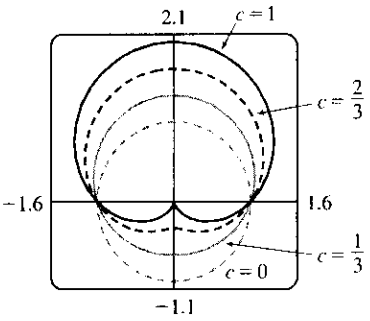
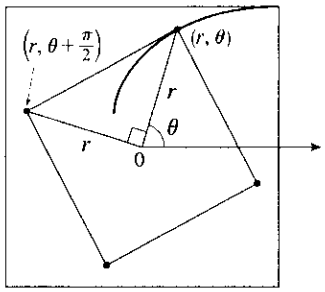
3. In terms of x and y, we have $x = r\cos\theta = (1 + c\sin\theta)\cos\theta = \cos\theta + c\sin\theta\cos\theta = \cos\theta + \frac{1}{2}c\sin 2\theta$ and $y = r\sin\theta = (1 + c\sin\theta)\sin\theta = \sin\theta + c\sin^2\theta$. Now $-1 \le \sin\theta \le 1 \;\Rightarrow$ $-1 \le \sin\theta + c\sin^2\theta \le 1 + c \le 2$, so $-1 \le y \le 2$. Furthermore, $y = 2$ when $c = 1$ and $\theta = \frac{\pi}{2}$, while $y = -1$ for $c = 0$ and $\theta = \frac{3\pi}{2}$. Therefore, we need a viewing rectangle with $-1 \le y \le 2$.

To find the x-values, look at the equation $x = \cos\theta + \frac{1}{2}c\sin 2\theta$ and use the fact that $\sin 2\theta \ge 0$ for $0 \le \theta \le \frac{\pi}{2}$ and $\sin 2\theta \le 0$ for $-\frac{\pi}{2} \le \theta \le 0$. [Because $r = 1 + c\sin\theta$ is symmetric about the y-axis, we only need to consider $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.] So for $-\frac{\pi}{2} \le \theta \le 0$, x has a maximum value when $c = 0$ and then $x = \cos\theta$ has a maximum value of 1 at $\theta = 0$. Thus, the maximum value of x must occur on $\left[0, \frac{\pi}{2}\right]$ with $c = 1$. Then $x = \cos\theta + \frac{1}{2}\sin 2\theta \;\Rightarrow$ $\frac{dx}{d\theta} = -\sin\theta + \cos 2\theta = -\sin\theta + 1 - 2\sin^2\theta \;\Rightarrow\; \frac{dx}{d\theta} = -(2\sin\theta - 1)(\sin\theta + 1) = 0$ when $\sin\theta = -1$ or $\frac{1}{2}$ (but $\sin\theta \ne -1$ for $0 \le \theta \le \frac{\pi}{2}$). If $\sin\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$ and $x = \cos\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3} = \frac{3}{4}\sqrt{3}$. Thus, the maximum value of x is $\frac{3}{4}\sqrt{3}$, and, by symmetry, the minimum value is $-\frac{3}{4}\sqrt{3}$. Therefore, the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c\sin\theta$, where $0 \le c \le 1$, is $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$.



4. (a) Let us find the polar equation of the path of the bug that starts in the upper right corner of the square. If the polar coordinates of this bug, at a particular moment, are (r, θ), then the polar coordinates of the bug that it is crawling toward must be $\left(r, \theta + \frac{\pi}{2}\right)$. (The next bug must be the same distance from the origin and the angle between the lines joining the bugs to the pole must be $\frac{\pi}{2}$.) The Cartesian coordinates of the first bug are $(r\cos\theta, r\sin\theta)$ and for the second bug we have $x = r\cos\left(\theta + \frac{\pi}{2}\right) = -r\sin\theta$, $y = r\sin\left(\theta + \frac{\pi}{2}\right) = r\cos\theta$. So the slope of the line joining the bugs is $\dfrac{r\cos\theta - r\sin\theta}{-r\sin\theta - r\cos\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$. This must be equal to the slope of the tangent line at (r, θ), so by Equation 11.3.3 we have $\dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$. Solving for $\dfrac{dr}{d\theta}$, we get



$\dfrac{dr}{d\theta}\sin^2\theta + \dfrac{dr}{d\theta}\sin\theta\cos\theta + r\sin\theta\cos\theta + r\cos^2\theta = \dfrac{dr}{d\theta}\sin\theta\cos\theta - \dfrac{dr}{d\theta}\cos^2\theta - r\sin^2\theta + r\sin\theta\cos\theta \Rightarrow$

$\dfrac{dr}{d\theta}(\sin^2\theta + \cos^2\theta) + r(\cos^2\theta + \sin^2\theta) = 0 \;\Rightarrow\; \dfrac{dr}{d\theta} = -r$. Solving this differential equation as a separable equation (as in Section 10.3), or using Theorem 10.4.2 with $k = -1$, we get $r = Ce^{-\theta}$. To determine C we use the fact that, at its starting position, $\theta = \frac{\pi}{4}$ and $r = \frac{1}{\sqrt{2}}a$, so $\frac{1}{\sqrt{2}}a = Ce^{-\pi/4} \;\Rightarrow\; C = \frac{1}{\sqrt{2}}ae^{\pi/4}$. Therefore, a polar equation of the bug's path is $r = \frac{1}{\sqrt{2}}ae^{\pi/4}e^{-\theta}$ or $r = \frac{1}{\sqrt{2}}ae^{(\pi/4)-\theta}$.

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(b) The distance traveled by this bug is $L = \int_{\pi/4}^{\infty} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$, where $\dfrac{dr}{d\theta} = \dfrac{a}{\sqrt{2}} e^{\pi/4}\left(-e^{-\theta}\right)$ and so

$$r^2 + (dr/d\theta)^2 = \tfrac{1}{2}a^2 e^{\pi/2} e^{-2\theta} + \tfrac{1}{2}a^2 e^{\pi/2} e^{-2\theta} = a^2 e^{\pi/2} e^{-2\theta}$$

Thus

$$L = \int_{\pi/4}^{\infty} ae^{\pi/4} e^{-\theta}\,d\theta = ae^{\pi/4} \lim_{t\to\infty} \int_{\pi/4}^{t} e^{-\theta}\,d\theta = ae^{\pi/4} \lim_{t\to\infty} \left[-e^{-\theta}\right]_{\pi/4}^{t}$$

$$= ae^{\pi/4} \lim_{t\to\infty} \left[e^{-\pi/4} - e^{-t}\right] = ae^{\pi/4} e^{-\pi/4} = a$$

5. (a) If (a, b) lies on the curve, then there is some parameter value t_1 such that $\dfrac{3t_1}{1+t_1^3} = a$ and $\dfrac{3t_1^2}{1+t_1^3} = b$. If $t_1 = 0$, the point is $(0, 0)$, which lies on the line $y = x$. If $t_1 \neq 0$, then the point corresponding to $t = \dfrac{1}{t_1}$ is given by $x = \dfrac{3(1/t_1)}{1+(1/t_1)^3} = \dfrac{3t_1^2}{t_1^3+1} = b$, $y = \dfrac{3(1/t_1)^2}{1+(1/t_1)^3} = \dfrac{3t_1}{t_1^3+1} = a$. So (b, a) also lies on the curve. [Another way to see this is to do part (e) first; the result is immediate.] The curve intersects the line $y = x$ when

$\dfrac{3t}{1+t^3} = \dfrac{3t^2}{1+t^3} \quad \Rightarrow \quad t = t^2 \quad \Rightarrow \quad t = 0$ or 1, so the points are $(0, 0)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$.

(b) $\dfrac{dy}{dt} = \dfrac{(1+t^3)(6t) - 3t^2(3t^2)}{(1+t^3)^2} = \dfrac{6t - 3t^4}{(1+t^3)^2} = 0$ when $6t - 3t^4 = 3t(2 - t^3) = 0 \quad \Rightarrow \quad t = 0$ or $t = \sqrt[3]{2}$, so there are horizontal tangents at $(0, 0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using the symmetry from part (a), we see that there are vertical tangents at $(0, 0)$ and $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$.

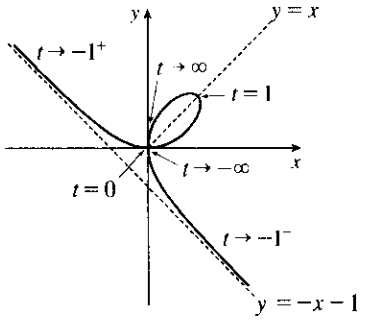
(c) Notice that as $t \to -1^+$, we have $x \to -\infty$ and $y \to \infty$. As $t \to -1^-$, we have $x \to \infty$ and $y \to -\infty$. Also

$y - (-x - 1) = y + x + 1 = \dfrac{3t + 3t^2 + (1+t^3)}{1+t^3} = \dfrac{(t+1)^3}{1+t^3} = \dfrac{(t+1)^2}{t^2 - t + 1} \to 0$ as $t \to -1$. So $y = -x - 1$ is a slant asymptote.

(d) $\dfrac{dx}{dt} = \dfrac{(1+t^3)(3) - 3t(3t^2)}{(1+t^3)^2} = \dfrac{3 - 6t^3}{(1+t^3)^2}$ and from part (b) we have $\dfrac{dy}{dt} = \dfrac{6t - 3t^4}{(1+t^3)^2}$.

So $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t(2 - t^3)}{1 - 2t^3}$. Also

$\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \dfrac{2(1+t^3)^4}{3(1-2t^3)^3} > 0 \quad \Leftrightarrow \quad t < \frac{1}{\sqrt[3]{2}}$. So the curve is concave upward there and has a minimum point at $(0, 0)$ and a maximum point at $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using this together with the information from parts (a), (b), and (c), we sketch the curve.



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(e) $x^3+y^3=\left(\dfrac{3t}{1+t^3}\right)^3+\left(\dfrac{3t^2}{1+t^3}\right)^3=\dfrac{27t^3+27t^6}{(1+t^3)^3}=\dfrac{27t^3(1+t^3)}{(1+t^3)^3}=\dfrac{27t^3}{(1+t^3)^2}$ and

$3xy=3\left(\dfrac{3t}{1+t^3}\right)\left(\dfrac{3t^2}{1+t^3}\right)=\dfrac{27t^3}{(1+t^3)^2}$, so $x^3+y^3=3xy$.

(f) We start with the equation from part (e) and substitute $x=r\cos\theta$, $y=r\sin\theta$. Then $x^3+y^3=3xy \;\Rightarrow$ $r^3\cos^3\theta+r^3\sin^3\theta=3r^2\cos\theta\sin\theta$. For $r\neq0$, this gives $r=\dfrac{3\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}$. Dividing numerator and denominator by $\cos^3\theta$, we obtain $r=\dfrac{3\left(\dfrac{1}{\cos\theta}\right)\dfrac{\sin\theta}{\cos\theta}}{1+\dfrac{\sin^3\theta}{\cos^3\theta}}=\dfrac{3\sec\theta\tan\theta}{1+\tan^3\theta}$.

(g) The loop corresponds to $\theta\in\left(0,\frac{\pi}{2}\right)$, so its area is

$$A=\int_0^{\pi/2}\frac{r^2}{2}d\theta=\frac{1}{2}\int_0^{\pi/2}\left(\frac{3\sec\theta\tan\theta}{1+\tan^3\theta}\right)^2d\theta=\frac{9}{2}\int_0^{\pi/2}\frac{\sec^2\theta\tan^2\theta}{(1+\tan^3\theta)^2}d\theta$$

$$=\frac{9}{2}\int_0^{\infty}\frac{u^2du}{(1+u^3)^2}\ \ [\text{let } u=\tan\theta]\ =\lim_{b\to\infty}\tfrac{9}{2}\left[-\tfrac{1}{3}\left(1+u^3\right)^{-1}\right]_0^b=\tfrac{3}{2}$$

(h) By symmetry, the area between the folium and the line $y=-x-1$ is equal to the enclosed area in the third quadrant, plus twice the enclosed area in the fourth quadrant. The area in the third quadrant is $\frac{1}{2}$, and since $y=-x-1 \;\Rightarrow\; r\sin\theta=-r\cos\theta-1 \;\Rightarrow\; r=-\dfrac{1}{\sin\theta+\cos\theta}$, the area in the fourth quadrant is $\dfrac{1}{2}\displaystyle\int_{-\pi/2}^{-\pi/4}\left[\left(-\frac{1}{\sin\theta+\cos\theta}\right)^2-\left(\frac{3\sec\theta\tan\theta}{1+\tan^3\theta}\right)^2\right]d\theta\overset{\text{CAS}}{=}\frac{1}{2}$. Therefore, the total area is $\frac{1}{2}+2\left(\frac{1}{2}\right)=\frac{3}{2}$.

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Chapter 12

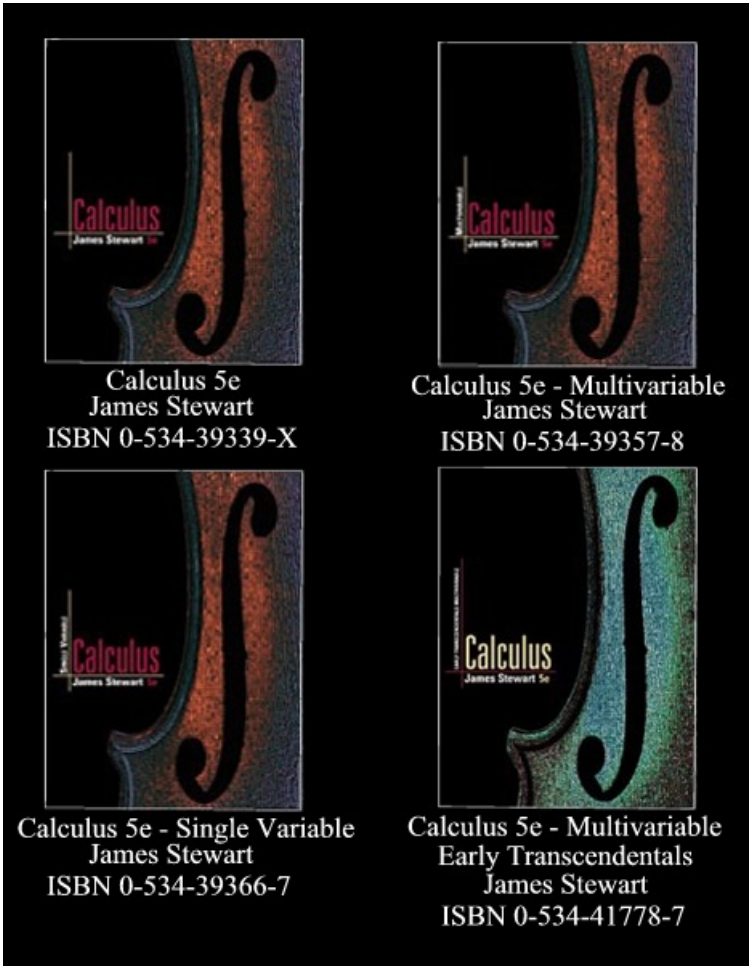
Adapted from the

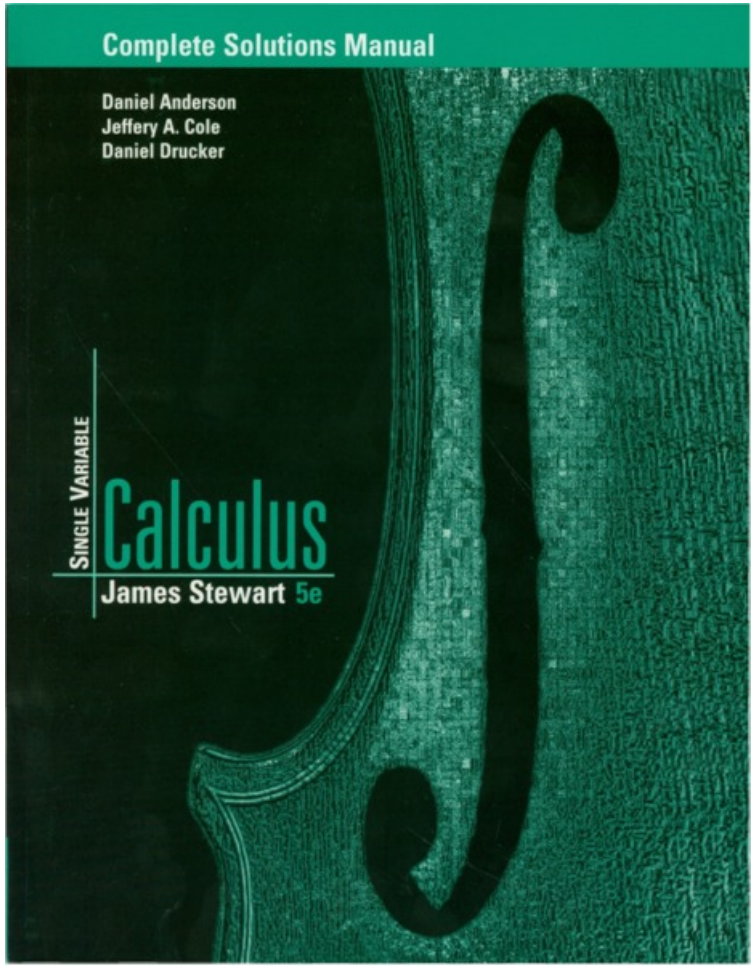
Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition





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12 □ INFINITE SEQUENCES AND SERIES

12.1 Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.

(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$

(b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. $a_n = 1-(0.2)^n$, so the sequence is $\{0.8, 0.96, 0.992, 0.9984, 0.99968, \dots\}$.

4. $a_n = \dfrac{n+1}{3n-1}$, so the sequence is $\left\{\dfrac{2}{2}, \dfrac{3}{5}, \dfrac{4}{8}, \dfrac{5}{11}, \dfrac{6}{14}, \dots\right\} = \left\{1, \dfrac{3}{5}, \dfrac{1}{2}, \dfrac{5}{11}, \dfrac{3}{7}, \dots\right\}$.

5. $a_n = \dfrac{3(-1)^n}{n!}$, so the sequence is $\left\{\dfrac{-3}{1}, \dfrac{3}{2}, \dfrac{-3}{6}, \dfrac{3}{24}, \dfrac{-3}{120}, \dots\right\} = \left\{-3, \dfrac{3}{2}, -\dfrac{1}{2}, \dfrac{1}{8}, -\dfrac{1}{40}, \dots\right\}$.

6. $a_n = 2\cdot 4\cdot 6\cdot \cdots \cdot (2n)$, so the sequence is
$\{2, 2\cdot 4, 2\cdot 4\cdot 6, 2\cdot 4\cdot 6\cdot 8, 2\cdot 4\cdot 6\cdot 8\cdot 10, \dots\} = \{2, 8, 48, 384, 3840, \dots\}$.

7. $a_1 = 3$, $a_{n+1} = 2a_n - 1$. Each term is defined in terms of the preceding term.
$a_2 = 2a_1 - 1 = 2(3) - 1 = 5$. $a_3 = 2a_2 - 1 = 2(5) - 1 = 9$. $a_4 = 2a_3 - 1 = 2(9) - 1 = 17$.
$a_5 = 2a_4 - 1 = 2(17) - 1 = 33$. The sequence is $\{3, 5, 9, 17, 33, \dots\}$.

8. $a_1 = 4$, $a_{n+1} = \dfrac{a_n}{a_n - 1}$. Each term is defined in terms of the preceding term.

$a_2 = \dfrac{a_1}{a_1 - 1} = \dfrac{4}{4-1} = \dfrac{4}{3}$. $a_3 = \dfrac{a_2}{a_2 - 1} = \dfrac{4/3}{\frac{4}{3} - 1} = \dfrac{4/3}{1/3} = 4$. Since $a_3 = a_1$, we can see that the terms of the sequence will alternately equal 4 and $4/3$, so the sequence is $\left\{4, \frac{4}{3}, 4, \frac{4}{3}, 4, \dots\right\}$.

9. The numerators are all 1 and the denominators are powers of 2, so $a_n = \dfrac{1}{2^n}$.

10. The numerators are all 1 and the denominators are multiples of 2, so $a_n = \dfrac{1}{2n}$.

11. $\{2, 7, 12, 17, \dots\}$. Each term is larger than the preceding one by 5, so
$a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n - 3$.

12. $\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \dots\right\}$. The numerator of the nth term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \dfrac{n}{(n+1)^2}$.

13. $\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \dots\right\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = \left(-\frac{2}{3}\right)^{n-1}$.

14. $\{5, 1, 5, 1, 5, 1, \dots\}$. The average of 5 and 1 is 3, so we can think of the sequence as alternately adding 2 and -2 to 3. Thus, $a_n = 3 + (-1)^{n+1}\cdot 2$.

15. $a_n = n(n-1)$. $a_n \to \infty$ as $n \to \infty$, so the sequence diverges.

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16. $a_n = \dfrac{n+1}{3n-1} = \dfrac{1+1/n}{3-1/n}$, so $a_n \to \dfrac{1+0}{3-0} = \dfrac{1}{3}$ as $n \to \infty$. Converges

17. $a_n = \dfrac{3+5n^2}{n+n^2} = \dfrac{(3+5n^2)/n^2}{(n+n^2)/n^2} = \dfrac{5+3/n^2}{1+1/n}$, so $a_n \to \dfrac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

18. $a_n = \dfrac{\sqrt{n}}{1+\sqrt{n}} = \dfrac{1}{1/\sqrt{n}+1}$, so $a_n \to \dfrac{1}{0+1} = 1$ as $n \to \infty$. Converges

19. $a_n = \dfrac{2^n}{3^{n+1}} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n$, so $\lim\limits_{n\to\infty} a_n = \frac{1}{3}\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\cdot 0 = 0$ by (8) with $r = \frac{2}{3}$. Converges

20. $a_n = \dfrac{n}{1+\sqrt{n}} = \dfrac{\sqrt{n}}{1/\sqrt{n}+1}$. The numerator approaches ∞ and the denominator approaches $0+1=1$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$ and the sequence diverges.

21. $a_n = \dfrac{(-1)^{n-1}\,n}{n^2+1} = \dfrac{(-1)^{n-1}}{n+1/n}$, so $0 \le |a_n| = \dfrac{1}{n+1/n} \le \dfrac{1}{n} \to 0$ as $n \to \infty$, so $a_n \to 0$ by the Squeeze Theorem and Theorem 6. Converges

22. $a_n = \dfrac{(-1)^n n^3}{n^3+2n^2+1}$. Now $|a_n| = \dfrac{n^3}{n^3+2n^2+1} = \dfrac{1}{1+\frac{2}{n}+\frac{1}{n^3}} \to 1$ as $n \to \infty$, but the terms of the sequence $\{a_n\}$ alternate in sign, so the sequence $a_1, a_3, a_5, \ldots$ converges to -1 and the sequence $a_2, a_4, a_6, \ldots$ converges to $+1$. This shows that the given sequence diverges since its terms don't approach a single real number.

23. $a_n = \cos(n/2)$. This sequence diverges since the terms don't approach any particular real number as $n \to \infty$. The terms take on values between -1 and 1.

24. $a_n = \cos(2/n)$. As $n \to \infty$, $2/n \to 0$, so $\cos(2/n) \to \cos 0 = 1$. Converges

25. $a_n = \dfrac{(2n-1)!}{(2n+1)!} = \dfrac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \dfrac{1}{(2n+1)(2n)} \to 0$ as $n \to \infty$. Converges

26. $2n \to \infty$ as $n \to \infty$, so since $\lim\limits_{x\to\infty} \arctan x = \frac{\pi}{2}$, we have $\lim\limits_{n\to\infty} \arctan 2n = \frac{\pi}{2}$. Converges

27. $a_n = \dfrac{e^n + e^{-n}}{e^{2n}-1}\cdot\dfrac{e^{-n}}{e^{-n}} = \dfrac{1+e^{-2n}}{e^n - e^{-n}} \to \dfrac{1+0}{e^n-0} \to 0$ as $n \to \infty$. Converges

28. $a_n = \dfrac{\ln n}{\ln 2n} = \dfrac{\ln n}{\ln 2 + \ln n} = \dfrac{1}{\frac{\ln 2}{\ln n}+1} \to \dfrac{1}{0+1} \to 1$ as $n \to \infty$. Converges

29. $a_n = n^2 e^{-n} = \dfrac{n^2}{e^n}$. Since $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{2x}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0$, it follows from Theorem 3 that $\lim\limits_{n\to\infty} a_n = 0$. Converges

30. $a_n = n\cos n\pi = n(-1)^n$. Since $|a_n| = n \to \infty$ as $n \to \infty$, the given sequence diverges.

31. $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$ [since $0 \le \cos^2 n \le 1$], so since $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, $\left\{\dfrac{\cos^2 n}{2^n}\right\}$ converges to 0 by the Squeeze Theorem.

32. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1+\dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$. Converges

33. $a_n = n\sin(1/n) = \dfrac{\sin(1/n)}{1/n}$. Since $\lim\limits_{x\to\infty} \dfrac{\sin(1/x)}{1/x} = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t}$ [where $t = 1/x$] $= 1$, it follows from Theorem 3 that $\{a_n\}$ converges to 1.

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34. $a_n = \sqrt{n} - \sqrt{n^2-1} = \sqrt{n^2 \cdot \frac{1}{n}} - \sqrt{n^2\left(1-\frac{1}{n^2}\right)} = n\left(\frac{1}{\sqrt{n}} - \sqrt{1-\frac{1}{n^2}}\right) \to n(0-1) \to -n$ as $n \to \infty$, so $a_n \to -\infty$ as $n \to \infty$. Diverges

35. $a_n = \left(1+\frac{2}{n}\right)^{1/n} \quad \Rightarrow \quad \ln a_n = \frac{1}{n}\ln\left(1+\frac{2}{n}\right)$. As $n \to \infty$, $\frac{1}{n} \to 0$ and $\ln\left(1+\frac{2}{n}\right) \to 0$, so $\ln a_n \to 0$. Thus, $a_n \to e^0 = 1$ as $n \to \infty$. Converges

36. $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$. $|a_n| \le \dfrac{1}{1+\sqrt{n}}$ and $\lim\limits_{n\to\infty} \dfrac{1}{1+\sqrt{n}} = 0$, so $\dfrac{-1}{1+\sqrt{n}} \le a_n \le \dfrac{1}{1+\sqrt{n}} \quad \Rightarrow \quad \lim\limits_{n\to\infty} a_n = 0$ by the Squeeze Theorem. Converges

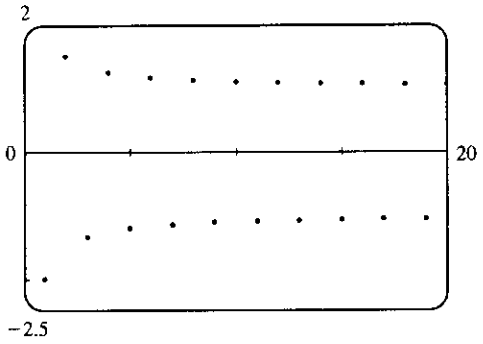
37. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$ diverges since the sequence takes on only two values, 0 and 1, and never stays arbitrarily close to either one (or any other value) for n sufficiently large.

38. $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \dots\right\}$. $a_{2n-1} = \dfrac{1}{n}$ and $a_{2n} = \dfrac{1}{n+2}$ for all positive integers n. $\lim\limits_{n\to\infty} a_n = 0$ since $\lim\limits_{n\to\infty} a_{2n-1} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ and $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} \dfrac{1}{n+2} = 0$. For n sufficiently large, a_n can be made as close to 0 as we like. Converges

39. $a_n = \dfrac{n!}{2^n} = \dfrac{1}{2}\cdot\dfrac{2}{2}\cdot\dfrac{3}{2}\cdot\cdots\cdot\dfrac{(n-1)}{2}\cdot\dfrac{n}{2} \ge \dfrac{1}{2}\cdot\dfrac{n}{2}$ [for $n > 1$] $= \dfrac{n}{4} \to \infty$ as $n \to \infty$, so $\{a_n\}$ diverges.

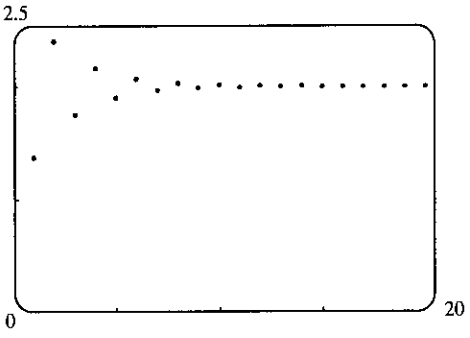
40. $0 < |a_n| = \dfrac{3^n}{n!} = \dfrac{3}{1}\cdot\dfrac{3}{2}\cdot\dfrac{3}{3}\cdot\cdots\cdot\dfrac{3}{(n-1)}\cdot\dfrac{3}{n} \le \dfrac{3}{1}\cdot\dfrac{3}{2}\cdot\dfrac{3}{n}$ [for $n > 2$] $= \dfrac{27}{2n} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\{(-3)^n/n\}$ converges to 0.

41.



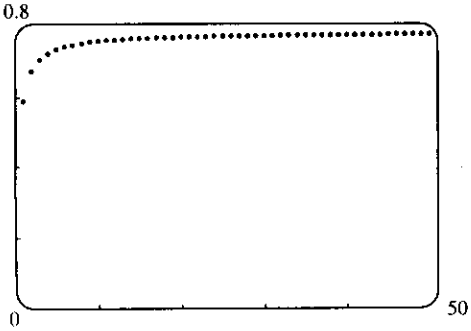
From the graph, we see that the sequence $\left\{(-1)^n \dfrac{n+1}{n}\right\}$ is divergent, since it oscillates between 1 and -1 (approximately).

42.



From the graph, it appears that the sequence converges to 2. $\left\{\left(-\frac{2}{\pi}\right)^n\right\}$ converges to 0 by (6), and hence $\left\{2+\left(-\frac{2}{\pi}\right)^n\right\}$ converges to $2+0=2$.

43.



From the graph, it appears that the sequence converges to about 0.78.

$$\lim_{n\to\infty} \frac{2n}{2n+1} = \lim_{n\to\infty} \frac{2}{2+1/n} = 1, \text{ so}$$

$$\lim_{n\to\infty} \arctan\left(\frac{2n}{2n+1}\right) = \arctan 1 = \frac{\pi}{4}.$$

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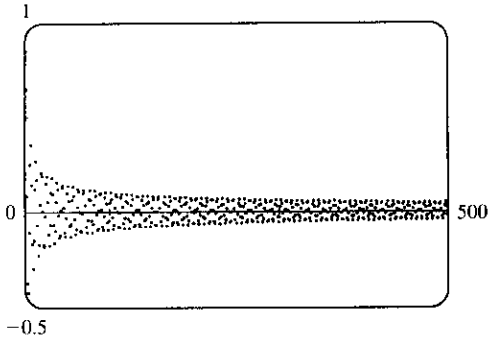
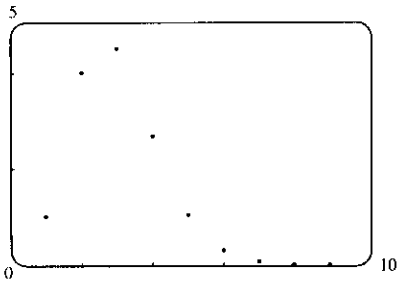
44.



From the graph, it appears that the sequence converges (slowly) to 0.

$0 \le \dfrac{|\sin n|}{\sqrt{n}} \le \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\left\{\dfrac{\sin n}{\sqrt{n}}\right\}$ converges to 0.

45.



From the graph, it appears that the sequence converges to 0.

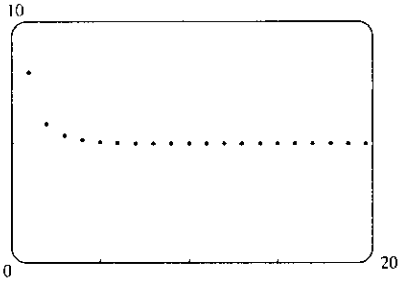
$$0 < a_n = \frac{n^3}{n!} = \frac{n}{n} \cdot \frac{n}{(n-1)} \cdot \frac{n}{(n-2)} \cdot \frac{1}{(n-3)} \cdots \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1}$$

$$\le \frac{n^2}{(n-1)(n-2)(n-3)} \quad [\text{for } n \ge 4]$$

$$= \frac{1/n}{(1-1/n)(1-2/n)(1-3/n)} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\{n^3/n!\}$ converges to 0.

46.



From the graph, it appears that the sequence converges to 5.

$$5 = \sqrt[n]{5^n} \le \sqrt[n]{3^n + 5^n} \le \sqrt[n]{5^n + 5^n} = \sqrt[n]{2}\sqrt[n]{5^n}$$

$$= \sqrt[n]{2} \cdot 5 \to 5 \text{ as } n \to \infty \quad [\lim_{n\to\infty} 2^{1/n} = 2^0 = 1]$$

Hence, $a_n \to 5$ by the Squeeze Theorem.

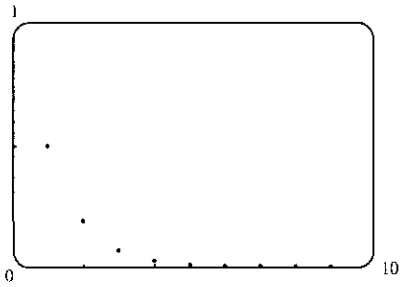
Alternate Solution: Let $y = (3^x + 5^x)^{1/x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(3^x + 5^x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3^x \ln 3 + 5^x \ln 5}{3^x + 5^x}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^x + 1} = \ln 5$$

so $\lim_{x\to\infty} y = e^{\ln 5} = 5$, and so $\left\{\sqrt[n]{3^n + 5^n}\right\}$ converges to 5.

47.



From the graph, it appears that the sequence approaches 0.

$$0 < a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n} = \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdots \frac{2n-1}{2n}$$

$$\le \frac{1}{2n} \cdot (1) \cdot (1) \cdot \cdots \cdot (1) = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{\dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n}\right\}$ converges to 0.

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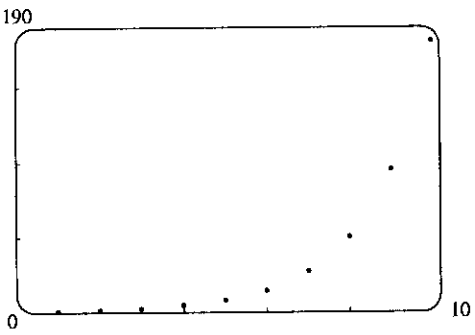
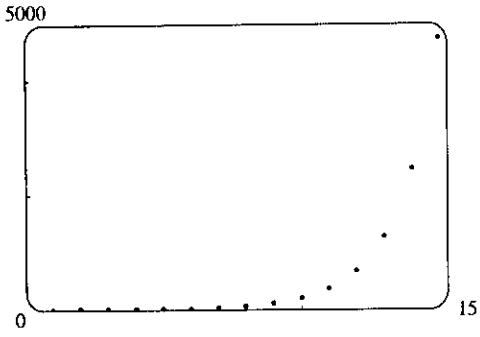
48.





From the graphs, it seems that the sequence diverges. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$. We first prove by induction that $a_n \geq \left(\dfrac{3}{2}\right)^{n-1}$ for all n. This is clearly true for $n = 1$, so let $P(n)$ be the statement that the above is true for n. We must show it is then true for $n+1$. $a_{n+1} = a_n \cdot \dfrac{2n+1}{n+1} \geq \left(\dfrac{3}{2}\right)^{n-1} \cdot \dfrac{2n+1}{n+1}$ (induction hypothesis).

But $\dfrac{2n+1}{n+1} \geq \dfrac{3}{2}$ [since $2(2n+1) \geq 3(n+1) \quad \Leftrightarrow \quad 4n+2 \geq 3n+3 \quad \Leftrightarrow \quad n \geq 1$], and so we get that $a_{n+1} \geq \left(\frac{3}{2}\right)^{n-1} \cdot \frac{3}{2} = \left(\frac{3}{2}\right)^n$ which is $P(n+1)$. Thus, we have proved our first assertion, so since $\left\{\left(\frac{3}{2}\right)^{n-1}\right\}$ diverges (by (8)), so does the given sequence $\{a_n\}$.

49. (a) $a_n = 1000(1.06)^n \quad \Rightarrow \quad a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim\limits_{n\to\infty} a_n = 1000 \lim\limits_{n\to\infty} (1.06)^n$, so the sequence diverges by (8) with $r = 1.06 > 1$.

50. $a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. When $a_1 = 25$, the first 40 terms are 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point a_1.

51. If $|r| \geq 1$, then $\{r^n\}$ diverges by (8), so $\{nr^n\}$ diverges also, since $|nr^n| = n\,|r^n| \geq |r^n|$. If $|r| < 1$ then $\lim\limits_{x\to\infty} xr^x = \lim\limits_{x\to\infty} \dfrac{x}{r^{-x}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{(-\ln r)\, r^{-x}} = \lim\limits_{x\to\infty} \dfrac{r^x}{-\ln r} = 0$, so $\lim\limits_{n\to\infty} nr^n = 0$, and hence $\{nr^n\}$ converges whenever $|r| < 1$.

52. (a) Let $\lim\limits_{n\to\infty} a_n = L$. By Definition 1, this means that for every $\varepsilon > 0$ there is an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. Thus, $|a_{n+1} - L| < \varepsilon$ whenever $n+1 > N \quad \Leftrightarrow \quad n > N-1$. It follows that $\lim\limits_{n\to\infty} a_{n+1} = L$ and so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$.

(b) If $L = \lim\limits_{n\to\infty} a_n$ then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 1/(1+L) \quad \Rightarrow \quad L^2 + L - 1 = 0 \quad \Rightarrow$ $L = \frac{-1+\sqrt{5}}{2}$ (since L has to be non-negative if it exists).

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53. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8, $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L. L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

54. $a_n = 1/5^n$ defines a decreasing geometric sequence since $a_{n+1} = \frac{1}{5}a_n < a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$.

55. $a_n = \dfrac{1}{2n+3}$ is decreasing since $a_{n+1} = \dfrac{1}{2(n+1)+3} = \dfrac{1}{2n+5} < \dfrac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.

56. $a_n = \dfrac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x) = \dfrac{2x-3}{3x+4}$,

$f'(x) = \dfrac{(3x+4)(2) - (2x-3)(3)}{(3x+4)^2} = \dfrac{17}{(3x+4)^2} > 0$. The sequence is bounded since $a_n \geq a_1 = -\frac{1}{7}$ for $n \geq 1$, and $a_n < \dfrac{2n-3}{3n} < \dfrac{2n}{3n} = \dfrac{2}{3}$ for $n \geq 1$.

57. $a_n = \cos(n\pi/2)$ is not monotonic. The first few terms are $0, -1, 0, 1, 0, -1, 0, 1, \ldots$. In fact, the sequence consists of the terms $0, -1, 0, 1$ repeated over and over again in that order. The sequence is bounded since $|a_n| \leq 1$ for all $n \geq 1$.

58. $a_n = ne^{-n}$ defines a positive decreasing sequence since the function $f(x) = xe^{-x}$ is decreasing for $x > 1$. [$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0$ for $x > 1$.] The sequence is bounded above by $a_1 = \frac{1}{e}$ and below by 0.

59. $a_n = \dfrac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \dfrac{x}{x^2+1}$,

$f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{2}$ for all $n \geq 1$.

60. $a_n = n + \dfrac{1}{n}$ defines an increasing sequence since the function $g(x) = x + \dfrac{1}{x}$ is increasing for $x > 1$. [$g'(x) = 1 - 1/x^2 > 0$ for $x > 1$.] The sequence is unbounded since $a_n \to \infty$ as $n \to \infty$. (It is, however, bounded below by $a_1 = 2$.)

61. $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, ..., so $a_n = 2^{(2^n-1)/2^n} = 2^{1-(1/2^n)}$. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2^{1-(1/2^n)} = 2^1 = 2$.

Alternate solution: Let $L = \lim\limits_{n\to\infty} a_n$. (We could show the limit exists by showing that $\{a_n\}$ is bounded and increasing.) Then L must satisfy $L = \sqrt{2 \cdot L} \quad \Rightarrow \quad L^2 = 2L \quad \Rightarrow \quad L(L-2) = 0$. $L \neq 0$ since the sequence increases, so $L = 2$.

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62. (a) Let P_n be the statement that $a_{n+1} \geq a_n$ and $a_n \leq 3$. P_1 is obviously true. We will assume that P_n is true and then show that as a consequence P_{n+1} must also be true. $a_{n+2} \geq a_{n+1} \quad \Leftrightarrow \quad \sqrt{2+a_{n+1}} \geq \sqrt{2+a_n} \quad \Leftrightarrow$ $2 + a_{n+1} \geq 2 + a_n \quad \Leftrightarrow \quad a_{n+1} \geq a_n$, which is the induction hypothesis. $a_{n+1} \leq 3 \quad \Leftrightarrow \quad \sqrt{2+a_n} \leq 3 \quad \Leftrightarrow$ $2 + a_n \leq 9 \quad \Leftrightarrow \quad a_n \leq 7$, which is certainly true because we are assuming that $a_n \leq 3$. So P_n is true for all n, and so $a_1 \leq a_n \leq 3$ (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim_{n\to\infty} a_n$ exists.

(b) If $L = \lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1} = L$ also, so $L = \sqrt{2+L} \quad \Rightarrow \quad L^2 = 2 + L \quad \Leftrightarrow \quad L^2 - L - 2 = 0 \quad \Leftrightarrow$ $(L+1)(L-2) = 0 \quad \Leftrightarrow \quad L = 2$ (since L can't be negative).

63. We show by induction that $\{a_n\}$ is increasing and bounded above by 3.

Let P_n be the proposition that $a_{n+1} > a_n$ and $0 < a_n < 3$. Clearly P_1 is true. Assume that P_n is true.

Then $a_{n+1} > a_n \quad \Rightarrow \quad \dfrac{1}{a_{n+1}} < \dfrac{1}{a_n} \quad \Rightarrow \quad -\dfrac{1}{a_{n+1}} > -\dfrac{1}{a_n}$.

Now $a_{n+2} = 3 - \dfrac{1}{a_{n+1}} > 3 - \dfrac{1}{a_n} = a_{n+1} \quad \Leftrightarrow \quad P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3, so $1 = a_1 < a_n < 3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem.

If $L = \lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 3 - 1/L \quad \Rightarrow \quad L^2 - 3L + 1 = 0 \quad \Rightarrow$ $L = \frac{3 \pm \sqrt{5}}{2}$. But $L > 1$, so $L = \frac{3+\sqrt{5}}{2}$.

64. We use induction. Let P_n be the statement that $0 < a_{n+1} \leq a_n \leq 2$. Clearly P_1 is true, since $a_2 = 1/(3-2) = 1$. Now assume that P_n is true. Then $a_{n+1} \leq a_n \quad \Rightarrow \quad -a_{n+1} \geq -a_n \quad \Rightarrow \quad 3 - a_{n+1} \geq 3 - a_n \quad \Rightarrow$ $a_{n+2} = \dfrac{1}{3 - a_{n+1}} \leq \dfrac{1}{3 - a_n} = a_{n+1}$. Also $a_{n+2} > 0$ (since $3 - a_{n+1}$ is positive) and $a_{n+1} \leq 2$ by the induction hypothesis, so P_{n+1} is true.

To find the limit, we use the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} \quad \Rightarrow \quad L = \frac{1}{3-L} \quad \Rightarrow \quad L^2 - 3L + 1 = 0 \quad \Rightarrow$ $L = \frac{3 \pm \sqrt{5}}{2}$. But $L \leq 2$, so we must have $L = \frac{3-\sqrt{5}}{2}$.

65. (a) Let a_n be the number of rabbit pairs in the nth month. Clearly $a_1 = 1 = a_2$. In the nth month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n = a_{n-1} + a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.

(b) $a_n = \dfrac{f_{n+1}}{f_n} \quad \Rightarrow \quad a_{n-1} = \dfrac{f_n}{f_{n-1}} = \dfrac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \dfrac{f_{n-2}}{f_{n-1}} = 1 + \dfrac{1}{f_{n-1}/f_{n-2}} = 1 + \dfrac{1}{a_{n-2}}$. If $L = \lim_{n\to\infty} a_n$, then $L = \lim_{n\to\infty} a_{n-1}$ and $L = \lim_{n\to\infty} a_{n-2}$, so L must satisfy $L = 1 + \dfrac{1}{L} \quad \Rightarrow$ $L^2 - L - 1 = 0 \quad \Rightarrow \quad L = \frac{1+\sqrt{5}}{2}$ (since L must be positive).

66. (a) If f is continuous, then $f(L) = f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_{n+1} = L$ by Exercise 52(a).

(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that $L \approx 0.73909$.

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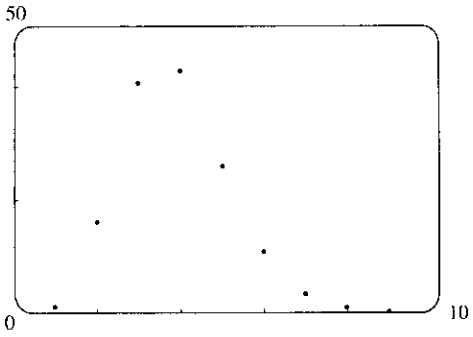
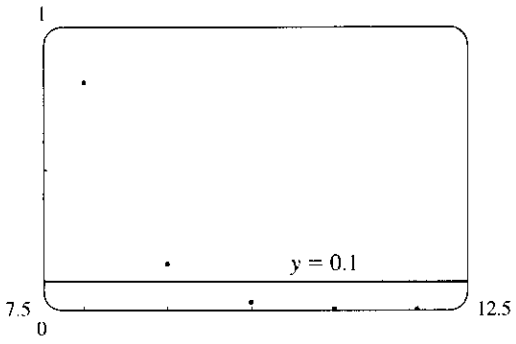
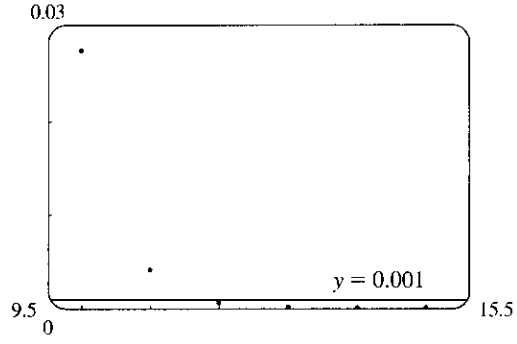
67. (a)



From the graph, it appears that the sequence $\left\{\dfrac{n^5}{n!}\right\}$ converges to 0, that is, $\lim\limits_{n\to\infty}\dfrac{n^5}{n!}=0$.

(b)





From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon=0.1$ is 9, since $n^5/n!<0.1$ whenever $n\geq 10$, but $9^5/9!>0.1$. From the second graph, it seems that for $\varepsilon=0.001$, the smallest possible value for N is 11.

68. Let $\varepsilon>0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n>N$ then $n>\ln(\varepsilon)/\ln|r| \;\Rightarrow\; n\ln|r|<\ln\varepsilon$ [since $|r|<1 \;\Rightarrow\; \ln|r|<0$] $\Rightarrow\; \ln(|r|^n)<\ln\varepsilon \;\Rightarrow\; |r|^n<\varepsilon \;\Rightarrow\; |r^n-0|<\varepsilon$, and so by Definition 1, $\lim\limits_{n\to\infty} r^n=0$.

69. If $\lim\limits_{n\to\infty}|a_n|=0$ then $\lim\limits_{n\to\infty}-|a_n|=0$, and since $-|a_n|\leq a_n\leq|a_n|$, we have that $\lim\limits_{n\to\infty}a_n=0$ by the Squeeze Theorem.

70. (a) $$\frac{b^{n+1}-a^{n+1}}{b-a}=b^n+b^{n-1}a+b^{n-2}a^2+b^{n-3}a^3+\cdots+ba^{n-1}+a^n$$
$$<b^n+b^{n-1}b+b^{n-2}b^2+b^{n-3}b^3+\cdots+bb^{n-1}+b^n=(n+1)b^n$$

(b) Since $b-a>0$, we have $b^{n+1}-a^{n+1}<(n+1)b^n(b-a) \;\Rightarrow\; b^{n+1}-(n+1)b^n(b-a)<a^{n+1} \;\Rightarrow\; b^n[(n+1)a-nb]<a^{n+1}$.

(c) With this substitution, $(n+1)a-nb=1$, and so $b^n=\left(1+\dfrac{1}{n}\right)^n<a^{n+1}=\left(1+\dfrac{1}{n+1}\right)^{n+1}$.

(d) With this substitution, we get $\left(1+\dfrac{1}{2n}\right)^n\left(\dfrac{1}{2}\right)<1 \;\Rightarrow\; \left(1+\dfrac{1}{2n}\right)^n<2 \;\Rightarrow\; \left(1+\dfrac{1}{2n}\right)^{2n}<4$.

(e) $a_n<a_{2n}$ since $\{a_n\}$ is increasing, so $a_n<a_{2n}<4$.

(f) Since $\{a_n\}$ is increasing and bounded above by 4, $a_1\leq a_n\leq 4$, and so $\{a_n\}$ is bounded and monotonic, and hence has a limit by Theorem 11.

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71. (a) First we show that $a > a_1 > b_1 > b$.

$a_1 - b_1 = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(a - 2\sqrt{ab} + b\right) = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$ (since $a > b$) $\Rightarrow$ $a_1 > b_1$. Also

$a - a_1 = a - \frac{1}{2}(a+b) = \frac{1}{2}(a-b) > 0$ and $b - b_1 = b - \sqrt{ab} = \sqrt{b}\left(\sqrt{b} - \sqrt{a}\right) < 0$, so $a > a_1 > b_1 > b$.

In the same way we can show that $a_1 > a_2 > b_2 > b_1$ and so the given assertion is true for $n = 1$. Suppose it is true for $n = k$, that is, $a_k > a_{k+1} > b_{k+1} > b_k$. Then

$$a_{k+2} - b_{k+2} = \tfrac{1}{2}(a_{k+1} + b_{k+1}) - \sqrt{a_{k+1}b_{k+1}} = \tfrac{1}{2}\left(a_{k+1} - 2\sqrt{a_{k+1}b_{k+1}} + b_{k+1}\right)$$

$$= \tfrac{1}{2}\left(\sqrt{a_{k+1}} - \sqrt{b_{k+1}}\right)^2 > 0$$

$$a_{k+1} - a_{k+2} = a_{k+1} - \tfrac{1}{2}(a_{k+1} + b_{k+1}) = \tfrac{1}{2}(a_{k+1} - b_{k+1}) > 0$$

and $b_{k+1} - b_{k+2} = b_{k+1} - \sqrt{a_{k+1}b_{k+1}} = \sqrt{b_{k+1}}\left(\sqrt{b_{k+1}} - \sqrt{a_{k+1}}\right) < 0$ $\Rightarrow$

$a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1}$, so the assertion is true for $n = k+1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a > a_n > a_{n+1} > b_{n+1} > b_n > b$, which shows that both sequences, $\{a_n\}$ and $\{b_n\}$, are monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

(c) Let $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$. Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n + b_n}{2}$ $\Rightarrow$ $\alpha = \frac{\alpha + \beta}{2}$ $\Rightarrow$

$2\alpha = \alpha + \beta$ $\Rightarrow$ $\alpha = \beta$.

72. (a) Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_{2n} = L$, there exists N_1 such that $|a_{2n} - L| < \varepsilon$ for $n > N_1$. Since $\lim_{n\to\infty} a_{2n+1} = L$, there exists N_2 such that $|a_{2n+1} - L| < \varepsilon$ for $n > N_2$. Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. If n is even, then $n = 2m$ where $m > N_1$, so $|a_n - L| = |a_{2m} - L| < \varepsilon$. If n is odd, then $n = 2m + 1$, where $m > N_2$, so $|a_n - L| = |a_{2m+1} - L| < \varepsilon$. Therefore $\lim_{n\to\infty} a_n = L$.

(b) $a_1 = 1$, $a_2 = 1 + \frac{1}{1+1} = \frac{3}{2} = 1.5$, $a_3 = 1 + \frac{1}{5/2} = \frac{7}{5} = 1.4$, $a_4 = 1 + \frac{1}{12/5} = \frac{17}{12} = 1.41\overline{6}$,

$a_5 = 1 + \frac{1}{29/12} = \frac{41}{29} \approx 1.413793$, $a_6 = 1 + \frac{1}{70/29} = \frac{99}{70} \approx 1.414286$, $a_7 = 1 + \frac{1}{169/70} = \frac{239}{169} \approx 1.414201$,

$a_8 = 1 + \frac{1}{408/169} = \frac{577}{408} \approx 1.414216$. Notice that $a_1 < a_3 < a_5 < a_7$ and $a_2 > a_4 > a_6 > a_8$. It appears that the odd terms are increasing and the even terms are decreasing. Let's prove that $a_{2n-2} > a_{2n}$ and $a_{2n-1} < a_{2n+1}$ by mathematical induction. Suppose that $a_{2k-2} > a_{2k}$. Then $1 + a_{2k-2} > 1 + a_{2k}$ $\Rightarrow$

$$\frac{1}{1+a_{2k-2}} < \frac{1}{1+a_{2k}} \quad \Rightarrow \quad 1 + \frac{1}{1+a_{2k-2}} < 1 + \frac{1}{1+a_{2k}} \quad \Rightarrow \quad a_{2k-1} < a_{2k+1} \quad \Rightarrow$$

$$1 + a_{2k-1} < 1 + a_{2k+1} \quad \Rightarrow \quad \frac{1}{1+a_{2k-1}} > \frac{1}{1+a_{2k+1}} \quad \Rightarrow \quad 1 + \frac{1}{1+a_{2k-1}} > 1 + \frac{1}{1+a_{2k+1}} \quad \Rightarrow$$

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$a_{2k} > a_{2k+2}$. We have thus shown, by induction, that the odd terms are increasing and the even terms are decreasing. Also all terms lie between 1 and 2, so both $\{a_n\}$ and $\{b_n\}$ are bounded monotonic sequences and are therefore convergent by Theorem 11. Let $\lim_{n\to\infty} a_{2n} = L$. Then $\lim_{n\to\infty} a_{2n+2} = L$ also. We have $a_{n+2} = 1 + \dfrac{1}{1+1+1/(1+a_n)} = 1 + \dfrac{1}{(3+2a_n)/(1+a_n)} = \dfrac{4+3a_n}{3+2a_n}$, so $a_{2n+2} = \dfrac{4+3a_{2n}}{3+2a_{2n}}$. Taking limits of both sides, we get $L = \dfrac{4+3L}{3+2L} \Rightarrow 3L + 2L^2 = 4 + 3L \Rightarrow L^2 = 2 \Rightarrow L = \sqrt{2}$ (since $L > 0$). Thus, $\lim_{n\to\infty} a_{2n} = \sqrt{2}$. Similarly we find that $\lim_{n\to\infty} a_{2n+1} = \sqrt{2}$. So, by part (a), $\lim_{n\to\infty} a_n = \sqrt{2}$.

73. (a) Suppose $\{p_n\}$ converges to p. Then $p_{n+1} = \dfrac{bp_n}{a+p_n} \Rightarrow \lim_{n\to\infty} p_{n+1} = \dfrac{b \lim_{n\to\infty} p_n}{a + \lim_{n\to\infty} p_n} \Rightarrow$

$$p = \frac{bp}{a+p} \quad \Rightarrow \quad p^2 + ap = bp \quad \Rightarrow \quad p(p+a-b) = 0 \quad \Rightarrow \quad p = 0 \text{ or } p = b - a.$$

(b) $p_{n+1} = \dfrac{bp_n}{a+p_n} = \dfrac{\frac{b}{a}p_n}{1+\frac{p_n}{a}} < \dfrac{b}{a}p_n$ since $1 + \dfrac{p_n}{a} > 1$.

(c) By part (b), $p_1 < \left(\dfrac{b}{a}\right)p_0$, $p_2 < \left(\dfrac{b}{a}\right)p_1 < \left(\dfrac{b}{a}\right)^2 p_0$, $p_3 < \left(\dfrac{b}{a}\right)p_2 < \left(\dfrac{b}{a}\right)^3 p_0$, etc. In general, $p_n < \left(\dfrac{b}{a}\right)^n p_0$, so $\lim_{n\to\infty} p_n \le \lim_{n\to\infty} \left(\dfrac{b}{a}\right)^n \cdot p_0 = 0$ since $b < a$. [By result 8, $\lim_{n\to\infty} r^n = 0$ if $-1 < r < 1$. Here $r = \dfrac{b}{a} \in (0,1)$.]

(d) Let $a < b$. We first show, by induction, that if $p_0 < b - a$, then $p_n < b - a$ and $p_{n+1} > p_n$.

For $n = 0$, we have $p_1 - p_0 = \dfrac{bp_0}{a+p_0} - p_0 = \dfrac{p_0(b-a-p_0)}{a+p_0} > 0$ since $p_0 < b - a$. So $p_1 > p_0$.

Now we suppose the assertion is true for $n = k$, that is, $p_k < b - a$ and $p_{k+1} > p_k$. Then

$$b - a - p_{k+1} = b - a - \frac{bp_k}{a+p_k} = \frac{a(b-a) + bp_k - ap_k - bp_k}{a+p_k} = \frac{a(b-a-p_k)}{a+p_k} > 0 \text{ because } p_k < b - a.$$

So $p_{k+1} < b - a$. And $p_{k+2} - p_{k+1} = \dfrac{bp_{k+1}}{a+p_{k+1}} - p_{k+1} = \dfrac{p_{k+1}(b-a-p_{k+1})}{a+p_{k+1}} > 0$ since $p_{k+1} < b - a$.

Therefore, $p_{k+2} > p_{k+1}$. Thus, the assertion is true for $n = k+1$. It is therefore true for all n by mathematical induction. A similar proof hy induction shows that if $p_0 > b - a$, then $p_n > b - a$ and $\{p_n\}$ is decreasing.

In either case the sequence $\{p_n\}$ is bounded and monotonic, so it is convergent by the Monotonic Sequence Theorem. It then follows from part (a) that $\lim_{n\to\infty} p_n = b - a$.

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LABORATORY PROJECT Logistic Sequences

1. To write such a program in Maple it is best to calculate all the points first and then graph them. One possible sequence of commands [taking $p_0 = \frac{1}{2}$ and $k = 1.5$ for the difference equation] is

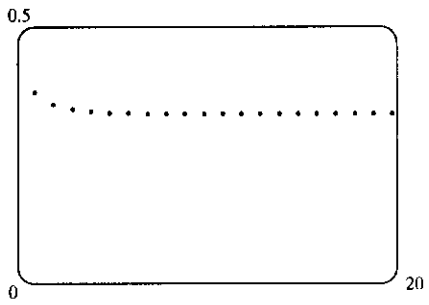
```
t:='t';p(0):=1/2;k:=1.5;
for j from 1 to 20  do p(j):=k*p(j-1)*(1-p(j-1)) od;
plot([seq([t,p(t)] t=0..20)],t=0..20,p=0..0.5,style=point);
```

In Mathematica, we can use the following program:

```
p[0]=1/2
k=1.5
p[j_]:=k*p[j-1]*(1-p[j-1])
P=Table[p[t],{t,20}]
ListPlot[P]
```

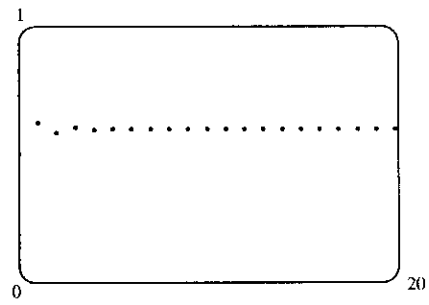
With $p_0 = \frac{1}{2}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.5	7	0.3338465076	14	0.3333373303
1	0.375	8	0.3335895255	15	0.3333353318
2	0.3515625	9	0.3334613309	16	0.3333343326
3	0.3419494629	10	0.3333973076	17	0.3333338329
4	0.3375300416	11	0.3333653143	18	0.3333335831
5	0.3354052689	12	0.3333493223	19	0.3333334582
6	0.3343628617	13	0.3333413274	20	0.3333333958



With $p_0 = \frac{1}{2}$ and $k = 2.5$:

n	p_n	n	p_n	n	p_n
0	0.5	7	0.6004164790	14	0.5999967417
1	0.625	8	0.5997913269	15	0.6000016291
2	0.5859375	9	0.6001042277	16	0.5999991854
3	0.6065368651	10	0.5999478590	17	0.6000004073
4	0.5966247409	11	0.6000260637	18	0.5999997964
5	0.6016591486	12	0.5999869664	19	0.6000001018
6	0.5991635437	13	0.6000065164	20	0.5999999491



Both of these sequences seem to converge (the first to about $\frac{1}{3}$, the second to about 0.60).

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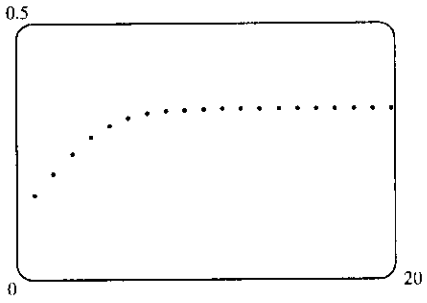
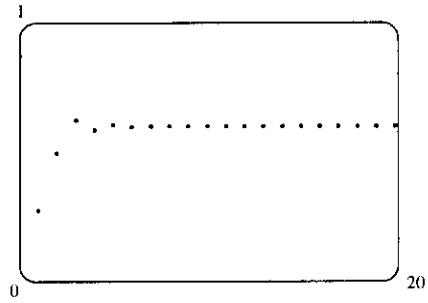
With $p_0 = \frac{7}{8}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.3239166554	14	0.3332554829
1	0.1640625	8	0.3284919837	15	0.3332943990
2	0.2057189941	9	0.3308775005	16	0.3333138639
3	0.2450980344	10	0.3320963702	17	0.3333235980
4	0.2775374819	11	0.3327125567	18	0.3333284655
5	0.3007656421	12	0.3330223670	19	0.3333308994
6	0.3154585059	13	0.3331777051	20	0.3333321164



With $p_0 = \frac{7}{8}$ and $k = 2.5$:

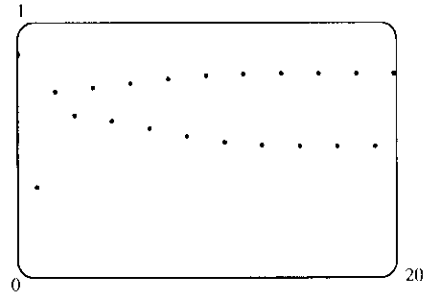
n	p_n	n	p_n	n	p_n
0	0.875	7	0.6016572368	14	0.5999869815
1	0.2734375	8	0.5991645155	15	0.6000065088
2	0.4966735840	9	0.6004159972	16	0.5999967455
3	0.6249723374	10	0.5997915688	17	0.6000016272
4	0.5859547872	11	0.6001041070	18	0.5999991864
5	0.6065294364	12	0.5999479194	19	0.6000004068
6	0.5966286980	13	0.6000260335	20	0.5999997966



The limit of the sequence seems to depend on k, but not on p_0.

2. With $p_0 = \frac{7}{8}$ and $k = 3.2$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.5830728495	14	0.7990633827
1	0.35	8	0.7779164854	15	0.5137954979
2	0.728	9	0.5528397669	16	0.7993909896
3	0.6336512	10	0.7910654689	17	0.5131681132
4	0.7428395416	11	0.5288988570	18	0.7994451225
5	0.6112926626	12	0.7973275394	19	0.5130643795
6	0.7603646184	13	0.5171082698	20	0.7994538304



It seems that eventually the terms fluctuate between two values (about 0.5 and 0.8 in this case).

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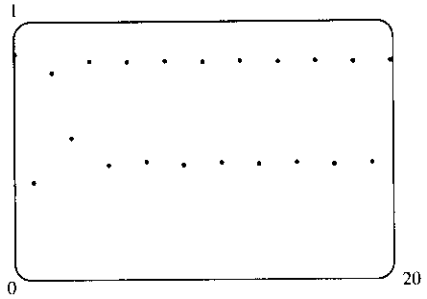
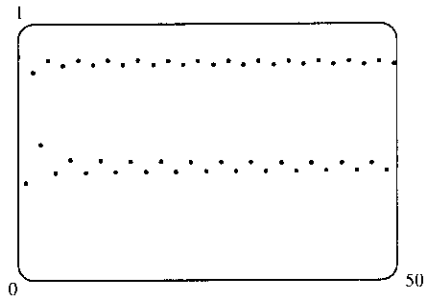
3. With $p_0 = \frac{7}{8}$ and $k = 3.42$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.4523028596	14	0.8442074951
1	0.3740625	8	0.8472194412	15	0.4498025048
2	0.8007579316	9	0.4426802161	16	0.8463823232
3	0.5456427596	10	0.8437633929	17	0.4446659586
4	0.8478752457	11	0.4508474156	18	0.8445284520
5	0.4411212220	12	0.8467373602	19	0.4490464985
6	0.8431438501	13	0.4438243545	20	0.8461207931



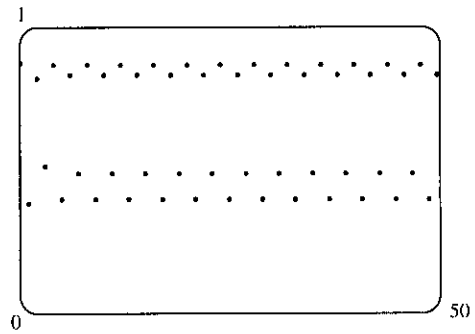
With $p_0 = \frac{7}{8}$ and $k = 3.45$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.4670259170	14	0.8403376122
1	0.37734375	8	0.8587488490	15	0.4628875685
2	0.8105962830	9	0.4184824586	16	0.8577482026
3	0.5296783241	10	0.8395743720	17	0.4209559716
4	0.8594612299	11	0.4646778983	18	0.8409445432
5	0.4167173034	12	0.8581956045	19	0.4614610237
6	0.8385707740	13	0.4198508858	20	0.8573758782



From the graphs above, it seems that for k between 3.4 and 3.5, the terms eventually fluctuate between four values. In the graph below, the pattern followed by the terms is $0.395, 0.832, 0.487, 0.869, 0.395, \ldots$. Note that even for $k = 3.42$ (as in the first graph), there are four distinct "branches; even after 1000 terms, the first and third terms in the pattern differ by about 2×10^{-9}, while the first and fifth terms differ by only 2×10^{-10}.

With $p_0 = \frac{7}{8}$ and $k = 3.48$:



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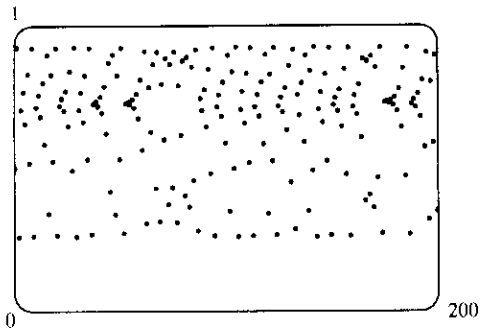
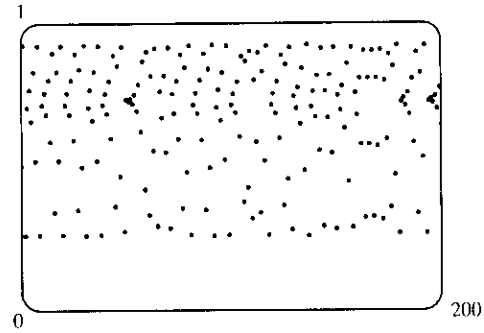
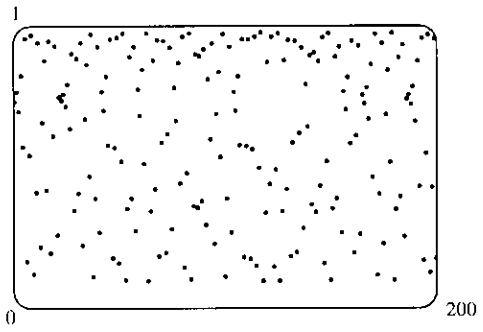
4.



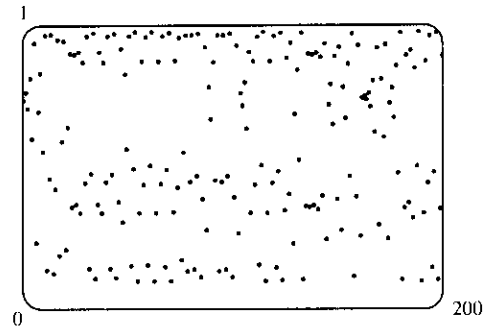
$p_0 = 0.5,\ k = 3.7$



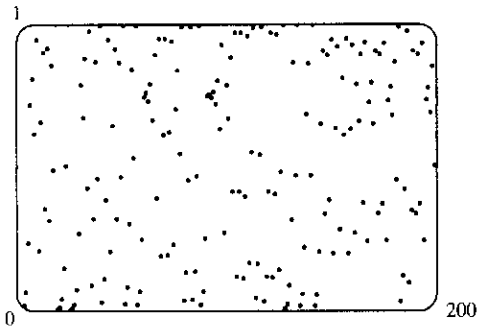
$p_0 = 0.501,\ k = 3.7$



$p_0 = 0.75,\ k = 3.9$



$p_0 = 0.749,\ k = 3.9$



$p_0 = 0.5,\ k = 3.999$

From the graphs, it seems that if p_0 is changed by 0.001, the whole graph changes completely. (Note, however, that this might be partially due to accumulated round-off error in the CAS. These graphs were generated by Maple with 100-digit accuracy, and different degrees of accuracy give different graphs.) There seem to be some some fleeting patterns in these graphs, but on the whole they are certainly very chaotic. As k increases, the graph spreads out vertically, with more extreme values close to 0 or 1.

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12.2 Series

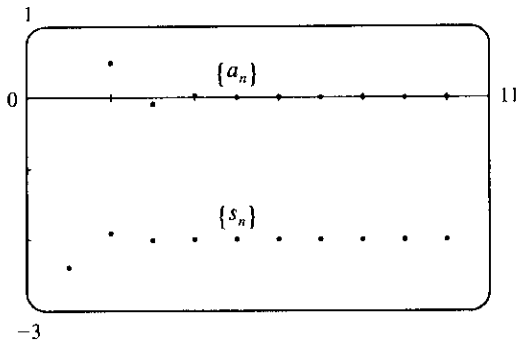
1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5. In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\sum_{i=1}^{n} a_i$.

3.

n	s_n
1	−2.40000
2	−1.92000
3	−2.01600
4	−1.99680
5	−2.00064
6	−1.99987
7	−2.00003
8	−1.99999
9	−2.00000
10	−2.00000



From the graph and the table, it seems that the series converges to −2. In fact, it is a geometric series with $a = -2.4$ and $r = -\frac{1}{5}$, so its sum is

$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{-2.4}{1-\left(-\frac{1}{5}\right)} = \frac{-2.4}{1.2} = -2$. Note that the dot corresponding to $n = 1$ is part of both $\{a_n\}$ and $\{s_n\}$.

TI-86 Note: To graph $\{a_n\}$ and $\{s_n\}$, set your calculator to Param mode and DrawDot mode. (DrawDot is under GRAPH, MORE, FORMT (F3).) Now under `E(t)=` make the assignments: `xt1=t, yt1=12/(-5)^t, xt2=t, yt2=sum seq(yt1,t,1,t,1)`. (sum and seq are under LIST, OPS (F5), MORE.) Under WIND use `1,10,1,0,10,1,-3,1,1` to obtain a graph similar to the one above. Then use TRACE (F4) to see the values.

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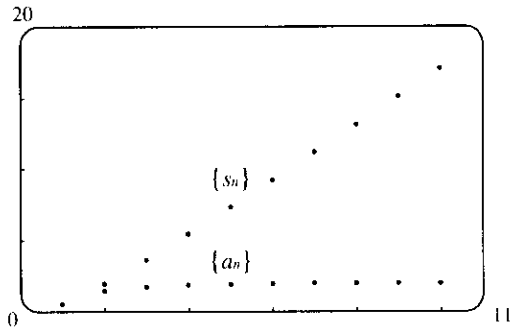
4.

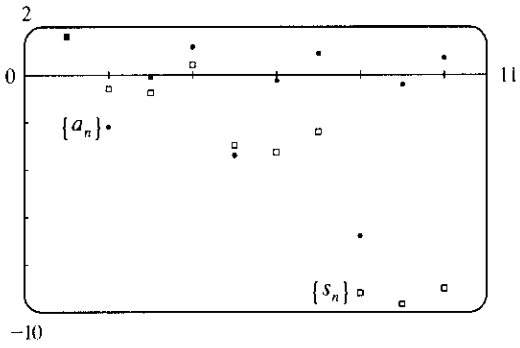
n	s_n
1	0.50000
2	1.90000
3	3.60000
4	5.42353
5	7.30814
6	9.22706
7	11.16706
8	13.12091
9	15.08432
10	17.05462



The series $\sum_{n=1}^{\infty} \dfrac{2n^2-1}{n^2+1}$ diverges, since its terms do not approach 0.

5.

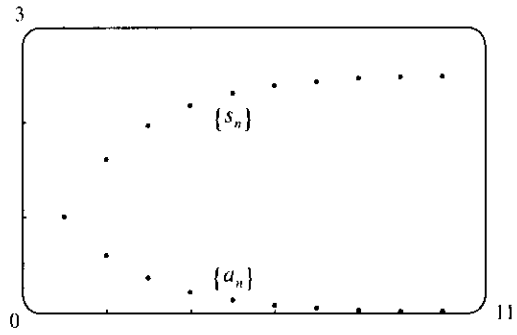
n	s_n
1	1.55741
2	−0.62763
3	−0.77018
4	0.38764
5	−2.99287
6	−3.28388
7	−2.41243
8	−9.21214
9	−9.66446
10	−9.01610



The series $\sum_{n=1}^{\infty} \tan n$ diverges, since its terms do not approach 0.

6.

n	s_n
1	1.00000
2	1.60000
3	1.96000
4	2.17600
5	2.30560
6	2.38336
7	2.43002
8	2.45801
9	2.47481
10	2.48488



From the graph and the table, it seems that the series converges to 2.5. In fact, it is a geometric series with $a = 1$ and $r = 0.6$, so its sum is

$$\sum_{n=1}^{\infty} (0.6)^{n-1} = \frac{1}{1-0.6} = \frac{1}{2/5} = 2.5.$$

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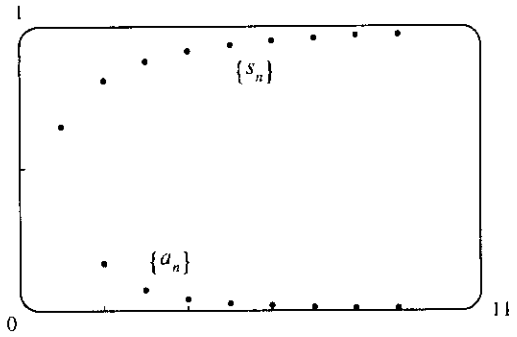
7.

n	s_n
1	0.64645
2	0.80755
3	0.87500
4	0.91056
5	0.93196
6	0.94601
7	0.95581
8	0.96296
9	0.96838
10	0.97259



From the graph, it seems that the series converges to 1. To find the sum, we write

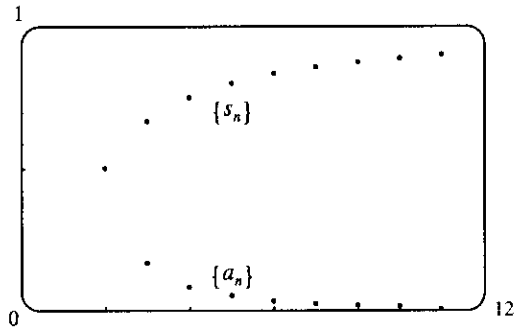
$$s_n = \sum_{i=1}^{n} \left(\frac{1}{i^{1.5}} - \frac{1}{(i+1)^{1.5}} \right)$$

$$= \left(1 - \frac{1}{2^{1.5}}\right) + \left(\frac{1}{2^{1.5}} - \frac{1}{3^{1.5}}\right) + \left(\frac{1}{3^{1.5}} - \frac{1}{4^{1.5}}\right) + \cdots + \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}}\right) = 1 - \frac{1}{(n+1)^{1.5}}$$

So the sum is $\lim_{n\to\infty} s_n = 1 - 0 = 1$.

8.

n	s_n
2	0.50000
3	0.66667
4	0.75000
5	0.80000
6	0.83333
7	0.85714
8	0.87500
9	0.88889
10	0.90000
11	0.90909
100	0.99000



From the graph and the table, it seems that the series converges to 1. To find the sum, we write

$$s_n = \sum_{i=2}^{n} \frac{1}{i(i-1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i} \right) \qquad \text{[partial fractions]}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n},$$

and so the sum is $\lim_{n\to\infty} s_n = 1 - 0 = 1$.

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9. (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{2n}{3n+1} = \dfrac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (12.1.1).

(b) Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence (7).

10. (a) Both $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the nth partial sum.

(b) $\sum_{i=1}^{n} a_j = \underbrace{a_j + a_j + \cdots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

11. $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$ is a geometric series with first term $a = 3$ and common ratio $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{3}{1-2/3} = \frac{3}{1/3} = 9$.

12. $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \cdots$ is a geometric series with $r = -2$. Since $|r| = 2 > 1$, the series diverges.

13. $-2 + \frac{5}{2} - \frac{25}{8} + \frac{125}{32} - \cdots$ is a geometric series with $a = -2$ and $r = \frac{5/2}{-2} = -\frac{5}{4}$. Since $|r| = \frac{5}{4} > 1$, the series diverges by (4).

14. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio 0.4. The series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$ since $|r| = \frac{2}{5} < 1$.

15. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ is a geometric series with $a = 5$ and $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{5}{1-2/3} = \frac{5}{1/3} = 15$.

16. $\sum_{n=1}^{\infty} \dfrac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{6}{5}$. The series diverges since $|r| = \frac{6}{5} > 1$.

17. $\sum_{n=1}^{\infty} \dfrac{(-3)^{n-1}}{4^n} = \dfrac{1}{4}\sum_{n=1}^{\infty} \left(-\dfrac{3}{4}\right)^{n-1}$. The latter series is geometric with $a = 1$ and $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.

18. $\sum_{n=0}^{\infty} \dfrac{1}{(\sqrt{2})^n}$ is a geometric series with ratio $r = \dfrac{1}{\sqrt{2}}$. Since $|r| = \dfrac{1}{\sqrt{2}} < 1$, the series converges. Its sum is

$$\frac{1}{1-1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2}(\sqrt{2}+1) = 2 + \sqrt{2}.$$

19. $\sum_{n=0}^{\infty} \dfrac{\pi^n}{3^{n+1}} = \dfrac{1}{3}\sum_{n=0}^{\infty} \left(\dfrac{\pi}{3}\right)^n$ is a geometric series with ratio $r = \dfrac{\pi}{3}$. Since $|r| > 1$, the series diverges.

20. $\sum_{n=1}^{\infty} \dfrac{e^n}{3^{n-1}} = 3\sum_{n=1}^{\infty} \left(\dfrac{e}{3}\right)^n$ is a geometric series with first term $3(e/3) = e$ and ratio $r = \dfrac{e}{3}$. Since $|r| < 1$, the series converges. Its sum is $\dfrac{e}{1-e/3} = \dfrac{3e}{3-e}$.

21. $\sum_{n=1}^{\infty} \dfrac{n}{n+5}$ diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n}{n+5} = 1 \neq 0$. [Use (7), the Test for Divergence.]

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22. $\sum_{n=1}^{\infty} \frac{3}{n} = 3\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. [If $\sum_{n=1}^{\infty} \frac{3}{n}$ were to converge, then $\sum_{n=1}^{\infty} \frac{1}{n}$ would also have to converge by Theorem 8(i).] In general, constant multiples of divergent series are divergent.

23. Using partial fractions, the partial sums are

$$s_n = \sum_{i=2}^{n} \frac{2}{(i-1)(i+1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1}\right)$$

$$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right)$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

Thus, $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right) = \frac{3}{2}$.

24. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = \lim_{n\to\infty} \left(1 + \frac{1}{n^2+2n}\right) = 1 \neq 0.$$

25. $\sum_{k-2}^{\infty} \frac{k^2}{k^2-1}$ diverges by the Test for Divergence since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1 \neq 0$.

26. Converges. $s_n = \sum_{i=1}^{n} \frac{2}{i^2+4i+3} = \sum_{i=1}^{n} \left(\frac{1}{i+1} - \frac{1}{i+3}\right)$ (using partial fractions). The latter sum is

$$\left(\tfrac{1}{2} - \tfrac{1}{4}\right) + \left(\tfrac{1}{3} - \tfrac{1}{5}\right) + \left(\tfrac{1}{4} - \tfrac{1}{6}\right) + \left(\tfrac{1}{5} - \tfrac{1}{7}\right) + \cdots + \left(\tfrac{1}{n} - \tfrac{1}{n+2}\right) + \left(\tfrac{1}{n+1} - \tfrac{1}{n+3}\right) = \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{n+2} - \tfrac{1}{n+3}$$

(telescoping series). Thus, $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

27. Converges. $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{3^n}{6^n} + \frac{2^n}{6^n}\right) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right] = \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

28. $\sum_{n=1}^{\infty} \left[(0.8)^{n-1} - (0.3)^n\right] = \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.3)^n$ [difference of two convergent geometric series]

$$= \frac{1}{1-0.8} - \frac{0.3}{1-0.3} = 5 - \frac{3}{7} = \frac{32}{7}.$$

29. $\sum_{n=1}^{\infty} \sqrt[n]{2} = 2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \cdots$ diverges by the Test for Divergence since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0.$$

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30. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n}{2n+5}\right) = \lim_{n\to\infty} \ln\left(\frac{1}{2+5/n}\right) = \ln\frac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

31. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$, so the series diverges by the Test for Divergence.

32. $\sum_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with ratio $r = \cos 1 \approx 0.540302$. It converges because $|r| < 1$. Its sum is $\dfrac{\cos 1}{1-\cos 1} \approx 1.175343$.

33. The first series is a telescoping sum:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3}{n(n+3)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3}\right) \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+3}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}\end{aligned}$$

The second series is geometric with first term $\frac{5}{4}$ and ratio $\frac{1}{4}$: $\sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{5/4}{1-1/4} = \frac{5}{3}$. Thus,

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+3)} + \frac{5}{4^n}\right) = \sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \sum_{n=1}^{\infty} \frac{5}{4^n} \text{ [sum of two convergent series]} = \frac{11}{6} + \frac{5}{3} = \frac{7}{2}.$$

34. $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$ diverges because $\sum_{n=1}^{\infty} \frac{2}{n} = 2\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (If it converged, then $\frac{1}{2} \cdot 2\sum_{n=1}^{\infty} \frac{1}{n}$ would also converge by Theorem 8(i), but we know from Example 7 that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.) If the given series converges, then the difference $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right) - \sum_{n=1}^{\infty} \frac{3}{5^n}$ must converge (since $\sum_{n=1}^{\infty} \frac{3}{5^n}$ is a convergent geometric series) and equal $\sum_{n=1}^{\infty} \frac{2}{n}$, but we have just seen that $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges, so the given series must also diverge.

35. $0.\overline{2} = \frac{2}{10} + \frac{2}{10^2} + \cdots$ is a geometric series with $a = \frac{2}{10}$ and $r = \frac{1}{10}$. It converges to $\frac{a}{1-r} = \frac{2/10}{1-1/10} = \frac{2}{9}$.

36. $0.\overline{73} = \frac{73}{10^2} + \frac{73}{10^4} + \cdots = \frac{73/10^2}{1-1/10^2} = \frac{73/100}{99/100} = \frac{73}{99}$

37. $3.\overline{417} = 3 + \frac{417}{10^3} + \frac{417}{10^6} + \cdots = 3 + \frac{417/10^3}{1-1/10^3} = 3 + \frac{417}{999} = \frac{3414}{999} = \frac{1138}{333}$

38. $6.2\overline{54} = 6.2 + \frac{54}{10^3} + \frac{54}{10^5} + \cdots = 6.2 + \frac{54/10^3}{1-1/10^2} = \frac{62}{10} + \frac{54}{990} = \frac{6192}{990} = \frac{344}{55}$

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39. $0.123\overline{456} = \dfrac{123}{1000} + \dfrac{0.000456}{1-0.001} = \dfrac{123}{1000} + \dfrac{456}{999{,}000} = \dfrac{123{,}333}{999{,}000} = \dfrac{41{,}111}{333{,}000}$

40. $5.\overline{6021} = 5 + \dfrac{6021}{10^4} + \dfrac{6021}{10^8} + \cdots = 5 + \dfrac{6021/10^4}{1-1/10^4} = 5 + \dfrac{6021}{9999} = \dfrac{56{,}016}{9999} = \dfrac{6224}{1111}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$ is a geometric series with $r = \dfrac{x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x|}{3} < 1$ $\Leftrightarrow$ $|x| < 3$; that is, $-3 < x < 3$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{x/3}{1-x/3} = \dfrac{x/3}{1-x/3} \cdot \dfrac{3}{3} = \dfrac{x}{3-x}$.

42. $\sum_{n=1}^{\infty} (x-4)^n$ is a geometric series with $r = x-4$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|x-4| < 1$ $\Leftrightarrow$ $3 < x < 5$. In that case, the sum of the series is $\dfrac{x-4}{1-(x-4)} = \dfrac{x-4}{5-x}$.

43. $\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n$ is a geometric series with $r = 4x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $4|x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{4}$. In that case, the sum of the series is $\dfrac{1}{1-4x}$.

44. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$ is a geometric series with $r = \dfrac{x+3}{2}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x+3|}{2} < 1$ $\Leftrightarrow$ $|x+3| < 2$ $\Leftrightarrow$ $-5 < x < -1$. For these values of x, the sum of the series is

$$\frac{1}{1-(x+3)/2} = \frac{2}{2-(x+3)} = -\frac{2}{x+1}.$$

45. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n}$ is a geometric series with first term 1 and ratio $r = \dfrac{\cos x}{2}$, so it converges $\Leftrightarrow$ $|r| < 1$. But $|r| = \dfrac{|\cos x|}{2} \le \dfrac{1}{2}$ for all x. Thus, the series converges for all real values of x and the sum of the series is

$$\frac{1}{1-(\cos x)/2} = \frac{2}{2-\cos x}.$$

46. Because $\dfrac{1}{n} \to 0$ and ln is continuous, we have $\displaystyle\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln 1 = 0$. We now show that the series

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} [\ln(n+1) - \ln n] \text{ diverges.}$$

$s_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n) = \ln(n+1) - \ln 1 = \ln(n+1)$. As $n \to \infty$, $s_n = \ln(n+1) \to \infty$, so the series diverges.

47. After defining f, We use `convert(f,parfrac);` in Maple, `Apart` in Mathematica, or `Expand Rational` and `Simplify` in Derive to find that the general term is $\dfrac{1}{(4n+1)(4n-3)} = -\dfrac{1/4}{4n+1} + \dfrac{1/4}{4n-3}$. So the

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nth partial sum is

$$s_n = \sum_{k=1}^{n}\left(-\frac{1/4}{4k+1}+\frac{1/4}{4k-3}\right) = \frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{4k-3}-\frac{1}{4k+1}\right)$$

$$= \frac{1}{4}\left[\left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)\right] = \frac{1}{4}\left(1-\frac{1}{4n+1}\right)$$

The series converges to $\lim_{n\to\infty} s_n = \frac{1}{4}$. This can be confirmed by directly computing the sum using `sum(f,1..infinity);` (in Maple), `Sum[f,{n,1,Infinity}]` (in Mathematica), or `Calculus Sum` (from 1 to ∞) and `Simplify` (in Derive).

48. See Exercise 47 for specific CAS commands. $\dfrac{n^2+3n+1}{(n^2+n)^2} = \dfrac{1}{n^2}+\dfrac{1}{n}-\dfrac{1}{(n+1)^2}-\dfrac{1}{n+1}$. So the nth partial sum is

$$s_n = \sum_{k=1}^{n}\left(\frac{1}{k^2}+\frac{1}{k}-\frac{1}{(k+1)^2}-\frac{1}{k+1}\right)$$

$$= \left(1+1-\frac{1}{2^2}-\frac{1}{2}\right)+\left(\frac{1}{2^2}+\frac{1}{2}-\frac{1}{3^2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n^2}+\frac{1}{n}-\frac{1}{(n+1)^2}-\frac{1}{n+1}\right)$$

$$= 1+1-\frac{1}{(n+1)^2}-\frac{1}{n+1}$$

The series converges to $\lim_{n\to\infty} s_n = 2$.

49. For $n=1$, $a_1 = 0$ since $s_1 = 0$. For $n>1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)n-(n+1)(n-2)}{(n+1)n} = \frac{2}{n(n+1)}$$

Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \dfrac{1-1/n}{1+1/n} = 1$.

50. $a_1 = s_1 = 3-\frac{1}{2} = \frac{5}{2}$. For $n \neq 1$,

$$a_n = s_n - s_{n-1} = \left(3-n2^{-n}\right) - \left[3-(n-1)2^{-(n-1)}\right] = -\frac{n}{2^n}+\frac{n-1}{2^{n-1}}\cdot\frac{2}{2} = \frac{2(n-1)}{2^n}-\frac{n}{2^n} = \frac{n-2}{2^n}$$

Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(3-\dfrac{n}{2^n}\right) = 3$ because $\lim_{x\to\infty}\dfrac{x}{2^x} \overset{\text{H}}{=} \lim_{x\to\infty}\dfrac{1}{2^x\ln 2} = 0$.

51. (a) The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is

$$S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \frac{D(1-c^n)}{1-c} \text{ by (3).}$$

(b) $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\dfrac{D(1-c^n)}{1-c} = \dfrac{D}{1-c}\lim_{n\to\infty}(1-c^n) = \dfrac{D}{1-c}$ (since $0<c<1 \;\Rightarrow\; \lim_{n\to\infty} c^n = 0$)

$$= \frac{D}{s} \text{ (since } c+s=1) = kD \text{ (since } k = 1/s)$$

If $c = 0.8$, then $s = 1-c = 0.2$ and the multiplier is $k = 1/s = 5$.

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52. (a) Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$H + 2rH + 2r^2H + 2r^3H + \cdots = H\left(1 + 2r + 2r^2 + 2r^3 + \cdots\right)$$
$$= H\left[1 + 2r\left(1 + r + r^2 + \cdots\right)\right] = H\left[1 + 2r\left(\frac{1}{1-r}\right)\right] = H\left(\frac{1+r}{1-r}\right) \text{ meters}$$

(b) From Example 3 in Section 2.1, we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t = \sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}r} + 2\sqrt{\frac{2H}{g}r^2} + 2\sqrt{\frac{2H}{g}r^3} + \cdots = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r} + 2\sqrt{r}^2 + 2\sqrt{r}^3 + \cdots\right]$$
$$= \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}\left[1 + \sqrt{r} + \sqrt{r}^2 + \cdots\right]\right) = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+\sqrt{r}}{1-\sqrt{r}}$$

(c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach tbe top of its bounce, where its velocity is 0. At that point, its height is k^2h. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$: $\dfrac{d^2y}{dt^2} = -g \;\Rightarrow\; v = \dfrac{dy}{dt} = v_0 - gt \;\Rightarrow\; y = y_0 + v_0t - \frac{1}{2}gt^2$.

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
...	...	...	...	...	...

The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + \cdots = \sqrt{\frac{2H}{g}}\left(1 + 2k + 2k^2 + 2k^3 + \cdots\right)$$
$$= \sqrt{\frac{2H}{g}}\left[1 + 2k\left(1 + k + k^2 + \cdots\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2k\left(\frac{1}{1-k}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+k}{1-k}$$

Another method: We could use part (b). At the top of the bounce, the height is $k^2h = rh$, so $\sqrt{r} = k$ and the result follows from part (b).

53. $\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a = (1+c)^{-2}$ and $r = (1+c)^{-1}$, so tbe series converges when $\left|(1+c)^{-1}\right| < 1 \;\Leftrightarrow\; |1+c| > 1 \;\Leftrightarrow\; 1+c > 1$ or $1+c < -1 \;\Leftrightarrow\; c > 0$ or $c < -2$. We calculate the sum of the series and set it equal to 2: $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}} = 2 \;\Leftrightarrow\; \left(\dfrac{1}{1+c}\right)^2 = 2 - 2\left(\dfrac{1}{1+c}\right) \;\Leftrightarrow$

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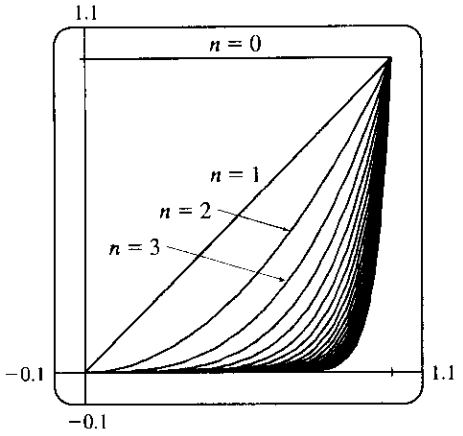
$1 = 2(1+c)^2 - 2(1+c) \quad \Leftrightarrow \quad 2c^2 + 2c - 1 = 0 \quad \Leftrightarrow \quad c = \frac{-2 \pm \sqrt{12}}{4} = \frac{\pm\sqrt{3} - 1}{2}$. However, the negative root is inadmissible because $-2 < \frac{-\sqrt{3}-1}{2} < 0$. So $c = \frac{\sqrt{3}-1}{2}$.

54. The area between $y = x^{n-1}$ and $y = x^n$ for $0 \le x \le 1$ is

$$\int_0^1 \left(x^{n-1} - x^n\right) dx = \left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{(n+1) - n}{n(n+1)} = \frac{1}{n(n+1)}$$

We can see from the diagram that as $n \to \infty$, the sum of the areas between the successive curves approaches the area of the unit square, that is, 1. So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.



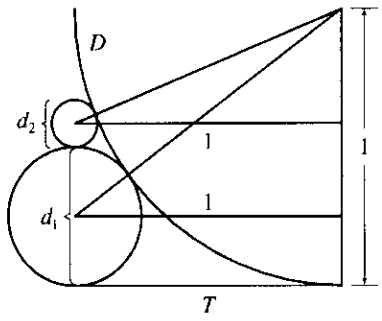
55. Let d_n be the diameter of C_n. We draw lines from the centers of the C_i to the center of D (or C), and using the Pythagorean Theorem, we can write $1^2 + \left(1 - \frac{1}{2}d_1\right)^2 = \left(1 + \frac{1}{2}d_1\right)^2 \quad \Leftrightarrow$

$1 = \left(1 + \frac{1}{2}d_1\right)^2 - \left(1 - \frac{1}{2}d_1\right)^2 = 2d_1$ (difference of squares)

$\Rightarrow \quad d_1 = \frac{1}{2}$. Similarly,

$$1 = \left(1 + \tfrac{1}{2}d_2\right)^2 - \left(1 - d_1 - \tfrac{1}{2}d_2\right)^2 = 2d_2 + 2d_1 - d_1^2 - d_1 d_2$$

$$= (2 - d_1)(d_1 + d_2) \quad \Leftrightarrow$$



$d_2 = \frac{1}{2 - d_1} - d_1 = \frac{(1 - d_1)^2}{2 - d_1}$, $1 = \left(1 + \frac{1}{2}d_3\right)^2 - \left(1 - d_1 - d_2 - \frac{1}{2}d_3\right)^2 \quad \Leftrightarrow \quad d_3 = \frac{[1 - (d_1 + d_2)]^2}{2 - (d_1 + d_2)}$, and in general, $d_{n+1} = \frac{\left(1 - \sum_{i=1}^{n} d_i\right)^2}{2 - \sum_{i=1}^{n} d_i}$. If we actually calculate d_2 and d_3 from the formulas above, we find that they are $\frac{1}{6} = \frac{1}{2 \cdot 3}$ and $\frac{1}{12} = \frac{1}{3 \cdot 4}$ respectively, so we suspect that in general, $d_n = \frac{1}{n(n+1)}$. To prove this, we use induction: Assume that for all $k \le n$, $d_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Then

$\sum_{i=1}^{n} d_i = 1 - \frac{1}{n+1} = \frac{n}{n+1}$ (telescoping sum). Substituting this into our formula for d_{n+1}, we get

$$d_{n+1} = \frac{\left[1 - \frac{n}{n+1}\right]^2}{2 - \left(\frac{n}{n+1}\right)} = \frac{\frac{1}{(n+1)^2}}{\frac{n+2}{n+1}} = \frac{1}{(n+1)(n+2)}$$, and the induction is complete.

Now, we observe that the partial sums $\sum_{i=1}^{n} d_i$ of the diameters of the circles approach 1 as $n \to \infty$; that is, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, which is what we wanted to prove.

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56. $|CD| = b\sin\theta$, $|DE| = |CD|\sin\theta = b\sin^2\theta$, $|EF| = |DE|\sin\theta = b\sin^3\theta, \ldots$. Therefore,

$|CD| + |DE| + |EF| + |FG| + \cdots = b\sum_{n=1}^{\infty}\sin^n\theta = b\left(\dfrac{\sin\theta}{1-\sin\theta}\right)$ since this is a geometric series with $r = \sin\theta$ and $|\sin\theta| < 1$ (because $0 < \theta < \frac{\pi}{2}$).

57. The series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges (geometric series with $r = -1$) so we cannot say that $0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$.

58. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim_{n\to\infty}\dfrac{1}{a_n} \neq 0$, and so $\sum_{n=1}^{\infty}\dfrac{1}{a_n}$ is divergent by the Test for Divergence.

59. $\sum_{n=1}^{\infty} ca_n = \lim_{n\to\infty}\sum_{i=1}^{n} ca_i = \lim_{n\to\infty} c\sum_{i=1}^{n} a_i = c\lim_{n\to\infty}\sum_{i=1}^{n} a_i = c\sum_{n=1}^{\infty} a_n$, which exists by hypothesis.

60. If $\sum ca_n$ were convergent, then $\sum(1/c)(ca_n) = \sum a_n$ would be also, by Theorem 8. But this is not the case, so $\sum ca_n$ must diverge.

61. Suppose on the contrary that $\sum(a_n + b_n)$ converges. Then $\sum(a_n + b_n)$ and $\sum a_n$ are convergent series. So by Theorem 8, $\sum[(a_n + b_n) - a_n]$ would also be convergent. But $\sum[(a_n + b_n) - a_n] = \sum b_n$, a contradiction, since $\sum b_n$ is given to be divergent.

62. No. For example, take $\sum a_n = \sum n$ and $\sum b_n = \sum(-n)$, which both diverge, yet $\sum(a_n + b_n) = \sum 0$, which converges with sum 0.

63. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n. Also, the sequence $\{s_n\}$ is bounded since $s_n \leq 1000$ for all n. So by Theorem 12.1.11, the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

64. (a) $\text{RHS} = \dfrac{1}{f_{n-1}f_n} - \dfrac{1}{f_nf_{n+1}} = \dfrac{f_nf_{n+1} - f_nf_{n-1}}{f_n^2f_{n-1}f_{n+1}} = \dfrac{f_{n+1} - f_{n-1}}{f_nf_{n-1}f_{n+1}} = \dfrac{(f_{n-1} + f_n) - f_{n-1}}{f_nf_{n-1}f_{n+1}}$

$= \dfrac{1}{f_{n-1}f_{n+1}} = \text{LHS}$

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}}\right)$ [from part (a)]

$\displaystyle= \lim_{n\to\infty}\left[\left(\frac{1}{f_1f_2} - \frac{1}{f_2f_3}\right) + \left(\frac{1}{f_2f_3} - \frac{1}{f_3f_4}\right) + \left(\frac{1}{f_3f_4} - \frac{1}{f_4f_5}\right) + \cdots + \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}}\right)\right]$

$\displaystyle= \lim_{n\to\infty}\left(\frac{1}{f_1f_2} - \frac{1}{f_nf_{n+1}}\right) = \frac{1}{f_1f_2} - 0 = \frac{1}{1\cdot 1} = 1$ because $f_n \to \infty$ as $n \to \infty$.

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(c) $\displaystyle\sum_{n=2}^{\infty}\frac{f_n}{f_{n-1}f_{n+1}}=\sum_{n=2}^{\infty}\left(\frac{f_n}{f_{n-1}f_n}-\frac{f_n}{f_nf_{n+1}}\right)$ (as above)

$$=\sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}}-\frac{1}{f_{n+1}}\right)$$

$$=\lim_{n\to\infty}\left[\left(\frac{1}{f_1}-\frac{1}{f_3}\right)+\left(\frac{1}{f_2}-\frac{1}{f_4}\right)+\left(\frac{1}{f_3}-\frac{1}{f_5}\right)+\left(\frac{1}{f_4}-\frac{1}{f_6}\right)+\cdots+\left(\frac{1}{f_{n-1}}-\frac{1}{f_{n+1}}\right)\right]$$

$$=\lim_{n\to\infty}\left(\frac{1}{f_1}+\frac{1}{f_2}-\frac{1}{f_n}-\frac{1}{f_{n+1}}\right)=1+1-0-0=2 \text{ because } f_n\to\infty \text{ as } n\to\infty.$$

65. (a) At the first step, only the interval $\left(\frac{1}{3},\frac{2}{3}\right)$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $\left(\frac{1}{9},\frac{2}{9}\right)$ and $\left(\frac{7}{9},\frac{8}{9}\right)$, which have a total length of $2\cdot\left(\frac{1}{3}\right)^2$. At the third step, we remove 2^2 intervals, each of length $\left(\frac{1}{3}\right)^3$. In general, at the nth step we remove 2^{n-1} intervals, each of length $\left(\frac{1}{3}\right)^n$, for a length of $2^{n-1}\cdot\left(\frac{1}{3}\right)^n=\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Thus, the total length of all removed intervals is $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}=\frac{1/3}{1-2/3}=1$ (geometric series with $a=\frac{1}{3}$ and $r=\frac{2}{3}$). Notice that at the nth step, the leftmost interval that is removed is $\left(\left(\frac{1}{3}\right)^n,\left(\frac{2}{3}\right)^n\right)$, so we never remove 0, and 0 is in the Cantor set. Also, the rightmost interval removed is $\left(1-\left(\frac{2}{3}\right)^n,1-\left(\frac{1}{3}\right)^n\right)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$.

(b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8\cdot\left(\frac{1}{9}\right)^2$; at the third step, $(8)^2\cdot\left(\frac{1}{9}\right)^3$. In general, the area removed at the nth step is $(8)^{n-1}\left(\frac{1}{9}\right)^n=\frac{1}{9}\left(\frac{8}{9}\right)^{n-1}$, so the total area of all removed squares is

$$\sum_{n=1}^{\infty}\frac{1}{9}\left(\frac{8}{9}\right)^{n-1}=\frac{1/9}{1-8/9}=1.$$

66. (a)

a_1	1	2	4	1	1	1000
a_2	2	3	1	4	1000	1
a_3	1.5	2.5	2.5	2.5	500.5	500.5
a_4	1.75	2.75	1.75	3.25	750.25	250.75
a_5	1.625	2.625	2.125	2.875	625.375	375.625
a_6	1.6875	2.6875	1.9375	3.0625	687.813	313.188
a_7	1.65625	2.65625	2.03125	2.96875	656.594	344.406
a_8	1.67188	2.67188	1.98438	3.01563	672.203	328.797
a_9	1.66406	2.66406	2.00781	2.99219	664.398	336.602
a_{10}	1.66797	2.66797	1.99609	3.00391	668.301	332.699
a_{11}	1.66602	2.66602	2.00195	2.99805	666.350	334.650
a_{12}	1.66699	2.66699	1.99902	3.00098	667.325	333.675

The limits seem to be $\frac{5}{3}$, $\frac{8}{3}$, 2, 3, 667, and 334. Note that the limits appear to be "weighted" more toward a_2. In general, we guess that the limit is $\dfrac{a_1+2a_2}{3}$.

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23. Use the Limit Comparison Test with $a_n = \dfrac{5+2n}{(1+n^2)^2}$ and $b_n = \dfrac{1}{n^3}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3(5+2n)}{(1+n^2)^2} = \lim\limits_{n\to\infty} \dfrac{5n^3+2n^4}{(1+n^2)^2} \cdot \dfrac{1/n^4}{1/(n^2)^2} = \lim\limits_{n\to\infty} \dfrac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2 > 0$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$), the series $\sum\limits_{n=1}^{\infty} \dfrac{5+2n}{(1+n^2)^2}$ also converges.

24. If $a_n = \dfrac{n^2-5n}{n^3+n+1}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3-5n^2}{n^3+n+1} = \lim\limits_{n\to\infty} \dfrac{1-5/n}{1+1/n^2+1/n^3} = 1 > 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^2-5n}{n^3+n+1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$. (Note that $a_n > 0$ for $n \geq 6$.)

25. If $a_n = \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \dfrac{1}{n}$, then

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim\limits_{n\to\infty} \dfrac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1 > 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$.

26. If $a_n = \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \dfrac{n}{\sqrt[3]{n^7}} = \dfrac{n}{n^{7/3}} = \dfrac{1}{n^{4/3}}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}} \cdot \frac{n^{-7/3}}{n^{-7/3}} = \lim_{n\to\infty} \frac{1+5/n}{\left[(n^7+n^2)/n^7\right]^{1/3}}$$

$$= \lim_{n\to\infty} \frac{1+5/n}{(1+1/n^5)^{1/3}} = \frac{1+0}{(1+0)^{1/3}} = 1 > 0,$$

so $\sum\limits_{n=1}^{\infty} \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{4/3}}$.

27. Use the Limit Comparison Test with $a_n = \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$: $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^2 = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} e^{-n} = \sum\limits_{n=1}^{\infty} \dfrac{1}{e^n}$ is a convergent geometric series $\left(|r| = \frac{1}{e} < 1\right)$, the series $\sum\limits_{n=1}^{\infty} \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ also converges.

28. Use the Limit Comparison Test with $a_n = \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ and $b_n = \dfrac{1}{3^n}$.

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2n^2+7n}{n^2+5n-1} = 2 > 0$, and since $\sum\limits_{n=1}^{\infty} b_n$ is a convergent geometric series ($|r| = \frac{1}{3} < 1$), $\sum\limits_{n=1}^{\infty} \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ converges also.

29. Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \geq 2\cdot 2\cdot 2\cdot\cdots\cdot 2\cdot 2 = 2^{n-1}$, so $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}$. $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^{n-1}}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n!}$ converges by the Comparison Test.

30. $\dfrac{n!}{n^n} = \dfrac{1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)n}{n\cdot n\cdot n\cdot\cdots\cdot n\cdot n} \leq \dfrac{1}{n}\cdot\dfrac{2}{n}\cdot 1\cdot 1\cdot\cdots\cdot 1$ for $n \geq 2$, so since $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^2}$ converges ($p = 2 > 1$), $\sum\limits_{n=1}^{\infty} \dfrac{n!}{n^n}$ converges also by the Comparison Test.

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31. Use the Limit Comparison Test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} \sin(1/n)$ also diverges. (Note that we could also use l'Hospital's Rule to evaluate the limit:

$$\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \cos\frac{1}{x} = \cos 0 = 1.)$$

32. Use the Limit Comparison Test with $a_n = \dfrac{1}{n^{1+1/n}}$ and $b_n = \dfrac{1}{n}$. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n}{n^{1+1/n}} = \lim_{n\to\infty} \dfrac{1}{n^{1/n}} = 1$ (since $\lim_{x\to\infty} x^{1/x} = 1$ by l'Hospital's Rule), so $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges (harmonic series) $\Rightarrow$ $\sum_{n=1}^{\infty} \dfrac{1}{n^{1+1/n}}$ diverges.

33. $\sum_{n=1}^{10} \dfrac{1}{n^4+n^2} = \dfrac{1}{2} + \dfrac{1}{20} + \dfrac{1}{90} + \cdots + \dfrac{1}{10{,}100} \approx 0.567975$. Now $\dfrac{1}{n^4+n^2} < \dfrac{1}{n^4}$, so using the reasoning and notation of Example 5, the error is $R_{10} \le T_{10} = \sum_{n=11}^{\infty} \dfrac{1}{n^4} \le \int_{10}^{\infty} \dfrac{dx}{x^4} = \lim_{t\to\infty} \left[-\dfrac{x^{-3}}{3}\right]_{10}^{t} = \dfrac{1}{3000} = 0.000\overline{3}$.

34. $\sum_{n=1}^{10} \dfrac{1+\cos n}{n^5} = 1 + \cos 1 + \dfrac{1+\cos 2}{32} + \dfrac{1+\cos 3}{243} + \cdots + \dfrac{1+\cos 10}{100{,}000} \approx 1.55972$. Now $\dfrac{1+\cos n}{n^5} \le \dfrac{2}{n^5}$, so as in Example 5, $R_{10} \le T_{10} \le \int_{10}^{\infty} \dfrac{2}{x^5}dx = 2\lim_{t\to\infty} \left[-\frac{1}{4}x^{-4}\right]_{10}^{t} = 0.00005$.

35. $\sum_{n=1}^{10} \dfrac{1}{1+2^n} = \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{9} + \cdots + \dfrac{1}{1025} \approx 0.76352$. Now $\dfrac{1}{1+2^n} < \dfrac{1}{2^n}$, so the error is

$R_{10} \le T_{10} = \sum_{n=11}^{\infty} \dfrac{1}{2^n} = \dfrac{1/2^{11}}{1-1/2}$ (geometric series) ≈ 0.00098.

36. $\sum_{n=1}^{10} \dfrac{n}{(n+1)3^n} = \dfrac{1}{6} + \dfrac{2}{27} + \dfrac{3}{108} + \cdots + \dfrac{10}{649{,}539} \approx 0.283597$. Now $\dfrac{n}{(n+1)3^n} < \dfrac{n}{n\cdot 3^n} = \dfrac{1}{3^n}$, so the error is

$R_{10} \le T_{10} = \sum_{n=11}^{\infty} \dfrac{1}{3^n} = \dfrac{1/3^{11}}{1-1/3} \approx 0.0000085$.

37. Since $\dfrac{d_n}{10^n} \le \dfrac{9}{10^n}$ for each n, and since $\sum_{n=1}^{\infty} \dfrac{9}{10^n}$ is a convergent geometric series ($|r| = \frac{1}{10} < 1$), $0.d_1d_2d_3\ldots = \sum_{n=1}^{\infty} \dfrac{d_n}{10^n}$ will always converge by the Comparison Test.

38. Clearly, if $p < 0$ then the series diverges, since $\lim_{n\to\infty} \dfrac{1}{n^p \ln n} = \infty$. If $0 \le p \le 1$, then $n^p \ln n \le n \ln n \Rightarrow \dfrac{1}{n^p \ln n} \ge \dfrac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges (Exercise 12.3.21), so $\sum_{n=2}^{\infty} \dfrac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison Test with $a_n = \dfrac{1}{n^p \ln n}$ and $b_n = \dfrac{1}{n^p}$. $\sum_{n=2}^{\infty} b_n$ converges, and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{1}{\ln n} = 0$, so $\sum_{n=2}^{\infty} \dfrac{1}{n^p \ln n}$ also converges. (Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.

39. Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N \Rightarrow 0 \le a_n < 1$ for all $n > N \Rightarrow 0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

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40. (a) Since $\lim_{n\to\infty}(a_n/b_n) = 0$, there is a number $N > 0$ such that $|a_n/b_n - 0| < 1$ for all $n > N$, and so $a_n < b_n$ since a_n and b_n are positive. Thus, since $\sum b_n$ converges, so does $\sum a_n$ by the Comparison Test.

(b) (i) If $a_n = \dfrac{\ln n}{n^3}$ and $b_n = \dfrac{1}{n^2}$, then $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\dfrac{\ln n}{n} = \lim_{x\to\infty}\dfrac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\dfrac{1/x}{1} = 0$, so $\sum_{n=1}^{\infty}\dfrac{\ln n}{n^3}$ converges by part (a).

(ii) If $a_n = \dfrac{\ln n}{\sqrt{n}e^n}$ and $b_n = \dfrac{1}{e^n}$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\ln n}{\sqrt{n}} = \lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0$$

Now $\sum b_n$ is a convergent geometric series with ratio $r = 1/e$ ($|r| < 1$), so $\sum a_n$ converges by part (a).

41. (a) Since $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \infty$, there is an integer N such that $\dfrac{a_n}{b_n} > 1$ whenever $n > N$. (Take $M = 1$ in Definition 12.1.5.) Then $a_n > b_n$ whenever $n > N$ and since $\sum b_n$ is divergent, $\sum a_n$ is also divergent by the Comparison Test.

(b) (i) If $a_n = \dfrac{1}{\ln n}$ and $b_n = \dfrac{1}{n}$ for $n \geq 2$, then

$\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\dfrac{n}{\ln n} = \lim_{x\to\infty}\dfrac{x}{\ln x} \overset{\text{H}}{=} \lim_{x\to\infty}\dfrac{1}{1/x} = \lim_{x\to\infty} x = \infty$, so by part (a), $\sum_{n=2}^{\infty}\dfrac{1}{\ln n}$ is divergent.

(ii) If $a_n = \dfrac{\ln n}{n}$ and $b_n = \dfrac{1}{n}$, then $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series and

$\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\ln n = \lim_{x\to\infty}\ln x = \infty$, so $\sum_{n=1}^{\infty} a_n$ diverges by part (a).

42. Let $a_n = \dfrac{1}{n^2}$ and $b_n = \dfrac{1}{n}$. Then $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\dfrac{1}{n} = 0$, but $\sum b_n$ diverges while $\sum a_n$ converges.

43. $\lim_{n\to\infty} na_n = \lim_{n\to\infty}\dfrac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n = \dfrac{1}{n}$. Since $\lim_{n\to\infty} na_n > 0$ we know that either botb series converge or both series diverge, and we also know that $\sum_{n=0}^{\infty}\dfrac{1}{n}$ diverges (p-series with $p = 1$). Therefore, $\sum a_n$ must be divergent.

44. First we observe that, by l'Hospital's Rule, $\lim_{x\to 0}\dfrac{\ln(1+x)}{x} = \lim_{x\to 0}\dfrac{1}{1+x} = 1$. Also, if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ by Theorem 12.2.6. Therefore, $\lim_{n\to\infty}\dfrac{\ln(1+a_n)}{a_n} = \lim_{x\to 0}\dfrac{\ln(1+x)}{x} = 1 > 0$. We are given that $\sum a_n$ is convergent and $a_n > 0$. Thus, $\sum \ln(1+a_n)$ is convergent by the Limit Comparison Test.

45. Yes. Since $\sum a_n$ is a convergent series with positive terms, $\lim_{n\to\infty} a_n = 0$ by Theorem 12.2.6, and $\sum b_n = \sum \sin(a_n)$ is a series with positive terms (for large enougb n). We have $\lim_{n\to\infty}\dfrac{b_n}{a_n} = \lim_{n\to\infty}\dfrac{\sin(a_n)}{a_n} = 1 > 0$ by Theorem 3.5.2. Thus, $\sum b_n$ is also convergent by the Limit Comparison Test.

46. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n \to \infty$, so for some intcger N, $a_n \leq 1$ for all $n \geq N$. But then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} a_n b_n \leq \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} b_n$. The first term is a finite sum, and the second term converges since $\sum_{n=1}^{\infty} b_n$ converges. So $\sum a_n b_n$ converges by the Comparison Test.

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12.5 Alternating Series

1. (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges if $0 < b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1}; that is, $|R_n| \le b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

2. $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots=\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+2}$. Here $a_n=(-1)^n\frac{n}{n+2}$. Since $\lim_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

3. $\frac{4}{7}-\frac{4}{8}+\frac{4}{9}-\frac{4}{10}+\frac{4}{11}-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4}{n+6}$. Now $b_n=\frac{4}{n+6}>0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

4. $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$. $b_n=\frac{1}{\ln n}$ is positive and $\{b_n\}$ is decreasing; $\lim_{n\to\infty}\frac{1}{\ln n}=0$, so the series converges by the Alternating Series Test.

5. $b_n=\frac{1}{\sqrt{n}}>0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

6. $b_n=\frac{1}{3n-1}>0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.

7. $\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}=\sum_{n=1}^{\infty}(-1)^n b_n$. Now $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{3-1/n}{2+1/n}=\frac{3}{2}\neq 0$. Since $\lim_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

8. $b_n=\frac{2n}{4n^2+1}>0$, $\{b_n\}$ is decreasing [since

$b_n - b_{n+1}=\frac{2n}{4n^2+1}-\frac{2n+2}{4n^2+8n+5}=\frac{8n^2+8n-2}{(4n^2+1)(4n^2+8n+5)}>0$ for $n\ge 1$], and

$\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{2/n}{4+1/n^2}=0$, so the series $\sum_{n=1}^{\infty}(-1)^n\frac{2n}{4n^2+1}$ converges by the Alternating Series Test.

Alternatively, to show that $\{b_n\}$ is decreasing, we could verify that $\frac{d}{dx}\left(\frac{2x}{4x^2+1}\right)<0$ for $x\ge 1$.

9. $b_n=\frac{1}{4n^2+1}>0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2+1}$ converges by the Alternating Series Test.

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10. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2+1/\sqrt{n}} = \frac{1}{2} \neq 0$. Since $\lim_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

11. $b_n = \frac{n^2}{n^3+4} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since

$\left(\frac{x^2}{x^3+4}\right)' = \frac{(x^3+4)(2x) - x^2(3x^2)}{(x^3+4)^2} = \frac{x(2x^3+8-3x^3)}{(x^3+4)^2} = \frac{x(8-x^3)}{(x^3+4)^2} < 0$ for $x > 2$. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1/n}{1+4/n^3} = 0$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ converges by the Alternating Series Test.

12. $b_n = \frac{e^{1/n}}{n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing since

$\left(\frac{e^{1/x}}{x}\right)' = \frac{x \cdot e^{1/x}(-1/x^2) - e^{1/x} \cdot 1}{x^2} = \frac{-e^{1/x}(1+x)}{x^3} < 0$ for $x > 0$. Also, $\lim_{n\to\infty} b_n = 0$ since

$\lim_{n\to\infty} e^{1/n} = 1$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{1/n}}{n}$ converges by the Alternating Series Test.

13. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$. $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{1/x} = \infty$, so the series diverges by the Test for Divergence.

14. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{\ln n}{n}\right) = 0 + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\frac{\ln n}{n}\right)$. $b_n = \frac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \frac{\ln x}{x}$, then $f'(x) = \frac{1-\ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

15. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.

16. $\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even and $(-1)^k$ if $n = 2k+1$, so the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. $b_n = \frac{1}{(2n+1)!} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.

17. $\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi}{n}$. $b_n = \sin\frac{\pi}{n} > 0$ for $n \geq 2$ and $\sin\frac{\pi}{n} \geq \sin\frac{\pi}{n+1}$, and $\lim_{n\to\infty} \sin\frac{\pi}{n} = \sin 0 = 0$, so the series converges by the Alternating Series Test.

18. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$. $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, so $\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.

19. $\frac{n^n}{n!} = \frac{n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n} \geq n \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n^n}{n!} = \infty \quad \Rightarrow \quad \lim_{n\to\infty} \frac{(-1)^n n^n}{n!}$ does not exist. So the series diverges by the Test for Divergence.

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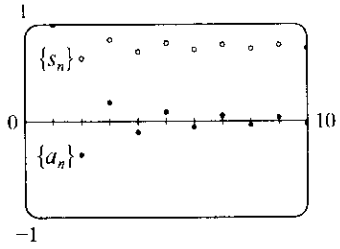
20. $\sum_{n=1}^{\infty} \left(-\frac{n}{5}\right)^n$ diverges by the Test for Divergence since $\lim_{n\to\infty} \left(\frac{n}{5}\right)^n = \infty \quad \Rightarrow \quad \lim_{n\to\infty} \left(-\frac{n}{5}\right)^n$ does not exist.

21.

n	a_n	s_n
1	1	1
2	−0.35355	0.64645
3	0.19245	0.83890
4	−0.125	0.71390
5	0.08944	0.80334
6	−0.06804	0.73530
7	0.05399	0.78929
8	−0.04419	0.74510
9	0.03704	0.78214
10	−0.03162	0.75051

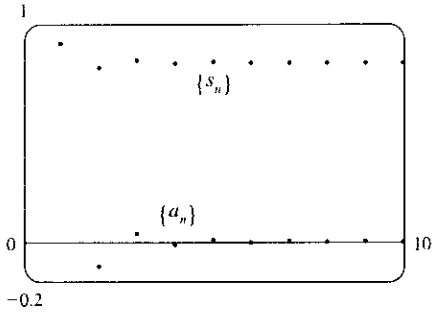


By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}} \approx 0.75051$ is

$|s - s_{10}| \le b_{11} = 1/(11)^{3/2} \approx 0.0275$ (to four decimal places, rounded up).

22.

n	a_n	s_n
1	1	1
2	−0.125	0.875
3	0.03704	0.91204
4	−0.01563	0.89641
5	0.008	0.90441
6	−0.00463	0.89978
7	0.00292	0.90270
8	−0.00195	0.90074
9	0.00137	0.90212
10	−0.001	0.90112



By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \approx 0.90112$ is

$|s - s_{10}| \le b_{11} = 1/11^3 \approx 0.0007513.$

23. The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ and

(ii) $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so the series is convergent. Now $b_{10} = \frac{1}{10^2} = 0.01$ and $b_{11} = \frac{1}{11^2} = \frac{1}{121} \approx 0.008 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 10$. (That is, since the 11th term is less than the desired error, we need to add the first 10 terms to get the sum to the desired accuracy.)

24. The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^4} < \frac{1}{n^4}$ and

(ii) $\lim_{n\to\infty} \frac{1}{n^4} = 0$, so the series is convergent. Now $b_5 = 1/5^4 = 0.0016 > 0.001$ and

$b_6 = 1/6^4 \approx 0.00077 < 0.001$, so by the Alternating Series Estimation Theorem, $n = 5$.

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25. The series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!} = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ satisfies (i) of the Alternating Series Test because $b_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2 \cdot 2^n}{(n+1)n!} = \frac{2}{n+1} \cdot \frac{2^n}{n!} = \frac{2}{n+1} \cdot b_n \le b_n$ and (ii) $\lim_{n\to\infty} \frac{2^n}{n!} = \frac{2}{n} \cdot \frac{2}{n-1} \cdot \cdots \cdot \frac{2}{2} \cdot \frac{2}{1} = 0$, so the series is convergent. Now $b_7 = 2^7/7! \approx 0.025 > 0.01$ and $b_8 = 2^8/8! \approx 0.006 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 7$. (That is, since the 8th term is less than the desired error, we need to add the first 7 terms to get the sum to the desired accuracy.)

26. The series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{4^n}$ satisfies (i) of the Alternating Series Test because $b_{n+1} = \frac{n+1}{4^{n+1}} < \frac{n+3n}{4^n \cdot 4^1} = \frac{4n}{4 \cdot 4^n} = \frac{n}{4^n} = b_n$ and (ii) $\lim_{n\to\infty} \frac{n}{4^n} = 0$, so the series is convergent. Now $b_5 = 5/4^5 \approx 0.0049 > 0.002$ and $b_6 = 6/4^6 \approx 0.0015 < 0.002$, so by the Alternating Series Estimation Theorem, $n = 5$.

27. $b_7 = \frac{1}{7^5} = \frac{1}{16{,}807} \approx 0.000\,059\,5$, so $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \approx s_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} \approx 0.972\,080$. Adding b_7 to s_6 does not change the fourth decimal place of s_6, so the sum of the series, correct to four decimal places, is 0.9721.

28. $b_6 = \frac{6}{8^6} = \frac{6}{262{,}144} \approx 0.000\,023$, so $\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n} \approx s_5 = \sum_{n=1}^{5} \frac{(-1)^n n}{8^n} = -\frac{1}{8} + \frac{2}{64} - \frac{3}{512} + \frac{4}{4096} - \frac{5}{32{,}768} \approx -0.098\,785$. Adding b_6 to s_5 does not change the fourth decimal place of s_5, so the sum of the series, correct to four decimal places, is -0.0988.

29. $b_7 = \frac{7^2}{10^7} = 0.000\,004\,9$, so $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n} \approx s_6 = \sum_{n=1}^{6} \frac{(-1)^{n-1} n^2}{10^n} = \frac{1}{10} - \frac{4}{100} + \frac{9}{1000} - \frac{16}{10{,}000} + \frac{25}{100{,}000} - \frac{36}{1{,}000{,}000} = 0.067\,614$. Adding b_7 to s_6 does not change the fourth decimal place of s_6, so the sum of the series, correct to four decimal places, is 0.0676.

30. $b_6 = \frac{1}{3^6 \cdot 6!} = \frac{1}{524{,}880} \approx 0.000\,001\,9$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!} \approx s_5 = \sum_{n=1}^{5} \frac{(-1)^n}{3^n n!} = -\frac{1}{3} + \frac{1}{18} - \frac{1}{162} + \frac{1}{1944} - \frac{1}{29{,}160} \approx -0.283\,471$. Adding b_6 to s_5 does not change the fourth decimal place of s_5, so the sum of the series, correct to four decimal places, is -0.2835.

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31. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{49} - \frac{1}{50} + \frac{1}{51} - \frac{1}{52} + \cdots$. The 50th partial sum of this series is an underestimate, since $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = s_{50} + \left(\frac{1}{51} - \frac{1}{52}\right) + \left(\frac{1}{53} - \frac{1}{54}\right) + \cdots$, and the terms in parentheses are all positive. The result can be seen geometrically in Figure 1.

32. If $p > 0$, $\frac{1}{(n+1)^p} \le \frac{1}{n^p}$ ($\{1/n^p\}$ is decreasing) and $\lim_{n\to\infty} \frac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. If $p \le 0$, $\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow$ $p > 0$.

33. Clearly $b_n = \frac{1}{n+p}$ is decreasing and eventually positive and $\lim_{n\to\infty} b_n = 0$ for any p. So the series converges (by the Alternating Series Test) for any p for which every b_n is defined, that is, $n + p \neq 0$ for $n \ge 1$, or p is not a negative integer.

34. Let $f(x) = \frac{(\ln x)^p}{x}$. Then $f'(x) = \frac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ if $x > e^p$ so f is eventually decreasing for every p. Clearly $\lim_{n\to\infty} \frac{(\ln n)^p}{n} = 0$ if $p \le 0$, and if $p > 0$ we can apply l'Hospital's Rule $[\![p+1]\!]$ times to get a limit of 0 as well. So the series converges for all p (by the Alternating Series Test).

35. $\sum b_{2n} = \sum 1/(2n)^2$ clearly converges (by comparison with the p-series for $p = 2$). So suppose that $\sum (-1)^{n-1} b_n$ converges. Then by Theorem 12.2.8(ii), so does $\sum \left[(-1)^{n-1} b_n + b_n\right] = 2\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = 2\sum \frac{1}{2n-1}$. But this diverges by comparison with the harmonic series, a contradiction. Therefore, $\sum (-1)^{n-1} b_n$ must diverge. The Alternating Series Test does not apply since $\{b_n\}$ is not decreasing.

36. (a) We will prove this by induction. Let $P(n)$ be the proposition that $s_{2n} = h_{2n} - h_n$. $P(1)$ is the statement $s_2 = h_2 - h_1$, which is true since $1 - \frac{1}{2} = \left(1 + \frac{1}{2}\right) - 1$. So suppose that $P(n)$ is true. We will show that $P(n+1)$ must be true as a consequence.

$$h_{2n+2} - h_{n+1} = \left(h_{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(h_n + \frac{1}{n+1}\right) = (h_{2n} - h_n) + \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$= s_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2} = s_{2n+2}$$

which is $P(n+1)$, and proves that $s_{2n} = h_{2n} - h_n$ for all n.

(b) We know that $h_{2n} - \ln(2n) \to \gamma$ and $h_n - \ln n \to \gamma$ as $n \to \infty$. So $s_{2n} = h_{2n} - h_n = [h_{2n} - \ln(2n)] - (h_n - \ln n) + [\ln(2n) - \ln n]$, and $\lim_{n\to\infty} s_{2n} = \gamma - \gamma + \lim_{n\to\infty} [\ln(2n) - \ln n] = \lim_{n\to\infty} (\ln 2 + \ln n - \ln n) = \ln 2$.

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12.6 Absolute Convergence and the Ratio and Root Tests

1. (a) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

2. The series $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ has positive terms and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right] = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot\frac{1}{2} = \frac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

3. $\sum_{n=0}^{\infty}\frac{(-10)^n}{n!}$. Using the Ratio Test, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-10)^n}\right| = \lim_{n\to\infty}\left|\frac{-10}{n+1}\right| = 0 < 1$, so the series is absolutely convergent.

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n\to\infty}\frac{2^n}{n^4} = \infty$, so $\lim_{n\to\infty}(-1)^{n-1}\frac{2^n}{n^4}$ does not exist.

5. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by the Alternating Series Test, but $\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n}}$ is a divergent p-series $\left(p = \frac{1}{4} \le 1\right)$, so the given series is conditionally convergent.

6. $\sum_{n=1}^{\infty}\frac{1}{n^4}$ is a convergent p-series $(p = 4 > 1)$, so $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ is absolutely convergent.

7. $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{n}{5+n} = \lim_{n\to\infty}\frac{1}{5/n+1} = 1$, so $\lim_{n\to\infty}a_n \neq 0$. Thus, the given series is divergent by the Test for Divergence.

8. $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series:

$\lim_{n\to\infty}\frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$. But $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ converges by the Alternating Series Test: $\left\{\frac{n}{n^2+1}\right\}$ has positive terms, is decreasing since $\left(\frac{x}{x^2+1}\right)' = \frac{1-x^2}{(x^2+1)^2} \le 0$ for $x \ge 1$, and $\lim_{n\to\infty}\frac{n}{n^2+1} = 0$. Thus, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$ is conditionally convergent.

9. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1/(2n+2)!}{1/(2n)!} = \lim_{n\to\infty}\frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty}\frac{(2n)!}{(2n+2)(2n+1)(2n)!}$

$= \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0 < 1$, so the series $\sum_{n=1}^{\infty}\frac{1}{(2n)!}$ is absolutely convergent by the Ratio Test. Of course, absolute convergence is the same as convergence for this series, since all of its terms are positive.

10. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!/e^{n+1}}{n!/e^n}\right| = \frac{1}{e}\lim_{n\to\infty}(n+1) = \infty$, so the series $\sum_{n=1}^{\infty}e^{-n}n!$ diverges by the Ratio Test.

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11. Since $0 \le \dfrac{e^{1/n}}{n^3} \le \dfrac{e}{n^3} = e\left(\dfrac{1}{n^3}\right)$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$), $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$ converges, and so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$ is absolutely convergent.

12. $\left|\dfrac{\sin 4n}{4^n}\right| \le \dfrac{1}{4^n}$, so $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$ ($|r| = \frac{1}{4} < 1$). Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is absolutely convergent.

13. $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{(n+1)\,3^{n+1}}{4^n} \cdot \frac{4^{n-1}}{n \cdot 3^n}\right] = \lim_{n\to\infty} \left(\frac{3}{4} \cdot \frac{n+1}{n}\right) = \frac{3}{4} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}}$ is absolutely convergent by the Ratio Test.

14. $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{(n+1)^2\,2^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2 2^n}\right] = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^2 \cdot \frac{2}{n+1}\right] = 0$, so the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$ is absolutely convergent by the Ratio Test.

15. $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{10^{n+1}}{(n+2)\,4^{2n+3}} \cdot \frac{(n+1)\,4^{2n+1}}{10^n}\right] = \lim_{n\to\infty} \left(\frac{10}{4^2} \cdot \frac{n+1}{n+2}\right) = \frac{5}{8} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

16. $n^{2/3} - 2 > 0$ for $n \ge 3$, so $\dfrac{3-\cos n}{n^{2/3}-2} > \dfrac{1}{n^{2/3}-2} > \dfrac{1}{n^{2/3}}$ for $n \ge 3$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges ($p = \frac{2}{3} \le 1$), so does $\displaystyle\sum_{n=1}^{\infty} \frac{3-\cos n}{n^{2/3}-2}$ by the Comparison Test.

17. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test since $\displaystyle\lim_{n\to\infty} \frac{1}{\ln n} = 0$ and $\left\{\dfrac{1}{\ln n}\right\}$ is decreasing. Now $\ln n < n$, so $\dfrac{1}{\ln n} > \dfrac{1}{n}$, and since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent (partial) harmonic series, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the Comparison Test. Thus, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent.

18. $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges absolutely by the Ratio Test.

19. $\dfrac{|\cos(n\pi/3)|}{n!} \le \dfrac{1}{n!}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ converges (use the Ratio Test or the result of Exercise 12.4.29), so the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ converges absolutely by the Comparison Test.

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20. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$, so the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ converges absolutely by the Root Test.

21. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n^n}{3^{1+3n}}\right)^{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{3}\cdot 3^3} = \infty$, so the series $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ is divergent by the Root Test.

Or: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{3^{4+3n}} \cdot \frac{3^{1+3n}}{n^n}\right] = \lim_{n\to\infty} \left[\frac{1}{3^3} \cdot \left(\frac{n+1}{n}\right)^n (n+1)\right]$

$= \frac{1}{27} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \lim_{n\to\infty} (n+1) = \frac{1}{27}e \lim_{n\to\infty} (n+1) = \infty,$

so the series is divergent by the Ratio Test.

22. Since $\left\{\frac{1}{n\ln n}\right\}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n\ln n} = 0$, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ converges by the Alternating Series Test. Since $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges by the Integral Test (Exercise 12.3.21), the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ is conditionally convergent.

23. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \lim_{n\to\infty} \frac{1+1/n^2}{2+1/n^2} = \frac{1}{2} < 1$, so the series $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

24. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\arctan n} = \frac{1}{\pi/2} = \frac{2}{\pi} < 1$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\arctan n)^n}$ is absolutely convergent by the Root Test.

25. Use the Ratio Test with the series $1 - \frac{1\cdot 3}{3!} + \frac{1\cdot 3\cdot 5}{5!} - \frac{1\cdot 3\cdot 5\cdot 7}{7!} + \cdots + (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n-1)!} + \cdots$

$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n-1)!}.$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n \cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)[2(n+1)-1]}{[2(n+1)-1]!} \cdot \frac{(2n-1)!}{(-1)^{n-1}\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(-1)(2n+1)(2n-1)!}{(2n+1)(2n)(2n-1)!}\right|$$

$$= \lim_{n\to\infty} \frac{1}{2n} = 0 < 1,$$

so the given series is absolutely convergent and therefore convergent.

26. Use the Ratio Test with the series $\frac{2}{5} + \frac{2\cdot 6}{5\cdot 8} + \frac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11} + \frac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14} + \cdots = \sum_{n=1}^{\infty} \frac{2\cdot 6\cdot 10\cdot 14\cdot\cdots\cdot(4n-2)}{5\cdot 8\cdot 11\cdot 14\cdot\cdots\cdot(3n+2)}.$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2\cdot 6\cdot 10\cdot\cdots\cdot(4n-2)[4(n+1)-2]}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)[3(n+1)+2]} \cdot \frac{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}{2\cdot 6\cdot 10\cdot\cdots\cdot(4n-2)}\right|$$

$$= \lim_{n\to\infty} \frac{4n+2}{3n+5} = \frac{4}{3} > 1,$$

so the given series is divergent.

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27. $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{n!} = \sum_{n=1}^{\infty} \frac{(2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdot \cdots \cdot (2 \cdot n)}{n!} = \sum_{n=1}^{\infty} \frac{2^n n!}{n!} = \sum_{n=1}^{\infty} 2^n$, which diverges by the Test for Divergence since $\lim_{n \to \infty} 2^n = \infty$.

28. $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\dfrac{2^{n+1}\,(n+1)!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+5)}}{\dfrac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)}} \right| = \lim_{n \to \infty} \frac{2(n+1)}{3n+5} = \frac{2}{3} < 1$, so the series converges absolutely by the Ratio Test.

29. By the recursive definition, $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{5n+1}{4n+3} \right| = \frac{5}{4} > 1$, so the series diverges by the Ratio Test.

30. By the recursive definition, $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{2 + \cos n}{\sqrt{n}} \right| = 0 < 1$, so the series converges absolutely by the Ratio Test.

31. (a) $\lim_{n \to \infty} \left| \frac{1/(n+1)^3}{1/n^3} \right| = \lim_{n \to \infty} \frac{n^3}{(n+1)^3} = \lim_{n \to \infty} \frac{1}{(1+1/n)^3} = 1$. Inconclusive.

(b) $\lim_{n \to \infty} \left| \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim_{n \to \infty} \frac{n+1}{2n} = \lim_{n \to \infty} \left(\frac{1}{2} + \frac{1}{2n} \right) = \frac{1}{2}$. Conclusive (convergent).

(c) $\lim_{n \to \infty} \left| \frac{(-3)^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-3)^{n-1}} \right| = 3 \lim_{n \to \infty} \sqrt{\frac{n}{n+1}} = 3 \lim_{n \to \infty} \sqrt{\frac{1}{1+1/n}} = 3$. Conclusive (divergent).

(d) $\lim_{n \to \infty} \left| \frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}} \right| = \lim_{n \to \infty} \left[\sqrt{1+\frac{1}{n}} \cdot \frac{1/n^2+1}{1/n^2+(1+1/n)^2} \right] = 1$. Inconclusive.

32. We use the Ratio Test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{[(n+1)!]^2}{[k(n+1)]!} \cdot \frac{(kn)!}{(n!)^2} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^2}{[k(n+1)]\,[k(n+1)-1] \cdots [kn+1]} \right|$$

Now if $k = 1$, then this is equal to $\lim_{n \to \infty} \left| \frac{(n+1)^2}{(n+1)} \right| = \infty$, so the series diverges; if $k = 2$, the limit is

$\lim_{n \to \infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| = \frac{1}{4} < 1$, so the series converges, and if $k > 2$, then the highest power of n in the denominator is larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.

33. (a) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \left| \frac{x}{n+1} \right| = |x| \lim_{n \to \infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1$, so by the Ratio Test the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

(b) Since the series of part (a) always converges, we must have $\lim_{n \to \infty} \frac{x^n}{n!} = 0$ by Theorem 12.2.6.

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34. (a) $R_n = a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots = a_{n+1}\left(1 + \dfrac{a_{n+2}}{a_{n+1}} + \dfrac{a_{n+3}}{a_{n+1}} + \dfrac{a_{n+4}}{a_{n+1}} + \cdots\right)$

$= a_{n+1}\left(1 + \dfrac{a_{n+2}}{a_{n+1}} + \dfrac{a_{n+3}}{a_{n+2}}\dfrac{a_{n+2}}{a_{n+1}} + \dfrac{a_{n+4}}{a_{n+3}}\dfrac{a_{n+3}}{a_{n+2}}\dfrac{a_{n+2}}{a_{n+1}} + \cdots\right)$

$= a_{n+1}(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots)$ $(\star)$

$\le a_{n+1}(1 + r_{n+1} + r_{n+1}^2 + r_{n+1}^3 + \cdots)$ [since $\{r_n\}$ is decreasing] $= \dfrac{a_{n+1}}{1 - r_{n+1}}$

(b) Note that since $\{r_n\}$ is increasing and $r_n \to L$ as $n \to \infty$, we have $r_n < L$ for all n. So, starting with equation $(\star)$,

$$R_n = a_{n+1}(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots) \le a_{n+1}(1 + L + L^2 + L^3 + \cdots) = \frac{a_{n+1}}{1 - L}$$

35. (a) $s_5 = \sum_{n=1}^{5} \dfrac{1}{n2^n} = \dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{24} + \dfrac{1}{64} + \dfrac{1}{160} = \dfrac{661}{960} \approx 0.68854$. Now the ratios

$r_n = \dfrac{a_{n+1}}{a_n} = \dfrac{n2^n}{(n+1)2^{n+1}} = \dfrac{n}{2(n+1)}$ form an increasing sequence, since

$r_{n+1} - r_n = \dfrac{n+1}{2(n+2)} - \dfrac{n}{2(n+1)} = \dfrac{(n+1)^2 - n(n+2)}{2(n+1)(n+2)} = \dfrac{1}{2(n+1)(n+2)} > 0$. So by Exercise 34(b), the error in using s_5 is $R_5 \le \dfrac{a_6}{1 - \lim\limits_{n\to\infty} r_n} = \dfrac{1/(6 \cdot 2^6)}{1 - 1/2} = \dfrac{1}{192} \approx 0.00521$.

(b) The error in using s_n as an approximation to the sum is $R_n = \dfrac{a_{n+1}}{1 - \frac{1}{2}} = \dfrac{2}{(n+1)2^{n+1}}$. We want

$R_n < 0.00005 \quad \Leftrightarrow \quad \dfrac{1}{(n+1)2^n} < 0.00005 \quad \Leftrightarrow \quad (n+1)2^n > 20{,}000$. To find such an n we can use trial and error or a graph. We calculate $(11+1)2^{11} = 24{,}576$, so $s_{11} = \sum_{n=1}^{11} \dfrac{1}{n2^n} \approx 0.693109$ is within 0.00005 of the actual sum.

36. $s_{10} = \sum_{n=1}^{10} \dfrac{n}{2^n} = \dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \cdots + \dfrac{10}{1024} \approx 1.988$. The ratios

$r_n = \dfrac{a_{n+1}}{a_n} = \dfrac{n+1}{2^{n+1}} \cdot \dfrac{2^n}{n} = \dfrac{n+1}{2n} = \dfrac{1}{2}\left(1 + \dfrac{1}{n}\right)$ form a decreasing sequence, and

$r_{11} = \dfrac{11+1}{2(11)} = \dfrac{12}{22} = \dfrac{6}{11} < 1$, so by Exercise 34(a), the error in using s_{10} to approximate the sum of the series $\sum_{n=1}^{\infty} \dfrac{n}{2^n}$ is $R_{10} \le \dfrac{a_{11}}{1 - r_{11}} = \dfrac{\frac{11}{2048}}{1 - \frac{6}{11}} = \dfrac{121}{10{,}240} \approx 0.0118$.

37. Summing the inequalities $-|a_i| \le a_i \le |a_i|$ for $i = 1, 2, \ldots, n$, we get $-\sum_{i=1}^{n} |a_i| \le \sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} |a_i|$

$\Rightarrow \quad -\lim\limits_{n\to\infty} \sum_{i=1}^{n} |a_i| \le \lim\limits_{n\to\infty} \sum_{i=1}^{n} a_i \le \lim\limits_{n\to\infty} \sum_{i=1}^{n} |a_i| \quad \Rightarrow \quad -\sum_{n=1}^{\infty} |a_n| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} |a_n| \quad \Rightarrow$

$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|$.

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38. (a) Following the hint, we get that $|a_n| < r^n$ for $n \geq N$, and so since the geometric series $\sum_{n=1}^{\infty} r^n$ converges $(0 < r < 1)$, the series $\sum_{n=N}^{\infty} |a_n|$ converges as well by the Comparison Test, and hence so does $\sum_{n=1}^{\infty} |a_n|$, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then there is an integer N such that $\sqrt[n]{|a_n|} > 1$ for all $n \geq N$, so $|a_n| > 1$ for $n \geq N$. Thus, $\lim_{n\to\infty} a_n \neq 0$, so $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

39. (a) Since $\sum a_n$ is absolutely convergent, and since $|a_n^+| \leq |a_n|$ and $|a_n^-| \leq |a_n|$ (because a_n^+ and a_n^- each equal either a_n or 0), we conclude by the Comparison Test that both $\sum a_n^+$ and $\sum a_n^-$ must be absolutely convergent. (Or use Theorem 12.2.8.)

(b) We will show by contradiction that both $\sum a_n^+$ and $\sum a_n^-$ must diverge. For suppose that $\sum a_n^+$ converged. Then so would $\sum \left(a_n^+ - \frac{1}{2}a_n\right)$ by Theorem 12.2.8. But $\sum \left(a_n^+ - \frac{1}{2}a_n\right) = \sum \left[\frac{1}{2}\left(a_n + |a_n|\right) - \frac{1}{2}a_n\right] = \frac{1}{2}\sum |a_n|$, which diverges because $\sum a_n$ is only conditionally convergent. Hence, $\sum a_n^+$ can't converge. Similarly, neither can $\sum a_n^-$.

40. Let $\sum b_n$ be the rearranged series constructed in the hint. [This series can be constructed by virtue of the result of Exercise 39(b).] This series will have partial sums s_n that oscillate in value back and forth across r. Since $\lim_{n\to\infty} a_n = 0$ (by Theorem 12.2.6), and since the size of the oscillations $|s_n - r|$ is always less than $|a_n|$ because of the way $\sum b_n$ was constructed, we have that $\sum b_n = \lim_{n\to\infty} s_n = r$.

12.7 Strategy for Testing Series

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^2-1}{n^2+1} = \lim_{n\to\infty} \dfrac{1-1/n^2}{1+1/n} = 1 \neq 0$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$ diverges by the Test for Divergence.

2. If $a_n = \dfrac{n-1}{n^2+n}$ and $b_n = \dfrac{1}{n}$, then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n^2-n}{n^2+n} = \lim_{n\to\infty} \dfrac{1-1/n}{1+1/n} = 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$ diverges by the Limit Comparison Test with the harmonic series.

3. $\dfrac{1}{n^2+n} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges by the Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p-series that converges because $p = 2 > 1$.

4. Let $b_n = \dfrac{n-1}{n^2+n}$. Then $b_1 = 0$, and $b_2 = b_3 = \frac{1}{6}$, but $b_n > b_{n+1}$ for $n \geq 3$ since $\left(\dfrac{x-1}{x^2+x}\right)' = \dfrac{(x^2+x)-(x-1)(2x+1)}{(x^2+x)^2} = \dfrac{-x^2+2x+1}{(x^2+x)^2} = \dfrac{2-(x-1)^2}{(x^2+x)^2} < 0$ for $x \geq 3$. Thus,

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$\{b_n \mid n \geq 3\}$ is decreasing and $\lim_{n\to\infty} b_n = 0$, so $\sum_{n=3}^{\infty} (-1)^{n-1} \dfrac{n-1}{n^2+n}$ converges by the Alternating Series Test.

Hence, the full series $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{n-1}{n^2+n}$ also converges.

5. $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \dfrac{2^{3n}}{(-3)^{n+1}}\right| = \lim_{n\to\infty} \left|\dfrac{-3 \cdot 2^{3n}}{2^{3n} \cdot 2^3}\right| = \lim_{n\to\infty} \dfrac{3}{2^3} = \dfrac{3}{8} < 1$, so the series

$\sum_{n=1}^{\infty} \dfrac{(-3)^{n+1}}{2^{3n}}$ is absolutely convergent by the Ratio Test.

6. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \dfrac{3n}{1+8n} = \lim_{n\to\infty} \dfrac{3}{1/n+8} = \dfrac{3}{8} < 1$, so $\sum_{n=1}^{\infty} \left(\dfrac{3n}{1+8n}\right)^n$ converges by the Root Test.

7. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2, \infty)$, so we can apply the Integral Test.

Since $\displaystyle\int \frac{1}{x\sqrt{\ln x}}\,dx \quad \left[\begin{array}{l} u = \ln x, \\ du = dx/x \end{array}\right] = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\displaystyle\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \left[2\sqrt{\ln x}\right]_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral

diverges, the given series $\sum_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$ diverges.

8. $\sum_{k=1}^{\infty} \dfrac{2^k k!}{(k+2)!} = \sum_{k=1}^{\infty} \dfrac{2^k}{(k+1)(k+2)}$. Using the Ratio Test, we get

$\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\dfrac{2^{k+1}}{(k+2)(k+3)} \cdot \dfrac{(k+1)(k+2)}{2^k}\right| = \lim_{k\to\infty} \left(2 \cdot \dfrac{k+1}{k+3}\right) = 2 > 1$, so the series diverges.

Or: Use the Test for Divergence.

9. $\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \dfrac{k^2}{e^k}$. Using the Ratio Test, we get

$\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\dfrac{(k+1)^2}{e^{k+1}} \cdot \dfrac{e^k}{k^2}\right| = \lim_{k\to\infty} \left[\left(\dfrac{k+1}{k}\right)^2 \cdot \dfrac{1}{e}\right] = 1^2 \cdot \dfrac{1}{e} = \dfrac{1}{e} < 1$, so the series converges.

10. Let $f(x) = x^2 e^{-x^3}$. Then f is continuous and positive on $[1, \infty)$, and $f'(x) = \dfrac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so f is

decreasing on $[1, \infty)$ as well, and we can apply the Integral Test. $\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{3}e^{-x^3}\right]_1^t = \frac{1}{3e}$, so the

integral converges, and hence, the series converges.

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11. $b_n = \dfrac{1}{n \ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the given series $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n+1}}{n \ln n}$ converges by the Alternating Series Test.

12. Let $b_n = \dfrac{n}{n^2+25}$. Then $b_n > 0$, $\lim\limits_{n\to\infty} b_n = 0$, and

$b_n - b_{n+1} = \dfrac{n}{n^2+25} - \dfrac{n+1}{n^2+2n+26} = \dfrac{n^2+n-25}{(n^2+25)(n^2+2n+26)}$, which is positive for $n \geq 5$, so the sequence $\{b_n\}$ decreases from $n = 5$ on. Hence, the given series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n}{n^2+25}$ converges by the Alternating Series Test.

13. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \dfrac{n!}{3^n n^2}\right| = \lim\limits_{n\to\infty} \left[\dfrac{3(n+1)^2}{(n+1)n^2}\right] = 3 \lim\limits_{n\to\infty} \dfrac{n+1}{n^2} = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{3^n n^2}{n!}$ converges by the Ratio Test.

14. The series $\sum_{n=1}^{\infty} \sin n$ diverges by the Test for Divergence since $\lim\limits_{n\to\infty} \sin n$ does not exist.

15. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)[3(n+1)+2]} \cdot \dfrac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}{n!}\right|$

$= \lim\limits_{n\to\infty} \dfrac{n+1}{3n+5} = \dfrac{1}{3} < 1$

so the series $\sum\limits_{n=0}^{\infty} \dfrac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}$ converges by the Ratio Test.

16. Using the Limit Comparison Test with $a_n = \dfrac{n^2+1}{n^3+1}$ and $b_n = \dfrac{1}{n}$, we have

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \left(\dfrac{n^2+1}{n^3+1} \cdot \dfrac{n}{1}\right) = \lim\limits_{n\to\infty} \dfrac{n^3+n}{n^3+1} = \lim\limits_{n\to\infty} \dfrac{1+1/n^2}{1+1/n^3} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

17. $\lim\limits_{n\to\infty} 2^{1/n} = 2^0 = 1$, so $\lim\limits_{n\to\infty} (-1)^n 2^{1/n}$ does not exist and the series $\sum\limits_{n=1}^{\infty} (-1)^n 2^{1/n}$ diverges by the Test for Divergence.

18. $b_n = \dfrac{1}{\sqrt{n}-1}$ for $n \geq 2$. $\{b_n\}$ is a decreasing sequence of positive numbers and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the Alternating Series Test.

19. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$. Then $f'(x) = \dfrac{2 - \ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\dfrac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{1/n}{1/(2\sqrt{n})} = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n}} = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

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20. $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{k+6}{5^{k+1}}\cdot\frac{5^k}{k+5}\right| = \frac{1}{5}\lim_{k\to\infty}\frac{k+6}{k+5} = \frac{1}{5} < 1$, so the series $\sum_{k=1}^{\infty}\frac{k+5}{5^k}$ converges by the Ratio Test.

21. $\sum_{n=1}^{\infty}\frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty}\left(\frac{4}{n}\right)^n$. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{4}{n} = 0 < 1$, so the given series is absolutely convergent by the Root Test.

22. $\frac{\sqrt{n^2-1}}{n^3+2n^2+5} < \frac{n}{n^3+2n^2+5} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \geq 1$, so $\sum_{n=1}^{\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ converges by the Comparison Test with the convergent p-series $\sum_{n=1}^{\infty}1/n^2$ $(p = 2 > 1)$.

23. Using the Limit Comparison Test with $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\tan(1/n)}{1/n} = \lim_{x\to\infty}\frac{\tan(1/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\sec^2(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty}\sec^2(1/x) = 1^2 = 1 > 0.$$

Since $\sum_{n=1}^{\infty}b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty}a_n$ is also divergent.

24. $\frac{|\cos(n/2)|}{n^2+4n} < \frac{1}{n^2+4n} < \frac{1}{n^2}$ and since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges $(p = 2 > 1)$, $\sum_{n=1}^{\infty}\frac{\cos(n/2)}{n^2+4n}$ converges absolutely by the Comparison Test.

25. Use the Ratio Test. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty}\frac{(n+1)n!\cdot e^{n^2}}{e^{n^2+2n+1}n!} = \lim_{n\to\infty}\frac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum_{n=1}^{\infty}\frac{n!}{e^{n^2}}$ converges.

26. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n^2+2n+2}{5^{n+1}}\cdot\frac{5^n}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1+2/n+2/n^2}{1+1/n^2}\cdot\frac{1}{5}\right) = \frac{1}{5} < 1$, so $\sum_{n=1}^{\infty}\frac{n^2+1}{5^n}$ converges by the Ratio Test.

27. $\int_2^{\infty}\frac{\ln x}{x^2}dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ (using integration by parts) $\overset{\text{H}}{=} 1$. So $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ converges by the Integral Test, and since $\frac{k\ln k}{(k+1)^3} < \frac{k\ln k}{k^3} = \frac{\ln k}{k^2}$, the given series $\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$ converges by the Comparison Test.

28. Since $\left\{\frac{1}{n}\right\}$ is a decreasing sequence, $e^{1/n} \leq e^{1/1} = e$ for all $n \geq 1$, and $\sum_{n=1}^{\infty}\frac{e}{n^2}$ converges $(p = 2 > 1)$, so $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$ converges by the Comparison Test. (Or use the Integral Test.)

29. $0 < \frac{\tan^{-1}n}{n^{3/2}} < \frac{\pi/2}{n^{3/2}}$. $\sum_{n=1}^{\infty}\frac{\pi/2}{n^{3/2}} = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ which is a convergent p-series $(p = \frac{3}{2} > 1)$, so $\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{n^{3/2}}$ converges by the Comparison Test.

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30. Let $f(x) = \dfrac{\sqrt{x}}{x+5}$. Then $f(x)$ is continuous and positive on $[1, \infty)$, and since $f'(x) = \dfrac{5-x}{2\sqrt{x}\,(x+5)^2} < 0$ for $x > 5$, $f(x)$ is eventually decreasing, so we can use the Alternating Series Test.

$\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n+5} = \lim\limits_{n\to\infty} \dfrac{1}{n^{1/2} + 5n^{-1/2}} = 0$, so the series $\sum\limits_{j=1}^{\infty} (-1)^j \dfrac{\sqrt{j}}{j+5}$ converges.

31. $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \dfrac{5^k}{3^k + 4^k} = [\text{divide by } 4^k] \lim\limits_{k\to\infty} \dfrac{(5/4)^k}{(3/4)^k + 1} = \infty$ since $\lim\limits_{k\to\infty} \left(\dfrac{3}{4}\right)^k = 0$ and $\lim\limits_{k\to\infty} \left(\dfrac{5}{4}\right)^k = \infty$.

Thus, $\sum\limits_{k=1}^{\infty} \dfrac{5^k}{3^k + 4^k}$ diverges by the Test for Divergence.

32. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{2n}{n^2} = \lim\limits_{n\to\infty} \dfrac{2}{n} = 0$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{(2n)^n}{n^{2n}}$ converges by the Root Test.

33. Let $a_n = \dfrac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \dfrac{1}{n\sqrt{n}}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\sin(1/n)}{1/n} = 1 > 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{\sin(1/n)}{\sqrt{n}}$ converges by limit comparison with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ $(p = 3/2 > 1)$.

34. $0 \le n\cos^2 n \le n$, so $\dfrac{1}{n + n\cos^2 n} \ge \dfrac{1}{n+n} = \dfrac{1}{2n}$. Thus, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n + n\cos^2 n}$ diverges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n}$, which is a constant multiple of the (divergent) harmonic series.

35. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^{n^2/n} = \lim\limits_{n\to\infty} \dfrac{1}{[(n+1)/n]^n} = \dfrac{1}{\lim\limits_{n\to\infty} (1+1/n)^n} = \dfrac{1}{e} < 1$, so the series $\sum\limits_{n=1}^{\infty} \left(\dfrac{n}{n+1}\right)^{n^2}$ converges by the Root Test.

36. Note that $(\ln n)^{\ln n} = \left(e^{\ln\ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln\ln n} = n^{\ln\ln n}$ and $\ln\ln n \to \infty$ as $n \to \infty$, so $\ln\ln n > 2$ for sufficiently large n. For these n we have $(\ln n)^{\ln n} > n^2$, so $\dfrac{1}{(\ln n)^{\ln n}} < \dfrac{1}{n^2}$. Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ converges $(p = 2 > 1)$, so does $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\ln n)^{\ln n}}$ by the Comparison Test.

37. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \left(2^{1/n} - 1\right) = 1 - 1 = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ converges by the Root Test.

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12. $f(x) = \dfrac{7x-1}{3x^2+2x-1} = \dfrac{7x-1}{(3x-1)(x+1)} = \dfrac{A}{3x-1} + \dfrac{B}{x+1} = \dfrac{1}{3x-1} + \dfrac{2}{x+1} = 2\cdot\dfrac{1}{1-(-x)} - \dfrac{1}{1-3x}$

$= 2\sum_{n=0}^{\infty}(-x)^n - \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}\left[2(-1)^n - 3^n\right]x^n$

The series $\sum(-x)^n$ converges for $x \in (-1,1)$ and the series $\sum(3x)^n$ converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$, so their sum converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right) = I$.

13. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$= \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(i)] $= \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R = 1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R = 1$.

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2\cdot\dfrac{1}{(1+x)^3} = x^2\cdot\dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$= \dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$. To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\dfrac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$.

14. (a) $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$ [geometric series with $R = 1$], so

$$f(x) = \ln(1+x) = \int\frac{dx}{1+x} = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$ [$C = 0$ since $f(0) = \ln 1 = 0$], with $R = 1$

(b) $f(x) = x\ln(1+x) = x\left[\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n}\right]$ [by part (a)] $= \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{n+1}}{n} = \sum_{n=2}^{\infty}\dfrac{(-1)^n x^n}{n-1}$

with $R = 1$.

(c) $f(x) = \ln(x^2+1) = \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}(x^2)^n}{n}$ [by part (a)] $= \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{2n}}{n}$ with $R = 1$.

15. $f(x) = \ln(5-x) = -\int\dfrac{dx}{5-x} = -\dfrac{1}{5}\int\dfrac{dx}{1-x/5}$

$= -\dfrac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\dfrac{x}{5}\right)^n\right]dx = C - \dfrac{1}{5}\sum_{n=0}^{\infty}\dfrac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\dfrac{x^n}{n5^n}$

Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \iff |x| < 5$, so $R = 5$.

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16. We know that $\dfrac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$. Differentiating, we get $\dfrac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1} = \sum_{n=0}^{\infty} 2^{n+1}(n+1)x^n$, so

$$f(x) = \frac{x^2}{(1-2x)^2} = \frac{x^2}{2} \cdot \frac{2}{(1-2x)^2} = \frac{x^2}{2} \sum_{n=0}^{\infty} 2^{n+1}(n+1)x^n = \sum_{n=0}^{\infty} 2^n(n+1)x^{n+2} \text{ or } \sum_{n=2}^{\infty} 2^{n-2}(n-1)x^n,$$

with $R = \frac{1}{2}$.

17. $\dfrac{1}{2-x} = \dfrac{1}{2(1-x/2)} = \dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n = \sum_{n=0}^{\infty} \dfrac{1}{2^{n+1}}x^n$ for $\left|\dfrac{x}{2}\right| < 1 \quad \Leftrightarrow \quad |x| < 2$. Now

$\dfrac{1}{(x-2)^2} = \dfrac{d}{dx}\left(\dfrac{1}{2-x}\right) = \dfrac{d}{dx}\left(\sum_{n=0}^{\infty} \dfrac{1}{2^{n+1}}x^n\right) = \sum_{n=1}^{\infty} \dfrac{n}{2^{n+1}}x^{n-1} = \sum_{n=0}^{\infty} \dfrac{n+1}{2^{n+2}}x^n$. So

$f(x) = \dfrac{x^3}{(x-2)^2} = x^3 \sum_{n=0}^{\infty} \dfrac{n+1}{2^{n+2}}x^n = \sum_{n=0}^{\infty} \dfrac{n+1}{2^{n+2}}x^{n+3}$ or $\sum_{n=3}^{\infty} \dfrac{n-2}{2^{n-1}}x^n$ for $|x| < 2$. Thus, $R = 2$ and $I = (-2, 2)$.

18. From Example 7, $g(x) = \arctan x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$. Thus,

$$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)}x^{2n+1} \text{ for } \left|\frac{x}{3}\right| < 1 \quad \Leftrightarrow \quad |x| < 3,$$

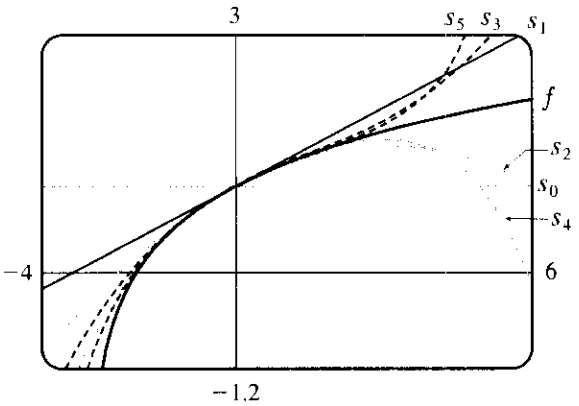
so $R = 3$.

19. $f(x) = \ln(3+x) = \displaystyle\int \frac{dx}{3+x} = \frac{1}{3}\int \frac{dx}{1+x/3} = \frac{1}{3}\int \frac{dx}{1-(-x/3)} = \frac{1}{3}\int \sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n dx$

$= C + \dfrac{1}{3}\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)3^n}x^{n+1} = \ln 3 + \frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^{n-1}}x^n \quad [C = f(0) = \ln 3]$

$= \ln 3 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^n}x^n$. The series converges when $|-x/3| < 1 \quad \Leftrightarrow \quad |x| < 3$, so $R = 3$.

The terms of the series are $a_0 = \ln 3, a_1 = \dfrac{x}{3}, a_2 = -\dfrac{x^2}{18}, a_3 = \dfrac{x^3}{81}, a_4 = -\dfrac{x^4}{324}, a_5 = \dfrac{x^5}{1215}, \ldots$.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-3, 3)$.

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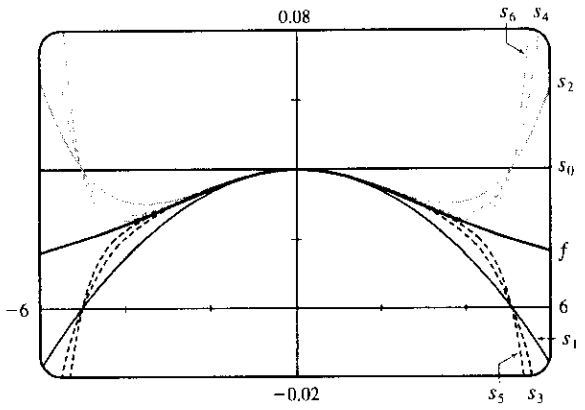
20. $f(x) = \dfrac{1}{x^2+25} = \dfrac{1}{25}\left(\dfrac{1}{1+x^2/25}\right) = \dfrac{1}{25}\left(\dfrac{1}{1-(-x^2/25)}\right) = \dfrac{1}{25}\sum_{n=0}^{\infty}\left(-\dfrac{x^2}{25}\right)^n = \dfrac{1}{25}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{x}{5}\right)^{2n}$.

The series converges when $|-x^2/25| < 1 \quad \Leftrightarrow \quad x^2 < 25 \quad \Leftrightarrow \quad |x| < 5$, so $R = 5$. The terms of the series are

$a_0 = \dfrac{1}{25}, a_1 = -\dfrac{x^2}{625}, a_2 = \dfrac{x^4}{15{,}625}, \ldots$.

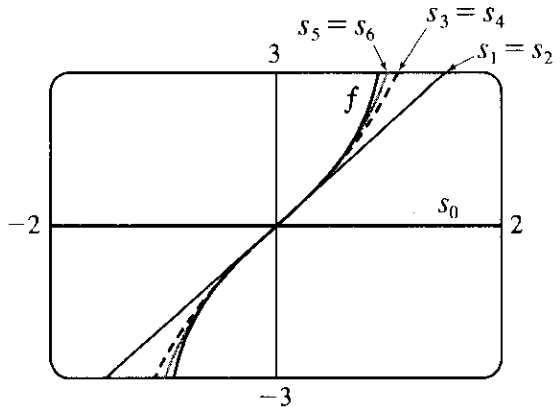


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-5, 5)$.

21.
$$f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \int \frac{dx}{1+x} + \int \frac{dx}{1-x}$$
$$= \int \frac{dx}{1-(-x)} + \int \frac{dx}{1-x} = \int \left[\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty} x^n\right] dx$$
$$= \int \left[(1 - x + x^2 - x^3 + x^4 - \cdots) + (1 + x + x^2 + x^3 + x^4 + \cdots)\right] dx$$
$$= \int \left(2 + 2x^2 + 2x^4 + \cdots\right) dx = \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$$

But $f(0) = \ln \frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then

$f(x) = \pm 2 \displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1}$, which both diverge by the Limit Comparison Test with $b_n = \dfrac{1}{n}$.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.

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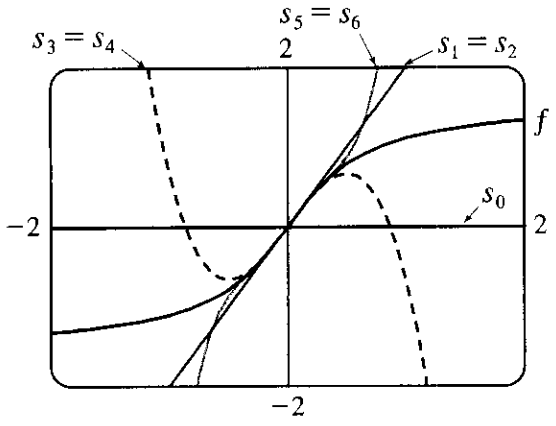
22. $f(x) = \tan^{-1}(2x) = 2\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty} (-1)^n \left(4x^2\right)^n \, dx = 2\int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}\, dx$

$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$ $[f(0) = \tan^{-1} 0 = 0, \text{ so } C = 0]$.

The series converges when $|4x^2| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm\frac{1}{2}$, then $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and $f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

23. $\frac{t}{1-t^8} = t \cdot \frac{1}{1-t^8} = t\sum_{n=0}^{\infty} (t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \quad \Rightarrow \quad \int \frac{t}{1-t^8}\, dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$. The series for $\frac{1}{1-t^8}$ converges when $|t^8| < 1 \quad \Leftrightarrow \quad |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\int \frac{t}{1-t^8}\, dt$ also has $R = 1$.

24. By Example 6, $\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$ for $|t| < 1$, so $\frac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty} \frac{t^{n-1}}{n}$ and

$\int \frac{\ln(1-t)}{t}\, dt = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}$. By Theorem 2, $R = 1$.

25. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ with $R = 1$, so

$$x - \tan^{-1} x = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

and $\frac{x - \tan^{-1} x}{x^3} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{2n+1}$, so

$$\int \frac{x - \tan^{-1} x}{x^3}\, dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{4n^2-1}.$$ By Theorem 2, $R = 1$.

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26. By Example 7, $\displaystyle\int \tan^{-1}(x^2)\,dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{2n+1}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$ with $R = 1$.

27. $\displaystyle\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty}(-x^5)^n = \sum_{n=0}^{\infty}(-1)^n x^{5n} \quad \Rightarrow$

$\displaystyle\int \frac{1}{1+x^5}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{5n}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{5n+1}}{5n+1}$. Thus,

$\displaystyle I = \int_0^{0.2} \frac{1}{1+x^5}\,dx = \left[x - \frac{x^6}{6} + \frac{x^{11}}{11} - \cdots\right]_0^{0.2} = 0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \cdots$. The series is alternating, so if we use the first two terms, the error is at most $(0.2)^{11}/11 \approx 1.9 \times 10^{-9}$. So $I \approx 0.2 - (0.2)^6/6 \approx 0.199989$ to six decimal places.

28. From Example 6 we know $\ln(1-x) = -\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}$, so

$\displaystyle\ln(1+x^4) = \ln[1-(-x^4)] = -\sum_{n=1}^{\infty}\frac{(-x^4)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n}}{n} \quad \Rightarrow$

$\displaystyle\int \ln(1+x^4)\,dx = \int \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n}}{n}\,dx = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n+1}}{n(4n+1)}$. Thus,

$\displaystyle I = \int_0^{0.4}\ln(1+x^4)\,dx = \left[\frac{x^5}{5} - \frac{x^9}{18} + \frac{x^{13}}{39} - \frac{x^{17}}{68} + \cdots\right]_0^{0.4} = \frac{(0.4)^5}{5} - \frac{(0.4)^9}{18} + \frac{(0.4)^{13}}{39} - \frac{(0.4)^{17}}{68} + \cdots$.

The series is alternating, so if we use the first three terms, the error is at most $(0.4)^{17}/68 \approx 2.5 \times 10^{-9}$. So $I \approx (0.4)^5/5 - (0.4)^9/18 + (0.9)^{13}/39 \approx 0.002034$ to six decimal places.

29. We substitute x^4 for x in Example 7, and find that

$$\int x^2 \tan^{-1}(x^4)\,dx = \int x^2 \sum_{n=0}^{\infty}(-1)^n \frac{(x^4)^{2n+1}}{2n+1}\,dx$$
$$= \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{8n+6}}{2n+1}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{8n+7}}{(2n+1)(8n+7)}$$

So $\displaystyle\int_0^{1/3} x^2\tan^{-1}(x^4)\,dx = \left[\frac{x^7}{7} - \frac{x^{15}}{45} + \cdots\right]_0^{1/3} = \frac{1}{7\cdot 3^7} - \frac{1}{45\cdot 3^{15}} + \cdots$. The series is alternating, so if we use only one term, the error is at most $1/(45\cdot 3^{15}) \approx 1.5 \times 10^{-9}$. So $\int_0^{1/3} x^2\tan^{-1}(x^4)\,dx \approx 1/(7\cdot 3^7) \approx 0.000065$ to six decimal places.

30. $\displaystyle\int_0^{0.5}\frac{dx}{1+x^6} = \int_0^{0.5}\sum_{n=0}^{\infty}(-1)^n x^{6n}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{6n+1}}{6n+1}\right]_0^{1/2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(6n+1)2^{6n+1}}$

$$= \frac{1}{2} - \frac{1}{7\cdot 2^7} + \frac{1}{13\cdot 2^{13}} - \frac{1}{19\cdot 2^{19}} + \cdots$$

The series is alternating, so if we use only three terms, the error is at most $\dfrac{1}{19\cdot 2^{19}} \approx 1.0 \times 10^{-7}$. So, to six decimal places, $\displaystyle\int_0^{0.5}\frac{dx}{1+x^6} \approx \frac{1}{2} - \frac{1}{7\cdot 2^7} + \frac{1}{13\cdot 2^{13}} \approx 0.498893$.

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31. Using the result of Example 6, $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, with $x = -0.1$, we have

$\ln 1.1 = \ln[1-(-0.1)] = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} - \cdots$. The series is alternating, so if we use only the first four terms, the error is at most $\frac{0.00001}{5} = 0.000002$. So

$\ln 1.1 \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} \approx 0.09531.$

32. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!}$ [the first term disappears], so

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1)x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \quad [\text{substituting } n+1 \text{ for } n]$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \quad \Rightarrow \quad f''(x) + f(x) = 0.$$

33. (a) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, $J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$, and $J_0''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$, so

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2^{2n-2}[(n-1)!]^2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^{-1} 2^2 n^2 x^{2n}}{2^{2n}(n!)^2}$$

$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{2n(2n-1) + 2n - 2^2 n^2}{2^{2n}(n!)^2}\right] x^{2n} = \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2 - 2n + 2n - 4n^2}{2^{2n}(n!)^2}\right] x^{2n} = 0$$

(b) $$\int_0^1 J_0(x)\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right] dx = \int_0^1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right) dx$$

$$= \left[x - \frac{x^3}{3 \cdot 4} + \frac{x^5}{5 \cdot 64} - \frac{x^7}{7 \cdot 2304} + \cdots\right]_0^1 = 1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16{,}128} + \cdots$$

Since $\frac{1}{16{,}128} \approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places, $\int_0^1 J_0(x)\,dx \approx 1 - \frac{1}{12} + \frac{1}{320} \approx 0.920$.

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34. (a) $J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$, $J_1'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)\,x^{2n}}{n!\,(n+1)!\,2^{2n+1}}$, and

$$J_1''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n)\,x^{2n-1}}{n!\,(n+1)!\,2^{2n+1}}.$$

$$\begin{aligned} x^2 J_1''(x) + x J_1'(x) + (x^2-1) J_1(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\ &\quad + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!\,(n+1)!\,2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\ &\quad - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(n-1)!\,n!\,2^{2n-1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \qquad \left[\begin{array}{c}\text{Replace } n \text{ with } n-1 \\ \text{in the third term}\end{array}\right] \\ &= \frac{x}{2} - \frac{x}{2} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{(2n+1)(2n) + (2n+1) - (n)(n+1)2^2 - 1}{n!\,(n+1)!\,2^{2n+1}}\right] x^{2n+1} = 0 \end{aligned}$$

(b) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \quad \Rightarrow$

$$\begin{aligned} J_0'(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2(n+1) x^{2n+1}}{2^{2n+2}[(n+1)!]^2} \qquad [\text{Replace } n \text{ with } n+1] \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(n+1)!\,n!} \quad [\text{cancel 2 and } n+1\text{; take } -1 \text{ outside sum}] \quad = -J_1(x) \end{aligned}$$

35. (a) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(b) By Theorem 10.4.2, the only solution to the differential equation $df(x)/dx = f(x)$ is $f(x) = Ke^x$, but $f(0) = 1$, so $K = 1$ and $f(x) = e^x$.

Or: We could solve the equation $df(x)/dx = f(x)$ as a separable differential equation.

36. $\frac{|\sin nx|}{n^2} \le \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges by the Comparison Test. $\frac{d}{dx}\left(\frac{\sin nx}{n^2}\right) = \frac{\cos nx}{n}$, so when $x = 2k\pi$ (k an integer), $\sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{\cos(2kn\pi)}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges (harmonic series). $f_n''(x) = -\sin nx$, so $\sum_{n=1}^{\infty} f_n''(x) = -\sum_{n=1}^{\infty} \sin nx$, which converges only if $\sin nx = 0$, or $x = k\pi$ (k an integer).

37. If $a_n = \frac{x^n}{n^2}$, then by the Ratio Test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| = |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = |x| < 1$ for convergence, so $R = 1$. When $x = \pm 1$, $\sum_{n=1}^{\infty} \left|\frac{x^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series ($p = 2 > 1$), so the interval of convergence for f is $[-1, 1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1, so we need

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only check the endpoints. $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2} \Rightarrow f'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n^2}=\sum_{n=0}^{\infty}\frac{x^n}{n+1}$, and this series diverges for $x=1$ (harmonic series) and converges for $x=-1$ (Alternating Series Test), so the interval of convergence is $[-1,1)$. $f''(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0$, so its interval of convergence is $(-1,1)$.

38. (a) $\sum_{n=1}^{\infty}nx^{n-1}=\sum_{n=0}^{\infty}\frac{d}{dx}x^n=\frac{d}{dx}\left[\sum_{n=0}^{\infty}x_n\right]=\frac{d}{dx}\left[\frac{1}{1-x}\right]=-\frac{1}{(1-x)^2}(-1)=\frac{1}{(1-x)^2},\ |x|<1.$

(b) (i) $\sum_{n=1}^{\infty}nx^n=x\sum_{n=1}^{\infty}nx^{n-1}=x\left[\frac{1}{(1-x)^2}\right]$ [from part (a)] $=\frac{x}{(1-x)^2}$ for $|x|<1$.

(ii) Put $x=\frac{1}{2}$ in (i): $\sum_{n=1}^{\infty}\frac{n}{2^n}=\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n=\frac{1/2}{(1-1/2)^2}=2.$

(c) (i) $\sum_{n=2}^{\infty}n(n-1)x^n=x^2\sum_{n=2}^{\infty}n(n-1)x^{n-2}=x^2\frac{d}{dx}\left[\sum_{n=1}^{\infty}nx^{n-1}\right]=x^2\frac{d}{dx}\frac{1}{(1-x)^2}$

$=x^2\frac{2}{(1-x)^3}=\frac{2x^2}{(1-x)^3}$ for $|x|<1$.

(ii) Put $x=\frac{1}{2}$ in (i): $\sum_{n=2}^{\infty}\frac{n^2-n}{2^n}=\sum_{n=2}^{\infty}n(n-1)\left(\frac{1}{2}\right)^n=\frac{2(1/2)^2}{(1-1/2)^3}=4.$

(iii) From (b)(ii) and (c)(ii), we have $\sum_{n=1}^{\infty}\frac{n^2}{2^n}=\sum_{n=1}^{\infty}\frac{n^2-n}{2^n}+\sum_{n=1}^{\infty}\frac{n}{2^n}=4+2=6.$

39. By Example 7, $\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ for $|x|<1$. In particular, for $x=\frac{1}{\sqrt{3}}$, we have

$$\frac{\pi}{6}=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\sum_{n=0}^{\infty}(-1)^n\frac{\left(1/\sqrt{3}\right)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{3}\right)^n\frac{1}{\sqrt{3}}\frac{1}{2n+1},\ \text{so}$$

$$\pi=\frac{6}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n}=2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n}.$$

40. (a) $\int_0^{1/2}\frac{dx}{x^2-x+1}=\int_0^{1/2}\frac{dx}{(x-1/2)^2+3/4}\quad\left[\begin{matrix}x-1/2=\left(\sqrt{3}/2\right)u,\ u=\left(2/\sqrt{3}\right)(x-1/2)\\ dx=\left(\sqrt{3}/2\right)du\end{matrix}\right]$

$=\int_{-1/\sqrt{3}}^{0}\frac{\left(\sqrt{3}/2\right)du}{(3/4)(u^2+1)}=\frac{2\sqrt{3}}{3}\quad\left[\tan^{-1}u\right]_{-1/\sqrt{3}}^{0}=\frac{2}{\sqrt{3}}\quad\left[0-\left(-\frac{\pi}{6}\right)\right]=\frac{\pi}{3\sqrt{3}}$

(b) $\frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}\Rightarrow\frac{1}{x^2-x+1}=(x+1)\left(\frac{1}{1+x^3}\right)=(x+1)\frac{1}{1-(-x^3)}$

$=(x+1)\sum_{n=0}^{\infty}(-1)^nx^{3n}=\sum_{n=0}^{\infty}(-1)^nx^{3n+1}+\sum_{n=0}^{\infty}(-1)^nx^{3n}$ for $|x|<1\Rightarrow\int\frac{dx}{x^2-x+1}$

$=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+2}}{3n+2}+\sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+1}}{3n+1}$ for $|x|<1\Rightarrow\int_0^{1/2}\frac{dx}{x^2-x+1}$

$=\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{4\cdot 8^n(3n+2)}+\frac{1}{2\cdot 8^n(3n+1)}\right]=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\left(\frac{2}{3n+1}+\frac{1}{3n+2}\right)$

By part (a), this equals $\frac{\pi}{3\sqrt{3}}$, so $\pi=\frac{3\sqrt{3}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\left(\frac{2}{3n+1}+\frac{1}{3n+2}\right)$.

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12.10 Taylor and Maclaurin Series

1. Using Theorem 5 with $\sum_{n=0}^{\infty} b_n(x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

2. (a) Using Formula 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $1.6 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

(b) A power series expansion of f at 2 must have the form $f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \cdots$. Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$, we must have $\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.

3.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
$\vdots$	$\vdots$	$\vdots$

We use Equation 7 with $f(x) = \cos x$.

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

If $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = x^2 \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x.$$

So $R = \infty$ (Ratio Test).

4.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-2^2 \sin 2x$	0
3	$-2^3 \cos 2x$	-2^3
4	$2^4 \sin 2x$	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(0) = 0$ if n is even and $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$, so

$$\sin 2x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^2|x|^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x,$$

so $R = \infty$ (Ratio Test).

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5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-3}$	1
1	$-3(1+x)^{-4}$	-3
2	$12(1+x)^{-5}$	12
3	$-60(1+x)^{-6}$	-60
4	$360(1+x)^{-7}$	360
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}(1+x)^{-3} &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots\\ &= 1-3x+\frac{4\cdot 3}{2!}x^2-\frac{5\cdot 4\cdot 3}{3!}x^3+\frac{6\cdot 5\cdot 4\cdot 3}{4!}x^4-\cdots\\ &= 1-3x+\frac{4\cdot 3\cdot 2}{2\cdot 2!}x^2-\frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 3!}x^3+\frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 4!}x^4-\cdots\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(n+2)!\,x^n}{2(n!)}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)(n+1)x^n}{2}\end{aligned}$$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+3)(n+2)x^{n+1}}{2}\cdot\frac{2}{(n+2)(n+1)x^n}\right|=|x|\lim_{n\to\infty}\frac{n+3}{n+1}=|x|<1$ for convergence, so $R=1$ (Ratio Test).

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\ln(1+x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3\\ &\quad+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\cdots\\ &= x-\tfrac{1}{2}x^2+\tfrac{2}{6}x^3-\tfrac{6}{24}x^4+\tfrac{24}{120}x^5-\cdots\\ &= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n\end{aligned}$$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n-1}}{n+1}\cdot\frac{n}{x^n}\right|=\lim_{n\to\infty}\frac{|x|}{1+1/n}=|x|<1$ for convergence, so $R=1$.

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	5^2e^{5x}	25
3	5^3e^{5x}	125
4	5^4e^{5x}	625
$\vdots$	$\vdots$	$\vdots$

$$e^{5x}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{5^n}{n!}x^n.$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\frac{5^{n+1}|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n|x|^n}\right]\\ &= \lim_{n\to\infty}\frac{5|x|}{n+1}=0<1 \text{ for all } x, \text{ so } R=\infty.\end{aligned}$$

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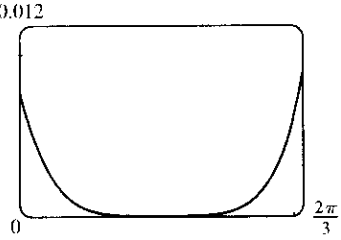

From the graph of $|R_4(x)| = |\cos x - T_4(x)|$, it seems that the error is less than 0.01 on $\left[0, \frac{2\pi}{3}\right]$.

17.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	

(a) $f(x) = \tan x \approx T_3(x) = x + \frac{1}{3}x^3$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now

$0 \le x \le \frac{\pi}{6} \quad \Rightarrow \quad x^4 \le \left(\frac{\pi}{6}\right)^4$, and letting $x = \frac{\pi}{6}$ $\left[\text{since } f^{(4)} \text{ is increasing on } \left(0, \frac{\pi}{6}\right)\right]$ gives

$$|R_3(x)| \le \frac{8\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)^3 + 16\left(\frac{2}{\sqrt{3}}\right)^4\left(\frac{1}{\sqrt{3}}\right)}{4!}\left(\frac{\pi}{6}\right)^4$$

$$= \tfrac{4\sqrt{3}}{9}\left(\tfrac{\pi}{6}\right)^4 \approx 0.057859$$

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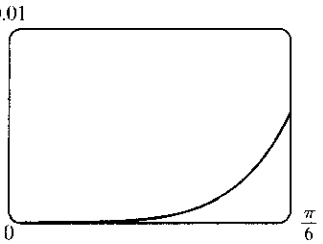

From the graph of $|R_3(x)| = |\tan x - T_3(3)|$, it seems that the error is less than 0.006 on $[0, \pi/6]$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(1+2x)$	$\ln 3$
1	$2/(1+2x)$	$\frac{2}{3}$
2	$-4/(1+2x)^2$	$-\frac{4}{9}$
3	$16/(1+2x)^3$	$\frac{16}{27}$
4	$-96/(1+2x)^4$	

(a) $f(x) = \ln(1+2x) \approx T_3(x) = \ln 3 + \frac{2}{3}(x-1) - \frac{4/9}{2!}(x-1)^2 + \frac{16/27}{3!}(x-1)^3$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0.5 \le x \le 1.5 \quad \Rightarrow \quad -0.5 \le x-1 \le 0.5 \quad \Rightarrow$ $|x-1| \le 0.5 \quad \Rightarrow \quad |x-1|^4 \le \frac{1}{16}$, and letting $x = 0.5$ gives $M = 6$, so

$$|R_3(x)| \le \frac{6}{4!} \cdot \frac{1}{16} = \frac{1}{64} = 0.015625.$$

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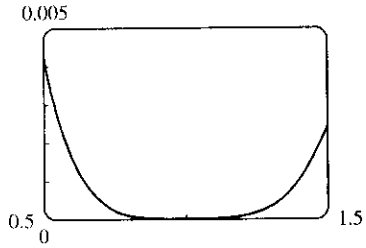

From the graph of $|R_3(x)| = |\ln(1+2x) - T_3(x)|$, it seems that the error is less than 0.005 on $[0.5, 1.5]$.

19.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{x^2}	1
1	$e^{x^2}(2x)$	0
2	$e^{x^2}(2+4x^2)$	2
3	$e^{x^2}(12x+8x^3)$	0
4	$e^{x^2}(12+48x^2+16x^4)$	

(a) $f(x) = e^{x^2} \approx T_3(x) = 1 + \frac{2}{2!}x^2 = 1 + x^2$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \le M$.

Now $0 \le x \le 0.1 \quad \Rightarrow \quad x^4 \le (0.1)^4$, and letting $x = 0.1$ gives

$|R_3(x)| \le \dfrac{e^{0.01}(12 + 0.48 + 0.0016)}{24}(0.1)^4 \approx 0.00006$.

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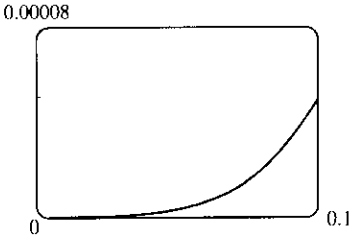

From the graph of $|R_3(x)| = \left|e^{x^2} - (1+x^2)\right|$, it appears that the error is less than 0.000051 on $[0, 0.1]$.

20.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x \ln x$	0
1	$\ln x + 1$	1
2	$1/x$	1
3	$-1/x^2$	-1
4	$2/x^3$	

(a) $f(x) = x\ln x \approx T_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$

(b) $|R_3(x)| \le \frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now

$0.5 \le x \le 1.5 \quad \Rightarrow \quad |x-1| \le \frac{1}{2} \quad \Rightarrow \quad |x-1|^4 \le \frac{1}{16}$. Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.5, 1.5]$, we can take $M = \left|f^{(4)}(0.5)\right| = 2/(0.5)^3 = 16$, so $|R_3(x)| \le \frac{16}{24}(1/16) = \frac{1}{24} = 0.041\overline{6}$.

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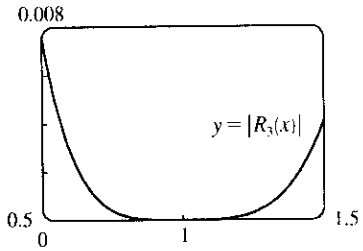
(c) From the graph of $|R_3(x)| = |x \ln x - T_3(x)|$, it seems that the error is less than 0.0076 on $[0.5, 1.5]$.



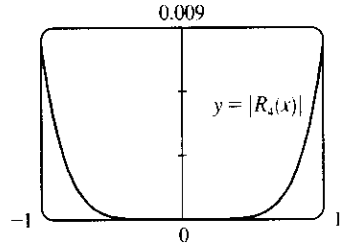
21.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x \sin x$	0
1	$\sin x + x \cos x$	0
2	$2 \cos x - x \sin x$	2
3	$-3 \sin x - x \cos x$	0
4	$-4 \cos x + x \sin x$	-4
5	$5 \sin x + x \cos x$	

(a) $f(x) = x \sin x \approx T_4(x) = \frac{2}{2!}(x-0)^2 + \frac{-4}{4!}(x-0)^4 = x^2 - \frac{1}{6}x^4$

(b) $|R_4(x)| \le \frac{M}{5!}|x|^5$, where $\left|f^{(5)}(x)\right| \le M$. Now $-1 \le x \le 1 \quad \Rightarrow \quad |x| \le 1$, and a graph of $f^{(5)}(x)$ shows that $\left|f^{(5)}(x)\right| \le 5$ for $-1 \le x \le 1$. Thus, we can take $M = 5$ and get $|R_4(x)| \le \frac{5}{5!} \cdot 1^5 = \dfrac{1}{24} = 0.041\overline{6}$.

(c) From the graph of $|R_4(x)| = |x \sin x - T_4(x)|$, it seems that the error is less than 0.0082 on $[-1, 1]$.



22.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh 2x$	0
1	$2 \cosh 2x$	2
2	$4 \sinh 2x$	0
3	$8 \cosh 2x$	8
4	$16 \sinh 2x$	0
5	$32 \cosh 2x$	32
6	$64 \sinh 2x$	

(a) $f(x) = \sinh 2x \approx T_5(x) = 2x + \frac{8}{3!}x^3 + \frac{32}{5!}x^5 = 2x + \frac{4}{3}x^3 + \frac{4}{15}x^5$

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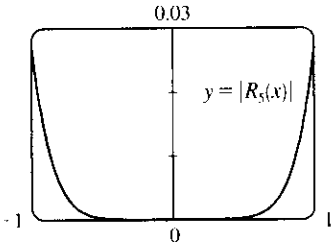
(b) $|R_5(x)| \le \frac{M}{6!}|x|^6$, where $\left|f^{(6)}(x)\right| \le M$. For x in $[-1,1]$, we have $|x| \le 1$. Since $f^{(6)}(x)$ is an increasing odd function on $[-1,1]$, we see that $\left|f^{(6)}(x)\right| \le f^{(6)}(1) = 64\sinh 2 = 32(e^2 - e^{-2}) \approx 232.119$, so we can take $M = 232.12$ and get $|R_5(x)| \le \frac{232.12}{720} \cdot 1^6 \approx 0.3224$.

(c) From the graph of $|R_5(x)| = |\sinh 2x - T_5(x)|$, it seems that the error is less than 0.027 on $[-1,1]$.



23. From Exercise 5, $\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + R_3(x)$, where $|R_3(x)| \le \frac{M}{4!}\left|x - \frac{\pi}{6}\right|^4$ with $\left|f^{(4)}(x)\right| = |\sin x| \le M = 1$. Now $x = 35^\circ = (30^\circ + 5^\circ) = \left(\frac{\pi}{6} + \frac{\pi}{36}\right)$ radians, so the error is $\left|R_3\left(\frac{\pi}{36}\right)\right| \le \frac{\left(\frac{\pi}{36}\right)^4}{4!} < 0.000003$. Therefore, to five decimal places, $\sin 35^\circ \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right) - \frac{1}{4}\left(\frac{\pi}{36}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3 \approx 0.57358$.

24. From Exercise 16, $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4 + R_4(x)$. Now since $x = 69^\circ = (60^\circ + 9^\circ) = \left(\frac{\pi}{3} + \frac{\pi}{20}\right)$ radians, the error is $|R_4(x)| \le \frac{\left(\frac{\pi}{20}\right)^5}{5!} < 8 \times 10^{-7}$. Therefore, to five decimal places, $\cos 69^\circ \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{20}\right) - \frac{1}{4}\left(\frac{\pi}{20}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{\pi}{20}\right)^3 + \frac{1}{48}\left(\frac{\pi}{20}\right)^4 \approx 0.35837$.

25. All derivatives of e^x are e^x, so $|R_n(x)| \le \frac{e^x}{(n+1)!}|x|^{n+1}$, where $0 < x < 0.1$. Letting $x = 0.1$, $R_n(0.1) \le \frac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.00001$, and by trial and error we find that $n = 3$ satisfies this inequality since $R_3(0.1) < 0.0000046$. Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n = 0, 1, 2$, and 3, we can estimate $e^{0.1}$ to within 0.00001. (In fact, this sum is $1.10516\overline{6}$ and $e^{0.1} \approx 1.10517$.)

26. Example 6 in Section 12.9 gives the Maclaurin series for $\ln(1-x)$ as $-\sum_{n=1}^{\infty}\frac{x^n}{n}$ for $|x| < 1$. Thus, $\ln 1.4 = \ln[1 - (-0.4)] = -\sum_{n=1}^{\infty}\frac{(-0.4)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.4)^n}{n}$. Since this is an alternating series, the error is less than the first neglected term by the Alternating Series Estimation Theorem, and we find that $|a_6| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (non-zero) terms of the Maclaurin series for the desired accuracy. (In fact, this sum is approximately 0.33698 and $\ln 1.4 \approx 0.33647$.)

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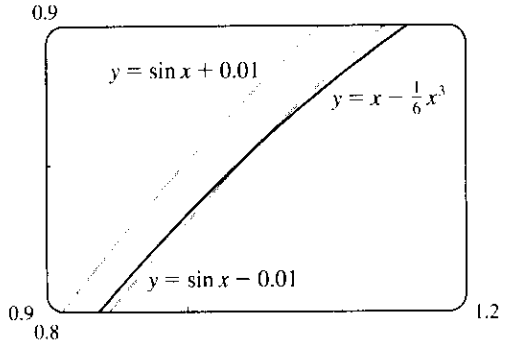
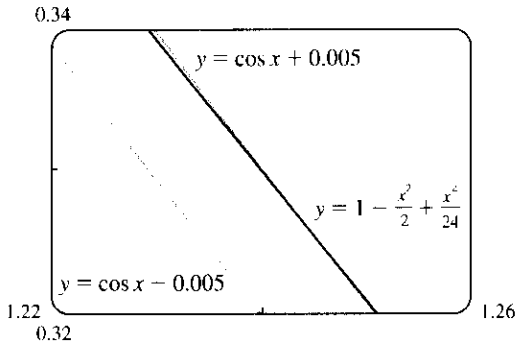
27. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$. By the Alternating Series Estimation Theorem, the error in the approximation $\sin x = x - \frac{1}{3!}x^3$ is less than $\left|\frac{1}{5!}x^5\right| < 0.01 \quad \Leftrightarrow \quad |x^5| < 120(0.01) \quad \Leftrightarrow$ $|x| < (1.2)^{1/5} \approx 1.037$. The curves $y = x - \frac{1}{6}x^3$ and $y = \sin x - 0.01$ intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and the given approximation are odd functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.037 < x < 1.037$.



28. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right| < 0.005 \quad \Leftrightarrow \quad x^6 < 720(0.005) \quad \Leftrightarrow$ $|x| < (3.6)^{1/6} \approx 1.238$. The curves $y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ and $y = \cos x + 0.005$ intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation are even functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.238 < x < 1.238$.



29. Let $s(t)$ be the position function of the car, and for convenience set $s(0) = 0$. The velocity of the car is $v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$, so the second degree Taylor polynomial is $T_2(t) = s(0) + v(0)t + \dfrac{a(0)}{2}t^2 = 20t + t^2$. We estimate the distance travelled during the next second to be $s(1) \approx T_2(1) = 20 + 1 = 21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m/s^2 for that long (if it did, its final speed would be 140 m/s $\approx$ 313 mi/h!)

30. (a)

n	$\rho^{(n)}(t)$	$\rho^{(n)}(20)$
0	$\rho_{20}e^{\alpha(t-20)}$	ρ_{20}
1	$\alpha\rho_{20}e^{\alpha(t-20)}$	$\alpha\rho_{20}$
2	$\alpha^2\rho_{20}e^{\alpha(t-20)}$	$\alpha^2\rho_{20}$

The linear approximation is $T_1(t) = \rho(20) + \rho'(20)(t-20) = \rho_{20}\,[1 + \alpha(t-20)]$. The quadratic approximation is

$$T_2(t) = \rho(20) + \rho'(20)(t-20) + \frac{\rho''(20)}{2}(t-20)^2 = \rho_{20}\left[1 + \alpha(t-20) + \tfrac{1}{2}\alpha^2(t-20)^2\right]$$

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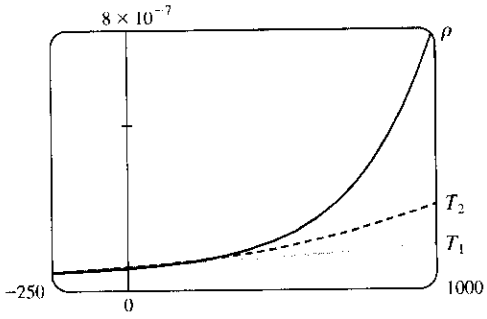
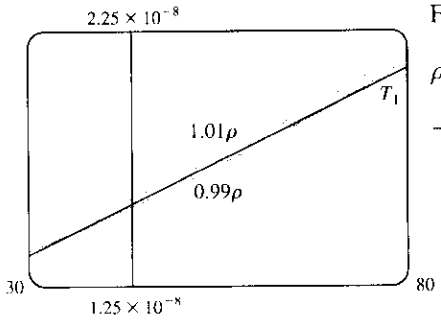
(b)



(c)



From the graph, it seems that $T_1(t)$ is within 1% of $\rho(t)$, that is, $0.99\rho(t) \le T_1(t) \le 1.01\rho(t)$, for $-14\ ^\circ\text{C} \le t \le 58\ ^\circ\text{C}$.

31. $E = \dfrac{q}{D^2} - \dfrac{q}{(D+d)^2} = \dfrac{q}{D^2} - \dfrac{q}{D^2(1+d/D)^2} = \dfrac{q}{D^2}\left[1 - \left(1 + \dfrac{d}{D}\right)^{-2}\right]$.

We use the Binomial Series to expand $(1 + d/D)^{-2}$:

$$E = \frac{q}{D^2}\left[1 - \left(1 - 2\left(\frac{d}{D}\right) + \frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2 - \frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3 + \cdots\right)\right]$$

$$= \frac{q}{D^2}\left[2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - \cdots\right] \approx \frac{q}{D^2}\cdot 2\left(\frac{d}{D}\right) = 2qd\cdot\frac{1}{D^3}$$

when D is much larger than d; that is, when P is far away from the dipole.

32. (a) $\dfrac{n_1}{\ell_o} + \dfrac{n_2}{\ell_i} = \dfrac{1}{R}\left(\dfrac{n_2 s_i}{\ell_i} - \dfrac{n_1 s_o}{\ell_o}\right)$ (Equation 1) where

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi} \quad \text{and} \quad \ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\phi} \quad (2)$$

Using $\cos\phi \approx 1$ gives

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)} = \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o - 2R^2} = \sqrt{s_o^2} = s_o$$

and similarly, $\ell_i = s_i$. Thus, Equation 1 becomes

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{1}{R}\left(\frac{n_2 s_i}{s_i} - \frac{n_1 s_o}{s_o}\right) \quad \Rightarrow \quad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

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(b) Using $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ in (2) gives us

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)(1 - \tfrac{1}{2}\phi^2)}$$

$$= \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o + Rs_o\phi^2 - 2R^2 + R^2\phi^2} = \sqrt{s_o^2 + Rs_o\phi^2 + R^2\phi^2}$$

Anticipating that we will use the binomial series expansion $(1+x)^k \approx 1 + kx$, we can write the last expression for ℓ_o as $s_o\sqrt{1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)}$ and similarly, $\ell_i = s_i\sqrt{1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)}$. Thus, from Equation 1,

$$\frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R}\left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o}\right) \quad \Leftrightarrow \quad n_1\ell_o^{-1} + n_2\ell_i^{-1} = \frac{n_2}{R}\cdot\frac{s_i}{\ell_i} - \frac{n_1}{R}\cdot\frac{s_o}{\ell_o} \quad \Leftrightarrow$$

$$\frac{n_1}{s_o}\left[1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right]^{-1/2} + \frac{n_2}{s_i}\left[1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]^{-1/2}$$

$$= \frac{n_2}{R}\left[1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]^{-1/2} - \frac{n_1}{R}\left[1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right]^{-1/2}$$

Approximating the expressions for ℓ_o^{-1} and ℓ_i^{-1} by the first two terms in their binomial series, we get

$$\frac{n_1}{s_o}\left[1 - \tfrac{1}{2}\phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right] + \frac{n_2}{s_i}\left[1 + \tfrac{1}{2}\phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]$$

$$= \frac{n_2}{R}\left[1 + \tfrac{1}{2}\phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right] - \frac{n_1}{R}\left[1 - \tfrac{1}{2}\phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right] \quad \Leftrightarrow$$

$$\frac{n_1}{s_o} - \frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) + \frac{n_2}{s_i} + \frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)$$

$$= \frac{n_2}{R} + \frac{n_2\phi^2}{2R}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right) - \frac{n_1}{R} + \frac{n_1\phi^2}{2R}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) \quad \Leftrightarrow$$

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2}{R} - \frac{n_1}{R} + \frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) + \frac{n_1\phi^2}{2R}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) + \frac{n_2\phi^2}{2R}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right) - \frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)$$

$$= \frac{n_2 - n_1}{R} + \frac{n_1\phi^2}{2}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\left(\frac{1}{s_o} + \frac{1}{R}\right) + \frac{n_2\phi^2}{2}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\left(\frac{1}{R} - \frac{1}{s_i}\right)$$

$$= \frac{n_2 - n_1}{R} + \frac{n_1\phi^2 R^2}{2s_o}\left(\frac{1}{R} + \frac{1}{s_o}\right)\left(\frac{1}{R} + \frac{1}{s_o}\right) + \frac{n_2\phi^2 R^2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)\left(\frac{1}{R} - \frac{1}{s_i}\right)$$

$$= \frac{n_2 - n_1}{R} + \phi^2 R^2\left[\frac{n_1}{2s_o}\left(\frac{1}{R} + \frac{1}{s_o}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right]$$

From Figure 8, we see that $\sin\phi = h/R$. So if we approximate $\sin\phi$ with ϕ, we get $h = R\phi$ and $h^2 = \phi^2 R^2$ and hence, Equation 4, as desired.

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33. (a) If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x \to 1$ as $x \to \infty$. So we can approximate $\tanh(2\pi d/L) \approx 1$, and so $v^2 \approx gL/(2\pi) \quad \Leftrightarrow \quad v \approx \sqrt{gL/(2\pi)}$.

(b) From the table, the first term in the Maclaurin series of $\tanh x$ is x, so if the water is shallow, we can approximate $\tanh \dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$, and so $v^2 \approx \dfrac{gL}{2\pi} \cdot \dfrac{2\pi d}{L} \quad \Leftrightarrow \quad v \approx \sqrt{gd}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tanh x$	0
1	$\operatorname{sech}^2 x$	1
2	$-2\operatorname{sech}^2 x \tanh x$	0
3	$2\operatorname{sech}^2 x\,(3\tanh^2 x - 1)$	-2

(c) Since $\tanh x$ is an odd function, its Maclaurin series is alternating, so the error in the approximation $\tanh \dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$ is less than the first neglected term, which is $\dfrac{|f'''(0)|}{3!}\left(\dfrac{2\pi d}{L}\right)^3 = \dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3$.

If $L > 10d$, then $\dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3 < \dfrac{1}{3}\left(2\pi \cdot \dfrac{1}{10}\right)^3 = \dfrac{\pi^3}{375}$, so the error in the approximation $v^2 = gd$ is less than $\dfrac{gL}{2\pi} \cdot \dfrac{\pi^3}{375} \approx 0.0132gL$.

34. (a)
$$4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \left[1+\left(-k^2\sin^2 x\right)\right]^{-1/2} dx$$
$$= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \left[1 - \frac{1}{2}\left(-k^2\sin^2 x\right) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(-k^2\sin^2 x\right)^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(-k^2\sin^2 x\right)^3 + \cdots\right] dx$$
$$= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \left[1 + \left(\frac{1}{2}\right)k^2\sin^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)k^4\sin^4 x + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)k^6\sin^6 x + \cdots\right] dx$$
$$= 4\sqrt{\frac{L}{g}}\left[\frac{\pi}{2} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\right)k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\right)k^6 + \cdots\right]$$

[split up the integral and use the result from Exercise 8.1.44]

$$= 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4 + \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6 + \cdots\right]$$

(b) The first of the two inequalities is true because all of the terms in the series are positive. For the second,

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4 + \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6 + \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}k^8 + \cdots\right]$$
$$\le 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{1}{4}k^4 + \frac{1}{4}k^6 + \frac{1}{4}k^8 + \cdots\right]$$

The terms in brackets (after the first) form a geometric series with $a = \frac{1}{4}k^2$ and $r = k^2 = \sin^2\left(\frac{1}{2}\theta_0\right) < 1$.

So $T \le 2\pi\sqrt{\dfrac{L}{g}}\left[1 + \dfrac{k^2/4}{1-k^2}\right] = 2\pi\sqrt{\dfrac{L}{g}}\,\dfrac{4-3k^2}{4-4k^2}$.

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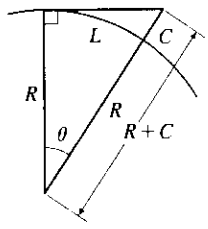
(c) We substitute $L = 1$, $g = 9.8$, and $k = \sin(10^\circ/2) \approx 0.08716$, and the inequality from part (b) becomes $2.01090 \le T \le 2.01093$, so $T \approx 2.0109$. The estimate $T \approx 2\pi\sqrt{L/g} \approx 2.0071$ differs by about 0.2%. If $\theta_0 = 42^\circ$, then $k \approx 0.35837$ and the inequality becomes $2.07153 \le T \le 2.08103$, so $T \approx 2.0763$. The one-term estimate is the same, and the discrepancy between the two estimates increases to about 3.4%.

35. (a) L is the length of the arc subtended by the angle θ, so $L = R\theta \Rightarrow \theta = L/R$. Now $\sec\theta = (R+C)/R \Rightarrow R\sec\theta = R + C \Rightarrow C = R\sec\theta - R = R\sec(L/R) - R$.



(b) From Exercise 11, $\sec x \approx T_4(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$. By part (a),

$$C \approx R\left[1 + \frac{1}{2}\left(\frac{L}{R}\right)^2 + \frac{5}{24}\left(\frac{L}{R}\right)^4\right] - R = R + \frac{1}{2}R\cdot\frac{L^2}{R^2} + \frac{5}{24}R\cdot\frac{L^4}{R^4} - R = \frac{L^2}{2R} + \frac{5L^4}{24R^3}.$$

(c) Taking $L = 100$ km and $R = 6370$ km, the formula in part (a) says that

$$C = R\sec(L/R) - R = 6370\sec(100/6370) - 6370 \approx 0.785\,009\,965\,44 \text{ km}$$

The formula in part (b) says that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3} = \frac{100^2}{2\cdot 6370} + \frac{5\cdot 100^4}{24\cdot 6370^3} \approx 0.785\,009\,957\,36 \text{ km}.$$

The difference between these two results is only $0.000\,000\,008\,08$ km, or $0.000\,008\,08$ m!

36. $T_n(x) = f(a) + \dfrac{f'(a)}{1!}(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. Let $0 \le m \le n$. Then

$$T_n^{(m)}(x) = m!\frac{f^{(m)}(a)}{m!}(x-a)^0 + (m+1)(m)\cdots(2)\frac{f^{(m+1)}(a)}{(m+1)!}(x-a)^1 + \cdots + n(n-1)\cdots(n-m+1)\frac{f^{(n)}(a)}{n!}(x-a)^{n-m}$$

For $x = a$, all terms in this sum except the first one are 0, so $T_n^{(m)}(a) = \dfrac{m!\,f^{(m)}(a)}{m!} = f^{(m)}(a)$.

37. Using $f(x) = T_n(x) + R_n(x)$ with $n = 1$ and $x = r$, we have $f(r) = T_1(r) + R_1(r)$, where T_1 is the first-degree Taylor polynomial of f at a. Because $a = x_n$, $f(r) = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. But r is a root of f, so $f(r) = 0$ and we have $0 = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. Taking the first two terms to the left side gives us $f'(x_n)(x_n - r) - f(x_n) = R_1(r)$. Dividing by $f'(x_n)$, we get $x_n - r - \dfrac{f(x_n)}{f'(x_n)} = \dfrac{R_1(r)}{f'(x_n)}$. By the formula for Newton's method, the left side of the preceding equation is $x_{n+1} - r$, so $|x_{n+1} - r| = \left|\dfrac{R_1(r)}{f'(x_n)}\right|$. Taylor's Inequality gives us $|R_1(r)| \le \dfrac{|f''(r)|}{2!}|r - x_n|^2$. Combining this inequality with the facts $|f''(x)| \le M$ and $|f'(x)| \ge K$ gives us $|x_{n+1} - r| \le \dfrac{M}{2K}|x_n - r|^2$.

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APPLIED PROJECT Radiation from the Stars

1. If we write $f(\lambda) = \dfrac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \dfrac{a\lambda^{-5}}{e^{b/(\lambda T)} - 1}$, then as $\lambda \to 0^+$, it is of the form ∞/∞, and as $\lambda \to \infty$ it is of the form $0/0$, so in either case we can use l'Hospital's Rule. First of all,

$$\lim_{\lambda\to\infty} f(\lambda) \overset{\text{H}}{=} \lim_{\lambda\to\infty} \frac{a\left(-5\lambda^{-6}\right)}{-\dfrac{bT}{(\lambda T)^2}e^{b/(\lambda T)}} = 5\frac{aT}{b}\lim_{\lambda\to\infty}\frac{\lambda^2\lambda^{-6}}{e^{b/(\lambda T)}} = 5\frac{aT}{b}\lim_{\lambda\to\infty}\frac{\lambda^{-4}}{e^{b/(\lambda T)}} = 0$$

Also,

$$\lim_{\lambda\to 0^+} f(\lambda) \overset{\text{H}}{=} 5\frac{aT}{b}\lim_{\lambda\to 0^+}\frac{\lambda^{-4}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} 5\frac{aT}{b}\lim_{\lambda\to 0^+}\frac{-4\lambda^{-5}}{-\dfrac{bT}{(\lambda T)^2}e^{b/(\lambda T)}} = 20\frac{aT^2}{b^2}\lim_{\lambda\to 0^+}\frac{\lambda^{-3}}{e^{b/(\lambda T)}}$$

This is still indeterminate, hut note that each time we use l'Hospital's Rule, we gain a factor of λ in the numerator, as well as a constant factor, and the denominator is unchanged. So if we use l'Hospital's Rule three more times, the exponent of λ in the numerator will become 0. That is, for some $\{k_i\}$, all constant,

$$\lim_{\lambda\to 0^+} f(\lambda) \overset{\text{H}}{=} k_1\lim_{\lambda\to 0^+}\frac{\lambda^{-3}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_2\lim_{\lambda\to 0^+}\frac{\lambda^{-2}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_3\lim_{\lambda\to 0^+}\frac{\lambda^{-1}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_4\lim_{\lambda\to 0^+}\frac{1}{e^{b/(\lambda T)}} = 0$$

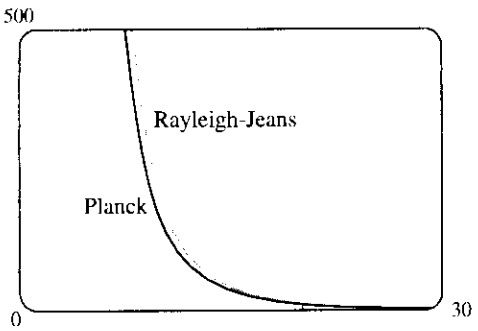
2. We expand the denominator of Planck's Law using the Taylor series $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ with $x = \dfrac{hc}{\lambda kT}$, and use the fact that if λ is large, then all subsequent terms in the Taylor expansion are very small compared to the first one, so we can approximate using the Taylor polynomial T_1:

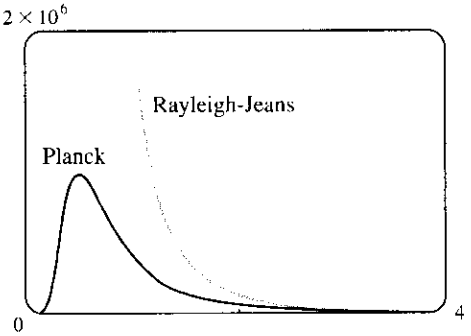
$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \frac{8\pi hc\lambda^{-5}}{\left[1 + \dfrac{hc}{\lambda kT} + \dfrac{1}{2!}\left(\dfrac{hc}{\lambda kT}\right)^2 + \dfrac{1}{3!}\left(\dfrac{hc}{\lambda kT}\right)^3 + \cdots\right] - 1}$$

$$\approx \frac{8\pi hc\lambda^{-5}}{\left(1 + \dfrac{hc}{\lambda kT}\right) - 1} = \frac{8\pi kT}{\lambda^4}$$

which is the Rayleigh-Jeans Law.

3. To convert to μm, we substitute $\lambda/10^6$ for λ in both laws. The first figure shows that the two laws are similar for large λ. The second figure shows that the two laws are very different for short wavelengths (Planck's Law gives a maximum at $\lambda \approx 0.51$ μm; the Rayleigh-Jeans Law gives no minimum or maximum.).





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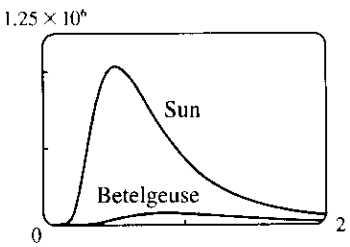
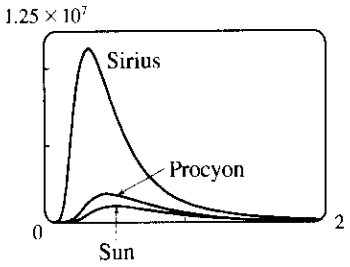
4. From the graph in Problem 3, $f(\lambda)$ has a maximum under Planck's Law at $\lambda \approx 0.51\ \mu$m.

5.





As T gets larger, the total area under the curve increases, as we would expect: the hotter the star, the more energy it emits. Also, as T increases, the λ-value of the maximum decreases, so the higher the temperature, the shorter the peak wavelength (and consequently the average wavelength) of light emitted. This is why Sirius is a blue star and Betelgeuse is a red star: most of Sirius's light is of a fairly short wavelength; that is, a higher frequency, toward the blue end of the spectrum, whereas most of Betelgeuse's light is of a lower frequency, toward the red end of the spectrum.

12 Review

CONCEPT CHECK

1. (a) See Definition 12.1.1.

(b) See Definition 12.2.2.

(c) The terms of the sequence $\{a_n\}$ approach 3 as n becomes large.

(d) By adding sufficiently many terms of the series, we can make the partial sums as close to 3 as we like.

2. (a) See Definition 12.1.10.

(b) A sequence is monotonic if it is either increasing or decreasing.

(c) By Theorem 12.1.11, every bounded, monotonic sequence is convergent.

3. (a) See (4) in Section 12.2.

(b) The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$.

4. If $\sum a_n = 3$, then $\displaystyle\lim_{n\to\infty} a_n = 0$ and $\displaystyle\lim_{n\to\infty} s_n = 3$.

5. (a) See the Test for Divergence on page 754.

(b) See the Integral Test on page 760.

(c) See the Comparison Test on page 767.

(d) See the Limit Comparison Test on page 768.

(e) See the Alternating Series Test on page 772.

(f) See the Ratio Test on page 778.

(g) See the Root Test on page 780.

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6. (a) A series $\sum a_n$ is called *absolutely convergent* if the series of absolute values $\sum |a_n|$ is convergent.

(b) If a series $\sum a_n$ is absolutely convergent, then it is convergent.

(c) A series $\sum a_n$ is called *conditionally convergent* if it is convergent but not absolutely convergent.

7. (a) Use (3) in Section 12.3.

(b) See Example 5 in Section 12.4.

(c) By adding terms until you reach the desired accuracy given by the Alternating Series Estimation Theorem on page 774.

8. (a) $\sum_{n=0}^{\infty} c_n(x-a)^n$

(b) Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(c) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (b), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers, that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a-R$ and $a+R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

9. (a), (b) See Theorem 12.9.2.

10. (a) $T_n(x) = \sum_{i=0}^{n} \dfrac{f^{(i)}(a)}{i!}(x-a)^i$

(b) $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

(c) $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!}x^n$ [$a = 0$ in part (b)]

(d) See Theorem 12.10.8.

(e) See Taylor's Inequality (12.10.9).

11. (a) – (e) See the table on page 803.

12. See the Binomial Series (12.11.2) for the expansion. The radius of convergence for the binomial series is 1.

TRUE-FALSE QUIZ

1. False. See Note 2 after Theorem 12.2.6.

2. False. The series $\sum_{n=1}^{\infty} n^{-\sin 1} = \sum_{n=1}^{\infty} \dfrac{1}{n^{\sin 1}}$ is a p-series with $p = \sin 1 \approx 0.84 \leq 1$, so the series diverges.

3. True. If $\lim_{n\to\infty} a_n = L$, then given any $\varepsilon > 0$, we can find a positive integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. If $n > N$, then $2n+1 > N$ and $|a_{2n+1} - L| < \varepsilon$. Thus, $\lim_{n\to\infty} a_{2n+1} = L$.

4. True by Theorem 12.8.3.

Or: Use the Comparison Test to show that $\sum c_n(-2)^n$ converges absolutely.

5. False. For example, take $c_n = (-1)^n / (n6^n)$.

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6. True by Theorem 12.8.3.

7. False, since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^3}\cdot\frac{n^3}{1}\right| = \lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\cdot\frac{1/n^3}{1/n^3}\right| = \lim_{n\to\infty}\frac{1}{(1+1/n)^3} = 1.$

8. True, since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)!}\cdot\frac{n!}{1}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1.$

9. False. See the note after Example 2 in Section 12.4.

10. True, since $\frac{1}{e} = e^{-1}$ and $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $e^{-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$.

11. True. See (8) in Section 12.1.

12. True, because if $\sum |a_n|$ is convergent, then so is $\sum a_n$ by Theorem 12.6.3.

13. True. By Theorem 12.10.5 the coefficient of x^3 is $\frac{f'''(0)}{3!} = \frac{1}{3} \Rightarrow f'''(0) = 2$.

Or: Use Theorem 12.9.2 to differentiate f three times.

14. False. Let $a_n = n$ and $b_n = -n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n + b_n = 0$, so $\{a_n + b_n\}$ is convergent.

15. False. For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n b_n = 1$, so $\{a_n b_n\}$ is convergent.

16. True by the Monotonic Sequence Theorem, since $\{a_n\}$ is decreasing and $0 < a_n \le a_1$ for all $n \Rightarrow \{a_n\}$ is bounded.

17. True by Theorem 12.6.3. [$\sum (-1)^n a_n$ is absolutely convergent and hence convergent.]

18. True. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n$ converges (Ratio Test) $\Rightarrow \lim_{n\to\infty} a_n = 0$ [Theorem 12.2.6].

EXERCISES

1. $\left\{\frac{2+n^3}{1+2n^3}\right\}$ converges since $\lim_{n\to\infty}\frac{2+n^3}{1+2n^3} = \lim_{n\to\infty}\frac{2/n^3+1}{1/n^3+2} = \frac{1}{2}$.

2. $a_n = \frac{9^{n+1}}{10^n} = 9\cdot\left(\frac{9}{10}\right)^n$, so $\lim_{n\to\infty} a_n = 9\lim_{n\to\infty}\left(\frac{9}{10}\right)^n = 9\cdot 0 = 0$ by (12.1.8).

3. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^3}{1+n^2} = \lim_{n\to\infty}\frac{n}{1/n^2+1} = \infty$, so the sequence diverges.

4. $a_n = \cos(n\pi/2)$, so $a_n = 0$ if n is odd and $a_n = \pm 1$ if n is even. As n increases, a_n keeps cycling through the values $0, 1, 0, -1$, so the sequence $\{a_n\}$ is divergent.

5. $|a_n| = \left|\frac{n\sin n}{n^2+1}\right| \le \frac{n}{n^2+1} < \frac{1}{n}$, so $|a_n| \to 0$ as $n \to \infty$. Thus, $\lim_{n\to\infty} a_n = 0$. The sequence $\{a_n\}$ is convergent.

6. $a_n = \frac{\ln n}{\sqrt{n}}$. Let $f(x) = \frac{\ln x}{\sqrt{x}}$ for $x > 0$. Then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0$.
Thus, by Theorem 3 in Section 12.1, $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = 0$.

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7. $\left\{\left(1+\frac{3}{n}\right)^{4n}\right\}$ is convergent. Let $y=\left(1+\frac{3}{x}\right)^{4x}$. Then

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}4x\ln(1+3/x)=\lim_{x\to\infty}\frac{\ln(1+3/x)}{1/(4x)}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\frac{1}{1+3/x}\left(-\frac{3}{x^2}\right)}{-1/(4x^2)}=\lim_{x\to\infty}\frac{12}{1+3/x}=12$$

so $\lim_{x\to\infty}y=\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{4n}=e^{12}$.

Or: Use Exercise 7.7.54.

8. $\left\{\frac{(-10)^n}{n!}\right\}$ converges, since $\frac{10^n}{n!}=\frac{10\cdot10\cdot10\cdot\cdots\cdot10}{1\cdot2\cdot3\cdot\cdots\cdot10}\cdot\frac{10\cdot10\cdot\cdots\cdot10}{11\cdot12\cdot\cdots\cdot n}\le10^{10}\left(\frac{10}{11}\right)^{n-10}\to0$ as $n\to\infty$,

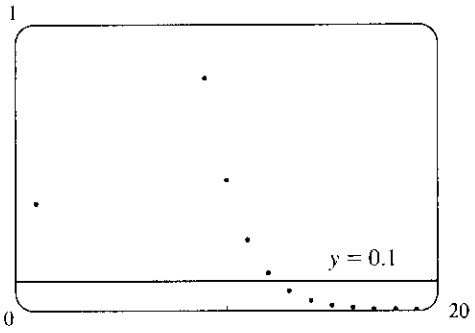
so $\lim_{n\to\infty}\frac{(-10)^n}{n!}=0$ (Squeeze Theorem). *Or:* Use (12.10.10).

9. We use induction, hypothesizing that $a_{n-1}<a_n<2$. Note first that $1<a_2=\frac{1}{3}(1+4)=\frac{5}{3}<2$, so the hypothesis holds for $n=2$. Now assume that $a_{k-1}<a_k<2$. Then $a_k=\frac{1}{3}(a_{k-1}+4)<\frac{1}{3}(a_k+4)<\frac{1}{3}(2+4)=2$. So $a_k<a_{k+1}<2$, and the induction is complete. To find the limit of the sequence, we note that $L=\lim_{n\to\infty}a_n=\lim_{n\to\infty}a_{n+1}\ \Rightarrow\ L=\frac{1}{3}(L+4)\ \Rightarrow\ L=2$.

10. $\lim_{x\to\infty}\frac{x^4}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{4x^3}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{12x^2}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{24x}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{24}{e^x}=0$

Then we conclude from Theorem 12.1.3 that $\lim_{n\to\infty}n^4e^{-n}=0$.

From the graph, it seems that $12^4e^{-12}>0.1$, hut $n^4e^{-n}<0.1$ whenever $n>12$. So the smallest value of N corresponding to $\varepsilon=0.1$ in the definition of the limit is $N=12$.



11. $\frac{n}{n^3+1}<\frac{n}{n^3}=\frac{1}{n^2}$, so $\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ converges by the Comparison Test with the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ $(p=2>1)$.

12. Let $a_n=\frac{n^2+1}{n^3+1}$ and $b_n=\frac{1}{n}$, so $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^3+n}{n^3+1}=\lim_{n\to\infty}\frac{1+1/n^2}{1+1/n^3}=1>0$. Since $\sum_{n=1}^{\infty}b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty}a_n$ also diverges by the Limit Comparison Test.

13. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{(n+1)^3}{5^{n+1}}\cdot\frac{5^n}{n^3}\right]=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3\cdot\frac{1}{5}=\frac{1}{5}<1$, so $\sum_{n=1}^{\infty}\frac{n^3}{5^n}$ converges by the Ratio Test.

14. Let $b_n=\frac{1}{\sqrt{n+1}}$. Then b_n is positive for $n\ge1$, the sequence $\{b_n\}$ is decreasing, and $\lim_{n\to\infty}b_n=0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series Test.

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15. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\int_2^\infty f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\sqrt{\ln x}}\,dx \quad \left[\begin{array}{c} u = \ln x, \\ du = \dfrac{1}{x}\,dx \end{array}\right] = \lim_{t\to\infty}\int_{\ln 2}^{\ln t} u^{-1/2}\,du = \lim_{t\to\infty}\left[2\sqrt{u}\right]_{\ln 2}^{\ln t}$$

$= \lim_{t\to\infty}(2\sqrt{\ln t} - 2\sqrt{\ln 2}) = \infty$, so the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ diverges.

16. $\lim_{n\to\infty}\dfrac{n}{3n+1} = \dfrac{1}{3}$, so $\lim_{n\to\infty}\ln\left(\dfrac{n}{3n+1}\right) = \ln\frac{1}{3} \neq 0$. Thus, the series $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n}{3n+1}\right)$ diverges by the Test for Divergence.

17. $|a_n| = \left|\dfrac{\cos 3n}{1+(1.2)^n}\right| \le \dfrac{1}{1+(1.2)^n} < \dfrac{1}{(1.2)^n} = \left(\dfrac{5}{6}\right)^n$, so $\displaystyle\sum_{n=1}^{\infty}|a_n|$ converges by comparison with the convergent geometric series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^n$ $\left(r = \frac{5}{6} < 1\right)$. It follows that $\displaystyle\sum_{n=1}^{\infty}a_n$ converges (by Theorem 3 in Section 12.6).

18. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\dfrac{n^{2n}}{(1+2n^2)^n}\right|} = \lim_{n\to\infty}\dfrac{n^2}{1+2n^2} = \lim_{n\to\infty}\dfrac{1}{1/n^2+2} = \dfrac{1}{2} < 1$, so $\displaystyle\sum_{n=1}^{\infty}\frac{n^{2n}}{(1+2n^2)^n}$ converges by the Root Test.

19. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}{5^{n+1}\,(n+1)!}\cdot\dfrac{5^n n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} = \lim_{n\to\infty}\dfrac{2n+1}{5(n+1)} = \dfrac{2}{5} < 1$, so the series converges by the Ratio Test.

20. $\displaystyle\sum_{n=1}^{\infty}\frac{(-5)^{2n}}{n^2 9^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{25}{9}\right)^n$. Now

$\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{25^{n+1}}{(n+1)^2\cdot 9^{n+1}}\cdot\dfrac{n^2\cdot 9^n}{25^n} = \lim_{n\to\infty}\dfrac{25n^2}{9(n+1)^2} = \dfrac{25}{9} > 1$, so the series diverges by the Ratio Test.

21. $b_n = \dfrac{\sqrt{n}}{n+1} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+1}$ converges by the Alternating Series Test.

22. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{n+1}-\sqrt{n-1}}{n} = \dfrac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$ (rationalizing the numerator) and $b_n = \dfrac{1}{n^{3/2}}$. $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\dfrac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}} = 1$, so since $\sum_{n=1}^{\infty}b_n$ converges $\left(p = \frac{3}{2} > 1\right)$, $\sum_{n=1}^{\infty}a_n$ converges also.

23. Consider the series of absolute values: $\sum_{n=1}^{\infty}n^{-1/3}$ is a p-series with $p = \frac{1}{3} \le 1$ and is therefore divergent. But if we apply the Alternating Series Test, we see that $b_n = \dfrac{1}{\sqrt[3]{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1/3}$ converges. Thus, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1/3}$ is conditionally convergent.

24. $\sum_{n=1}^{\infty}\left|(-1)^{n-1}n^{-3}\right| = \sum_{n=1}^{\infty}n^{-3}$ is a convergent p-series $(p = 3 > 1.)$ Therefore, $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-3}$ is absolutely convergent.

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25. $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(-1)^{n+1}(n+2)3^{n+1}}{2^{2n+3}} \cdot \dfrac{2^{2n+1}}{(-1)^n(n+1)3^n}\right| = \dfrac{n+2}{n+1}\cdot\dfrac{3}{4} = \dfrac{1+(2/n)}{1+(1/n)}\cdot\dfrac{3}{4} \to \dfrac{3}{4} < 1$ as $n \to \infty$, so by the Ratio Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)3^n}{2^{2n+1}}$ is absolutely convergent.

26. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{\ln x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/(2\sqrt{x})}{1/x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{2} = \infty$. Therefore, $\lim\limits_{n\to\infty} \dfrac{(-1)^n\sqrt{n}}{\ln n} \neq 0$, so the given series is divergent by the Test for Divergence.

27. $\dfrac{2^{2n+1}}{5^n} = \dfrac{2^{2n}\cdot 2^1}{5^n} = \dfrac{(2^2)^n\cdot 2}{5^n} = 2\left(\dfrac{4}{5}\right)^n$, so $\displaystyle\sum_{n=1}^{\infty}\frac{2^{2n+1}}{5^n} = 2\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$ is a geometric series with $a = \dfrac{8}{5}$ and $r = \dfrac{4}{5}$. Since $|r| = \dfrac{4}{5} < 1$, the series converges to $\dfrac{a}{1-r} = \dfrac{8/5}{1-4/5} = \dfrac{8/5}{1/5} = 8$.

28. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+3)} = \sum_{n=1}^{\infty}\left[\frac{1}{3n} - \frac{1}{3(n+3)}\right]$ (partial fractions).

$s_n = \displaystyle\sum_{i=1}^{n}\left[\frac{1}{3i} - \frac{1}{3(i+3)}\right] = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} - \frac{1}{3(n+1)} - \frac{1}{3(n+2)} - \frac{1}{3(n+3)}$ (telescoping sum), so

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+3)} = \lim_{n\to\infty} s_n = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}$.

29. $\sum_{n=1}^{\infty}\left[\tan^{-1}(n+1) - \tan^{-1} n\right] = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left[\left(\tan^{-1}2 - \tan^{-1}1\right) + \left(\tan^{-1}3 - \tan^{-1}2\right) + \cdots + \left(\tan^{-1}(n+1) - \tan^{-1} n\right)\right]$

$= \lim\limits_{n\to\infty}\left[\tan^{-1}(n+1) - \tan^{-1}1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

30. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^n}{3^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\cdot\frac{\pi^n}{3^{2n}} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\cdot\left(\frac{\sqrt{\pi}}{3}\right)^{2n} = \cos\left(\frac{\sqrt{\pi}}{3}\right)$ since $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ for all x.

31. $1 - e + \dfrac{e^2}{2!} - \dfrac{e^3}{3!} + \dfrac{e^4}{4!} - \cdots = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{e^n}{n!} = \sum_{n=0}^{\infty}\frac{(-e)^n}{n!} = e^{-e}$ since $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x.

32. $4.17\overline{326} = 4.17 + \dfrac{326}{10^5} + \dfrac{326}{10^8} + \cdots = 4.17 + \dfrac{326/10^5}{1-1/10^3} = \dfrac{417}{100} + \dfrac{326}{99{,}900} = \dfrac{416{,}909}{99{,}900}$

33. $\cosh x = \dfrac{1}{2}(e^x + e^{-x}) = \dfrac{1}{2}\left(\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)$

$= \dfrac{1}{2}\left[\left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots\right) + \left(1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots\right)\right]$

$= \dfrac{1}{2}\left(2 + 2\cdot\dfrac{x^2}{2!} + 2\cdot\dfrac{x^4}{4!} + \cdots\right)$

$= 1 + \dfrac{1}{2}x^2 + \displaystyle\sum_{n=2}^{\infty}\frac{x^{2n}}{(2n)!}$

$\geq 1 + \dfrac{1}{2}x^2$ for all x

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34. $\sum_{n=1}^{\infty} (\ln x)^n$ is a geometric series which converges whenever $|\ln x| < 1 \quad \Rightarrow \quad -1 < \ln x < 1 \quad \Rightarrow \quad e^{-1} < x < e$.

35. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} + \frac{1}{16{,}807} - \frac{1}{32{,}768} + \cdots$.

Since $b_8 = \frac{1}{8^5} = \frac{1}{32{,}768} < 0.000031$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \approx \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n^5} \approx 0.9721$.

36. (a) $s_5 = \sum_{n=1}^{5} \frac{1}{n^6} = 1 + \frac{1}{2^6} + \cdots + \frac{1}{5^6} \approx 1.017305$. The series $\sum_{n=1}^{\infty} \frac{1}{n^6}$ converges by the Integral Test, so we estimate the remainder R_5 with (12.3.2): $R_5 \le \int_5^{\infty} \frac{dx}{x^6} = \left[-\frac{x^{-5}}{5} \right]_5^{\infty} = \frac{5^{-5}}{5} = 0.000064$. So the error is at most 0.000064.

(b) In general, $R_n \le \int_n^{\infty} \frac{dx}{x^6} = \frac{1}{5n^5}$. If we take $n = 9$, then $s_9 \approx 1.01734$ and $R_9 \le \frac{1}{5 \cdot 9^5} \approx 3.4 \times 10^{-6}$. So to five decimal places, $\sum_{n=1}^{\infty} \frac{1}{n^5} \approx \sum_{n=1}^{9} \frac{1}{n^5} \approx 1.01734$.

Another Method: Use (12.3.3) instead of (12.3.2).

37. $\sum_{n=1}^{\infty} \frac{1}{2+5^n} \approx \sum_{n=1}^{8} \frac{1}{2+5^n} \approx 0.18976224$. To estimate the error, note that $\frac{1}{2+5^n} < \frac{1}{5^n}$, so the remainder term is $R_8 = \sum_{n=9}^{\infty} \frac{1}{2+5^n} < \sum_{n=9}^{\infty} \frac{1}{5^n} = \frac{1/5^9}{1-1/5} = 6.4 \times 10^{-7}$ (geometric series with $a = \frac{1}{5^9}$ and $r = \frac{1}{5}$).

38. (a) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{[2(n+1)]!} \cdot \frac{(2n)!}{n^n} \right| = \lim_{n\to\infty} \frac{(n+1)^n (n+1)^1}{(2n+2)(2n+1)n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n \frac{1}{2(2n+1)}$

$= \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n \frac{1}{2(2n+1)} = e \cdot 0 = 0 < 1$

so the series converges by the Ratio Test.

(b) The series in part (a) is convergent, so $\lim_{n\to\infty} a_n = 0$ by Theorem 12.2.6.

39. Use the Limit Comparison Test. $\lim_{n\to\infty} \left| \frac{\left(\frac{n+1}{n}\right) a_n}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right) = 1 > 0$.

Since $\sum |a_n|$ is convergent, so is $\sum \left| \left(\frac{n+1}{n} \right) a_n \right|$, by the Limit Comparison Test.

40. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{x^n} \right| = \lim_{n\to\infty} \frac{1}{(1+1/n)^2} \frac{|x|}{5} = \frac{|x|}{5}$, so by the Ratio Test, $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$ converges when $\frac{|x|}{5} < 1 \quad \Leftrightarrow \quad |x| < 5$. $R = 5$. When $x = -5$, the series becomes the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $p = 2 > 1$. When $x = 5$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. Thus, $I = [-5, 5]$.

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41. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x+2|^{n+1}}{(n+1)\,4^{n+1}}\cdot\frac{n4^n}{|x+2|^n}\right] = \lim_{n\to\infty}\left[\frac{n}{n+1}\,\frac{|x+2|}{4}\right] = \frac{|x+2|}{4} < 1 \quad\Leftrightarrow\quad |x+2| < 4,$

so $R = 4$. $|x+2| < 4 \quad\Leftrightarrow\quad -4 < x+2 < 4 \quad\Leftrightarrow\quad -6 < x < 2$. If $x = -6$, then the series $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n4^n}$ becomes $\sum_{n=1}^{\infty}\frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. When $x = 2$, the series becomes the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges. Thus, $I = [-6, 2)$.

42. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{2^n(x-2)^n}\right| = \lim_{n\to\infty}\frac{2}{n+3}|x-2| = 0 < 1$, so the series $\sum_{n=1}^{\infty}\frac{2^n(x-2)^n}{(n+2)!}$ converges for all x. $R = \infty$ and $I = (-\infty, \infty)$.

43. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}\cdot\frac{\sqrt{n+3}}{2^n(x-3)^n}\right| = 2|x-3|\lim_{n\to\infty}\sqrt{\frac{n+3}{n+4}} = 2|x-3| < 1 \quad\Leftrightarrow$

$|x-3| < \frac{1}{2}$, so $R = \frac{1}{2}$. $|x-3| < \frac{1}{2} \quad\Leftrightarrow\quad -\frac{1}{2} < x-3 < \frac{1}{2} \quad\Leftrightarrow\quad \frac{5}{2} < x < \frac{7}{2}$. For $x = \frac{7}{2}$, the series $\sum_{n=1}^{\infty}\frac{2^n(x-3)^n}{\sqrt{n+3}}$ becomes $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+3}} = \sum_{n=3}^{\infty}\frac{1}{n^{1/2}}$, which diverges ($p = \frac{1}{2} \le 1$), but for $x = \frac{5}{2}$, we get $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+3}}$, which is a convergent alternating series, so $I = \left[\frac{5}{2}, \frac{7}{2}\right)$.

44. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!x^{n+1}}{[(n+1)!]^2}\cdot\frac{(n!)^2}{(2n)!x^n}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)}|x| = 4|x|.$

To converge, we must have $4|x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{4}$, so $R = \frac{1}{4}$.

45.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$
4	$\sin x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\sin x &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{6}\right)}{4!}\left(x-\frac{\pi}{6}\right)^4 + \cdots \\ &= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{6}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[\left(x-\frac{\pi}{6}\right) - \frac{1}{3!}\left(x-\frac{\pi}{6}\right)^3 + \cdots\right] \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}\end{aligned}$$

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16.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\cos x = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{3}\right)}{4!}\left(x-\frac{\pi}{3}\right)^4 + \cdots$$

$$= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[-\left(x-\frac{\pi}{3}\right) + \frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3 - \cdots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x-\frac{\pi}{3}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}$$

47. $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x|<1 \quad\Rightarrow\quad \dfrac{x^2}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^{n+2}$ with $R=1$.

48. $\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$ with interval of convergence $[-1,1]$, so

$\tan^{-1}\left(x^2\right) = \sum_{n=0}^{\infty}(-1)^n\dfrac{\left(x^2\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{4n+2}}{2n+1}$, which converges when $x^2\in[-1,1] \quad\Leftrightarrow$

$x\in[-1,1]$. Therefore, $R=1$.

49. $\dfrac{1}{1-x} = \sum_{n=0}^{\infty}x^n$ for $|x|<1 \quad\Rightarrow\quad \ln(1-x) = -\int\dfrac{dx}{1-x} = -\int\sum_{n=0}^{\infty}x^n\,dx = C - \sum_{n=0}^{\infty}\dfrac{x^{n+1}}{n+1}$.

$\ln(1-0) = C-0 \quad\Rightarrow\quad C=0 \quad\Rightarrow\quad \ln(1-x) = -\sum_{n=0}^{\infty}\dfrac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\dfrac{-x^n}{n}$ with $R=1$.

50. $e^x = \sum_{n=0}^{\infty}\dfrac{x^n}{n!} \quad\Rightarrow\quad e^{2x} = \sum_{n=0}^{\infty}\dfrac{(2x)^n}{n!} \quad\Rightarrow\quad xe^{2x} = x\sum_{n=0}^{\infty}\dfrac{2^n x^n}{n!} = \sum_{n=0}^{\infty}\dfrac{2^n x^{n+1}}{n!}$, $R=\infty$

51. $\sin x = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{(2n+1)!} \quad\Rightarrow\quad \sin\left(x^4\right) = \sum_{n=0}^{\infty}\dfrac{(-1)^n\left(x^4\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{8n+4}}{(2n+1)!}$ for all x, so the radius of convergence is ∞.

52. $e^x = \sum_{n=0}^{\infty}\dfrac{x^n}{n!} \quad\Rightarrow\quad 10^x = e^{(\ln 10)x} = \sum_{n=0}^{\infty}\dfrac{[(\ln 10)\,x]^n}{n!} = \sum_{n=0}^{\infty}\dfrac{(\ln 10)^n x^n}{n!}$, $R=\infty$

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53. $f(x) = 1/\sqrt[4]{16 - x} = 1/\left(\sqrt[4]{16}\sqrt[4]{1 - \frac{1}{16}x}\right) = \frac{1}{2}\left(1 - \frac{1}{16}x\right)^{-1/4}$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{4}\right)\left(-\frac{x}{16}\right) + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}\left(-\frac{x}{16}\right)^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}\left(-\frac{x}{16}\right)^3 + \cdots\right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2 \cdot 4^n \cdot n! \cdot 16^n} x^n = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2^{6n+1}\, n!} x^n$$

for $\left|-\frac{x}{16}\right| < 1 \quad \Leftrightarrow \quad |x| < 16$, so $R = 16$.

54. $(1 - 3x)^{-5} = \sum_{n=0}^{\infty} \binom{-5}{n} (-3x)^n = 1 + (-5)(-3x) + \frac{(-5)(-6)}{2!}(-3x)^2 + \frac{(-5)(-6)(-7)}{3!}(-3x)^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{5 \cdot 6 \cdot 7 \cdot \cdots \cdot (n+4) \cdot 3^n x^n}{n!} \quad \text{for } |-3x| < 1 \quad \Leftrightarrow \quad |x| < \tfrac{1}{3}, \text{ so } R = \tfrac{1}{3}.$$

55. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $\frac{e^x}{x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$ and $\int \frac{e^x}{x}\, dx = C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$.

56. $\left(1 + x^4\right)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(x^4\right)^n = 1 + \left(\frac{1}{2}\right)x^4 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(x^4\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(x^4\right)^3 + \cdots$

$$= 1 + \tfrac{1}{2}x^4 - \tfrac{1}{8}x^8 + \tfrac{1}{16}x^{12} - \cdots$$

so $\int_0^1 \left(1 + x^4\right)^{1/2} dx = \left[x + \frac{1}{10}x^5 - \frac{1}{72}x^9 + \frac{1}{208}x^{13} - \cdots\right]_0^1 = 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} - \cdots$.

This is an alternating series, so by the Alternating Series Test, the error in the approximation $\int_0^1 \left(1 + x^4\right)^{1/2} dx \approx 1 + \frac{1}{10} - \frac{1}{72} \approx 1.086$ is less than $\frac{1}{208}$, sufficient for the desired accuracy.

Thus, correct to two decimal places, $\int_0^1 \left(1 + x^4\right)^{1/2} dx \approx 1.09$.

57. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}$
⋮	⋮	⋮

$$\sqrt{x} \approx T_3(x) = 1 + \frac{1/2}{1!}(x-1) - \frac{1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3$$

$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3$$

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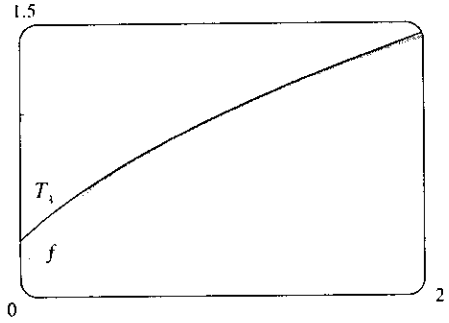
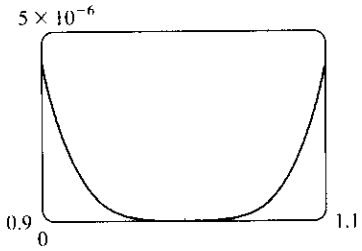
(b)



(c) $|R_3(x)| \le \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$ with $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$. Now $0.9 \le x \le 1.1 \Rightarrow -0.1 \le x-1 \le 0.1 \Rightarrow (x-1)^4 \le (0.1)^4$, and letting $x = 0.9$ gives $M = \dfrac{15}{16(0.9)^{7/2}}$, so

$$|R_3(x)| \le \frac{15}{16(0.9)^{7/2}4!}(0.1)^4 \approx 0.000005648.$$

(d)



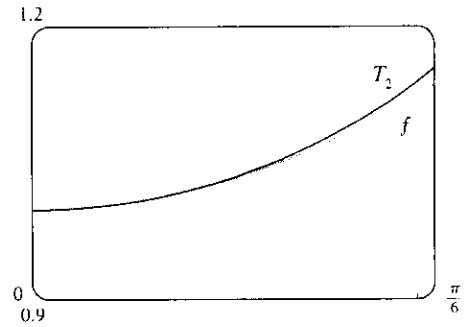
From the graph of $|R_3(x)| = |\sqrt{x} - T_3(x)|$, it appears that the error is less than 5×10^{-6} on $[0.9, 1.1]$.

58. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sec x$	1
1	$\sec x \tan x$	0
2	$\sec x \tan^2 x + \sec^3 x$	1
3	$\sec x \tan^3 x + 5\sec^3 x \tan x$	0
⋮	⋮	⋮

$$\sec x \approx T_2(x) = 1 + \tfrac{1}{2}x^2$$

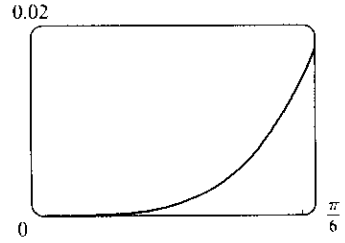
(b)



(c) $|R_2(x)| \le \dfrac{M}{3!}|x|^3$, where $\left|f^{(3)}(x)\right| \le M$ with $f^{(3)}(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$. Now $0 \le x \le \frac{\pi}{6} \Rightarrow x^3 \le \left(\frac{\pi}{6}\right)^3$, and letting $x = \frac{\pi}{6}$ gives $M = \frac{14}{3}$, so

$$|R_2(x)| \le \tfrac{14}{3 \cdot 6}\left(\tfrac{\pi}{6}\right)^3 \approx 0.111648.$$

(d)



From the graph of $|R_2(x)| = |\sec x - T_2(x)|$, it appears that the error is less than 0.02 on $\left[0, \frac{\pi}{6}\right]$.

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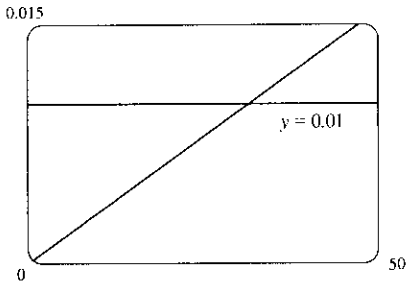
59. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, so $\sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ and $\frac{\sin x - x}{x^3} = -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots$. Thus, $\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \left(-\frac{1}{6} + \frac{x^2}{120} - \frac{x^4}{5040} + \cdots \right) = -\frac{1}{6}$.

60. (a) $F = \dfrac{mgR^2}{(R+h)^2} = \dfrac{mg}{(1+h/R)^2} = mg \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{h}{R}\right)^n$ (Binomial Series)

(b) We expand $F = mg\left[1 - 2(h/R) + 3(h/R)^2 - \cdots\right]$.

This is an alternating series, so by the Alternating Series Estimation Theorem, the error in the approximation $F = mg$ is less than $2mgh/R$, so for accuracy within 1% we want $\left| \dfrac{2mgh/R}{mgR^2/(R+h)^2} \right| < 0.01 \quad \Leftrightarrow \quad \dfrac{2h(R+h)^2}{R^3} < 0.01$. This inequality would be difficult to solve for h, so we substitute $R = 6{,}400$ km and plot both sides of the inequality. It appears that the approximation is accurate to within 1% for $h < 31$ km.



61. $f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad f(-x) = \sum_{n=0}^{\infty} c_n (-x)^n = \sum_{n=0}^{\infty} (-1)^n c_n x^n$

(a) If f is an odd function, then $f(-x) = -f(x) \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n c_n x^n = \sum_{n=0}^{\infty} -c_n x^n$. The coefficients of any power series are uniquely determined (by Theorem 12.10.5), so $(-1)^n c_n = -c_n$. If n is even, then $(-1)^n = 1$, so $c_n = -c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all even coefficients are 0, that is, $c_0 = c_2 = c_4 = \cdots = 0$.

(b) If f is even, then $f(-x) = f(x) \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n c_n x^n = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad (-1)^n c_n = c_n$.

If n is odd, then $(-1)^n = -1$, so $-c_n = c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all odd coefficients are 0, that is, $c_1 = c_3 = c_5 = \cdots = 0$.

62. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad f(x) = e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$. By Theorem 12.10.6 with $a = 0$, we also have $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$. Comparing coefficients for $k = 2n$, we have $\frac{f^{(2n)}(0)}{(2n)!} = \frac{1}{n!} \quad \Rightarrow \quad f^{(2n)}(0) = \frac{(2n)!}{n!}$.

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PROBLEMS PLUS

1. It would be far too much work to compute 15 derivatives of f. The key idea is to remember that $f^{(n)}(0)$ occurs in the coefficient of x^n in the Maclaurin series of f. We start with the Maclaurin series for sin: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$. Then $\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots$, and so the coefficient of x^{15} is $\frac{f^{(15)}(0)}{15!} = \frac{1}{5!}$. Therefore, $f^{(15)}(0) = \frac{15!}{5!} = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 = 10{,}897{,}286{,}400$.

2. We use the problem-solving strategy of taking cases:

Case (i): If $|x| < 1$, then $0 \le x^2 < 1$, so $\lim_{n\to\infty} x^{2n} = 0$ (see Example 9 in Section 12.1)

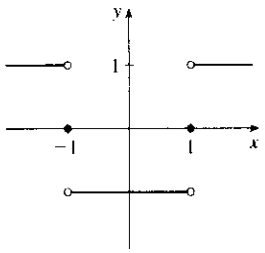
and $f(x) = \lim_{n\to\infty} \dfrac{x^{2n} - 1}{x^{2n} + 1} = \dfrac{0 - 1}{0 + 1} = -1$.

Case (ii): If $|x| = 1$, that is, $x = \pm 1$, then $x^2 = 1$, so $f(x) = \lim_{n\to\infty} \dfrac{x^{2n} - 1}{x^{2n} + 1} = \lim_{n\to\infty} \dfrac{1 - 1}{1 + 1} = 0$.

Case (iii): If $|x| > 1$, then $x^2 > 1$, so $\lim_{n\to\infty} x^{2n} = \infty$ and

$$f(x) = \lim_{n\to\infty} \frac{x^{2n} - 1}{x^{2n} + 1} = \lim_{n\to\infty} \frac{1 - (1/x^{2n})}{1 + (1/x^{2n})} = \frac{1 - 0}{1 + 0} = 1.$$

Thus, $f(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ -1 & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$



The graph shows that f is continuous everywhere except at $x = \pm 1$.

3. (a) From Formula 14a in Appendix D, with $x = y = \theta$, we get $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, so $\cot 2\theta = \dfrac{1 - \tan^2\theta}{2\tan\theta}$ $\Rightarrow$

$2\cot 2\theta = \dfrac{1 - \tan^2\theta}{\tan\theta} = \cot\theta - \tan\theta$. Replacing θ by $\frac{1}{2}x$, we get $2\cot x = \cot\frac{1}{2}x - \tan\frac{1}{2}x$, or

$$\tan\tfrac{1}{2}x = \cot\tfrac{1}{2}x - 2\cot x$$

(b) From part (a) with $\dfrac{x}{2^{n-1}}$ in place of x, $\tan\dfrac{x}{2^n} = \cot\dfrac{x}{2^n} - 2\cot\dfrac{x}{2^{n-1}}$, so the nth partial sum of $\sum_{n=1}^{\infty} \dfrac{1}{2^n}\tan\dfrac{x}{2^n}$ is

$$\begin{aligned} s_n &= \frac{\tan(x/2)}{2} + \frac{\tan(x/4)}{4} + \frac{\tan(x/8)}{8} + \cdots + \frac{\tan(x/2^n)}{2^n} \\ &= \left[\frac{\cot(x/2)}{2} - \cot x\right] + \left[\frac{\cot(x/4)}{4} - \frac{\cot(x/2)}{2}\right] + \left[\frac{\cot(x/8)}{8} - \frac{\cot(x/4)}{4}\right] + \cdots \\ &\qquad + \left[\frac{\cot(x/2^n)}{2^n} - \frac{\cot(x/2^{n-1})}{2^{n-1}}\right] = -\cot x + \frac{\cot(x/2^n)}{2^n} \quad \text{(telescoping sum)} \end{aligned}$$

Now $\dfrac{\cot(x/2^n)}{2^n} = \dfrac{\cos(x/2^n)}{2^n \sin(x/2^n)} = \dfrac{\cos(x/2^n)}{x} \cdot \dfrac{x/2^n}{\sin(x/2^n)} \to \dfrac{1}{x} \cdot 1 = \dfrac{1}{x}$ as $n \to \infty$ since $x/2^n \to 0$

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for $x \neq 0$. Therefore, if $x \neq 0$ and $x \neq k\pi$ where k is any integer, then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan\frac{x}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(-\cot x + \frac{1}{2^n}\cot\frac{x}{2^n} \right) = -\cot x + \frac{1}{x}$$

If $x = 0$, then all terms in the series are 0, so the sum is 0.

4. $|AP_2|^2 = 2$, $|AP_3|^2 = 2 + 2^2$, $|AP_4|^2 = 2 + 2^2 + \left(2^2\right)^2$, $|AP_5|^2 = 2 + 2^2 + \left(2^2\right)^2 + \left(2^3\right)^2, \ldots,$

$$|AP_n|^2 = 2 + 2^2 + \left(2^2\right)^2 + \cdots + \left(2^{n-2}\right)^2 \quad [\text{for } n \geq 3] \quad = 2 + \left(4 + 4^2 + 4^3 + \cdots + 4^{n-2}\right)$$

$$= 2 + \frac{4(4^{n-2} - 1)}{4 - 1} \quad [\text{finite geometric sum with } a = 4,\ r = 4] \quad = \frac{6}{3} + \frac{4^{n-1} - 4}{3} = \frac{2}{3} + \frac{4^{n-1}}{3}$$

$$\text{So } \tan \angle P_n A P_{n+1} = \frac{|P_n P_{n+1}|}{|AP_n|} = \frac{2^{n-1}}{\sqrt{\frac{2}{3} + \frac{4^{n-1}}{3}}} = \frac{\sqrt{4^{n-1}}}{\sqrt{\frac{2}{3} + \frac{4^{n-1}}{3}}} = \frac{1}{\sqrt{\frac{2}{3 \cdot 4^{n-1}} + \frac{1}{3}}} \to \sqrt{3} \text{ as } n \to \infty.$$

Thus, $\angle P_n A P_{n+1} \to \frac{\pi}{3}$ as $n \to \infty$.

5. (a) At each stage, each side is replaced by four shorter sides, each of length $\frac{1}{3}$ of the side length at the preceding stage. Writing s_0 and ℓ_0 for the number of sides and the length of the side of the initial triangle, we generate the table at right. In general, we have $s_n = 3 \cdot 4^n$ and $\ell_n = \left(\frac{1}{3}\right)^n$, so the length of the perimeter at the nth stage of construction is $p_n = s_n \ell_n = 3 \cdot 4^n \cdot \left(\frac{1}{3}\right)^n = 3 \cdot \left(\frac{4}{3}\right)^n$.

$s_0 = 3$	$\ell_0 = 1$
$s_1 = 3 \cdot 4$	$\ell_1 = 1/3$
$s_2 = 3 \cdot 4^2$	$\ell_2 = 1/3^2$
$s_3 = 3 \cdot 4^3$	$\ell_3 = 1/3^3$
$\ldots$	$\ldots$

(b) $p_n = \dfrac{4^n}{3^{n-1}} = 4\left(\dfrac{4}{3}\right)^{n-1}$. Since $\frac{4}{3} > 1$, $p_n \to \infty$ as $n \to \infty$.

(c) The area of each of the small triangles added at a given stage is one-ninth of the area of the triangle added at the preceding stage. Let a be the area of the original triangle. Then the area a_n of each of the small triangles added at stage n is $a_n = a \cdot \dfrac{1}{9^n} = \dfrac{a}{9^n}$. Since a small triangle is added to each side at every stage, it follows that the total area A_n added to the figure at the nth stage is $A_n = s_{n-1} \cdot a_n = 3 \cdot 4^{n-1} \cdot \dfrac{a}{9^n} = a \cdot \dfrac{4^{n-1}}{3^{2n-1}}$. Then the total area enclosed by the snowflake curve is $A = a + A_1 + A_2 + A_3 + \cdots = a + a \cdot \dfrac{1}{3} + a \cdot \dfrac{4}{3^3} + a \cdot \dfrac{4^2}{3^5} + a \cdot \dfrac{4^3}{3^7} + \cdots$. After the first term, this is a geometric series with common ratio $\frac{4}{9}$, so $A = a + \dfrac{a/3}{1 - \frac{4}{9}} = a + \dfrac{a}{3} \cdot \dfrac{9}{5} = \dfrac{8a}{5}$. But the area of the original equilateral triangle with side 1 is $a = \frac{1}{2} \cdot 1 \cdot \sin\frac{\pi}{3} = \frac{\sqrt{3}}{4}$. So the area enclosed by the snowflake curve is $\frac{8}{5} \cdot \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{5}$.

6. Let the series $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$. Then every term in S is of the form $\dfrac{1}{2^m 3^n}$, $m, n \geq 0$, and furthermore each term occurs only once. So we can write

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^m 3^n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^m} \frac{1}{3^n} = \sum_{m=0}^{\infty} \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{2}} \cdot \frac{1}{1 - \frac{1}{3}} = 2 \cdot \frac{3}{2} = 3$$

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7. (a) Let $a = \arctan x$ and $b = \arctan y$. Then, from Formula 14b in Appendix D,

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} = \frac{\tan(\arctan x) - \tan(\arctan y)}{1 + \tan(\arctan x)\tan(\arctan y)} = \frac{x-y}{1+xy}.$$

Now $\arctan x - \arctan y = a - b = \arctan(\tan(a-b)) = \arctan\dfrac{x-y}{1+xy}$ since $-\frac{\pi}{2} < a - b < \frac{\pi}{2}$.

(b) From part (a) we have

$$\arctan\tfrac{120}{119} - \arctan\tfrac{1}{239} = \arctan\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\cdot\frac{1}{239}} = \arctan\frac{\frac{28{,}561}{28{,}441}}{\frac{28{,}561}{28{,}441}} = \arctan 1 = \tfrac{\pi}{4}$$

(c) Replacing y by $-y$ in the formula of part (a), we get $\arctan x + \arctan y = \arctan\dfrac{x+y}{1-xy}$. So

$$4\arctan\tfrac{1}{5} = 2\left(\arctan\tfrac{1}{5} + \arctan\tfrac{1}{5}\right) = 2\arctan\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}} = 2\arctan\tfrac{5}{12} = \arctan\tfrac{5}{12} + \arctan\tfrac{5}{12}$$

$$= \arctan\frac{\frac{5}{12}+\frac{5}{12}}{1-\frac{5}{12}\cdot\frac{5}{12}} = \arctan\tfrac{120}{119}$$

Thus, from part (b), we have $4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan\frac{120}{119} - \arctan\frac{1}{239} = \frac{\pi}{4}$.

(d) From Example 7 in Section 12.9 we have $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \dfrac{x^{11}}{11} + \cdots$, so

$$\arctan\frac{1}{5} = \frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \frac{1}{9\cdot 5^9} - \frac{1}{11\cdot 5^{11}} + \cdots$$

This is an alternating series and the size of the terms decreases to 0, so by the Alternating Series Estimation Theorem, the sum lies between s_5 and s_6, that is, $0.197395560 < \arctan\frac{1}{5} < 0.197395562$.

(e) From the series in part (d) we get $\arctan\dfrac{1}{239} = \dfrac{1}{239} - \dfrac{1}{3\cdot 239^3} + \dfrac{1}{5\cdot 239^5} - \cdots$. The third term is less than 2.6×10^{-13}, so by the Alternating Series Estimation Theorem, we have, to nine decimal places, $\arctan\frac{1}{239} \approx s_2 \approx 0.004184076$. Thus, $0.004184075 < \arctan\frac{1}{239} < 0.004184077$.

(f) From part (c) we have $\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239}$, so from parts (d) and (e) we have

$16(0.197395560) - 4(0.004184077) < \pi < 16(0.197395562) - 4(0.004184075) \Rightarrow$

$3.141592652 < \pi < 3.141592692$. So, to 7 decimal places, $\pi \approx 3.1415927$.

8. (a) Let $a = \operatorname{arccot} x$ and $b = \operatorname{arccot} y$ where $0 < a - b < \pi$. Then

$$\cot(a-b) = \frac{1}{\tan(a-b)} = \frac{1+\tan a\tan b}{\tan a - \tan b} = \frac{1 + \dfrac{1}{\cot a}\cdot\dfrac{1}{\cot b}}{\dfrac{1}{\cot a} - \dfrac{1}{\cot b}} \cdot \frac{\cot a\cot b}{\cot a\cot b}$$

$$= \frac{\cot a\cot b + 1}{\cot b - \cot a}, \text{ so}$$

$$\cot(a-b) = \frac{1+\cot a\cot b}{\cot b - \cot a} = \frac{1+\cot(\operatorname{arccot} x)\cot(\operatorname{arccot} y)}{\cot(\operatorname{arccot} y) - \cot(\operatorname{arccot} x)} = \frac{1+xy}{y-x}.$$

Now $\operatorname{arccot} x - \operatorname{arccot} y = a - b = \operatorname{arccot}(\cot(a-b))$

$$= \operatorname{arccot}\frac{1+xy}{y-x} \text{ since } 0 < a - b < \pi.$$

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(b) Applying the identity in part (a) with $x = n$ and $y = n + 1$, we have

$$\operatorname{arccot}(n^2 + n + 1) = \operatorname{arccot}(1 + n(n+1)) = \operatorname{arccot}\frac{1 + n(n+1)}{(n+1) - n} = \operatorname{arccot} n - \operatorname{arccot}(n+1)$$

Thus, we have a telescoping series with nth partial sum

$$\begin{aligned} s_n &= [\operatorname{arccot} 0 - \operatorname{arccot} 1] + [\operatorname{arccot} 1 - \operatorname{arccot} 2] + \cdots + [\operatorname{arccot} n - \operatorname{arccot}(n+1)] \\ &= \operatorname{arccot} 0 - \operatorname{arccot}(n+1) \end{aligned}$$

Thus, $\sum_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} [\operatorname{arccot} 0 - \operatorname{arccot}(n+1)] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$.

9. We start with the geometric series $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, $|x| < 1$, and differentiate:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} \text{ for } |x| < 1 \quad \Rightarrow$$

$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \dfrac{x}{(1-x)^2}$ for $|x| < 1$. Differentiate again:

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{(1-x)^2 - x\cdot 2(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^2 x^n = \frac{x^2+x}{(1-x)^3} \quad \Rightarrow$$

$$\sum_{n=1}^{\infty} n^3 x^{n-1} = \frac{d}{dx}\frac{x^2+x}{(1-x)^3} = \frac{(1-x)^3(2x+1) - (x^2+x)\,3(1-x)^2(-1)}{(1-x)^6} = \frac{x^2+4x+1}{(1-x)^4} \quad \Rightarrow$$

$\sum_{n=1}^{\infty} n^3 x^n = \dfrac{x^3+4x^2+x}{(1-x)^4}$, $|x| < 1$. The radius of convergence is 1 because that is the radius of convergence for the geometric series we started with. If $x = \pm 1$, the series is $\sum n^3 (\pm 1)^n$, which diverges by the Test For Divergence, so the interval of convergence is $(-1, 1)$.

10. Let's first try the case $k = 1$: $a_0 + a_1 = 0 \quad \Rightarrow \quad a_1 = -a_0 \quad \Rightarrow$

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1}\right) = \lim_{n\to\infty}\left(a_0\sqrt{n} - a_0\sqrt{n+1}\right) = a_0 \lim_{n\to\infty}\left(\sqrt{n} - \sqrt{n+1}\right)\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}+\sqrt{n+1}}$$

$$= a_0 \lim_{n\to\infty} \frac{-1}{\sqrt{n}+\sqrt{n+1}} = 0$$

In general we have $a_0 + a_1 + \cdots + a_k = 0 \quad \Rightarrow \quad a_k = -a_0 - a_1 - \cdots - a_{k-1} \quad \Rightarrow$

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right)$$

$$= \lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + \cdots + a_{k-1}\sqrt{n+k-1} - a_0\sqrt{n+k} - a_1\sqrt{n+k} - \cdots - a_{k-1}\sqrt{n+k}\right)$$

$$= a_0 \lim_{n\to\infty}\left(\sqrt{n} - \sqrt{n+k}\right) + a_1 \lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n+k}\right) + \cdots + a_{k-1} \lim_{n\to\infty}\left(\sqrt{n+k-1} - \sqrt{n+k}\right)$$

Each of these limits is 0 by the same type of simplification as in the case $k = 1$. So we have

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right) = a_0(0) + a_1(0) + \cdots + a_{k-1}(0) = 0$$

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11. $\ln\left(1-\frac{1}{n^2}\right) = \ln\left(\frac{n^2-1}{n^2}\right) = \ln\frac{(n+1)(n-1)}{n^2} = \ln[(n+1)(n-1)] - \ln n^2$

$$= \ln(n+1) + \ln(n-1) - 2\ln n$$
$$= \ln(n-1) - \ln n - \ln n + \ln(n+1)$$
$$= \ln\frac{n-1}{n} - [\ln n - \ln(n+1)] = \ln\frac{n-1}{n} - \ln\frac{n}{n+1}.$$

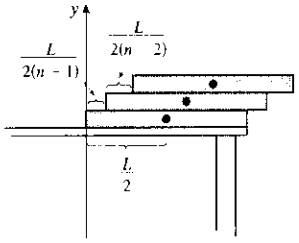
Let $s_k = \sum_{n=2}^{k} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n=2}^{k}\left(\ln\frac{n-1}{n} - \ln\frac{n}{n+1}\right)$ for $k \geq 2$. Then

$s_k = \left(\ln\frac{1}{2} - \ln\frac{2}{3}\right) + \left(\ln\frac{2}{3} - \ln\frac{3}{4}\right) + \cdots + \left(\ln\frac{k-1}{k} - \ln\frac{k}{k+1}\right) = \ln\frac{1}{2} - \ln\frac{k}{k+1}$, so

$$\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right) = \lim_{k\to\infty} s_k = \lim_{k\to\infty}\left(\ln\frac{1}{2} - \ln\frac{k}{k+1}\right) = \ln\frac{1}{2} - \ln 1 = \ln 1 - \ln 2 - \ln 1 = -\ln 2.$$

12. Place the y-axis as shown and let the length of each book be L. We want to show that the center of mass of the system of n books lies above the table, that is, $\overline{x} < L$. The x-coordinates of the centers of mass of the books are

$x_1 = \frac{L}{2}$, $x_2 = \frac{L}{2(n-1)} + \frac{L}{2}$, $x_3 = \frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \frac{L}{2}$, and so on.



Each book has the same mass m, so if there are n books, then

$$\overline{x} = \frac{mx_1 + mx_2 + \cdots + mx_n}{mn} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$= \frac{1}{n}\left[\frac{L}{2} + \left(\frac{L}{2(n-1)} + \frac{L}{2}\right) + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \frac{L}{2}\right) + \cdots \right.$$
$$\left. + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \cdots + \frac{L}{4} + \frac{L}{2} + \frac{L}{2}\right)\right]$$

$$= \frac{L}{n}\left[\frac{n-1}{2(n-1)} + \frac{n-2}{2(n-2)} + \cdots + \frac{2}{4} + \frac{1}{2} + \frac{n}{2}\right] = \frac{L}{n}\left[(n-1)\frac{1}{2} + \frac{n}{2}\right] = \frac{2n-1}{2n}L < L$$

This shows that, no matter how many books are added according to the given scheme, the center of mass lies above the table. It remains to observe that the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \frac{1}{2}\sum(1/n)$ is divergent (harmonic series), so we can make the top hook extend as far as we like beyond the edge of the table if we add enough books.

13. $u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots$, $v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots$, $w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$.

Use the Ratio Test to show that the series for u, v, and w have positive radii of convergence (∞ in each case), so Theorem 12.9.2 applies, and hence, we may differentiate each of these series:

$$\frac{du}{dx} = \frac{3x^2}{3!} + \frac{6x^5}{6!} + \frac{9x^8}{9!} + \cdots = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots = w$$

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Similarly, $\dfrac{dv}{dx} = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots = u$, and $\dfrac{dw}{dx} = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots = v$.

So $u' = w$, $v' = u$, and $w' = v$. Now differentiate the left hand side of the desired equation:

$$\frac{d}{dx}\left(u^3 + v^3 + w^3 - 3uvw\right) = 3u^2u' + 3v^2v' + 3w^2w' - 3(u'vw + uv'w + uvw')$$

$$= 3u^2w + 3v^2u + 3w^2v - 3(vw^2 + u^2w + uv^2) = 0 \quad \Rightarrow$$

$u^3 + v^3 + w^3 - 3uvw = C$. To find the value of the constant C, we put $x = 0$ in the last equation and get $1^3 + 0^3 + 0^3 - 3(1 \cdot 0 \cdot 0) = C \quad \Rightarrow \quad C = 1$, so $u^3 + v^3 + w^3 - 3uvw = 1$.

14. First notice that both series are absolutely convergent (p-series with $p > 1$.) Let the given expression be called x. Then

$$x = \frac{1 + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \dfrac{1}{4^p} + \cdots}{1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdots} = \frac{1 + \left(2 \cdot \dfrac{1}{2^p} - \dfrac{1}{2^p}\right) + \dfrac{1}{3^p} + \left(2 \cdot \dfrac{1}{4^p} - \dfrac{1}{4^p}\right) + \cdots}{1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdots}$$

$$= \frac{\left(1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdots\right) + \left(2 \cdot \dfrac{1}{2^p} + 2 \cdot \dfrac{1}{4^p} + 2 \cdot \dfrac{1}{6^p} + \cdots\right)}{1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdots}$$

$$= 1 + \frac{2\left(\dfrac{1}{2^p} + \dfrac{1}{4^p} + \dfrac{1}{6^p} + \dfrac{1}{8^p} + \cdots\right)}{1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdots} = 1 + \frac{\dfrac{1}{2^{p-1}}\left(1 + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \dfrac{1}{4^p} + \cdots\right)}{1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdots} = 1 + 2^{1-p}x$$

Therefore, $x = 1 + 2^{1-p}x \quad \Leftrightarrow \quad x - 2^{1-p}x = 1 \quad \Leftrightarrow \quad x(1 - 2^{1-p}) = 1 \quad \Leftrightarrow \quad x = \dfrac{1}{1 - 2^{1-p}}$.

15. If L is the length of a side of the equilateral triangle, then the area is $A = \frac{1}{2}L \cdot \frac{\sqrt{3}}{2}L = \frac{\sqrt{3}}{4}L^2$ and so $L^2 = \frac{4}{\sqrt{3}}A$.

Let r be the radius of one of the circles. When there are n rows of circles, the figure shows that

$$L = \sqrt{3}r + r + (n-2)(2r) + r + \sqrt{3}r = r\left(2n - 2 + 2\sqrt{3}\right), \text{ so } r = \frac{L}{2\left(n + \sqrt{3} - 1\right)}.$$

The number of circles is $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$, and so the total area of the circles is

$$A_n = \frac{n(n+1)}{2}\pi r^2 = \frac{n(n+1)}{2}\pi\frac{L^2}{4\left(n + \sqrt{3} - 1\right)^2} = \frac{n(n+1)}{2}\pi\frac{4A/\sqrt{3}}{4\left(n + \sqrt{3} - 1\right)^2}$$

$$= \frac{n(n+1)}{\left(n + \sqrt{3} - 1\right)^2}\frac{\pi A}{2\sqrt{3}} \quad \Rightarrow$$

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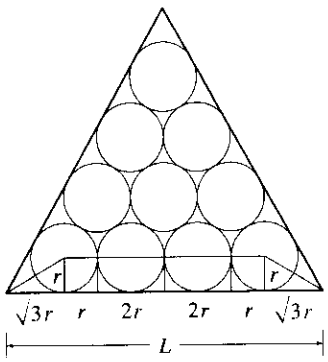
$$\frac{A_n}{A} = \frac{n(n+1)}{\left(n+\sqrt{3}-1\right)^2}\frac{\pi}{2\sqrt{3}}$$
$$= \frac{1+1/n}{\left[1+\left(\sqrt{3}-1\right)/n\right]^2}\frac{\pi}{2\sqrt{3}} \to \frac{\pi}{2\sqrt{3}} \text{ as } n \to \infty$$



16. Given $a_0 = a_1 = 1$ and $a_n = \dfrac{(n-1)(n-2)a_{n-1} - (n-3)a_{n-2}}{n(n-1)}$, we calculate the next few terms of the sequence: $a_2 = \dfrac{1\cdot 0\cdot a_1 - (-1)a_0}{2\cdot 1} = \dfrac{1}{2}$, $a_3 = \dfrac{2\cdot 1\cdot a_2 - 0\cdot a_1}{3\cdot 2} = \dfrac{1}{6}$, $a_4 = \dfrac{3\cdot 2\cdot a_3 - 1\cdot a_2}{4\cdot 3} = \dfrac{1}{24}$.

It seems that $a_n = \dfrac{1}{n!}$, so we try to prove this by induction. The first step is done, so assume $a_k = \dfrac{1}{k!}$ and $a_{k-1} = \dfrac{1}{(k-1)!}$. Then

$$a_{k+1} = \frac{k(k-1)a_k - (k-2)a_{k-1}}{(k+1)k} = \frac{\dfrac{k(k-1)}{k!} - \dfrac{k-2}{(k-1)!}}{(k+1)k} = \frac{(k-1)-(k-2)}{[(k+1)(k)]\,(k-1)!} = \frac{1}{(k+1)!}$$

and the induction is complete. Therefore, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} 1/n! = e$.

17. As in Section 12.9 we have to integrate the function x^x by integrating series. Writing $x^x = (e^{\ln x})^x = e^{x\ln x}$ and using the Maclaurin series for e^x, we have $x^x = (e^{\ln x})^x = e^{x\ln x} = \displaystyle\sum_{n=0}^{\infty} \frac{(x\ln x)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n(\ln x)^n}{n!}$.

As with power series, we can integrate this series term-by-term:

$$\int_0^1 x^x\,dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^n(\ln x)^n}{n!}\,dx = \sum_{n=0}^{\infty} \frac{1}{n!}\int_0^1 x^n(\ln x)^n\,dx$$

We integrate by parts with $u = (\ln x)^n$, $dv = x^n\,dx$, so $du = \dfrac{n(\ln x)^{n-1}}{x}\,dx$ and $v = \dfrac{x^{n+1}}{n+1}$:

$$\int_0^1 x^n(\ln x)^n\,dx = \lim_{t\to 0^+}\int_t^1 x^n(\ln x)^n\,dx = \lim_{t\to 0^+}\left[\frac{x^{n+1}}{n+1}(\ln x)^n\right]_t^1 - \lim_{t\to 0^+}\int_t^1 \frac{n}{n+1}x^n(\ln x)^{n-1}\,dx$$
$$= 0 - \frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}\,dx$$

(where l'Hospital's Rule was used to help evaluate the first limit).

Further integration by parts gives $\displaystyle\int_0^1 x^n(\ln x)^k\,dx = -\frac{k}{n+1}\int_0^1 x^n(\ln x)^{k-1}\,dx$ and, combining these steps, we get $\displaystyle\int_0^1 x^n(\ln x)^n\,dx = \frac{(-1)^n\,n!}{(n+1)^n}\int_0^1 x^n\,dx = \frac{(-1)^n\,n!}{(n+1)^{n+1}}$ $\Rightarrow$

$$\int_0^1 x^x\,dx = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 x^n(\ln x)^n\,dx = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{(-1)^n\,n!}{(n+1)^{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n}.$$

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13.2 Vectors ET 12.2

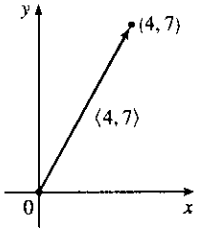
1. (a) The cost of a theater ticket is a scalar, because it has only magnitude.

(b) The current in a river is a vector, because it has both magnitude (the speed of the current) and direction at any given location.

(c) If we assume that the initial path is linear, the initial flight path from Houston to Dallas is a vector, because it has both magnitude (distance) and direction.

(d) The population of the world is a scalar, because it has only magnitude.

2. If the initial point of the vector $\langle 4, 7\rangle$ is placed at the origin, then $\langle 4, 7\rangle$ is the position vector of the point $(4, 7)$.



3. Vectors are equal when they share the same length and direction (but not necessarily location). Using the symmetry of the parallelogram as a guide, we see that $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, and $\overrightarrow{EA} = \overrightarrow{CE}$.

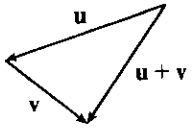
4. (a) The initial point of $\overrightarrow{QR}$ is positioned at the terminal point of $\overrightarrow{PQ}$, so by the Triangle Law the sum $\overrightarrow{PQ} + \overrightarrow{QR}$ is the vector with initial point P and terminal point R, namely $\overrightarrow{PR}$.

(b) By the Triangle Law, $\overrightarrow{RP} + \overrightarrow{PS}$ is the vector with initial point R and terminal point S, namely $\overrightarrow{RS}$.

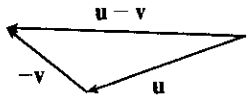
(c) First we consider $\overrightarrow{QS} - \overrightarrow{PS}$ as $\overrightarrow{QS} + \left(-\overrightarrow{PS}\right)$. Then since $-\overrightarrow{PS}$ has the same length as $\overrightarrow{PS}$ but points in the opposite direction, we have $-\overrightarrow{PS} = \overrightarrow{SP}$ and so $\overrightarrow{QS} - \overrightarrow{PS} = \overrightarrow{QS} + \overrightarrow{SP} = \overrightarrow{QP}$.

(d) We use the Triangle Law twice: $\overrightarrow{RS} + \overrightarrow{SP} + \overrightarrow{PQ} = \left(\overrightarrow{RS} + \overrightarrow{SP}\right) + \overrightarrow{PQ} = \overrightarrow{RP} + \overrightarrow{PQ} = \overrightarrow{RQ}$

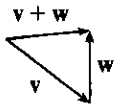
5. (a)



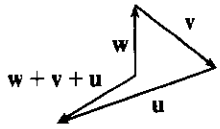
(b)



(c)



(d)



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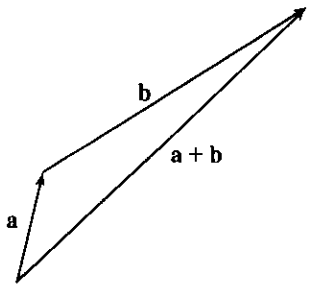
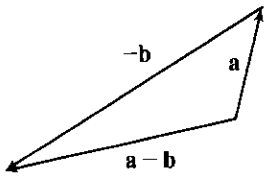
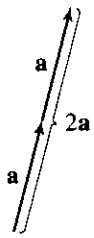
6. (a)



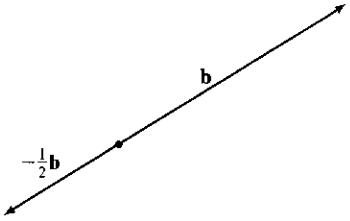
(b)



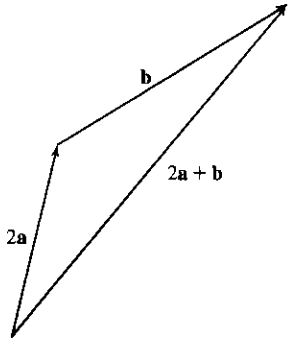
(c)



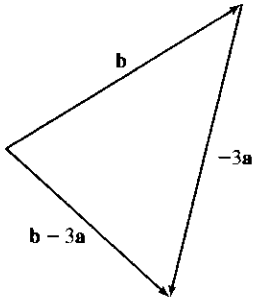
d)



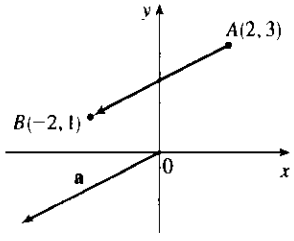
(e)



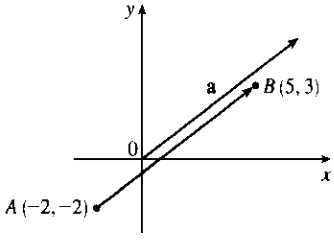
(f)



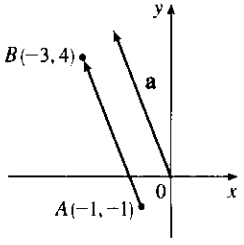
7. $\mathbf{a} = \langle -2-2, 1-3 \rangle = \langle -4, -2 \rangle$



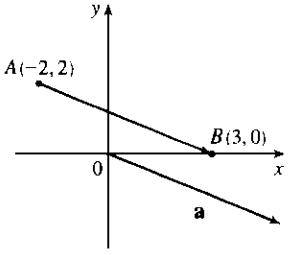
8. $\mathbf{a} = \langle 5-(-2), 3-(-2) \rangle = \langle 7, 5 \rangle$



9. $\mathbf{a} = \langle -3-(-1), 4-(-1) \rangle = \langle -2, 5 \rangle$



10. $\mathbf{a} = \langle 3-(-2), 0-2 \rangle = \langle 5, -2 \rangle$



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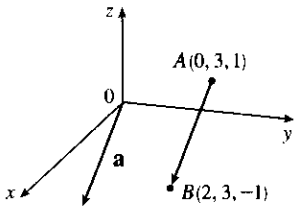
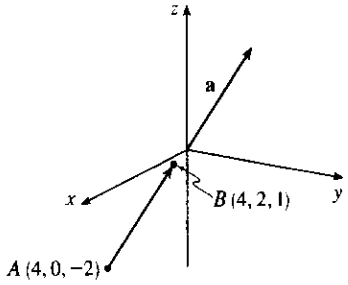
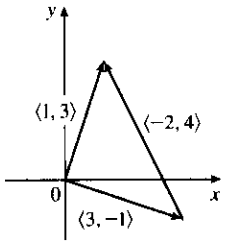
11. $\mathbf{a} = \langle 2-0, 3-3, -1-1 \rangle = \langle 2, 0, -2 \rangle$



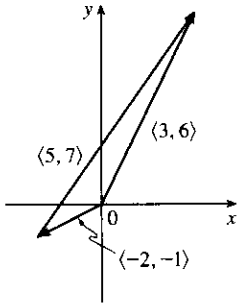
12. $\mathbf{a} = \langle 4-4, 2-0, 1-(-2) \rangle = \langle 0, 2, 3 \rangle$



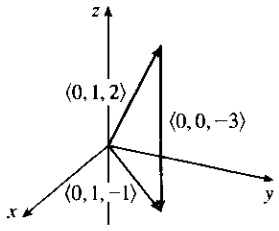
13. $\langle 3, -1 \rangle + \langle -2, 4 \rangle = \langle 3 + (-2), -1 + 4 \rangle$
$= \langle 1, 3 \rangle$



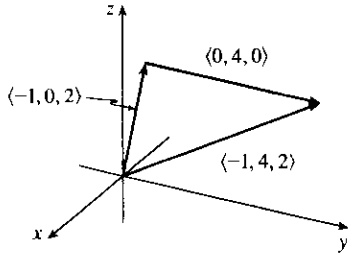
14. $\langle -2, -1 \rangle + \langle 5, 7 \rangle = \langle -2 + 5, -1 + 7 \rangle$
$= \langle 3, 6 \rangle$



15. $\langle 0, 1, 2 \rangle + \langle 0, 0, -3 \rangle = \langle 0 + 0, 1 + 0, 2 + (-3) \rangle$
$= \langle 0, 1, -1 \rangle$



16. $\langle -1, 0, 2 \rangle + \langle 0, 4, 0 \rangle = \langle -1 + 0, 0 + 4, 2 + 0 \rangle$
$= \langle -1, 4, 2 \rangle$



17. $|\mathbf{a}| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$

$\mathbf{a} + \mathbf{b} = \langle -4 + 6, 3 + 2 \rangle = \langle 2, 5 \rangle$

$\mathbf{a} - \mathbf{b} = \langle -4 - 6, 3 - 2 \rangle = \langle -10, 1 \rangle$

$2\mathbf{a} = \langle 2(-4), 2(3) \rangle = \langle -8, 6 \rangle$

$3\mathbf{a} + 4\mathbf{b} = \langle -12, 9 \rangle + \langle 24, 8 \rangle = \langle 12, 17 \rangle$

18. $|\mathbf{a}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$\mathbf{a} + \mathbf{b} = (2\,\mathbf{i} - 3\,\mathbf{j}) + (\mathbf{i} + 5\,\mathbf{j}) = 3\,\mathbf{i} + 2\,\mathbf{j}$

$\mathbf{a} - \mathbf{b} = (2\,\mathbf{i} - 3\,\mathbf{j}) - (\mathbf{i} + 5\,\mathbf{j}) = \mathbf{i} - 8\,\mathbf{j}$

$2\mathbf{a} = 2(2\,\mathbf{i} - 3\,\mathbf{j}) = 4\,\mathbf{i} - 6\,\mathbf{j}$

$3\mathbf{a} + 4\mathbf{b} = 3(2\,\mathbf{i} - 3\,\mathbf{j}) + 4(\mathbf{i} + 5\,\mathbf{j})$
$= 6\,\mathbf{i} - 9\,\mathbf{j} + 4\,\mathbf{i} + 20\,\mathbf{j} = 10\,\mathbf{i} + 11\,\mathbf{j}$

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19. $|\mathbf{a}| = \sqrt{6^2 + 2^2 + 3^2} = \sqrt{49} = 7$

$\mathbf{a} + \mathbf{b} = \langle 6 + (-1), 2 + 5, 3 + (-2) \rangle = \langle 5, 7, 1 \rangle$

$\mathbf{a} - \mathbf{b} = \langle 6 - (-1), 2 - 5, 3 - (-2) \rangle$

$= \langle 7, -3, 5 \rangle$

$2\mathbf{a} = \langle 2(6), 2(2), 2(3) \rangle = \langle 12, 4, 6 \rangle$

$3\mathbf{a} + 4\mathbf{b} = \langle 18, 6, 9 \rangle + \langle -4, 20, -8 \rangle$

$= \langle 14, 26, 1 \rangle$

20. $|\mathbf{a}| = \sqrt{(-3)^2 + (-4)^2 + (-1)^2} = \sqrt{26}$

$\mathbf{a} + \mathbf{b} = \langle -3 + 6, -4 + 2, -1 + (-3) \rangle$

$= \langle 3, -2, -4 \rangle$

$\mathbf{a} - \mathbf{b} = \langle -3 - 6, -4 - 2, -1 - (-3) \rangle$

$= \langle -9, -6, 2 \rangle$

$2\mathbf{a} = \langle 2(-3), 2(-4), 2(-1) \rangle = \langle -6, -8, -2 \rangle$

$3\mathbf{a} + 4\mathbf{b} = \langle -9, -12, -3 \rangle + \langle 24, 8, -12 \rangle$

$= \langle 15, -4, -15 \rangle$

21. $|\mathbf{a}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

$\mathbf{a} + \mathbf{b} = (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + (\mathbf{j} + 2\mathbf{k}) = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$

$\mathbf{a} - \mathbf{b} = (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (\mathbf{j} + 2\mathbf{k}) = \mathbf{i} - 3\mathbf{j} - \mathbf{k}$

$2\mathbf{a} = 2(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$3\mathbf{a} + 4\mathbf{b} = 3(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + 4(\mathbf{j} + 2\mathbf{k})$

$= 3\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} + 4\mathbf{j} + 8\mathbf{k}$

$= 3\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}$

22. $|\mathbf{a}| = \sqrt{3^2 + 0^2 + (-2)^2} = \sqrt{13}$

$\mathbf{a} + \mathbf{b} = (3\mathbf{i} - 2\mathbf{k}) + (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$

$\mathbf{a} - \mathbf{b} = (3\mathbf{i} - 2\mathbf{k}) - (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$2\mathbf{a} = 2(3\mathbf{i} - 2\mathbf{k}) = 6\mathbf{i} - 4\mathbf{k}$

$3\mathbf{a} + 4\mathbf{b} = 3(3\mathbf{i} - 2\mathbf{k}) + 4(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$= 9\mathbf{i} - 6\mathbf{k} + 4\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

$= 13\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$

23. $|\langle 9, -5 \rangle| = \sqrt{9^2 + (-5)^2} = \sqrt{106}$, so $\mathbf{u} = \dfrac{1}{\sqrt{106}} \langle 9, -5 \rangle = \left\langle \dfrac{9}{\sqrt{106}}, \dfrac{-5}{\sqrt{106}} \right\rangle$.

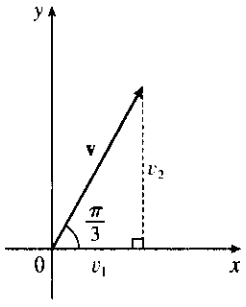
24. $|12\mathbf{i} - 5\mathbf{j}| = \sqrt{12^2 + (-5)^2} = \sqrt{169} = 13$, so $\mathbf{u} = \frac{1}{13}(12\mathbf{i} - 5\mathbf{j}) = \frac{12}{13}\mathbf{i} - \frac{5}{13}\mathbf{j}$.

25. The vector $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ has length $|8\mathbf{i} - \mathbf{j} + 4\mathbf{k}| = \sqrt{8^2 + (-1)^2 + 4^2} = \sqrt{81} = 9$, so by Equation 4 the unit vector with the same direction is $\frac{1}{9}(8\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = \frac{8}{9}\mathbf{i} - \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k}$.

26. $|\langle -2, 4, 2 \rangle| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{24} = 2\sqrt{6}$, so a unit vector in the direction of $\langle -2, 4, 2 \rangle$ is $\mathbf{u} = \dfrac{1}{2\sqrt{6}} \langle -2, 4, 2 \rangle$. A vector in the same direction but with length 6 is

$6\mathbf{u} = 6 \cdot \dfrac{1}{2\sqrt{6}} \langle -2, 4, 2 \rangle = \left\langle -\dfrac{6}{\sqrt{6}}, \dfrac{12}{\sqrt{6}}, \dfrac{6}{\sqrt{6}} \right\rangle$ or $\langle -\sqrt{6}, 2\sqrt{6}, \sqrt{6} \rangle$.

27. From the figure, we see that the x-component of $\mathbf{v}$ is $v_1 = |\mathbf{v}| \cos(\pi/3) = 4 \cdot \frac{1}{2} = 2$ and the y-component is $v_2 = |\mathbf{v}| \sin(\pi/3) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$. Thus $\mathbf{v} = \langle v_1, v_2 \rangle = \langle 2, 2\sqrt{3} \rangle$.



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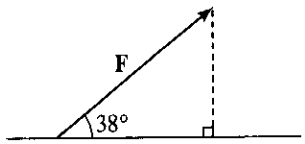
28. From the figure, we see that the horizontal component of the force $\mathbf{F}$ is $|\mathbf{F}|\cos 38^\circ = 50\cos 38^\circ \approx 39.4$ N, and the vertical component is $|\mathbf{F}|\sin 38^\circ = 50\sin 38^\circ \approx 30.8$ N.



29. $|\mathbf{F}_1| = 10$ lb and $|\mathbf{F}_2| = 12$ lb.

$$\mathbf{F}_1 = -|\mathbf{F}_1|\cos 45^\circ\,\mathbf{i} + |\mathbf{F}_1|\sin 45^\circ\,\mathbf{j} = -10\cos 45^\circ\,\mathbf{i} + 10\sin 45^\circ\,\mathbf{j}$$
$$= -5\sqrt{2}\,\mathbf{i} + 5\sqrt{2}\,\mathbf{j}$$

$$\mathbf{F}_2 = |\mathbf{F}_2|\cos 30^\circ\,\mathbf{i} + |\mathbf{F}_2|\sin 30^\circ\,\mathbf{j} = 12\cos 30^\circ\,\mathbf{i} + 12\sin 30^\circ\,\mathbf{j} = 6\sqrt{3}\,\mathbf{i} + 6\,\mathbf{j}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \left(6\sqrt{3} - 5\sqrt{2}\right)\mathbf{i} + \left(6 + 5\sqrt{2}\right)\mathbf{j} \approx 3.32\,\mathbf{i} + 13.07\,\mathbf{j}$$

$|\mathbf{F}| \approx \sqrt{(3.32)^2 + (13.07)^2} \approx 13.5$ lb. $\tan\theta = \dfrac{6 + 5\sqrt{2}}{6\sqrt{3} - 5\sqrt{2}} \Rightarrow \theta = \tan^{-1}\dfrac{6 + 5\sqrt{2}}{6\sqrt{3} - 5\sqrt{2}} \approx 76^\circ$.

30. Set up the coordinate axes so that north is the positive y-direction, and east is the positive x-direction. The wind is blowing at 50 km/h from the direction N45° W, so that its velocity vector is 50 km/h S45° E, which can be written as $\mathbf{v}_{\text{wind}} = 50(\cos 45^\circ\,\mathbf{i} - \sin 45^\circ\,\mathbf{j})$. With respect to the still air, the velocity vector of the plane is 250 km/h N 60° E, or equivalently $\mathbf{v}_{\text{plane}} = 250(\cos 30^\circ\,\mathbf{i} + \sin 30^\circ\,\mathbf{j})$. The velocity of the plane relative to the ground is

$$\mathbf{v} = \mathbf{v}_{\text{wind}} + \mathbf{v}_{\text{plane}} = (50\cos 45^\circ + 250\cos 30^\circ)\,\mathbf{i} + (-50\sin 45^\circ + 250\sin 30^\circ)\,\mathbf{j}$$
$$= \left(25\sqrt{2} + 125\sqrt{3}\right)\mathbf{i} + \left(125 - 25\sqrt{2}\right)\mathbf{j} \approx 251.9\,\mathbf{i} + 89.6\,\mathbf{j}$$

The ground speed is $|\mathbf{v}| \approx \sqrt{(251.9)^2 + (89.6)^2} \approx 267$ km/h. The angle the velocity vector makes with the x-axis is $\theta \approx \tan^{-1}\left(\frac{89.6}{251.9}\right) \approx 20^\circ$. Therefore, the true course of the plane is about N$(90 - 20)^\circ$ E = N 70° E.

31. With respect to the water's surface, the woman's velocity is the vector sum of the velocity of the ship with respect to the water, and the woman's velocity with respect to the ship. If we let north be the positive y-direction, then $\mathbf{v} = \langle 0, 22\rangle + \langle -3, 0\rangle = \langle -3, 22\rangle$. The woman's speed is $|\mathbf{v}| = \sqrt{9 + 484} \approx 22.2$ mi/h. The vector $\mathbf{v}$ makes an angle θ with the east, where $\theta = \tan^{-1}\left(\frac{22}{-3}\right) \approx 98^\circ$. Therefore, the woman's direction is about N$(98 - 90)^\circ$ W = N8° W.

32. Call the two tensile forces $\mathbf{T}_3$ and $\mathbf{T}_5$, corresponding to the ropes of length 3 m and 5 m. In terms of vertical and horizontal components,

$$\mathbf{T}_3 = -|\mathbf{T}_3|\cos 52^\circ\,\mathbf{i} + |\mathbf{T}_3|\sin 52^\circ\,\mathbf{j} \quad (1) \qquad \text{and} \qquad \mathbf{T}_5 = |\mathbf{T}_5|\cos 40^\circ\,\mathbf{i} + |\mathbf{T}_5|\sin 40^\circ\,\mathbf{j} \quad (2)$$

The resultant of these forces, $\mathbf{T}_3 + \mathbf{T}_5$, counterbalances the force of gravity acting on the decoration [which is $-5g\,\mathbf{j} \approx -5(9.8)\,\mathbf{j} = -49\,\mathbf{j}$]. So $\mathbf{T}_3 + \mathbf{T}_5 = 49\,\mathbf{j}$. Hence

$\mathbf{T}_3 + \mathbf{T}_5 = (-|\mathbf{T}_3|\cos 52^\circ + |\mathbf{T}_5|\cos 40^\circ)\,\mathbf{i} + (|\mathbf{T}_3|\sin 52^\circ + |\mathbf{T}_5|\sin 40^\circ)\,\mathbf{j} = 49\,\mathbf{j}$. Thus

$-|\mathbf{T}_3|\cos 52^\circ + |\mathbf{T}_5|\cos 40^\circ = 0$ and $|\mathbf{T}_3|\sin 52^\circ + |\mathbf{T}_5|\sin 40^\circ = 49$.

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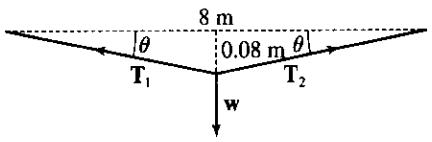
From the first of these two equations $|\mathbf{T}_3| = |\mathbf{T}_5| \dfrac{\cos 40^\circ}{\cos 52^\circ}$. Substituting this into the second equation gives $|\mathbf{T}_5| = \dfrac{49}{\cos 40^\circ \tan 52^\circ + \sin 40^\circ} \approx 30$ N. Therefore, $|\mathbf{T}_3| = |\mathbf{T}_5| \dfrac{\cos 40^\circ}{\cos 52^\circ} \approx 38$ N. Finally, from (1) and (2), $\mathbf{T}_3 \approx -23\,\mathbf{i} + 30\,\mathbf{j}$, and $\mathbf{T}_5 \approx 23\,\mathbf{i} + 19\,\mathbf{j}$.

33. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ represent the tension vectors in each side of the clothesline as shown in the figure. $\mathbf{T}_1$ and $\mathbf{T}_2$ have equal vertical components and opposite horizontal components, so we can write $\mathbf{T}_1 = -a\,\mathbf{i} + b\,\mathbf{j}$ and $\mathbf{T}_2 = a\,\mathbf{i} + b\,\mathbf{j}$ $(a, b > 0)$. By similar triangles, $\dfrac{b}{a} = \dfrac{0.08}{4} \quad \Rightarrow \quad a = 50b$. The force due to gravity acting on the shirt has magnitude $0.8g \approx (0.8)(9.8) = 7.84$ N, hence we have $\mathbf{w} = -7.84\,\mathbf{j}$. The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensile forces counterbalances $\mathbf{w}$, so $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} \quad \Rightarrow \quad (-a\,\mathbf{i} + b\,\mathbf{j}) + (a\,\mathbf{i} + b\,\mathbf{j}) = 7.84\,\mathbf{j}$ $\Rightarrow \quad (-50b\,\mathbf{i} + b\,\mathbf{j}) + (50b\,\mathbf{i} + b\,\mathbf{j}) = 2b\,\mathbf{j} = 7.84\,\mathbf{j} \quad \Rightarrow \quad b = \frac{7.84}{2} = 3.92$ and $a = 50b = 196$. Thus the tensions are $\mathbf{T}_1 = -a\,\mathbf{i} + b\,\mathbf{j} = -196\,\mathbf{i} + 3.92\,\mathbf{j}$ and $\mathbf{T}_2 = a\,\mathbf{i} + b\,\mathbf{j} = 196\,\mathbf{i} + 3.92\,\mathbf{j}$.

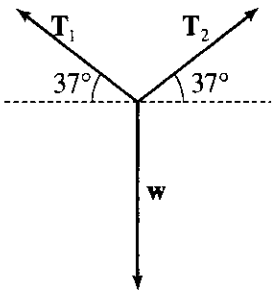


Alternatively, we can find the value of θ and proceed as in Example 7.

34. We can consider the weight of the chain to be concentrated at its midpoint. The forces acting on the chain then are the tension vectors $\mathbf{T}_1$, $\mathbf{T}_2$ in each end of the chain and the weight $\mathbf{w}$, as shown in the figure. We know $|\mathbf{T}_1| = |\mathbf{T}_2| = 25$ N so, in terms of vertical and horizontal components, we have

$$\mathbf{T}_1 = -25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}$$

$$\mathbf{T}_2 = 25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}$$



The resultant vector $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$, giving $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w}$. Since $\mathbf{w} = -|\mathbf{w}|\,\mathbf{j}$, we have $(-25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}) + (25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}) = |\mathbf{w}|\,\mathbf{j} \quad \Rightarrow$ $50\sin 37^\circ\,\mathbf{j} = |\mathbf{w}|\,\mathbf{j} \quad \Rightarrow \quad |\mathbf{w}| = 50\sin 37^\circ \approx 30.1$. So the weight is 30.1 N, and since $w = mg$, the mass is $\frac{30.1}{9.8} \approx 3.07$ kg.

35. By the Triangle Law, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. Then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA}$, but $\overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AC} + \left(-\overrightarrow{AC}\right) = \mathbf{0}$. So $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$.

36. $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{BC} = \frac{2}{3}\overrightarrow{BA}$. $\mathbf{c} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \frac{1}{3}\overrightarrow{AB} \quad \Rightarrow \quad \overrightarrow{AB} = 3\,\mathbf{c} - 3\,\mathbf{a}$.

$\mathbf{c} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{BA} \quad \Rightarrow \quad \overrightarrow{BA} = \frac{3}{2}\,\mathbf{c} - \frac{3}{2}\,\mathbf{b}$. $\overrightarrow{BA} = -\overrightarrow{AB}$, so $\frac{3}{2}\,\mathbf{c} - \frac{3}{2}\,\mathbf{b} = 3\,\mathbf{a} - 3\,\mathbf{c} \quad \Leftrightarrow$ $\mathbf{c} + 2\,\mathbf{c} = 2\,\mathbf{a} + \mathbf{b} \quad \Leftrightarrow \quad \mathbf{c} = \frac{2}{3}\,\mathbf{a} + \frac{1}{3}\,\mathbf{b}$.

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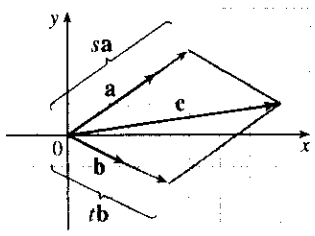
37. (a), (b)

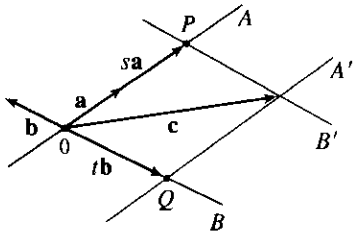


(c) From the sketch, we estimate that $s \approx 1.3$ and $t \approx 1.6$.

(d) $\mathbf{c} = s\,\mathbf{a} + t\,\mathbf{b} \quad \Leftrightarrow \quad 7 = 3s + 2t$ and $1 = 2s - t$.

Solving these equations gives $s = \frac{9}{7}$ and $t = \frac{11}{7}$.

38. Draw $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ emanating from the origin. Extend $\mathbf{a}$ and $\mathbf{b}$ to form lines A and B, and draw lines A' and B' parallel to these two lines through the terminal point of $\mathbf{c}$.



Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, A and B' must meet (at P), and A' and B must also meet (at Q). Now we see that $\overrightarrow{OP} + \overrightarrow{OQ} = \mathbf{c}$, so if $s = \dfrac{|\overrightarrow{OP}|}{|\mathbf{a}|}$ $\left(\text{or its negative, if } \mathbf{a} \text{ points in the direction opposite } \overrightarrow{OP}\right)$ and $t = \dfrac{|\overrightarrow{OQ}|}{|\mathbf{b}|}$ (or its negative, as in the diagram), then $c = s\mathbf{a} + t\mathbf{b}$, as required.

Argument using components: Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ all lie in the same plane, we can consider them to be vectors in two dimensions. Let $\mathbf{a} = \langle a_1, a_2 \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$, and $\mathbf{c} = \langle c_1, c_2 \rangle$. We need $sa_1 + tb_1 = c_1$ and $sa_2 + tb_2 = c_2$. Multiplying the first equation by a_2 and the second by a_1 and subtracting, we get $t = \dfrac{c_2a_1 - c_1a_2}{b_2a_1 - b_1a_2}$. Similarly $s = \dfrac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2}$. Since $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, the denominator is not zero.

39. $|\mathbf{r} - \mathbf{r}_0|$ is the distance between the points (x, y, z) and (x_0, y_0, z_0), so the set of points is a sphere with radius 1 and center (x_0, y_0, z_0).

Alternate method: $|\mathbf{r} - \mathbf{r}_0| = 1 \quad \Leftrightarrow \quad \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = 1 \quad \Leftrightarrow$

$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1$, which is the equation of a sphere with radius 1 and center (x_0, y_0, z_0).

40. Let P_1 and P_2 be the points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2|$ is the sum of the distances from (x, y) to P_1 and P_2. Since this sum is constant, the set of points (x, y) represents an ellipse with foci P_1 and P_2. The condition $k > |\mathbf{r}_1 - \mathbf{r}_2|$ assures us that the ellipse is not degenerate.

41.
$$\begin{aligned}
\mathbf{a} + (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2 \rangle + (\langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle) = \langle a_1, a_2 \rangle + \langle b_1 + c_1, b_2 + c_2 \rangle \\
&= \langle a_1 + b_1 + c_1, a_2 + b_2 + c_2 \rangle = \langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2 \rangle \\
&= \langle a_1 + b_1, a_2 + b_2 \rangle + \langle c_1, c_2 \rangle = (\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle) + \langle c_1, c_2 \rangle \\
&= (\mathbf{a} + \mathbf{b}) + \mathbf{c}
\end{aligned}$$

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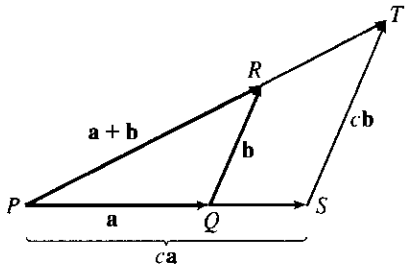
42. *Algebraically:*

$$\begin{aligned} c(\mathbf{a}+\mathbf{b}) &= c\left(\langle a_1,a_2,a_3\rangle + \langle b_1,b_2,b_3\rangle\right) = c\,\langle a_1+b_1, a_2+b_2, a_3+b_3\rangle \\ &= \langle c\,(a_1+b_1)\,, c\,(a_2+b_2)\,, c\,(a_3+b_3)\rangle = \langle ca_1+cb_1, ca_2+cb_2, ca_3+cb_3\rangle \\ &= \langle ca_1, ca_2, ca_3\rangle + \langle cb_1, cb_2, cb_3\rangle = c\,\mathbf{a}+c\,\mathbf{b} \end{aligned}$$

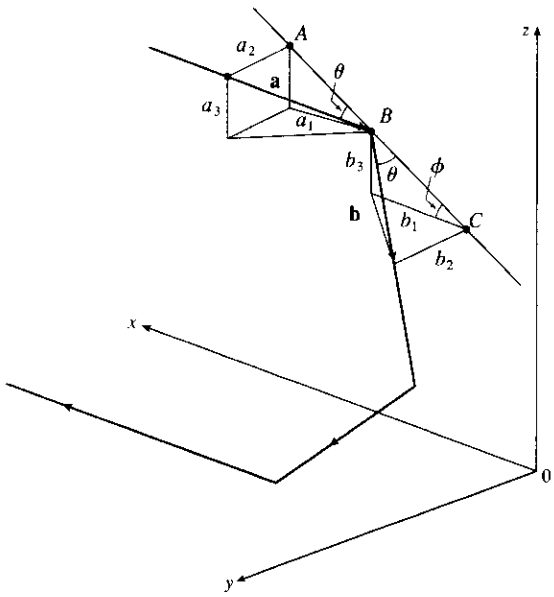
Geometrically:

According to the Triangle Law, if $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{QR}$, then $\mathbf{a}+\mathbf{b} = \overrightarrow{PR}$. Construct triangle PST as shown so that $\overrightarrow{PS} = c\,\mathbf{a}$ and $\overrightarrow{ST} = c\,\mathbf{b}$. (We have drawn the case where $c > 1$.) By the Triangle Law, $\overrightarrow{PT} = c\,\mathbf{a} + c\,\mathbf{b}$. But triangle PQR and triangle PST are similar triangles because $c\,\mathbf{b}$ is parallel to $\mathbf{b}$. Therefore, $\overrightarrow{PR}$ and $\overrightarrow{PT}$ are parallel and, in fact, $\overrightarrow{PT} = c\overrightarrow{PR}$. Thus, $c\,\mathbf{a} + c\,\mathbf{b} = c(\mathbf{a}+\mathbf{b})$.



43. Consider triangle ABC, where D and E are the midpoints of AB and BC. We know that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (1) and $\overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DE}$ (2). However, $\overrightarrow{DB} = \frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BC}$. Substituting these expressions for $\overrightarrow{DB}$ and $\overrightarrow{BE}$ into (2) gives $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{DE}$. Comparing this with (1) gives $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}$. Therefore $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel and $\left|\overrightarrow{DE}\right| = \frac{1}{2}\left|\overrightarrow{AC}\right|$.

44.



The question states that the light ray strikes all three mirrors, so it is not parallel to any of them and $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$. Let $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, as in the diagram. We can let $|\mathbf{b}| = |\mathbf{a}|$, since only its direction is important. Then $\dfrac{|b_2|}{|\mathbf{b}|} = \sin\theta = \dfrac{|a_2|}{|\mathbf{a}|} \quad \Rightarrow \quad |b_2| = |a_2|$.

From the diagram $b_2\,\mathbf{j}$ and $a_2\,\mathbf{j}$ point in opposite directions, so $b_2 = -a_2$. $|AB| = |BC|$, so $|b_3| = \sin\phi\ |BC| = \sin\phi\ |AB| = |a_3|$, and $|b_1| = \cos\phi\ |BC| = \cos\phi\ |AB| = |a_1|$.

$b_3\,\mathbf{k}$ and $a_3\,\mathbf{k}$ have the same direction, as do $b_1\,\mathbf{i}$ and $a_1\,\mathbf{i}$, so $\mathbf{b} = \langle a_1, -a_2, a_3\rangle$. When the ray hits the other mirrors, similar arguments show that these reflections will reverse the signs of the other two coordinates, so the final reflected ray will be $\langle -a_1, -a_2, -a_3\rangle = -\mathbf{a}$, which is parallel to $\mathbf{a}$.

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13.3 The Dot Product

ET 12.3

1. (a) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ has no meaning.

(b) $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$ is a scalar multiple of a vector, so it does have meaning.

(c) Both $|\mathbf{a}|$ and $\mathbf{b} \cdot \mathbf{c}$ are scalars, so $|\mathbf{a}|\,(\mathbf{b} \cdot \mathbf{c})$ is an ordinary product of real numbers, and has meaning.

(d) Both $\mathbf{a}$ and $\mathbf{b} + \mathbf{c}$ are vectors, so the dot product $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ has meaning.

(e) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, but $\mathbf{c}$ is a vector, and so the two quantities cannot be added and this expression has no meaning.

(f) $|\mathbf{a}|$ is a scalar, and the dot product is defined only for vectors, so $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$ has no meaning.

2. Let the vectors be $\mathbf{a}$ and $\mathbf{b}$. Then by Theorem 3, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = (6)\left(\frac{1}{3}\right)\cos\frac{\pi}{4} = \frac{6}{3\sqrt{2}} = \sqrt{2}$.

3. $\mathbf{a} \cdot \mathbf{b} = \langle 4, -1\rangle \cdot \langle 3, 6\rangle = (4)\,(3) + (-1)\,(6) = 6$

4. $\mathbf{a} \cdot \mathbf{b} = \left\langle \frac{1}{2}, 4\right\rangle \cdot \langle -8, -3\rangle = \left(\frac{1}{2}\right)(-8) + (4)(-3) = -16$

5. $\mathbf{a} \cdot \mathbf{b} = \langle 5, 0, -2\rangle \cdot \langle 3, -1, 10\rangle = (5)(3) + (0)(-1) + (-2)(10) = -5$

6. $\mathbf{a} \cdot \mathbf{b} = \langle s, 2s, 3s\rangle \cdot \langle t, -t, 5t\rangle = (s)(t) + (2s)(-t) + (3s)(5t) = st - 2st + 15st = 14st$

7. $\mathbf{a} \cdot \mathbf{b} = (\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}) \cdot (5\,\mathbf{i} + 9\,\mathbf{k}) = (1)(5) + (-2)(0) + (3)(9) = 32$

8. $\mathbf{a} \cdot \mathbf{b} = (4\,\mathbf{j} - 3\,\mathbf{k}) \cdot (2\,\mathbf{i} + 4\,\mathbf{j} + 6\,\mathbf{k}) = (0)(2) + (4)(4) + (-3)(6) = -2$

9. Use Theorem 3: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = (12)(15)\cos\frac{\pi}{6} = 180 \cdot \frac{\sqrt{3}}{2} = 90\sqrt{3} \approx 155.9$

10. Use Theorem 3: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = (4)(10)\cos 120^\circ = 40\left(-\frac{1}{2}\right) = -20$

11. $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all unit vectors, so the triangle is an equilateral triangle. Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is 60° and $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos 60^\circ = (1)(1)\left(\frac{1}{2}\right) = \frac{1}{2}$. If $\mathbf{w}$ is moved so it has the same initial point as $\mathbf{u}$, we can see that the angle between them is 120° and we have $\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}|\,|\mathbf{w}| \cos 120^\circ = (1)(1)\left(-\frac{1}{2}\right) = -\frac{1}{2}$.

12. $\mathbf{u}$ is a unit vector, so $\mathbf{w}$ is also a unit vector, and $|\mathbf{v}|$ can be determined by examining the right triangle formed by $\mathbf{u}$ and $\mathbf{v}$. Since the angle between $\mathbf{u}$ and $\mathbf{v}$ is 45°, we have $|\mathbf{v}| = |\mathbf{u}| \cos 45^\circ = \frac{\sqrt{2}}{2}$. Then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos 45^\circ = (1)\left(\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = \frac{1}{2}$. Since $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, $\mathbf{u} \cdot \mathbf{w} = 0$.

13. (a) $\mathbf{i} \cdot \mathbf{j} = \langle 1, 0, 0\rangle \cdot \langle 0, 1, 0\rangle = (1)(0) + (0)(1) + (0)(0) = 0$. Similarly $\mathbf{j} \cdot \mathbf{k} = (0)(0) + (1)(0) + (0)(1) = 0$ and $\mathbf{k} \cdot \mathbf{i} = (0)(1) + (0)(0) + (1)(0) = 0$.

Another method: Because $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular, the cosine factor in each dot product (see Theorem 3) is $\cos\frac{\pi}{2} = 0$.

(h) By Property 1 of the dot product, $\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}|^2 = 1^2 = 1$ since $\mathbf{i}$ is a unit vector. Similarly, $\mathbf{j} \cdot \mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k} \cdot \mathbf{k} = |\mathbf{k}|^2 = 1$.

14. The dot product $\mathbf{A} \cdot \mathbf{P}$ is

$$\begin{aligned}\langle a, b, c\rangle \cdot \langle 2, 1.5, 1\rangle &= a(2) + b(1.5) + c(1)\\ &= (\text{number of hamburgers sold})(\text{price per hamburger})\\ &\quad + (\text{number of hot dogs sold})(\text{price per hot dog})\\ &\quad + (\text{number of soft drinks sold})(\text{price per soft drink})\end{aligned}$$

so it is equal to the vendor's total revenue for that day.

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15. $|\mathbf{a}| = \sqrt{3^2 + 4^2} = 5$, $|\mathbf{b}| = \sqrt{5^2 + 12^2} = 13$, and $\mathbf{a} \cdot \mathbf{b} = (3)(5) + (4)(12) = 63$. Using Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{63}{5 \cdot 13} = \dfrac{63}{65}$. So the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\frac{63}{65}\right) \approx 14°$.

16. $|\mathbf{a}| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$, $|\mathbf{b}| = \sqrt{0 + 25} = 5$, and $\mathbf{a} \cdot \mathbf{b} = \left(\sqrt{3}\right)(0) + (1)(5) = 5$. Using Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{5}{2 \cdot 5} = \dfrac{1}{2}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\cos^{-1}\left(\frac{1}{2}\right) = 60°$.

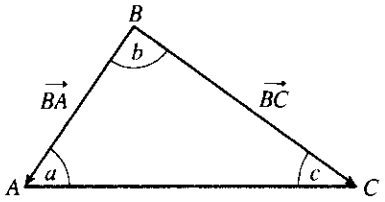
17. $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, $|\mathbf{b}| = \sqrt{4^2 + 0^2 + (-1)^2} = \sqrt{17}$, and $\mathbf{a} \cdot \mathbf{b} = (1)(4) + (2)(0) + (3)(-1) = 1$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{1}{\sqrt{14} \cdot \sqrt{17}} = \dfrac{1}{\sqrt{238}}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\frac{1}{\sqrt{238}}\right) \approx 86°$.

18. $|\mathbf{a}| = \sqrt{6^2 + (-3)^2 + 2^2} = 7$, $|\mathbf{b}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$, and $\mathbf{a} \cdot \mathbf{b} = (6)(2) + (-3)(1) + (2)(-2) = 5$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{5}{7 \cdot 3} = \dfrac{5}{21}$ and $\theta = \cos^{-1}\left(\frac{5}{21}\right) \approx 76°$.

19. $|\mathbf{a}| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$, $|\mathbf{b}| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$, and $\mathbf{a} \cdot \mathbf{b} = (0)(1) + (1)(2) + (1)(-3) = -1$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{2} \cdot \sqrt{14}} = \dfrac{-1}{2\sqrt{7}}$ and $\theta = \cos^{-1}\left(-\dfrac{1}{2\sqrt{7}}\right) \approx 101°$.

20. $|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$, $|\mathbf{b}| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$, and $\mathbf{a} \cdot \mathbf{b} = (2)(3) + (-1)(2) + (1)(-1) = 3$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{3}{\sqrt{6} \cdot \sqrt{14}} = \dfrac{3}{2\sqrt{21}}$ and $\theta = \cos^{-1}\left(\dfrac{3}{2\sqrt{21}}\right) \approx 71°$.

21. Let a, b, and c be the angles at vertices A, B, and C respectively. Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.



Thus $\cos a = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \dfrac{\langle 2, 6\rangle \cdot \langle -2, 4\rangle}{\sqrt{2^2 + 6^2}\sqrt{(-2)^2 + 4^2}} = \dfrac{1}{\sqrt{40}\sqrt{20}}(-4 + 24) = \dfrac{20}{\sqrt{800}} = \dfrac{\sqrt{2}}{2}$ and $a = \cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = 45°$. Similarly, $\cos b = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} = \dfrac{\langle -2, -6\rangle \cdot \langle -4, -2\rangle}{\sqrt{4 + 36}\sqrt{16 + 4}} = \dfrac{1}{\sqrt{40}\sqrt{20}}(8 + 12) = \dfrac{20}{\sqrt{800}} = \dfrac{\sqrt{2}}{2}$ so $b = \cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = 45°$ and $c = 180° - (45° + 45°) = 90°$.

Alternate solution: Apply the Law of Cosines three times as follows: $\cos a = \dfrac{\left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2}{2\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$, $\cos b = \dfrac{\left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|}$, and $\cos c = \dfrac{\left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AC}\right|\left|\overrightarrow{BC}\right|}$.

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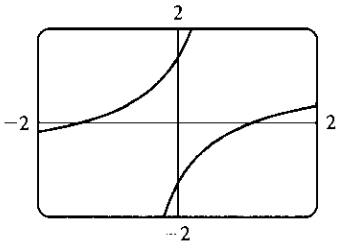
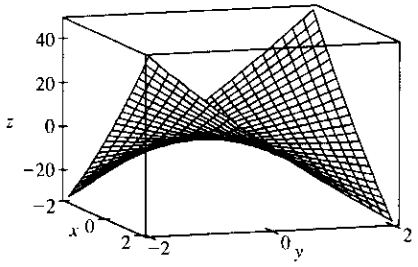
2. $z = x^2 + y^2 + cxy$. When $c < -2$, the surface intersects the plane $z = k \neq 0$ in a hyperbola. (See graph below.) It intersects the plane $x = y$ in the parabola $z = (2 + c)x^2$, and the plane $x = -y$ in the parabola $z = (2 - c)x^2$. These parabolas open in opposite directions, so the surface is a hyperbolic paraboloid.

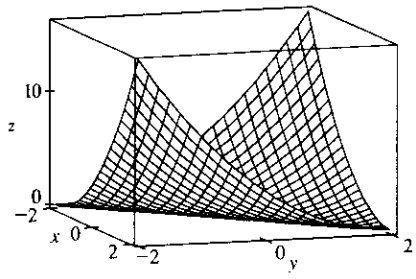
When $c = -2$ the surface is $z = x^2 + y^2 - 2xy = (x - y)^2$. So the surface is constant along each line $x - y = k$. That is, the surface is a cylinder with axis $x - y = 0$, $z = 0$. The shape of the cylinder is determined by its intersection with the plane $x + y = 0$, where $z = 4x^2$, and hence the cylinder is parabolic with minima of 0 on the line $y = x$.



$c = -5$, $z = 2$



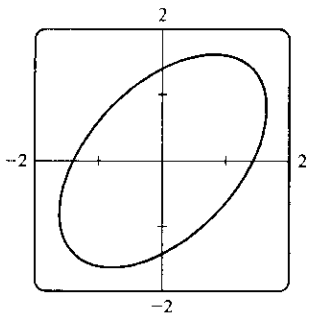
$c = -10$



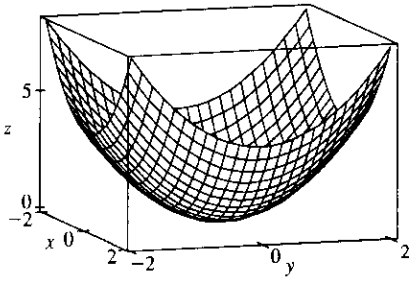
$c = -2$

When $-2 < c \leq 0$, $z \geq 0$ for all x and y. If x and y have the same sign, then $x^2 + y^2 + cxy \geq x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. If they have opposite signs, then $cxy \geq 0$. The intersection with the surface and the plane $z = k > 0$ is an ellipse (see graph below). The intersection with the surface and the planes $x = 0$ and $y = 0$ are parabolas $z = y^2$ and $z = x^2$ respectively, so the surface is an elliptic paraboloid.

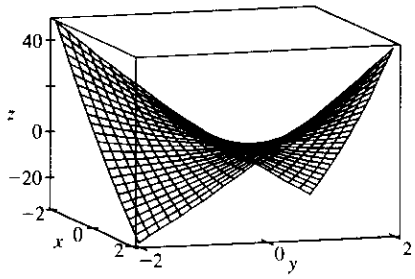
When $c > 0$ the graphs have the same shape, but are reflected in the plane $x = 0$, because $x^2 + y^2 + cxy = (-x)^2 + y^2 + (-c)(-x)y$. That is, the value of z is the same for c at (x, y) as it is for $-c$ at $(-x, y)$.



$c = -1$, $z = 2$



$c = 0$



$c = 10$

So the surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$.

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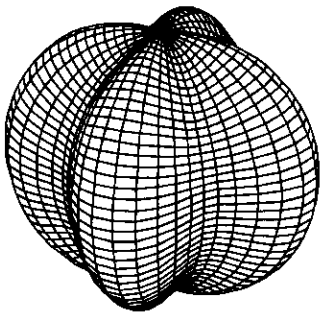
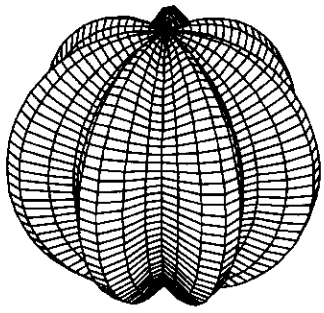
3. $\rho = 1 + 0.2 \sin m\theta \sin n\phi$. If we start with $m = 1, n = 1$ the equation is $\rho = 1 + 0.2 \sin\theta \sin\phi$, whose graph appears spherical or nearly spherical in shape. First we investigate varying just m. Values of $m > 1$ produce vertical ridges in the sphere, the number of ridges corresponding to the value of m. We graph two examples.

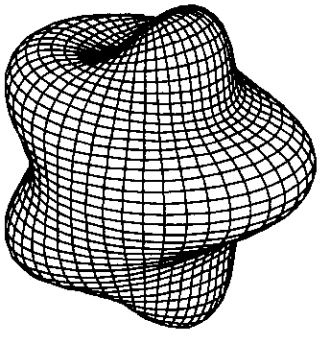


$m = 4, n = 1$

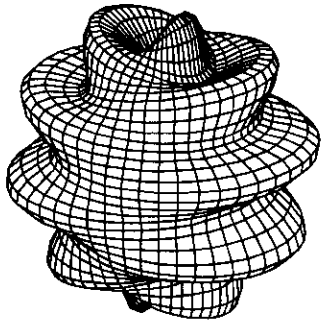


$m = 7, n = 1$

If we leave m fixed at 1 and vary n, we see horizontal ridges that span half the sphere arranged in a staggered fashion. Again, the number of "bumps" coincides with the value of n.

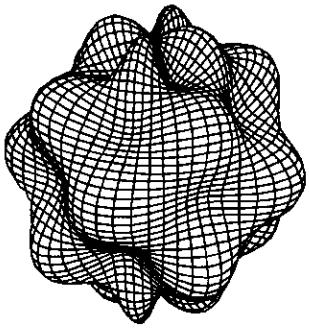


$m = 1, n = 5$

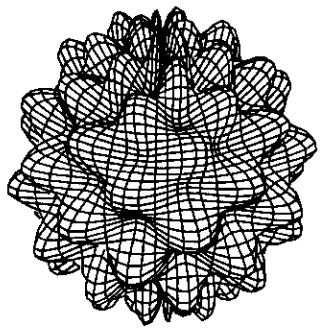


$m = 1, n = 10$

If we allow both m and n to vary, we get combinations of the vertical and horizontal bumps.



$m = 4, n = 5$



$m = 7, n = 10$

The graph on the left shows $m = 4, n = 5$. Looking at the top of the bumpy sphere, we can see the 4 vertical ridges which become perturbed horizontally as they progress down the sphere. We can also see the 5 horizontal rows of bumps. (Consequently, there are 20 bumps on the surface.) The graph on the right shows $m = 7, n = 10$ which should have 70 bumps.

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13 Review

ET 12

CONCEPT CHECK

1. A scalar is a real number, while a vector is a quantity that has both a real-valued magnitude and a direction.
2. To add two vectors geometrically, we can use either the Triangle Law or the Parallelogram Law, as illustrated in Figures 3 and 4 in Section 13.2 [ET 12.2]. Algebraically, we add the corresponding components of the vectors.
3. For $c > 0$, $c\,\mathbf{a}$ is a vector with the same direction as $\mathbf{a}$ and length c times the length of $\mathbf{a}$. If $c < 0$, $c\mathbf{a}$ points in the opposite direction as $\mathbf{a}$ and has length $|c|$ times the length of $\mathbf{a}$. (See Figures 7 and 15 in Section 13.2 [ET 12.2].) Algebraically, to find $c\,\mathbf{a}$ we multiply each component of $\mathbf{a}$ by c.
4. See (1) in Section 13.2 [ET 12.2].
5. See Theorem 13.3.3 [ET 12.3.3] and Definition 13.3.1 [ET 12.3.1].
6. The dot product can be used to find the angle between two vectors and the scalar projection of one vector onto another. In particular, the dot product can determine if two vectors are orthogonal. Also, the dot product can be used to determine the work done moving an object given the force and displacement vectors.
7. See the boxed equations on page 847 [ET 811] as well as Figures 4 and 5 and the accompanying discussion on pages 846–47 [ET 810–11].
8. See Theorem 13.4.6 [ET 12.4.6] and the preceding discussion; use either (1) or (4) in Section 13.4 [ET 12.4].
9. The cross product can be used to create a vector orthogonal to two given vectors as well as to determine if two vectors are parallel. The cross product can also be used to find the area of a parallelogram determined by two vectors. In addition, the cross product can be used to determine torque if the force and position vectors are known.
10. (a) The area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is the length of the cross product: $|\mathbf{a} \times \mathbf{b}|$.

    (b) The volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product: $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.
11. If an equation of the plane is known, it can be written as $ax + by + cz + d = 0$. A normal vector, which is perpendicular to the plane, is $\langle a, b, c \rangle$ (or any scalar multiple of $\langle a, b, c \rangle$). If an equation is not known, we can use points on the plane to find two non-parallel vectors which lie in the plane. The cross product of these vectors is a vector perpendicular to the plane.
12. The angle between two intersecting planes is defined as the acute angle between their normal vectors. We can find this angle using Corollary 13.3.6 [ET 12.3.6].
13. See (1), (2), and (3) in Section 13.5 [ET 12.5].
14. See (5), (6), and (7) in Section 13.5 [ET 12.5].
15. (a) Two (nonzero) vectors are parallel if and only if one is a scalar multiple of the other. In addition, two nonzero vectors are parallel if and only if their cross product is $\mathbf{0}$.

    (b) Two vectors are perpendicular if and only if their dot product is 0.

    (c) Two planes are parallel if and only if their normal vectors are parallel.
16. (a) Determine the vectors $\overrightarrow{PQ} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{PR} = \langle b_1, b_2, b_3 \rangle$. If there is a scalar t such that $\langle a_1, a_2, a_3 \rangle = t\,\langle b_1, b_2, b_3 \rangle$, then the vectors are parallel and the points must all lie on the same line.
    Alternatively, if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$, then $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, so P, Q, and R are collinear.
    Thirdly, an algebraic method is to determine an equation of the line joining two of the points, and then check whether or not the third point satisfies this equation.

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(b) Find the vectors $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$, $\overrightarrow{PS} = \mathbf{c}$. $\mathbf{a} \times \mathbf{b}$ is normal to the plane formed by P, Q and R, and so S lies on this plane if $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ are orthogonal, that is, if $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$. (Or use the reasoning in Example 5 in Section 13.4 [ET 12.4].)

Alternatively, find an equation for the plane determined by three of the points and check whether or not the fourth point satisfies this equation.

17. (a) See Exercise 13.4.39 [ET 12.4.39].

(b) See Example 8 in Section 13.5 [ET 12.5].

(c) See Example 10 in Section 13.5 [ET 12.5].

18. The traces of a surface are the curves of intersection of the surface with planes parallel to the coordinate planes. We can find the trace in the plane $x = k$ (parallel to the yz-plane) by setting $x = k$ and determining the curve represented by the resulting equation. Traces in the planes $y = k$ (parallel to the xz-plane) and $z = k$ (parallel to the xy-plane) are found similarly.

19. See Table 1 in Section 13.6 [ET 12.6].

20. (a) See (1) and the discussion accompanying Figure 3 in Section 13.7 [ET 12.7].

(b) See (3) and Figures 6–8, and the accompanying discussion, in Section 13.7 [ET 12.7].

TRUE-FALSE QUIZ

1. True, by Theorem 13.3.2 [ET 12.3.2] #2.

2. False. Theorem 13.4.8 [ET 12.4.8] #1 says that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

3. True. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then by Theorem 13.4.6 [ET 12.4.6], $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta = |\mathbf{v}|\,|\mathbf{u}| \sin\theta = |\mathbf{v} \times \mathbf{u}|$.
(Or, by Theorem 13.4.8 [ET 12.4.8], $|\mathbf{u} \times \mathbf{v}| = |-\mathbf{v} \times \mathbf{u}| = |-1|\,|\mathbf{v} \times \mathbf{u}| = |\mathbf{v} \times \mathbf{u}|$.)

4. This is true by Theorem 13.3.2 [ET 12.3.2] #4.

5. Theorem 13.4.8 [ET 12.4.8] #2 tells us that this is true.

6. This is true by Theorem 13.4.8 [ET 12.4.8] #4.

7. This is true by Theorem 13.4.8 [ET 12.4.8] #5.

8. In general, this assertion is false; a counterexample is $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$. (See the paragraph preceding Theorem 13.4.8 [ET 12.4.8].)

9. This is true because $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ (see Theorem 13.4.5 [ET 12.4.5]), and the dot product of two orthogonal vectors is 0.

10. $(\mathbf{u} + \mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{v}$ (by Theorem 13.4.8 [ET 12.4.8] #4)

$= \mathbf{u} \times \mathbf{v} + \mathbf{0}$ (by Example 13.4.2 [ET 12.4.2])

$= \mathbf{u} \times \mathbf{v}$, so this is true.

11. If $|\mathbf{u}| = 1$, $|\mathbf{v}| = 1$ and θ is the angle between these two vectors (so $0 \le \theta \le \pi$), then by Theorem 13.4.6 [ET 12.4.6], $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta = \sin\theta$, which is equal to 1 if and only if $\theta = \frac{\pi}{2}$ (that is, if and only if the two vectors are orthogonal). Therefore, the assertion that the cross product of two unit vectors is a unit vector is false.

12. This is false, because according to Equation 13.5.8 [ET 12.5.8], $ax + by + cz + d = 0$ is the general equation of a plane.

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13. This is false. In $\mathbb{R}^2$, $x^2 + y^2 = 1$ represents a circle, but $\{(x, y, z) \mid x^2 + y^2 = 1\}$ represents a *three-dimensional surface*, namely, a circular cylinder with axis the z-axis.

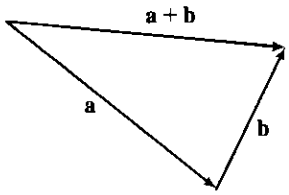
14. This is false, as the dot product of two vectors is a scalar, not a vector.

EXERCISES

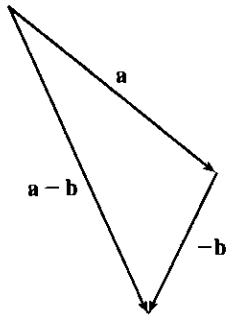
1. (a) By the formula for an equation of a sphere (see Section 13.1 [ET 12.1]), an equation of the sphere with center $(1, -1, 2)$ and radius 3 is $(x-1)^2 + (y+1)^2 + (z-2)^2 = 9$.

(b) Completing squares gives $(x+2)^2 + (y+3)^2 + (z-5)^2 = -2 + 4 + 9 + 25 = 36$. Thus, the sphere is centered at $(-2, -3, 5)$ and has radius 6.

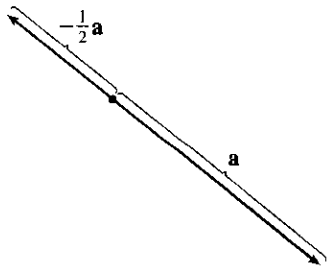
2. (a)



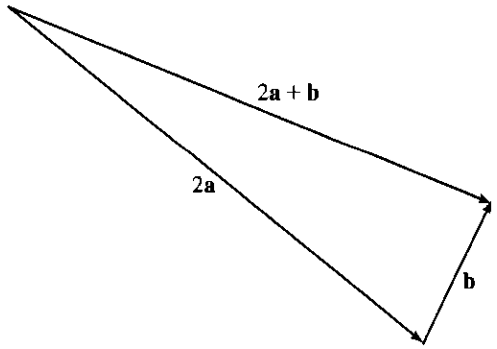
(b)



(c)



(d)



3. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed out of the page.

4. (a) $2\,\mathbf{a} + 3\,\mathbf{b} = 2\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k} + 9\,\mathbf{i} - 6\,\mathbf{j} + 3\,\mathbf{k} = 11\,\mathbf{i} - 4\,\mathbf{j} - \mathbf{k}$

(b) $|\mathbf{b}| = \sqrt{9 + 4 + 1} = \sqrt{14}$

(c) $\mathbf{a} \cdot \mathbf{b} = (1)(3) + (1)(-2) + (-2)(1) = -1$

(d) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = (1-4)\,\mathbf{i} - (1+6)\,\mathbf{j} + (-2-3)\,\mathbf{k} = -3\,\mathbf{i} - 7\,\mathbf{j} - 5\,\mathbf{k}$

(e) $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 9\,\mathbf{i} + 15\,\mathbf{j} + 3\,\mathbf{k}$, $|\mathbf{b} \times \mathbf{c}| = 3\sqrt{9 + 25 + 1} = 3\sqrt{35}$

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(f) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -5 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 9 + 15 - 6 = 18$

(g) $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ for any $\mathbf{c}$.

(h) From part (e),

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times(9\,\mathbf{i}+15\,\mathbf{j}+3\,\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 9 & 15 & 3 \end{vmatrix} = (3+30)\,\mathbf{i} - (3+18)\,\mathbf{j} + (15-9)\,\mathbf{k}$$

$$= 33\,\mathbf{i} - 21\,\mathbf{j} + 6\,\mathbf{k}.$$

(i) The scalar projection is $\text{comp}_{\mathbf{a}}\,\mathbf{b} = |\mathbf{b}|\cos\theta = \mathbf{a}\cdot\mathbf{b}/|\mathbf{a}| = -\frac{1}{\sqrt{6}}$.

(j) The vector projection is $\text{proj}_{\mathbf{a}}\,\mathbf{b} = -\frac{1}{\sqrt{6}}(\mathbf{a}/|\mathbf{a}|) = -\frac{1}{6}(\mathbf{i}+\mathbf{j}-2\,\mathbf{k})$.

(k) $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{6}\sqrt{14}} = \dfrac{-1}{2\sqrt{21}}$ and $\theta = \cos^{-1}\left(\dfrac{-1}{2\sqrt{21}}\right) \approx 96^\circ$.

5. For the two vectors to be orthogonal, we need $\langle 3,2,x\rangle\cdot\langle 2x,4,x\rangle = 0 \quad\Leftrightarrow$ $(3)(2x)+(2)(4)+(x)(x) = 0 \quad\Leftrightarrow\quad x^2+6x+8=0 \quad\Leftrightarrow\quad (x+2)(x+4)=0 \quad\Leftrightarrow$ $x=-2$ or $x=-4$.

6. We know that the cross product of two vectors is orthogonal to both. So we calculate $(\mathbf{j}+2\,\mathbf{k})\times(\mathbf{i}-2\,\mathbf{j}+3\,\mathbf{k}) = [3-(-4)]\,\mathbf{i} - (0-2)\,\mathbf{j} + (0-1)\,\mathbf{k} = 7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k}$. Then two unit vectors orthogonal to both given vectors are $\pm\dfrac{7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k}}{\sqrt{7^2+2^2+(-1)^2}} = \pm\dfrac{1}{3\sqrt{6}}(7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k})$, that is,

$\frac{7}{3\sqrt{6}}\,\mathbf{i}+\frac{2}{3\sqrt{6}}\,\mathbf{j}-\frac{1}{3\sqrt{6}}\,\mathbf{k}$ and $-\frac{7}{3\sqrt{6}}\,\mathbf{i}-\frac{2}{3\sqrt{6}}\,\mathbf{j}+\frac{1}{3\sqrt{6}}\,\mathbf{k}$.

7. (a) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 2$

(b) $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = \mathbf{u}\cdot[-(\mathbf{v}\times\mathbf{w})] = -\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -2$

(c) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = (\mathbf{v}\times\mathbf{u})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -2$

(d) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = \mathbf{u}\cdot\mathbf{0} = 0$

8. $(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})] = (\mathbf{a}\times\mathbf{b})\cdot([(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} - [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}]\,\mathbf{a})$

(see Exercise 13.4.42 [ET 12.4.42])

$$= (\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

$$= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})] = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2$$

9. For simplicity, consider a unit cube positioned with its back left corner at the origin. Vector representations of the diagonals joining the points $(0,0,0)$ to $(1,1,1)$ and $(1,0,0)$ to $(0,1,1)$ are $\langle 1,1,1\rangle$ and $\langle -1,1,1\rangle$. Let θ he the angle between these two vectors. $\langle 1,1,1\rangle\cdot\langle -1,1,1\rangle = -1+1+1 = 1 = |\langle 1,1,1\rangle|\,|\langle -1,1,1\rangle|\cos\theta = 3\cos\theta$ $\Rightarrow\quad \cos\theta = \frac{1}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 71^\circ$.

10. $\overrightarrow{AB} = \langle 1,3,-1\rangle$, $\overrightarrow{AC} = \langle -2,1,3\rangle$ and $\overrightarrow{AD} = \langle -1,3,1\rangle$. By Equation 13.4.10 [ET 12.4.10],

$$\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right) = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix} = -8-3+5 = -6.$$

The volume is $\left|\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right| = 6$ cubic units.

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11. $\overrightarrow{AB} = \langle 1, 0, -1\rangle$, $\overrightarrow{AC} = \langle 0, 4, 3\rangle$, so

(a) a vector perpendicular to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0+4, -(3+0), 4-0\rangle = \langle 4, -3, 4\rangle$.

(b) $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{16+9+16} = \frac{\sqrt{41}}{2}$.

12. $\mathbf{D} = 4\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}$, $W = \mathbf{F} \cdot \mathbf{D} = 12 + 15 + 60 = 87$ joules

13. Let F_1 be the magnitude of the force directed 20° away from the direction of shore, and let F_2 be the magnitude of the other force. Separating these forces into components parallel to the direction of the resultant force and perpendicular to it gives $F_1 \cos 20^\circ + F_2 \cos 30^\circ = 255$ (1), and $F_1 \sin 20^\circ - F_2 \sin 30^\circ = 0 \;\Rightarrow$ $F_1 = F_2 \dfrac{\sin 30^\circ}{\sin 20^\circ}$ (2). Substituting (2) into (1) gives $F_2(\sin 30^\circ \cot 20^\circ + \cos 30^\circ) = 255 \;\Rightarrow\; F_2 \approx 114$ N. Substituting this into (2) gives $F_1 \approx 166$ N.

14. $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.40)(50)\sin(90^\circ - 30^\circ) \approx 17.3$ joules

15. The line has direction $\mathbf{v} = \langle -3, 2, 3\rangle$. Letting $P_0 = (4, -1, 2)$, parametric equations are $x = 4 - 3t$, $y = -1 + 2t$, $z = 2 + 3t$.

16. A direction vector for the line is $\mathbf{v} = \langle 3, 2, 1\rangle$, so parametric equations for the line are $x = 1 + 3t$, $y = 2t$, $z = -1 + t$.

17. A direction vector for the line is a normal vector for the plane, $\mathbf{n} = \langle 2, -1, 5\rangle$, and parametric equations for the line are $x = -2 + 2t$, $y = 2 - t$, $z = 4 + 5t$.

18. Since the two planes are parallel, they will have the same normal vectors. Then we can take $\mathbf{n} = \langle 1, 4, -3\rangle$ and an equation of the plane is $1(x-2) + 4(y-1) - 3(z-0) = 0$ or $x + 4y - 3z = 6$.

19. Here the vectors $\mathbf{a} = \langle 4-3, 0-(-1), 2-1\rangle = \langle 1, 1, 1\rangle$ and $\mathbf{b} = \langle 6-3, 3-(-1), 1-1\rangle = \langle 3, 4, 0\rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -4, 3, 1\rangle$ is a normal vector to the plane and an equation of the plane is $-4(x-3) + 3(y-(-1)) + 1(z-1) = 0$ or $-4x + 3y + z = -14$.

20. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle 2, -1, 3\rangle$ is one vector in the plane. We can verify that the given point $(1, 2, -2)$ does not lie on this line. The point $(0, 3, 1)$ is on the line (obtained by putting $t = 0$) and hence in the plane, so the vector $\mathbf{b} = \langle 0-1, 3-2, 1-(-2)\rangle = \langle -1, 1, 3\rangle$ lies in the plane, and a normal vector is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -6, -9, 1\rangle$. Thus an equation of the plane is $-6(x-1) - 9(y-2) + (z+2) = 0$ or $6x + 93y - z = 26$.

21. Substitution of the parametric equations into the equation of the plane gives $2x - y + z = 2(2-t) - (1+3t) + 4t = 2 \;\Rightarrow\; -t + 3 = 2 \;\Rightarrow\; t = 1$. When $t = 1$, the parametric equations give $x = 2 - 1 = 1$, $y = 1 + 3 = 4$ and $z = 4$. Therefore, the point of intersection is $(1, 4, 4)$.

22. Use the formula proven in Exercise 13.4.39 [ET 12.4.39]. In the notation used in that exercise, $\mathbf{a}$ is just the direction of the line; that is, $\mathbf{a} = \langle 1, -1, 2\rangle$. A point on the line is $(1, 2, -1)$ (setting $t = 0$), and therefore $\mathbf{b} = \langle 1-0, 2-0, -1-0\rangle = \langle 1, 2, -1\rangle$. Hence

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -1, 2\rangle \times \langle 1, 2, -1\rangle|}{\sqrt{1+1+4}} = \frac{|\langle -3, 3, 3\rangle|}{\sqrt{6}} = \sqrt{\frac{27}{6}} = \frac{3}{\sqrt{2}}.$$

23. Since the direction vectors $\langle 2, 3, 4\rangle$ and $\langle 6, -1, 2\rangle$ aren't parallel, neither are the lines. For the lines to intersect, the three equations $1 + 2t = -1 + 6s$, $2 + 3t = 3 - s$, $3 + 4t = -5 + 2s$ must be satisfied simultaneously. Solving the first two equations gives $t = \frac{1}{5}$, $s = \frac{2}{5}$ and checking we see these values don't satisfy the third equation. Thus the lines aren't parallel and they don't intersect, so they must be skew.

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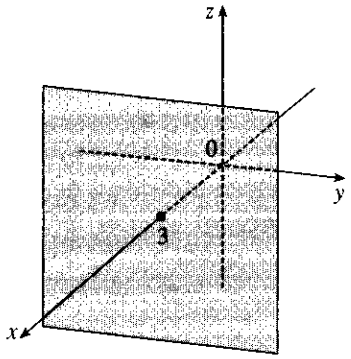
24. (a) The normal vectors are $\langle 1, 1, -1\rangle$ and $\langle 2, -3, 4\rangle$. Since these vectors aren't parallel, neither are the planes parallel. Also $\langle 1, 1, -1\rangle \cdot \langle 2, -3, 4\rangle = 2 - 3 - 4 = -5 \neq 0$ so the normal vectors, and thus the planes, are not perpendicular.

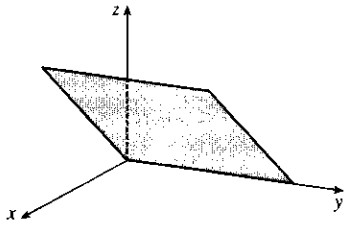
(b) $\cos\theta = \dfrac{\langle 1, 1, -1\rangle \cdot \langle 2, -3, 4\rangle}{\sqrt{3}\sqrt{29}} = -\dfrac{5}{\sqrt{87}}$ and $\theta = \cos^{-1}\left(-\frac{5}{\sqrt{87}}\right) \approx 122^\circ$ (or we can say $\approx 58^\circ$).

25. By Exercise 13.5.69 [ET 12.5.69], $D = \dfrac{|2 - 24|}{\sqrt{26}} = \dfrac{22}{\sqrt{26}}$.

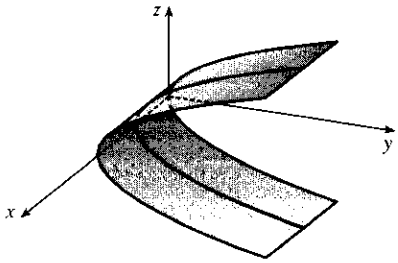
26. The equation $x = 3$ represents a plane parallel to the yz-plane and 3 units in front of it.



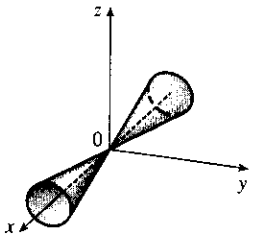
27. The equation $x = z$ represents a plane perpendicular to the xz-plane and intersecting the xz-plane in the line $x = z$, $y = 0$.



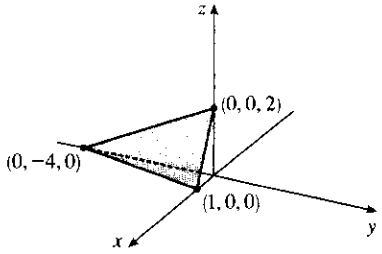
28. A parabolic cylinder whose trace in the xz-plane is the x-axis and which opens to the right.



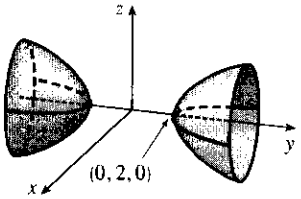
29. A (right elliptical) cone with vertex at the origin and axis the x-axis.



30. $4x - y + 2z = 4$ is a plane with intercepts $(1, 0, 0)$, $(0, -4, 0)$, and $(0, 0, 2)$.



31. An equivalent equation is $-x^2 + \dfrac{y^2}{4} - z^2 = 1$, a hyperboloid of two sheets with axis the y-axis. For $|y| > 2$, traces parallel to the xz-plane are circles.



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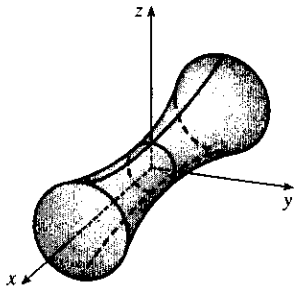
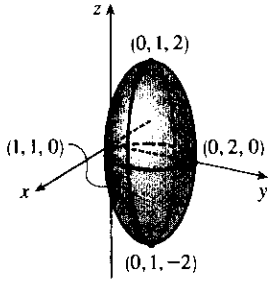
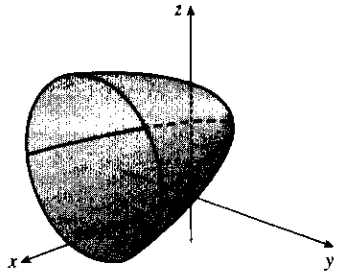
32. An equivalent equation is $-x^2 + y^2 + z^2 = 1$, a hyperboloid of one sheet with axis the x-axis.



33. Completing the square in y gives $4x^2 + 4(y-1)^2 + z^2 = 4$ or $x^2 + (y-1)^2 + \frac{z^2}{4} = 1$, an ellipsoid centered at $(0, 1, 0)$.



34. Completing the square in y and z gives $x = (y-1)^2 + (z-2)^2$, a circular paraboloid with vertex $(0, 1, 2)$ and axis the horizontal line $y = 1, z = 2$.



35. $4x^2 + y^2 = 16 \;\Leftrightarrow\; \frac{x^2}{4} + \frac{y^2}{16} = 1$. The equation of the ellipsoid is $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{c^2} = 1$, since the horizontal trace in the plane $z = 0$ must be the original ellipse. The traces of the ellipsoid in the yz-plane must be circles since the surface is obtained by rotation about the x-axis. Therefore, $c^2 = 16$ and the equation of the ellipsoid is $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \;\Leftrightarrow\; 4x^2 + y^2 + z^2 = 16$.

36. The distance from a point $P(x, y, z)$ to the plane $y = 1$ is $|y-1|$, so the given condition becomes $|y-1| = 2\sqrt{(x-0)^2 + (y+1)^2 + (z-0)^2} \;\Rightarrow\; |y-1| = 2\sqrt{x^2 + (y+1)^2 + z^2} \;\Rightarrow$ $(y-1)^2 = 4x^2 + 4(y+1)^2 + 4z^2 \;\Leftrightarrow\; -3 = 4x^2 + (3y^2 + 10y) + 4z^2 \;\Leftrightarrow$ $\frac{16}{3} = 4x^2 + 3\left(y + \frac{5}{3}\right)^2 + 4z^2 \;\Rightarrow\; \frac{3}{4}x^2 + \frac{9}{16}\left(y + \frac{5}{3}\right)^2 + \frac{3}{4}z^2 = 1$. This is the equation of an ellipsoid whose center is $\left(0, -\frac{5}{3}, 0\right)$.

37. $x = r\cos\theta = 2\sqrt{3}\cos\frac{\pi}{3} = 2\sqrt{3}\cdot\frac{1}{2} = \sqrt{3}$, $y = r\sin\theta = 2\sqrt{3}\sin\frac{\pi}{3} = 2\sqrt{3}\cdot\frac{\sqrt{3}}{2} = 3$, $z = 2$, so in rectangular coordinates the point is $(\sqrt{3}, 3, 2)$. $\rho = \sqrt{r^2 + z^2} = \sqrt{12 + 4} = 4$, $\theta = \frac{\pi}{3}$, and $\cos\phi = \frac{z}{\rho} = \frac{1}{2}$, so $\phi = \frac{\pi}{3}$ and spherical coordinates are $\left(4, \frac{\pi}{3}, \frac{\pi}{3}\right)$.

38. $r = \sqrt{4+4} = 2\sqrt{2}$, $z = -1$, $\cos\theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$ so $\theta = \frac{\pi}{4}$ and in cylindrical coordinates the point is $\left(2\sqrt{2}, \frac{\pi}{4}, -1\right)$. $\rho = \sqrt{4+4+1} = 3$, $\cos\phi = -\frac{1}{3}$, so the spherical coordinates are $\left(3, \frac{\pi}{4}, \cos^{-1}\left(-\frac{1}{3}\right)\right)$.

39. $x = \rho\sin\phi\cos\theta = 8\sin\frac{\pi}{6}\cos\frac{\pi}{4} = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2} = 2\sqrt{2}$, $y = \rho\sin\phi\sin\theta = 8\sin\frac{\pi}{6}\sin\frac{\pi}{4} = 2\sqrt{2}$, and $z = \rho\cos\phi = 8\cos\frac{\pi}{6} = 8\cdot\frac{\sqrt{3}}{2} = 4\sqrt{3}$. Thus rectangular coordinates for the point are $\left(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3}\right)$. $r^2 = x^2 + y^2 = 8 + 8 = 16 \;\Rightarrow\; r = 4$, $\theta = \frac{\pi}{4}$, and $z = 4\sqrt{3}$, so cylindrical coordinates are $\left(4, \frac{\pi}{4}, 4\sqrt{3}\right)$.

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40. $\phi = \frac{\pi}{4}$. This is one frustum of a circular cone with vertex the origin and axis the positive z-axis.

41. $\theta = \frac{\pi}{4}$. In spherical coordinates, this is a half-plane including the z-axis and intersecting the xy-plane in the half-line $x = y$, $x > 0$.

42. Since $r = \cos\theta$ and $x = r\cos\theta$, $x = \cos^2\theta$. Also $r^2 = x^2 + y^2$ so $x^2 + y^2 = 2\cos^2\theta$. Thus $x^2 + y^2 - 2x = 0$ or $(x-1)^2 + y^2 = 1$. Thus the surface is a circular cylinder with axis the line $x = 1$, $y = 0$, $z = z$.

43. Since $\rho = 3\sec\phi$, $\rho\cos\phi = 3$ or $z = 3$. Thus the surface is a plane parallel to the xy-plane and through the point $(0, 0, 3)$.

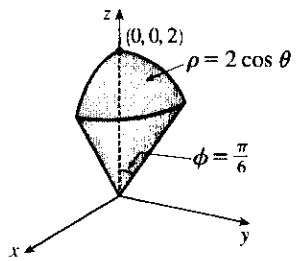
44. $x^2 + y^2 = 4$. In cylindrical coordinates: $r^2 = 4$. In spherical coordinates: $\rho^2 - z^2 = 4$ or $\rho^2 - \rho^2\cos^2\phi = 4$ or $\rho^2\sin^2\phi = 4$ or $\rho\sin\phi = 2$.

45. $x^2 + y^2 + z^2 = 4$. In cylindrical coordinates, this becomes $r^2 + z^2 = 4$. In spherical coordinates, it becomes $\rho^2 = 4$ or $\rho = 2$.

46. In cylindrical coordinates: $r^2 + z^2 = 2r\cos\theta$ or $z^2 = r(2\cos\theta - r)$.
In spherical coordinates: $\rho^2 = 2\rho\sin\phi\cos\theta$ or $\rho = 2\sin\phi\cos\theta$.

47. The resulting surface is a circular paraboloid with equation $z = 4x^2 + 4y^2$. Changing to cylindrical coordinates we have $z = 4(x^2 + y^2) = 4r^2$.

48. $\rho = 2\cos\phi \;\Rightarrow\; \rho^2 = 2\rho\cos\phi \;\Rightarrow\; x^2 + y^2 + z^2 = 2z \;\Rightarrow$ $x^2 + y^2 + (z-1)^2 = 1$. This is the equation of a sphere with radius 1, centered at $(0, 0, 1)$. Therefore, $0 \le \rho \le 2\cos\phi$ is the solid ball whose boundary is this sphere. $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{6}$ restrict the solid to the section of this ball that lies above the cone $\phi = \frac{\pi}{6}$ and is in the first octant.



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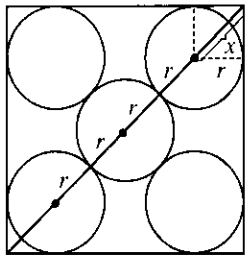
PROBLEMS PLUS

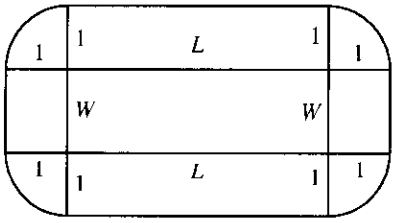
1. Since three-dimensional situations are often difficult to visualize and work with, let us first try to find an analogous problem in two dimensions. The analogue of a cube is a square and the analogue of a sphere is a circle. Thus a similar problem in two dimensions is the following: if five circles with the same radius r are contained in a square of side 1 m so that the circles touch each other and four of the circles touch two sides of the square, find r.



The diagonal of the square is $\sqrt{2}$. The diagonal is also $4r + 2x$. But x is the diagonal of a smaller square of side r. Therefore $x = \sqrt{2}\,r \quad \Rightarrow \quad \sqrt{2} = 4r + 2x = 4r + 2\sqrt{2}\,r = (4 + 2\sqrt{2})r \quad \Rightarrow \quad r = \frac{\sqrt{2}}{4 + 2\sqrt{2}}$.

Let us use these ideas to solve the original three-dimensional problem. The diagonal of the cube is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. The diagonal of the cube is also $4r + 2x$ where x is the diagonal of a smaller cube with edge r. Therefore $x = \sqrt{r^2 + r^2 + r^2} = \sqrt{3}\,r \quad \Rightarrow \quad \sqrt{3} = 4r + 2x = 4r + 2\sqrt{3}\,r = (4 + 2\sqrt{3})r$. Thus $r = \dfrac{\sqrt{3}}{4 + 2\sqrt{3}} = \dfrac{2\sqrt{3} - 3}{2}$. The radius of each ball is $\left(\sqrt{3} - \frac{3}{2}\right)$ m.

2. Try an analogous problem in two dimensions. Consider a rectangle with length L and width W and find the area of S in terms of L and W. Since S contains B, it has area



$$A(S) = LW + \text{the area of two } L \times 1 \text{ rectangles}$$
$$+ \text{ tbe area of two } 1 \times W \text{ rectangles}$$
$$+ \text{ the area of four quarter-circles of radius 1}$$

as seen in the diagram. So $A(S) = LW + 2L + 2W + \pi \cdot 1^2$.

Now in three dimensions, the volume of S is

$$LWH + 2(L \times W \times 1) + 2(1 \times W \times H) + 2(L \times 1 \times H)$$
$$+ \text{ the volume of 4 quarter-cylinders with radius 1 and height } W$$
$$+ \text{ the volume of 4 quarter-cylinders with radius 1 and height } L$$
$$+ \text{ the volume of 4 quarter-cylinders with radius 1 and height } H$$
$$+ \text{ the volume of 8 eighths of a sphere of radius 1}$$

So

$$V(S) = LWH + 2LW + 2WH + 2LH + \pi \cdot 1^2 \cdot W + \pi \cdot 1^2 \cdot L + \pi \cdot 1^2 \cdot H + \tfrac{4}{3}\pi \cdot 1^3$$
$$= LWH + 2(LW + WH + LH) + \pi(L + W + H) + \tfrac{4}{3}\pi.$$

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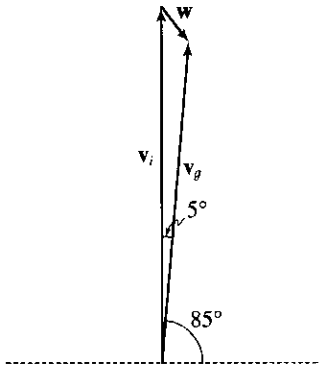
3. (a) We find the line of intersection L as in Example 13.5.7(b) [ET 12.5.7(b)]. Observe that the point $(-1, c, c)$ lies on both planes. Now since L lies in both planes, it is perpendicular to both of the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, and thus parallel to their cross product $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c & 1 & 1 \\ 1 & -c & c \end{vmatrix} = \langle 2c, -c^2+1, -c^2-1 \rangle$. So symmetric equations of L can be written as $\dfrac{x+1}{-2c} = \dfrac{y-c}{c^2-1} = \dfrac{z-c}{c^2+1}$, provided that $c \neq 0, \pm 1$.

If $c = 0$, then the two planes are given by $y + z = 0$ and $x = -1$, so symmetric equations of L are $x = -1$, $y = -z$. If $c = -1$, then the two planes are given by $-x + y + z = -1$ and $x + y + z = -1$, and they intersect in the line $x = 0$, $y = -z - 1$. If $c = 1$, then the two planes are given by $x + y + z = 1$ and $x - y + z = 1$, and they intersect in the line $y = 0$, $x = 1 - z$.

(b) If we set $z = t$ in the symmetric equations and solve for x and y separately, we get $x + 1 = \dfrac{(t-c)(-2c)}{c^2+1}$, $y - c = \dfrac{(t-c)(c^2-1)}{c^2+1} \quad \Rightarrow \quad x = \dfrac{-2ct + (c^2-1)}{c^2+1}$, $y = \dfrac{(c^2-1)t + 2c}{c^2+1}$. Eliminating c from these equations, we have $x^2 + y^2 = t^2 + 1$. So the curve traced out by L in the plane $z = t$ is a circle with center at $(0, 0, t)$ and radius $\sqrt{t^2+1}$.

(c) The area of a horizontal cross-section of the solid is $A(z) = \pi(z^2+1)$, so $V = \int_0^1 A(z)\,dz = \pi\left[\frac{1}{3}z^3 + z\right]_0^1 = \frac{4\pi}{3}$.

4. (a) We consider velocity vectors for the plane and the wind. Let $\mathbf{v}_i$ be the initial, intended velocity for the plane and $\mathbf{v}_g$ the actual velocity relative to the ground. If $\mathbf{w}$ is the velocity of the wind, $\mathbf{v}_g$ is the resultant, that is, the vector sum $\mathbf{v}_i + \mathbf{w}$ as shown in the figure. We know $\mathbf{v}_i = 180\,\mathbf{j}$, and since the plane actually flew 80 km in $\frac{1}{2}$ hour, $|\mathbf{v}_g| = 160$. Thus $\mathbf{v}_g = (160\cos 85^\circ)\,\mathbf{i} + (160\sin 85^\circ)\,\mathbf{j} \approx 13.9\,\mathbf{i} + 159.4\,\mathbf{j}$. Finally, $\mathbf{v}_i + \mathbf{w} = \mathbf{v}_g$, so $\mathbf{w} = \mathbf{v}_g - \mathbf{v}_i \approx 13.9\,\mathbf{i} - 20.6\,\mathbf{j}$. Thus, the wind velocity is about $13.9\,\mathbf{i} - 20.6\,\mathbf{j}$, and the wind speed is $|\mathbf{w}| \approx \sqrt{(13.9)^2 + (-20.6)^2} \approx 24.9$ km/h.



(b) Let $\mathbf{v}$ be the velocity the pilot should take. With the effect of wind, the actual velocity (with respect to the ground) will be $\mathbf{v} + \mathbf{w}$, which we want to be $\mathbf{v}_i$. Thus $\mathbf{v} = \mathbf{v}_i - \mathbf{w} \approx 180\,\mathbf{j} - (13.9\,\mathbf{i} - 20.6\,\mathbf{j}) \approx -13.9\,\mathbf{i} + 200.6\,\mathbf{j}$. The angle for this vector can be found by $\tan\theta \approx \frac{200.6}{-13.9} \quad \Rightarrow \quad \theta \approx 94.0^\circ$, or 4.0° west of north.

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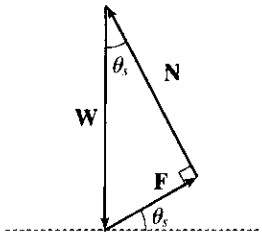
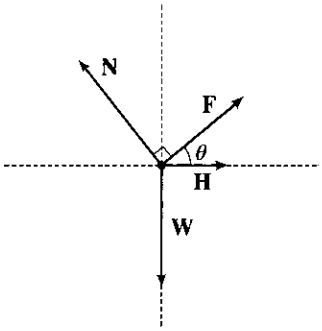
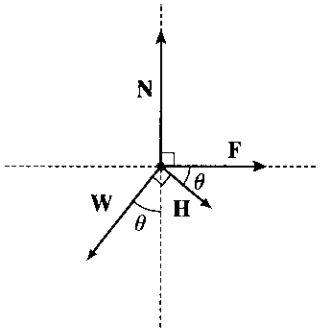
5. (a) When $\theta = \theta_s$, the block is not moving, so the sum of the forces on the block must be $\mathbf{0}$, thus $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$. This relationship is illustrated geometrically in the figure. Since the vectors form a right triangle, we have $\tan(\theta_s) = \dfrac{|\mathbf{F}|}{|\mathbf{N}|} = \dfrac{\mu_s n}{n} = \mu_s$.



(b) We place the block at the origin and sketch the force vectors acting on the block, including the additional horizontal force $\mathbf{H}$, with initial points at the origin. We then rotate this system so that $\mathbf{F}$ lies along the positive x-axis and the inclined plane is parallel to the x-axis.





$|\mathbf{F}|$ is maximal, so $|\mathbf{F}| = \mu_s n$ for $\theta > \theta_s$. Then the vectors, in terms of components parallel and perpendicular to the inclined plane, are

$$\mathbf{N} = n\,\mathbf{j} \qquad \mathbf{F} = (\mu_s n)\,\mathbf{i}$$

$$\mathbf{W} = (-mg\sin\theta)\,\mathbf{i} + (-mg\cos\theta)\,\mathbf{j}$$

$$\mathbf{H} = (h_{\min}\cos\theta)\,\mathbf{i} + (-h_{\min}\sin\theta)\,\mathbf{j}$$

Equating components, we have

$$\mu_s n - mg\sin\theta + h_{\min}\cos\theta = 0 \quad \Rightarrow \quad h_{\min}\cos\theta + \mu_s n = mg\sin\theta \tag{1}$$

$$n - mg\cos\theta - h_{\min}\sin\theta = 0 \quad \Rightarrow \quad h_{\min}\sin\theta + mg\cos\theta = n \tag{2}$$

(c) Since (2) is solved for n, we substitute into (1):

$$h_{\min}\cos\theta + \mu_s(h_{\min}\sin\theta + mg\cos\theta) = mg\sin\theta \quad \Rightarrow$$

$$h_{\min}\cos\theta + h_{\min}\mu_s\sin\theta = mg\sin\theta - mg\mu_s\cos\theta \quad \Rightarrow$$

$$h_{\min} = mg\left(\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)$$

From part (a) we know $\mu_s = \tan\theta_s$, so this becomes $h_{\min} = mg\left(\dfrac{\tan\theta - \tan\theta_s}{1 + \tan\theta_s\tan\theta}\right)$ and using a trigonometric identity, this is $mg\tan(\theta - \theta_s)$ as desired.

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Note for $\theta = \theta_s$, $h_{\min} = mg \tan 0 = 0$, which makes sense since the block is at rest for θ_s, thus no additional force $\mathbf{H}$ is necessary to prevent it from moving. As θ increases, the factor $\tan(\theta - \theta_s)$, and hence the value of $h_{\min}$, increases slowly for small values of $\theta - \theta_s$ but much more rapidly as $\theta - \theta_s$ becomes significant. This seems reasonable, as the steeper the inclined plane, the less the horizontal components of the various forces affect the movement of the block, so we would need a much larger magnitude of horizontal force to keep the block motionless. If we allow $\theta \to 90^\circ$, corresponding to the inclined plane being placed vertically, the value of $h_{\min}$ is quite large; this is to be expected, as it takes a great amount of horizontal force to keep an object from moving vertically. In fact, without friction (so $\theta_s = 0$), we would have $\theta \to 90^\circ \quad \Rightarrow \quad h_{\min} \to \infty$, and it would be impossible to keep the block from slipping.

(d) Since $h_{\max}$ is the largest value of h that keeps the block from slipping, the force of friction is keeping the block from moving *up* the inclined plane; thus, $\mathbf{F}$ is directed *down* the plane. Our system of forces is similar to that in part (b), then, except that we have $\mathbf{F} = -(\mu_s n)\,\mathbf{i}$. (Note that $|\mathbf{F}|$ is again maximal.) Following our procedure in parts (b) and (c), we equate components:

$$-\mu_s n - mg \sin\theta + h_{\max} \cos\theta = 0 \quad \Rightarrow \quad h_{\max} \cos\theta - \mu_s n = mg \sin\theta$$

$$n - mg \cos\theta - h_{\max} \sin\theta = 0 \quad \Rightarrow \quad h_{\max} \sin\theta + mg \cos\theta = n$$

Then substituting,

$$h_{\max} \cos\theta - \mu_s (h_{\max} \sin\theta + mg \cos\theta) = mg \sin\theta \quad \Rightarrow$$

$$h_{\max} \cos\theta - h_{\max} \mu_s \sin\theta = mg \sin\theta + mg\mu_s \cos\theta \quad \Rightarrow$$

$$h_{\max} = mg\left(\frac{\sin\theta + \mu_s \cos\theta}{\cos\theta - \mu_s \sin\theta}\right) = mg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta}\right)$$

$$= mg\left(\frac{\tan\theta + \tan\theta_s}{1 - \tan\theta_s \tan\theta}\right) = mg \tan(\theta + \theta_s)$$

We would expect $h_{\max}$ to increase as θ increases, with similar behavior as we established for $h_{\min}$, but with $h_{\max}$ values always larger than $h_{\min}$. We can see that this is the case if we graph $h_{\max}$ as a function of θ, as the curve is the graph of $h_{\min}$ translated $2\theta_s$ to the left, so the equation does seem reasonable. Notice that the equation predicts $h_{\max} \to \infty$ as $\theta \to (90^\circ - \theta_s)$. In fact, as $h_{\max}$ increases, the normal force increases as well. When $(90^\circ - \theta_s) \le \theta \le 90^\circ$, the horizontal force is completely counteracted by the sum of the normal and frictional forces, so no part of the horizontal force contributes to moving the block up the plane no matter how large its magnitude.

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Chapter 14

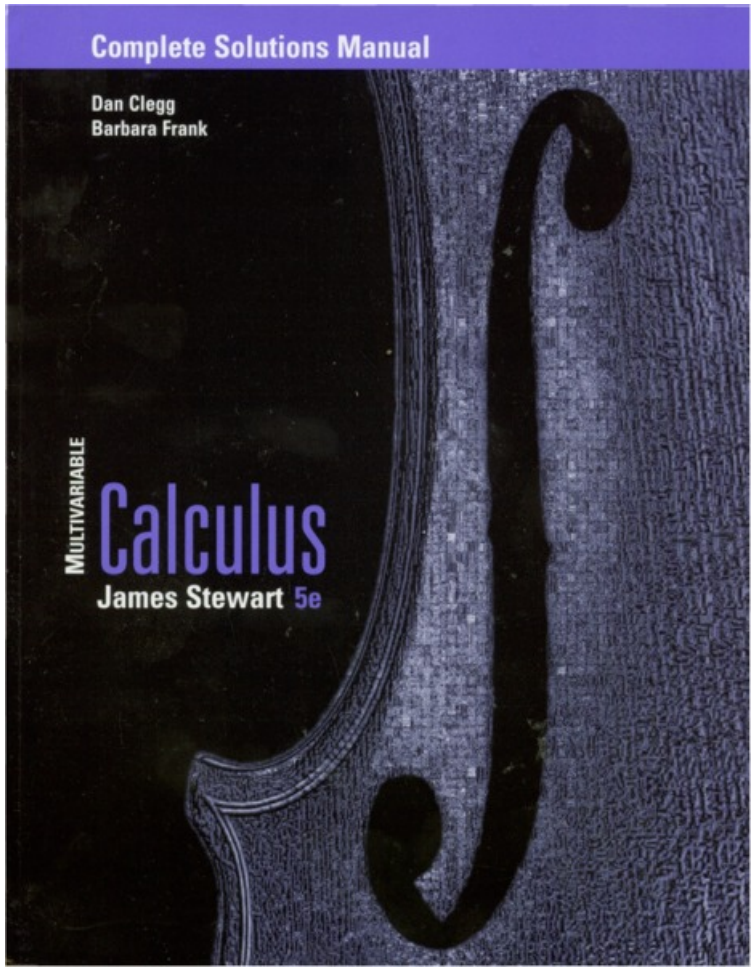
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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14.1 Vector Functions and Space Curves ET 13.1

1. The component functions t^2, $\sqrt{t-1}$, and $\sqrt{5-t}$ are all defined when $t-1 \geq 0 \Rightarrow t \geq 1$ and $5-t \geq 0 \Rightarrow t \leq 5$, so the domain of $\mathbf{r}(t)$ is $[1,5]$.

2. The component functions $\dfrac{t-2}{t+2}$, $\sin t$, and $\ln(9-t^2)$ are all defined when $t \neq -2$ and $9-t^2 > 0 \Rightarrow -3 < t < 3$, so the domain of $\mathbf{r}(t)$ is $(-3,-2) \cup (-2,3)$.

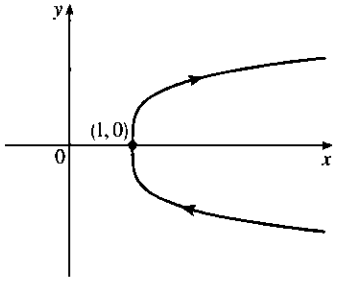
3. $\lim\limits_{t\to 0^+} \cos t = \cos 0 = 1$, $\lim\limits_{t\to 0^+} \sin t = \sin 0 = 0$, $\lim\limits_{t\to 0^+} t\ln t = \lim\limits_{t\to 0^+} \dfrac{\ln t}{1/t} = \lim\limits_{t\to 0^+} \dfrac{1/t}{-1/t^2} = \lim\limits_{t\to 0^+} -t = 0$ [by l'Hospital's Rule]. Thus $\lim\limits_{t\to 0^+} \langle \cos t, \sin t, t\ln t\rangle = \left\langle \lim\limits_{t\to 0^+} \cos t, \lim\limits_{t\to 0^+} \sin t, \lim\limits_{t\to 0^+} t\ln t \right\rangle = \langle 1,0,0\rangle$.

4. $\lim\limits_{t\to 0} \dfrac{e^t-1}{t} = \lim\limits_{t\to 0} \dfrac{e^t}{1} = 1$ [using l'Hospital's Rule],
$\lim\limits_{t\to 0} \dfrac{\sqrt{1+t}-1}{t} = \lim\limits_{t\to 0} \dfrac{\sqrt{1+t}-1}{t} \cdot \dfrac{\sqrt{1+t}+1}{\sqrt{1+t}+1} = \lim\limits_{t\to 0} \dfrac{1}{\sqrt{1+t}+1} = \dfrac{1}{2}$, $\lim\limits_{t\to 0} \dfrac{3}{1+t} = 3$.
Thus the given limit equals $\langle 1, \frac{1}{2}, 3\rangle$.

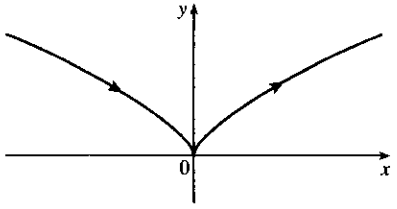
5. $\lim\limits_{t\to 1} \sqrt{t+3} = 2$, $\lim\limits_{t\to 1} \dfrac{t-1}{t^2-1} = \lim\limits_{t\to 1} \dfrac{1}{t+1} = \dfrac{1}{2}$, $\lim\limits_{t\to 1} \left(\dfrac{\tan t}{t}\right) = \tan 1$.
Thus the given limit equals $\langle 2, \frac{1}{2}, \tan 1\rangle$.

6. $\lim\limits_{t\to\infty} \arctan t = \frac{\pi}{2}$, $\lim\limits_{t\to\infty} e^{-2t} = 0$, $\lim\limits_{t\to\infty} \dfrac{\ln t}{t} = \lim\limits_{t\to\infty} \dfrac{1/t}{1} = 0$ [by l'Hospital's Rule].
Thus $\lim\limits_{t\to\infty} \left\langle \arctan t, e^{-2t}, \dfrac{\ln t}{t} \right\rangle = \langle \frac{\pi}{2}, 0, 0\rangle$.

7. The corresponding parametric equations for this curve are $x = t^4+1$, $y = t$. We can make a table of values, or we can eliminate the parameter: $t = y \Rightarrow x = y^4+1$, with $y \in \mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



8. The corresponding parametric equations for this curve are $x = t^3$, $y = t^2$. We can make a table of values, or we can eliminate the parameter: $x = t^3 \Rightarrow t = \sqrt[3]{x} \Rightarrow y = t^2 = (\sqrt[3]{x})^2 = x^{2/3}$, with $t \in \mathbb{R} \Rightarrow x \in \mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



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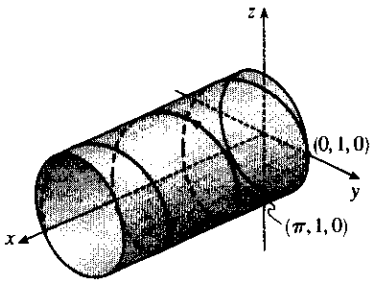
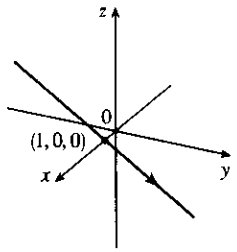
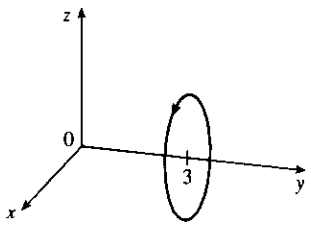
9. The corresponding parametric equations are $x = t$, $y = \cos 2t$, $z = \sin 2t$. Note that $y^2 + z^2 = \cos^2 2t + \sin^2 2t = 1$, so the curve lies on the circular cylinder $y^2 + z^2 = 1$. Since $x = t$, the curve is a helix.



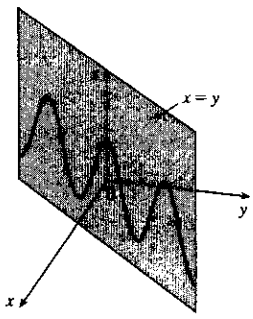
10. The corresponding parametric equations are $x = 1 + t$, $y = 3t$, $z = -t$, which are parametric equations of a line through the point $(1, 0, 0)$ and with direction vector $\langle 1, 3, -1 \rangle$.



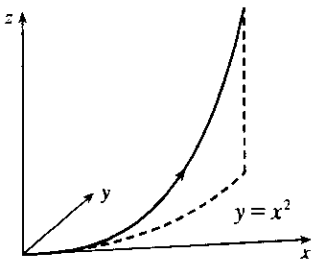
11. The parametric equations give $x^2 + z^2 = \sin^2 t + \cos^2 t = 1$, $y = 3$, which is a circle of radius 1, center $(0, 3, 0)$ in the plane $y = 3$.



12. The parametric equations are $x = t$, $y = t$, $z = \cos t$. Thus $x = y$, so the curve must lie in the plane $x = y$. Combine this with $z = \cos t$ to determine that the curve traces out the cosine curve in the vertical plane $x = y$.



13. The parametric equations are $x = t^2$, $y = t^4$, $z = t^6$. These are positive for $t \neq 0$ and 0 when $t = 0$. So the curve lies entirely in the first quadrant. The projection of the graph onto the xy-plane is $y = x^2$, $y > 0$, a half parabola. On the xz-plane $z = x^3$, $z > 0$, a half cubic, and the yz-plane, $y^3 = z^2$.



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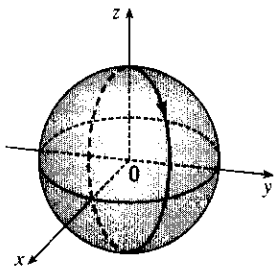
14. The parametric equations give $x^2 + y^2 + z^2 = 2\sin^2 t + 2\cos^2 t = 2$, so the curve lies on the sphere with radius $\sqrt{2}$ and center $(0, 0, 0)$. Furthermore $x = y = \sin t$, so the curve is the intersection of this sphere with the plane $x = y$, that is, the curve is the circle of radius $\sqrt{2}$, center $(0, 0, 0)$ in the plane $x = y$.



15. Taking $\mathbf{r}_0 = \langle 0, 0, 0\rangle$ and $\mathbf{r}_1 = \langle 1, 2, 3\rangle$, we have from Equation 13.5.4 [ET 12.5.4]
$\mathbf{r}(t) = (1 - t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1 - t)\,\langle 0, 0, 0\rangle + t\,\langle 1, 2, 3\rangle$, $0 \le t \le 1$ or $\mathbf{r}(t) = \langle t, 2t, 3t\rangle$, $0 \le t \le 1$.
Parametric equations are $x = t$, $y = 2t$, $z = 3t$, $0 \le t \le 1$.

16. Taking $\mathbf{r}_0 = \langle 1, 0, 1\rangle$ and $\mathbf{r}_1 = \langle 2, 3, 1\rangle$, we have from Equation 13.5.4 [ET 12.5.4]
$\mathbf{r}(t) = (1 - t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1 - t)\,\langle 1, 0, 1\rangle + t\,\langle 2, 3, 1\rangle$, $0 \le t \le 1$ or $\mathbf{r}(t) = \langle 1 + t, 3t, 1\rangle$, $0 \le t \le 1$.
Parametric equations are $x = 1 + t$, $y = 3t$, $z = 1$, $0 \le t \le 1$.

17. Taking $\mathbf{r}_0 = \langle 1, -1, 2\rangle$ and $\mathbf{r}_1 = \langle 4, 1, 7\rangle$, we have $\mathbf{r}(t) = (1 - t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1 - t)\,\langle 1, -1, 2\rangle + t\,\langle 4, 1, 7\rangle$, $0 \le t \le 1$ or $\mathbf{r}(t) = \langle 1 + 3t, -1 + 2t, 2 + 5t\rangle$, $0 \le t \le 1$. Parametric equations are $x = 1 + 3t$, $y = -1 + 2t$, $z = 2 + 5t$, $0 \le t \le 1$.

18. Taking $\mathbf{r}_0 = \langle -2, 4, 0\rangle$ and $\mathbf{r}_1 = \langle 6, -1, 2\rangle$, we have $\mathbf{r}(t) = (1 - t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1 - t)\,\langle -2, 4, 0\rangle + t\,\langle 6, -1, 2\rangle$, $0 \le t \le 1$ or $\mathbf{r}(t) = \langle -2 + 8t, 4 - 5t, 2t\rangle$, $0 \le t \le 1$. Parametric equations are $x = -2 + 8t$, $y = 4 - 5t$, $z = 2t$, $0 \le t \le 1$.

19. $x = \cos 4t$, $y = t$, $z = \sin 4t$. At any point (x, y, z) on the curve, $x^2 + z^2 = \cos^2 4t + \sin^2 4t = 1$. So the curve lies on a circular cylinder with axis the y-axis. Since $y = t$, this is a helix. So the graph is VI.

20. $x = t$, $y = t^2$, $z = e^{-t}$. At any point on the curve, $y = x^2$. So the curve lies on the parabolic cylinder $y = x^2$. Note that y and z are positive for all t, and the point $(0, 0, 1)$ is on the curve (when $t = 0$). As $t \to \infty$, $(x, y, z) \to (\infty, \infty, 0)$, while as $t \to -\infty$, $(x, y, z) \to (-\infty, \infty, \infty)$, so the graph must be II.

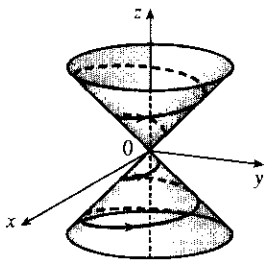
21. $x = t$, $y = 1/(1 + t^2)$, $z = t^2$. Note that y and z are positive for all t. The curve passes through $(0, 1, 0)$ when $t = 0$. As $t \to \infty$, $(x, y, z) \to (\infty, 0, \infty)$, and as $t \to -\infty$, $(x, y, z) \to (-\infty, 0, \infty)$. So the graph is IV.

22. $x = e^{-t}\cos 10t$, $y = e^{-t}\sin 10t$, $z = e^{-t}$.
$x^2 + y^2 = e^{-2t}\cos^2 10t + e^{-2t}\sin^2 10t = e^{-2t}(\cos^2 10t + \sin^2 10t) = e^{-2t} = z^2$, so the curve lies on the cone $x^2 + y^2 = z^2$. Also, z is always positive; the graph must be **I**.

23. $x = \cos t$, $y = \sin t$, $z = \sin 5t$. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the z-axis. Each of x, y and z is periodic, and at $t = 0$ and $t = 2\pi$ the curve passes through the same point, so the curve repeats itself and the graph is V.

24. $x = \cos t$, $y = \sin t$, $z = \ln t$. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the z-axis. As $t \to 0$, $z \to -\infty$, so the graph is III.

25. If $x = t\cos t$, $y = t\sin t$, and $z = t$, then $x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z^2$, so the curve lies on the cone $z^2 = x^2 + y^2$. Since $z = t$, the curve is a spiral on this cone.



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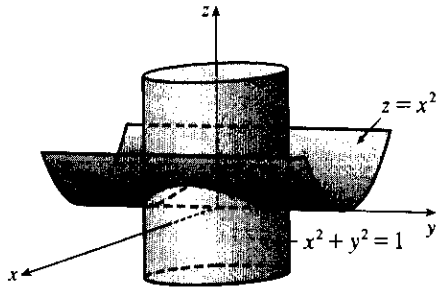
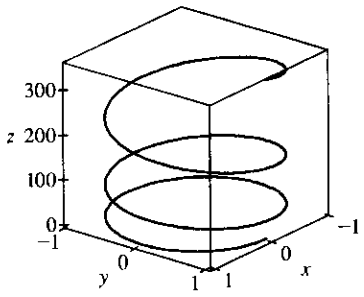
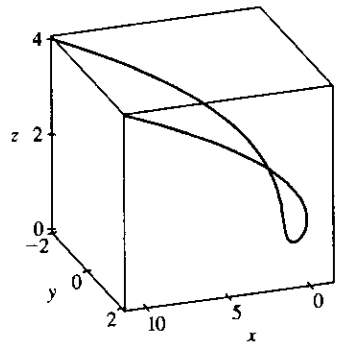
26. Here $x^2 = \sin^2 t = z$ and $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$, so the curve is the intersection of the parabolic cylinder $z = x^2$ with the circular cylinder $x^2 + y^2 = 1$.



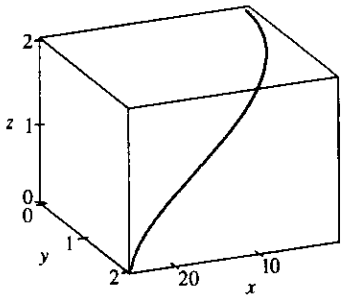
27. $\mathbf{r}(t) = \langle \sin t, \cos t, t^2 \rangle$



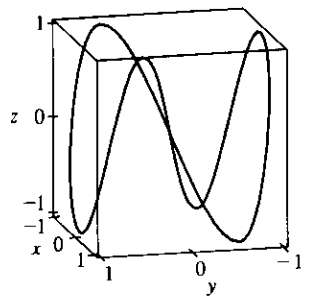
28. $\mathbf{r}(t) = \langle t^4 - t^2 + 1, t, t^2 \rangle$

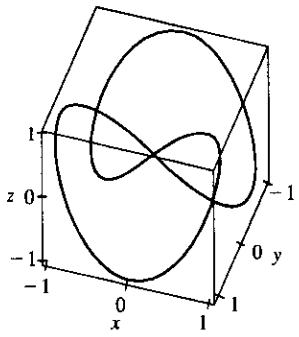


29. $\mathbf{r}(t) = \langle t^2, \sqrt{t-1}, \sqrt{5-t} \rangle$

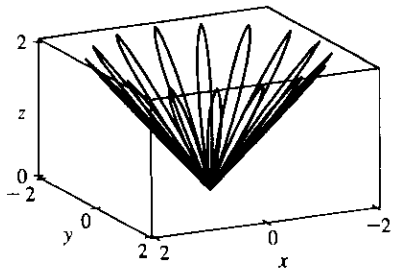


30. We have the computer plot the parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$, $0 \le t \le 2\pi$. The shape of the curve is not clear from just one viewpoint, so we include a second plot drawn from a different angle.





31.



$x = (1 + \cos 16t)\cos t$, $y = (1 + \cos 16t)\sin t$, $z = 1 + \cos 16t$. At any point on the graph,

$x^2 + y^2 = (1 + \cos 16t)^2 \cos^2 t + (1 + \cos 16t)^2 \sin^2 t$

$= (1 + \cos 16t)^2 = z^2$, so the graph lies on the cone $x^2 + y^2 = z^2$. From the graph at left, we see that this curve looks like the projection of a leaved two-dimensional curve onto a cone.

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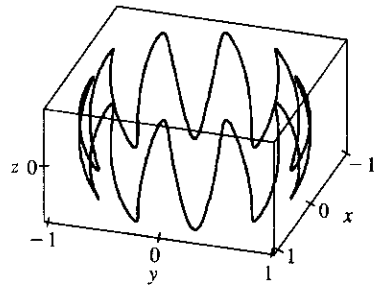
32.



$x = \sqrt{1 - 0.25\cos^2 10t}\cos t$, $y = \sqrt{1 - 0.25\cos^2 10t}\sin t$, $z = 0.5\cos 10t$. At any point on the graph,

$$\begin{aligned} x^2 + y^2 + z^2 &= (1 - 0.25\cos^2 10t)\cos^2 t \\ &\quad + (1 - 0.25\cos^2 10t)\sin^2 t + 0.25\cos^2 t \\ &= 1 - 0.25\cos^2 10t + 0.25\cos^2 10t = 1, \end{aligned}$$

so the graph lies on the sphere $x^2 + y^2 + z^2 = 1$, and since $z = 0.5\cos 10t$ the graph resembles a trigonometric curve with ten peaks projected onto the sphere. The graph is generated by $t \in [0, 2\pi]$.

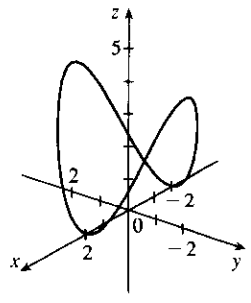
33. If $t = -1$, then $x = 1, y = 4, z = 0$, so the curve passes through the point $(1, 4, 0)$. If $t = 3$, then $x = 9, y = -8, z = 28$, so the curve passes through the point $(9, -8, 28)$. For the point $(4, 7, -6)$ to be on the curve, we require $y = 1 - 3t = 7 \;\Rightarrow\; t = -2$. But then $z = 1 + (-2)^3 = -7 \neq -6$, so $(4, 7, -6)$ is not on the curve.

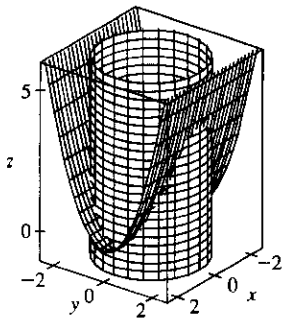
34. The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 4, z = 0$. Then we can write $x = 2\cos t, y = 2\sin t, 0 \le t \le 2\pi$. Since C also lies on the surface $z = xy$, we have $z = xy = (2\cos t)(2\sin t) = 4\cos t \sin t$, or $2\sin(2t)$. Then parametric equations for C are $x = 2\cos t, y = 2\sin t, z = 2\sin(2t), 0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\sin(2t)\,\mathbf{k}, 0 \le t \le 2\pi$.

35. Both equations are solved for z, so we can substitute to eliminate z: $\sqrt{x^2 + y^2} = 1 + y \;\Rightarrow$ $x^2 + y^2 = 1 + 2y + y^2 \;\Rightarrow\; x^2 = 1 + 2y \;\Rightarrow\; y = \frac{1}{2}(x^2 - 1)$. We can form parametric equations for the curve C of intersection by choosing a parameter $x = t$, then $y = \frac{1}{2}(t^2 - 1)$ and $z = 1 + y = 1 + \frac{1}{2}(t^2 - 1) = \frac{1}{2}(t^2 + 1)$. Thus a vector function representing C is $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$.

36. The projection of the curve C of intersection onto the xy-plane is the parabola $y = x^2, z = 0$. Then we can choose the parameter $x = t \;\Rightarrow\; y = t^2$. Since C also lies on the surface $z = 4x^2 + y^2$, we have $z = 4x^2 + y^2 = 4t^2 + (t^2)^2$. Then parametric equations for C are $x = t, y = t^2, z = 4t^2 + t^4$, and the corresponding vector function is $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + (4t^2 + t^4)\,\mathbf{k}$.

37.





The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 4, z = 0$. Then we can write $x = 2\cos t, y = 2\sin t$, $0 \le t \le 2\pi$. Since C also lies on the surface $z = x^2$, we have $z = x^2 = (2\cos t)^2 = 4\cos^2 t$. Then parametric equations for C are $x = 2\cos t$, $y = 2\sin t, z = 4\cos^2 t, 0 \le t \le 2\pi$.

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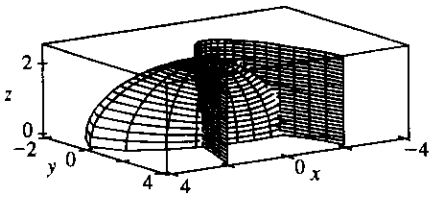
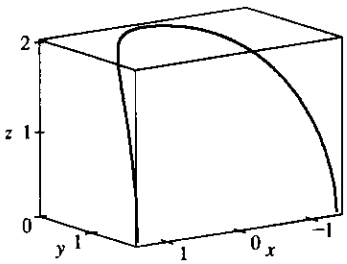
38.





$x = t \;\Rightarrow\; y = t^2 \;\Rightarrow\; 4z^2 = 16 - x^2 - 4y^2 = 16 - t^2 - 4t^4 \;\Rightarrow\; z = \sqrt{4 - \left(\frac{1}{2}t\right)^2 - t^4}$. Note that z is positive because the intersection is with the top half of the ellipsoid. Hence the curve is given by $x = t$, $y = t^2$, $z = \sqrt{4 - \frac{1}{4}t^2 - t^4}$.

39. For the particles to collide, we require $\mathbf{r}_1(t) = \mathbf{r}_2(t) \;\Leftrightarrow\; \langle t^2, 7t - 12, t^2 \rangle = \langle 4t - 3, t^2, 5t - 6 \rangle$. Equating components gives $t^2 = 4t - 3$, $7t - 12 = t^2$, and $t^2 = 5t - 6$. From the first equation, $t^2 - 4t + 3 = 0 \;\Leftrightarrow\; (t - 3)(t - 1) = 0$ so $t = 1$ or $t = 3$. $t = 1$ does not satisfy the other two equations, but $t = 3$ does. The particles collide when $t = 3$, at the point $(9, 9, 9)$.

40. The particles collide provided $\mathbf{r}_1(t) = \mathbf{r}_2(t) \;\Leftrightarrow\; \langle t, t^2, t^3 \rangle = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$. Equating components gives $t = 1 + 2t$, $t^2 = 1 + 6t$, and $t^3 = 1 + 14t$. The first equation gives $t = -1$, but this does not satisfy the other equations, so the particles do not collide. For the paths to intersect, we need to find a value for t and a value for s where $\mathbf{r}_1(t) = \mathbf{r}_2(s) \;\Leftrightarrow\; \langle t, t^2, t^3 \rangle = \langle 1 + 2s, 1 + 6s, 1 + 14s \rangle$. Equating components, $t = 1 + 2s$, $t^2 = 1 + 6s$, and $t^3 = 1 + 14s$. Substituting the first equation into the second gives $(1 + 2s)^2 = 1 + 6s \;\Rightarrow\; 4s^2 - 2s = 0 \;\Rightarrow\; 2s(2s - 1) = 0 \;\Rightarrow\; s = 0$ or $s = \frac{1}{2}$. From the first equation, $s = 0 \;\Rightarrow\; t = 1$ and $s = \frac{1}{2} \;\Rightarrow\; t = 2$. Checking, we see that both pairs of values satisfy the third equation. Thus the paths intersect twice, at the point $(1, 1, 1)$ when $s = 0$ and $t = 1$, and at $(2, 4, 8)$ when $s = \frac{1}{2}$ and $t = 2$.

41. Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t) \rangle$. In each part of this problem the basic procedure is to use Equation 1 and then analyze the individual component functions using the limit properties we have already developed for real-valued functions.

(a) $\lim_{t \to a} \mathbf{u}(t) + \lim_{t \to a} \mathbf{v}(t) = \left\langle \lim_{t \to a} u_1(t), \lim_{t \to a} u_2(t), \lim_{t \to a} u_3(t) \right\rangle + \left\langle \lim_{t \to a} v_1(t), \lim_{t \to a} v_2(t), \lim_{t \to a} v_3(t) \right\rangle$ and the limits of these component functions must each exist since the vector functions both possess limits as $t \to a$. Then adding the two vectors and using the addition property of limits for real-valued functions, we have that

$$\begin{aligned}
\lim_{t \to a} \mathbf{u}(t) + \lim_{t \to a} \mathbf{v}(t) &= \left\langle \lim_{t \to a} u_1(t) + \lim_{t \to a} v_1(t), \lim_{t \to a} u_2(t) + \lim_{t \to a} v_2(t), \lim_{t \to a} u_3(t) + \lim_{t \to a} v_3(t) \right\rangle \\
&= \left\langle \lim_{t \to a} [u_1(t) + v_1(t)], \lim_{t \to a} [u_2(t) + v_2(t)], \lim_{t \to a} [u_3(t) + v_3(t)] \right\rangle \\
&= \lim_{t \to a} \langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t) \rangle \quad \text{[using (1) backward]} \\
&= \lim_{t \to a} [\mathbf{u}(t) + \mathbf{v}(t)]
\end{aligned}$$

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(b) $\lim_{t\to a} c\mathbf{u}(t) = \lim_{t\to a} \langle cu_1(t), cu_2(t), cu_3(t)\rangle = \left\langle \lim_{t\to a} cu_1(t), \lim_{t\to a} cu_2(t), \lim_{t\to a} cu_3(t)\right\rangle$

$$= \left\langle c\lim_{t\to a} u_1(t), c\lim_{t\to a} u_2(t), c\lim_{t\to a} u_3(t)\right\rangle = c\left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle$$

$$= c\lim_{t\to a}\langle u_1(t), u_2(t), u_3(t)\rangle = c\lim_{t\to a}\mathbf{u}(t)$$

(c) $\lim_{t\to a}\mathbf{u}(t)\cdot\lim_{t\to a}\mathbf{v}(t) = \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \cdot \left\langle \lim_{t\to a} v_1(t), \lim_{t\to a} v_2(t), \lim_{t\to a} v_3(t)\right\rangle$

$$= \left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_1(t)\right] + \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_2(t)\right] + \left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_3(t)\right]$$

$$= \lim_{t\to a} u_1(t)v_1(t) + \lim_{t\to a} u_2(t)v_2(t) + \lim_{t\to a} u_3(t)v_3(t)$$

$$= \lim_{t\to a}\left[u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)\right] = \lim_{t\to a}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]$$

(d) $\lim_{t\to a}\mathbf{u}(t)\times\lim_{t\to a}\mathbf{v}(t) = \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \times \left\langle \lim_{t\to a} v_1(t), \lim_{t\to a} v_2(t), \lim_{t\to a} v_3(t)\right\rangle$

$$= \left\langle \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_3(t)\right] - \left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_2(t)\right],\right.$$
$$\left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_1(t)\right] - \left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_3(t)\right],$$
$$\left.\left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_2(t)\right] - \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_1(t)\right]\right\rangle$$

$$= \left\langle \lim_{t\to a}\left[u_2(t)v_3(t) - u_3(t)v_2(t)\right], \lim_{t\to a}\left[u_3(t)v_1(t) - u_1(t)v_3(t)\right],\right.$$
$$\left.\lim_{t\to a}\left[u_1(t)v_2(t) - u_2(t)v_1(t)\right]\right\rangle$$

$$= \lim_{t\to a}\langle u_2(t)v_3(t) - u_3(t)v_2(t), u_3(t)v_1(t) - u_1(t)v_3(t),$$
$$u_1(t)v_2(t) - u_2(t)v_1(t)\rangle$$

$$= \lim_{t\to a}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]$$

42. The projection of the curve onto the xy-plane is given by the parametric equations $x = (2+\cos 1.5t)\cos t$, $y = (2+\cos 1.5t)\sin t$. If we convert to polar coordinates, we have

$$r^2 = x^2 + y^2 = [(2+\cos 1.5t)\cos t]^2 + [(2+\cos 1.5t)\sin t]^2$$
$$= (2+\cos 1.5t)^2(\cos^2 t + \sin^2 t)$$
$$= (2+\cos 1.5t)^2$$

$\Rightarrow\quad r = 2+\cos 1.5t$. Also, $\tan\theta = \dfrac{y}{x} = \dfrac{(2+\cos 1.5t)\sin t}{(2+\cos 1.5t)\cos t} = \tan t \quad\Rightarrow\quad \theta = t$.

Thus the polar equation of the curve is $r = 2+\cos 1.5\theta$. At $\theta = 0$, we have $r = 3$, and r decreases to 1 as θ increases to $\frac{2\pi}{3}$. For $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$, r increases to 3; r decreases to 1 again at $\theta = 2\pi$, increases to 3 at $\theta = \frac{8\pi}{3}$,

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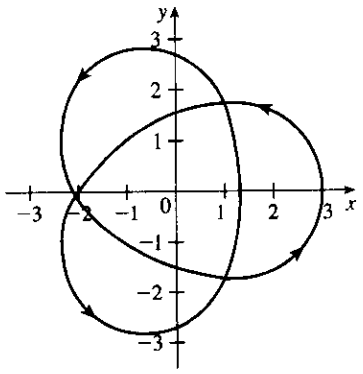
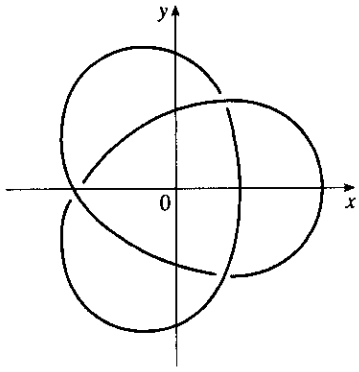
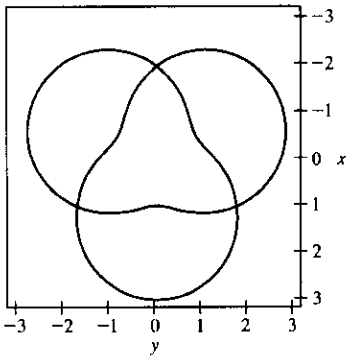
decreases to 1 at $\theta = \frac{10\pi}{3}$, and completes the closed curve by increasing to 3 at $\theta = 4\pi$. We sketch an approximate graph as shown in the figure.



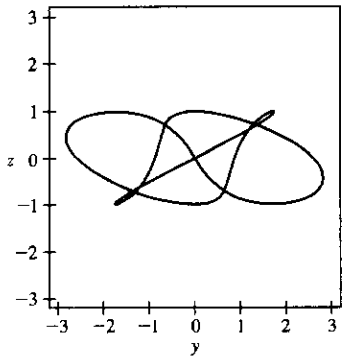
We can determine how the curve passes over itself by investigating the maximum and minimum values of z for $t = \theta \in [0, 4\pi]$. Since $z = \sin 1.5t$, z is maximized where $\sin 1.5t = 1 \Rightarrow 1.5t = \frac{\pi}{2}, \frac{5\pi}{2}$, or $\frac{9\pi}{2} \Rightarrow t = \frac{\pi}{3}, \frac{5\pi}{3}$, or 3π. z is minimized where $\sin 1.5t = -1 \Rightarrow 1.5t = \frac{3\pi}{2}, \frac{7\pi}{2}$, or $\frac{11\pi}{2} \Rightarrow t = \pi, \frac{7\pi}{3}$, or $\frac{11\pi}{3}$. Note that these are precisely the values for which $\cos 1.5t = 0 \Rightarrow r = 2$, and on the graph of the projection, these six points appear to be at the three self-intersections we see. Comparing the maximum and minimum values of z at these intersections, we can determine where the curve passes over itself, as indicated in the figure.



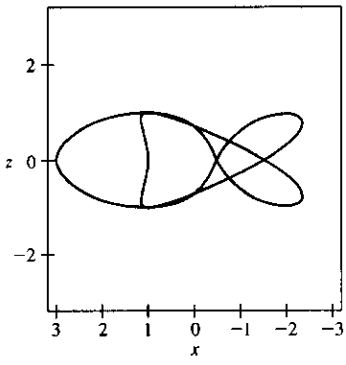
We show a computer-drawn graph of the curve from above, as well as views from the front and from the right side.



Top view



Front view



Side view

The top view graph shows a more accurate representation of the projection of the trefoil knot on the xy-plane (the axes are rotated 90°). Notice the indentations the graph exhibits at the points corresponding to $r = 1$. Finally, we graph several additional viewpoints of the trefoil knot, along with two plots showing a tube of radius 0.2 around the curve.

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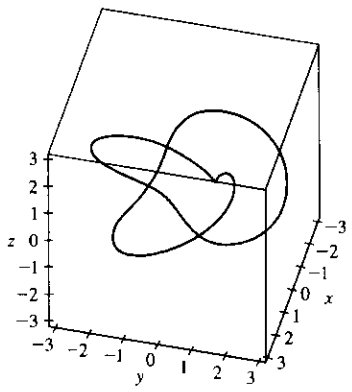
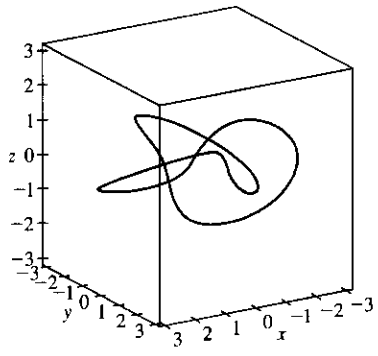
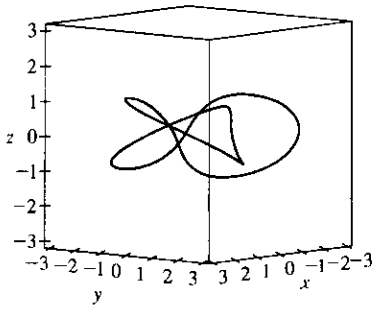
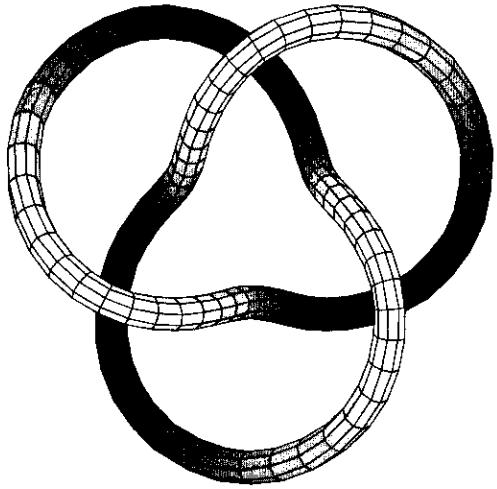
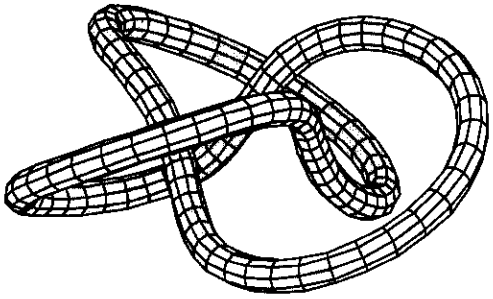

43. Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. If $\lim_{t \to a} \mathbf{r}(t) = \mathbf{b}$, then $\lim_{t \to a} \mathbf{r}(t)$ exists, so by (1), $\mathbf{b} = \lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$. By the definition of equal vectors we have $\lim_{t \to a} f(t) = b_1$, $\lim_{t \to a} g(t) = b_2$ and $\lim_{t \to a} h(t) = b_3$. But these are limits of real-valued functions, so by the definition of limits, for every $\epsilon > 0$ there exists $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ so $|f(t) - b_1| < \epsilon/3$ whenever $0 < |t - a| < \delta_1$, $|g(t) - b_2| < \epsilon/3$ whenever $0 < |t - a| < \delta_2$, and $|h(t) - b_3| < \epsilon/3$ whenever $0 < |t - a| < \delta_3$. Letting $\delta =$ minimum of $\{\delta_1, \delta_2, \delta_3\}$, we have $|f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$ whenever $0 < |t - a| < \delta$. But $|\mathbf{r}(t) - \mathbf{b}| = |\langle f(t) - b_1, g(t) - b_2, h(t) - b_3 \rangle|$ $= \sqrt{(f(t) - b_1)^2 + (g(t) - b_2)^2 + (h(t) - b_3)^2} \leq \sqrt{[f(t) - b_1]^2} + \sqrt{[g(t) - b_2]^2} + \sqrt{[h(t) - b_3]^2}$ $= |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3|$. Thus for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{r}(t) - b| \leq |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \epsilon$ whenever $0 < |t - a| < \delta$. Conversely, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\mathbf{r}(t) - \mathbf{b}| < \epsilon$ whenever $0 < |t - a| < \delta$, then

$|\langle f(t) - b_1, g(t) - b_2, h(t) - b_3 \rangle| < \epsilon \quad \Leftrightarrow \quad \sqrt{[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2} < \epsilon \quad \Leftrightarrow$

$[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2 < \epsilon^2$ whenever $0 < |t - a| < \delta$. But each term on the left side of this inequality is positive so $[f(t) - b_1]^2 < \epsilon^2$, $[g(t) - b_2]^2 < \epsilon^2$ and $[h(t) - b_3]^2 < \epsilon^2$ whenever $0 < |t - a| < \delta$, or

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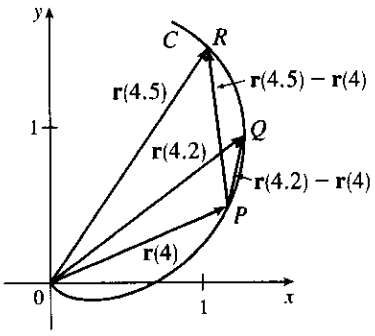
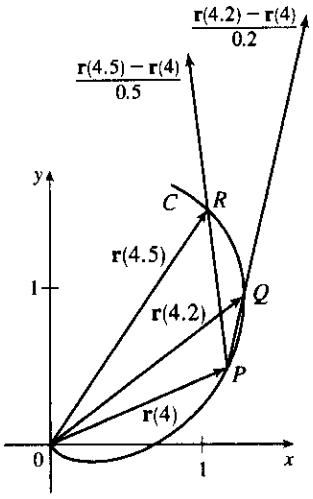
taking the square root of both sides in each of the above we have $|f(t) - b_1| < \epsilon$, $|g(t) - b_2| < \epsilon$ and $|h(t) - b_3| < \epsilon$ whenever $0 < |t - a| < \delta$. And by definition of limits of real-valued functions we have $\lim_{t \to a} f(t) = b_1$, $\lim_{t \to a} g(t) = b_2$ and $\lim_{t \to a} h(t) = b_3$. But by (1), $\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$, so $\lim_{t \to a} \mathbf{r}(t) = \langle b_1, b_2, b_3 \rangle = \mathbf{b}$.

14.2 Derivatives and Integrals of Vector Functions ET 13.2

1. (a)



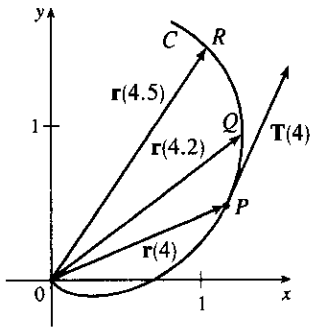
(b) $\dfrac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} = 2[\mathbf{r}(4.5) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\mathbf{r}(4.5) - \mathbf{r}(4)$. $\dfrac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2} = 5[\mathbf{r}(4.2) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2) - \mathbf{r}(4)$.



(c) By Definition 1, $\mathbf{r}'(4) = \lim_{h \to 0} \dfrac{\mathbf{r}(4 + h) - \mathbf{r}(4)}{h}$.

$\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$.

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$ with length 1.



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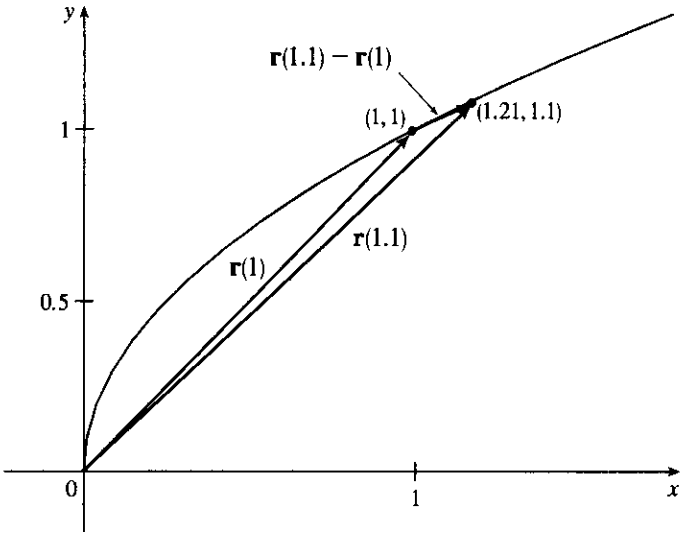
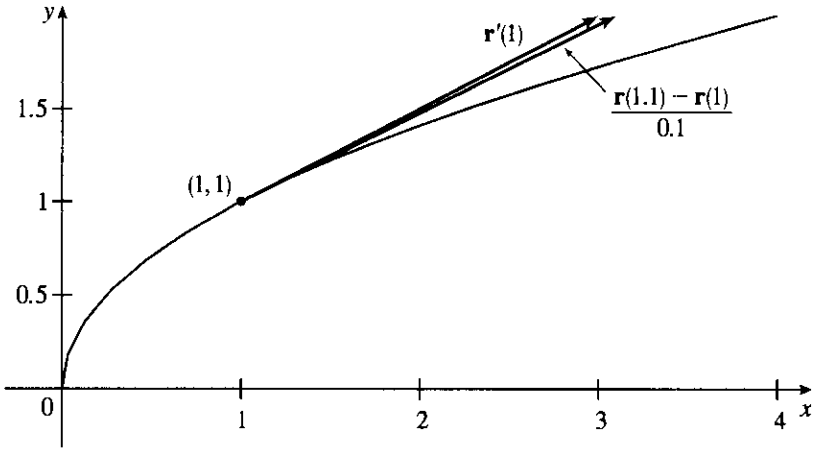
2. (a) The curve can be represented by the parametric equations $x = t^2$, $y = t$, $0 \le t \le 2$. Eliminating the parameter, we have $x = y^2$, $0 \le y \le 2$, a portion of which we graph here, along with the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.



(b) Since $\mathbf{r}(t) = \langle t^2, t \rangle$, we differentiate components, giving $\mathbf{r}'(t) = \langle 2t, 1 \rangle$, so $\mathbf{r}'(1) = \langle 2, 1 \rangle$.

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1} = \frac{\langle 1.21, 1.1 \rangle - \langle 1, 1 \rangle}{0.1} = 10 \langle 0.21, 0.1 \rangle = \langle 2.1, 1 \rangle.$$



As we can see from the graph, these vectors are very close in length and direction. $\mathbf{r}'(1)$ is defined to be $\lim_{h \to 0} \dfrac{\mathbf{r}(1+h) - \mathbf{r}(1)}{h}$, and we recognize $\dfrac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$ as the expression after the limit sign with $h = 0.1$.

Since h is close to 0, we would expect $\dfrac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$ to be a vector close to $\mathbf{r}'(1)$.

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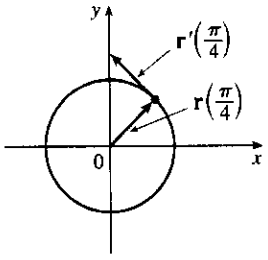
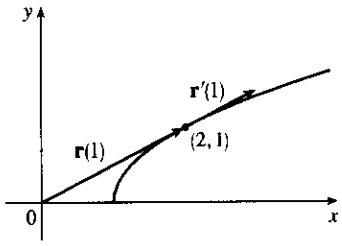
3. (a), (c)



(b) $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$

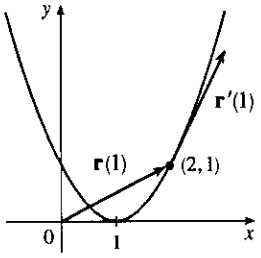
4. (a), (c)



(b) $\mathbf{r}'(t) = \left\langle 1, \dfrac{1}{2\sqrt{t}} \right\rangle$

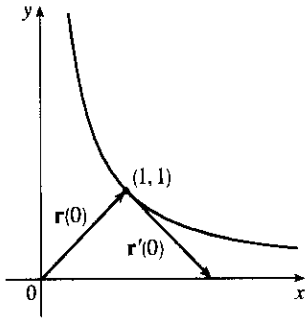
5. Since $(x-1)^2 = t^2 = y$, the curve is a parabola.

(a), (c)



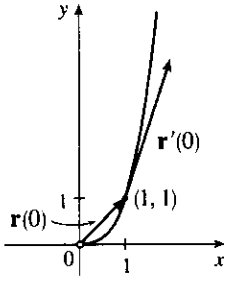
(b) $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j}$

6. (a), (c)



(b) $\mathbf{r}'(t) = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}$

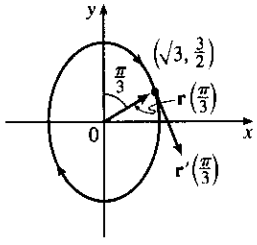
7. (a), (c)



(b) $\mathbf{r}'(t) = e^t\,\mathbf{i} + 3e^{3t}\,\mathbf{j}$

8. $x = 2\sin t$, $y = 3\cos t$, so $(x/2)^2 + (y/3)^2 = \sin^2 t + \cos^2 t = 1$ and the curve is an ellipse.

(a), (c)



(b) $\mathbf{r}'(t) = 2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$

9. $\mathbf{r}'(t) = \left\langle \dfrac{d}{dt}\left[t^2\right], \dfrac{d}{dt}\left[1-t\right], \dfrac{d}{dt}\left[\sqrt{t}\right] \right\rangle = \left\langle 2t, -1, \dfrac{1}{2\sqrt{t}} \right\rangle$

10. $\mathbf{r}(t) = \langle \cos 3t, t, \sin 3t \rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle -3\sin 3t, 1, 3\cos 3t \rangle$

11. $\mathbf{r}(t) = \mathbf{i} - \mathbf{j} + e^{4t}\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 0\,\mathbf{i} + 0\,\mathbf{j} + 4e^{4t}\,\mathbf{k} = 4e^{4t}\,\mathbf{k}$

12. $\mathbf{r}(t) = \sin^{-1} t\,\mathbf{i} + \sqrt{1-t^2}\,\mathbf{j} + \mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \dfrac{1}{\sqrt{1-t^2}}\,\mathbf{i} - \dfrac{t}{\sqrt{1-t^2}}\,\mathbf{j}$

13. $\mathbf{r}(t) = e^{t^2}\,\mathbf{i} - \mathbf{j} + \ln(1+3t)\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 2te^{t^2}\,\mathbf{i} + \dfrac{3}{1+3t}\,\mathbf{k}$

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14. $\mathbf{r}'(t) = [at(-3\sin 3t) + a\cos 3t]\,\mathbf{i} + b\cdot 3\sin^2 t\cos t\,\mathbf{j} + c\cdot 3\cos^2 t(-\sin t)\,\mathbf{k}$

$= (a\cos 3t - 3at\sin 3t)\,\mathbf{i} + 3b\sin^2 t\cos t\,\mathbf{j} - 3c\cos^2 t\sin t\,\mathbf{k}$

15. $\mathbf{r}'(t) = \mathbf{0} + \mathbf{b} + 2t\,\mathbf{c} = \mathbf{b} + 2t\,\mathbf{c}$ by Formulas 1 and 3 of Theorem 3.

16. To find $\mathbf{r}'(t)$, we first expand $\mathbf{r}(t) = t\,\mathbf{a}\times(\mathbf{b}+t\,\mathbf{c}) = t(\mathbf{a}\times\mathbf{b}) + t^2(\mathbf{a}\times\mathbf{c})$, so $\mathbf{r}'(t) = \mathbf{a}\times\mathbf{b} + 2t(\mathbf{a}\times\mathbf{c})$.

17. $\mathbf{r}'(t) = \langle 30t^4, 12t^2, 2\rangle \;\Rightarrow\; \mathbf{r}'(1) = \langle 30, 12, 2\rangle$. So $|\mathbf{r}'(1)| = \sqrt{30^2+12^2+2^2} = \sqrt{1048} = 2\sqrt{262}$ and

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{2\sqrt{262}}\langle 30, 12, 2\rangle = \left\langle \frac{15}{\sqrt{262}}, \frac{6}{\sqrt{262}}, \frac{1}{\sqrt{262}}\right\rangle$.

18. $\mathbf{r}'(t) = \dfrac{2}{\sqrt{t}}\,\mathbf{i} + 2t\,\mathbf{j} + \mathbf{k} \;\Rightarrow\; \mathbf{r}'(1) = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$. Thus

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \dfrac{1}{\sqrt{2^2+2^2+1^2}}(2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k}) = \frac{1}{3}(2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k}) = \frac{2}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{1}{3}\,\mathbf{k}$.

19. $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + 3\,\mathbf{j} + 4\cos 2t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(0) = 3\,\mathbf{j} + 4\,\mathbf{k}$. Thus

$\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \dfrac{1}{\sqrt{0^2+3^2+4^2}}(3\,\mathbf{j}+4\,\mathbf{k}) = \frac{1}{5}(3\,\mathbf{j}+4\,\mathbf{k}) = \frac{3}{5}\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$.

20. $\mathbf{r}'(t) = 2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'\left(\frac{\pi}{4}\right) = \sqrt{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j} + 2\,\mathbf{k}$ and $\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right| = \sqrt{2+2+4} = 2\sqrt{2}$.

Thus $\mathbf{T}\left(\frac{\pi}{4}\right) = \dfrac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|} = \dfrac{1}{2\sqrt{2}}\left(\sqrt{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j} + 2\,\mathbf{k}\right) = \dfrac{1}{2}\,\mathbf{i} - \dfrac{1}{2}\,\mathbf{j} + \dfrac{1}{\sqrt{2}}\,\mathbf{k}$.

21. $\mathbf{r}(t) = \langle t, t^2, t^3\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$. Then $\mathbf{r}'(1) = \langle 1,2,3\rangle$ and $|\mathbf{r}'(1)| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$, so

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{14}}\langle 1,2,3\rangle = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right\rangle$. $\mathbf{r}''(t) = \langle 0, 2, 6t\rangle$, so

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}\mathbf{k}$$

$$= (12t^2 - 6t^2)\,\mathbf{i} - (6t-0)\,\mathbf{j} + (2-0)\,\mathbf{k} = \langle 6t^2, -6t, 2\rangle.$$

22. $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t}\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t}\rangle \;\Rightarrow$

$\mathbf{r}'(0) = \langle 2e^0, -2e^0, (0+1)e^0\rangle = \langle 2, -2, 1\rangle$ and $|\mathbf{r}'(0)| = \sqrt{2^2+(-2)^2+1^2} = 3$. Then

$\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{3}\langle 2,-2,1\rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle$.

$\mathbf{r}''(t) = \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t}\rangle \;\Rightarrow\; \mathbf{r}''(0) = \langle 4e^0, 4e^0, (0+4)e^0\rangle = \langle 4,4,4\rangle$.

$\mathbf{r}'(t)\cdot\mathbf{r}''(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t}\rangle \cdot \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t}\rangle$

$= (2e^{2t})(4e^{2t}) + (-2e^{-2t})(4e^{-2t}) + ((2t+1)e^{2t})((4t+4)e^{2t})$

$= 8e^{4t} - 8e^{-4t} + (8t^2+12t+4)e^{4t} = (8t^2+12t+12)e^{4t} - 8e^{-4t}$

23. The vector equation for the curve is $\mathbf{r}(t) = \langle t^5, t^4, t^3\rangle$, so $\mathbf{r}'(t) = \langle 5t^4, 4t^3, 3t^2\rangle$. The point $(1,1,1)$ corresponds to $t=1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 5,4,3\rangle$. Thus, the tangent line goes through the point $(1,1,1)$ and is parallel to the vector $\langle 5,4,3\rangle$. Parametric equations are $x = 1+5t$, $y = 1+4t$, $z = 1+3t$.

24. The vector equation for the curve is $\mathbf{r}(t) = \langle t^2-1, t^2+1, t+1\rangle$, so $\mathbf{r}'(t) = \langle 2t, 2t, 1\rangle$. The point $(-1,1,1)$ corresponds to $t=0$, so the tangent vector there is $\mathbf{r}'(0) = \langle 0,0,1\rangle$. Thus, the tangent line is parallel to the vector $\langle 0,0,1\rangle$ and parametric equations are $x = -1+0\cdot t = -1$, $y = 1+0\cdot t = 1$, $z = 1+1\cdot t = 1+t$.

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25. The vector equation for the curve is $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$, so

$\mathbf{r}'(t) = \langle e^{-t}(-\sin t) + (\cos t)(-e^{-t}), e^{-t}\cos t + (\sin t)(-e^{-t}), (-e^{-t})\rangle$

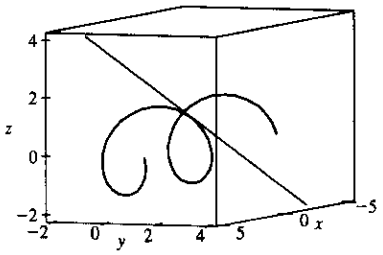
$= \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t}\rangle$.

The point $(1, 0, 1)$ corresponds to $t = 0$, so the tangent vector there is $\mathbf{r}'(0) = \langle -e^0(\cos 0 + \sin 0), e^0(\cos 0 - \sin 0), -e^0\rangle = \langle -1, 1, -1\rangle$. Thus, the tangent line is parallel to the vector $\langle -1, 1, -1\rangle$ and parametric equations are $x = 1 + (-1)t = 1 - t$, $y = 0 + 1 \cdot t = t$, $z = 1 + (-1)t = 1 - t$.

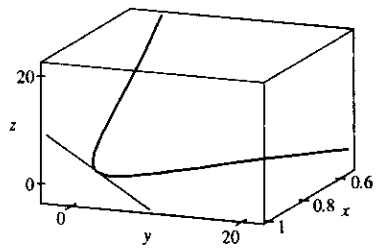
26. $\mathbf{r}(t) = \langle \ln t, 2\sqrt{t}, t^2\rangle$, $\mathbf{r}'(t) = \langle 1/t, 1/\sqrt{t}, 2t\rangle$. At $(0, 2, 1)$, $t = 1$ and $\mathbf{r}'(1) = \langle 1, 1, 2\rangle$. Thus, parametric equations of the tangent line are $x = t$, $y = 2 + t$, $z = 1 + 2t$.

27. $\mathbf{r}(t) = \langle t, \sqrt{2}\cos t, \sqrt{2}\sin t\rangle \Rightarrow$

$\mathbf{r}'(t) = \langle 1, -\sqrt{2}\sin t, \sqrt{2}\cos t\rangle$. At $\left(\frac{\pi}{4}, 1, 1\right)$, $t = \frac{\pi}{4}$ and $\mathbf{r}'\left(\frac{\pi}{4}\right) = \langle 1, -1, 1\rangle$. Thus, parametric equations of the tangent line are $x = \frac{\pi}{4} + t$, $y = 1 - t$, $z = 1 + t$.



28. $\mathbf{r}(t) = \langle \cos t, 3e^{2t}, 3e^{-2t}\rangle$, $\mathbf{r}'(t) = \langle -\sin t, 6e^{2t}, -6e^{-2t}\rangle$. At $(1, 3, 3)$, $t = 0$ and $\mathbf{r}'(0) = \langle 0, 6, -6\rangle$. Thus, parametric equations of the tangent line are $x = 1$, $y = 3 + 6t$, $z = 3 - 6t$.



29. (a) $\mathbf{r}(t) = \langle t^3, t^4, t^5\rangle \Rightarrow \mathbf{r}'(t) = \langle 3t^2, 4t^3, 5t^4\rangle$, and since $\mathbf{r}'(0) = \langle 0, 0, 0\rangle = \mathbf{0}$, the curve is not smooth.

(b) $\mathbf{r}(t) = \langle t^3 + t, t^4, t^5\rangle \Rightarrow \mathbf{r}'(t) = \langle 3t^2 + 1, 4t^3, 5t^4\rangle$. $\mathbf{r}'(t)$ is continuous since its component functions are continuous. Also, $\mathbf{r}'(t) \neq \mathbf{0}$, as the y- and z-components are 0 only for $t = 0$, but $\mathbf{r}'(0) = \langle 1, 0, 0\rangle \neq \mathbf{0}$. Thus, the curve is smooth.

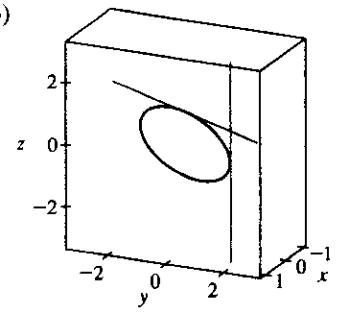
(c) $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t\rangle \Rightarrow \mathbf{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t\rangle$. Since $\mathbf{r}'(0) = \langle -3\cos^2 0 \sin 0, 3\sin^2 0 \cos 0\rangle = \langle 0, 0\rangle = \mathbf{0}$, the curve is not smooth.

30. (a) The tangent line at $t = 0$ is the line through the point with position vector $\mathbf{r}(0) = \langle \sin 0, 2\sin 0, \cos 0\rangle = \langle 0, 0, 1\rangle$, and in the direction of the tangent vector, $\mathbf{r}'(0) = \langle \pi\cos 0, 2\pi\cos 0, -\pi\sin 0\rangle = \langle \pi, 2\pi, 0\rangle$.

So an equation of the line is

$\langle x, y, z\rangle = \mathbf{r}(0) + u\,\mathbf{r}'(0) = \langle 0 + \pi u, 0 + 2\pi u, 1\rangle = \langle \pi u, 2\pi u, 1\rangle$.

(b)



$\mathbf{r}\left(\frac{1}{2}\right) = \left\langle \sin\frac{\pi}{2}, 2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}\right\rangle = \langle 1, 2, 0\rangle$, $\mathbf{r}'\left(\frac{1}{2}\right) = \left\langle \pi\cos\frac{\pi}{2}, 2\pi\cos\frac{\pi}{2}, -\pi\sin\frac{\pi}{2}\right\rangle = \langle 0, 0, -\pi\rangle$.

So the equation of the second line is $\langle x, y, z\rangle = \langle 1, 2, 0\rangle + v\,\langle 0, 0, -\pi\rangle = \langle 1, 2, -\pi v\rangle$. The lines intersect where $\langle \pi u, 2\pi u, 1\rangle = \langle 1, 2, -\pi v\rangle$, so the point of intersection is $(1, 2, 1)$.

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31. The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\mathbf{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$ and $t = 0$ at $(0,0,0)$, $\mathbf{r}_1'(0) = \langle 1,0,0 \rangle$ is a tangent vector to $\mathbf{r}_1$ at $(0,0,0)$. Similarly, $\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 1 \rangle$ and since $\mathbf{r}_2(0) = \langle 0,0,0 \rangle$, $\mathbf{r}_2'(0) = \langle 1,2,1 \rangle$ is a tangent vector to $\mathbf{r}_2$ at $(0,0,0)$. If θ is the angle between these two tangent vectors, then $\cos\theta = \frac{1}{\sqrt{1}\sqrt{6}} \langle 1,0,0 \rangle \cdot \langle 1,2,1 \rangle = \frac{1}{\sqrt{6}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66°$.

32. To find the point of intersection, we must find the values of t and s which satisfy the following three equations simultaneously: $t = 3 - s$, $1 - t = s - 2$, $3 + t^2 = s^2$. Solving the last two equations gives $t = 1$, $s = 2$ (check these in the first equation). Thus the point of intersection is $(1,0,4)$. To find the angle θ of intersection, we proceed as in Exercise 31. The tangent vectors to the respective curves at $(1,0,4)$ are $\mathbf{r}_1'(1) = \langle 1,-1,2 \rangle$ and $\mathbf{r}_2'(2) = \langle -1,1,4 \rangle$. So $\cos\theta = \frac{1}{\sqrt{6}\sqrt{18}}(-1-1+8) = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55°$.

Note: In Exercise 31, the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

33. $\int_0^1 (16t^3\,\mathbf{i} - 9t^2\,\mathbf{j} + 25t^4\,\mathbf{k})\,dt = \left(\int_0^1 16t^3\,dt\right)\mathbf{i} - \left(\int_0^1 9t^2\,dt\right)\mathbf{j} + \left(\int_0^1 25t^4\,dt\right)\mathbf{k}$

$= \left[4t^4\right]_0^1 \mathbf{i} - \left[3t^3\right]_0^1 \mathbf{j} + \left[5t^5\right]_0^1 \mathbf{k} = 4\,\mathbf{i} - 3\,\mathbf{j} + 5\,\mathbf{k}$

34. $\int_0^1 \left(\frac{4}{1+t^2}\,\mathbf{j} + \frac{2t}{1+t^2}\,\mathbf{k}\right) dt = \left[4\tan^{-1} t\,\mathbf{j} + \ln(1+t^2)\,\mathbf{k}\right]_0^1$

$= \left[4\tan^{-1} 1\,\mathbf{j} + \ln 2\,\mathbf{k}\right] - \left[4\tan^{-1} 0\,\mathbf{j} + \ln 1\,\mathbf{k}\right] = 4\left(\frac{\pi}{4}\right)\mathbf{j} + \ln 2\,\mathbf{k} - 0\,\mathbf{j} - 0\,\mathbf{k} = \pi\,\mathbf{j} + \ln 2\,\mathbf{k}$

35. $\int_0^{\pi/2} (3\sin^2 t\cos t\,\mathbf{i} + 3\sin t\cos^2 t\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k})\,dt$

$= \left(\int_0^{\pi/2} 3\sin^2 t\cos t\,dt\right)\mathbf{i} + \left(\int_0^{\pi/2} 3\sin t\cos^2 t\,dt\right)\mathbf{j} + \left(\int_0^{\pi/2} 2\sin t\cos t\,dt\right)\mathbf{k}$

$= \left[\sin^3 t\right]_0^{\pi/2}\,\mathbf{i} + \left[-\cos^3 t\right]_0^{\pi/2}\,\mathbf{j} + \left[\sin^2 t\right]_0^{\pi/2}\,\mathbf{k}$

$= (1-0)\,\mathbf{i} + (0+1)\,\mathbf{j} + (1-0)\,\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

36. $\int_1^4 \left(\sqrt{t}\,\mathbf{i} + te^{-t}\,\mathbf{j} + t^{-2}\,\mathbf{k}\right) dt = \left[\tfrac{2}{3}t^{3/2}\,\mathbf{i} - t^{-1}\,\mathbf{k}\right]_1^4 + \left(\left[-te^{-t}\right]_1^4 + \int_1^4 e^{-t}\,dt\right)\mathbf{j}$

$= \left(\tfrac{16}{3} - \tfrac{2}{3}\right)\mathbf{i} - \left(\tfrac{1}{4} - 1\right)\mathbf{k} + (-4e^{-4} + e^{-1} - e^{-4} + e^{-1})\,\mathbf{j} = \tfrac{14}{3}\,\mathbf{i} + e^{-1}(2 - 5e^{-3})\,\mathbf{j} + \tfrac{3}{4}\,\mathbf{k}$

37. $\int \left(e^t\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}\right)\,dt = \left(\int e^t\,dt\right)\mathbf{i} + \left(\int 2t\,dt\right)\mathbf{j} + \left(\int \ln t\,dt\right)\mathbf{k}$

$= e^t\,\mathbf{i} + t^2\,\mathbf{j} + (t\ln t - t)\,\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a vector constant of integration.

38. $\int (\cos \pi t\,\mathbf{i} + \sin \pi t\,\mathbf{j} + t\,\mathbf{k})\,dt = \left(\int \cos \pi t\,dt\right)\mathbf{i} + \left(\int \sin \pi t\,dt\right)\mathbf{j} + \left(\int t\,dt\right)\mathbf{k}$

$= \frac{1}{\pi}\sin \pi t\,\mathbf{i} - \frac{1}{\pi}\cos \pi t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k} + \mathbf{C}$

39. $\mathbf{r}'(t) = t^2\,\mathbf{i} + 4t^3\,\mathbf{j} - t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{r}(t) = \frac{1}{3}t^3\,\mathbf{i} + t^4\,\mathbf{j} - \frac{1}{3}t^3\,\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a constant vector.
But $\mathbf{j} = \mathbf{r}(0) = (0)\,\mathbf{i} + (0)\,\mathbf{j} - (0)\,\mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = \mathbf{j}$ and
$\mathbf{r}(t) = \frac{1}{3}t^3\,\mathbf{i} + t^4\,\mathbf{j} - \frac{1}{3}t^3\,\mathbf{k} + \mathbf{j} = \frac{1}{3}t^3\,\mathbf{i} + (t^4+1)\,\mathbf{j} - \frac{1}{3}t^3\,\mathbf{k}$.

40. $\mathbf{r}'(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + 2t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}(t) = (-\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{C}$.
But $\mathbf{i} + \mathbf{j} + 2\mathbf{k} = \mathbf{r}(0) = -\mathbf{i} + (0)\,\mathbf{j} + (0)\,\mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = 2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$ and
$\mathbf{r}(t) = (2 - \cos t)\,\mathbf{i} + (1 - \sin t)\,\mathbf{j} + (2 + t^2)\,\mathbf{k}$.

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For Exercises 41–44, let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. In each of these exercises, the procedure is to apply Theorem 2 so that the corresponding properties of derivatives of real-valued functions can be used.

41. $$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] &= \frac{d}{dt}\langle u_1(t)+v_1(t), u_2(t)+v_2(t), u_3(t)+v_3(t)\rangle \\ &= \left\langle \frac{d}{dt}[u_1(t)+v_1(t)], \frac{d}{dt}[u_2(t)+v_2(t)], \frac{d}{dt}[u_3(t)+v_3(t)]\right\rangle \\ &= \langle u_1'(t)+v_1'(t), u_2'(t)+v_2'(t), u_3'(t)+v_3'(t)\rangle \\ &= \langle u_1'(t), u_2'(t), u_3'(t)\rangle + \langle v_1'(t), v_2'(t), v_3'(t)\rangle = \mathbf{u}'(t)+\mathbf{v}'(t).\end{aligned}$$

42. $$\begin{aligned}\frac{d}{dt}[f(t)\,\mathbf{u}(t)] &= \frac{d}{dt}\langle f(t)u_1(t), f(t)u_2(t), f(t)u_3(t)\rangle \\ &= \left\langle \frac{d}{dt}[f(t)u_1(t)], \frac{d}{dt}[f(t)u_2(t)], \frac{d}{dt}[f(t)u_3(t)]\right\rangle \\ &= \langle f'(t)u_1(t)+f(t)u_1'(t), f'(t)u_2(t)+f(t)u_2'(t), f'(t)u_3(t)+f(t)u_3'(t)\rangle \\ &= f'(t)\langle u_1(t), u_2(t), u_3(t)\rangle + f(t)\langle u_1'(t), u_2'(t), u_3'(t)\rangle \\ &= f'(t)\,\mathbf{u}(t)+f(t)\,\mathbf{u}'(t)\end{aligned}$$

43. $$\begin{aligned}&\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] \\ &= \frac{d}{dt}\langle u_2(t)v_3(t)-u_3(t)v_2(t), u_3(t)v_1(t)-u_1(t)v_3(t), u_1(t)v_2(t)-u_2(t)v_1(t)\rangle \\ &= \langle u_2'v_3(t)+u_2(t)v_3'(t)-u_3'(t)v_2(t)-u_3(t)v_2'(t), \\ &\qquad u_3'(t)v_1(t)+u_3(t)v_1'(t)-u_1'(t)v_3(t)-u_1(t)v_3'(t), \\ &\qquad\qquad u_1'(t)v_2(t)+u_1(t)v_2'(t)-u_2'(t)v_1(t)-u_2(t)v_1'(t)\rangle \\ &= \langle u_2'(t)v_3(t)-u_3'(t)v_2(t), u_3'(t)v_1(t)-u_1'(t)v_3(t), u_1'(t)v_2(t)-u_2'(t)v_1(t)\rangle \\ &\qquad + \langle u_2(t)v_3'(t)-u_3(t)v_2'(t), u_3(t)v_1'(t)-u_1(t)v_3'(t), u_1(t)v_2'(t)-u_2(t)v_1'(t)\rangle \\ &= \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)\end{aligned}$$

Alternate solution: Let $\mathbf{r}(t) = \mathbf{u}(t)\times\mathbf{v}(t)$. Then

$$\begin{aligned}\mathbf{r}(t+h)-\mathbf{r}(t) &= [\mathbf{u}(t+h)\times\mathbf{v}(t+h)]-[\mathbf{u}(t)\times\mathbf{v}(t)] \\ &= [\mathbf{u}(t+h)\times\mathbf{v}(t+h)]-[\mathbf{u}(t)\times\mathbf{v}(t)]+[\mathbf{u}(t+h)\times\mathbf{v}(t)]-[\mathbf{u}(t+h)\times\mathbf{v}(t)] \\ &= \mathbf{u}(t+h)\times[\mathbf{v}(t+h)-\mathbf{v}(t)]+[\mathbf{u}(t+h)-\mathbf{u}(t)]\times\mathbf{v}(t)\end{aligned}$$

(Be careful of the order of the cross product.)

Dividing through by h and taking the limit as $h\to 0$ we have

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times[\mathbf{v}(t+h)-\mathbf{v}(t)]}{h}+\lim_{h\to 0}\frac{[\mathbf{u}(t+h)-\mathbf{u}(t)]\times\mathbf{v}(t)}{h} \\ &= \mathbf{u}(t)\times\mathbf{v}'(t)+\mathbf{u}'(t)\times\mathbf{v}(t)\end{aligned}$$

by Exercise 14.1.41(a) [ET 13.1.41(a)] and Definition 1.

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44. $\dfrac{d}{dt}\left[\mathbf{u}(f(t))\right] = \dfrac{d}{dt}\left\langle u_1(f(t)), u_2(f(t)), u_3(f(t))\right\rangle = \left\langle \dfrac{d}{dt}\left[u_1(f(t))\right], \dfrac{d}{dt}\left[u_2(f(t))\right], \dfrac{d}{dt}\left[u_3(f(t))\right]\right\rangle$

$= \left\langle f'(t)u_1'(f(t)), f'(t)u_2'(f(t)), f'(t)u_3'(f(t))\right\rangle = f'(t)\,\mathbf{u}'(t)$

45. $\dfrac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$ [by Formula 4 of Theorem 3]

$= (-4t\,\mathbf{j} + 9t^2\,\mathbf{k})\cdot(t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) + (\mathbf{i} - 2t^2\,\mathbf{j} + 3t^3\,\mathbf{k})\cdot(\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k})$

$= -4t\cos t + 9t^2\sin t + 1 + 2t^2\sin t + 3t^3\cos t$

$= 1 - 4t\cos t + 11t^2\sin t + 3t^3\cos t$

46. $\dfrac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$ [by Formula 5 of Theorem 3]

$= (-4t\,\mathbf{j} + 9t^2\,\mathbf{k})\times(t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) + (\mathbf{i} - 2t^2\,\mathbf{j} + 3t^3\,\mathbf{k})\times(\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k})$

$= (-4t\sin t - 9t^2\cos t)\,\mathbf{i} + (9t^3 - 0)\,\mathbf{j} + (0 + 4t^2)\,\mathbf{k}$

$+ (-2t^2\cos t + 3t^3\sin t)\,\mathbf{i} + (3t^3 - \cos t)\,\mathbf{j} + (-\sin t + 2t^2)\,\mathbf{k}$

$= [(\sin t)(3t^3 - 4t) - 11t^2\cos t]\,\mathbf{i} + (12t^3 - \cos t)\,\mathbf{j} + (6t^2 - \sin t)\,\mathbf{k}$

47. $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\mathbf{r}''(t)$ by Formula 5 of Theorem 3. But $\mathbf{r}'(t)\times\mathbf{r}'(t) = \mathbf{0}$ (see Example 13.4.2 [ET 12.4.2]). Thus, $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \mathbf{r}(t)\times\mathbf{r}''(t)$.

48. $\dfrac{d}{dt}\left(\mathbf{u}(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)]\right) = \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot\dfrac{d}{dt}\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]$

$= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}'(t)\times\mathbf{w}(t) + \mathbf{v}(t)\times\mathbf{w}'(t)]$

$= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}'(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}(t)\times\mathbf{w}'(t)]$

$= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] - \mathbf{v}'(t)\cdot[\mathbf{u}(t)\times\mathbf{w}(t)] + \mathbf{w}'(t)\cdot[\mathbf{u}(t)\times\mathbf{v}(t)]$

49. $\dfrac{d}{dt}\left|\mathbf{r}(t)\right| = \dfrac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{1/2} = \frac{1}{2}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]^{-1/2}\left[2\mathbf{r}(t)\cdot\mathbf{r}'(t)\right] = \dfrac{1}{|\mathbf{r}(t)|}\,\mathbf{r}(t)\cdot\mathbf{r}'(t)$

50. Since $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, we have $0 = 2\mathbf{r}(t)\cdot\mathbf{r}'(t) = \dfrac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right] = \dfrac{d}{dt}\left|\mathbf{r}(t)\right|^2$. Thus $|\mathbf{r}(t)|^2$, and so $|\mathbf{r}(t)|$, is a constant, and hence the curve lies on a sphere with center the origin.

51. Since $\mathbf{u}(t) = \mathbf{r}(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}''(t)]$,

$\mathbf{u}'(t) = \mathbf{r}'(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}''(t)] + \mathbf{r}(t)\cdot\dfrac{d}{dt}\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]$

$= 0 + \mathbf{r}(t)\cdot[\mathbf{r}''(t)\times\mathbf{r}''(t) + \mathbf{r}'(t)\times\mathbf{r}'''(t)]$ [since $\mathbf{r}'(t)\perp\mathbf{r}'(t)\times\mathbf{r}''(t)$]

$= \mathbf{r}(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}'''(t)]$ [since $\mathbf{r}''(t)\times\mathbf{r}''(t) = \mathbf{0}$]

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14.3 Arc Length and Curvature ET 13.3

1. $\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{(2\cos t)^2 + 5^2 + (-2\sin t)^2} = \sqrt{29}$. Then using Formula 3, we have $L = \int_{-10}^{10} |\mathbf{r}'(t)|\,dt = \int_{-10}^{10} \sqrt{29}\,dt = \sqrt{29}\,t\big]_{-10}^{10} = 20\sqrt{29}$.

2. $\mathbf{r}'(t) = \langle 2t, \cos t + t\sin t - \cos t, -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle \Rightarrow$
$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (t\sin t)^2 + (t\cos t)^2} = \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5}\,|t| = \sqrt{5}\,t$ for $0 \le t \le \pi$. Then using Formula 3, we have $L = \int_0^{\pi} |\mathbf{r}'(t)|\,dt = \int_0^{\pi} \sqrt{5}\,t\,dt = \sqrt{5}\,\dfrac{t^2}{2}\Big]_0^{\pi} = \dfrac{\sqrt{5}}{2}\,\pi^2$.

3. $\mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \Rightarrow$
$|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ (since $e^t + e^{-t} > 0$).
Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}$.

4. $\mathbf{r}'(t) = \langle 2t, 2, 1/t\rangle$, $|\mathbf{r}'(t)| = \sqrt{4t^2 + 4 + (1/t)^2} = \dfrac{1 + 2t^2}{|t|} = \dfrac{1 + 2t^2}{t}$ for $1 \le t \le e$.

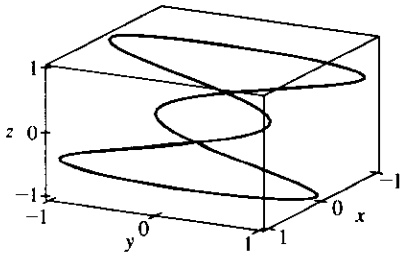
$$L = \int_1^e \frac{1 + 2t^2}{t}\,dt = \int_1^e \left(\frac{1}{t} + 2t\right) dt = \left[\ln t + t^2\right]_1^e = e^2$$

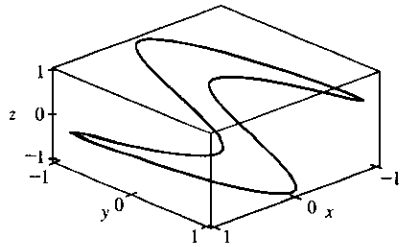
5. $\mathbf{r}'(t) = 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \Rightarrow |\mathbf{r}'(t)| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$ (since $t \ge 0$). Then
$L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 t\sqrt{4 + 9t^2}\,dt = \frac{1}{18} \cdot \frac{2}{3}(4 + 9t^2)^{3/2}\Big]_0^1 = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{1}{27}(13^{3/2} - 8)$.

6. $\mathbf{r}'(t) = 12\,\mathbf{i} + 12\sqrt{t}\,\mathbf{j} + 6t\,\mathbf{k} \Rightarrow |\mathbf{r}'(t)| = \sqrt{144 + 144t + 36t^2} = \sqrt{36(t+2)^2} = 6\,|t+2| = 6(t+2)$ for $0 \le t \le 1$. Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 6(t+2)\,dt = \left[3t^2 + 12t\right]_0^1 = 15$.

7. The point $(2, 4, 8)$ corresponds to $t = 2$, so by Equation 2, $L = \int_0^2 \sqrt{(1)^2 + (2t)^2 + (3t^2)^2}\,dt$.
If $f(t) = \sqrt{1 + 4t^2 + 9t^4}$, then Simpson's Rule gives
$L \approx \dfrac{2 - 0}{10 \cdot 3}\left[f(0) + 4f(0.2) + 2f(0.4) + \cdots + 4f(1.8) + f(2)\right] \approx 9.5706$.

8. Here are two views of the curve with parametric equations $x = \cos t$, $y = \sin 3t$, $z = \sin t$:





The complete curve is given by the parameter interval $[0, 2\pi]$, so
$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (3\cos 3t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi} \sqrt{1 + 9\cos^2 3t}\,dt \approx 13.9744$.

9. $\mathbf{r}'(t) = 2\,\mathbf{i} - 3\,\mathbf{j} + 4\,\mathbf{k}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{4 + 9 + 16} = \sqrt{29}$. Then
$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{29}\,du = \sqrt{29}\,t$. Therefore, $t = \frac{1}{\sqrt{29}}\,s$, and substituting for t in the original equation, we have $\mathbf{r}(t(s)) = \frac{2}{\sqrt{29}}\,s\,\mathbf{i} + \left(1 - \frac{3}{\sqrt{29}}\,s\right)\mathbf{j} + \left(5 + \frac{4}{\sqrt{29}}\,s\right)\mathbf{k}$.

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10. $\mathbf{r}'(t) = 2e^{2t}(\cos 2t - \sin 2t)\,\mathbf{i} + 2e^{2t}(\cos 2t + \sin 2t)\,\mathbf{k}$,

$\frac{ds}{dt} = |\mathbf{r}'(t)| = 2e^{2t}\sqrt{(\cos 2t - \sin 2t)^2 + (\cos 2t + \sin 2t)^2} = 2e^{2t}\sqrt{2\cos^2 2t + 2\sin^2 2t} = 2\sqrt{2}\,e^{2t}$.

$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 2\sqrt{2}\,e^{2u}\,du = \sqrt{2}\,e^{2u}\big]_0^t = \sqrt{2}\,(e^{2t} - 1) \quad \Rightarrow$

$\frac{s}{\sqrt{2}} + 1 = e^{2t} \quad \Rightarrow \quad t = \frac{1}{2}\ln\left(\frac{s}{\sqrt{2}} + 1\right)$. Substituting, we have

$$\mathbf{r}(t(s)) = e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\cos 2\left(\tfrac{1}{2}\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right)\mathbf{i} + 2\mathbf{j} + e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\sin 2\left(\tfrac{1}{2}\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right)\mathbf{k}$$
$$= \left(\tfrac{s}{\sqrt{2}}+1\right)\cos\left(\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right)\mathbf{i} + 2\mathbf{j} + \left(\tfrac{s}{\sqrt{2}}+1\right)\sin\left(\ln\left(\tfrac{s}{\sqrt{2}}+1\right)\right)\mathbf{k}.$$

11. $|\mathbf{r}'(t)| = \sqrt{(3\cos t)^2 + 16 + (-3\sin t)^2} = \sqrt{9+16} = 5$ and $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t \quad \Rightarrow$ $t(s) = \frac{1}{5}s$. Therefore, $\mathbf{r}(t(s)) = 3\sin\left(\frac{1}{5}s\right)\mathbf{i} + \frac{4}{5}s\,\mathbf{j} + 3\cos\left(\frac{1}{5}s\right)\mathbf{k}$.

12. $\mathbf{r}'(t) = \dfrac{-4t}{(t^2+1)^2}\,\mathbf{i} + \dfrac{-2t^2+2}{(t^2+1)^2}\,\mathbf{j}$,

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left[\frac{-4t}{(t^2+1)^2}\right]^2 + \left[\frac{-2t^2+2}{(t^2+1)^2}\right]^2} = \sqrt{\frac{4t^4+8t^2+4}{(t^2+1)^4}} = \sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}}$$
$$= \sqrt{\frac{4}{(t^2+1)^2}} = \frac{2}{t^2+1}$$

Since the initial point $(1,0)$ corresponds to $t = 0$, the arc length function

$s(t) = \displaystyle\int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \frac{2}{u^2+1}\,du = 2\arctan t$. Then $\arctan t = \frac{1}{2}s \quad \Rightarrow \quad t = \tan\frac{1}{2}s$. Substituting, we have

$$\mathbf{r}(t(s)) = \left[\frac{2}{\tan^2\left(\frac{1}{2}s\right)+1} - 1\right]\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right)+1}\,\mathbf{j} = \frac{1-\tan^2\left(\frac{1}{2}s\right)}{1+\tan^2\left(\frac{1}{2}s\right)}\,\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\mathbf{j}$$
$$= \frac{1-\tan^2\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\mathbf{i} + 2\tan\left(\tfrac{1}{2}s\right)\cos^2\left(\tfrac{1}{2}s\right)\mathbf{j}$$
$$= \left[\cos^2\left(\tfrac{1}{2}s\right) - \sin^2\left(\tfrac{1}{2}s\right)\right]\mathbf{i} + 2\sin\left(\tfrac{1}{2}s\right)\cos\left(\tfrac{1}{2}s\right)\mathbf{j} = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1,0)$, since $\cos s = -1$ for $s = \pi + 2k\pi$ (k an integer) but then $t = \tan\left(\frac{1}{2}s\right)$ is undefined.

13. (a) $\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t} = \sqrt{29}$. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{29}}\langle 2\cos t, 5, -2\sin t\rangle$ or $\left\langle \frac{2}{\sqrt{29}}\cos t, \frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\sin t\right\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle \quad \Rightarrow \quad |\mathbf{T}'(t)| = \frac{1}{\sqrt{29}}\sqrt{4\sin^2 t + 0 + 4\cos^2 t} = \frac{2}{\sqrt{29}}$. Thus

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{29}}{2/\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle = \langle -\sin t, 0, -\cos t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/\sqrt{29}}{\sqrt{29}} = \dfrac{2}{29}$.

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14. (a) $\mathbf{r}'(t) = \langle 2t, t\sin t, t\cos t\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{5t^2} = \sqrt{5}\,t$

(since $t > 0$). Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}\,t}\langle 2t, t\sin t, t\cos t\rangle = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \;\Rightarrow\; |\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{0 + \cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}$. Thus

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0, \cos t, -\sin t\rangle = \langle 0, \cos t, -\sin t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{5t}$.

15. (a) $\mathbf{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t}\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{e^t + e^{-t}}\langle \sqrt{2}, e^t, -e^{-t}\rangle = \dfrac{1}{e^{2t}+1}\langle \sqrt{2}e^t, e^{2t}, -1\rangle$ $\left(\text{after multiplying by } \dfrac{e^t}{e^t}\right)$ and

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{e^{2t}+1}\left\langle \sqrt{2}e^t, 2e^{2t}, 0\right\rangle - \frac{2e^{2t}}{(e^{2t}+1)^2}\left\langle \sqrt{2}e^t, e^{2t}, -1\right\rangle \\ &= \frac{1}{(e^{2t}+1)^2}\left[(e^{2t}+1)\left\langle \sqrt{2}e^t, 2e^{2t}, 0\right\rangle - 2e^{2t}\left\langle \sqrt{2}e^t, e^{2t}, -1\right\rangle\right] \\ &= \frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}e^t\left(1 - e^{2t}\right), 2e^{2t}, 2e^{2t}\right\rangle\end{aligned}$$

Then

$$\begin{aligned}|\mathbf{T}'(t)| &= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1 - 2e^{2t} + e^{4t}) + 4e^{4t} + 4e^{4t}} = \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1 + 2e^{2t} + e^{4t})} \\ &= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}\left(1 + e^{2t}\right)^2} = \frac{\sqrt{2}e^t(1 + e^{2t})}{(e^{2t}+1)^2} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{e^{2t}+1}{\sqrt{2}\,e^t}\frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t}\right\rangle \\ &= \frac{1}{\sqrt{2}\,e^t(e^{2t}+1)}\left\langle \sqrt{2}\,e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t}\right\rangle \\ &= \frac{1}{e^{2t}+1}\left\langle 1 - e^{2t}, \sqrt{2}\,e^t, \sqrt{2}\,e^t\right\rangle\end{aligned}$$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{2}\,e^t}{e^{2t}+1}\cdot\dfrac{1}{e^t + e^{-t}} = \dfrac{\sqrt{2}\,e^t}{e^{3t} + 2e^t + e^{-t}} = \dfrac{\sqrt{2}\,e^{2t}}{e^{4t} + 2e^{2t} + 1} = \dfrac{\sqrt{2}\,e^{2t}}{(e^{2t}+1)^2}$.

16. (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{4t^2 + 4 + (1/t)^2}}\langle 2t, 2, 1/t\rangle = \dfrac{|t|}{2t^2+1}\langle 2t, 2, 1/t\rangle$. But since the

k-component is $\ln t$, t is positive, $|t| = t$ and $\mathbf{T}(t) = \dfrac{1}{2t^2+1}\langle 2t^2, 2t, 1\rangle$. Then

$\mathbf{T}'(t) = \dfrac{1}{2t^2+1}\langle 4t, 2, 0\rangle - \left(2t^2+1\right)^{-2}(4t)\langle 2t^2, 2t, 1\rangle = \dfrac{1}{(2t^2+1)^2}\langle 4t, 2 - 4t^2, -4t\rangle$, so

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\langle 4t, 2 - 4t^2, -4t\rangle}{\sqrt{(4t)^2 + (2 - 4t^2)^2 + (-4t)^2}} = \dfrac{1}{2t^2+1}\langle 2t, 1 - 2t^2, -2t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2}{2t^2+1}\left(\dfrac{t}{2t^2+1}\right) = \dfrac{2t}{(2t^2+1)^2}$

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17. $\mathbf{r}'(t) = 2t\,\mathbf{i} + \mathbf{k}$, $\mathbf{r}''(t) = 2\,\mathbf{i}$, $|\mathbf{r}'(t)| = \sqrt{(2t)^2 + 0^2 + 1^2} = \sqrt{4t^2+1}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\,\mathbf{j}$, $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2$.

Then $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{2}{\left(\sqrt{4t^2+1}\right)^3} = \dfrac{2}{(4t^2+1)^{3/2}}$.

18. $\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{r}''(t) = 2\,\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1^2 + 1^2 + (2t)^2} = \sqrt{4t^2+2}$,

$\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\,\mathbf{i} - 2\,\mathbf{j}$, $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{2^2 + 2^2 + 0^2} = \sqrt{8} = 2\sqrt{2}$. Then

$\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{2\sqrt{2}}{\left(\sqrt{4t^2+2}\right)^3} = \dfrac{2\sqrt{2}}{\left(\sqrt{2}\sqrt{2t^2+1}\right)^3} = \dfrac{1}{(2t^2+1)^{3/2}}$.

19. $\mathbf{r}'(t) = 3\,\mathbf{i} + 4\cos t\,\mathbf{j} - 4\sin t\,\mathbf{k}$, $\mathbf{r}''(t) = -4\sin t\,\mathbf{j} - 4\cos t\,\mathbf{k}$,

$|\mathbf{r}'(t)| = \sqrt{9 + 16\cos^2 t + 16\sin^2 t} = \sqrt{9+16} = 5$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = -16\,\mathbf{i} + 12\cos t\,\mathbf{j} - 12\sin t\,\mathbf{k}$,

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{256 + 144\cos^2 t + 144\sin^2 t} = \sqrt{400} = 20$. Then $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{20}{5^3} = \dfrac{4}{25}$.

20. $\mathbf{r}'(t) = \langle e^t\cos t - e^t\sin t,\, e^t\cos t + e^t\sin t,\, 1\rangle$. The point $(1,0,0)$ corresponds to $t = 0$, and

$\mathbf{r}'(0) = \langle 1,1,1\rangle \;\Rightarrow\; |\mathbf{r}'(0)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.

$\mathbf{r}''(t) = \langle e^t\cos t - e^t\sin t - e^t\cos t - e^t\sin t,\, e^t\cos t - e^t\sin t + e^t\cos t + e^t\sin t,\, 0\rangle$

$= \langle -2e^t\sin t,\, 2e^t\cos t,\, 0\rangle \;\Rightarrow\; \mathbf{r}''(0) = \langle 0,2,0\rangle$.

$\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle -2,0,2\rangle$. $\quad |\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{(-2)^2+0^2+2^2} = \sqrt{8} = 2\sqrt{2}$.

Then $\kappa(0) = \dfrac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \dfrac{2\sqrt{2}}{\left(\sqrt{3}\right)^3} = \dfrac{2\sqrt{2}}{3\sqrt{3}}$ or $\dfrac{2\sqrt{6}}{9}$.

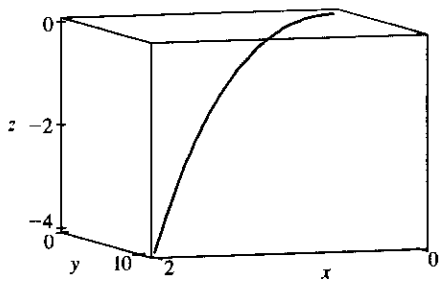
21. $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$. The point $(1,1,1)$ corresponds to $t = 1$, and

$\mathbf{r}'(1) = \langle 1,2,3\rangle \;\Rightarrow\; |\mathbf{r}'(1)| = \sqrt{1+4+9} = \sqrt{14}$. $\mathbf{r}''(t) = \langle 0,2,6t\rangle \;\Rightarrow\; \mathbf{r}''(1) = \langle 0,2,6\rangle$.

$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 6,-6,2\rangle$, so $|\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{36+36+4} = \sqrt{76}$. Then

$\kappa(1) = \dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{\sqrt{76}}{\sqrt{14}^3} = \dfrac{1}{7}\sqrt{\dfrac{19}{14}}$.

22.



$\mathbf{r}(t) = \left\langle t, 4t^{3/2}, -t^2\right\rangle \;\Rightarrow\; \mathbf{r}'(t) = \left\langle 1, 6t^{1/2}, -2t\right\rangle$,

$\mathbf{r}''(t) = \left\langle 0, 3t^{-1/2}, -2\right\rangle$, $|\mathbf{r}'(t)|^3 = \left(1 + 36t + 4t^2\right)^{3/2}$,

$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left\langle -12t^{1/2} + 6t^{1/2}, 2, 3t^{-1/2}\right\rangle \;\Rightarrow$

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t + 4 + 9t^{-1}} = \left[\dfrac{36t^2 + 4t + 9}{t}\right]^{1/2}$

$\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \left(\dfrac{36t^2+4t+9}{t}\right)^{1/2} \dfrac{1}{(1+36t+4t^2)^{3/2}} = \dfrac{\sqrt{36t^2+4t+9}}{t^{1/2}(1+36t+4t^2)^{3/2}}$.

The point $(1,4,-1)$ corresponds to $t = 1$, so the curvature at this point is $\kappa(1) = \dfrac{\sqrt{36+4+9}}{(1+36+4)^{3/2}} = \dfrac{7}{41\sqrt{41}}$.

23. $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, $\kappa(x) = \dfrac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} = \dfrac{6|x|}{(1+9x^4)^{3/2}}$

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24. $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$,

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|-\cos x|}{[1 + (-\sin x)^2]^{3/2}} = \frac{|\cos x|}{(1 + \sin^2 x)^{3/2}}$$

25. $f(x) = 4x^{5/2}$, $f'(x) = 10x^{3/2}$, $f''(x) = 15x^{1/2}$,

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{\left|15x^{1/2}\right|}{[1 + (10x^{3/2})^2]^{3/2}} = \frac{15\sqrt{x}}{(1 + 100x^3)^{3/2}}$$

26. $y' = \dfrac{1}{x}$, $y'' = -\dfrac{1}{x^2}$,

$$\kappa(x) = \frac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{3/2}} = \left|\frac{-1}{x^2}\right| \frac{1}{(1 + 1/x^2)^{3/2}} = \frac{1}{x^2} \frac{(x^2)^{3/2}}{(x^2+1)^{3/2}} = \frac{|x|}{(x^2+1)^{3/2}} = \frac{x}{(x^2+1)^{3/2}}$$

(since $x > 0$). To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = \frac{(x^2+1)^{3/2} - x\left(\frac{3}{2}\right)(x^2+1)^{1/2}(2x)}{[(x^2+1)^{3/2}]^2} = \frac{(x^2+1)^{1/2}[(x^2+1) - 3x^2]}{(x^2+1)^3} = \frac{1 - 2x^2}{(x^2+1)^{5/2}};$$

$\kappa'(x) = 0 \quad \Rightarrow \quad 1 - 2x^2 = 0$, so the only critical number in the domain is $x = \frac{1}{\sqrt{2}}$. Since $\kappa'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2}}$ and $\kappa'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x = \frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}}, \ln\frac{1}{\sqrt{2}}\right)$. Since $\lim_{x\to\infty} \dfrac{x}{(x^2+1)^{3/2}} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

27. Since $y' = y'' = e^x$, the curvature is $\kappa(x) = \dfrac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \dfrac{e^x}{(1 + e^{2x})^{3/2}} = e^x(1 + e^{2x})^{-3/2}$.

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

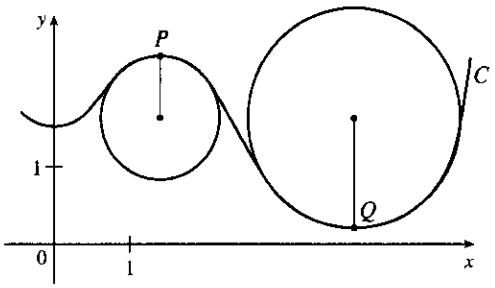
$$\kappa'(x) = e^x(1 + e^{2x})^{-3/2} + e^x\left(-\tfrac{3}{2}\right)(1 + e^{2x})^{-5/2}(2e^{2x}) = e^x \frac{1 + e^{2x} - 3e^{2x}}{(1 + e^{2x})^{5/2}} = e^x \frac{1 - 2e^{2x}}{(1 + e^{2x})^{5/2}}.$$

$\kappa'(x) = 0$ when $1 - 2e^{2x} = 0$, so $e^{2x} = \frac{1}{2}$ or $x = -\frac{1}{2}\ln 2$. And since $1 - 2e^{2x} > 0$ for $x < -\frac{1}{2}\ln 2$ and $1 - 2e^{2x} < 0$ for $x > -\frac{1}{2}\ln 2$, the maximum curvature is attained at the point $\left(-\frac{1}{2}\ln 2, e^{(-\ln 2)/2}\right) = \left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$. Since $\lim_{x\to\infty} e^x(1 + e^{2x})^{-3/2} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

28. We can take the parabola as having its vertex at the origin and opening upward, so the equation is $f(x) = ax^2, a > 0$. Then by Equation 11, $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \dfrac{|2a|}{[1 + (2ax)^2]^{3/2}} = \dfrac{2a}{(1 + 4a^2x^2)^{3/2}}$, thus $\kappa(0) = 2a$. We want $\kappa(0) = 4$, so $a = 2$ and the equation is $y = 2x^2$.

29. (a) C appears to be changing direction more quickly at P than Q, so we would expect the curvature to be greater at P.

(b) First we sketch approximate osculating circles at P and Q. Using the axes scale as a guide, we measure the radius of the osculating circle at P to be approximately 0.8 units, thus $\rho = \dfrac{1}{\kappa} \quad \Rightarrow \quad \kappa = \dfrac{1}{\rho} \approx \dfrac{1}{0.8} \approx 1.3$. Similarly, we estimate the radius of the osculating circle at Q to be 1.4 units, so $\kappa = \dfrac{1}{\rho} \approx \dfrac{1}{1.4} \approx 0.7$.



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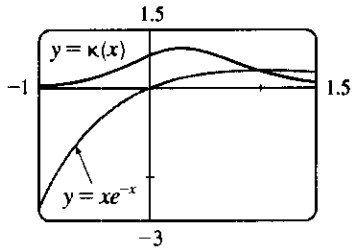
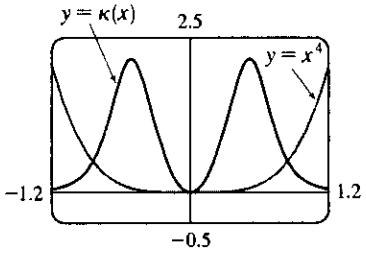
30. $y = xe^{-x} \Rightarrow y' = e^{-x}(1-x), y'' = e^{-x}(x-2)$, and $\kappa(x) = \dfrac{|y''|}{[1+(y')^2]^{3/2}} = \dfrac{e^{-x}|x-2|}{[1+e^{-2x}(1-x)^2]^{3/2}}$. The graph of the curvature here is what we would expect. The graph of xe^{-x} is bending most sharply slightly to the right of the origin. As $x \to \infty$, the graph of xe^{-x} is asymptotic to the x-axis, and so the curvature approaches zero.



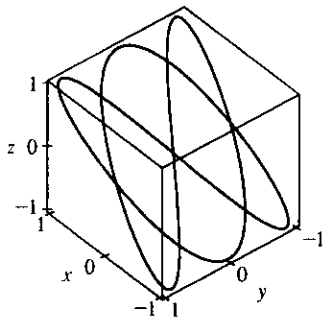
31. $y = x^4 \Rightarrow y' = 4x^3, y'' = 12x^2$, and $\kappa(x) = \dfrac{|y''|}{[1+(y')^2]^{3/2}} = \dfrac{12x^2}{(1+16x^6)^{3/2}}$. The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y = x^4$ is very flat around the origin, and so here the curvature is zero.



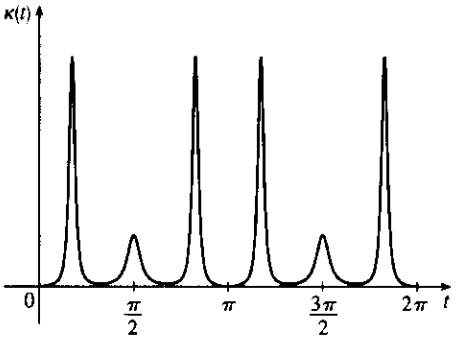
32. Notice that the curve a is highest for the same x-values at which curve b is turning more sharply, and a is 0 or near 0 where b is nearly straight. So, a must be the graph of $y = \kappa(x)$, and b is the graph of $y = f(x)$.

33. Notice that the curve b has two inflection points at which the graph appears almost straight. We would expect the curvature to be 0 or nearly 0 at these values, but the curve a isn't near 0 there. Thus, a must be the graph of $y = f(x)$ rather than the graph of curvature, and b is the graph of $y = \kappa(x)$.

34. (a) The complete curve is given by $0 \le t \le 2\pi$. Curvature appears to have a local (or absolute) maximum at 6 points. (Look at points where the curve appears to turn more sharply.)



(b) Using a CAS, we find (after simplifying) $\kappa(t) = \dfrac{3\sqrt{2}\sqrt{(5\sin t + \sin 5t)^2}}{(9\cos 6t + 2\cos 4t + 11)^{3/2}}$. (To compute cross products in Maple, use the `Linalg` package and the `crossprod(a,b)` command; in Mathematica, use `Cross[a,b]`.) The graph shows 6 local (or absolute) maximum points for $0 \le t \le 2\pi$, as observed in part (a).



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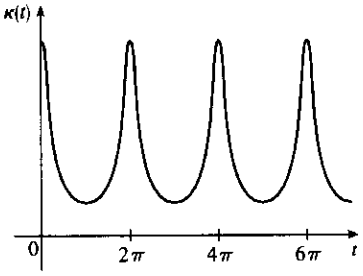
35. Using a CAS, we find (after simplifying) $\kappa(t) = \dfrac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{3/2}}$. (To compute cross products in Maple, use the `Linalg` package and the `crossprod(a,b)` command; in Mathematica, use `Cross[a,b]`.) Curvature is largest at integer multiples of 2π.



36. Here $\mathbf{r}(t) = \langle f(t), g(t)\rangle$, $\mathbf{r}'(t) = \langle f'(t), g'(t)\rangle$, $\mathbf{r}''(t) = \langle f''(t), g''(t)\rangle$,

$|\mathbf{r}'(t)|^3 = \left[\sqrt{(f'(t))^2 + (g'(t))^2}\right]^3 = [(f'(t))^2 + (g'(t))^2]^{3/2} = (\dot{x}^2 + \dot{y}^2)^{3/2}$, and

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle 0, 0, f'(t)\,g''(t) - f''(t)\,g'(t)\rangle| = \left[(\dot{x}\ddot{y} - \ddot{x}\dot{y})^2\right]^{1/2} = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|$.

Thus $\kappa(t) = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

37. $x = e^t\cos t \;\Rightarrow\; \dot{x} = e^t(\cos t - \sin t) \;\Rightarrow\; \ddot{x} = e^t(-\sin t - \cos t) + e^t(\cos t - \sin t) = -2e^t\sin t$,
$y = e^t\sin t \;\Rightarrow\; \dot{y} = e^t(\cos t + \sin t) \;\Rightarrow\; \ddot{y} = e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) = 2e^t\cos t$. Then

$$\begin{aligned}\kappa(t) &= \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{\left|e^t(\cos t - \sin t)(2e^t\cos t) - e^t(\cos t + \sin t)(-2e^t\sin t)\right|}{\left([e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2\right)^{3/2}} \\ &= \frac{\left|2e^{2t}(\cos^2 t - \sin t\cos t + \sin t\cos t + \sin^2 t)\right|}{\left[e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\cos t\sin t + \sin^2 t)\right]^{3/2}} \\ &= \frac{\left|2e^{2t}(1)\right|}{[e^{2t}(1+1)]^{3/2}} = \frac{2e^{2t}}{e^{3t}(2)^{3/2}} = \frac{1}{\sqrt{2}\,e^t}\end{aligned}$$

38. $x = 1 + t^3 \;\Rightarrow\; \dot{x} = 3t^2 \;\Rightarrow\; \ddot{x} = 6t$, $y = t + t^2 \;\Rightarrow\; \dot{y} = 1 + 2t \;\Rightarrow\; \ddot{y} = 2$. Then

$$\begin{aligned}\kappa(t) &= \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{\left|(3t^2)(2) - (1+2t)(6t)\right|}{[(3t^2)^2 + (1+2t)^2]^{3/2}} = \frac{\left|-6t^2 - 6t\right|}{(9t^4 + 4t^2 + 4t + 1)^{3/2}} \\ &= \frac{6\left|t^2 + t\right|}{(9t^4 + 4t^2 + 4t + 1)^{3/2}}\end{aligned}$$

39. $(1, \frac{2}{3}, 1)$ corresponds to $t = 1$. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle 2t, 2t^2, 1\rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \dfrac{\langle 2t, 2t^2, 1\rangle}{2t^2 + 1}$, so $\mathbf{T}(1) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle$.

$$\begin{aligned}\mathbf{T}'(t) &= -4t(2t^2 + 1)^{-2}\langle 2t, 2t^2, 1\rangle + (2t^2 + 1)^{-1}\langle 2, 4t, 0\rangle \qquad \text{[by Theorem 14.2.3 [ET 13.2.3] \#3]} \\ &= (2t^2 + 1)^{-2}\langle -8t^2 + 4t^2 + 2, -8t^3 + 8t^3 + 4t, -4t\rangle = 2(2t^2 + 1)^{-2}\langle 1 - 2t^2, 2t, -2t\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2(2t^2 + 1)^{-2}\langle 1 - 2t^2, 2t, -2t\rangle}{2(2t^2 + 1)^{-2}\sqrt{(1 - 2t^2)^2 + (2t)^2 + (-2t)^2}} = \frac{\langle 1 - 2t^2, 2t, -2t\rangle}{\sqrt{1 - 4t^2 + 4t^4 + 8t^2}} \\ &= \frac{\langle 1 - 2t^2, 2t, -2t\rangle}{1 + 2t^2}\end{aligned}$$

$\mathbf{N}(1) = \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle$ and $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \langle -\frac{4}{9} - \frac{2}{9}, -\left(-\frac{4}{9} + \frac{1}{9}\right), \frac{4}{9} + \frac{2}{9}\rangle = \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\rangle$.

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40. $(1,0,1)$ corresponds to $t=0$. $\mathbf{r}(t)=e^t\langle 1,\sin t,\cos t\rangle$, so

$\mathbf{r}'(t)=e^t\langle 1,\sin t,\cos t\rangle+e^t\langle 0,\cos t,-\sin t\rangle=e^t\langle 1,\sin t+\cos t,\cos t-\sin t\rangle$ and

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{e^t\langle 1,\sin t+\cos t,\cos t-\sin t\rangle}{e^t\sqrt{1+\sin^2 t+2\sin t\cos t+\cos^2 t+\cos^2 t-2\sin t\cos t+\sin^2 t}}$$

$$=\frac{\langle 1,\sin t+\cos t,\cos t-\sin t\rangle}{\sqrt{3}},$$

$\mathbf{T}(0)=\left\langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle$. $\mathbf{T}'(t)=\frac{1}{\sqrt{3}}\langle 0,\cos t-\sin t,-\sin t-\cos t\rangle$, so

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=\frac{\frac{1}{\sqrt{3}}\langle 0,\cos t-\sin t,-\sin t-\cos t\rangle}{\frac{1}{\sqrt{3}}\sqrt{0^2+\cos^2 t-2\cos t\sin t+\sin^2 t+\sin^2 t+2\sin t\cos t+\cos^2 t}}$$

$$=\tfrac{1}{\sqrt{2}}\langle 0,\cos t-\sin t,-\sin t-\cos t\rangle.$$

$\mathbf{N}(0)=\left\langle 0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\rangle$ and $\mathbf{B}(0)=\mathbf{T}(0)\times\mathbf{N}(0)=\left\langle -\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right\rangle$.

41. $(0,\pi,-2)$ corresponds to $t=\pi$. $\mathbf{r}(t)=\langle 2\sin 3t,t,2\cos 3t\rangle\ \Rightarrow$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{\langle 6\cos 3t,1,-6\sin 3t\rangle}{\sqrt{36\cos^2 3t+1+36\sin^2 3t}}=\tfrac{1}{\sqrt{37}}\langle 6\cos 3t,1,-6\sin 3t\rangle.$$

$\mathbf{T}(\pi)=\frac{1}{\sqrt{37}}\langle -6,1,0\rangle$ is a normal vector for the normal plane, and so $\langle -6,1,0\rangle$ is also normal. Thus an equation for the plane is $-6(x-0)+1(y-\pi)+0(z+2)=0$ or $y-6x=\pi$.

$$\mathbf{T}'(t)=\tfrac{1}{\sqrt{37}}\langle -18\sin 3t,0,-18\cos 3t\rangle\ \Rightarrow\ |\mathbf{T}'(t)|=\frac{\sqrt{18^2\sin^2 3t+18^2\cos^2 3t}}{\sqrt{37}}=\frac{18}{\sqrt{37}}\ \Rightarrow$$

$\mathbf{N}(t)=\dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=\langle -\sin 3t,0,-\cos 3t\rangle$. So $\mathbf{N}(\pi)=\langle 0,0,1\rangle$ and

$\mathbf{B}(\pi)=\frac{1}{\sqrt{37}}\langle -6,1,0\rangle\times\langle 0,0,1\rangle=\frac{1}{\sqrt{37}}\langle 1,6,0\rangle$. Since $\mathbf{B}(\pi)$ is a normal to the osculating plane, so is $\langle 1,6,0\rangle$ and an equation for the plane is $1(x-0)+6(y-\pi)+0(z+2)=0$ or $x+6y=6\pi$.

42. $t=1$ at $(1,1,1)$. $\mathbf{r}'(t)=\langle 1,2t,3t^2\rangle$. $\mathbf{r}'(1)=\langle 1,2,3\rangle$ is normal to the normal plane, so an equation for this plane is $1(x-1)+2(y-1)+3(z-1)=0$, or $x+2y+3z=6$.

$\mathbf{T}(t)=\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\dfrac{1}{\sqrt{1+4t^2+9t^4}}\langle 1,2t,3t^2\rangle$. Using the product rule on each term of $\mathbf{T}(t)$ gives

$$\mathbf{T}'(t)=\frac{1}{(1+4t^2+9t^4)^{3/2}}\left\langle -\tfrac{1}{2}(8t+36t^3),2(1+4t^2+9t^4)-\tfrac{1}{2}(8t+36t^3)2t,\right.$$
$$\left.6t(1+4t^2+9t^4)-\tfrac{1}{2}(8t+36t^3)3t^2\right\rangle$$

$$=\frac{1}{(1+4t^2+9t^4)^{3/2}}\langle -4t-18t^3,2-18t^4,6t+12t^3\rangle=\frac{-2}{(14)^{3/2}}\langle 11,8,-9\rangle \text{ when } t=1.$$

$\mathbf{N}(1)\parallel\mathbf{T}'(1)\parallel\langle 11,8,-9\rangle$ and $\mathbf{T}(1)\parallel\mathbf{r}'(1)=\langle 1,2,3\rangle\ \Rightarrow$ a normal vector to the osculating plane is $\langle 11,8,-9\rangle\times\langle 1,2,3\rangle=\langle 42,-42,14\rangle$ or equivalently $\langle 3,-3,1\rangle$. An equation for the plane is $3(x-1)-3(y-1)+(z-1)=0$ or $3x-3y+z=1$.

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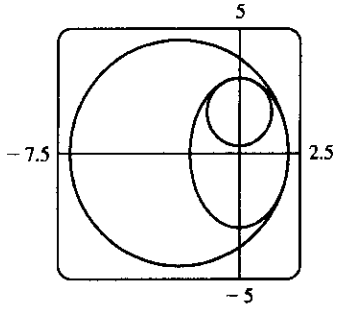
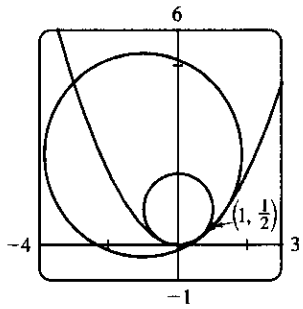
43. The ellipse is given by the parametric equations $x = 2\cos t$, $y = 3\sin t$, so using the result from Exercise 36,

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|(-2\sin t)(-3\sin t) - (3\cos t)(-2\cos t)|}{(4\sin^2 t + 9\cos^2 t)^{3/2}} = \frac{6}{(4\sin^2 t + 9\cos^2 t)^{3/2}}$$

At $(2,0)$, $t = 0$. Now $\kappa(0) = \frac{6}{27} = \frac{2}{9}$, so the radius of the osculating circle is $1/\kappa(0) = \frac{9}{2}$ and its center is $\left(-\frac{5}{2}, 0\right)$. Its equation is therefore $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}$. At $(0,3)$, $t = \frac{\pi}{2}$, and $\kappa\left(\frac{\pi}{2}\right) = \frac{6}{8} = \frac{3}{4}$. So the radius of the osculating circle is $\frac{4}{3}$ and its center is $\left(0, \frac{5}{3}\right)$. Hence its equation is $x^2 + \left(y - \frac{5}{3}\right)^2 = \frac{16}{9}$.



44. $y = \frac{1}{2}x^2 \Rightarrow y' = x$ and $y'' = 1$, so Formula 11 gives $\kappa(x) = \dfrac{1}{(1 + x^2)^{3/2}}$. So the curvature at $(0,0)$ is $\kappa(0) = 1$ and the osculating circle has radius 1 and center $(0,1)$, and hence equation $x^2 + (y-1)^2 = 1$. The curvature at $\left(1, \frac{1}{2}\right)$ is $\kappa(1) = \dfrac{1}{(1+1^2)^{3/2}} = \dfrac{1}{2\sqrt{2}}$. The tangent line to the parabola at $\left(1, \frac{1}{2}\right)$ has slope 1, so the normal line has slope -1. Thus the center of the osculating circle lies in the direction of the unit vector $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. The circle has radius $2\sqrt{2}$, so its center has position vector $\left\langle 1, \frac{1}{2} \right\rangle + 2\sqrt{2}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -1, \frac{5}{2} \right\rangle$. So the equation of the circle is $(x+1)^2 + \left(y - \frac{5}{2}\right)^2 = 8$.



45. The tangent vector is normal to the normal plane, and the vector $\langle 6, 6, -8\rangle$ is normal to the given plane. But $\mathbf{T}(t) \parallel \mathbf{r}'(t)$ and $\langle 6,6,-8\rangle \parallel \langle 3,3,-4\rangle$, so we need to find t such that $\mathbf{r}'(t) \parallel \langle 3,3,-4\rangle$. $\mathbf{r}(t) = \langle t^3, 3t, t^4\rangle \Rightarrow \mathbf{r}'(t) = \langle 3t^2, 3, 4t^3\rangle \parallel \langle 3,3,-4\rangle$ when $t = -1$. So the planes are parallel at the point $\mathbf{r}(-1) = (-1,-3,1)$.

46. To find the osculating plane, we first calculate the tangent and normal vectors.

In Maple, we set `x:=t^3; y:=3*t;` and `z:=t^4;` and then calculate the components of the tangent vector $\mathbf{T}(t)$ using the `diff` command. We find that $\mathbf{T}(t) = \dfrac{\langle 3t^2, 3, 4t^3\rangle}{\sqrt{16t^6 + 9t^4 + 9}}$. Differentiating the components of $\mathbf{T}(t)$, we find that $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\langle -6t(8t^6 - 9), 3(48t^5 + 18t^3), 36t^2(t^4+3)\rangle}{\sqrt{144t(8t^6-9)^2 + 9(96t^5 + 36t^3)^2 + 5{,}184t^{12} + 31{,}104t^8 + 46{,}656t^4}}$.

In Maple, we can calculate $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ using the `linalg` package. First we define $\mathbf{T}$ and $\mathbf{N}$ using `T:=array([f,g,h]);` and `N:=array([F,G,H]);` where `f`, `g`, `h`, `F`, `G`, and `H` are the components of $\mathbf{T}$ and $\mathbf{N}$. Then we use the command `B:=crossprod(T,N);`. After normalization and simplification, we find that $\mathbf{B}(t) = b\langle 6t, -2t^3, -3\rangle$, where

$$b = \frac{t\sqrt{16t^6 + 9t^4 + 9}}{\sqrt{16t^2(8t^6-9)^2 + (96t^5 + 36t^3)^2 + 576t^{12} + 3456t^8 + 5184t^4}}$$

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In Mathematica, we use the command `Dt` to differentiate the components of $\mathbf{r}(t)$ and subsequently $\mathbf{T}(t)$, and then load the vector analysis package with the command `<<Calculus`VectorAnalysis``. After setting `T={f,g,h}` and `N={F,G,H}`, we use `CrossProduct[T,N]` to find $\mathbf{B}$ (before normalization).

Now $\mathbf{B}(t)$ is parallel to $\langle 6t, -2t^3, -3\rangle$, so if $\mathbf{B}(t)$ is parallel to $\langle 1,1,1\rangle$ for some t, then $6t = 1 \Rightarrow t = \frac{1}{6}$, but $-2\left(\frac{1}{6}\right)^3 \neq 1$. So there is no such osculating plane.

47. $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \left|\dfrac{d\mathbf{T}/dt}{ds/dt}\right| = \dfrac{|d\mathbf{T}/dt|}{ds/dt}$ and $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$, so $\kappa\mathbf{N} = \dfrac{\left|\dfrac{d\mathbf{T}}{dt}\right|\dfrac{d\mathbf{T}}{dt}}{\left|\dfrac{d\mathbf{T}}{dt}\right|\dfrac{ds}{dt}} = \dfrac{d\mathbf{T}/dt}{ds/dt} = \dfrac{d\mathbf{T}}{ds}$ by the Chain Rule.

48. For a plane curve, $\mathbf{T} = |\mathbf{T}|\cos\phi\,\mathbf{i} + |\mathbf{T}|\sin\phi\,\mathbf{j} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$. Then $\dfrac{d\mathbf{T}}{ds} = \left(\dfrac{d\mathbf{T}}{d\phi}\right)\left(\dfrac{d\phi}{ds}\right) = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j})\left(\dfrac{d\phi}{ds}\right)$ and $\left|\dfrac{d\mathbf{T}}{ds}\right| = |-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}|\left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{d\phi}{ds}\right|$. Hence for a plane curve, the curvature is $\kappa = |d\phi/ds|$.

49. (a) $|\mathbf{B}| = 1 \Rightarrow \mathbf{B}\cdot\mathbf{B} = 1 \Rightarrow \dfrac{d}{ds}(\mathbf{B}\cdot\mathbf{B}) = 0 \Rightarrow 2\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{B} = 0 \Rightarrow \dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$

(b) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \Rightarrow$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\frac{1}{ds/dt} = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\frac{1}{|\mathbf{r}'(t)|}$$

$$= [(\mathbf{T}'\times\mathbf{N}) + (\mathbf{T}\times\mathbf{N}')]\frac{1}{|\mathbf{r}'(t)|} = \left[\left(\mathbf{T}'\times\frac{\mathbf{T}'}{|\mathbf{T}'|}\right) + (\mathbf{T}\times\mathbf{N}')\right]\frac{1}{|\mathbf{r}'(t)|} = \frac{\mathbf{T}\times\mathbf{N}'}{|\mathbf{r}'(t)|}$$

$$\Rightarrow \frac{d\mathbf{B}}{ds} \perp \mathbf{T}$$

(c) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \Rightarrow \mathbf{T}\perp\mathbf{N}, \mathbf{B}\perp\mathbf{T}$ and $\mathbf{B}\perp\mathbf{N}$. So $\mathbf{B}$, $\mathbf{T}$ and $\mathbf{N}$ form an orthogonal set of vectors in the three-dimensional space $\mathbb{R}^3$. From parts (a) and (b), $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$, so $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$. Therefore, $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$, where $\tau(s)$ is a scalar.

(d) Since $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, $\mathbf{T}\perp\mathbf{N}$ and both $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is a unit vector mutually perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. For a plane curve, $\mathbf{T}$ and $\mathbf{N}$ always lie in the plane of the curve, so that $\mathbf{B}$ is a constant unit vector always perpendicular to the plane. Thus $d\mathbf{B}/ds = \mathbf{0}$, but $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$ and $\mathbf{N} \neq \mathbf{0}$, so $\tau(s) = 0$.

50. $\mathbf{N} = \mathbf{B}\times\mathbf{T} \Rightarrow$

$$\frac{d\mathbf{N}}{ds} = \frac{d}{ds}(\mathbf{B}\times\mathbf{T}) = \frac{d\mathbf{B}}{ds}\times\mathbf{T} + \mathbf{B}\times\frac{d\mathbf{T}}{ds} \qquad \text{[by Theorem 14.2.3 [ET 13.2.3] \#5]}$$

$$= -\tau\mathbf{N}\times\mathbf{T} + \mathbf{B}\times\kappa\mathbf{N} \qquad \text{[by Formulas 3 and 1]}$$

$$= -\tau(\mathbf{N}\times\mathbf{T}) + \kappa(\mathbf{B}\times\mathbf{N}) \qquad \text{[by Theorem 13.4.8 [ET 12.4.8] \#2]}$$

But $\mathbf{B}\times\mathbf{N} = \mathbf{B}\times(\mathbf{B}\times\mathbf{T}) = (\mathbf{B}\cdot\mathbf{T})\mathbf{B} - (\mathbf{B}\cdot\mathbf{B})\mathbf{T}$ [by Theorem 13.4.8 [ET 12.4.8] \#6] $= -\mathbf{T} \Rightarrow$

$d\mathbf{N}/ds = \tau(\mathbf{T}\times\mathbf{N}) - \kappa\mathbf{T} = -\kappa\mathbf{T} + \tau\mathbf{B}$.

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51. (a) $\mathbf{r}' = s'\,\mathbf{T} \;\Rightarrow\; \mathbf{r}'' = s''\,\mathbf{T} + s'\,\mathbf{T}' = s''\,\mathbf{T} + s'\,\dfrac{d\mathbf{T}}{ds}s' = s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}$ by the first Serret-Frenet formula.

(b) Using part (a), we have

$$\begin{aligned}\mathbf{r}' \times \mathbf{r}'' &= (s'\,\mathbf{T}) \times [s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}] \\ &= [(s'\,\mathbf{T}) \times (s''\,\mathbf{T})] + [(s'\mathbf{T}) \times (\kappa(s')^2\,\mathbf{N})] \qquad \text{[by Theorem 13.4.8 [ET 12.4.8] \#3]} \\ &= (s's'')(\mathbf{T} \times \mathbf{T}) + \kappa(s')^3(\mathbf{T} \times \mathbf{N}) = \mathbf{0} + \kappa(s')^3\,\mathbf{B} = \kappa(s')^3\,\mathbf{B}\end{aligned}$$

(c) Using part (a), we have

$$\begin{aligned}\mathbf{r}''' &= [s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}]' = s'''\,\mathbf{T} + s''\,\mathbf{T}' + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^2\,\mathbf{N}' \\ &= s'''\,\mathbf{T} + s''\frac{d\mathbf{T}}{ds}\,s' + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^2\,\frac{d\mathbf{N}}{ds}\,s' \\ &= s'''\,\mathbf{T} + s''s'\kappa\,\mathbf{N} + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^3(-\kappa\,\mathbf{T} + \tau\,\mathbf{B}) \qquad \text{[by the second formula]} \\ &= [s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}\end{aligned}$$

(d) Using parts (b) and (c) and the facts that $\mathbf{B}\cdot\mathbf{T} = 0$, $\mathbf{B}\cdot\mathbf{N} = 0$, and $\mathbf{B}\cdot\mathbf{B} = 1$, we get

$$\begin{aligned}\frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} &= \frac{\kappa(s')^3\,\mathbf{B}\cdot\{[s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}\}}{|\kappa(s')^3\,\mathbf{B}|^2} \\ &= \frac{\kappa(s')^3\kappa\tau(s')^3}{[\kappa(s')^3]^2} = \tau\end{aligned}$$

52. First we find the quantities required to compute κ:

$$\mathbf{r}'(t) = \langle -a\sin t, a\cos t, b\rangle \Rightarrow \mathbf{r}''(t) = \langle -a\cos t, -a\sin t, 0\rangle \Rightarrow \mathbf{r}'''(t) = \langle a\sin t, -a\cos t, 0\rangle$$

$$|\mathbf{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = ab\sin t\,\mathbf{i} - ab\cos t\,\mathbf{j} + a^2\,\mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2} = \sqrt{a^2b^2 + a^4}$$

$$(\mathbf{r}'(t) \times \mathbf{r}''(t))\cdot\mathbf{r}'''(t) = (ab\sin t)(a\sin t) + (-ab\cos t)(-a\cos t) + (a^2)(0) = a^2b$$

Then by Theorem 10,

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{a^2b^2 + a^4}}{\left(\sqrt{a^2 + b^2}\right)^3} = \frac{a\sqrt{a^2 + b^2}}{\left(\sqrt{a^2 + b^2}\right)^3} = \frac{a}{a^2 + b^2}$$

which is a constant.

From Exercise 51(d), the torsion τ is given by

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{a^2b}{\left(\sqrt{a^2b^2 + a^4}\right)^2} = \frac{b}{a^2 + b^2}$$

which is also a constant.

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53. $\mathbf{r} = \langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle \Rightarrow \mathbf{r}' = \langle 1, t, t^2 \rangle, \mathbf{r}'' = \langle 0, 1, 2t \rangle, \mathbf{r}''' = \langle 0, 0, 2 \rangle \Rightarrow \mathbf{r}' \times \mathbf{r}'' = \langle t^2, -2t, 1 \rangle \Rightarrow$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle t^2, -2t, 1 \rangle \cdot \langle 0, 0, 2 \rangle}{t^4 + 4t^2 + 1} = \frac{2}{t^4 + 4t^2 + 1}$$

54. $\mathbf{r} = \langle \sinh t, \cosh t, t \rangle \Rightarrow \mathbf{r}' = \langle \cosh t, \sinh t, 1 \rangle, \mathbf{r}'' = \langle \sinh t, \cosh t, 0 \rangle, \mathbf{r}''' = \langle \cosh t, \sinh t, 0 \rangle \Rightarrow$

$\mathbf{r}' \times \mathbf{r}'' = \langle -\cosh t, \sinh t, \cosh^2 t - \sinh^2 t \rangle = \langle -\cosh t, \sinh t, 1 \rangle \Rightarrow$

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|\langle -\cosh t, \sinh t, 1 \rangle|}{|\langle \cosh t, \sinh t, 1 \rangle|^3} = \frac{\sqrt{\cosh^2 t + \sinh^2 t + 1}}{(\cosh^2 t + \sinh^2 t + 1)^{3/2}} = \frac{1}{\cosh^2 t + \sinh^2 t + 1} = \frac{1}{2\cosh^2 t},$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle -\cosh t, \sinh t, 1 \rangle \cdot \langle \cosh t, \sinh t, 0 \rangle}{\cosh^2 t + \sinh^2 t + 1} = \frac{-\cosh^2 t + \sinh^2 t}{2\cosh^2 t} = \frac{-1}{2\cosh^2 t}$$

So at the point $(0, 1, 0)$, $t = 0$, and $\kappa = \frac{1}{2}$ and $\tau = -\frac{1}{2}$.

55. For one helix, the vector equation is $\mathbf{r}(t) = \langle 10\cos t, 10\sin t, 34t/(2\pi) \rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and z increases by 34 angstroms for each increase of 2π in t. Using the arc length formula, letting t go from 0 to $2.9 \times 10^8 \times 2\pi$, we find the approximate length of each helix to be

$$L = \int_0^{2.9\times10^8\times2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2.9\times10^8\times2\pi} \sqrt{(-10\sin t)^2 + (10\cos t)^2 + \left(\tfrac{34}{2\pi}\right)^2}\, dt$$

$$= \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2}\, t \Big]_0^{2.9\times10^8\times2\pi} = 2.9 \times 10^8 \times 2\pi \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2}$$

$\approx 2.07 \times 10^{10}$ Å — more than two meters!

56. (a) For the function $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$ to be continuous, we must have $P(0) = 0$ and $P(1) = 1$.

For F' to be continuous, we must have $P'(0) = P'(1) = 0$. The curvature of the curve $y = F(x)$ at the point $(x, F(x))$ is $\kappa(x) = \dfrac{|F''(x)|}{(1 + [F'(x)]^2)^{3/2}}$. For $\kappa(x)$ to be continuous, we must have $P''(0) = P''(1) = 0$.

Write $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. Then $P'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$ and $P''(x) = 20ax^3 + 12bx^2 + 6cx + 2d$. Our six conditions are:

$$P(0) = 0 \quad \Rightarrow \quad f = 0 \tag{1}$$

$$P(1) = 1 \quad \Rightarrow \quad a + b + c + d + e + f = 1 \tag{2}$$

$$P'(0) = 0 \quad \Rightarrow \quad e = 0 \tag{3}$$

$$P'(1) = 0 \quad \Rightarrow \quad 5a + 4b + 3c + 2d + e = 0 \tag{4}$$

$$P''(0) = 0 \quad \Rightarrow \quad d = 0 \tag{5}$$

$$P''(1) = 0 \quad \Rightarrow \quad 20a + 12b + 6c + 2d = 0 \tag{6}$$

From (1), (3), and (5), we have $d = e = f = 0$. Thus (2), (4) and (6) become (7) $a + b + c = 1$,

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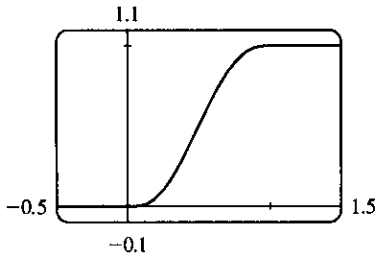
(8) $5a + 4b + 3c = 0$, and (9) $10a + 6b + 3c = 0$. Subtracting (8) from (9) gives (10) $5a + 2b = 0$. Multiplying (7) by 3 and subtracting from (8) gives (11) $2a + b = -3$. Multiplying (11) by 2 and subtracting from (10) gives $a = 6$. By (10), $b = -15$. By (7), $c = 10$. Thus, $P(x) = 6x^5 - 15x^4 + 10x^3$.

(b)



14.4 Motion in Space: Velocity and Acceleration ET 13.4

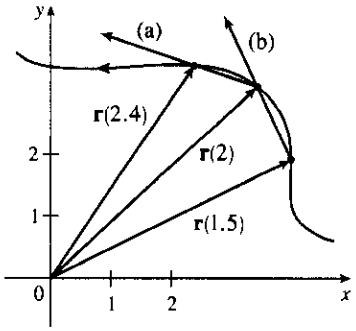
1. (a) If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is the position vector of the particle at time t, then the average velocity over the time interval $[0, 1]$ is

$\mathbf{v}_{\text{ave}} = \dfrac{\mathbf{r}(1) - \mathbf{r}(0)}{1 - 0} = \dfrac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (2.7\,\mathbf{i} + 9.8\,\mathbf{j} + 3.7\,\mathbf{k})}{1} = 1.8\,\mathbf{i} - 3.8\,\mathbf{j} - 0.7\,\mathbf{k}$. Similarly, over the other intervals we have

$$[0.5, 1]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1) - \mathbf{r}(0.5)}{1 - 0.5} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (3.5\,\mathbf{i} + 7.2\,\mathbf{j} + 3.3\,\mathbf{k})}{0.5}$$
$$= 2.0\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}$$
$$[1, 2]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(2) - \mathbf{r}(1)}{2 - 1} = \frac{(7.3\,\mathbf{i} + 7.8\,\mathbf{j} + 2.7\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{1}$$
$$= 2.8\,\mathbf{i} + 1.8\,\mathbf{j} - 0.3\,\mathbf{k}$$
$$[1, 1.5]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1.5) - \mathbf{r}(1)}{1.5 - 1} = \frac{(5.9\,\mathbf{i} + 6.4\,\mathbf{j} + 2.8\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{0.5}$$
$$= 2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k}$$

(b) We can estimate the velocity at $t = 1$ by averaging the average velocities over the time intervals $[0.5, 1]$ and $[1, 1.5]$: $\mathbf{v}(1) \approx \frac{1}{2}[(2\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}) + (2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k})] = 2.4\,\mathbf{i} - 0.8\,\mathbf{j} - 0.5\,\mathbf{k}$. Then the speed is $|\mathbf{v}(1)| \approx \sqrt{(2.4)^2 + (-0.8)^2 + (-0.5)^2} \approx 2.58$.

2. (a) The average velocity over $2 \le t \le 2.4$ is $\dfrac{\mathbf{r}(2.4) - \mathbf{r}(2)}{2.4 - 2} = 2.5\,[\mathbf{r}(2.4) - \mathbf{r}(2)]$, so we sketch a vector in the same direction but 2.5 times the length of $[\mathbf{r}(2.4) - \mathbf{r}(2)]$.



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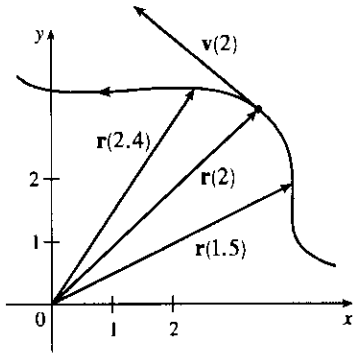
(b) The average velocity over $1.5 \le t \le 2$ is $\dfrac{\mathbf{r}(2)-\mathbf{r}(1.5)}{2-1.5} = 2[\mathbf{r}(2)-\mathbf{r}(1.5)]$, so we sketch a vector in the same direction but twice the length of $[\mathbf{r}(2)-\mathbf{r}(1.5)]$.

(c) Using Equation 2 we have $\mathbf{v}(2) = \lim\limits_{h\to 0} \dfrac{\mathbf{r}(2+h)-\mathbf{r}(2)}{h}$.

(d) $\mathbf{v}(2)$ is tangent to the curve at $\mathbf{r}(2)$ and points in the direction of increasing t. Its length is the speed of the particle at $t = 2$. We can estimate the speed by averaging the lengths of the vectors found in parts (a) and (b) which represent the average speed over $2 \le t \le 2.4$ and $1.5 \le t \le 2$ respectively. Using the axes scale as a guide, we estimate the vectors to have lengths 2.8 and 2.7. Thus, we estimate the speed at $t = 2$ to be $|\mathbf{v}(2)| \approx \frac{1}{2}(2.8+2.7) = 2.75$ and we draw the velocity vector $\mathbf{v}(2)$ with this length.

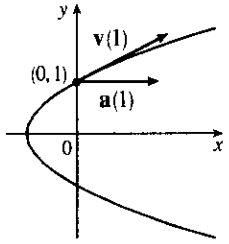


3. $\mathbf{r}(t) = \langle t^2-1, t\rangle \Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 1\rangle$, $\mathbf{v}(1) = \langle 2, 1\rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 2, 0\rangle$, $\mathbf{a}(1) = \langle 2, 0\rangle$

$|\mathbf{v}(t)| = \sqrt{4t^2+1}$

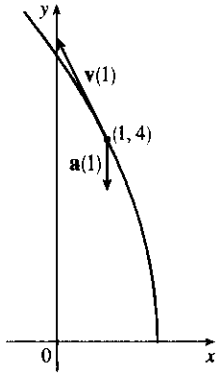


4. $\mathbf{r}(t) = \langle 2-t, 4\sqrt{t}\,\rangle \Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -1, 2/\sqrt{t}\,\rangle$, $\mathbf{v}(1) = \langle -1, 2\rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, -1/t^{3/2}\rangle$, $\mathbf{a}(1) = \langle 0, -1\rangle$

$|\mathbf{v}(t)| = \sqrt{1+4/t}$



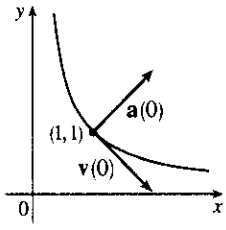
5. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j} \Rightarrow$ At $t = 0$:

$\mathbf{v}(t) = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}$, $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$,

$\mathbf{a}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$ $\mathbf{a}(0) = \mathbf{i} + \mathbf{j}$

$|\mathbf{v}(t)| = \sqrt{e^{2t}+e^{-2t}} = e^{-t}\sqrt{e^{4t}+1}$

Since $x = e^t$, $t = \ln x$ and $y = e^{-t} = e^{-\ln x} = 1/x$, and $x > 0$, $y > 0$.



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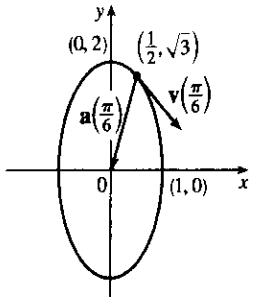
6. $\mathbf{r}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} \Rightarrow$

$\mathbf{v}(t) = \cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$, $\mathbf{v}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\,\mathbf{i} - \mathbf{j}$

$\mathbf{a}(t) = -\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j}$, $\mathbf{a}\left(\frac{\pi}{6}\right) = -\frac{1}{2}\,\mathbf{i} - \sqrt{3}\,\mathbf{j}$

$|\mathbf{v}(t)| = \sqrt{\cos^2 t + 4\sin^2 t} = \sqrt{1 + 3\sin^2 t}$

And $x^2 + y^2/4 = \sin^2 t + \cos^2 t = 1$, an ellipse.



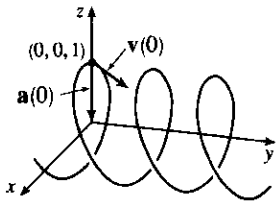
7. $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + \cos t\,\mathbf{k} \Rightarrow$

$\mathbf{v}(t) = \cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k}$, $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$, $\mathbf{a}(0) = -\mathbf{k}$

$|\mathbf{v}(t)| = \sqrt{\cos^2 t + 1 + \sin^2 t} = \sqrt{2}$

Since $x^2 + z^2 = 1$, $y = t$, the path of the particle is a helix about the y-axis.



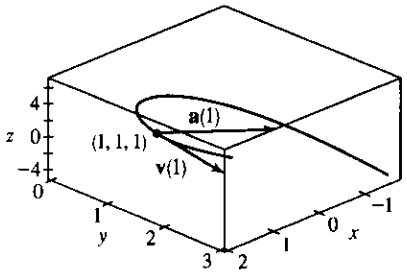
8. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k} \Rightarrow$

$\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$, $\mathbf{v}(1) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$

$\mathbf{a}(t) = 2\,\mathbf{j} + 6t\,\mathbf{k}$, $\mathbf{a}(1) = 2\,\mathbf{j} + 6\,\mathbf{k}$

$|\mathbf{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$

The path is a "twisted cubic" (see Example 14.1.7 [ET 13.1.7]).



9. $\mathbf{r}(t) = \langle t^2 + 1, t^3, t^2 - 1\rangle \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 3t^2, 2t\rangle$, $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 6t, 2\rangle$,
$|\mathbf{v}(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 8t^2} = |t|\sqrt{9t^2 + 8}$.

10. $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t\rangle \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin t, 3, 2\cos t\rangle$, $\mathbf{a}(t) = \mathbf{v}'(t) = \langle -2\cos t, 0, -2\sin t\rangle$,
$|\mathbf{v}(t)| = \sqrt{4\sin^2 t + 9 + 4\cos^2 t} = \sqrt{13}$.

11. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{v}'(t) = e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$.

12. $\mathbf{r}(t) = t^2\,\mathbf{i} + \ln t\,\mathbf{j} + t\,\mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = 2t\,\mathbf{i} + t^{-1}\,\mathbf{j} + \mathbf{k}$, $\mathbf{a}(t) = \mathbf{v}'(t) = 2\,\mathbf{i} - t^{-2}\,\mathbf{j}$,
$|\mathbf{v}(t)| = \sqrt{4t^2 + t^{-2} + 1}$.

13. $\mathbf{r}(t) = e^t\langle\cos t, \sin t, t\rangle \Rightarrow$

$\mathbf{v}(t) = \mathbf{r}'(t) = e^t\langle\cos t, \sin t, t\rangle + e^t\langle -\sin t, \cos t, 1\rangle = e^t\langle\cos t - \sin t, \sin t + \cos t, t + 1\rangle$

$\mathbf{a}(t) = \mathbf{v}'(t) = e^t\langle\cos t - \sin t - \sin t - \cos t, \sin t + \cos t + \cos t - \sin t, t + 1 + 1\rangle$

$= e^t\langle -2\sin t, 2\cos t, t + 2\rangle$

$|\mathbf{v}(t)| = e^t\sqrt{\cos^2 t + \sin^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\sin t\cos t + t^2 + 2t + 1}$

$= e^t\sqrt{t^2 + 2t + 3}$

14. $\mathbf{r}(t) = t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j} + t^2\,\mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = (\sin t + t\cos t)\,\mathbf{i} + (\cos t - t\sin t)\,\mathbf{j} + 2t\,\mathbf{k}$,
$\mathbf{a}(t) = \mathbf{v}'(t) = (2\cos t - t\sin t)\,\mathbf{i} + (-2\sin t - t\cos t)\,\mathbf{j} + 2\,\mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{(\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + (\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + 4t^2} = \sqrt{5t^2 + 1}$.

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15. $\mathbf{a}(t) = \mathbf{k} \Rightarrow \mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \mathbf{k}\,dt = t\,\mathbf{k} + \mathbf{c}_1$ and $\mathbf{i} - \mathbf{j} = \mathbf{v}(0) = 0\,\mathbf{k} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ and $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + t\,\mathbf{k}$. $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (\mathbf{i} - \mathbf{j} + t\,\mathbf{k})\,dt = t\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k} + \mathbf{c}_2$. But $\mathbf{0} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k}$.

16. $\mathbf{a}(t) = -10\,\mathbf{k} \Rightarrow \mathbf{v}(t) = \int(-10\mathbf{k})\,dt = -10t\,\mathbf{k} + \mathbf{c}_1$, and $\mathbf{i} + \mathbf{j} - \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - (10t + 1)\,\mathbf{k}$.
$\mathbf{r}(t) = \int [\mathbf{i} + \mathbf{j} - (10t + 1)\,\mathbf{k}]\,dt = t\,\mathbf{i} + t\,\mathbf{j} - (5t^2 + t)\,\mathbf{k} + \mathbf{c}_2$. But $2\,\mathbf{i} + 3\,\mathbf{j} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = 2\,\mathbf{i} + 3\,\mathbf{j}$ and $\mathbf{r}(t) = (t + 2)\,\mathbf{i} + (t + 3)\,\mathbf{j} - (5t^2 + t)\,\mathbf{k}$.

17. (a) $\mathbf{a}(t) = \mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k} \Rightarrow$

$\mathbf{v}(t) = \int(\mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k})\,dt = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{c}_1$, and

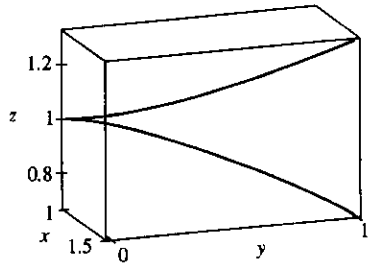
$\mathbf{0} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{0}$ and $\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}$.

$\mathbf{r}(t) = \int(t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k})\,dt = \frac{1}{2}t^2\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k} + \mathbf{c}_2$.

But $\mathbf{i} + \mathbf{k} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{i} + \mathbf{k}$ and

$\mathbf{r}(t) = \left(1 + \frac{1}{2}t^2\right)\mathbf{i} + t^2\,\mathbf{j} + \left(1 + \frac{1}{3}t^3\right)\mathbf{k}$.

(b)



18. (a) $\mathbf{a}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \cos 2t\,\mathbf{k} \Rightarrow$

$$\mathbf{v}(t) = \int(t\,\mathbf{i} + t^2\,\mathbf{j} + \cos 2t\,\mathbf{k})\,dt$$
$$= \frac{t^2}{2}\,\mathbf{i} + \frac{t^3}{3}\,\mathbf{j} + \frac{\sin 2t}{2}\,\mathbf{k} + \mathbf{c}_1$$

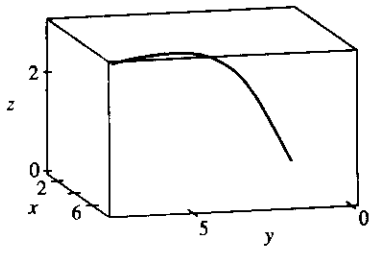
and $\mathbf{i} + \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} + \mathbf{k}$ and

$\mathbf{v}(t) = \left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + \frac{1}{3}t^3\,\mathbf{j} + \left(1 + \frac{1}{2}\sin 2t\right)\mathbf{k}$.

$\mathbf{r}(t) = \int\left[\left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + \frac{1}{3}t^3\,\mathbf{j} + \left(1 + \frac{1}{2}\sin 2t\right)\mathbf{k}\right]dt = \left(\frac{1}{6}t^3 + t\right)\mathbf{i} + \frac{1}{12}t^4\,\mathbf{j} + \left(t - \frac{1}{4}\cos 2t\right)\mathbf{k} + \mathbf{c}_2$

But $\mathbf{j} = \mathbf{r}(0) = -\frac{1}{4}\,\mathbf{k} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{j} + \frac{1}{4}\,\mathbf{k}$ and $\mathbf{r}(t) = \left(\frac{1}{6}t^3 + t\right)\mathbf{i} + \left(1 + \frac{1}{12}t^4\right)\mathbf{j} + \left(\frac{1}{4} + t - \frac{1}{4}\cos 2t\right)\mathbf{k}$.

(b)



19. $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t\rangle \Rightarrow \mathbf{v}(t) = \langle 2t, 5, 2t - 16\rangle$,
$|\mathbf{v}(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$ and
$\frac{d}{dt}|\mathbf{v}(t)| = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$. This is zero if and only if the numerator is zero, that is, $16t - 64 = 0$ or $t = 4$. Since $\frac{d}{dt}|\mathbf{v}(t)| < 0$ for $t < 4$ and $\frac{d}{dt}|\mathbf{v}(t)| > 0$ for $t > 4$, the minimum speed of $\sqrt{153}$ is attained at $t = 4$ units of time.

20. Since $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}$. By Newton's Second Law, $\mathbf{F}(t) = m\,\mathbf{a}(t) = 6mt\,\mathbf{i} + 2m\,\mathbf{j} + 6mt\,\mathbf{k}$ is the required force.

21. $|\mathbf{F}(t)| = 20$ N in the direction of the positive z-axis, so $\mathbf{F}(t) = 20\,\mathbf{k}$. Also $m = 4$ kg, $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Since $20\mathbf{k} = \mathbf{F}(t) = 4\,\mathbf{a}(t)$, $\mathbf{a}(t) = 5\,\mathbf{k}$. Then $\mathbf{v}(t) = 5t\,\mathbf{k} + \mathbf{c}_1$ where $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ so $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + 5t\,\mathbf{k}$ and the speed is $|\mathbf{v}(t)| = \sqrt{1 + 1 + 25t^2} = \sqrt{25t^2 + 2}$. Also $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k} + \mathbf{c}_2$ and $\mathbf{0} = \mathbf{r}(0)$, so $\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$.

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22. The argument here is the same as that in Example 14.2.5 [ET 13.2.5] with $\mathbf{r}(t)$ replaced by $\mathbf{v}(t)$ and $\mathbf{r}'(t)$ replaced by $\mathbf{a}(t)$.

23. $|\mathbf{v}(0)| = 500$ m/s and since the angle of elevation is $30°$, the direction of the velocity is $\frac{1}{2}(\sqrt{3}\,\mathbf{i} + \mathbf{j})$. Thus $\mathbf{v}(0) = 250(\sqrt{3}\,\mathbf{i} + \mathbf{j})$ and if we set up the axes so the projectile starts at the origin, then $\mathbf{r}(0) = \mathbf{0}$. Ignoring air resistance, the only force is that due to gravity, so $\mathbf{F}(t) = -mg\,\mathbf{j}$ where $g \approx 9.8$ m/s^2. Thus $\mathbf{a}(t) = -g\,\mathbf{j}$ and $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{c}_1$. But $250(\sqrt{3}\,\mathbf{i} + \mathbf{j}) = \mathbf{v}(0) = \mathbf{c}_1$, so $\mathbf{v}(t) = 250\sqrt{3}\,\mathbf{i} + (250 - gt)\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + (250t - \frac{1}{2}gt^2)\,\mathbf{j} + \mathbf{c}_2$ where $\mathbf{0} = \mathbf{r}(0) = \mathbf{c}_2$. Thus $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + (250t - \frac{1}{2}gt^2)\,\mathbf{j}$.

(a) Setting $250t - \frac{1}{2}gt^2 = 0$ gives $t = 0$ or $t = \frac{500}{g} \approx 51.0$ s. So the range is $250\sqrt{3} \cdot \frac{500}{g} \approx 22$ km.

(b) $0 = \dfrac{d}{dt}\left(250t - \frac{1}{2}gt^2\right) = 250 - gt$ implies that the maximum height is attained when $t = 250/g \approx 25.5$ s. Thus, the maximum height is $(250)(250/g) - g(250/g)^2\frac{1}{2} = (250)^2/(2g) \approx 3.2$ km.

(c) From part (a), impact occurs at $t = 500/g \approx 51.0$. Thus, the velocity at impact is $\mathbf{v}(500/g) = 250\sqrt{3}\,\mathbf{i} + [250 - g(500/g)]\,\mathbf{j} = 250\sqrt{3}\,\mathbf{i} - 250\,\mathbf{j}$ and the speed is $|\mathbf{v}(500/g)| = 250\sqrt{3+1} = 500$ m/s.

24. As in Exercise 23, $\mathbf{v}(t) = 250\sqrt{3}\,\mathbf{i} + (250 - gt)\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + (250t - \frac{1}{2}gt^2)\,\mathbf{j} + \mathbf{c}_2$. But $\mathbf{r}(0) = 200\,\mathbf{j}$, so $\mathbf{c}_2 = 200\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + (200 + 250t - \frac{1}{2}gt^2)\,\mathbf{j}$.

(a) $200 + 250t - \frac{1}{2}gt^2 = 0$ implies that $gt^2 - 500t - 400 = 0$ or $t = \dfrac{500 \pm \sqrt{500^2 + 1600g}}{2g}$. Taking the positive t-value gives $t = \dfrac{500 + \sqrt{250{,}000 + 1600g}}{2g} \approx 51.8$ s. Thus the range is $(250\sqrt{3})\,\dfrac{500 + \sqrt{250{,}000 + 1600g}}{2g} \approx 22.4$ km.

(b) $0 = \dfrac{d}{dt}\left(200 + 250t - \frac{1}{2}gt^2\right) = 250 - gt$ implies that the maximum height is attained when $t = 250/g \approx 25.5$ s and thus the maximum height is

$$\left[200 + (250)\left(\frac{250}{g}\right) - \frac{g}{2}\left(\frac{250}{g}\right)^2\right] = 200 + \frac{(250)^2}{2g} \approx 3.4 \text{ km}.$$

Alternate solution: Because the projectile is fired in the same direction and with the same velocity as in Exercise 23, but from a point 200 m higher, the maximum height reached is 200 m higher than that found in Exercise 23, that is, 3.2 km + 200 m = 3.4 km.

(c) From part (a), impact occurs at $t = \dfrac{500 + \sqrt{250{,}000 + 1600g}}{2g}$. Thus the velocity at impact is $250\sqrt{3}\,\mathbf{i} + \left[250 - g\,\dfrac{500 + \sqrt{250{,}000 + 1600g}}{2g}\right]\mathbf{j}$, so $|\mathbf{v}| \approx \sqrt{(250)^2(3) + (250 - 51.8g)^2} \approx 504$ m/s.

25. As in Example 5, $\mathbf{r}(t) = (v_0 \cos 45°)t\,\mathbf{i} + \left[(v_0 \sin 45°)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{1}{2}\left[v_0\sqrt{2}\,t\,\mathbf{i} + \left(v_0\sqrt{2}\,t - gt^2\right)\mathbf{j}\right]$. Then the ball lands at $t = \dfrac{v_0\sqrt{2}}{g}$ s. Now since it lands 90 m away, $90 = \frac{1}{2}v_0\sqrt{2}\,\dfrac{v_0\sqrt{2}}{g}$ or $v_0^2 = 90g$ and the initial velocity is $v_0 = \sqrt{90g} \approx 30$ m/s.

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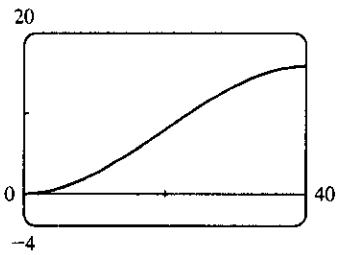
26. As in Example 5, $\mathbf{r}(t) = (v_0 \cos 30^\circ)t\,\mathbf{i} + \left[(v_0 \sin 30^\circ)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{1}{2}\left[v_0\sqrt{3}\,t\,\mathbf{i} + (v_0\,t - gt^2)\,\mathbf{j}\right]$ and then $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{1}{2}\left[v_0\sqrt{3}\,\mathbf{i} + (v_0 - 2gt)\,\mathbf{j}\right]$. The shell reaches its maximum height when the vertical component of velocity is zero, so $\frac{1}{2}(v_0 - 2gt) = 0 \;\Rightarrow\; t = \dfrac{v_0}{2g}$. The vertical height of the shell at that time is 500 m, so

$$\frac{1}{2}\left[v_0\left(\frac{v_0}{2g}\right) - g\left(\frac{v_0}{2g}\right)^2\right] = 500 \;\Rightarrow\; \frac{v_0^2}{8g} = 500 \;\Rightarrow\; v_0 = \sqrt{4000g} = \sqrt{4000(9.8)} \approx 198 \text{ m/s}.$$

27. Let α be the angle of elevation. Then $v_0 = 150$ m/s and from Example 5, the horizontal distance traveled by the projectile is $d = \dfrac{v_0^2 \sin 2\alpha}{g}$. Thus $\dfrac{150^2 \sin 2\alpha}{g} = 800 \;\Rightarrow\; \sin 2\alpha = \dfrac{800g}{150^2} \approx 0.3484 \;\Rightarrow\; 2\alpha \approx 20.4^\circ$ or $180 - 20.4 = 159.6^\circ$. Two angles of elevation then are $\alpha \approx 10.2^\circ$ and $\alpha \approx 79.8^\circ$.

28. Here $v_0 = 115$ ft/s, the angle of elevation is $\alpha = 50^\circ$, and if we place the origin at home plate, then $\mathbf{r}(0) = 3\,\mathbf{j}$. As in Example 5, we have $\mathbf{r}\,(t) = -\frac{1}{2}gt^2\,\mathbf{j} + t\,\mathbf{v}_0 + \mathbf{D}$ where $\mathbf{D} = \mathbf{r}(0) = 3\,\mathbf{j}$ and $\mathbf{v}_0 = v_0 \cos\alpha\,\mathbf{i} + v_0 \sin\alpha\,\mathbf{j}$, so $\mathbf{r}(t) = (v_0 \cos\alpha)t\,\mathbf{i} + \left[(v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 3\right]\mathbf{j}$. Thus, parametric equations for the trajectory of the ball are $x = (v_0 \cos\alpha)t$, $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 3$. The ball reaches the fence when $x = 400 \;\Rightarrow\; (v_0 \cos\alpha)t = 400$ $\Rightarrow\; t = \dfrac{400}{v_0 \cos\alpha} = \dfrac{400}{115\cos 50^\circ} \approx 5.41$ s. At this time, the height of the ball is $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 3 \approx (115 \sin 50^\circ)(5.41) - \frac{1}{2}(32)(5.41)^2 + 3 \approx 11.2$ ft. Since the fence is 10 ft high, the ball clears the fence.

29. (a) After t seconds, the boat will be $5t$ meters west of point A. The velocity of the water at that location is $\frac{3}{400}(5t)(40 - 5t)\,\mathbf{j}$. The velocity of the boat in still water is $5\,\mathbf{i}$, so the resultant velocity of the boat is $\mathbf{v}(t) = 5\,\mathbf{i} + \frac{3}{400}(5t)(40 - 5t)\,\mathbf{j} = 5\mathbf{i} + \left(\frac{3}{2}t - \frac{3}{16}t^2\right)\mathbf{j}$.



Integrating, we obtain $\mathbf{r}(t) = 5t\,\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j} + \mathbf{C}$.

If we place the origin at A (and consider $\mathbf{j}$ to coincide with the northern direction) then $\mathbf{r}(0) = \mathbf{0} \;\Rightarrow\; \mathbf{C} = \mathbf{0}$ and we have $\mathbf{r}(t) = 5t\,\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j}$. The boat reaches the east bank after 8 s, and it is located at $\mathbf{r}(8) = 5(8)\mathbf{i} + \left(\frac{3}{4}(8)^2 - \frac{1}{16}(8)^3\right)\mathbf{j} = 40\,\mathbf{i} + 16\,\mathbf{j}$. Thus the boat is 16 m downstream.

(b) Let α be the angle north of east that the boat heads. Then the velocity of the boat in still water is given by $5(\cos\alpha)\,\mathbf{i} + 5(\sin\alpha)\,\mathbf{j}$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $\frac{3}{400}[5(\cos\alpha)t][40 - 5(\cos\alpha)t]\,\mathbf{j}$. The resultant velocity of the boat is given by

$$\begin{aligned}\mathbf{v}(t) &= 5(\cos\alpha)\,\mathbf{i} + \left[5\sin\alpha + \tfrac{3}{400}(5t\cos\alpha)(40 - 5t\cos\alpha)\right]\mathbf{j} \\ &= (5\cos\alpha)\,\mathbf{i} + \left(5\sin\alpha + \tfrac{3}{2}t\cos\alpha - \tfrac{3}{16}t^2\cos^2\alpha\right)\mathbf{j}\end{aligned}$$

Integrating, $\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left(5t\sin\alpha + \frac{3}{4}t^2\cos\alpha - \frac{1}{16}t^3\cos^2\alpha\right)\mathbf{j}$ (where we have again placed

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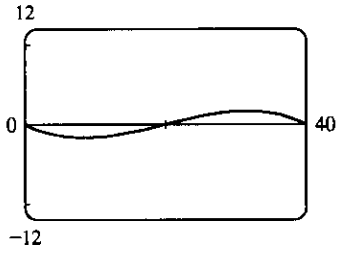
the origin at A). The boat will reach the east bank when $5t\cos\alpha = 40 \Rightarrow t = \dfrac{40}{5\cos\alpha} = \dfrac{8}{\cos\alpha}$.

In order to land at point $B(40, 0)$ we need $5t\sin\alpha + \frac{3}{4}t^2\cos\alpha - \frac{1}{16}t^3\cos^2\alpha = 0 \Rightarrow$

$5\left(\dfrac{8}{\cos\alpha}\right)\sin\alpha + \frac{3}{4}\left(\dfrac{8}{\cos\alpha}\right)^2\cos\alpha - \frac{1}{16}\left(\dfrac{8}{\cos\alpha}\right)^3\cos^2\alpha = 0 \Rightarrow \dfrac{1}{\cos\alpha}(40\sin\alpha + 48 - 32) = 0 \Rightarrow$

$40\sin\alpha + 16 = 0 \Rightarrow \sin\alpha = -\frac{2}{5}$. Thus $\alpha = \sin^{-1}\left(-\frac{2}{5}\right) \approx -23.6°$, so the boat should head $23.6°$ south of east (upstream).

The path does seem realistic. The boat initially heads upstream to counteract the effect of the current. Near the center of the river, the current is stronger and the boat is pushed downstream. When the boat nears the eastern bank, the current is slower and the boat is able to progress upstream to arrive at point B.



30. As in Exercise 29(b), let α be the angle north of east that the boat heads, so the velocity of the boat in still water is given by $5(\cos\alpha)\,\mathbf{i} + 5(\sin\alpha)\,\mathbf{j}$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $3\sin(\pi x/40)\,\mathbf{j} = 3\sin[\pi\cdot 5(\cos\alpha)t/40]\,\mathbf{j} = 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\mathbf{j}$. The resultant velocity of the boat then is given by $\mathbf{v}(t) = 5(\cos\alpha)\,\mathbf{i} + \left[5\sin\alpha + 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j}$. Integrating,

$$\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left[5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j} + \mathbf{C}.$$

If we place the origin at A then $\mathbf{r}(0) = \mathbf{0} \Rightarrow -\dfrac{24}{\pi\cos\alpha}\,\mathbf{j} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = \dfrac{24}{\pi\cos\alpha}\,\mathbf{j}$ and

$$\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left[5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right) + \frac{24}{\pi\cos\alpha}\right]\mathbf{j}.$$

The boat will reach the east bank when $5t\cos\alpha = 40 \Rightarrow t = \dfrac{8}{\cos\alpha}$. In order to land at point $B(40, 0)$ we need $5t\sin\alpha - \dfrac{24}{\pi\cos\alpha}\cos\left(\frac{\pi}{8}t\cos\alpha\right) + \dfrac{24}{\pi\cos\alpha} = 0$

$$\Rightarrow 5\left(\frac{8}{\cos\alpha}\right)\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left[\frac{\pi}{8}\left(\frac{8}{\cos\alpha}\right)\cos\alpha\right] + \frac{24}{\pi\cos\alpha} = 0 \Rightarrow$$

$$\frac{1}{\cos\alpha}\left(40\sin\alpha - \frac{24}{\pi}\cos\pi + \frac{24}{\pi}\right) = 0 \Rightarrow 40\sin\alpha + \frac{48}{\pi} = 0 \Rightarrow \sin\alpha = -\frac{6}{5\pi}.$$ Thus

$\alpha = \sin^{-1}\left(-\dfrac{6}{5\pi}\right) \approx -22.5°$, so the boat should head $22.5°$ south of east.

31. $\mathbf{r}(t) = (3t - t^3)\,\mathbf{i} + 3t^2\,\mathbf{j} \Rightarrow \mathbf{r}'(t) = (3 - 3t^2)\,\mathbf{i} + 6t\,\mathbf{j}$,

$|\mathbf{r}'(t)| = \sqrt{(3-3t^2)^2 + (6t)^2} = \sqrt{9 + 18t^2 + 9t^4} = \sqrt{(3+3t^2)^2} = 3 + 3t^2$,

$\mathbf{r}''(t) = -6t\,\mathbf{i} + 6\,\mathbf{j}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = (18 + 18t^2)\,\mathbf{k}$. Then Equation 9 gives

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{(3-3t^2)(-6t) + (6t)(6)}{3+3t^2} = \frac{18t + 18t^3}{3+3t^2} = \frac{18t(1+t^2)}{3(1+t^2)} = 6t$$ [or by Equation 8,

$a_T = v' = \dfrac{d}{dt}\left[3 + 3t^2\right] = 6t$] and Equation 10 gives $a_N = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{18 + 18t^2}{3 + 3t^2} = \dfrac{18(1+t^2)}{3(1+t^2)} = 6$.

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32. $\mathbf{r}(t) = (1+t)\,\mathbf{i} + (t^2-2t)\,\mathbf{j} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} + (2t-2)\,\mathbf{j}$, $|\mathbf{r}'(t)| = \sqrt{1^2+(2t-2)^2} = \sqrt{4t^2-8t+5}$, $\mathbf{r}''(t) = 2\,\mathbf{j}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\,\mathbf{k}$. Then Equation 9 gives $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{2(2t-2)}{\sqrt{4t^2-8t+5}}$ and Equation 10 gives $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{2}{\sqrt{4t^2-8t+5}}$.

33. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$, $\mathbf{r}''(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$, $\mathbf{r}'(t)\times\mathbf{r}''(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}$.

Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{\sin t\cos t - \sin t\cos t}{\sqrt{2}} = 0$ and

$a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{\sin^2 t+\cos^2 t+1}}{\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1.$

34. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3\,\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1^2+(2t)^2+3^2} = \sqrt{4t^2+10}$, $\mathbf{r}''(t) = 2\mathbf{j}$, $\mathbf{r}'(t)\times\mathbf{r}''(t) = -6\mathbf{i} + 2\mathbf{k}$. Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{4t}{\sqrt{4t^2+10}}$ and $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{2\sqrt{10}}{\sqrt{4t^2+10}}$.

35. $\mathbf{r}(t) = e^t\,\mathbf{i} + \sqrt{2}\,t\,\mathbf{j} + e^{-t}\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = e^t\,\mathbf{i} + \sqrt{2}\,\mathbf{j} - e^{-t}\,\mathbf{k}$, $|\mathbf{r}(t)| = \sqrt{e^{2t}+2+e^{-2t}} = \sqrt{(e^t+e^{-t})^2} = e^t + e^{-t}$, $\mathbf{r}''(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{k}$.

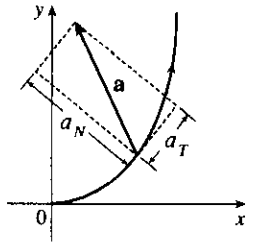
Then $a_T = \dfrac{e^{2t}-e^{-2t}}{e^t+e^{-t}} = \dfrac{(e^t+e^{-t})(e^t-e^{-t})}{e^t+e^{-t}} = e^t - e^{-t} = 2\sinh t$ and

$a_N = \dfrac{|\sqrt{2}e^{-t}\,\mathbf{i} - 2\,\mathbf{j} - \sqrt{2}e^t\,\mathbf{k}|}{e^t+e^{-t}} = \dfrac{\sqrt{2(e^{-2t}+2+e^{2t})}}{e^t+e^{-t}} = \sqrt{2}\,\dfrac{e^t+e^{-t}}{e^t+e^{-t}} = \sqrt{2}.$

36. $\mathbf{r}(t) = t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \sin^2 t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} - 2\cos t\sin t\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k} = \mathbf{i} - \sin 2t\,\mathbf{j} + \sin 2t\,\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1+2\sin^2 2t}$, $\mathbf{r}''(t) = 2(\sin^2 t - \cos^2 t)\,\mathbf{j} + 2(\cos^2 t - \sin^2 t)\,\mathbf{k} = -2\cos 2t\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$. So

$a_T = \dfrac{2\sin 2t\cos 2t + 2\sin 2t\cos 2t}{\sqrt{1+2\sin^2 2t}} = \dfrac{4\sin 2t\cos 2t}{\sqrt{1+2\sin^2 2t}}$ and $a_N = \dfrac{|-2\cos 2t\,\mathbf{j} - 2\cos 2t\,\mathbf{k}|}{\sqrt{1+2\sin^2 2t}} = \dfrac{2\sqrt{2}\,|\cos 2t|}{\sqrt{1+2\sin^2 t}}$.

37. The tangential component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{T}$, so we sketch the scalar projection of $\mathbf{a}$ in the tangential direction to the curve and estimate its length to be 4.5 (using the fact that $\mathbf{a}$ has length 10 as a guide). Similarly, the normal component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{N}$, so we sketch the scalar projection of $\mathbf{a}$ in the normal direction to the curve and estimate its length to be 9.0. Thus $a_T \approx 4.5$ cm/s^2 and $a_N \approx 9.0$ cm/s^2.



38. $\mathbf{L}(t) = m\,\mathbf{r}(t)\times\mathbf{v}(t) \;\Rightarrow$

$$\mathbf{L}'(t) = m[\mathbf{r}'(t)\times\mathbf{v}(t) + \mathbf{r}(t)\times\mathbf{v}'(t)] \qquad \text{[by Theorem 14.2.3 [ET 13.2.3] \#5]}$$

$$= m[\mathbf{v}(t)\times\mathbf{v}(t) + \mathbf{r}(t)\times\mathbf{v}'(t)] = m[\mathbf{0} + \mathbf{r}(t)\times\mathbf{a}(t)] = \boldsymbol{\tau}(t)$$

So if the torque is always $\mathbf{0}$, then $\mathbf{L}'(t) = \mathbf{0}$ for all t, and so $\mathbf{L}(t)$ is constant.

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39. If the engines are turned off at time t, then the spacecraft will continue to travel in the direction of $\mathbf{v}(t)$, so we need a t such that for some scalar $s > 0$, $\mathbf{r}(t) + s\,\mathbf{v}(t) = \langle 6, 4, 9\rangle$. $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + \dfrac{1}{t}\,\mathbf{j} + \dfrac{8t}{(t^2+1)^2}\,\mathbf{k} \;\Rightarrow$

$$\mathbf{r}(t) + s\,\mathbf{v}(t) = \left\langle 3 + t + s,\ 2 + \ln t + \frac{s}{t},\ 7 - \frac{4}{t^2+1} + \frac{8st}{(t^2+1)^2} \right\rangle \;\Rightarrow\; 3 + t + s = 6 \;\Rightarrow\; s = 3 - t,$$

so $7 - \dfrac{4}{t^2+1} + \dfrac{8(3-t)t}{(t^2+1)^2} = 9 \;\Leftrightarrow\; \dfrac{24t - 12t^2 - 4}{(t^2+1)^2} = 2 \;\Leftrightarrow\; t^4 + 8t^2 - 12t + 3 = 0$. It is easily seen that $t = 1$ is a root of this polynomial. Also $2 + \ln 1 + \dfrac{3-1}{1} = 4$, so $t = 1$ is the desired solution.

40. (a) $m\dfrac{d\mathbf{v}}{dt} = \dfrac{dm}{dt}\,\mathbf{v}_e \;\Leftrightarrow\; \dfrac{d\mathbf{v}}{dt} = \dfrac{1}{m}\dfrac{dm}{dt}\,\mathbf{v}_e$. Integrating both sides of this equation with respect to t gives

$$\int_0^t \frac{d\mathbf{v}}{du}\,du = \mathbf{v}_e \int_0^t \frac{1}{m}\frac{dm}{du}\,du \;\Rightarrow\; \int_{\mathbf{v}(0)}^{\mathbf{v}(t)} d\mathbf{v} = \mathbf{v}_e \int_{m(0)}^{m(t)} \frac{dm}{m} \quad \text{[Substitution Rule]} \;\Rightarrow$$

$$\mathbf{v}(t) - \mathbf{v}(0) = \ln\left(\frac{m(t)}{m(0)}\right)\mathbf{v}_e \;\Rightarrow\; \mathbf{v}(t) = \mathbf{v}(0) - \ln\left(\frac{m(0)}{m(t)}\right)\mathbf{v}_e.$$

(b) $|\mathbf{v}(t)| = 2\,|\mathbf{v}_e|$, and $|\mathbf{v}(0)| = 0$. Therefore, by part (a), $2\,|\mathbf{v}_e| = \left|-\ln\left(\dfrac{m(0)}{m(t)}\right)\mathbf{v}_e\right| \;\Rightarrow$

$2\,|\mathbf{v}_e| = \ln\left(\dfrac{m(0)}{m(t)}\right)|\mathbf{v}_e|$. $\left[\text{Note: } m(0) > m(t) \text{ so that } \ln\left(\dfrac{m(0)}{m(t)}\right) > 0\right] \;\Rightarrow\; m(t) = e^{-2}m(0)$.

Thus $\dfrac{m(0) - e^{-2}m(0)}{m(0)} = 1 - e^{-2}$ is the fraction of the initial mass that is burned as fuel.

APPLIED PROJECT Kepler's Laws

1. With $\mathbf{r} = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}$ and $\mathbf{h} = \alpha\,\mathbf{k}$ where $\alpha > 0$,

(a) $\mathbf{h} = \mathbf{r} \times \mathbf{r}' = [(r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}] \times \left[\left(r'\cos\theta - r\sin\theta\,\dfrac{d\theta}{dt}\right)\mathbf{i} + \left(r'\sin\theta + r\cos\theta\,\dfrac{d\theta}{dt}\right)\mathbf{j}\right]$

$$= \left[rr'\cos\theta\sin\theta + r^2\cos^2\theta\,\frac{d\theta}{dt} - rr'\cos\theta\sin\theta + r^2\sin^2\theta\,\frac{d\theta}{dt}\right]\mathbf{k} = r^2\frac{d\theta}{dt}\,\mathbf{k}$$

(b) Since $\mathbf{h} = \alpha\,\mathbf{k}$, $\alpha > 0$, $\alpha = |\mathbf{h}|$. But by part (a), $\alpha = |\mathbf{h}| = r^2\,(d\theta/dt)$.

(c) $A(t) = \frac{1}{2}\int_{\theta_0}^{\theta} |\mathbf{r}|^2\,d\theta = \frac{1}{2}\int_{t_0}^{t} r^2\,(d\theta/dt)\,dt$ in polar coordinates. Thus, by the Fundamental Theorem of Calculus, $\dfrac{dA}{dt} = \dfrac{r^2}{2}\dfrac{d\theta}{dt}$.

(d) $\dfrac{dA}{dt} = \dfrac{r^2}{2}\dfrac{d\theta}{dt} = \dfrac{h}{2} =$ constant since $\mathbf{h}$ is a constant vector and $h = |\mathbf{h}|$.

2. (a) Since $dA/dt = \frac{1}{2}h$, a constant, $A(t) = \frac{1}{2}ht + c_1$. But $A(0) = 0$, so $A(t) = \frac{1}{2}ht$. But $A(T)$ = area of the ellipse $= \pi ab$ and $A(T) = \frac{1}{2}hT$, so $T = 2\pi ab/h$.

(b) $h^2/(GM) = ed$ where e is the eccentricity of the ellipse. But $a = ed/(1-e^2)$ or $ed = a(1-e^2)$ and $1 - e^2 = b^2/a^2$. Hence $h^2/(GM) = ed = b^2/a$.

(c) $T^2 = \dfrac{4\pi a^2 b^2}{h^2} = 4\pi^2 a^2 b^2 \dfrac{a}{GMb^2} = \dfrac{4\pi^2}{GM}a^3$.

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3. From Problem 2, $T^2 = \dfrac{4\pi^2}{GM}a^3$. $T \approx 365.25$ days $\times\, 24 \cdot 60^2 \dfrac{\text{seconds}}{\text{day}} \approx 3.1558 \times 10^7$ seconds. Therefore

$$a^3 = \frac{GMT^2}{4\pi^2} \approx \frac{(6.67 \times 10^{-11})(1.99 \times 10^{30})(3.1558 \times 10^7)^2}{4\pi^2} \approx 3.348 \times 10^{33}\ \text{m}^3 \quad \Rightarrow \quad a \approx 1.496 \times 10^{11}\ \text{m}.$$

Thus, the length of the major axis of Earth's orbit (that is, $2a$) is approximately 2.99×10^{11} m $= 2.99 \times 10^8$ km.

4. We can adapt the equation $T^2 = \dfrac{4\pi^2}{GM}a^3$ from Problem 2(c) with Earth at the center of the system, so T is the period of the satellite's orbit about Earth, M is the mass of Earth, and a is the length of the semimajor axis of the satellite's orbit (measured from Earth's center). Since we want the satellite to remain fixed above a particular point on Earth's equator, T must coincide with the period of Earth's own rotation, so $T = 24$ h $= 86{,}400$ s. The mass of Earth is $M = 5.98 \times 10^{24}$ kg, so

$a = \left(\dfrac{T^2GM}{4\pi^2}\right)^{1/3} \approx \left[\dfrac{(86{,}400)^2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{4\pi^2}\right]^{1/3} \approx 4.23 \times 10^7$ m. If we assume a circular orbit, the radius of the orbit is a, and since the radius of Earth is 6.37×10^6 m, the required altitude above Earth's surface for the satellite is $4.23 \times 10^7 - 6.37 \times 10^6 \approx 3.59 \times 10^7$ m, or 35,900 km.

14 Review — ET 13

CONCEPT CHECK

1. A vector function is a function whose domain is a set of real numbers and whose range is a set of vectors. To find the derivative or integral, we can differentiate or integrate each component of the vector function.

2. The tip of the moving vector $\mathbf{r}(t)$ of a continuous vector function traces out a space curve.

3. (a) A curve represented by the vector function $\mathbf{r}(t)$ is smooth if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on its parametric domain (except possibly at the endpoints).

(b) The tangent vector to a smooth curve at a point P with position vector $\mathbf{r}(t)$ is the vector $\mathbf{r}'(t)$. The tangent line at P is the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

4. (a)–(f) See Theorem 14.2.3 [ET 13.2.3].

5. Use Formula 14.3.2 [ET 13.3.2], or equivalently 14.3.3 [ET 13.3.3].

6. (a) The curvature of a curve is $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$ where $\mathbf{T}$ is the unit tangent vector.

(b) $\kappa(t) = \left|\dfrac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right|$ (c) $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ (d) $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$

7. (a) The unit normal vector: $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. The binormal vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

(b) See the discussion preceding Example 7 in Section 14.3 [ET 13.3].

8. (a) If $\mathbf{r}(t)$ is the position vector of the particle on the space curve, the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$, the speed is given by $|\mathbf{v}(t)|$, and the acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

(b) $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ where $a_T = v'$ and $a_N = \kappa v^2$.

9. See the statement of Kepler's Laws on page 912 [ET 876].

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TRUE-FALSE QUIZ

1. True. If we reparametrize the curve by replacing $u = t^3$, we have $\mathbf{r}(u) = u\,\mathbf{i} + 2u\,\mathbf{j} + 3u\,\mathbf{k}$, which is a line through the origin with direction vector $\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$.

2. True. $\mathbf{r}'(t) = \langle 1, 3t^2, 5t^4 \rangle$ is continuous for all t (since its component functions are each continuous) and since $x'(t) = 1$, we have $\mathbf{r}'(t) \neq \mathbf{0}$ for all t.

3. False. $\mathbf{r}'(t) = \langle -\sin t, 2t, 4t^3 \rangle$, and since $\mathbf{r}'(0) = \langle 0, 0, 0 \rangle = \mathbf{0}$, the curve is not smooth.

4. True. See Theorem 14.2.2 [ET 13.2.2].

5. False. By Formula 5 of Theorem 14.2.3 [ET 13.2.3], $\dfrac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

6. False. For example, let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then $|\mathbf{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1 \;\Rightarrow\; \dfrac{d}{dt}\,|\mathbf{r}(t)| = 0$, but $|\mathbf{r}'(t)| = |\langle -\sin t, \cos t \rangle| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$.

7. False. κ is the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length s, not with respect to t.

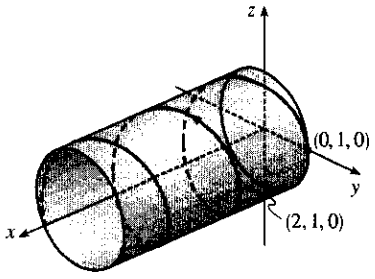
8. False. The binormal vector, by the definition given in Section 14.3 [ET 13.3], is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = -\left[\mathbf{N}(t) \times \mathbf{T}(t)\right]$.

9. True. See the discussion preceding Example 7 in Section 14.3 [ET 13.3].

10. False. For example, $\mathbf{r}_1(t) = \langle t, t \rangle$ and $\mathbf{r}_2(t) = \langle 2t, 2t \rangle$ both represent the same plane curve (the line $y = x$), but the tangent vector $\mathbf{r}_1'(t) = \langle 1, 1 \rangle$ for all t, while $\mathbf{r}_2'(t) = \langle 2, 2 \rangle$. In fact, different parametrizations give parallel tangent vectors at a point, but their magnitudes may differ.

EXERCISES

1. (a) The corresponding parametric equations for the curve are $x = t$, $y = \cos \pi t$, $z = \sin \pi t$. Since $y^2 + z^2 = 1$, the curve is contained in a circular cylinder with axis the x-axis. Since $x = t$, the curve is a helix.



(b) $\mathbf{r}(t) = t\,\mathbf{i} + \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\,\mathbf{j} + \pi \cos \pi t\,\mathbf{k} \;\Rightarrow$
$\mathbf{r}''(t) = -\pi^2 \cos \pi t\,\mathbf{j} - \pi^2 \sin \pi t\,\mathbf{k}$

2. (a) The expressions $\sqrt{2 - t}$, $(e^t - 1)/t$, and $\ln(t + 1)$ are all defined when $2 - t \geq 0 \;\Rightarrow\; t \leq 2$, $t \neq 0$, and $t + 1 > 0 \;\Rightarrow\; t > -1$. Thus the domain of $\mathbf{r}$ is $(-1, 0) \cup (0, 2]$.

(b) $\lim\limits_{t \to 0} \mathbf{r}(t) = \left\langle \lim\limits_{t \to 0} \sqrt{2 - t}, \lim\limits_{t \to 0} \dfrac{e^t - 1}{t}, \lim\limits_{t \to 0} \ln(t + 1) \right\rangle = \left\langle \sqrt{2 - 0}, \lim\limits_{t \to 0} \dfrac{e^t}{1}, \ln(0 + 1) \right\rangle = \langle \sqrt{2}, 1, 0 \rangle$
(using l'Hospital's Rule in the y-component).

(c) $\mathbf{r}'(t) = \left\langle \dfrac{d}{dt}\sqrt{2 - t}, \dfrac{d}{dt}\dfrac{e^t - 1}{t}, \dfrac{d}{dt}\ln(t + 1) \right\rangle = \left\langle -\dfrac{1}{2\sqrt{2 - t}}, \dfrac{te^t - e^t + 1}{t^2}, \dfrac{1}{t + 1} \right\rangle$

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34. $f(x,y,z) = \sqrt{x+y+z} = h(g(x,y,z))$ where $g(x,y,z) = x+y+z$, continuous everywhere, and $h(t) = \sqrt{t}$ is continuous on its domain $\{t \mid t \geq 0\}$. Thus f is continuous on its domain $\{(x,y,z) \mid x+y+z \geq 0\}$, so f is continuous on and above the plane $z = -x-y$.

35. $f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. Since $x^2 \leq 2x^2+y^2$, we have $\left|x^2y^3/(2x^2+y^2)\right| \leq \left|y^3\right|$. We know that $\left|y^3\right| \to 0$ as $(x,y) \to (0,0)$. So, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^3}{2x^2+y^2} = 0$. But $f(0,0) = 1$, so f is discontinuous at $(0,0)$. Therefore, f is continuous on the set $\{(x,y) \mid (x,y) \neq (0,0)\}$.

36. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But $f(x,x) = x^2/(3x^2) = \frac{1}{3}$ for $x \neq 0$, so $f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along the line $y = x$. Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is not continuous at $(0,0)$ and the largest set on which f is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

37. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = \lim\limits_{r\to0^+} \dfrac{(r\cos\theta)^3 + (r\sin\theta)^3}{r^2} = \lim\limits_{r\to0^+} (r\cos^3\theta + r\sin^3\theta) = 0$

38. $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim\limits_{r\to0^+} r^2 \ln r^2 = \lim\limits_{r\to0^+} \dfrac{\ln r^2}{1/r^2}$

$= \lim\limits_{r\to0^+} \dfrac{(1/r^2)(2r)}{-2/r^3}$ [using l'Hospital's Rule] $= \lim\limits_{r\to0^+} (-r^2) = 0$

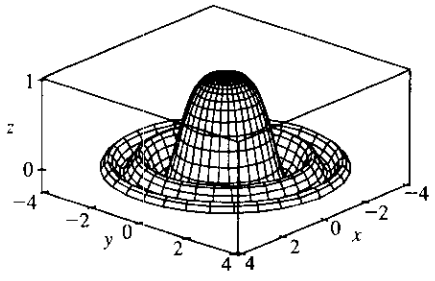
39. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2} = \lim\limits_{\rho\to0^+} \dfrac{(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)}{\rho^2}$

$= \lim\limits_{\rho\to0^+} (\rho\sin^2\phi\cos\phi\sin\theta\cos\theta) = 0$

40. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{r\to0^+} \dfrac{\sin(r^2)}{r^2}$, which is an indeterminate form of type $0/0$. Using l'Hospital's Rule, we get

$\lim\limits_{r\to0^+} \dfrac{\sin(r^2)}{r^2} \overset{\text{H}}{=} \lim\limits_{r\to0^+} \dfrac{2r\cos(r^2)}{2r} = \lim\limits_{r\to0^+} \cos(r^2) = 1.$

Or: Use the fact that $\lim\limits_{\theta\to0} \dfrac{\sin\theta}{\theta} = 1$.



41. Since $|\mathbf{x}-\mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}||\mathbf{a}|\cos\theta \geq |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}||\mathbf{a}| = (|\mathbf{x}| - |\mathbf{a}|)^2$, we have $||\mathbf{x}| - |\mathbf{a}|| \leq |\mathbf{x}-\mathbf{a}|$. Let $\epsilon > 0$ be given and set $\delta = \epsilon$. Then whenever $0 < |\mathbf{x}-\mathbf{a}| < \delta$, $||\mathbf{x}| - |\mathbf{a}|| \leq |\mathbf{x}-\mathbf{a}| < \delta = \epsilon$. Hence $\lim_{\mathbf{x}\to\mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

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42. Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$ or $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$. But $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| = |\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})|$ and $|\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})| \leq |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}|$ by Exercise 13.3.57 [ET 12.3.57] (the Cauchy-Schwartz Inequality). Let $\epsilon > 0$ be given and set $\delta = \epsilon / |\mathbf{c}|$. Then whenever $0 < |\mathbf{x} - \mathbf{a}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| \leq |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}| < |\mathbf{c}|\,\delta = |\mathbf{c}|\,(\epsilon / |\mathbf{c}|) = \epsilon$. So f is continuous on $\mathbb{R}^n$.

15.3 Partial Derivatives ET 14.3

1. (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude 158°W, latitude 21°N at 9:00 A.M. when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.

2. By Definition 4, $f_T(92, 60) = \lim_{h \to 0} \dfrac{f(92 + h, 60) - f(92, 60)}{h}$, which we can approximate by considering $h = 2$ and $h = -2$ and using the values given in Table 1: $f_T(92, 60) \approx \dfrac{f(94, 60) - f(92, 60)}{2} = \dfrac{111 - 105}{2} = 3$, $f_T(92, 60) \approx \dfrac{f(90, 60) - f(92, 60)}{-2} = \dfrac{100 - 105}{-2} = 2.5$. Averaging these values, we estimate $f_T\,(92, 60)$ to be approximately 2.75. Thus, when the actual temperature is 92°F and the relative humidity is 60%, the apparent temperature rises by about 2.75°F for every degree that the actual temperature rises.

Similarly, $f_H(92, 60) = \lim_{h \to 0} \dfrac{f(92, 60 + h) - f(92, 60)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$: $f_H(92, 60) \approx \dfrac{f(92, 65) - f(92, 60)}{5} = \dfrac{108 - 105}{5} = 0.6$, $f_H(92, 60) \approx \dfrac{f(92, 55) - f(92, 60)}{-5} = \dfrac{103 - 105}{-5} = 0.4$. Averaging these values, we estimate $f_H\,(92, 60)$ to be approximately 0.5. Thus, when the actual temperature is 92°F and the relative humidity is 60%, the apparent temperature rises by about 0.5°F for every percent that the relative humidity increases.

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3. (a) By Definition 4, $f_T(-15,30) = \lim_{h\to 0} \dfrac{f(-15+h,30) - f(-15,30)}{h}$, which we can approximate by considering $h = 5$ and $h = -5$ and using the values given in the table:

$f_T(-15,30) \approx \dfrac{f(-10,30) - f(-15,30)}{5} = \dfrac{-20-(-26)}{5} = \dfrac{6}{5} = 1.2,$

$f_T(-15,30) \approx \dfrac{f(-20,30) - f(-15,30)}{-5} = \dfrac{-33-(-26)}{-5} = \dfrac{-7}{-5} = 1.4.$ Averaging these values, we estimate $f_T(-15,30)$ to be approximately 1.3. Thus, when the actual temperature is $-15°\text{C}$ and the wind speed is 30 km/h, the apparent temperature rises by about $1.3°\text{C}$ for every degree that the actual temperature rises.

Similarly, $f_v(-15,30) = \lim_{h\to 0} \dfrac{f(-15,30+h) - f(-15,30)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$: $f_v(-15,30) \approx \dfrac{f(-15,40) - f(-15,30)}{10} = \dfrac{-27-(-26)}{10} = \dfrac{-1}{10} = -0.1,$

$f_v(-15,30) \approx \dfrac{f(-15,20) - f(-15,30)}{-10} = \dfrac{-24-(-26)}{-10} = \dfrac{2}{-10} = -0.2.$ Averaging these values, we estimate $f_v(-15,30)$ to be approximately -0.15. Thus, when the actual temperature is $-15°\text{C}$ and the wind speed is 30 km/h, the apparent temperature decreases by about $0.15°\text{C}$ for every km/h that the wind speed increases.

(b) For a fixed wind speed v, the values of the wind-chill index W increase as temperature T increases (look at a column of the table), so $\dfrac{\partial W}{\partial T}$ is positive. For a fixed temperature T, the values of W decrease (or remain constant) as v increases (look at a row of the table), so $\dfrac{\partial W}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of T, the function values $f(T,v)$ appear to become constant (or nearly constant) as v increases, so the corresponding rate of change is 0 or near 0 as v increases. This suggests that $\lim_{v\to\infty} (\partial W/\partial v) = 0$.

4. (a) $\partial h/\partial v$ represents the rate of change of h when we fix t and consider h as a function of v, which describes how quickly the wave heights change when the wind speed changes for a fixed time duration. $\partial h/\partial t$ represents the rate of change of h when we fix v and consider h as a function of t, which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

(b) By Definition 4, $f_v(40,15) = \lim_{h\to 0} \dfrac{f(40+h,15) - f(40,15)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$ and using the values given in the table:

$f_v(40,15) \approx \dfrac{f(50,15) - f(40,15)}{10} = \dfrac{36-25}{10} = 1.1,$

$f_v(40,15) \approx \dfrac{f(30,15) - f(40,15)}{-10} = \dfrac{16-25}{-10} = 0.9.$ Averaging these values, we have $f_v(40,15) \approx 1.0$.

Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights should increase by about 1 foot for every knot that the wind speed increases (with the same time duration). Similarly,

$f_t(40,15) = \lim_{h\to 0} \dfrac{f(40,15+h) - f(40,15)}{h}$ which we can approximate by considering

$h = 5$ and $h = -5$: $f_t(40,15) \approx \dfrac{f(40,20) - f(40,15)}{5} = \dfrac{28-25}{5} = 0.6,$

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$f_t(40,15) \approx \dfrac{f(40,10)-f(40,15)}{-5} = \dfrac{21-25}{-5} = 0.8$. Averaging these values, we have $f_t(40,15) \approx 0.7$. Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights increase by about 0.7 feet for every additional hour that the wind blows.

(c) For fixed values of v, the function values $f(v,t)$ appear to increase in smaller and smaller increments, becoming nearly constant as t increases. Thus, the corresponding rate of change is nearly 0 as t increases, suggesting that $\lim_{t\to\infty} (\partial h/\partial t) = 0$.

5. (a) If we start at $(1,2)$ and move in the positive x-direction, the graph of f increases. Thus $f_x(1,2)$ is positive.

(b) If we start at $(1,2)$ and move in the positive y-direction, the graph of f decreases. Thus $f_y(1,2)$ is negative.

6. (a) The graph of f decreases if we start at $(-1,2)$ and move in the positive x-direction, so $f_x(-1,2)$ is negative.

(b) The graph of f decreases if we start at $(-1,2)$ and move in the positive y-direction, so $f_y(-1,2)$ is negative.

(c) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so f_{xx} is the rate of change of f_x in the x-direction. f_x is negative at $(-1,2)$ and if we move in the positive x-direction, the surface becomes less steep. Thus the values of f_x are increasing and $f_{xx}(-1,2)$ is positive.

(d) f_{yy} is the rate of change of f_y in the y-direction. f_y is negative at $(-1,2)$ and if we move in the positive y-direction, the surface becomes steeper. Thus the values of f_y are decreasing, and $f_{yy}(-1,2)$ is negative.

7. First of all, if we start at the point $(3,-3)$ and move in the positive y-direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3,-1.5)$, while a is 0 at this point. So a is definitely the graph of f_y, and one of b and c is the graph of f. To see which is which, we start at the point $(-3,-1.5)$ and move in the positive x-direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x-derivative of c. So c is the graph of f, b is the graph of f_x, and a is the graph of f_y.

8. $f_x(2,1)$ is the rate of change of f at $(2,1)$ in the x-direction. If we start at $(2,1)$, where $f(2,1) = 10$, and move in the positive x-direction, we reach the next contour line (where $f(x,y) = 12$) after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2,1)$ from the left (moving in the positive x-direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2,1)$ would be the average of these two, so $f_x(2,1) \approx 2.8$. Similarly, $f_y(2,1)$ is the rate of change of f at $(2,1)$ in the y-direction. If we approach $(2,1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2. If we start at $(2,1)$ and move in the positive y-direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2,1) \approx -2.1$.

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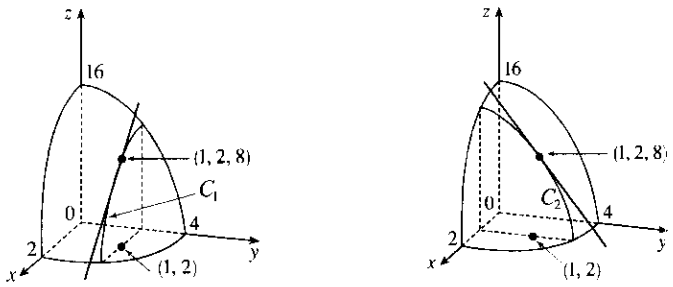
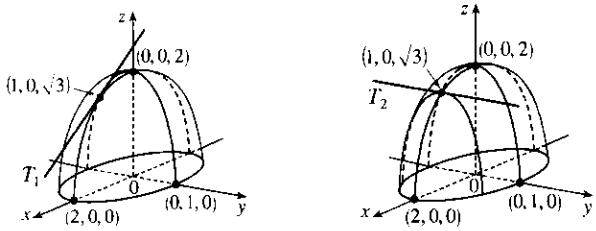
9. $f(x,y) = 16 - 4x^2 - y^2 \;\Rightarrow\; f_x(x,y) = -8x$ and $f_y(x,y) = -2y \;\Rightarrow\; f_x(1,2) = -8$ and $f_y(1,2) = -4$.

The graph of f is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$ (the curve C_1 in the first figure).

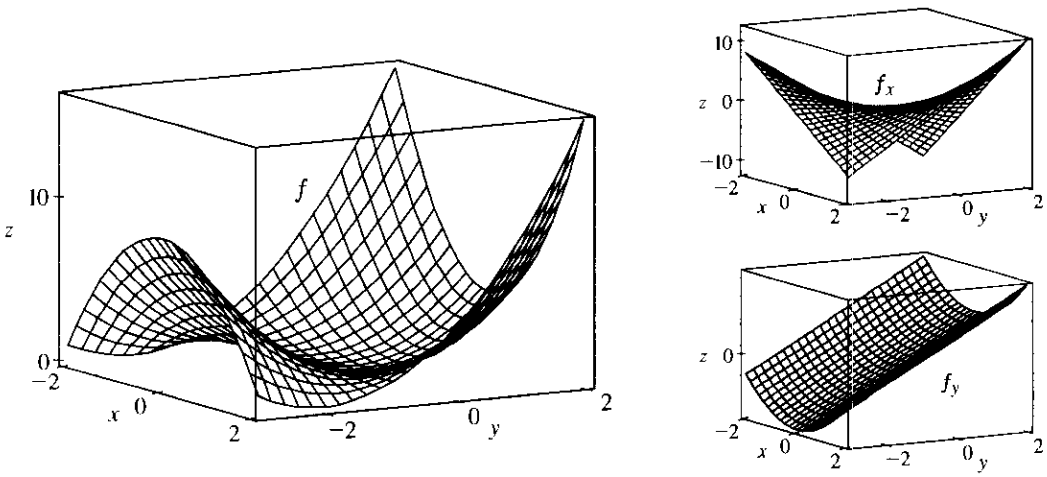
The slope of the tangent line to this parabola at $(1, 2, 8)$ is $f_x(1,2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1, 2, 8)$ is $f_y(1,2) = -4$.



10. $f(x,y) = (4 - x^2 - 4y^2)^{1/2} \;\Rightarrow\; f_x(x,y) = -x(4 - x^2 - 4y^2)^{-1/2}$ and $f_y(x,y) = -4y(4 - x^2 - 4y^2)^{-1/2}$ $\Rightarrow\; f_x(1,0) = -\frac{1}{\sqrt{3}}$, $f_y(1,0) = 0$. The graph of f is the upper half of the ellipsoid $z^2 + x^2 + 4y^2 = 4$ and the plane $y = 0$ intersects the graph in the semicircle $x^2 + z^2 = 4$, $z \geq 0$ and the slope of the tangent line T_1 to this semicircle at $\left(1, 0, \sqrt{3}\right)$ is $f_x(1,0) = -\frac{1}{\sqrt{3}}$. Similarly the plane $x = 1$ intersects the graph in the semi-ellipse $z^2 + 4y^2 = 3$, $z \geq 0$ and the slope of the tangent line T_2 to this semi-ellipse at $\left(1, 0, \sqrt{3}\right)$ is $f_y(1,0) = 0$.



11. $f(x,y) = x^2 + y^2 + x^2y \;\Rightarrow\; f_x = 2x + 2xy$, $f_y = 2y + x^2$



Note that the traces of f in planes parallel to the xz-plane are parabolas which open downward for $y < -1$ and

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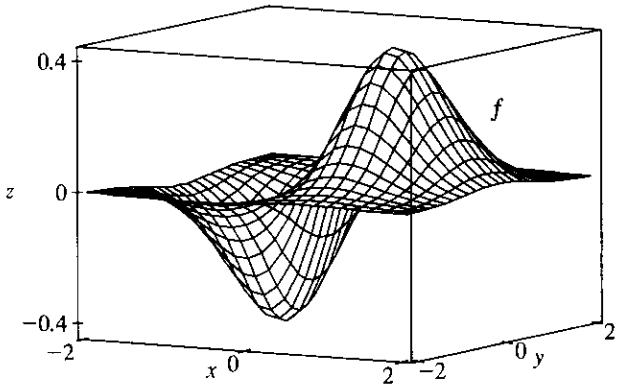
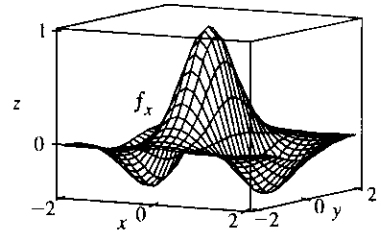
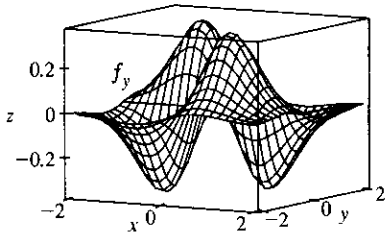
upward for $y > -1$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y < -1$ and positive slopes for $y > -1$. The traces of f in planes parallel to the yz-plane are parabolas which always open upward, and the traces of f_y in these planes are straight lines with positive slopes.

12. $f(x,y) = xe^{-x^2-y^2} \Rightarrow f_x = x\left(-2xe^{-x^2-y^2}\right) + e^{-x^2-y^2} = e^{-x^2-y^2}(1-2x^2)$, $f_y = -2xye^{-x^2-y^2}$







Note that traces of f in planes parallel to the xz-plane have two extreme values, while traces of f_x in these planes have two zeros. Traces of f in planes parallel to the yz-plane have only one extreme value (a minimum if $x < 0$, a maximum if $x > 0$), and traces of f_y in these planes have only one zero (going from negative to positive if $x < 0$ and from positive to negative if $x > 0$).

13. $f(x,y) = 3x - 2y^4 \Rightarrow f_x(x,y) = 3 - 0 = 3$, $f_y(x,y) = 0 - 8y^3 = -8y^3$

14. $f(x,y) = x^5 + 3x^3y^2 + 3xy^4 \Rightarrow f_x(x,y) = 5x^4 + 3 \cdot 3x^2 \cdot y^2 + 3 \cdot 1 \cdot y^4 = 5x^4 + 9x^2y^2 + 3y^4$,
$f_y(x,y) = 0 + 3x^3 \cdot 2y + 3x \cdot 4y^3 = 6x^3y + 12xy^3$.

15. $z = xe^{3y} \Rightarrow \dfrac{\partial z}{\partial x} = e^{3y}$, $\dfrac{\partial z}{\partial y} = 3xe^{3y}$

16. $z = y \ln x \Rightarrow \dfrac{\partial z}{\partial x} = \dfrac{y}{x}$, $\dfrac{\partial z}{\partial y} = \ln x$

17. $f(x,y) = \dfrac{x-y}{x+y} \Rightarrow f_x(x,y) = \dfrac{(1)(x+y) - (x-y)(1)}{(x+y)^2} = \dfrac{2y}{(x+y)^2}$,

$f_y(x,y) = \dfrac{(-1)(x+y) - (x-y)(1)}{(x+y)^2} = -\dfrac{2x}{(x+y)^2}$

18. $f(x,y) = x^y \Rightarrow f_x(x,y) = yx^{y-1}$, $f_y(x,y) = x^y \ln x$

19. $w = \sin\alpha\cos\beta \Rightarrow \dfrac{\partial w}{\partial \alpha} = \cos\alpha\cos\beta$, $\dfrac{\partial w}{\partial \beta} = -\sin\alpha\sin\beta$

20. $f(s,t) = \dfrac{st^2}{s^2+t^2} \Rightarrow$

$f_s(s,t) = \dfrac{t^2(s^2+t^2) - st^2(2s)}{(s^2+t^2)^2} = \dfrac{t^4 - s^2t^2}{(s^2+t^2)^2}$, $f_t(s,t) = \dfrac{2st(s^2+t^2) - st^2(2t)}{(s^2+t^2)^2} = \dfrac{2s^3t}{(s^2+t^2)^2}$

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21. $f(r,s) = r\ln(r^2+s^2) \;\Rightarrow\; f_r(r,s) = r\cdot\dfrac{2r}{r^2+s^2} + \ln(r^2+s^2)\cdot 1 = \dfrac{2r^2}{r^2+s^2} + \ln(r^2+s^2)$,

$f_s(r,s) = r\cdot\dfrac{2s}{r^2+s^2} + 0 = \dfrac{2rs}{r^2+s^2}$

22. $f(x,t) = \arctan\left(x\sqrt{t}\,\right) \;\Rightarrow\; f_x(x,t) = \dfrac{1}{1+\left(x\sqrt{t}\,\right)^2}\cdot\sqrt{t} = \dfrac{\sqrt{t}}{1+x^2t}$,

$f_t(x,t) = \dfrac{1}{1+\left(x\sqrt{t}\,\right)^2}\cdot x\left(\dfrac{1}{2}t^{-1/2}\right) = \dfrac{x}{2\sqrt{t}\,(1+x^2t)}$

23. $u = te^{w/t} \;\Rightarrow\; \dfrac{\partial u}{\partial t} = t\cdot e^{w/t}(-wt^{-2}) + e^{w/t}\cdot 1 = e^{w/t} - \dfrac{w}{t}e^{w/t} = e^{w/t}\left(1-\dfrac{w}{t}\right)$, $\dfrac{\partial u}{\partial w} = te^{w/t}\cdot\dfrac{1}{t} = e^{w/t}$

24. $f(x,y) = \displaystyle\int_y^x \cos(t^2)\,dt \;\Rightarrow\; f_x(x,y) = \frac{\partial}{\partial x}\int_y^x \cos\left(t^2\right)\,dt = \cos(x^2)$ by the Fundamental Theorem of Calculus, Part 1; $f_y(x,y) = \dfrac{\partial}{\partial y}\displaystyle\int_y^x \cos\left(t^2\right)\,dt = -\frac{\partial}{\partial y}\cos(t^2)\,dt = -\cos(y^2)$.

25. $f(x,y,z) = xy^2z^3 + 3yz \;\Rightarrow\; f_x(x,y,z) = y^2z^3$, $f_y(x,y,z) = 2xyz^3 + 3z$, $f_z(x,y,z) = 3xy^2z^2 + 3y$

26. $f(x,y,z) = x^2e^{yz} \;\Rightarrow\; f_x(x,y,z) = 2xe^{yz}$, $f_y(x,y,z) = x^2e^{yz}(z) = x^2ze^{yz}$,
$f_z(x,y,z) = x^2e^{yz}(y) = x^2ye^{yz}$

27. $w = \ln(x+2y+3z) \;\Rightarrow\; \dfrac{\partial w}{\partial x} = \dfrac{1}{x+2y+3z}$, $\dfrac{\partial w}{\partial y} = \dfrac{2}{x+2y+3z}$, $\dfrac{\partial w}{\partial z} = \dfrac{3}{x+2y+3z}$

28. $w = \sqrt{r^2+s^2+t^2} \;\Rightarrow\; \dfrac{\partial w}{\partial r} = \dfrac{1}{2}(r^2+s^2+t^2)^{-1/2}(2r) = \dfrac{r}{\sqrt{r^2+s^2+t^2}}$, $\dfrac{\partial w}{\partial s} = \dfrac{s}{\sqrt{r^2+s^2+t^2}}$,

$\dfrac{\partial w}{\partial t} = \dfrac{t}{\sqrt{r^2+s^2+t^2}}$

29. $u = xe^{-t}\sin\theta \;\Rightarrow\; \dfrac{\partial u}{\partial x} = e^{-t}\sin\theta$, $\dfrac{\partial u}{\partial t} = -xe^{-t}\sin\theta$, $\dfrac{\partial u}{\partial \theta} = xe^{-t}\cos\theta$

30. $u = x^{y/z} \;\Rightarrow\; u_x = \dfrac{y}{z}x^{(y/z)-1}$, $u_y = x^{y/z}\ln x\cdot\dfrac{1}{z} = \dfrac{x^{y/z}}{z}\ln x$, $u_z = x^{y/z}\ln x\cdot\dfrac{-y}{z^2} = -\dfrac{yx^{y/z}}{z^2}\ln x$

31. $f(x,y,z,t) = xyz^2\tan(yt) \;\Rightarrow\; f_x(x,y,z,t) = yz^2\tan(yt)$,
$f_y(x,y,z,t) = xyz^2\cdot\sec^2(yt)\cdot t + xz^2\tan(yt) = xyz^2t\sec^2(yt) + xz^2\tan(yt)$,
$f_z(x,y,z,t) = 2xyz\tan(yt)$, $f_t(x,y,z,t) = xyz^2\sec^2(yt)\cdot y = xy^2z^2\sec^2(yt)$.

32. $f(x,y,z,t) = \dfrac{xy^2}{t+2z} \;\Rightarrow$

$f_x(x,y,z,t) = \dfrac{y^2}{t+2z}$, $f_y(x,y,z,t) = \dfrac{2xy}{t+2z}$,

$f_z(x,y,z,t) = xy^2(-1)(t+2z)^{-2}(2) = -\dfrac{2xy^2}{(t+2z)^2}$, $f_t(x,y,z,t) = xy^2(-1)(t+2z)^{-2}(1) = -\dfrac{xy^2}{(t+2z)^2}$.

33. $u = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$. For each $i = 1,\ldots,n$,

$u_{x_i} = \frac{1}{2}\left(x_1^2+x_2^2+\cdots+x_n^2\right)^{-1/2}(2x_i) = \dfrac{x_i}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}$.

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34. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$. For each $i = 1, \ldots, n$, $u_{x_i} = i\cos(x_1 + 2x_2 + \cdots + nx_n)$.

35. $f(x,y) = \sqrt{x^2+y^2} \quad \Rightarrow \quad f_x(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+y^2}}$, so $f_x(3,4) = \dfrac{3}{\sqrt{3^2+4^2}} = \dfrac{3}{5}$.

36. $f(x,y) = \sin(2x+3y) \quad \Rightarrow \quad f_y(x,y) = \cos(2x+3y)\cdot 3 = 3\cos(2x+3y)$, so
$f_y(-6,4) = 3\cos[2(-6)+3(4)] = 3\cos 0 = 3$.

37. $f(x,y,z) = \dfrac{x}{y+z} = x(y+z)^{-1} \quad \Rightarrow \quad f_z(x,y,z) = x(-1)(y+z)^{-2} = -\dfrac{x}{(y+z)^2}$, so
$f_z(3,2,1) = -\dfrac{3}{(2+1)^2} = -\dfrac{1}{3}$.

38. $f(u,v,w) = w\tan(uv) \quad \Rightarrow \quad f_v(u,v,w) = w\sec^2(uv)\cdot u = uw\sec^2(uv)$, so
$f_v(2,0,3) = (2)(3)\sec^2(2\cdot 0) = 6$.

39. $f(x,y) = x^2 - xy + 2y^2 \quad \Rightarrow$

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x+h)y+2y^2-(x^2-xy+2y^2)}{h}$$

$$= \lim_{h\to 0}\frac{h(2x-y+h)}{h} = \lim_{h\to 0}(2x-y+h) = 2x-y$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0}\frac{x^2-x(y+h)+2(y+h)^2-(x^2-xy+2y^2)}{h}$$

$$= \lim_{h\to 0}\frac{h(4y-x+2h)}{h} = \lim_{h\to 0}(4y-x+2h) = 4y-x$$

40. $f(x,y) = \sqrt{3x-y} \quad \Rightarrow$

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3(x+h)-y}-\sqrt{3x-y}}{h}\cdot\frac{\sqrt{3(x+h)-y}+\sqrt{3x-y}}{\sqrt{3(x+h)-y}+\sqrt{3x-y}}$$

$$= \lim_{h\to 0}\frac{3}{\sqrt{3(x+h)-y}+\sqrt{3x-y}} = \frac{3}{2\sqrt{3x-y}}$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3x-(y+h)}-\sqrt{3x-y}}{h}\cdot\frac{\sqrt{3x-(y+h)}+\sqrt{3x-y}}{\sqrt{3x-(y+h)}+\sqrt{3x-y}}$$

$$= \lim_{h\to 0}\frac{-1}{\sqrt{3x-(y+h)}+\sqrt{3x-y}} = \frac{-1}{2\sqrt{3x-y}}$$

41. $x^2+y^2+z^2 = 3xyz \quad \Rightarrow \quad \dfrac{\partial}{\partial x}(x^2+y^2+z^2) = \dfrac{\partial}{\partial x}(3xyz) \quad \Rightarrow \quad 2x+0+2z\dfrac{\partial z}{\partial x} = 3y\left(x\dfrac{\partial z}{\partial x}+z\cdot 1\right)$

$\Leftrightarrow \quad 2z\dfrac{\partial z}{\partial x} - 3xy\dfrac{\partial z}{\partial x} = 3yz-2x \quad \Leftrightarrow \quad (2z-3xy)\dfrac{\partial z}{\partial x} = 3yz-2x$, so $\dfrac{\partial z}{\partial x} = \dfrac{3yz-2x}{2z-3xy}$.

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$\dfrac{\partial}{\partial y}\left(x^2+y^2+z^2\right)=\dfrac{\partial}{\partial y}(3xyz) \;\Rightarrow\; 0+2y+2z\dfrac{\partial z}{\partial y}=3x\left(y\dfrac{\partial z}{\partial y}+z\cdot 1\right) \;\Leftrightarrow$

$2z\dfrac{\partial z}{\partial y}-3xy\dfrac{\partial z}{\partial y}=3xz-2y \;\Leftrightarrow\; (2z-3xy)\dfrac{\partial z}{\partial y}=3xz-2y$, so $\dfrac{\partial z}{\partial y}=\dfrac{3xz-2y}{2z-3xy}$.

42. $yz=\ln(x+z) \;\Rightarrow\; \dfrac{\partial}{\partial x}(yz)=\dfrac{\partial}{\partial x}(\ln(x+z)) \;\Rightarrow\; y\dfrac{\partial z}{\partial x}=\dfrac{1}{x+z}\left(1+\dfrac{\partial z}{\partial x}\right) \;\Leftrightarrow$

$\left(y-\dfrac{1}{x+z}\right)\dfrac{\partial z}{\partial x}=\dfrac{1}{x+z}$, so $\dfrac{\partial z}{\partial x}=\dfrac{1/(x+z)}{y-1/(x+z)}=\dfrac{1}{y(x+z)-1}$.

$\dfrac{\partial}{\partial y}(yz)=\dfrac{\partial}{\partial y}(\ln(x+z)) \;\Rightarrow\; y\dfrac{\partial z}{\partial y}+z\cdot 1=\dfrac{1}{x+z}\left(0+\dfrac{\partial z}{\partial y}\right) \;\Leftrightarrow\; \left(y-\dfrac{1}{x+z}\right)\dfrac{\partial z}{\partial y}=-z$, so

$\dfrac{\partial z}{\partial y}=\dfrac{-z}{y-1/(x+z)}=\dfrac{z(x+z)}{1-y(x+z)}$.

43. $x-z=\arctan(yz) \;\Rightarrow\; \dfrac{\partial}{\partial x}(x-z)=\dfrac{\partial}{\partial x}(\arctan(yz)) \;\Rightarrow\; 1-\dfrac{\partial z}{\partial x}=\dfrac{1}{1+(yz)^2}\cdot y\dfrac{\partial z}{\partial x} \;\Leftrightarrow$

$1=\left(\dfrac{y}{1+y^2z^2}+1\right)\dfrac{\partial z}{\partial x} \;\Leftrightarrow\; 1=\left(\dfrac{y+1+y^2z^2}{1+y^2z^2}\right)\dfrac{\partial z}{\partial x}$, so $\dfrac{\partial z}{\partial x}=\dfrac{1+y^2z^2}{1+y+y^2z^2}$.

$\dfrac{\partial}{\partial y}(x-z)=\dfrac{\partial}{\partial y}(\arctan(yz)) \;\Rightarrow\; 0-\dfrac{\partial z}{\partial y}=\dfrac{1}{1+(yz)^2}\cdot\left(y\dfrac{\partial z}{\partial y}+z\cdot 1\right) \;\Leftrightarrow$

$-\dfrac{z}{1+y^2z^2}=\left(\dfrac{y}{1+y^2z^2}+1\right)\dfrac{\partial z}{\partial y} \;\Leftrightarrow\; -\dfrac{z}{1+y^2z^2}=\left(\dfrac{y+1+y^2z^2}{1+y^2z^2}\right)\dfrac{\partial z}{\partial y} \;\Leftrightarrow\; \dfrac{\partial z}{\partial y}=-\dfrac{z}{1+y+y^2z^2}$.

44. $\sin(xyz)=x+2y+3z \;\Rightarrow\; \dfrac{\partial}{\partial x}(\sin(xyz))=\dfrac{\partial}{\partial x}(x+2y+3z) \;\Rightarrow$

$\cos(xyz)\cdot y\left(x\dfrac{\partial z}{\partial x}+z\right)=1+3\dfrac{\partial z}{\partial x} \;\Leftrightarrow\; (xy\cos(xyz)-3)\dfrac{\partial z}{\partial x}=1-yz\cos(xyz)$, so

$\dfrac{\partial z}{\partial x}=\dfrac{1-yz\cos(xyz)}{xy\cos(xyz)-3}$.

$\dfrac{\partial}{\partial y}(\sin(xyz))=\dfrac{\partial}{\partial y}(x+2y+3z) \;\Rightarrow\; \cos(xyz)\cdot x\left(y\dfrac{\partial z}{\partial y}+z\right)=2+3\dfrac{\partial z}{\partial y} \;\Leftrightarrow$

$(xy\cos(xyz)-3)\dfrac{\partial z}{\partial y}=2-xz\cos(xyz)$, so $\dfrac{\partial z}{\partial y}=\dfrac{2-xz\cos(xyz)}{xy\cos(xyz)-3}$.

45. (a) $z=f(x)+g(y) \;\Rightarrow\; \dfrac{\partial z}{\partial x}=f'(x),\ \dfrac{\partial z}{\partial y}=g'(y)$

(b) $z=f(x+y)$. Let $u=x+y$. Then $\dfrac{\partial z}{\partial x}=\dfrac{df}{du}\dfrac{\partial u}{\partial x}=\dfrac{df}{du}(1)=f'(u)=f'(x+y)$,

$\dfrac{\partial z}{\partial y}=\dfrac{df}{du}\dfrac{\partial u}{\partial y}=\dfrac{df}{du}(1)=f'(u)=f'(x+y)$.

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46. (a) $z = f(x)g(y) \Rightarrow \dfrac{\partial z}{\partial x} = f'(x)g(y), \dfrac{\partial z}{\partial y} = f(x)g'(y)$

(b) $z = f(xy)$. Let $u = xy$. Then $\dfrac{\partial u}{\partial x} = y$ and $\dfrac{\partial u}{\partial y} = x$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du} \cdot y = yf'(u) = yf'(xy)$ and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du} \cdot x = xf'(u) = xf'(xy)$.

(c) $z = f\left(\dfrac{x}{y}\right)$. Let $u = \dfrac{x}{y}$. Then $\dfrac{\partial u}{\partial x} = \dfrac{1}{y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{x}{y^2}$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = f'(u)\dfrac{1}{y} = \dfrac{f'(x/y)}{y}$ and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = f'(u)\left(-\dfrac{x}{y^2}\right) = -\dfrac{xf'(x/y)}{y^2}$.

47. $f(x,y) = x^4 - 3x^2y^3 \Rightarrow f_x(x,y) = 4x^3 - 6xy^3$, $f_y(x,y) = -9x^2y^2$. Then $f_{xx}(x,y) = 12x^2 - 6y^3$, $f_{xy}(x,y) = -18xy^2$, $f_{yx}(x,y) = -18xy^2$, and $f_{yy}(x,y) = -18x^2y$.

48. $f(x,y) = \ln(3x+5y) \Rightarrow f_x(x,y) = \dfrac{3}{3x+5y}$, $f_y(x,y) = \dfrac{5}{3x+5y}$. Then

$f_{xx}(x,y) = 3(-1)(3x+5y)^{-2}(3) = -\dfrac{9}{(3x+5y)^2}$, $f_{xy}(x,y) = -\dfrac{15}{(3x+5y)^2}$, $f_{yx}(x,y) = -\dfrac{15}{(3x+5y)^2}$,

and $f_{yy}(x,y) = -\dfrac{25}{(3x+5y)^2}$.

49. $z = \dfrac{x}{x+y} = x(x+y)^{-1} \Rightarrow z_x = \dfrac{1(x+y) - 1(x)}{(x+y)^2} = \dfrac{y}{(x+y)^2}$, $z_y = x(-1)(x+y)^{-2} = -\dfrac{x}{(x+y)^2}$.

Then $z_{xx} = y(-2)(x+y)^{-3} = -\dfrac{2y}{(x+y)^3}$, $z_{xy} = \dfrac{1(x+y)^2 - y(2)(x+y)}{[(x+y)^2]^2} = \dfrac{x+y-2y}{(x+y)^3} = \dfrac{x-y}{(x+y)^3}$,

$z_{yx} = -\dfrac{1(x+y)^2 - x(2)(x+y)}{[(x+y)^2]^2} = -\dfrac{-x^2+xy+y^2}{(x+y)^2} = \dfrac{(x+y)(x-y)}{(x+y)^2} = \dfrac{x-y}{(x+y)^3}$, and

$z_{yy} = -x(-2)(x+y)^{-3} = \dfrac{2x}{(x+y)^3}$.

50. $z = y\tan 2x \Rightarrow z_x = y\sec^2(2x) \cdot 2 = 2y\sec^2(2x)$, $z_y = \tan 2x$. Then

$z_{xx} = 2y(2)\sec(2x) \cdot \sec(2x)\tan(2x) \cdot 2 = 8y\sec^2(2x)\tan(2x)$, $z_{xy} = 2\sec^2(2x)$,

$z_{yx} = \sec^2(2x) \cdot 2 = 2\sec^2(2x)$, and $z_{yy} = 0$.

51. $u = e^{-s}\sin t \Rightarrow u_s = -e^{-s}\sin t$, $u_t = e^{-s}\cos t$. Then $u_{ss} = e^{-s}\sin t$, $u_{st} = -e^{-s}\cos t$, $u_{ts} = -e^{-s}\cos t$, and $u_{tt} = -e^{-s}\sin t$.

52. $v = \sqrt{x+y^2} \Rightarrow v_x = \frac{1}{2}(x+y^2)^{-1/2} = \dfrac{1}{2\sqrt{x+y^2}}$,

$v_y = \frac{1}{2}(x+y^2)^{-1/2}(2y) = \dfrac{y}{\sqrt{x+y^2}}$. Then $v_{xx} = \frac{1}{2}\left(-\frac{1}{2}\right)(x+y^2)^{-3/2} = -\dfrac{1}{4(x+y^2)^{3/2}}$,

$v_{xy} = \frac{1}{2}\left(-\frac{1}{2}\right)(x+y^2)^{-3/2}(2y) = -\dfrac{y}{2(x+y^2)^{3/2}}$, $v_{yx} = y\left(-\frac{1}{2}\right)(x+y^2)^{-3/2} = -\dfrac{y}{2(x+y^2)^{3/2}}$,

and $v_{yy} = \dfrac{1\sqrt{x+y^2} - y\left(\frac{1}{2}\right)(x+y^2)^{-1/2}(2y)}{\left(\sqrt{x+y^2}\right)^2} = \dfrac{(x+y^2)-y^2}{(x+y^2)^{3/2}} = \dfrac{x}{(x+y^2)^{3/2}}$.

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53. $u = x\sin(x+2y) \Rightarrow u_x = x\cdot\cos(x+2y)(1)+\sin(x+2y)\cdot 1 = x\cos(x+2y)+\sin(x+2y)$,
$u_{xy} = x(-\sin(x+2y)(2))+\cos(x+2y)(2) = 2\cos(x+2y)-2x\sin(x+2y)$ and
$u_y = x\cos(x+2y)(2) = 2x\cos(x+2y)$,
$u_{yx} = 2x\cdot(-\sin(x+2y)(1))+\cos(x+2y)\cdot 2 = 2\cos(x+2y)-2x\sin(x+2y)$. Thus $u_{xy} = u_{yx}$.

54. $u = x^4y^2-2xy^5 \Rightarrow u_x = 4x^3y^2-2y^5$, $u_{xy} = 8x^3y-10y^4$ and $u_y = 2x^4y-10xy^4$, $u_{yx} = 8x^3y-10y^4$.
Thus $u_{xy} = u_{yx}$.

55. $u = \ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2) \Rightarrow u_x = \frac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2x = \dfrac{x}{x^2+y^2}$,

$u_{xy} = x(-1)(x^2+y^2)^{-2}(2y) = -\dfrac{2xy}{(x^2+y^2)^2}$ and $u_y = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2y = \dfrac{y}{x^2+y^2}$,

$u_{yx} = y(-1)(x^2+y^2)^{-2}(2x) = -\dfrac{2xy}{(x^2+y^2)^2}$. Thus $u_{xy} = u_{yx}$.

56. $u = xye^y \Rightarrow u_x = ye^y$, $u_{xy} = ye^y+e^y = (y+1)e^y$ and $u_y = x(ye^y+e^y) = x(y+1)e^y$,
$u_{yx} = (y+1)e^y$. Thus $u_{xy} = u_{yx}$.

57. $f(x,y) = 3xy^4+x^3y^2 \Rightarrow f_x = 3y^4+3x^2y^2$, $f_{xx} = 6xy^2$, $f_{xxy} = 12xy$ and $f_y = 12xy^3+2x^3y$,
$f_{yy} = 36xy^2+2x^3$, $f_{yyy} = 72xy$.

58. $f(x,t) = x^2e^{-ct} \Rightarrow f_t = x^2(-ce^{-ct})$, $f_{tt} = x^2(c^2e^{-ct})$, $f_{ttt} = x^2(-c^3e^{-ct}) = -c^3x^2e^{-ct}$ and
$f_{tx} = 2x(-ce^{-ct})$, $f_{txx} = 2(-ce^{-ct}) = -2ce^{-ct}$.

59. $f(x,y,z) = \cos(4x+3y+2z) \Rightarrow$
$f_x = -\sin(4x+3y+2z)(4) = -4\sin(4x+3y+2z)$,
$f_{xy} = -4\cos(4x+3y+2z)(3) = -12\cos(4x+3y+2z)$,
$f_{xyz} = -12(-\sin(4x+3y+2z))(2) = 24\sin(4x+3y+2z)$ and
$f_y = -\sin(4x+3y+2z)(3) = -3\sin(4x+3y+2z)$,
$f_{yz} = -3\cos(4x+3y+2z)(2) = -6\cos(4x+3y+2z)$,
$f_{yzz} = -6(-\sin(4x+3y+2z))(2) = 12\sin(4x+3y+2z)$.

60. $f(r,s,t) = r\ln(rs^2t^3) \Rightarrow$

$f_r = r\cdot\dfrac{1}{rs^2t^3}(s^2t^3)+\ln(rs^2t^3)\cdot 1 = \dfrac{rs^2t^3}{rs^2t^3}+\ln(rs^2t^3) = 1+\ln(rs^2t^3)$,

$f_{rs} = \dfrac{1}{rs^2t^3}(2rst^3) = \dfrac{2}{s} = 2s^{-1}$, $f_{rss} = -2s^{-2} = -\dfrac{2}{s^2}$ and $f_{rst} = 0$.

61. $u = e^{r\theta}\sin\theta \Rightarrow \dfrac{\partial u}{\partial\theta} = e^{r\theta}\cos\theta+\sin\theta\cdot e^{r\theta}(r) = e^{r\theta}(\cos\theta+r\sin\theta)$,

$\dfrac{\partial^2 u}{\partial r\,\partial\theta} = e^{r\theta}(\sin\theta)+(\cos\theta+r\sin\theta)e^{r\theta}(\theta) = e^{r\theta}(\sin\theta+\theta\cos\theta+r\theta\sin\theta)$,

$\dfrac{\partial^3 u}{\partial r^2\,\partial\theta} = e^{r\theta}(\theta\sin\theta)+(\sin\theta+\theta\cos\theta+r\theta\sin\theta)\cdot e^{r\theta}(\theta) = \theta e^{r\theta}(2\sin\theta+\theta\cos\theta+r\theta\sin\theta)$.

62. $z = u\sqrt{v-w} = u(v-w)^{1/2} \Rightarrow \dfrac{\partial z}{\partial w} = u\left[\frac{1}{2}(v-w)^{-1/2}(-1)\right] = -\frac{1}{2}u(v-w)^{-1/2}$,

$\dfrac{\partial^2 z}{\partial v\,\partial w} = -\frac{1}{2}u\left(-\frac{1}{2}(v-w)^{-3/2}(1)\right) = \frac{1}{4}u(v-w)^{-3/2}$, $\dfrac{\partial^3 z}{\partial u\,\partial v\,\partial w} = \frac{1}{4}(v-w)^{-3/2}$.

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63. $w = \dfrac{x}{y+2z} = x(y+2z)^{-1} \quad \Rightarrow \quad \dfrac{\partial w}{\partial x} = (y+2z)^{-1}$, $\dfrac{\partial^2 w}{\partial y\,\partial x} = -(y+2z)^{-2}(1) = -(y+2z)^{-2}$,

$\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x} = -(-2)(y+2z)^{-3}(2) = 4(y+2z)^{-3} = \dfrac{4}{(y+2z)^3}$ and

$\dfrac{\partial w}{\partial y} = x(-1)(y+2z)^{-2}(1) = -x(y+2z)^{-2}$, $\dfrac{\partial^2 w}{\partial x\,\partial y} = -(y+2z)^{-2}$, $\dfrac{\partial^3 w}{\partial x^2\,\partial y} = 0$.

64. $u = x^a y^b z^c$. If $a = 0$, or if $b = 0$ or 1, or if $c = 0$, 1, or 2, then $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = 0$. Otherwise $\dfrac{\partial u}{\partial z} = cx^a y^b z^{c-1}$,

$\dfrac{\partial^2 u}{\partial z^2} = c(c-1)x^a y^b z^{c-2}$, $\dfrac{\partial^3 u}{\partial z^3} = c(c-1)(c-2)x^a y^b z^{c-3}$, $\dfrac{\partial^4 u}{\partial y\,\partial z^3} = bc(c-1)(c-2)x^a y^{b-1} z^{c-3}$,

$\dfrac{\partial^5 u}{\partial y^2\,\partial z^3} = b(b-1)c(c-1)(c-2)x^a y^{b-2} z^{c-3}$, and $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = ab(b-1)c(c-1)(c-2)x^{a-1} y^{b-2} z^{c-3}$.

65. By Definition 4, $f_x(3,2) = \lim\limits_{h\to 0} \dfrac{f(3+h,2) - f(3,2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2) \approx \dfrac{f(3.5,2) - f(3,2)}{0.5} = \dfrac{22.4 - 17.5}{0.5} = 9.8$,

$f_x(3,2) \approx \dfrac{f(2.5,2) - f(3,2)}{-0.5} = \dfrac{10.2 - 17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2. Similarly, $f_x(3,2.2) = \lim\limits_{h\to 0} \dfrac{f(3+h,2.2) - f(3,2.2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2.2) \approx \dfrac{f(3.5,2.2) - f(3,2.2)}{0.5} = \dfrac{26.1 - 15.9}{0.5} = 20.4$,

$f_x(3,2.2) \approx \dfrac{f(2.5,2.2) - f(3,2.2)}{-0.5} = \dfrac{9.3 - 15.9}{-0.5} = 13.2$. Averaging these values, we have $f_x(3,2.2) \approx 16.8$.

To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:

$f_x(3,1.8) \approx \dfrac{f(3.5,1.8) - f(3,1.8)}{0.5} = \dfrac{20.0 - 18.1}{0.5} = 3.8$,

$f_x(3,1.8) \approx \dfrac{f(2.5,1.8) - f(3,1.8)}{-0.5} = \dfrac{12.5 - 18.1}{-0.5} = 11.2$. Averaging these values, we get $f_x(3,1.8) \approx 7.5$.

Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[f_x(x,y)]$ and $f_x(x,y)$ is itself a function of 2 variables, so Definition 4 says that

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\,[f_x(x,y)] = \lim_{h\to 0} \frac{f_x(x,y+h) - f_x(x,y)}{h} \quad \Rightarrow \quad f_{xy}(3,2) = \lim_{h\to 0} \frac{f_x(3,2+h) - f_x(3,2)}{h}.$$

We can estimate this value using our previous work with $h = 0.2$ and $h = -0.2$:

$f_{xy}(3,2) \approx \dfrac{f_x(3,2.2) - f_x(3,2)}{0.2} = \dfrac{16.8 - 12.2}{0.2} = 23$,

$f_{xy}(3,2) \approx \dfrac{f_x(3,1.8) - f_x(3,2)}{-0.2} = \dfrac{7.5 - 12.2}{-0.2} = 23.5$. Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25.

66. (a) If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x-direction, so f_x is negative at P.

(b) If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y-direction, so f_y is positive at P.

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(c) $f_{xx} = \dfrac{\partial}{\partial x}(f_x)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x-direction) than at points to the left of P, demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x-direction the (negative) value of f_x increases, hence $\dfrac{\partial}{\partial x}(f_x) = f_{xx}$ is positive at P.

(d) $f_{xy} = \dfrac{\partial}{\partial y}(f_x)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x-direction) at points above P than at those below P, demonstrating that f decreases more quickly with respect to x for y-values above P. So as we move through P in the positive y-direction, the (negative) value of f_x decreases, hence f_{xy} is negative.

(e) $f_{yy} = \dfrac{\partial}{\partial y}(f_y)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to y above P. So as we move through P in the positive y-direction the (positive) value of f_y increases, hence $\dfrac{\partial}{\partial y}(f_y) = f_{yy}$ is positive at P.

67. $u = e^{-\alpha^2 k^2 t}\sin kx \;\Rightarrow\; u_x = ke^{-\alpha^2 k^2 t}\cos kx$, $u_{xx} = -k^2 e^{-\alpha^2 k^2 t}\sin kx$, and $u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t}\sin kx$. Thus $\alpha^2 u_{xx} = u_t$.

68. (a) $u = x^2 + y^2 \;\Rightarrow\; u_x = 2x$, $u_{xx} = 2$; $u_y = 2y$, $u_{yy} = 2$. Thus $u_{xx} + u_{yy} \neq 0$ and $u = x^2 + y^2$ does not satisfy Laplace's Equation.

(b) $u = x^2 - y^2$ is a solution: $u_{xx} = 2$, $u_{yy} = -2$ so $u_{xx} + u_{yy} = 0$.

(c) $u = x^3 + 3xy^2$ is not a solution: $u_x = 3x^2 + 3y^2$, $u_{xx} = 6x$; $u_y = 6xy$, $u_{yy} = 6x$.

(d) $u = \ln\sqrt{x^2+y^2}$ is a solution: $u_x = \dfrac{1}{\sqrt{x^2+y^2}}\left(\dfrac{1}{2}\right)(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{x^2+y^2}$,

$u_{xx} = \dfrac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$. By symmetry, $u_{yy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$, so $u_{xx} + u_{yy} = 0$.

(e) $u = \sin x\cosh y + \cos x\sinh y$ is a solution:
$u_x = \cos x\cosh y - \sin x\sinh y$, $u_{xx} = -\sin x\cosh y - \cos x\sinh y$, and $u_y = \sin x\sinh y + \cos x\cosh y$, $u_{yy} = \sin x\cosh y + \cos x\sinh y$.

(f) $u = e^{-x}\cos y - e^{-y}\cos x$ is a solution: $u_x = -e^{-x}\cos y + e^{-y}\sin x$, $u_{xx} = e^{-x}\cos y + e^{-y}\cos x$, and $u_y = -e^{-x}\sin y + e^{-y}\cos x$, $u_{yy} = -e^{-x}\cos y - e^{-y}\cos x$.

69. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \;\Rightarrow\; u_x = \left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$ and

$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) = \dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$.

By symmetry, $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$.

Thus $u_{xx} + u_{yy} + u_{zz} = \dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$.

70. (a) $u = \sin(kx)\sin(akt) \;\Rightarrow\; u_t = ak\sin(kx)\cos(akt)$, $u_{tt} = -a^2k^2\sin(kx)\sin(akt)$, $u_x = k\cos(kx)\sin(akt)$, $u_{xx} = -k^2\sin(kx)\sin(akt)$. Thus $u_{tt} = a^2u_{xx}$.

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(b) $u = \dfrac{t}{a^2t^2 - x^2} \quad \Rightarrow \quad u_t = \dfrac{(a^2t^2 - x^2) - t(2a^2t)}{(a^2t^2 - x^2)^2} = -\dfrac{a^2t^2 + x^2}{(a^2t^2 - x^2)^2}$,

$u_{tt} = \dfrac{-2a^2t(a^2t^2 - x^2)^2 + (a^2t^2 - x^2)(2)(a^2t^2 - x^2)(2a^2t)}{(a^2t^2 - x^2)^4} = \dfrac{2a^4t^3 + 6a^2tx^2}{a^2t^2 - x^2}$,

$u_x = t(-1)(a^2t^2 - x^2)^{-2}(2x) = \dfrac{2tx}{(a^2t^2 - x^2)^2}$,

$u_{xx} = \dfrac{2t(a^2t^2 - x^2)^2 - 2tx\,(2)\,(a^2t^2 - x^2)(-2x)}{(a^2t^2 - x^2)^4} = \dfrac{2a^2t^3 - 2tx^2 + 8tx^2}{(a^2t^2 - x^2)^3} = \dfrac{2a^2t^3 + 6tx^2}{(a^2t^2 - x^2)^4}$.

Thus $u_{tt} = a^2u_{xx}$.

(c) $u = (x - at)^6 + (x + at)^6 \quad \Rightarrow \quad u_t = -6a(x - at)^5 + 6a(x + at)^5$,
$u_{tt} = 30a^2(x - at)^4 + 30a^2(x + at)^4$, $u_x = 6(x - at)^5 + 6(x + at)^5$, $u_{xx} = 30(x - at)^4 + 30(x + at)^4$.
Thus $u_{tt} = a^2u_{xx}$.

(d) $u = \sin(x - at) + \ln(x + at) \quad \Rightarrow \quad u_t = -a\cos(x - at) + \dfrac{a}{x + at}$, $u_{tt} = -a^2\sin(x - at) - \dfrac{a^2}{(x + at)^2}$,
$u_x = \cos(x - at) + \dfrac{1}{x + at}$, $u_{xx} = -\sin(x - at) - \dfrac{1}{(x + at)^2}$. Thus $u_{tt} = a^2u_{xx}$.

71. Let $v = x + at$, $w = x - at$. Then $u_t = \dfrac{\partial[f(v) + g(w)]}{\partial t} = \dfrac{df(v)}{dv}\dfrac{\partial v}{\partial t} + \dfrac{dg(w)}{dw}\dfrac{\partial w}{\partial t} = af'(v) - ag'(w)$ and
$u_{tt} = \dfrac{\partial[af'(v) - ag'(w)]}{\partial t} = a[af''(v) + ag''(w)] = a^2[f''(v) + g''(w)]$. Similarly, by using the Chain Rule we have $u_x = f'(v) + g'(w)$ and $u_{xx} = f''(v) + g''(w)$. Thus $u_{tt} = a^2u_{xx}$.

72. For each i, $i = 1, \ldots, n$, $\partial u/\partial x_i = a_ie^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$ and $\partial^2u/\partial x_i^2 = a_i^2e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$. Then
$\dfrac{\partial^2u}{\partial x_1^2} + \dfrac{\partial^2u}{\partial x_2^2} + \cdots + \dfrac{\partial^2u}{\partial x_n^2} = \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n} = e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n} = u$ since
$a_1^2 + a_2^2 + \cdots + a_n^2 = 1$.

73. $z_x = e^y + ye^x$, $z_{xx} = ye^x$, $\partial^3z/\partial x^3 = ye^x$. By symmetry $z_y = xe^y + e^x$, $z_{yy} = xe^y$ and $\partial^3z/\partial y^3 = xe^y$.
Then $\partial^3z/\partial x\partial y^2 = e^y$ and $\partial^3z/\partial x^2\partial y = e^x$. Thus $z = xe^y + ye^x$ satisfies the given partial differential equation.

74. $P = bL^\alpha K^\beta$, so $\dfrac{\partial P}{\partial L} = \alpha bL^{\alpha-1}K^\beta$ and $\dfrac{\partial P}{\partial K} = \beta bL^\alpha K^{\beta-1}$. Then

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = L\left(\alpha bL^{\alpha-1}K^\beta\right) + K\left(\beta bL^\alpha K^{\beta-1}\right) = \alpha bL^{1+\alpha-1}K^\beta + \beta bL^\alpha K^{1+\beta-1}$$
$$= (\alpha + \beta)bL^\alpha K^\beta = (\alpha + \beta)P$$

75. If we fix $K = K_0$, $P(L, K_0)$ is a function of a single variable L, and $\dfrac{dP}{dL} = \alpha\dfrac{P}{L}$ is a separable differential equation. Then $\dfrac{dP}{P} = \alpha\dfrac{dL}{L} \quad \Rightarrow \quad \displaystyle\int\frac{dP}{P} = \int\alpha\frac{dL}{L} \quad \Rightarrow \quad \ln|P| = \alpha\ln|L| + C(K_0)$, where $C(K_0)$ can depend on K_0. Then $|P| = e^{\alpha\ln|L| + C(K_0)}$, and since $P > 0$ and $L > 0$, we have
$P = e^{\alpha\ln L}e^{C(K_0)} = e^{C(K_0)}e^{\ln L^\alpha} = C_1(K_0)L^\alpha$ where $C_1(K_0) = e^{C(K_0)}$.

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76. (a) $\partial T/\partial x = -60(2x)/(1+x^2+y^2)^2$, so at $(2,1)$, $T_x = -240/(1+4+1)^2 = -\frac{20}{3}$.

(b) $\partial T/\partial y = -60(2y)/(1+x^2+y^2)^2$, so at $(2,1)$, $T_y = -120/36 = -\frac{10}{3}$. Thus from the point $(2,1)$ the temperature is decreasing at a rate of $\frac{20}{3}$ °C/m in the x-direction and is decreasing at a rate of $\frac{10}{3}$ °C/m in the y-direction.

77. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives $\dfrac{\partial R^{-1}}{\partial R}\dfrac{\partial R}{\partial R_1} = \dfrac{\partial\left[(1/R_1)+(1/R_2)+(1/R_3)\right]}{\partial R_1}$ or $-R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2}$. Thus $\dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$.

78. $P = \dfrac{mRT}{V}$ so $\dfrac{\partial P}{\partial V} = \dfrac{-mRT}{V^2}$; $V = \dfrac{mRT}{P}$, so $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$; $T = \dfrac{PV}{mR}$, so $\dfrac{\partial T}{\partial P} = \dfrac{V}{mR}$.

Thus $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P} = \dfrac{-mRT}{V^2}\dfrac{mR}{P}\dfrac{V}{mR} = \dfrac{-mRT}{PV} = -1$, since $PV = mRT$.

79. By Exercise 78, $PV = mRT \;\Rightarrow\; P = \dfrac{mRT}{V}$, so $\dfrac{\partial P}{\partial T} = \dfrac{mR}{V}$. Also, $PV = mRT \;\Rightarrow\; V = \dfrac{mRT}{P}$ and $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$. Since $T = \dfrac{PV}{mR}$, we have $T\dfrac{\partial P}{\partial T}\dfrac{\partial V}{\partial T} = \dfrac{PV}{mR}\cdot\dfrac{mR}{V}\cdot\dfrac{mR}{P} = mR$.

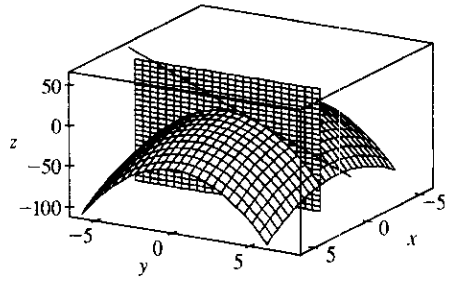
80. $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965v^{0.16}$. When $T = -15$°C and $v = 30$ km/h, $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965(30)^{0.16} \approx 1.3048$, so we would expect the apparent temperature to drop by approximately 1.3°C if the actual temperature decreases by 1°C. $\dfrac{\partial W}{\partial v} = -11.37(0.16)v^{-0.84} + 0.3965T(0.16)v^{-0.84}$ and when $T = -15$°C and $v = 30$ km/h, $\dfrac{\partial W}{\partial v} = -11.37(0.16)(30)^{-0.84} + 0.3965(-15)(0.16)(30)^{-0.84} \approx -0.1592$, so we would expect the apparent temperature to drop by approximately 0.16°C if the wind speed increases by 1 km/h.

81. $\dfrac{\partial K}{\partial m} = \frac{1}{2}V^2$, $\dfrac{\partial K}{\partial V} = mV$, $\dfrac{\partial^2 K}{\partial V^2} = m$. Thus $\dfrac{\partial K}{\partial m}\cdot\dfrac{\partial^2 K}{\partial V^2} = \frac{1}{2}V^2 m = K$.

82. The Law of Cosines says that $a^2 = b^2 + c^2 - 2bc\cos A$. Thus $\dfrac{\partial(a^2)}{\partial a} = \dfrac{\partial(b^2+c^2-2ab\cos A)}{\partial a}$ or $2a = -2bc(-\sin A)\dfrac{\partial A}{\partial a}$, implying that $\dfrac{\partial A}{\partial a} = \dfrac{a}{bc\sin A}$. Taking the partial derivative of both sides with respect to b gives $0 = 2b - 2c(\cos A) - 2bc(-\sin A)\dfrac{\partial A}{\partial b}$. Thus $\dfrac{\partial A}{\partial b} = \dfrac{c\cos A - b}{bc\sin A}$. By symmetry $\dfrac{\partial A}{\partial c} = \dfrac{b\cos A - c}{bc\sin A}$.

83. $f_x(x,y) = x + 4y \;\Rightarrow\; f_{xy}(x,y) = 4$ and $f_y(x,y) = 3x - y \;\Rightarrow\; f_{yx}(x,y) = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y) \neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

84. Setting $x = 1$, the equation of the parabola of intersection is $z = 6 - 1 - 1 - 2y^2 = 4 - 2y^2$. The slope of the tangent is $\partial z/\partial y = -4y$, so at $(1, 2, -4)$ the slope is -8. Parametric equations for the line are therefore $x = 1$, $y = 2 + t$, $z = -4 - 8t$.



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85. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z\,(\partial z/\partial x) = 0 \;\Rightarrow\; \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1,2) = -2$. Thus the tangent line is given by $z - 2 = -2(x-1)$, $y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t$, $y = 2$, $z = 2 - 2t$.

86. $T(x,t) = T_0 + T_1 e^{-\lambda x}\sin(\omega t - \lambda x)$

(a) $\partial T/\partial x = T_1 e^{-\lambda x}\left[\cos(\omega t - \lambda x)(-\lambda)\right] + T_1(-\lambda e^{-\lambda x})\sin(\omega t - \lambda x)$

$= -\lambda T_1 e^{-\lambda x}\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]$

This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t.

(b) $\partial T/\partial t = T_1 e^{-\lambda x}\left[\cos(\omega t - \lambda x)(\omega)\right] = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x.

(c) $T_{xx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial T}{\partial x}\right)$

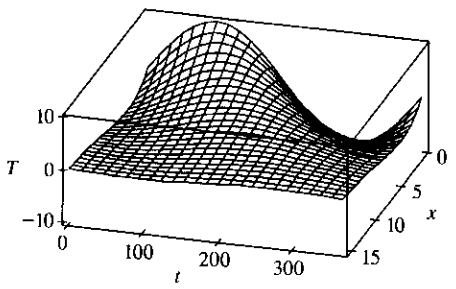
$= -\lambda T_1\big(e^{-\lambda x}\left[\cos(\omega t - \lambda x)(-\lambda) - \sin(\omega t - \lambda x)(-\lambda)\right]$

$+ e^{-\lambda x}(-\lambda)\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]\big)$

$= 2\lambda^2 T_1 e^{-\lambda x}\cos(\omega t - \lambda x)$

But from part (b), $T_t = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x) = \dfrac{\omega}{2\lambda^2}T_{xx}$. So with $k = \dfrac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.

(d)



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

(e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, at the surface the highest temperature is reached at $t \approx 100$, whereas at a depth of 5 feet the peak temperature is attained at $t \approx 150$, and at a depth of 10 feet, at $t \approx 220$.

87. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

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88. (a) Since we are differentiating n times, with two choices of variable at each differentiation, there are 2^n nth order partial derivatives.

(b) If these partial derivatives are all continuous, then the order in which the partials are taken doesn't affect the value of the result, that is, all nth order partial derivatives with p partials with respect to x and $n-p$ partials with respect to y are equal. Since the number of partials taken with respect to x for an nth order partial derivative can range from 0 to n, a function of two variables has $n+1$ distinct partial derivatives of order n if these partial derivatives are all continuous.

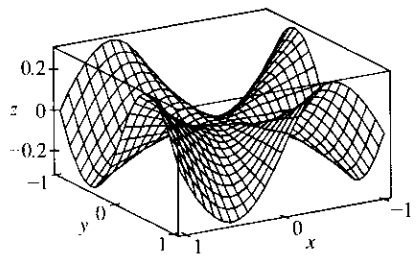
(c) Since n differentiations are to be performed with three choices of variable at each differentiation, there are 3^n nth order partial derivatives of a function of three variables.

89. Let $g(x) = f(x,0) = x(x^2)^{-3/2}e^0 = x\,|x|^{-3}$. But we are using the point $(1,0)$, so near $(1,0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using (1) we have $f_x(1,0) = g'(1) = -2$.

90. $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(h^3+0)^{1/3} - 0}{h} = \lim\limits_{h\to 0} \dfrac{h}{h} = 1.$

Or: Let $g(x) = f(x,0) = \sqrt[3]{x^3+0} = x$. Then $g'(x) = 1$ and $g'(0) = 1$ so, by (1), $f_x(0,0) = g'(0) = 1$.

91. (a)



(b) For $(x,y) \neq (0,0)$, $f_x(x,y) = \dfrac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$, and by symmetry $f_y(x,y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$.

(c) $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(0/h^2) - 0}{h} = 0$ and $f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = 0.$

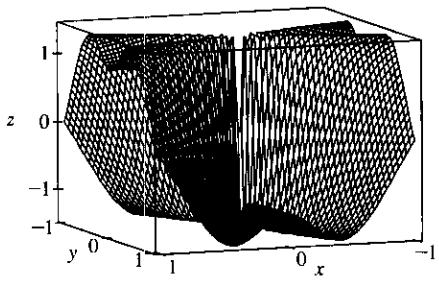
(d) By (3), $f_{xy}(0,0) = \dfrac{\partial f_x}{\partial y} = \lim\limits_{h\to 0} \dfrac{f_x(0,h) - f_x(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(-h^5 - 0)/h^4}{h} = -1$ while by (2),

$f_{yx}(0,0) = \dfrac{\partial f_y}{\partial x} = \lim\limits_{h\to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{h^5/h^4}{h} = 1.$

(e) For $(x,y) \neq (0,0)$, we use a CAS to compute

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 4x^2y^4 + 4y^6}{(x^2+y^2)^3}.$$

Now as $(x,y) \to (0,0)$ along the x-axis, $f_{xy}(x,y) \to 1$ while as $(x,y) \to (0,0)$ along the y-axis, $f_{xy}(x,y) \to 4$. Thus f_{xy} isn't continuous at $(0,0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.



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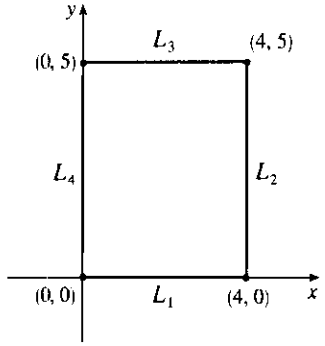
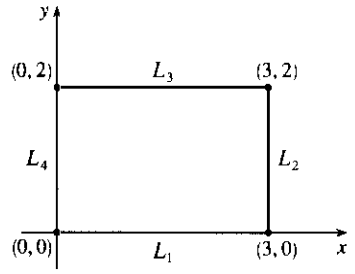
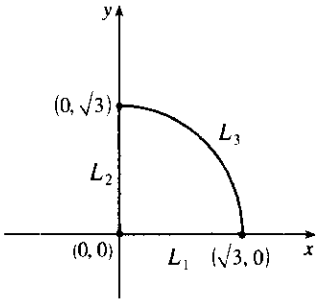
Along L_4: $x = 0$, so $f(0, y) = 6y - y^2 = -(y-3)^2 + 9$, $0 \le y \le 5$, which has a maximum value when $y = 3$ where $f(0,3) = 9$ and a minimum value when $y = 0$ where $f(0,0) = 0$. Thus the absolute maximum is $f(2,3) = 13$ and the absolute minimum is attained at both $(0,0)$ and $(4,0)$, where $f(0,0) = f(4,0) = 0$.



31. $f(x,y) = x^4 + y^4 - 4xy + 2$ is a polynomial and hence continuous on D, so it has an absolute maximum and minimum on D. In Exercise 7, we found the critical points of f; only $(1,1)$ with $f(1,1) = 0$ is inside D. On L_1: $y = 0$, $f(x,0) = x^4 + 2$, $0 \le x \le 3$, a polynomial in x which attains its maximum at $x = 3$, $f(3,0) = 83$, and its minimum at $x = 0$, $f(0,0) = 2$. On L_2: $x = 3$, $f(3,y) = y^4 - 12y + 83$, $0 \le y \le 2$, a polynomial in y which attains its minimum at $y = \sqrt[3]{3}$, $f\left(3, \sqrt[3]{3}\right) = 83 - 9\sqrt[3]{3} \approx 70.0$, and its maximum at $y = 0$, $f(3,0) = 83$. On L_3: $y = 2$, $f(x,2) = x^4 - 8x + 18$, $0 \le x \le 3$, a polynomial in x which attains its minimum at $x = \sqrt[3]{2}$, $f\left(\sqrt[3]{2}, 2\right) = 18 - 6\sqrt[3]{2} \approx 10.4$, and its maximum at $x = 3$, $f(3,2) = 75$. On L_4: $x = 0$, $f(0,y) = y^4 + 2$, $0 \le y \le 2$, a polynomial in y which attains its maximum at $y = 2$, $f(0,2) = 18$, and its minimum at $y = 0$, $f(0,0) = 2$. Thus the absolute maximum of f on D is $f(3,0) = 83$ and the absolute minimum is $f(1,1) = 0$.



32. $f_x = y^2$ and $f_y = 2xy$, and since $f_x = 0 \Leftrightarrow y = 0$, there are no critical points in the interior of D. Along L_1, $y = 0$ and $f(x,0) = 0$. Along L_2, $x = 0$ and $f(0,y) = 0$. Along L_3, $y = \sqrt{3 - x^2}$, so let $g(x) = f\left(x, \sqrt{3-x^2}\right) = 3x - x^3$ for $0 \le x \le \sqrt{3}$. Then $g'(x) = 3 - 3x^2 = 0 \Leftrightarrow x = 1$. The maximum value is $f\left(1, \sqrt{2}\right) = 2$ and the minimum occurs both at $x = 0$ and $x = \sqrt{3}$ where $f\left(0, \sqrt{3}\right) = f\left(\sqrt{3}, 0\right) = 0$. Thus the absolute maximum of f on D is $f\left(1, \sqrt{2}\right) = 2$, and the absolute minimum is 0 which occurs at all points along L_1 and L_2.



33. $f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0,0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x,y) = 2x^3 + (1-x^2)^2 = x^4 + 2x^3 - 2x^2 + 1, -1 \le x \le 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \Rightarrow x = 0, -2$, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2,-3)$ is not in D. Checking the endpoints, we get $f(-1,0) = g(-1) = -2$ and $f(1,0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1,0) = 2$ and $f(-1,0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta, 0 \le \theta \le 2\pi$.

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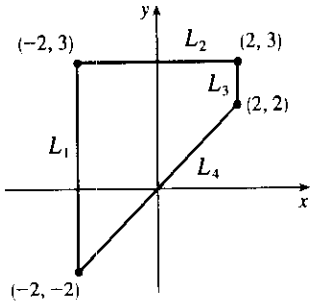
34. $f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 12$ and the critical points are $(1,2)$, $(1,-2)$, $(-1,2)$, and $(-1,-2)$. But only $(1,2)$ and $(-1,2)$ are in D and $f(1,2) = 14$, $f(-1,2) = 18$. Along L_1: $x = -2$ and $f(-2,y) = -2 - y^3 + 12y$, $-2 \le y \le 3$, which has a maximum at $y = 2$ where $f(-2,2) = 14$ and a minimum at $y = -2$ where $f(-2,-2) = -18$. Along L_2: $x = 2$ and $f(2,y) = 2 - y^3 + 12y$, $2 \le y \le 3$, which has a maximum at $y = 2$ where $f(2,2) = 18$ and a minimum at $y = 3$ where $f(2,3) = 11$. Along L_3: $y = 3$ and $f(x,3) = x^3 - 3x + 9$, $-2 \le x \le 2$, which has a maximum at $x = -1$ and $x = 2$ where $f(-1,3) = f(2,3) = 11$ and a minimum at $x = 1$ and $x = -2$ where $f(1,3) = f(-2,3) = 7$. Along L_4: $y = x$ and $f(x,x) = 9x$, $-2 \le x \le 2$, which has a maximum at $x = 2$ where $f(2,2) = 18$ and a minimum at $x = -2$ where $f(-2,-2) = -18$. So the absolute maximum value of f on D is $f(2,2) = 18$ and the minimum is $f(-2,-2) = -18$.

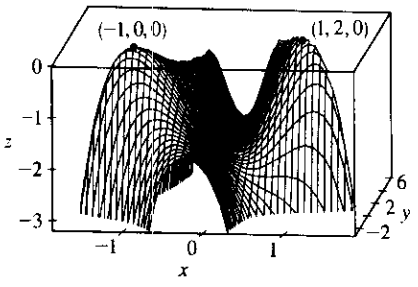


35. $f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 \Rightarrow f_x(x,y) = -2(x^2-1)(2x) - 2(x^2y - x - 1)(2xy - 1)$ and $f_y(x,y) = -2(x^2y - x - 1)x^2$. Setting $f_y(x,y) = 0$ gives either $x = 0$ or $x^2y - x - 1 = 0$. There are no critical points for $x = 0$, since $f_x(0,y) = -2$, so we set $x^2y - x - 1 = 0 \Leftrightarrow y = \dfrac{x+1}{x^2}$ $(x \neq 0)$, so $f_x\left(x, \dfrac{x+1}{x^2}\right) = -2(x^2-1)(2x) - 2\left(x^2\,\dfrac{x+1}{x^2} - x - 1\right)\left(2x\,\dfrac{x+1}{x^2} - 1\right) = -4x(x^2-1)$. Therefore $f_x(x,y) = f_y(x,y) = 0$ at the points $(1,2)$ and $(-1,0)$. To classify these critical points, we calculate $f_{xx}(x,y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2$, $f_{yy}(x,y) = -2x^4$, and $f_{xy}(x,y) = -8x^3y + 6x^2 + 4x$.

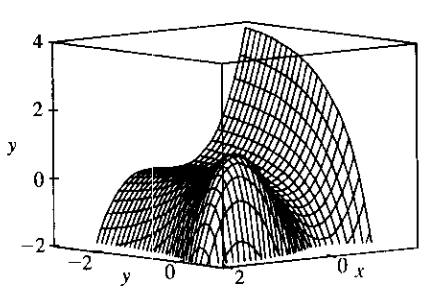
In order to use the Second Derivatives Test we calculate

$$D(-1,0) = f_{xx}(-1,0)\,f_{yy}(-1,0) - [f_{xy}(-1,0)]^2$$
$$= 16 > 0,$$

$f_{xx}(-1,0) = -10 < 0$, $D(1,2) = 16 > 0$, and $f_{xx}(1,2) = -26 < 0$, so both $(-1,0)$ and $(1,2)$ give local maxima.



36. $f(x,y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that (1) $f_x = 3e^y - 3x^2 = 0$ and (2) $f_y = 3xe^y - 3e^{3y} = 0$. From (1) we obtain $e^y = x^2$, and then (2) gives $3x^3 - 3x^6 = 0 \Rightarrow x = 1$ or 0, but only $x = 1$ is valid, since $x = 0$ makes (1) impossible. So substituting $x = 1$ into (1) gives $y = 0$, and the only critical point is $(1,0)$.



The Second Derivatives Test shows that this gives a local maximum, since

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$D(1,0) = \left[-6x(3xe^y - 9e^{3y}) - (3e^y)^2\right]_{(1,0)} = 27 > 0$ and $f_{xx}(1,0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1,0) = 1$ is not an absolute maximum because, for instance, $f(-3,0) = 17$. This can also be seen from the graph.

37. Let d be the distance from $(2,1,-1)$ to any point (x,y,z) on the plane $x+y-z=1$, so $d = \sqrt{(x-2)^2+(y-1)^2+(z+1)^2}$ where $z = x+y-1$, and we minimize $d^2 = f(x,y) = (x-2)^2+(y-1)^2+(x+y)^2$. Then $f_x(x,y) = 2(x-2)+2(x+y) = 4x+2y-4$, $f_y(x,y) = 2(y-1)+2(x+y) = 2x+4y-2$. Solving $4x+2y-4=0$ and $2x+4y-2=0$ simultaneously gives $x=1$, $y=0$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x=1$, $y=0$ for which $d = \sqrt{(1-2)^2+(0-1)^2+(1+0)^2} = \sqrt{3}$.

38. Here the distance d from a point on the plane to the point $(1,2,3)$ is $d = \sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$, where $z = 4-x+y$. We can minimize $d^2 = f(x,y) = (x-1)^2+(y-2)^2+(1-x+y)^2$, so $f_x(x,y) = 2(x-1)+2(1-x+y)(-1) = 4x-2y-4$ and $f_y(x,y) = 2(y-2)+2(1-x+y) = 4y-2x-2$. Solving $4x-2y-4=0$ and $4y-2x-2=0$ simultaneously gives $x=\frac{5}{3}$ and $y=\frac{4}{3}$, so the only critical point is $\left(\frac{5}{3},\frac{4}{3}\right)$. This point must correspond to the minimum distance, so the point on the plane closest to $(1,2,3)$ is $\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$.

39. Minimize $d^2 = x^2+y^2+z^2 = x^2+y^2+xy+1$. Then $f_x = 2x+y$, $f_y = 2y+x$ so the critical point is $(0,0)$ and $D(0,0) = 4-1>0$ with $f_{xx}(0,0) = 2$ so this is a minimum. Thus $z^2 = 1$ or $z = \pm 1$ and the points on the surface are $(0,0,\pm 1)$.

40. Since $z = 1/(x^2y^2)$ on the surface, we minimize $d^2 = x^2+y^2+z^2 = x^2+y^2+x^{-4}y^{-4} = f(x,y)$. $f_x = 2x - \dfrac{4}{x^5y^4}$, $f_y = 2y - \dfrac{4}{x^4y^5}$, so the critical points occur when $2x = \dfrac{4}{x^5y^4}$ and $2y = \dfrac{4}{x^4y^5}$ or $x^6y^4 = 2 = x^4y^6$, so $x^2 = y^2 \;\Rightarrow\; x = \pm y$ and $x^{10} = 2 \;\Rightarrow\; x = \pm 2^{1/10}$, $y = \pm 2^{1/10}$. The four critical points are $(\pm 2^{1/10}, \pm 2^{1/10})$. The absolute minimum must occur at these points (there is no maximum since the surface is infinite in extent). Thus the points on the surface closest to the origin are $(\pm 2^{1/10}, \pm 2^{1/10}, 2^{-2/5})$.

41. $x+y+z = 100$, so maximize $f(x,y) = xy(100-x-y)$. $f_x = 100y-2xy-y^2$, $f_y = 100x-x^2-2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100-2x-2y$. Then $f_x = 0$ implies $y=0$ or $y = 100-2x$. Substituting $y=0$ into $f_y = 0$ gives $x=0$ or $x=100$ and substituting $y = 100-2x$ into $f_y = 0$ gives $3x^2-100x = 0$ so $x=0$ or $\frac{100}{3}$. Thus the critical points are $(0,0)$, $(100,0)$, $(0,100)$ and $\left(\frac{100}{3},\frac{100}{3}\right)$.

$D(0,0) = D(100,0) = D(0,100) = -10{,}000$ while $D\left(\frac{100}{3},\frac{100}{3}\right) = \frac{10{,}000}{3}$ and $f_{xx}\left(\frac{100}{3},\frac{100}{3}\right) = -\frac{200}{3} < 0$. Thus $(0,0)$, $(100,0)$ and $(0,100)$ are saddle points whereas $f\left(\frac{100}{3},\frac{100}{3}\right)$ is a local maximum. Thus the numbers are $x = y = z = \frac{100}{3}$.

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42. Maximize $f(x, y) = x^a y^b (100 - x - y)^c$.

$f_x = ax^{a-1}y^b(100 - x - y)^c - cx^a y^b(100 - x - y)^{c-1} = x^{a-1}y^b(100 - x - y)^{c-1}[a(100 - x - y) - cx]$ and $f_y = x^a y^{b-1}(100 - x - y)^{c-1}[b(100 - x - y) - cy]$. Since x, y and z are all positive, the only critical point occurs when $x = a\,\dfrac{100 - y}{a + c}$ and $y = \dfrac{100b}{a + b + c}$. Thus the point is $\left(\dfrac{100a}{a + b + c}, \dfrac{100b}{a + b + c}\right)$ and the numbers are $x = \dfrac{100a}{a + b + c}$, $y = \dfrac{100b}{a + b + c}$, $z = \dfrac{100c}{a + b + c}$.

43. Maximize $f(x, y) = xy\left(36 - 9x^2 - 36y^2\right)^{1/2}/2$ with (x, y, z) in first octant. Then

$f_x = \dfrac{y(36 - 9x^2 - 36y^2)^{1/2}}{2} + \dfrac{-9x^2y(36 - 9x^2 - 36y^2)^{-1/2}}{2} = \dfrac{(36y - 18x^2y - 36y^3)}{2(36 - 9x^2 - 36y^2)^{1/2}}$ and

$f_y = \dfrac{36x - 9x^3 - 72xy^2}{2(36 - 9x^2 - 36y^2)^{1/2}}$. Setting $f_x = 0$ gives $y = 0$ or $y^2 = \dfrac{2 - x^2}{2}$ but $y > 0$, so only the latter solution applies. Substituting this y into $f_y = 0$ gives $x^2 = \frac{4}{3}$ or $x = \frac{2}{\sqrt{3}}$, $y = \frac{1}{\sqrt{3}}$ and then $z^2 = (36 - 12 - 12)/4 = 3$. The fact that this gives a maximum volume follows from the geometry. This maximum volume is

$V = (2x)(2y)(2z) = 8\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\left(\sqrt{3}\right) = \frac{16}{\sqrt{3}}$.

44. Here maximize $f(x, y) = xy\dfrac{(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2)^{1/2}}{a^2b^2}$. Then

$f_x = yc^2\,\dfrac{a^2b^2 - 2b^2x^2 - a^2y^2}{a^2b^2(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2)^{1/2}}$ and $f_y = xc^2\,\dfrac{a^2b^2 - 2a^2y^2 - b^2x^2}{a^2b^2(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2)^{1/2}}$. Then $f_x = 0$ (with $x, y > 0$) implies $y^2 = \dfrac{a^2b^2 - 2b^2x^2}{a^2}$ and substituting into $f_y = 0$ implies $3b^2x^2 = a^2b^2$ or $x = \frac{1}{\sqrt{3}}\,a$, $y = \frac{1}{\sqrt{3}}\,b$ and then $z = \frac{1}{\sqrt{3}}\,c$. Thus the maximum volume of such a rectangle is $V = (2x)(2y)(2z) = \frac{8}{3\sqrt{3}}\,abc$.

45. Maximize $f(x, y) = \dfrac{xy}{3}\,(6 - x - 2y)$, then the maximum volume is $V = xyz$.

$f_x = \frac{1}{3}(6y - 2xy - y^2) = \frac{1}{3}y(6 - 2x - 2y)$ and $f_y = \frac{1}{3}x\,(6 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives the critical point $(2, 1)$ which geometrically must yield a maximum. Thus the volume of the largest such box is $V = (2)(1)\left(\frac{2}{3}\right) = \frac{4}{3}$.

46. Surface area $= 2(xy + xz + yz) = 64$ cm^2, so $xy + xz + yz = 32$ or $z = \dfrac{32 - xy}{x + y}$. Maximize the volume $f(x, y) = xy\,\dfrac{32 - xy}{x + y}$. Then $f_x = \dfrac{32y^2 - 2xy^3 - x^2y^2}{(x + y)^2} = y^2\,\dfrac{32 - 2xy - x^2}{(x + y)^2}$ and

$f_y = x^2\,\dfrac{32 - 2xy - y^2}{(x + y)^2}$. Setting $f_x = 0$ implies $y = \dfrac{32 - x^2}{2x}$ and substituting into $f_y = 0$ gives $32(4x^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0$ or $3x^4 + 64x^2 - (32)^2 = 0$. Thus $x^2 = \frac{64}{6}$ or $x = \frac{8}{\sqrt{6}}$, $y = \frac{64/3}{16/\sqrt{6}} = \frac{8}{\sqrt{6}}$ and $z = \frac{8}{\sqrt{6}}$. Thus the box is a cube with edge length $\frac{8}{\sqrt{6}}$ cm.

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47. Let the dimensions be x, y, and z; then $4x + 4y + 4z = c$ and the volume is $V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2$, $x > 0$, $y > 0$. Then $V_x = \frac{1}{4}cy - 2xy - y^2$ and $V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

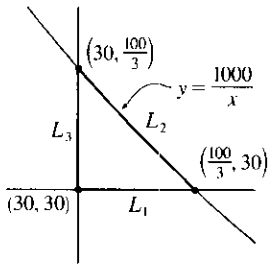
48. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})$. Then $C_x = 5y - 2Vx^{-2}$, $C_y = 5x - 2Vy^{-2}$, $f_x = 0$ implies $y = 2V/(5x^2)$, $f_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus the dimensions of the aquarium which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

49. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000$ m^3. Then $f(x, y) = xy + [64{,}000(x + y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x, y) = [(2)(64{,}000)]^2x^{-3}y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

50. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given hy $h = 10(2yz) + 8(2xz) + 1(xy) + 5(xy) = 6xy + 16xz + 20yz$. The volume is 4000 m^3, so $xyz = 4000$, and we substitute $z = \frac{4000}{xy}$ to obtain the heat loss function $h(x, y) = 6xy + 80{,}000/x + 64{,}000/y$.

(a) Since $z = \frac{4000}{xy} \geq 4$, $xy \leq 1000 \quad \Rightarrow \quad y \leq 1000/x$.

Also $x \geq 30$ and $y \geq 30$, so the domain of h is $D = \{(x, y) \mid x \geq 30, 30 \leq y \leq 1000/x\}$.



(b) $h(x, y) = 6xy + 80{,}000x^{-1} + 64{,}000y^{-1} \quad \Rightarrow \quad h_x = 6y - 80{,}000x^{-2}$, $h_y = 6x - 64{,}000y^{-2}$.

$h_x = 0$ implies $6x^2y = 80{,}000 \quad \Rightarrow \quad y = \dfrac{80{,}000}{6x^2}$ and substituting into $h_y = 0$ gives

$$6x = 64{,}000\left(\frac{6x^2}{80{,}000}\right)^2 \quad \Rightarrow \quad x^3 = \frac{80{,}000^2}{6 \cdot 64{,}000} = \frac{50{,}000}{3}, \text{ so } x = \sqrt[3]{\frac{50{,}000}{3}} = 10\sqrt[3]{\frac{50}{3}} \quad \Rightarrow \quad y = \frac{80}{\sqrt[3]{60}},$$

and the only critical point of h is $\left(10\sqrt[3]{\frac{50}{3}}, \frac{80}{\sqrt[3]{60}}\right) \approx (25.54, 20.43)$ which is not in D. Next we check the boundary of D. On L_1: $y = 30$, $h(x, 30) = 180x + 80{,}000/x + 6400/3$, $30 \leq x \leq \frac{100}{3}$. Since $h'(x, 30) = 180 - 80{,}000/x^2 > 0$ for $30 \leq x \leq \frac{100}{3}$, $h(x, 30)$ is an increasing function with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$. On L_2: $y = 1000/x$,

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$h(x, 1000/x) = 6000 + 64x + 80{,}000/x$, $30 \le x \le \frac{100}{3}$. Since $h'(x, 1000/x) = 64 - 80{,}000/x^2 < 0$ for $30 \le x \le \frac{100}{3}$, $h(x, 1000/x)$ is a decreasing function with minimum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$. On L_3: $x = 30$, $h(30, y) = 180y + 64{,}000/y + 8000/3$, $30 \le y \le \frac{100}{3}$. $h'(30, y) = 180 - 64{,}000/y^2 > 0$ for $30 \le y \le \frac{100}{3}$, so $h(30, y)$ is an increasing function of y with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$. Thus the absolute minimum of h is $h(30, 30) = 10{,}200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\frac{4000}{30^2} = \frac{40}{9} \approx 4.44$ m.

(c) From part (b), the only critical point of h, which gives a local (and absolute) minimum, is approximately $h(25.54, 20.43) \approx 9396$. So a building of volume 4000 m^2 with dimensions $x \approx 25.54$ m, $y \approx 20.43$ m, $z \approx \frac{4000}{(25.54)(20.43)} \approx 7.67$ m has the least amount of heat loss.

51. Let x, y, z be the dimensions of the rectangular box. Then the volume of the box is xyz and $L = \sqrt{x^2 + y^2 + z^2} \;\Rightarrow\; L^2 = x^2 + y^2 + z^2 \;\Rightarrow\; z = \sqrt{L^2 - x^2 - y^2}$. Substituting, we have volume $V(x, y) = xy\sqrt{L^2 - x^2 - y^2}$, $x, y > 0$.

$$V_x = xy \cdot \tfrac{1}{2}(L^2 - x^2 - y^2)^{-1/2}(-2x) + y\sqrt{L^2 - x^2 - y^2} = y\sqrt{L^2 - x^2 - y^2} - \frac{x^2 y}{\sqrt{L^2 - x^2 - y^2}},$$

$$V_y = x\sqrt{L^2 - x^2 - y^2} - \frac{xy^2}{\sqrt{L^2 - x^2 - y^2}},$$

$V_x = 0$ implies $y(L^2 - x^2 - y^2) = x^2 y \;\Rightarrow\; y(L^2 - 2x^2 - y^2) = 0 \;\Rightarrow\; 2x^2 + y^2 = L^2$ (since $y > 0$), and $V_y = 0$ implies $x(L^2 - x^2 - y^2) = xy^2 \;\Rightarrow\; x(L^2 - x^2 - 2y^2) = 0 \;\Rightarrow\; x^2 + 2y^2 = L^2$ (since $x > 0$). Substituting $y^2 = L^2 - 2x^2$ into $x^2 + 2y^2 = L^2$ gives $x^2 + 2L^2 - 4x^2 = L^2 \;\Rightarrow\; 3x^2 = L^2 \;\Rightarrow\; x = L/\sqrt{3}$ (since $x > 0$) and then $y = \sqrt{L^2 - 2\left(L/\sqrt{3}\right)^2} = L/\sqrt{3}$. So the only critical point is $\left(L/\sqrt{3}, L/\sqrt{3}\right)$ which, from the geometrical nature of the problem, must give an absolute maximum. Thus the maximum volume is $V\left(L/\sqrt{3}, L/\sqrt{3}\right) = \left(L/\sqrt{3}\right)^2\sqrt{L^2 - \left(L/\sqrt{3}\right)^2 - \left(L/\sqrt{3}\right)^2} = L^3/\left(3\sqrt{3}\right)$ cubic units.

52. Since $p + q + r = 1$ we can substitute $p = 1 - r - q$ into P giving $P = P(q, r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$. Since p, q and r represent proportions and $p + q + r = 1$, we know $q \ge 0$, $r \ge 0$, and $q + r \le 1$. Thus, we want to find the absolute maximum of the continuous function $P(q, r)$ on tbe closed set D enclosed by the lines $q = 0$, $r = 0$, and $q + r = 1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q, r) = 2 - 4q - 2r = 0$ and $P_r(q, r) = 2 - 4r - 2q = 0$. The first equation gives $r = 1 - 2q$, and substituting into the second equation we have

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$2-4(1-2q)-2q=0 \Rightarrow q=\frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3},\frac{1}{3}\right)$, where $P\left(\frac{1}{3},\frac{1}{3}\right)=\frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r=0, 0\le q\le 1$, $P(q,r)=P(q,0)=2q-2q^2, 0\le q\le 1$. This represents a parabola with maximum value $P\left(\frac{1}{2},0\right)=\frac{1}{2}$. On the segment $q=0, 0\le r\le 1$ we have $P(0,r)=2r-2r^2, 0\le r\le 1$. This represents a parabola with maximum value $P\left(0,\frac{1}{2}\right)=\frac{1}{2}$. Finally, on the segment $q+r=1, 0\le q\le 1$, $P(q,r)=P(q,1-q)=2q-2q^2, 0\le q\le 1$ which has a maximum value of $P\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q,r)$ on D is $\frac{2}{3}$.

53. Note that here the variables are m and b, and $f(m,b)=\sum_{i=1}^{n}[y_i-(mx_i+b)]^2$. Then $f_m=\sum_{i=1}^{n}-2x_i[y_i-(mx_i+b)]=0$ implies $\sum_{i=1}^{n}\left(x_iy_i-mx_i^2-bx_i\right)=0$ or $\sum_{i=1}^{n}x_iy_i=m\sum_{i=1}^{n}x_i^2+b\sum_{i=1}^{n}x_i$ and $f_b=\sum_{i=1}^{n}-2[y_i-(mx_i+b)]=0$ implies $\sum_{i=1}^{n}y_i=m\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}b=m\left(\sum_{i=1}^{n}x_i\right)+nb$. Thus we have the two desired equations. Now $f_{mm}=\sum_{i=1}^{n}2x_i^2$, $f_{bb}=\sum_{i=1}^{n}2=2n$ and $f_{mb}=\sum_{i=1}^{n}2x_i$. And $f_{mm}(m,b)>0$ always and $D(m,b)=4n\left(\sum_{i=1}^{n}x_i^2\right)-4\left(\sum_{i=1}^{n}x_i\right)^2=4\left[n\left(\sum_{i=1}^{n}x_i^2\right)-\left(\sum_{i=1}^{n}x_i\right)^2\right]>0$ always so the solutions of these two equations do indeed minimize $\sum_{i=1}^{n}d_i^2$.

54. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1,2,3)$. Writing the equation of the plane as $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the volume of the tetrahedron is given by $V=\frac{abc}{6}$. But $(1,2,3)$ must lie on the plane, so we need $\frac{1}{a}+\frac{2}{b}+\frac{3}{c}=1$ $(*)$ and thus can think of c as a function of a and b. Then $V_a=\frac{b}{6}\left(c+a\frac{\partial c}{\partial a}\right)$ and $V_b=\frac{a}{6}\left(c+b\frac{\partial c}{\partial b}\right)$. Differentiating $(*)$ with respect to a we get $-a^{-2}-3c^{-2}\frac{\partial c}{\partial a}=0 \Rightarrow \frac{\partial c}{\partial a}=\frac{-c^2}{3a^2}$, and differentiating $(*)$ with respect to b gives $-2b^{-2}-3c^{-2}\frac{\partial c}{\partial b}=0 \Rightarrow \frac{\partial c}{\partial b}=\frac{-2c^2}{3b^2}$. Then $V_a=\frac{b}{6}\left(c+a\frac{-c^2}{3a^2}\right)=0 \Rightarrow c=3a$, and $V_b=\frac{a}{6}\left(c+b\frac{-2c^2}{3b^2}\right)=0 \Rightarrow c=\frac{3}{2}b$. Thus $3a=\frac{3}{2}b$ or $b=2a$. Putting these into $(*)$ gives $\frac{3}{a}=1$ or $a=3$ and then $b=6, c=9$. Thus the equation of the required plane is $\frac{x}{3}+\frac{y}{6}+\frac{z}{9}=1$ or $6x+3y+2z=18$.

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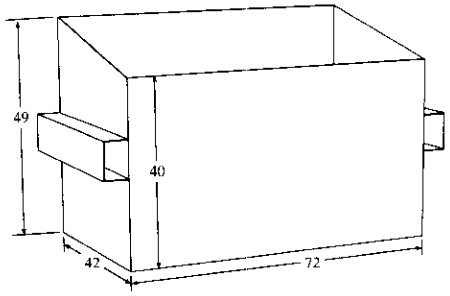
APPLIED PROJECT Designing a Dumpster

Note: The difficulty and results of this project vary widely with the type of container studied. In addition to the variation of basic shapes of containers, dumpsters may include additional constructed parts such as supports, lift pockets, wheels, etc. Also, a CAS or graphing utility may be needed to solve the resulting equations.

Here we present a typical solution for one particular trash dumpster.

1. The basic shape and dimensions (in inches) of an actual trash dumpster are as shown in the figure.



The front and back, as well as both sides, have an extra one-inch-wide flap that is folded under and welded to the base. In addition, the side panels each fold over one inch onto the front and back pieces where they are welded. Each side has a rectangular lift pocket, with cross-section 5 by 8 inches, made of the same material. These are attached with an extra one-inch width of steel on both top and bottom where each pocket is welded to the side sheet. All four sides have a "lip" at the top; the front and back panels have an extra 5 inches of steel at the top which is folded outward in three creases to form a rectangular tube. The edge is then welded back to the main sheet. The two sides form a top lip with separate sheets of steel 5 inches wide, similarly bent into three sides and welded to the main sheets (requiring two welds each). These extend beyond the main side sheets by 1.5 inches at each end in order to join with the lips on the front and back panels. The container has a hinged lid, extra steel supports on the base at each corner, metal "fins" serving as extra support for the side lift pockets, and wheels underneath. The volume of the container is $V = \frac{1}{2}(40 + 49) \times 42 \times 72 = 134{,}568$ in^3 or 77.875 ft^3.

2. First, we assume that some aspects of the construction do not change with different dimensions, so they may be considered fixed costs. This includes the lid (with hinges), wheels, and extra steel supports. Also, the upper "lip" we previously described extends beyond the side width to connect to the other pieces. We can safely assume that this extra portion, including any associated welds, costs the same regardless of the container's dimensions, so we will consider just the portion matching the measurement of the side panels in our calculations. We will further assume that the angle of the top of the container should he preserved. Then to compute the variable costs, let x be the width, y the length, and z the height of the front of the container. The back of the container is 9 inches, or $\frac{3}{4}$ ft, taller than the front, so using similar triangles we can say the back panel has height $z + \frac{3}{14}x$. Measuring in feet, we want the volume to remain constant, so $V = \frac{1}{2}\left(z + z + \frac{3}{14}x\right)(x)(y) = xyz + \frac{3}{28}x^2y = 77.875$. To determine a function for the variable cost, we first find the area of each sheet of metal needed. The base has area xy ft^2. The front panel has visible area yz plus $\frac{1}{12}y$ for the portion folded onto the base and $\frac{5}{12}y$ for the steel at the top used to form the lip, so $\left(yz + \frac{1}{2}y\right)$ ft^2 in total. Similarly, the back sheet has area $y\left(z + \frac{3}{14}x\right) + \frac{1}{12}y + \frac{5}{12}y = yz + \frac{3}{14}xy + \frac{1}{2}y$.

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Each side has visible area $\frac{1}{2}\left[z+\left(z+\frac{3}{14}x\right)\right](x)$, and the sheet includes one-inch flaps folding onto the front and back panels, so with area $\frac{1}{12}z$ and $\frac{1}{12}\left(z+\frac{3}{14}x\right)$, and a one-inch flap to fold onto the base with area $\frac{1}{12}x$. The lift pocket is constructed of a piece of steel 20 inches by x ft (including the 2 extra inches used by the welds). The additional metal used to make the lip at the top of the panel has width 5 inches and length that we can determine using the Pythagorean Theorem: $x^2+\left(\frac{3}{14}x\right)^2=\text{length}^2$, so length $=\frac{\sqrt{205}}{14}x\approx 1.0227x$. Thus the area of steel needed for each side panel is approximately

$$\tfrac{1}{2}\left[z+\left(z+\tfrac{3}{14}x\right)\right](x)+\tfrac{1}{12}z+\tfrac{1}{12}\left(z+\tfrac{3}{14}x\right)+\tfrac{1}{12}x+\tfrac{5}{3}x+\tfrac{5}{12}(1.0227x)\approx xz+\tfrac{3}{28}x^2+\tfrac{1}{6}z+2.194x$$

We also have the following welds:

Weld	Length
Front, back welded to base	$2y$
Sides welded to base	$2x$
Sides welded to front	$2z$
Sides welded to back	$2\left(z+\frac{3}{14}x\right)$
Weld on front and back lip	$2y$
Two welds on each side lip	$4(1.0227x)$
Two welds for each lift pocket	$4x$

Thus the total length of welds needed is

$$2y+2x+2z+2\left(z+\tfrac{3}{14}x\right)+2y+4(1.0227x)+4x\approx 10.519x+4y+4z$$

Finally, the total variable cost is approximately

$$\begin{aligned}0.90(xy)+0.70\left[\left(yz+\tfrac{1}{2}y\right)+\left(yz+\tfrac{3}{14}xy+\tfrac{1}{2}y\right)+2\left(xz+\tfrac{3}{28}x^2+\tfrac{1}{6}z+2.194x\right)\right]&\\ +0.18(10.519x+4y+4z)&\\ \approx 1.05xy+1.4yz+1.42y+1.4xz+0.15x^2+0.953z+4.965x&\end{aligned}$$

We would like to minimize this function while keeping volume constant, so since $xyz+\frac{3}{28}x^2y=77.875$ we can substitute $z=\dfrac{77.875}{xy}-\dfrac{3}{28}x$ giving variable cost as a function of x and y:

$C(x,y)\approx 0.9xy+\dfrac{109.0}{x}+1.42y+\dfrac{109.0}{y}+\dfrac{74.2}{xy}+4.86x$. Using a CAS, we solve the system of equations $C_x(x,y)=0$ and $C_y(x,y)=0$; the only critical point within an appropriate domain is approximately $(3.58,5.29)$. From the nature of the function C (or from a graph) we can determine that C has an absolute minimum at $(3.58,5.29)$, and so the minimum cost is attained for $x\approx 3.58$ ft (or 43.0 in), $y\approx 5.29$ ft (or 63.5 in), and $z\approx\frac{77.875}{3.58(5.29)}-\frac{3}{28}(3.58)\approx 3.73$ ft (or 44.8 in).

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3. The fixed cost aspects of the container which we did not include in our calculations, such as the wheels and lid, don't affect the validity of our results. Some of our other assumptions, however, may influence the accuracy of our findings. We simplified the price of the steel sheets to include cuts and bends, and we simplified the price of welding to include the labor and materials. This may not be accurate for areas of the container, such as the lip and lift pockets, that require several cuts, bends, and welds in a relatively small surface area. Consequently, increasing some dimensions of the container may not increase the cost in the same manner as our computations predict. If we do not assume that the angle of the sloped top of the container must he preserved, it is likely that we could further improve our cost. Finally, our results show that the length of the container should be changed to minimize cost; this may not be possible if the two lift pockets must remain a fixed distance apart for handling by machinery.

4. The minimum variable cost using our values found in Problem 2 is $C(3.58, 5.29) \approx \$96.95$, while the current dimensions give an estimated variable cost of $C(3.5, 6.0) \approx \$97.30$. If we determine that our assumptions and simplifications are acceptable, our work shows that a slight savings can be gained by adjusting the dimensions of the container. However, the difference in cost is modest, and may not justify changes in the manufacturing process.

DISCOVERY PROJECT Quadratic Approximations and Critical Points

1. $$Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \tfrac{1}{2}f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \tfrac{1}{2}f_{yy}(a,b)(y-b)^2,$$

so

$$\begin{aligned} Q_x(x,y) &= f_x(a,b) + \tfrac{1}{2}f_{xx}(a,b)(2)(x-a) + f_{xy}(a,b)(y-b) \\ &= f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b) \end{aligned}$$

At (a,b) we have $Q_x(a,b) = f_x(a,b) + f_{xx}(a,b)\,(a-a) + f_{xy}(a,b)(b-b) = f_x(a,b)$.

Similarly, $Q_y(x,y) = f_y(a,b) + f_{xy}(a,b)(x-a) + f_{yy}(a,b)(y-b) \quad \Rightarrow$

$Q_y(a,b) = f_y(a,b) + f_{xy}(a,b)(a-a) + f_{yy}(a,b)(b-b) = f_y(a,b)$.

For the second-order partial derivatives we have

$$Q_{xx}(x,y) = \frac{\partial}{\partial x}\left[f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)\right] = f_{xx}(a,b)$$

$$\Rightarrow \quad Q_{xx}(a,b) = f_{xx}(a,b)$$

$$Q_{xy}(x,y) = \frac{\partial}{\partial y}\left[f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)\right] = f_{xy}(a,b)$$

$$\Rightarrow \quad Q_{xy}(a,b) = f_{xy}(a,b)$$

$$Q_{yy}(x,y) = \frac{\partial}{\partial y}\left[f_y(a,b) + f_{xy}(a,b)(x-a) + f_{yy}(a,b)(y-b)\right] = f_{yy}(a,b)$$

$$\Rightarrow \quad Q_{yy}(a,b) = f_{yy}(a,b)$$

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2. (a) First we find the partial derivatives and values that will be needed:

$$\begin{aligned}
f(x,y) &= e^{-x^2-y^2} & f(0,0) &= 1\\
f_x(x,y) &= -2xe^{-x^2-y^2} & f_x(0,0) &= 0\\
f_y(x,y) &= -2ye^{-x^2-y^2} & f_y(0,0) &= 0\\
f_{xx}(x,y) &= (4x^2-2)e^{-x^2-y^2} & f_{xx}(0,0) &= -2\\
f_{xy}(x,y) &= 4xye^{-x^2-y^2} & f_{xy}(0,0) &= 0\\
f_{yy}(x,y) &= (4y^2-2)e^{-x^2-y^2} & f_{yy}(0,0) &= -2
\end{aligned}$$

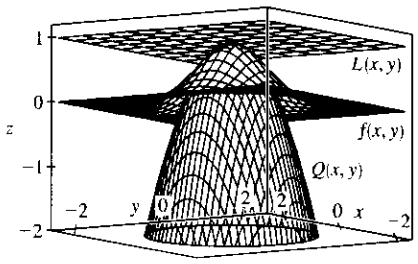
Then the first-degree Taylor polynomial of f at $(0,0)$ is

$$\begin{aligned}
L(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 + (0)(x-0) + (0)(y-0)\\
&= 1
\end{aligned}$$

The second-degree Taylor polynomial is given by

$$\begin{aligned}
Q(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \tfrac{1}{2}f_{xx}(0,0)(x-0)^2\\
&\qquad + f_{xy}(0,0)(x-0)(y-0) + \tfrac{1}{2}f_{yy}(0,0)(y-0)^2\\
&= 1 - x^2 - y^2
\end{aligned}$$

(b)



As we see from the graph, L approximates f well only for points (x,y) extremely close to the origin. Q is a much better approximation; the shape of its graph looks similar to that of the graph of f near the origin, and the values of Q appear to be good estimates for the values of f within a significant radius of the origin.

3. (a) First we find the partial derivatives and values that will be needed:

$$\begin{aligned}
f(x,y) &= xe^y & f(1,0) &= 1 & f_{xx}(x,y) &= 0 & f_{xx}(1,0) &= 0\\
f_x(x,y) &= e^y & f_x(1,0) &= 1 & f_{xy}(x,y) &= e^y & f_{xy}(1,0) &= 1\\
f_y(x,y) &= xe^y & f_y(1,0) &= 1 & f_{yy}(x,y) &= xe^y & f_{yy}(1,0) &= 1
\end{aligned}$$

Then the first-degree Taylor polynomial of f at $(1,0)$ is

$$\begin{aligned}
L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)\\
&= 1 + (1)(x-1) + (1)(y-0)\\
&= x + y
\end{aligned}$$

The second-degree Taylor polynomial is given by

$$\begin{aligned}
Q(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \tfrac{1}{2}f_{xx}(1,0)(x-1)^2\\
&\qquad + f_{xy}(1,0)(x-1)(y-0) + \tfrac{1}{2}f_{yy}(1,0)(y-0)^2\\
&= \tfrac{1}{2}y^2 + x + xy
\end{aligned}$$

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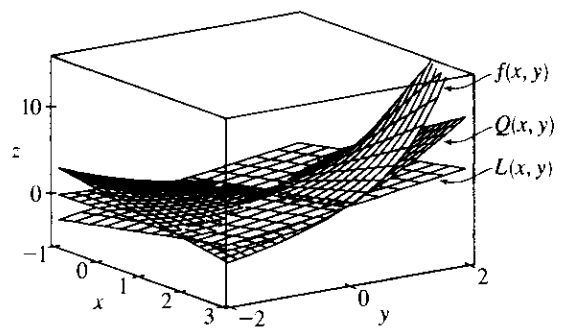
(b) $L(0.9, 0.1) = 0.9 + 0.1 = 1.0$

$Q(0.9, 0.1) = \frac{1}{2}(0.1)^2 + 0.9 + (0.9)(0.1) = 0.995$

$f(0.9, 0.1) = 0.9e^{0.1} \approx 0.9947$

(c)



As we see from the graph, L and Q both approximate f reasonably well near the point $(1, 0)$. As we venture farther from the point, the graph of Q follows the shape of the graph of f more closely than L.

4. (a) $f(x, y) = ax^2 + bxy + cy^2 = a\left[x^2 + \frac{b}{a}xy + \frac{c}{a}y^2\right]$

$$= a\left[x^2 + \frac{b}{a}xy + \left(\frac{b}{2a}y\right)^2 - \left(\frac{b}{2a}y\right)^2 + \frac{c}{a}y^2\right]$$

$$= a\left[\left(x + \frac{b}{2a}y\right)^2 - \frac{b^2}{4a^2}y^2 + \frac{c}{a}y^2\right] = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right]$$

(b) For $D = 4ac - b^2$, from part (a) we have $f(x, y) = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right]$. If $D > 0$, $\left(\frac{D}{4a^2}\right)y^2 \geq 0$ and $\left(x + \frac{b}{2a}y\right)^2 \geq 0$, so $\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] \geq 0$. Here $a > 0$, thus $f(x, y) = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] \geq 0$. We know $f(0, 0) = 0$, so $f(0, 0) \leq f(x, y)$ for all (x, y), and by definition f has a local minimum at $(0, 0)$.

(c) As in part (b), $\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] \geq 0$, and since $a < 0$ we have $f(x, y) = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] \leq 0$. Since $f(0, 0) = 0$, we must have $f(0, 0) \geq f(x, y)$ for all (x, y), so by definition f has a local maximum at $(0, 0)$.

(d) $f(x, y) = ax^2 + bxy + cy^2$, so $f_x(x, y) = 2ax + by \Rightarrow f_x(0, 0) = 0$ and $f_y(x, y) = bx + 2cy \Rightarrow f_y(0, 0) = 0$. Since $f(0, 0) = 0$ and f and its partial derivatives are continuous, we know from Equation 15.4.2 [ET 14.4.2] that the tangent plane to the graph of f at $(0, 0)$ is the plane $z = 0$. Then f has a saddle point at $(0, 0)$ if the grapb of f crosses the tangent plane at $(0, 0)$, or equivalently, if some paths to the origin have positive function values while other paths have negative function values. Suppose we approach the origin along the x-axis; then we have $y = 0 \Rightarrow f(x, 0) = ax^2$ which has the same sign as a. We must now find at least

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one path to the origin where $f(x, y)$ gives values with sign opposite that of a. Since $f(x, y) = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right]$, if we approach the origin along the line $x = -\frac{b}{2a}y$, we have $f\left(-\frac{b}{2a}y, y\right) = a\left[\left(-\frac{b}{2a}y + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] = \frac{D}{4a}y^2$. Since $D < 0$, these values have signs opposite that of a. Thus, f has a saddle point at $(0, 0)$.

5. (a) Since the partial derivatives of f exist at $(0, 0)$ and $(0, 0)$ is a critical point, we know $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. Then the second-degree Taylor polynomial of f at $(0, 0)$ can be expressed as

$$\begin{aligned} Q(x, y) &= f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) + \tfrac{1}{2}f_{xx}(0, 0)(x - 0)^2 \\ &\qquad + f_{xy}(0, 0)(x - 0)(y - 0) + \tfrac{1}{2}f_{yy}(0, 0)(y - 0)^2 \\ &= \tfrac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \tfrac{1}{2}f_{yy}(0, 0)y^2. \end{aligned}$$

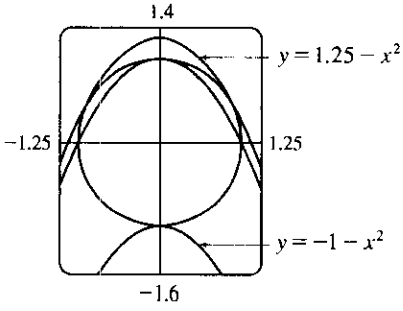
(b) $Q(x, y) = \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2$ fits the form of the polynomial function in Problem 4 with $a = \frac{1}{2}f_{xx}(0, 0)$, $b = f_{xy}(0, 0)$, and $c = \frac{1}{2}f_{yy}(0, 0)$. Then we know Q is a paraboloid, and that Q has a local maximum, local minimum, or saddle point at $(0, 0)$. Here, $D = 4ac - b^2 = 4\left(\frac{1}{2}\right)f_{xx}(0, 0)\left(\frac{1}{2}\right)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2$, and if $D > 0$ with $a = \frac{1}{2}f_{xx}(0, 0) > 0 \Rightarrow f_{xx}(0, 0) > 0$, we know from Problem 4 that Q has a local minimum at $(0, 0)$. Similarly, if $D > 0$ and $a < 0 \Rightarrow f_{xx}(0, 0) < 0$, Q has a local maximum at $(0, 0)$, and if $D < 0$, Q has a saddle point at $(0, 0)$.

(c) Since $f(x, y) \approx Q(x, y)$ near $(0, 0)$, part (b) suggests that for $D = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2$, if $D > 0$ and $f_{xx}(0, 0) > 0$, f has a local minimum at $(0, 0)$. If $D > 0$ and $f_{xx}(0, 0) < 0$, f has a local maximum at $(0, 0)$, and if $D < 0$, f has a saddle point at $(0, 0)$. Together with the conditions given in part (a), this is precisely the Second Derivatives Test from Section 15.7 [ET 14.7].

15.8 Lagrange Multipliers ET 14.8

1. At the extreme values of f, the level curves of f just touch the curve $g(x, y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x, y) = c$ with the largest value of c which still intersects the curve $g(x, y) = 8$ is approximately $c = 59$, and the smallest value of c corresponding to a level curve which intersects $g(x, y) = 8$ appears to be $c = 30$. Thus we estimate the maximum value of f subject to the constraint $g(x, y) = 8$ to be about 59 and the minimum to be 30.

2. (a) The values $c = \pm 1$ and $c = 1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function $x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.



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(b) $\nabla f = \langle 2x, 1\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$. So $2x = 2\lambda x \Rightarrow$ either $\lambda = 1$ or $x = 0$. If $\lambda = 1$, then $y = \frac{1}{2}$ and so $x = \pm\frac{\sqrt{3}}{2}$ (from the constraint). If $x = 0$, then $y = \pm 1$. Therefore f has possible extreme values at the points $(0, \pm 1)$ and $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. We calculate $f\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{5}{4}$ (the maximum value), $f(0, 1) = 1$, and $f(0, -1) = -1$ (the minimum value). These are our answers from (a).

3. $f(x, y) = x^2 - y^2$, $g(x, y) = x^2 + y^2 = 1 \Rightarrow \nabla f = \langle 2x, -2y\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$. Then $2x = 2\lambda x$ implies $x = 0$ or $\lambda = 1$. If $x = 0$, then $x^2 + y^2 = 1$ implies $y = \pm 1$ and if $\lambda = 1$, then $-2y = 2\lambda y$ implies $y = 0$ and thus $x = \pm 1$. Thus the possible points for the extreme values of f are $(\pm 1, 0)$, $(0, \pm 1)$. But $f(\pm 1, 0) = 1$ while $f(0, \pm 1) = -1$ so the maximum value of f on $x^2 + y^2 = 1$ is $f(\pm 1, 0) = 1$ and the minimum value is $f(0, \pm 1) = -1$.

4. $f(x, y) = 4x + 6y$, $g(x, y) = x^2 + y^2 = 13 \Rightarrow \nabla f = \langle 4, 6\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$. Then $2\lambda x = 4$ and $2\lambda y = 6$ imply $x = \dfrac{2}{\lambda}$ and $y = \dfrac{3}{\lambda}$. But $13 = x^2 + y^2 = \left(\dfrac{2}{\lambda}\right)^2 + \left(\dfrac{3}{\lambda}\right)^2 \Rightarrow 13 = \dfrac{13}{\lambda^2} \Rightarrow \lambda = \pm 1$, so f has possible extreme values at the points $(2, 3)$, $(-2, -3)$. We compute $f(2, 3) = 26$ and $f(-2, -3) = -26$, so the maximum value of f on $x^2 + y^2 = 13$ is $f(2, 3) = 26$ and the minimum value is $f(-2, -3) = -26$.

5. $f(x, y) = x^2 y$, $g(x, y) = x^2 + 2y^2 = 6 \Rightarrow \nabla f = \langle 2xy, x^2\rangle$, $\lambda\nabla g = \langle 2\lambda x, 4\lambda y\rangle$. Then $2xy = 2\lambda x$ implies $x = 0$ or $\lambda = y$. If $x = 0$, then $x^2 = 4\lambda y$ implies $\lambda = 0$ or $y = 0$. However, if $y = 0$ then $g(x, y) = 0$, a contradiction. So $\lambda = 0$ and then $g(x, y) = 6 \Rightarrow y = \pm\sqrt{3}$. If $\lambda = y$, then $x^2 = 4\lambda y$ implies $x^2 = 4y^2$, and so $g(x, y) = 6 \Rightarrow 4y^2 + 2y^2 = 6 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$. Thus f has possible extreme values at the points $(0, \pm\sqrt{3})$, $(\pm 2, 1)$, and $(\pm 2, -1)$. After evaluating f at these points, we find the maximum value to be $f(\pm 2, 1) = 4$ and the minimum to be $f(\pm 2, -1) = -4$.

6. $f(x, y) = x^2 + y^2$, $g(x, y) = x^4 + y^4 = 1 \Rightarrow \nabla f = \langle 2x, 2y\rangle$, $\lambda\nabla g = \langle 4\lambda x^3, 4\lambda y^3\rangle$. Then $x = 2\lambda x^3$ implies $x = 0$ or $\lambda = \dfrac{1}{2x^2}$. If $x = 0$, then $x^4 + y^4 = 1$ implies $y = \pm 1$. But $y = 2\lambda y^3$ implies $y = 0$ so $x = \pm 1$ or $\lambda = \dfrac{1}{2y^2}$ and $x^2 = y^2$ and $2x^4 = 1$ so $x = \pm\frac{1}{\sqrt[4]{2}}$. Hence the possible points are $(0, \pm 1)$, $(\pm 1, 0)$, $\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right)$, with the maximum value of f on $x^4 + y^4 = 1$ being $f\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$ and the minimum value being $f(0, \pm 1) = f(\pm 1, 0) = 1$.

7. $f(x, y, z) = 2x + 6y + 10z$, $g(x, y, z) = x^2 + y^2 + z^2 = 35 \Rightarrow \nabla f = \langle 2, 6, 10\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $2\lambda x = 2$, $2\lambda y = 6$, $2\lambda z = 10$ imply $x = \dfrac{1}{\lambda}$, $y = \dfrac{3}{\lambda}$, and $z = \dfrac{5}{\lambda}$. But $35 = x^2 + y^2 + z^2 = \left(\dfrac{1}{\lambda}\right)^2 + \left(\dfrac{3}{\lambda}\right)^2 + \left(\dfrac{5}{\lambda}\right)^2 \Rightarrow 35 = \dfrac{35}{\lambda^2} \Rightarrow \lambda = \pm 1$, so f has possible extreme values at the points $(1, 3, 5)$, $(-1, -3, -5)$. The maximum value of f on $x^2 + y^2 + z^2 = 35$ is $f(1, 3, 5) = 70$, and the minimum is $f(-1, -3, -5) = -70$.

8. $f(x, y, z) = 8x - 4z$, $g(x, y, z) = x^2 + 10y^2 + z^2 = 5 \Rightarrow \nabla f = \langle 8, 0, -4\rangle$, $\lambda\nabla g = \langle 2\lambda x, 20\lambda y, 2\lambda z\rangle$. Then $2\lambda x = 8$, $20\lambda y = 0$, $2\lambda z = -4$ imply $x = \dfrac{4}{\lambda}$, $y = 0$, and $z = -\dfrac{2}{\lambda}$. But $5 = x^2 + 10y^2 + z^2 = \left(\dfrac{4}{\lambda}\right)^2 + 10\,(0)^2 + \left(-\dfrac{2}{\lambda}\right)^2 \Rightarrow 5 = \dfrac{20}{\lambda^2} \Rightarrow \lambda = \pm 2$, so f has possible extreme values at the points $(2, 0, -1)$, $(-2, 0, 1)$. The maximum of f on $x^2 + 10y^2 + z^2 = 5$ is $f(2, 0, -1) = 20$, and the minimum is $f(-2, 0, 1) = -20$.

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9. $f(x,y,z) = xyz$, $g(x,y,z) = x^2 + 2y^2 + 3z^2 = 6 \;\Rightarrow\; \nabla f = \langle yz, xz, xy\rangle$, $\lambda\nabla g = \langle 2\lambda x, 4\lambda y, 6\lambda z\rangle$. Then $\nabla f = \lambda\nabla g$ implies $\lambda = (yz)/(2x) = (xz)/(4y) = (xy)/(6z)$ or $x^2 = 2y^2$ and $z^2 = \frac{2}{3}y^2$. Thus $x^2 + 2y^2 + 3z^2 = 6$ implies $6y^2 = 6$ or $y = \pm 1$. Then the possible points are $\left(\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

10. $f(x,y,z) = x^2y^2z^2$, $g(x,y,z) = x^2 + y^2 + z^2 = 1 \;\Rightarrow\; \nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $\nabla f = \lambda\nabla g$ implies (1) $\lambda = y^2z^2 = x^2z^2 = x^2y^2$ and $\lambda \neq 0$, or (2) $\lambda = 0$ and one or two (hut not three) of the coordinates are 0. If (1) then $x^2 = y^2 = z^2 = \frac{1}{3}$. The minimum value of f on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

11. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^4 + y^4 + z^4 = 1 \;\Rightarrow$
$\nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda\nabla g = \langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3\rangle$.
Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$, then $\nabla f = \lambda\nabla g$ implies $\lambda = 1/(2x^2) = 1/(2y^2) = 1/(2z^2)$ or $x^2 = y^2 = z^2$ and $3x^4 = 1$ or $x = \pm\frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$ all with an f-value of $\sqrt{3}$.
Case 2: If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f value of $\sqrt{2}$.
Case 3: If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f value of 1. Thus on $x^4 + y^4 + z^4 = 1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1.

12. $f(x,y,z) = x^4 + y^4 + z^4$, $g(x,y,z) = x^2 + y^2 + z^2 = 1 \;\Rightarrow$
$\nabla f = \langle 4x^3, 4y^3, 4z^3\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$.
Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$ then $\nabla f = \lambda\nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ yielding 8 points each with an f-value of $\frac{1}{3}$.
Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.
Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1. Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

13. $f(x,y,z,t) = x + y + z + t$, $g(x,y,z,t) = x^2 + y^2 + z^2 + t^2 = 1 \;\Rightarrow\; \langle 1,1,1,1\rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z, 2\lambda t\rangle$, so $\lambda = 1/(2x) = 1/(2y) = 1/(2z) = 1/(2t)$ and $x = y = z = t$. But $x^2 + y^2 + z^2 + t^2 = 1$, so the possible points are $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. Thus the maximum value of f is $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$ and the minimum value is $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$.

14. $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \;\Rightarrow$
$\langle 1, 1, \ldots, 1\rangle = \langle 2\lambda x_1, 2\lambda x_2, \ldots, 2\lambda x_n\rangle$, so $\lambda = 1/(2x_1) = 1/(2x_2) = \cdots = 1/(2x_n)$ and $x_1 = x_2 = \cdots = x_n$. But $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, so $x_i = \pm 1/\sqrt{n}$ for $i = 1, \ldots, n$. Thus the maximum value of f is $f(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}) = \sqrt{n}$ and the minimum value is $f(-1/\sqrt{n}, -1/\sqrt{n}, \ldots, -1/\sqrt{n}) = -\sqrt{n}$.

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15. $f(x,y,z) = x + 2y$, $g(x,y,z) = x + y + z = 1$, $h(x,y,z) = y^2 + z^2 = 4 \;\Rightarrow\; \nabla f = \langle 1, 2, 0\rangle$, $\lambda\nabla g = \langle \lambda, \lambda, \lambda\rangle$ and $\mu\nabla h = \langle 0, 2\mu y, 2\mu z\rangle$. Then $1 = \lambda$, $2 = \lambda + 2\mu y$ and $0 = \lambda + 2\mu z$ so $\mu y = \frac{1}{2} = -\mu z$ or $y = 1/(2\mu)$, $z = -1/(2\mu)$. Thus $x + y + z = 1$ implies $x = 1$ and $y^2 + z^2 = 4$ implies $\mu = \pm\frac{1}{2\sqrt{2}}$. Then the possible points are $\left(1, \pm\sqrt{2}, \mp\sqrt{2}\right)$ and the maximum value is $f\left(1, \sqrt{2}, -\sqrt{2}\right) = 1 + 2\sqrt{2}$ and the minimum value is $f\left(1, -\sqrt{2}, \sqrt{2}\right) = 1 - 2\sqrt{2}$.

16. $f(x,y,z) = 3x - y - 3z$, $g(x,y,z) = x + y - z = 0$, $h(x,y,z) = x^2 + 2z^2 = 1 \;\Rightarrow\; \nabla f = \langle 3, -1, -3\rangle$, $\lambda\nabla g = \langle \lambda, \lambda, -\lambda\rangle$, $\mu\nabla h = (2\mu x, 0, 4\mu z)$. Then $3 = \lambda + 2\mu x$, $-1 = \lambda$ and $-3 = -\lambda + 4\mu z$, so $\lambda = -1$, $\mu z = -1$, $\mu x = 2$. Thus $h(x,y,z) = 1$ implies $\dfrac{4}{\mu^2} + 2\left(\dfrac{1}{\mu^2}\right) = 1$ or $\mu = \pm\sqrt{6}$, so $z = \mp\frac{1}{\sqrt{6}}$; $x = \pm\frac{2}{\sqrt{6}}$; and $g(x,y,z) = 0$ implies $y = \mp\frac{3}{\sqrt{6}}$. Hence the maximum of f subject to the constraints is $f\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) = 2\sqrt{6}$ and the minimum is $f\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) = -2\sqrt{6}$.

17. $f(x,y,z) = yz + xy$, $g(x,y,z) = xy = 1$, $h(x,y,z) = y^2 + z^2 = 1 \;\Rightarrow\; \nabla f = \langle y, x + z, y\rangle$, $\lambda\nabla g = \langle \lambda y, \lambda x, 0\rangle$, $\mu\nabla h = \langle 0, 2\mu y, 2\mu z\rangle$. Then $y = \lambda y$ implies $\lambda = 1$ [$y \neq 0$ since $g(x,y,z) = 1$], $x + z = \lambda x + 2\mu y$ and $y = 2\mu z$. Thus $\mu = z/(2y) = y/(2y)$ or $y^2 = z^2$, and so $y^2 + z^2 = 1$ implies $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm\frac{1}{\sqrt{2}}$. Then $xy = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Hence the maximum of f subject to the constraints is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2}$ and the minimum is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$.

Note: Since $xy = 1$ is one of the constraints we could have solved the problem by solving $f(y,z) = yz + 1$ subject to $y^2 + z^2 = 1$.

18. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \;\Rightarrow\; \nabla f = \langle 4x - 4, 6y\rangle = \langle 0, 0\rangle \;\Rightarrow\; x = 1, y = 0$. Thus $(1,0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \;\Rightarrow$ $\lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$, so $6y = 2\lambda y \;\Rightarrow$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \;\Rightarrow\; x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \leq 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1,0) = -7$.

19. $f(x,y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0) = 1$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + 4y^2 = 1 \;\Rightarrow\; \lambda\nabla g = \langle 2\lambda x, 8\lambda y\rangle$, so setting $\nabla f = \lambda\nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \;\Rightarrow$ $x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$. The former are the maxima on the region and the latter are the minima.

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$(f_x)(x_2 - x_1) + (f_y)(y_2 - y_1) = -f$ and $(g_x)(x_2 - x_1) + (g_y)(y_2 - y_1) = -g$ simultaneously for $(x_2 - x_1)$ and $(y_2 - y_1)$. Then $y_2 - y_1 = \dfrac{gf_x - fg_x}{g_x f_y - f_x g_y}$ or

$y_2 = y_1 - \dfrac{gf_x - fg_x}{f_x g_y - g_x f_y}$ and $(f_x)(x_2 - x_1) + \dfrac{(f_y)(gf_x - fg_x)}{g_x f_y - f_x g_y} = -f$ so

$x_2 - x_1 = \dfrac{-f - [(f_y)(gf_x - fg_x)/(g_x f_y - f_x g_y)]}{f_x} = \dfrac{fg_y - f_y g}{g_x f_y - f_x g_y}$. Hence $x_2 = x_1 - \dfrac{fg_y - f_y g}{f_x g_y - g_x f_y}$.

(b) Let $f(x, y) = x^x + y^y - 1000$ and $g(x, y) = x^y + y^x - 100$. Then we wish to solve the system of equations $f(x, y) = 0$, $g(x, y) = 0$. Recall $\dfrac{d}{dx}[x^x] = x^x(1 + \ln x)$ (differentiate logarithmically), so $f_x(x, y) = x^x(1 + \ln x)$, $f_y(x, y) = y^y(1 + \ln y)$, $g_x(x, y) = yx^{y-1} + y^x \ln y$, and $g_y(x, y) = x^y \ln x + xy^{x-1}$. Looking at the graph, we estimate the first point of intersection of the curves, and thus the solution to the system, to be approximately $(2.5, 4.5)$. Then following the method of part (a), $x_1 = 2.5$, $y_1 = 4.5$ and

$$x_2 = 2.5 - \frac{f(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g(2.5, 4.5)}{f_x(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g_x(2.5, 4.5)} \approx 2.447674117$$

$$y_2 = 4.5 - \frac{f_x(2.5, 4.5)\, g(2.5, 4.5) - f(2.5, 4.5)\, g_x(2.5, 4.5)}{f_x(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g_x(2.5, 4.5)} \approx 4.555657467$$

Continuing this procedure, we arrive at the following values. (If you use a CAS, you may need to increase its computational precision.)

$x_1 = 2.5$	$y_1 = 4.5$
$x_2 = 2.447674117$	$y_2 = 4.555657467$
$x_3 = 2.449614877$	$y_3 = 4.551969333$
$x_4 = 2.449624628$	$y_4 = 4.551951420$
$x_5 = 2.449624628$	$y_5 = 4.551951420$

Thus, to six decimal places, the point of intersection is $(2.449625, 4.551951)$. The second point of intersection can be found similarly, or, by symmetry it is approximately $(4.551951, 2.449625)$.

7. (a) $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Then $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ and

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial r}\right] + \sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial r}\right] \\ &= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\sin\theta\end{aligned}$$

Similarly $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}\, r\sin\theta + \dfrac{\partial u}{\partial y}\, r\cos\theta$ and

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$\dfrac{\partial^2 u}{\partial \theta^2} = \dfrac{\partial^2 u}{\partial x^2} r^2 \sin^2\theta + \dfrac{\partial^2 u}{\partial y^2} r^2 \cos^2\theta - 2\dfrac{\partial^2 u}{\partial y\,\partial x} r^2 \sin\theta\cos\theta - \dfrac{\partial u}{\partial x} r\cos\theta - \dfrac{\partial u}{\partial y} r\sin\theta$. So

$$\begin{aligned}
&\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \\
&\quad = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\sin\theta + \frac{\partial u}{\partial x}\frac{\cos\theta}{r} + \frac{\partial u}{\partial y}\frac{\sin\theta}{r} \\
&\qquad + \frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial y\,\partial x}\sin\theta\cos\theta - \frac{\partial u}{\partial x}\frac{\cos\theta}{r} - \frac{\partial u}{\partial y}\frac{\sin\theta}{r} + \frac{\partial^2 u}{\partial z^2} \\
&\quad = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}
\end{aligned}$$

(b) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Then

$\dfrac{\partial u}{\partial \rho} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \rho} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \rho} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial \rho} = \dfrac{\partial u}{\partial x}\sin\phi\cos\theta + \dfrac{\partial u}{\partial y}\sin\phi\sin\theta + \dfrac{\partial u}{\partial z}\cos\phi$, and

$$\begin{aligned}
\frac{\partial^2 u}{\partial \rho^2} &= \sin\phi\cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial \rho}\right] \\
&\quad + \sin\phi\sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial \rho}\right] \\
&\quad + \cos\phi\left[\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial \rho} + \frac{\partial^2 u}{\partial x\,\partial z}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\,\partial z}\frac{\partial y}{\partial \rho}\right] \\
&= 2\frac{\partial^2 u}{\partial y\,\partial x}\sin^2\phi\sin\theta\cos\theta + 2\frac{\partial^2 u}{\partial z\,\partial x}\sin\phi\cos\phi\cos\theta + 2\frac{\partial^2 u}{\partial y\,\partial z}\sin\phi\cos\phi\sin\theta \\
&\quad + \frac{\partial^2 u}{\partial x^2}\sin^2\phi\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\phi\sin^2\theta + \frac{\partial^2 u}{\partial z^2}\cos^2\phi
\end{aligned}$$

Similarly $\dfrac{\partial u}{\partial \phi} = \dfrac{\partial u}{\partial x}\rho\cos\phi\cos\theta + \dfrac{\partial u}{\partial y}\rho\cos\phi\sin\theta - \dfrac{\partial u}{\partial z}\rho\sin\phi$, and

$$\begin{aligned}
\frac{\partial^2 u}{\partial \phi^2} &= 2\frac{\partial^2 u}{\partial y\,\partial x}\rho^2\cos^2\phi\sin\theta\cos\theta - 2\frac{\partial^2 u}{\partial x\,\partial z}\rho^2\sin\phi\cos\phi\cos\theta \\
&\quad - 2\frac{\partial^2 u}{\partial y\,\partial z}\rho^2\sin\phi\cos\phi\sin\theta + \frac{\partial^2 u}{\partial x^2}\rho^2\cos^2\phi\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\rho^2\cos^2\phi\sin^2\theta \\
&\quad + \frac{\partial^2 u}{\partial z^2}\rho^2\sin^2\phi - \frac{\partial u}{\partial x}\rho\sin\phi\cos\theta - \frac{\partial u}{\partial y}\rho\sin\phi\sin\theta - \frac{\partial u}{\partial z}\rho\cos\phi
\end{aligned}$$

And $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}\rho\sin\phi\sin\theta + \dfrac{\partial u}{\partial y}\rho\sin\phi\cos\theta$, while

$$\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= -2\frac{\partial^2 u}{\partial y\,\partial x}\rho^2\sin^2\phi\cos\theta\sin\theta + \frac{\partial^2 u}{\partial x^2}\rho^2\sin^2\phi\sin^2\theta \\
&\quad + \frac{\partial^2 u}{\partial y^2}\rho^2\sin^2\phi\cos^2\theta - \frac{\partial u}{\partial x}\rho\sin\phi\cos\theta - \frac{\partial u}{\partial y}\rho\sin\phi\sin\theta
\end{aligned}$$

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Therefore

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{\partial^2 u}{\partial x^2}\left[(\sin^2\phi\cos^2\theta) + (\cos^2\phi\cos^2\theta) + \sin^2\theta\right]$$

$$+ \frac{\partial^2 u}{\partial y^2}\left[(\sin^2\phi\sin^2\theta) + (\cos^2\phi\sin^2\theta) + \cos^2\theta\right] + \frac{\partial^2 u}{\partial z^2}\left[\cos^2\phi + \sin^2\phi\right]$$

$$+ \frac{\partial u}{\partial x}\left[\frac{2\sin^2\phi\cos\theta + \cos^2\phi\cos\theta - \sin^2\phi\cos\theta - \cos\theta}{\rho\sin\phi}\right]$$

$$+ \frac{\partial u}{\partial y}\left[\frac{2\sin^2\phi\sin\theta + \cos^2\phi\sin\theta - \sin^2\phi\sin\theta - \sin\theta}{\rho\sin\phi}\right]$$

But $2\sin^2\phi\cos\theta + \cos^2\phi\cos\theta - \sin^2\phi\cos\theta - \cos\theta = (\sin^2\phi + \cos^2\phi - 1)\cos\theta = 0$ and similarly the coefficient of $\partial u/\partial y$ is 0. Also $\sin^2\phi\cos^2\theta + \cos^2\phi\cos^2\theta + \sin^2\theta = \cos^2\theta(\sin^2\phi + \cos^2\phi) + \sin^2\theta = 1$, and similarly the coefficient of $\partial^2 u/\partial y^2$ is 1. So Laplace's Equation in spherical coordinates is as stated.

8. The tangent plane to the surface $xy^2z^2 = 1$, at the point (x_0, y_0, z_0) is

$y_0^2 z_0^2(x - x_0) + 2x_0y_0z_0^2(y - y_0) + 2x_0y_0^2z_0(z - z_0) = 0 \;\Rightarrow$

$(y_0^2z_0^2)x + (2x_0y_0z_0^2)y + (2x_0y_0^2z_0)z = 5x_0y_0^2z_0^2 = 5$. Using the formula derived in Example 13.5.8 [ET 12.5.8], we find that the distance from $(0, 0, 0)$ to this tangent plane is

$$D(x_0, y_0, z_0) = \frac{|5x_0y_0^2z_0^2|}{\sqrt{(y_0^2z_0^2)^2 + (2x_0y_0z_0^2)^2 + (2x_0y_0^2z_0)^2}}.$$

When D is a maximum, D^2 is a maximum and $\nabla D^2 = \mathbf{0}$. Dropping the subscripts, let

$f(x, y, z) = D^2 = \dfrac{25(xyz)^2}{y^2z^2 + 4x^2z^2 + 4x^2y^2}$. Now use the fact that for points on the surface $xy^2z^2 = 1$ we have

$z^2 = \dfrac{1}{xy^2}$, to get $f(x, y) = D^2 = \dfrac{25x}{\dfrac{1}{x} + \dfrac{4x}{y^2} + 4x^2y^2} = \dfrac{25x^2y^2}{y^2 + 4x^2 + 4x^3y^4}$. Now $\nabla D^2 = \mathbf{0} \;\Rightarrow\; f_x = 0$

and $f_y = 0$. $f_x = 0 \;\Rightarrow\; \dfrac{50xy^2(y^2 + 4x^2 + 4x^3y^4) - (8x + 12x^2y^4)(25x^2y^2)}{(y^2 + 4x^2 + 4x^3y^4)^2} = 0 \;\Rightarrow$

$xy^2(y^2 + 4x^2 + 4x^3y^4) - (4x + 6x^2y^4)x^2y^2 = 0 \;\Rightarrow\; xy^4 - 2x^4y^6 = 0 \;\Rightarrow\; xy^4(1 - 2x^3y^2) = 0 \;\Rightarrow$

$1 = 2y^2x^3$ (since $x = 0$, $y = 0$ both give a minimum distance of 0). Also $f_y = 0 \;\Rightarrow$

$\dfrac{50x^2y(y^2 + 4x^2 + 4x^3y^4) - (2y + 16x^3y^3)25x^2y^2}{(y^2 + 4x^2 + 4x^3y^4)^2} = 0 \;\Rightarrow\; 4x^4y - 4x^5y^5 = 0 \;\Rightarrow\; x^4y(1 - xy^4) = 0$

$\Rightarrow\; 1 = xy^4$. Now substituting $x = 1/y^4$ into $1 = 2y^2x^3$, we get $1 = 2y^{-10} \;\Rightarrow\; y = \pm 2^{1/10} \;\Rightarrow$

$x = 2^{-2/5} \;\Rightarrow\; z^2 = \dfrac{1}{xy^2} = \dfrac{1}{(2^{-2/5})(2^{1/5})} = 2^{1/5} \;\Rightarrow\; z = \pm 2^{1/10}$.

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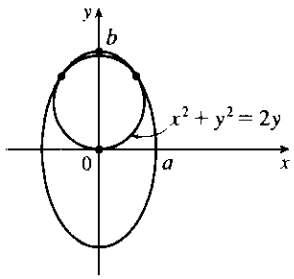
Therefore the tangent planes that are farthest from the origin are at the four points $(2^{-2/5}, \pm 2^{1/10}, \pm 2^{1/10})$. These points all give a maximum since the minimum distance occurs when $x_0 = 0$ or $y_0 = 0$ in which case $D = 0$. The equations are $(2^{1/5}2^{1/5})x \pm [(2)(2^{-2/5})(2^{1/10})(2^{1/5})]y \pm [(2)(2^{-2/5})(2^{1/5})(2^{1/10})]z = 5 \quad \Rightarrow$ $(2^{2/5})x \pm (2^{9/10})y \pm (2^{9/10})z = 5$.

9. Since we are minimizing the area of the ellipse, and the circle lies above the x-axis, the ellipse will intersect the circle for only one value of y. This y-value must satisfy both the equation of the circle and the equation of the ellipse. Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad x^2 = \dfrac{a^2}{b^2}(b^2 - y^2)$. Substituting into the equation of the circle gives $\dfrac{a^2}{b^2}(b^2 - y^2) + y^2 - 2y = 0 \quad \Rightarrow$



$\left(\dfrac{b^2 - a^2}{b^2}\right)y^2 - 2y + a^2 = 0$. In order for there to be only one solution to this quadratic equation, the discriminant must be 0, so $4 - 4a^2\dfrac{b^2 - a^2}{b^2} = 0 \quad \Rightarrow \quad b^2 - a^2b^2 + a^4 = 0$. The area of the ellipse is $A(a, b) = \pi ab$, and we minimize this function subject to the constraint $g(a, b) = b^2 - a^2b^2 + a^4 = 0$.

Now $\nabla A = \lambda \nabla g \quad \Leftrightarrow \quad \pi b = \lambda(4a^3 - 2ab^2), \pi a = \lambda(2b - 2ba^2) \quad \Rightarrow \quad$ (1) $\lambda = \dfrac{\pi b}{2a(2a^2 - b^2)}$,

(2) $\lambda = \dfrac{\pi a}{2b(1 - a^2)}$, (3) $b^2 - a^2b^2 + a^4 = 0$. Comparing (1) and (2) gives $\dfrac{\pi b}{2a(2a^2 - b^2)} = \dfrac{\pi a}{2b(1 - a^2)} \quad \Rightarrow$

$2\pi b^2 = 4\pi a^4 \quad \Leftrightarrow \quad a^2 = \frac{1}{\sqrt{2}}\, b$. Substitute this into (3) to get $b = \frac{3}{\sqrt{2}} \quad \Rightarrow \quad a = \sqrt{\frac{3}{2}}$.

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Chapter 16

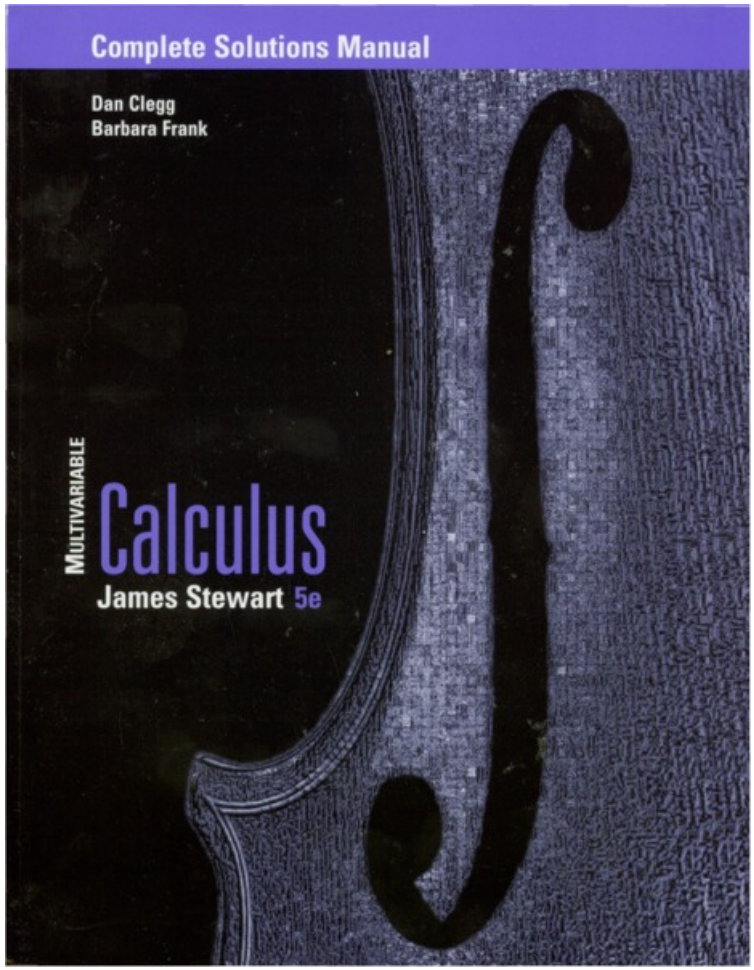
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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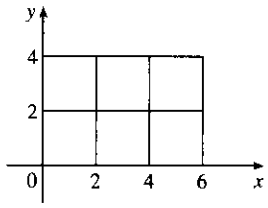
16.1 Double Integrals over Rectangles ET 15.1

1. (a) The subrectangles are shown in the figure.



The surface is the graph of $f(x, y) = xy$ and $\Delta A = 4$,

so we estimate

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i, y_j)\,\Delta A$$

$$= f(2,2)\,\Delta A + f(2,4)\,\Delta A + f(4,2)\,\Delta A + f(4,4)\,\Delta A + f(6,2)\,\Delta A + f(6,4)\,\Delta A$$

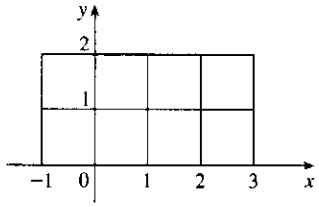
$$= 4(4) + 8(4) + 8(4) + 16(4) + 12(4) + 24(4) = 288$$

(b) $V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\,\Delta A$

$$= f(1,1)\,\Delta A + f(1,3)\,\Delta A + f(3,1)\,\Delta A + f(3,3)\,\Delta A + f(5,1)\,\Delta A + f(5,3)\,\Delta A$$

$$= 1(4) + 3(4) + 3(4) + 9(4) + 5(4) + 15(4) = 144$$

2. The subrectangles are shown in the figure.



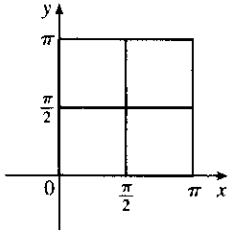
Since $\Delta A = 1$, we estimate

$$\iint_R (y^2 - 2x^2)\,dA \approx \sum_{i=1}^{4} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

$$= f(-1,1)\,\Delta A + f(-1,2)\,\Delta A + f(0,1)\,\Delta A + f(0,2)\,\Delta A$$
$$+ f(1,1)\,\Delta A + f(1,2)\,\Delta A + f(2,1)\,\Delta A + f(2,2)\,\Delta A$$

$$= -1(1) + 2(1) + 1(1) + 4(1) - 1(1) + 2(1) - 7(1) - 4(1) = -4$$

3. (a) The subrectangles are shown in the figure. Since $\Delta A = \pi^2/4$, we estimate

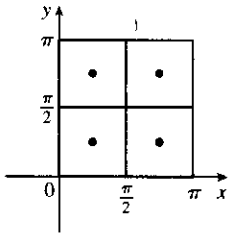


$$\iint_R \sin(x+y)\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

$$= f(0,0)\,\Delta A + f\left(0, \tfrac{\pi}{2}\right)\Delta A + f\left(\tfrac{\pi}{2}, 0\right)\Delta A + f\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)\Delta A$$

$$= 0\left(\tfrac{\pi^2}{4}\right) + 1\left(\tfrac{\pi^2}{4}\right) + 1\left(\tfrac{\pi^2}{4}\right) + 0\left(\tfrac{\pi^2}{4}\right) = \tfrac{\pi^2}{2} \approx 4.935$$

(b) $\iint_R \sin(x+y)\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\,\Delta A$



$$= f\left(\tfrac{\pi}{4}, \tfrac{\pi}{4}\right)\Delta A + f\left(\tfrac{\pi}{4}, \tfrac{3\pi}{4}\right)\Delta A$$
$$+ f\left(\tfrac{3\pi}{4}, \tfrac{\pi}{4}\right)\Delta A + f\left(\tfrac{3\pi}{4}, \tfrac{3\pi}{4}\right)\Delta A$$

$$= 1\left(\tfrac{\pi^2}{4}\right) + 0\left(\tfrac{\pi^2}{4}\right) + 0\left(\tfrac{\pi^2}{4}\right) + (-1)\left(\tfrac{\pi^2}{4}\right) = 0$$

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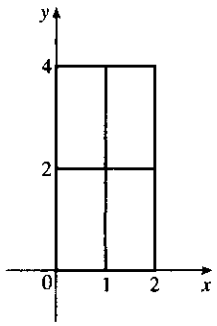
4. (a) The subrectangles are shown in the figure.



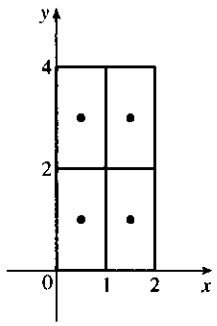
The surface is the graph of $f(x, y) = x + 2y^2$ and $\Delta A = 2$, so we estimate

$$V = \iint_R (x + 2y^2)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

$$= f(1,0)\, \Delta A + f(1,2)\, \Delta A + f(2,0)\, \Delta A + f(2,2)\, \Delta A$$

$$= 1(2) + 9(2) + 2(2) + 10(2) = 44$$

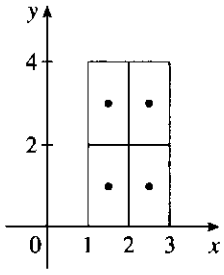
(b) $V = \iint_R (x + 2y^2)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\, \Delta A$



$$= f\left(\tfrac{1}{2}, 1\right) \Delta A + f\left(\tfrac{1}{2}, 3\right) \Delta A + f\left(\tfrac{3}{2}, 1\right) \Delta A + f\left(\tfrac{3}{2}, 3\right) \Delta A$$

$$= \tfrac{5}{2}(2) + \tfrac{37}{2}(2) + \tfrac{7}{2}(2) + \tfrac{39}{2}(2) = 88$$

5. (a) Each subrectangle and its midpoint are shown in the figure. The area of each subrectangle is $\Delta A = 2$, so we evaluate f at each midpoint and estimate

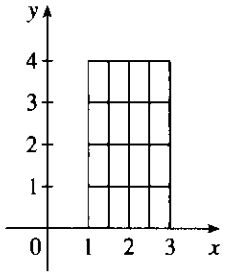


$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\, \Delta A$$

$$= f(1.5, 1)\, \Delta A + f(1.5, 3)\, \Delta A + f(2.5, 1)\, \Delta A + f(2.5, 3)\, \Delta A$$

$$= 1(2) + (-8)(2) + 5(2) + (-1)(2) = -6$$

(b) The subrectangles are shown in the figure. In each subrectangle, the sample point farthest from the origin is the upper right corner, and the area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus we estimate



$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{4} \sum_{j=1}^{4} f(x_i, y_j)\, \Delta A$$

$$= f(1.5, 1)\, \Delta A + f(1.5, 2)\, \Delta A + f(1.5, 3)\, \Delta A + f(1.5, 4)\, \Delta A$$
$$+ f(2, 1)\, \Delta A + f(2, 2)\, \Delta A + f(2, 3)\, \Delta A + f(2, 4)\, \Delta A$$
$$+ f(2.5, 1)\, \Delta A + f(2.5, 2)\, \Delta A + f(2.5, 3)\, \Delta A + f(2.5, 4)\, \Delta A$$
$$+ f(3, 1)\, \Delta A + f(3, 2)\, \Delta A + f(3, 3)\, \Delta A + f(3, 4)\, \Delta A$$

$$= 1\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right) + (-6)\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right) + (-5)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right)$$
$$+ 5\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + 8\left(\tfrac{1}{2}\right) + 6\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right)$$

$$= -3.5$$

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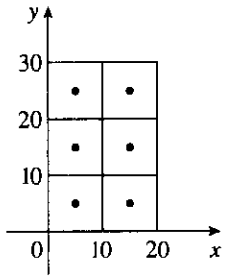
6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x, y)$ to be the depth of the water at (x, y), so the volume of water in the pool is the volume of the solid that lies above the rectangle $R = [0, 20] \times [0, 30]$ and below the graph of $f(x, y)$. We can estimate this volume using the Midpoint Rule with $m = 2$ and $n = 3$, so $\Delta A = 100$. Each subrectangle with its midpoint is shown in the figure. Then



$$\begin{aligned} V &\approx \sum_{i=1}^{2} \sum_{j=1}^{3} f(\bar{x}_i, \bar{y}_j)\, \Delta A \\ &= \Delta A[f(5,5) + f(5,15) + f(5,25) + f(15,5) + f(15,15) + f(15,25)] \\ &= 100(3 + 7 + 10 + 3 + 5 + 8) = 3600 \end{aligned}$$

Thus, we estimate that the pool contains 3600 cubic feet of water.

Alternatively, we can approximate the volume with a Riemann sum where $m = 4$, $n = 6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A = 25$ and

$$\begin{aligned} V &\approx \sum_{i=1}^{4} \sum_{j=1}^{6} f(x_i, y_j)\, \Delta A \\ &= 25[3 + 4 + 7 + 8 + 10 + 8 + 4 + 6 + 8 + 10 + 12 + 10 + 3 + 4 \\ &\qquad + 5 + 6 + 8 + 7 + 2 + 2 + 2 + 3 + 4 + 4] \\ &= 25(140) = 3500 \end{aligned}$$

So we estimate that the pool contains 3500 ft^3 of water.

7. The values of $f(x, y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)

8. From the level curves we see that $f\left(\frac{1}{2}, \frac{1}{2}\right) \approx 11$. So, using the Midpoint Rule with only one subrectangle, we get $\iint_R f(x, y)\, dA \approx 1 \cdot f\left(\frac{1}{2}, \frac{1}{2}\right) \approx 11$. Dividing R into four squares of equal size, we get $\iint_R f(x, y)\, dA \approx \frac{1}{4}\left[f\left(\frac{1}{4}, \frac{1}{4}\right) + f\left(\frac{1}{4}, \frac{3}{4}\right) + f\left(\frac{3}{4}, \frac{1}{4}\right) + f\left(\frac{3}{4}, \frac{3}{4}\right)\right] \approx \frac{1}{4}(11 + 13 + 9.5 + 11) \approx 11$.
Using sixteen squares we get the same result. So $\iint_R f(x, y)\, dA \approx 11$.

9. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

$$\begin{aligned} \iint_R f(x, y)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A = \Delta A[f(1,1) + f(1,3) + f(3,1) + f(3,3)] \\ &\approx 4(27 + 4 + 14 + 17) = 248 \end{aligned}$$

(b) $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA \approx \frac{1}{16}(248) = 15.5$

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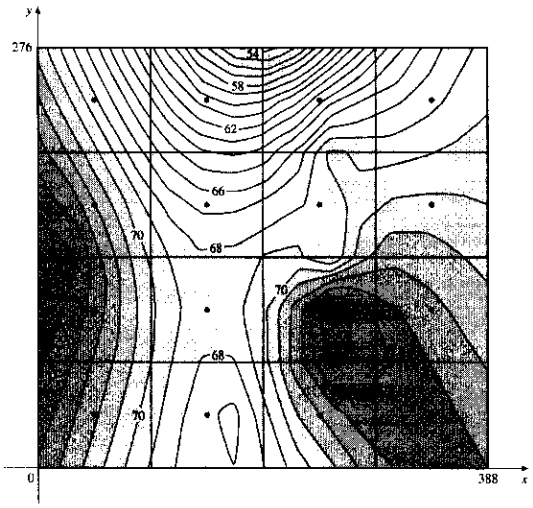
10. As in Example 4, we place the origin at the southwest corner of the state. Then $R = [0, 388] \times [0, 276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x, y)$ to be the temperature at the location (x, y). The average temperature is given by

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA = \frac{1}{388 \cdot 276} \iint_R f(x, y)\, dA$$

We can use the Midpoint Rule with $m = n = 4$ to give a reasonable estimate of the value of the double integral.



Thus, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated. The area of each subrectangle is $\Delta A = \frac{388}{4} \cdot \frac{276}{4} = 6693$, so using the contour map to estimate the function values at each midpoint, we have

$$\begin{aligned}\iint_R f(x, y)\, dA &\approx \sum_{i=1}^{4} \sum_{j=1}^{4} f(\overline{x}_i, \overline{y}_j)\, \Delta A \\ &\approx \Delta A\,[72.2 + 73.6 + 72.1 + 68.2 + 67.4 + 68.5 + 66.7 + 60.3 \\ &\qquad + 72.0 + 74.9 + 68.4 + 63.7 + 73.2 + 72.3 + 70.3 + 67.7] \\ &= 6693(1111.5)\end{aligned}$$

Therefore, $f_{\text{ave}} \approx \dfrac{6693 \cdot 1111.5}{388 \cdot 276} \approx 69.5$, so the average temperature in Colorado on May 1, 1996, was approximately 69.5°F.

Alternatively, we can use the Midpoint Rule with $m = n = 2$ which is easier computationally but will most likely be less accurate since we have fewer subrectangles. In this case, $\Delta A = \frac{388}{2} \cdot \frac{276}{2} = 26{,}772$ and we can use the same grid to estimate the function values at the midpoints of the four subrectangles. Then

$$\begin{aligned}\iint_R f(x, y)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\, \Delta A \approx 26{,}772[70.0 + 66.5 + 74.3 + 68.5] \\ &= 26{,}772 \cdot 279.3\end{aligned}$$

and $f_{\text{ave}} \approx \dfrac{26{,}772 \cdot 279.3}{388 \cdot 276} \approx 69.8$°F.

11. $z = 3 > 0$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 3$ and above the rectangle $[-2, 2] \times [1, 6]$. S is a rectangular solid, thus $\iint_R 3\, dA = 4 \cdot 5 \cdot 3 = 60$.

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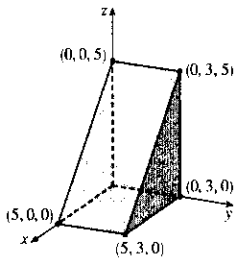
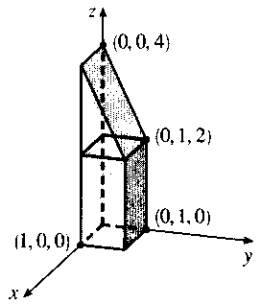
12. $z = 5 - x \geq 0$ for $0 \leq x \leq 5$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 5 - x$ and above the rectangle $[0, 5] \times [0, 3]$. S is a triangular cylinder whose volume is $3(\text{area of triangle}) = 3\left(\frac{1}{2} \cdot 5 \cdot 5\right) = 37.5$. Thus,

$$\iint_R (5 - x)\, dA = 37.5$$

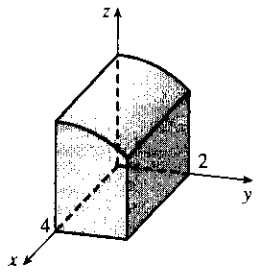


13. $z = f(x, y) = 4 - 2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0, 1] \times [0, 1] \times [0, 4]$ which lies below the plane $z = 4 - 2y$. So

$$\iint_R (4 - 2y)\, dA = (1)(1)(2) + \tfrac{1}{2}(1)(1)(2) = 3$$



14. Here $z = \sqrt{9 - y^2}$, so $z^2 + y^2 = 9$, $z \geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2 + y^2 = 9$ that lies above the rectangle $[0, 4] \times [0, 2]$.



15. To calculate the estimates using a programmable calculator, we can use an algorithm similar to that of Exercise 5.1.7 [ET 5.1.7]. In Maple, we can define the function $f(x, y) = e^{-x^2-y^2}$ (calling it `f`), load the `student` package, and then use the command

```
middlesum(middlesum(f,x=0..1,m),
                    y=0..1,m);
```

to get the estimate with $n = m^2$ squares of equal size. Mathematica has no special Riemann sum command, but we can define `f` and then use nested `Sum` commands to calculate the estimates.

n	estimate
1	0.6065
4	0.5694
16	0.5606
64	0.5585
256	0.5579
1024	0.5578

16.

n	estimate
1	0.9922
4	0.9262
16	0.8797

n	estimate
64	0.8660
256	0.8625
1024	0.8616

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17. If we divide R into mn subrectangles, $\iint_R k\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$ for any choice of sample points (x_{ij}^*, y_{ij}^*). But $f(x_{ij}^*, y_{ij}^*) = k$ always and $\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta A = \text{area of } R = (b-a)(d-c)$. Thus, no matter how we choose the sample points, $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A = k\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta A = k(b-a)(d-c)$ and so

$$\iint_R k\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A = \lim_{m,n\to\infty} k\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta A$$
$$= \lim_{m,n\to\infty} k(b-a)(d-c) = k(b-a)(d-c)$$

18. On R, $0 \le x+y \le 2 < \pi$ and $\sin\theta \ge 0$ for $0 \le \theta \le \pi$. Thus $f(x,y) = \sin(x+y) \ge 0$ for all $(x,y) \in R$. Since $0 \le \sin(x+y) \le 1$, Property (9) gives $\iint_R 0\,dA \le \iint_R \sin(x+y)\,dA \le \iint_R 1\,dA$, so by Exercise 17 we have $0 \le \iint_R \sin(x+y)\,dA \le 1$.

16.2 Iterated Integrals ET 15.2

1. $\int_0^3 (2x + 3x^2y)\,dx = \left[x^2 + x^3y\right]_{x=0}^{x=3} = (9+27y) - (0+0) = 9+27y,$

$$\int_0^4 (2x+3x^2y)\,dy = \left[2xy + 3x^2\frac{y^2}{2}\right]_{y=0}^{y=4} = \left(8x + 3x^2\cdot\frac{16}{2}\right) - (0+0) = 8x + 24x^2$$

2. $\displaystyle\int_0^3 \frac{y}{x+2}\,dx = y\ln|x+2|\,\Big]_{x=0}^{x=3} = y\ln 5 - y\ln 2 = y\ln\frac{5}{2},$

$$\int_0^4 \frac{y}{x+2}\,dy = \frac{1}{x+2}\left[\frac{y^2}{2}\right]_{y=0}^{y=4} = \frac{1}{x+2}\left(\frac{16}{2} - 0\right) = \frac{8}{x+2}$$

3. $\int_1^3 \int_0^1 (1+4xy)\,dx\,dy = \int_1^3 \left[x + 2x^2y\right]_{x=0}^{x=1} dy = \int_1^3 (1+2y)\,dy = \left[y+y^2\right]_1^3 = (3+9) - (1+1) = 10$

4. $\int_2^4 \int_{-1}^1 (x^2+y^2)\,dy\,dx = \int_2^4 \left[x^2y + \frac{1}{3}y^3\right]_{y=-1}^{y=1} dx = \int_2^4 \left[\left(x^2+\frac{1}{3}\right) - \left(-x^2-\frac{1}{3}\right)\right] dx$
$$= \int_2^4 \left(2x^2 + \frac{2}{3}\right) dx = \left[\frac{2}{3}x^3 + \frac{2}{3}x\right]_2^4 = \left(\frac{128}{3} + \frac{8}{3}\right) - \left(\frac{16}{3} + \frac{4}{3}\right) = \frac{116}{3}$$

5. $\int_0^2 \int_0^{\pi/2} x\sin y\,dy\,dx = \int_0^2 x\,dx \int_0^{\pi/2} \sin y\,dy$ [as in Example 5]

$$= \left[\frac{x^2}{2}\right]_0^2 \left[-\cos y\right]_0^{\pi/2} = (2-0)(0+1) = 2.$$

6. $\int_1^4 \int_0^2 \left(x + \sqrt{y}\right) dx\,dy = \int_1^4 \left[\frac{1}{2}x^2 + x\sqrt{y}\right]_{x=0}^{x=2} dy = \int_1^4 \left(2 + 2\sqrt{y}\right) dy$
$$= \left[2y + 2\cdot\frac{2}{3}y^{3/2}\right]_1^4 = \left(8 + \frac{4}{3}\cdot 8\right) - \left(2 + \frac{4}{3}\right) = \frac{46}{3}$$

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7. $\int_0^2 \int_0^1 (2x+y)^8\,dx\,dy = \int_0^2 \left[\frac{1}{2}\frac{(2x+y)^9}{9}\right]_{x=0}^{x=1} dy$ [substitute $u = 2x+y \Rightarrow dx = \frac{1}{2}du$]

$= \frac{1}{18}\int_0^2 [(2+y)^9 - (0+y)^9]\,dy = \frac{1}{18}\left[\frac{(2+y)^{10}}{10} - \frac{y^{10}}{10}\right]_0^2$

$= \frac{1}{180}[(4^{10} - 2^{10}) - (2^{10} - 0^{10})] = \frac{1{,}046{,}528}{180} = \frac{261{,}632}{45}$

8. $\int_0^1 \int_1^2 \frac{xe^x}{y}\,dy\,dx = \int_0^1 xe^x\,dx \int_1^2 \frac{1}{y}\,dy$ [as in Example 5]

$= \left[xe^x - e^x\right]_0^1 \left[\ln|y|\right]_1^2$ [by integrating by parts]

$= [(e-e) - (0-1)](\ln 2 - 0) = \ln 2$

9. $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\,dx = \int_1^4 \left[x\ln|y| + \frac{1}{x}\cdot\frac{1}{2}y^2\right]_{y=1}^{y=2} dx = \int_1^4 \left(x\ln 2 + \frac{3}{2x}\right) dx$

$= \left[\frac{1}{2}x^2\ln 2 + \frac{3}{2}\ln|x|\right]_1^4 = 8\ln 2 + \frac{3}{2}\ln 4 - \frac{1}{2}\ln 2$

$= \frac{15}{2}\ln 2 + 3\ln 4^{1/2} = \frac{21}{2}\ln 2$

10. $\int_1^2 \int_0^1 (x+y)^{-2}\,dx\,dy = \int_1^2 \left[-(x+y)^{-1}\right]_{x=0}^{x=1} dy = \int_1^2 \left[y^{-1} - (1+y)^{-1}\right] dy$

$= \left[\ln y - \ln(1+y)\right]_1^2 = \ln 2 - \ln 3 - 0 + \ln 2 = \ln\frac{4}{3}$

11. $\int_0^{\ln 2} \int_0^{\ln 5} e^{2x-y}\,dx\,dy = \left(\int_0^{\ln 5} e^{2x}\,dx\right)\left(\int_0^{\ln 2} e^{-y}\,dy\right) = \left[\frac{1}{2}e^{2x}\right]_0^{\ln 5}\left[-e^{-y}\right]_0^{\ln 2}$

$= \left(\frac{25}{2} - \frac{1}{2}\right)\left(-\frac{1}{2} + 1\right) = 6$

12. $\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx = \int_0^1 \left[x\sqrt{x^2+y^2+1}\right]_{y=0}^{y=1} dx = \int_0^1 x\left(\sqrt{x^2+2} - \sqrt{x^2+1}\right) dx$

$= \frac{1}{3}\left[(x^2+2)^{3/2} - (x^2+1)^{3/2}\right]_0^1 = \frac{1}{3}\left[(3^{3/2} - 2^{3/2}) - (2^{3/2} - 1)\right]$

$= \frac{1}{3}\left(3\sqrt{3} - 4\sqrt{2} + 1\right)$

13. $\iint_R (6x^2y^3 - 5y^4)\,dA = \int_0^3 \int_0^1 (6x^2y^3 - 5y^4)\,dy\,dx = \int_0^3 \left[\frac{3}{2}x^2y^4 - y^5\right]_{y=0}^{y=1} dx$

$= \int_0^3 \left(\frac{3}{2}x^2 - 1\right) dx = \left[\frac{1}{2}x^3 - x\right]_0^3 = \frac{27}{2} - 3 = \frac{21}{2}$

14. $\iint_R \cos(x+2y)\,dA = \int_0^{\pi} \int_0^{\pi/2} \cos(x+2y)\,dy\,dx$

$= \int_0^{\pi} \left[\frac{1}{2}\sin(x+2y)\right]_{y=0}^{y=\pi/2} dx = \frac{1}{2}\int_0^{\pi} (\sin(x+\pi) - \sin x)\,dx$

$= \frac{1}{2}\left[-\cos(x+\pi) + \cos x\right]_0^{\pi} = \frac{1}{2}\left[-\cos 2\pi + \cos\pi - (-\cos\pi + \cos 0)\right]$

$= \frac{1}{2}(-1 - 1 - (1+1)) = -2$

15. $\iint_R \frac{xy^2}{x^2+1}\,dA = \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1 \frac{x}{x^2+1}\,dx \int_{-3}^3 y^2\,dy$

$= \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 \left[\frac{1}{3}y^3\right]_{-3}^3 = \frac{1}{2}(\ln 2 - \ln 1)\cdot\frac{1}{3}(27+27) = 9\ln 2$

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16. $\displaystyle\iint_R \frac{1+x^2}{1+y^2}\,dA = \int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\,dy\,dx = \int_0^1 (1+x^2)\,dx \int_0^1 \frac{1}{1+y^2}\,dy$

$= \left[x+\frac{1}{3}x^3\right]_0^1 \left[\tan^{-1} y\right]_0^1 = \left(1+\frac{1}{3}-0\right)\left(\frac{\pi}{4}-0\right) = \frac{\pi}{3}$

17. $\int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\,dy\,dx$

$= \int_0^{\pi/6} \left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3} dx = \int_0^{\pi/6} \left[x\cos x - x\cos\left(x+\frac{\pi}{3}\right)\right] dx$

$= x\left[\sin x - \sin\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6} \left[\sin x - \sin\left(x+\frac{\pi}{3}\right)\right] dx$

[by integrating by parts separately for each term]

$= \frac{\pi}{6}\left[\frac{1}{2}-1\right] - \left[-\cos x + \cos\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6} = -\frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2}+0-\left(-1+\frac{1}{2}\right)\right]$

$= \frac{\sqrt{3}-1}{2} - \frac{\pi}{12}$

18. $\displaystyle\iint_R \frac{x}{1+xy}\,dA = \int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx$

$= \int_0^1 \left[\ln(1+xy)\right]_{y=0}^{y=1} dx = \int_0^1 \left[\ln(1+x) - \ln 1\right] dx$

$= \int_0^1 \ln(1+x)\,dx = \left[(1+x)\ln(1+x) - x\right]_0^1$ [by integrating by parts]

$= (2\ln 2 - 1) - (\ln 1 - 0) = 2\ln 2 - 1$

19. $\iint_R xye^{x^2y}\,dA = \int_0^2\int_0^1 xye^{x^2y}\,dx\,dy = \int_0^2 \left[\frac{1}{2}e^{x^2y}\right]_{x=0}^{x=1} dy = \frac{1}{2}\int_0^2 (e^y - 1)\,dy$

$= \frac{1}{2}\left[e^y - y\right]_0^2 = \frac{1}{2}[(e^2-2)-(1-0)] = \frac{1}{2}(e^2-3)$

20. $\displaystyle\int_0^1\int_1^2 \frac{x}{x^2+y^2}\,dx\,dy = \int_0^1 \left[\frac{1}{2}\ln(x^2+y^2)\right]_{x=1}^{x=2} dy = \frac{1}{2}\int_0^1 \left[\ln(4+y^2) - \ln(1+y^2)\right] dy$

To evaluate the first term, we integrate by parts with $u = \ln(4+y^2) \;\Rightarrow\; du = \dfrac{2y}{4+y^2}\,dy$ and $dv = dy \;\Rightarrow\; v = y$. Then

$$\int \ln(4+y^2)\,dy = y\ln(4+y^2) - \int \frac{2y^2}{4+y^2}\,dy = y\ln(4+y^2) - \int \left(2 - \frac{8}{4+y^2}\right) dy$$

$$= y\ln(4+y^2) - 2y + 8\cdot\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = y\ln(4+y^2) - 2y + 4\tan^{-1}\left(\frac{y}{2}\right)$$

Similarly, $\int \ln(1+y^2)\,dy = y\ln(1+y^2) - 2y + 2\tan^{-1} y$. Thus,

$$\int_0^1\int_1^2 \frac{x}{x^2+y^2}\,dx\,dy = \frac{1}{2}\int_0^1 \left[\ln(4+y^2) - \ln(1+y^2)\right] dy$$

$$= \frac{1}{2}\left[y\ln(4+y^2) - 2y + 4\tan^{-1}\left(\frac{y}{2}\right) - y\ln(1+y^2) + 2y - 2\tan^{-1} y\right]_0^1$$

$$= \frac{1}{2}\left[\left(\ln 5 + 4\tan^{-1}\left(\tfrac{1}{2}\right) - \ln 2 - 2\tan^{-1} 1\right) - 0\right]$$

$$= \frac{1}{2}\left[\ln 5 - \ln 2 + 4\tan^{-1}\left(\tfrac{1}{2}\right) - 2\left(\tfrac{\pi}{4}\right)\right] = \frac{1}{2}\ln\tfrac{5}{2} + 2\tan^{-1}\left(\tfrac{1}{2}\right) - \frac{\pi}{4}$$

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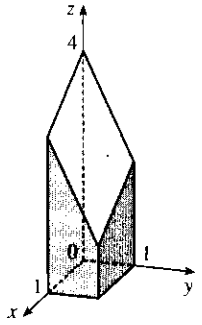
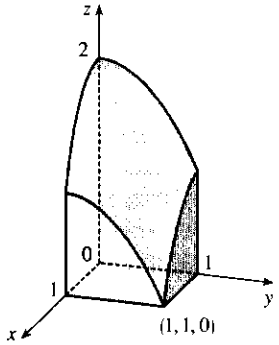
21. $z = f(x, y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0, 1] \times [0, 1]$.



22. $z = 2 - x^2 - y^2 \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z = 2 - x^2 - y^2$ and above $[0, 1] \times [0, 1]$.



23. $V = \iint_R (12 - 3x - 2y)\,dA = \int_{-2}^{3} \int_{0}^{1} (12 - 3x - 2y)\,dx\,dy = \int_{-2}^{3} \left[12x - \frac{3}{2}x^2 - 2xy\right]_{x=0}^{x=1} dy$
$= \int_{-2}^{3} \left(\frac{21}{2} - 2y\right) dy = \left[\frac{21}{2}y - y^2\right]_{-2}^{3} = \frac{95}{2}$

24. $V = \iint_R (4 + x^2 - y^2)\,dA = \int_{-1}^{1} \int_{0}^{2} (4 + x^2 - y^2)\,dy\,dx = \int_{-1}^{1} \left[4y + x^2 y - \frac{1}{3}y^3\right]_{y=0}^{y=2} dx$
$= \int_{-1}^{1} \left(2x^2 + \frac{16}{3}\right) dx = \left[\frac{2}{3}x^3 + \frac{16}{3}x\right]_{-1}^{1} = \frac{2}{3} + \frac{16}{3} + \frac{2}{3} + \frac{16}{3} = 12$

25. $V = \int_{-2}^{2} \int_{-1}^{1} \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dx\,dy = 4 \int_{0}^{2} \int_{0}^{1} \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dx\,dy$
$= 4 \int_{0}^{2} \left[x - \frac{1}{12}x^3 - \frac{1}{9}y^2 x\right]_{x=0}^{x=1} dy = 4 \int_{0}^{2} \left(\frac{11}{12} - \frac{1}{9}y^2\right) dy = 4\left[\frac{11}{12}y - \frac{1}{27}y^3\right]_{0}^{2} = 4 \cdot \frac{83}{54} = \frac{166}{27}$

26. $V = \int_{-1}^{1} \int_{0}^{\pi} (1 + e^x \sin y)\,dy\,dx = \int_{-1}^{1} \left[y - e^x \cos y\right]_{y=0}^{y=\pi} dx = \int_{-1}^{1} (\pi + e^x - 0 + e^x)\,dx$
$= \int_{-1}^{1} (\pi + 2e^x)\,dx = \left[\pi x + 2e^x\right]_{-1}^{1} = 2\pi + 2e - \frac{2}{e}$

27. Here we need the volume of the solid lying under the surface $z = x\sqrt{x^2 + y}$ and above the square $R = [0, 1] \times [0, 1]$ in the xy-plane.

$$V \int_0^1 \int_0^1 x\sqrt{x^2 + y}\,dx\,dy = \int_0^1 \tfrac{1}{3}\left[(x^2 + y)^{3/2}\right]_{x=0}^{x=1} dy = \tfrac{1}{3}\int_0^1 \left[(1 + y)^{3/2} - y^{3/2}\right] dy$$
$$= \tfrac{1}{3} \cdot \tfrac{2}{5}\left[(1 + y)^{5/2} - y^{5/2}\right]_0^1 = \tfrac{4}{15}\left(2\sqrt{2} - 1\right)$$

28. Here we need the volume of the solid lying under the surface $z = 1 + (x - 1)^2 + 4y^2$ and above the rectangle $R = [0, 3] \times [0, 2]$ in the xy-plane.

$$V = \int_0^3 \int_0^2 \left[1 + (x - 1)^2 + 4y^2\right] dy\,dx = \int_0^3 \left[y + (x - 1)^2 y + \tfrac{4}{3}y^3\right]_{y=0}^{y=2} dx$$
$$= \int_0^3 \left[2 + 2(x - 1)^2 + \tfrac{32}{3}\right] dx = \left[\tfrac{38}{3}x + \tfrac{2}{3}(x - 1)^3\right]_0^3 = 44$$

29. In the first octant, $z \geq 0 \quad \Rightarrow \quad y \leq 3$, so
$V = \int_0^3 \int_0^2 (9 - y^2)\,dx\,dy = \int_0^3 \left[9x - y^2 x\right]_{x=0}^{x=2} dy = \int_0^3 (18 - 2y^2)\,dy = \left[18y - \frac{2}{3}y^3\right]_0^3 = 36$

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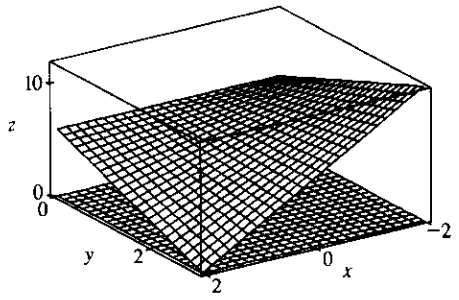
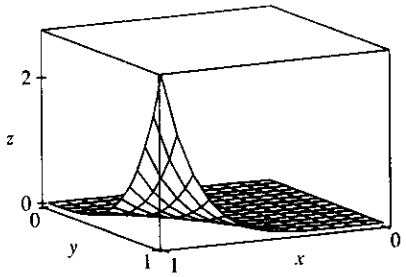
30. (a) Here we need the volume of the solid lying under the surface $z = 6 - xy$ and above the rectangle $R = [-2, 2] \times [0, 3]$ in the xy-plane.

$$\begin{aligned} V &= \int_{-2}^{2} \int_{0}^{3} (6 - xy)\, dy\, dx \\ &= \int_{-2}^{2} \left[6y - \tfrac{1}{2}xy^2\right]_{y=0}^{y=3} dx \\ &= \int_{-2}^{2} \left(18 - \tfrac{9}{2}x\right) dx \\ &= \left[18x - \tfrac{9}{4}x^2\right]_{-2}^{2} = 72 \end{aligned}$$

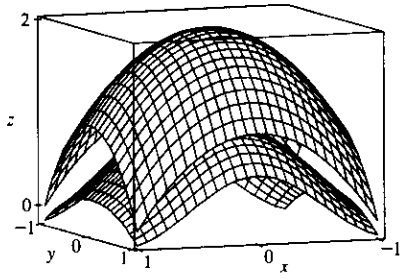
(b) The solid occupies the region between the two surfaces shown.



31. In Maple, we can calculate the integral by defining the integrand as `f` and then using the command `int(int(f,x=0..1),y=0..1);`. In Mathematica, we can use the command `Integrate[Integrate[f,{x,0,1}],{y,0,1}]`. We find that $\iint_R x^5 y^3 e^{xy}\, dA = 21e - 57 \approx 0.0839$. We can use `plot3d` (in Maple) or `Plot3d` (in Mathematica) to graph the function.



32. In Maple, we can calculate the integral by defining `f:=exp(-x^2)*cos(x^2+y^2);` and `g:=2-x^2-y^2;` and then [since $2 - x^2 - y^2 > e^{-x^2}\cos(x^2 + y^2)$ for $-1 \le x \le 1$, $-1 \le y \le 1$] using the command `evalf(int(int(g-f,x=-1..1),y=-1..1),5);`. In Mathematica, we can use the command `N[Integrate[Integrate[f,{x,0,1}],{y,0,1}],5]`.



In each of these commands, the `5` indicates that we want only five significant digits; this speeds up the calculation considerably. We find that $\iint_R \left[(2 - x^2 - y^2) - \left(e^{-x^2}\cos(x^2 + y^2)\right)\right] dA \approx 3.0271$. We can use the `plot3d` command (in Maple) or `Plot3d` (in Mathematica) to graph both functions on the same screen.

33. R is the rectangle $[-1, 1] \times [0, 5]$. Thus, $A(R) = 2 \cdot 5 = 10$ and

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA = \tfrac{1}{10} \int_0^5 \int_{-1}^1 x^2 y\, dx\, dy = \tfrac{1}{10} \int_0^5 \left[\tfrac{1}{3}x^3 y\right]_{x=-1}^{x=1} dy = \tfrac{1}{10} \int_0^5 \tfrac{2}{3} y\, dy$$

$$= \tfrac{1}{10}\left[\tfrac{1}{3}y^2\right]_0^5 = \tfrac{5}{6}$$

34. $A(R) = 4 \cdot 1 = 4$, so

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA = \tfrac{1}{4} \int_0^4 \int_0^1 e^y \sqrt{x + e^y}\, dy\, dx = \tfrac{1}{4} \int_0^4 \left[\tfrac{2}{3}(x + e^y)^{3/2}\right]_{y=0}^{y=1} dx$$

$$= \tfrac{1}{4} \cdot \tfrac{2}{3} \int_0^4 \left[(x + e)^{3/2} - (x + 1)^{3/2}\right] dx = \tfrac{1}{6}\left[\tfrac{2}{5}(x + e)^{5/2} - \tfrac{2}{5}(x + 1)^{5/2}\right]_0^4$$

$$= \tfrac{1}{6} \cdot \tfrac{2}{5}[(4 + e)^{5/2} - 5^{5/2} - e^{5/2} + 1] = \tfrac{1}{15}[(4 + e)^{5/2} - e^{5/2} - 5^{5/2} + 1] \approx 3.327$$

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35. Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{1}{2}$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\frac{1}{2}$.

To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

36. (a) Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second-order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).

(b) To find g_{xy}, we first hold y constant and use the single-variable Fundamental Theorem of Calculus, Part 1: $g_x = \dfrac{d}{dx}\,g(x,y) = \dfrac{d}{dx}\displaystyle\int_a^x \left(\int_c^y f(s,t)\,dt\right) ds = \int_c^y f(x,t)\,dt$. Now we use the Fundamental Theorem again: $g_{xy} = \dfrac{d}{dy}\displaystyle\int_c^y f(x,t)\,dt = f(x,y)$.

To find g_{yx}, we first use Fubini's Theorem to find that $\int_a^x \int_c^y f(s,t)\,dt\,ds = \int_c^y \int_a^x f(s,t)\,dt\,ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx} = f(x,y)$. So $g_{xy} = g_{yx} = f(x,y)$.

16.3 Double Integrals over General Regions ET 15.3

1. $\displaystyle\int_0^1 \int_0^{x^2} (x+2y)\,dy\,dx = \int_0^1 \left[xy+y^2\right]_{y=0}^{y=x^2} dx = \int_0^1 \left[x(x^2)+(x^2)^2-0-0\right] dx$

$= \displaystyle\int_0^1 (x^3+x^4)\,dx = \left[\tfrac{1}{4}x^4+\tfrac{1}{5}x^5\right]_0^1 = \tfrac{9}{20}$

2. $\displaystyle\int_1^2 \int_y^2 xy\,dx\,dy = \int_1^2 \left[\tfrac{1}{2}x^2y\right]_{x=y}^{x=2} dy = \int_1^2 \tfrac{1}{2}y(4-y^2)\,dy = \tfrac{1}{2}\int_1^2 (4y-y^3)\,dy$

$= \tfrac{1}{2}\left[2y^2-\tfrac{1}{4}y^4\right]_1^2 = \tfrac{1}{2}\left(8-4-2+\tfrac{1}{4}\right) = \tfrac{9}{8}$

3. $\displaystyle\int_0^1 \int_y^{e^y} \sqrt{x}\,dx\,dy = \int_0^1 \left[\tfrac{2}{3}x^{3/2}\right]_{x=y}^{x=e^y} dy = \tfrac{2}{3}\int_0^1 (e^{3y/2}-y^{3/2})\,dy = \tfrac{2}{3}\left[\tfrac{2}{3}e^{3y/2}-\tfrac{2}{5}y^{5/2}\right]_0^1$

$= \tfrac{2}{3}\left(\tfrac{2}{3}e^{3/2}-\tfrac{2}{5}-\tfrac{2}{3}e^0+0\right) = \tfrac{4}{9}e^{3/2}-\tfrac{32}{45}$

4. $\displaystyle\int_0^1 \int_x^{2-x} (x^2-y)\,dy\,dx = \int_0^1 \left[x^2y-\tfrac{1}{2}y^2\right]_{y=x}^{y=2-x} dx = \int_0^1 \left[x^2(2-x)-\tfrac{1}{2}(2-x)^2-x^2(x)+\tfrac{1}{2}x^2\right] dx$

$= \displaystyle\int_0^1 (-2x^3+2x^2+2x-2)\,dx = \left[-\tfrac{1}{2}x^4+\tfrac{2}{3}x^3+x^2-2x\right]_0^1 = -\tfrac{5}{6}$

5. $\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta} dr\,d\theta = \int_0^{\pi/2} \left[re^{\sin\theta}\right]_{r=0}^{r=\cos\theta} d\theta = \int_0^{\pi/2} (\cos\theta)\,e^{\sin\theta} d\theta = e^{\sin\theta}\Big]_0^{\pi/2}$

$= e^{\sin(\pi/2)} - e^0 = e-1$

6. $\displaystyle\int_0^1 \int_0^v \sqrt{1-v^2}\,du\,dv = \int_0^1 \left[u\sqrt{1-v^2}\right]_{u=0}^{u=v} dv = \int_0^1 v\sqrt{1-v^2}\,dv = -\tfrac{1}{3}(1-v^2)^{3/2}\Big]_0^1$

$= -\tfrac{1}{3}(0-1) = \tfrac{1}{3}$

7. $\displaystyle\iint_D x^3y^2\,dA = \int_0^2 \int_{-x}^{x} x^3y^2\,dy\,dx = \int_0^2 \left[\tfrac{1}{3}x^3y^3\right]_{y=-x}^{y=x} dx = \tfrac{1}{3}\int_0^2 2x^6\,dx$

$= \tfrac{2}{3}\left[\tfrac{1}{7}x^7\right]_0^2 = \tfrac{2}{21}\left[2^7-0\right] = \tfrac{256}{21}$

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8. $\displaystyle\iint_D \frac{4y}{x^3+2}\,dA = \int_1^2 \int_0^{2x} \frac{4y}{x^3+2}\,dy\,dx = \int_1^2 \left[\frac{2y^2}{x^3+2}\right]_{y=0}^{y=2x} dx = \int_1^2 \frac{8x^2}{x^3+2}\,dx$

$= \frac{8}{3}\ln\left|x^3+2\right|\Big]_1^2 = \frac{8}{3}(\ln 10 - \ln 3) = \frac{8}{3}\ln\frac{10}{3}$

9. $\displaystyle\int_0^1 \int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx = \int_0^1 \left[\frac{y^2}{x^2+1}\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \frac{x}{x^2+1}\,dx$

$= \frac{1}{2}\ln\left|x^2+1\right|\Big]_0^1 = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$

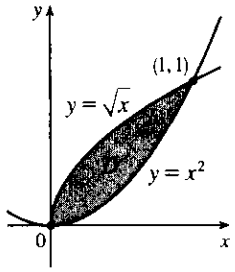
10. $\int_0^1 \int_0^y e^{y^2}\,dx\,dy = \int_0^1 \left[xe^{y^2}\right]_{x=0}^{x=y} dy = \int_0^1 ye^{y^2}\,dy = \frac{1}{2}e^{y^2}\Big]_0^1 = \frac{1}{2}(e-1)$

11. $\int_1^2 \int_y^{y^3} e^{x/y}\,dx\,dy = \int_1^2 \left[ye^{x/y}\right]_{x=y}^{x=y^3} dy = \int_1^2 \left(ye^{y^2} - ey\right) dy = \left[\frac{1}{2}e^{y^2} - \frac{1}{2}ey^2\right]_1^2 = \frac{1}{2}(e^4 - 4e)$

12. $\int_0^1 \int_0^y x\sqrt{y^2-x^2}\,dx\,dy = \int_0^1 \left[-\frac{1}{3}(y^2-x^2)^{3/2}\right]_{x=0}^{x=y} dy = \frac{1}{3}\int_0^1 y^3\,dy = \frac{1}{3}\cdot\frac{1}{4}y^4\Big]_0^1 = \frac{1}{12}$

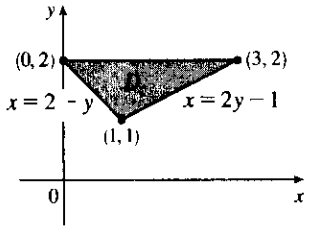
13. $\int_0^1 \int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 \left[x\sin y\right]_{y=0}^{y=x^2} dx = \int_0^1 x\sin x^2\,dx = -\frac{1}{2}\cos x^2\Big]_0^1 = \frac{1}{2}(1-\cos 1)$

14.



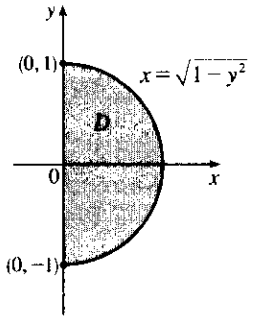
$$\int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 \left[xy + \tfrac{1}{2}y^2\right]_{y=x^2}^{y=\sqrt{x}} dx$$
$$= \int_0^1 \left(x^{3/2} + \tfrac{1}{2}x - x^3 - \tfrac{1}{2}x^4\right) dx$$
$$= \left[\tfrac{2}{5}x^{5/2} + \tfrac{1}{4}x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{10}x^5\right]_0^1 = \tfrac{3}{10}$$

15.



$$\int_1^2 \int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2 \left[xy^3\right]_{x=2-y}^{x=2y-1} dy$$
$$= \int_1^2 \left[(2y-1) - (2-y)\right] y^3\,dy$$
$$= \int_1^2 (3y^4 - 3y^3)\,dy = \left[\tfrac{3}{5}y^5 - \tfrac{3}{4}y^4\right]_1^2$$
$$= \tfrac{96}{5} - 12 - \tfrac{3}{5} + \tfrac{3}{4} = \tfrac{147}{20}$$

16.



$$\iint_D xy^2\,dA = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} xy^2\,dx\,dy$$
$$= \int_{-1}^1 y^2\left[\tfrac{1}{2}x^2\right]_{x=0}^{x=\sqrt{1-y^2}} dy = \tfrac{1}{2}\int_{-1}^1 y^2(1-y^2)\,dy$$
$$= \tfrac{1}{2}\int_{-1}^1 (y^2 - y^4)\,dy = \tfrac{1}{2}\left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5\right]_{-1}^1$$
$$= \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{2}{15}$$

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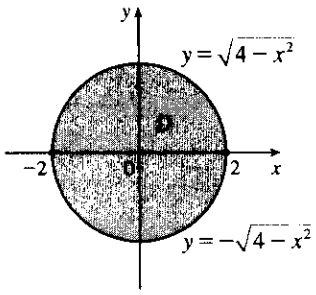
17.



$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x - y)\, dy\, dx$$

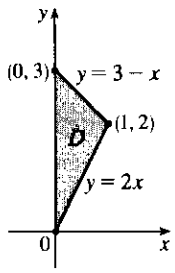
$$= \int_{-2}^{2} \left[2xy - \frac{1}{2}y^2 \right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left[2x\sqrt{4-x^2} - \tfrac{1}{2}(4 - x^2) + 2x\sqrt{4-x^2} + \tfrac{1}{2}(4 - x^2) \right] dx$$

$$= \int_{-2}^{2} 4x\sqrt{4-x^2}\, dx = -\tfrac{4}{3}(4 - x^2)^{3/2} \Big]_{-2}^{2} = 0$$

(Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^{2} 4x\sqrt{4-x^2}\, dx = 0$.)

18.



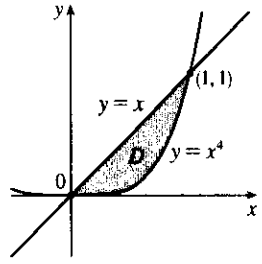
$$\iint_D 2xy\, dA = \int_0^1 \int_{2x}^{3-x} 2xy\, dy\, dx = \int_0^1 \left[xy^2 \right]_{y=2x}^{y=3-x} dx$$

$$= \int_0^1 x[(3 - x)^2 - (2x)^2]\, dx$$

$$= \int_0^1 (-3x^3 - 6x^2 + 9x)\, dx$$

$$= \left[-\tfrac{3}{4}x^4 - 2x^3 + \tfrac{9}{2}x^2 \right]_0^1 = -\tfrac{3}{4} - 2 + \tfrac{9}{2} = \tfrac{7}{4}$$

19.

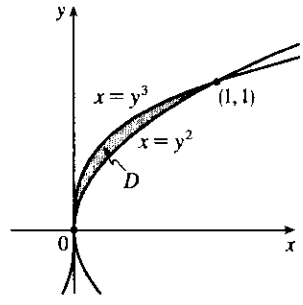


$$V = \int_0^1 \int_{x^4}^{x} (x + 2y)\, dy\, dx$$

$$= \int_0^1 \left[xy + y^2 \right]_{y=x^4}^{y=x} dx = \int_0^1 (2x^2 - x^5 - x^8)\, dx$$

$$= \left[\tfrac{2}{3}x^3 - \tfrac{1}{6}x^6 - \tfrac{1}{9}x^9 \right]_0^1 = \tfrac{2}{3} - \tfrac{1}{6} - \tfrac{1}{9} = \tfrac{7}{18}$$

20.

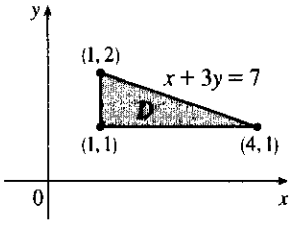


$$V = \int_0^1 \int_{y^3}^{y^2} (2x + y^2)\, dx\, dy$$

$$= \int_0^1 \left[x^2 + xy^2 \right]_{x=y^3}^{x=y^2} dy = \int_0^1 (2y^4 - y^6 - y^5)\, dy$$

$$= \left[\tfrac{2}{5}y^5 - \tfrac{1}{7}y^7 - \tfrac{1}{6}y^6 \right]_0^1 = \tfrac{19}{210}$$

21.



$$V = \int_1^2 \int_1^{7-3y} xy\, dx\, dy = \int_1^2 \left[\tfrac{1}{2}x^2 y \right]_{x=1}^{x=7-3y} dy$$

$$= \tfrac{1}{2} \int_1^2 (48y - 42y^2 + 9y^3)\, dy$$

$$= \tfrac{1}{2} \left[24y^2 - 14y^3 + \tfrac{9}{4}y^4 \right]_1^2 = \tfrac{31}{8}$$

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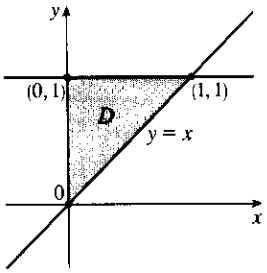
22.

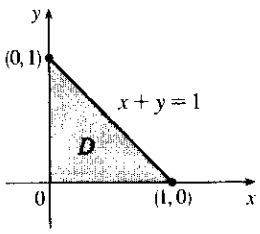


$$V = \int_0^1 \int_x^1 (x^2 + 3y^2)\, dy\, dx$$

$$= \int_0^1 \left[x^2 y + y^3\right]_{y=x}^{y=1} dx = \int_0^1 (x^2 + 1 - 2x^3)\, dx$$

$$= \left[\tfrac{1}{3}x^3 + x - \tfrac{1}{2}x^4\right]_0^1 = \tfrac{5}{6}$$

23.



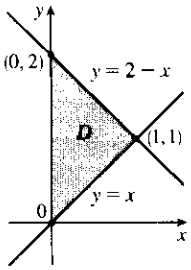
$$V = \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left[(1-x)^2 - \tfrac{1}{2}(1-x)^2\right] dx$$

$$= \int_0^1 \tfrac{1}{2}(1-x)^2\, dx = \left[-\tfrac{1}{6}(1-x)^3\right]_0^1 = \tfrac{1}{6}$$

24.

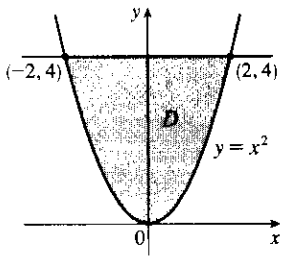


$$V = \int_0^1 \int_x^{2-x} x\, dy\, dx$$

$$= \int_0^1 x\left[y\right]_{y=x}^{y=2-x} dx = \int_0^1 (2x - 2x^2)\, dx$$

$$= \left[x^2 - \tfrac{2}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

25.

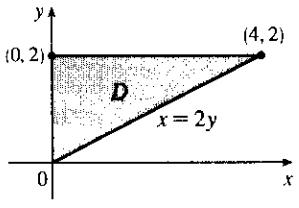


$$V = \int_{-2}^2 \int_{x^2}^4 x^2\, dy\, dx$$

$$= \int_{-2}^2 x^2\left[y\right]_{y=x^2}^{y=4} dx = \int_{-2}^2 (4x^2 - x^4)\, dx$$

$$= \left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_{-2}^2 = \tfrac{32}{3} - \tfrac{32}{5} + \tfrac{32}{3} - \tfrac{32}{5} = \tfrac{128}{15}$$

26.

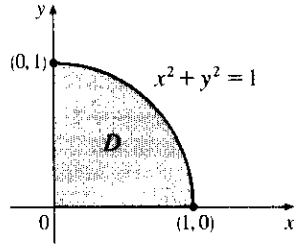


$$V = \int_0^2 \int_0^{2y} \sqrt{4 - y^2}\, dx\, dy$$

$$= \int_0^2 \left[x\sqrt{4 - y^2}\right]_{x=0}^{x=2y} dy = \int_0^2 2y\sqrt{4 - y^2}\, dy$$

$$= \left[-\tfrac{2}{3}(4 - y^2)^{3/2}\right]_0^2 = 0 + \tfrac{16}{3} = \tfrac{16}{3}$$

27.



$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} y\, dy\, dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1 - x^2}{2}\, dx = \tfrac{1}{2}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

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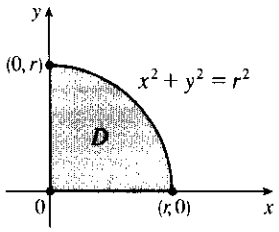
28.



By symmetry, the desired volume V is 8 times the volume V_1 in the first octant. Now

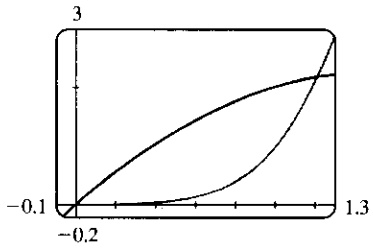
$$V_1 = \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\,dx\,dy$$

$$= \int_0^r \left[x\sqrt{r^2-y^2}\right]_{x=0}^{x=\sqrt{r^2-y^2}} dy$$

$$= \int_0^r (r^2-y^2)\,dy = \left[r^2y - \tfrac{1}{3}y^3\right]_0^r = \tfrac{2}{3}r^3$$

Thus $V = \frac{16}{3}r^3$.

29.



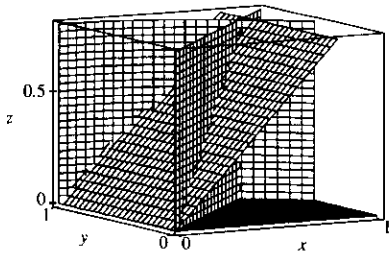
From the graph, it appears that the two curves intersect at $x = 0$ and at $x \approx 1.213$. Thus the desired integral is

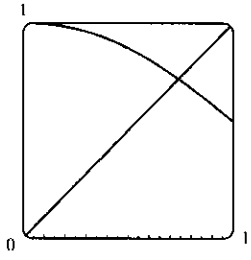
$$\iint_D x\,dA \approx \int_0^{1.213} \int_{x^4}^{3x-x^2} x\,dy\,dx = \int_0^{1.213} \left[xy\right]_{y=x^4}^{y=3x-x^2} dx$$

$$= \int_0^{1.213} (3x^2 - x^3 - x^5)\,dx = \left[x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^{1.213}$$

$$\approx 0.713$$

30.





The desired solid is shown in the first graph. From the second graph, we estimate that $y = \cos x$ intersects $y = x$ at $x \approx 0.7391$. Therefore the volume of the solid is

$$V \approx \int_0^{0.7391} \int_x^{\cos x} z\,dy\,dx = \int_0^{0.7391} \int_x^{\cos x} x\,dy\,dx$$

$$= \int_0^{0.7391} \left[xy\right]_{y=x}^{y=\cos x} dx = \int_0^{0.7391} (x\cos x - x^2)\,dx$$

$$= \left[\cos x + x\sin x - \tfrac{1}{3}x^3\right]_0^{0.7391} \approx 0.1024$$

Note: There is a different solid which can also be construed to satisfy the conditions stated in the exercise. This is the solid bounded by all of the given surfaces, as well as the plane $y = 0$. In case you calculated the volume of this solid and want to check your work, its volume is $V \approx \int_0^{0.7391} \int_0^x x\,dy\,dx + \int_{0.7391}^{\pi/2} \int_0^{\cos x} x\,dy\,dx \approx 0.4684$.

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31. The two bounding curves $y = 1 - x^2$ and $y = x^2 - 1$ intersect at $(\pm 1, 0)$ with $1 - x^2 \geq x^2 - 1$ on $[-1, 1]$. Within this region, the plane $z = 2x + 2y + 10$ is above the plane $z = 2 - x - y$, so

$$V = \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2x + 2y + 10)\, dy\, dx - \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2 - x - y)\, dy\, dx$$

$$= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2x + 2y + 10 - (2 - x - y))\, dy\, dx = \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (3x + 3y + 8)\, dy\, dx$$

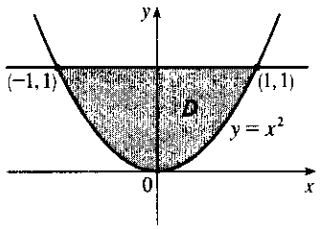
$$= \int_{-1}^{1} \left[3xy + \tfrac{3}{2}y^2 + 8y\right]_{y=x^2-1}^{y=1-x^2} dx$$

$$= \int_{-1}^{1} \left[3x(1 - x^2) + \tfrac{3}{2}(1 - x^2)^2 + 8(1 - x^2) - 3x(x^2 - 1) - \tfrac{3}{2}(x^2 - 1)^2 - 8(x^2 - 1)\right] dx$$

$$= \int_{-1}^{1} (-6x^3 - 16x^2 + 6x + 16)\, dx = \left[-\tfrac{3}{2}x^4 - \tfrac{16}{3}x^3 + 3x^2 + 16x\right]_{-1}^{1}$$

$$= -\tfrac{3}{2} - \tfrac{16}{3} + 3 + 16 + \tfrac{3}{2} - \tfrac{16}{3} - 3 + 16 = \tfrac{64}{3}$$

32. The two planes intersect in the line $y = 1$, $z = 3$, so the region of integration is the plane region enclosed by the parabola $y = x^2$ and the line $y = 1$. We have $2 + y \geq 3y$ for $0 \leq y \leq 1$, so the solid region is bounded above by $z = 2 + y$ and bounded below by $z = 3y$.



$$V = \int_{-1}^{1} \int_{x^2}^{1} (2 + y)\, dy\, dx - \int_{-1}^{1} \int_{x^2}^{1} (3y)\, dy\, dx = \int_{-1}^{1} \int_{x^2}^{1} (2 + y - 3y)\, dy\, dx = \int_{-1}^{1} \int_{x^2}^{1} (2 - 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[2y - y^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^{1} (1 - 2x^2 + x^4)\, dx = x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\Big]_{-1}^{1} = \tfrac{16}{15}$$

33. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume is

$$V = \int_0^2 \int_{x^3-x}^{x^2+x} z\, dy\, dx = \int_0^2 \int_{x^3-x}^{x^2+x} (x^3y^4 + xy^2)\, dy\, dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

34. For $|x| \leq 1$ and $|y| \leq 1$, $2x^2 + y^2 < 8 - x^2 - 2y^2$. Also, the cylinder is described by the inequalities $-1 \leq x \leq 1$, $-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$. So the volume is given by

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[(8 - x^2 - 2y^2) - (2x^2 + y^2)\right] dy\, dx = \frac{13\pi}{2} \qquad \text{[using a CAS]}$$

35. The two surfaces intersect in the circle $x^2 + y^2 = 1$, $z = 0$ and the region of integration is the disk D: $x^2 + y^2 \leq 1$. Using a CAS, the volume is $\iint_D (1 - x^2 - y^2)\, dA = \displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \frac{\pi}{2}$.

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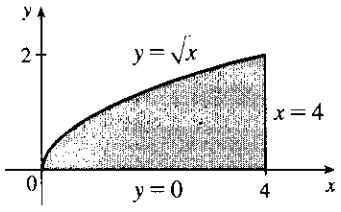
36. The projection onto the xy-plane of the intersection of the two surfaces is the circle $x^2 + y^2 = 2y \Rightarrow x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y-1)^2 = 1$, so the region of integration is given by $-1 \le x \le 1$, $1 - \sqrt{1-x^2} \le y \le 1 + \sqrt{1-x^2}$. In this region, $2y \ge x^2 + y^2$ so, using a CAS, the volume is

$$V = \int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} [2y - (x^2 + y^2)]\, dy\, dx = \frac{\pi}{2}.$$

37.



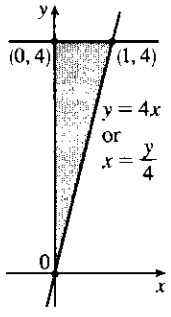
Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le \sqrt{x}, 0 \le x \le 4\}$$
$$= \{(x,y) \mid y^2 \le x \le 4, 0 \le y \le 2\}$$

we have

$$\int_0^4 \int_0^{\sqrt{x}} f(x,y)\, dy\, dx = \iint_D f(x,y)\, dA = \int_0^2 \int_{y^2}^4 f(x,y)\, dx\, dy.$$

38.



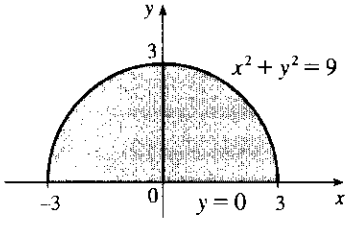
Because the region of integration is

$$D = \{(x,y) \mid 4x \le y \le 4, 0 \le x \le 1\}$$
$$= \left\{(x,y) \mid 0 \le x \le \tfrac{y}{4}, 0 \le y \le 4\right\}$$

we have

$$\int_0^1 \int_{4x}^4 f(x,y)\, dy\, dx = \iint_D f(x,y)\, dA$$
$$= \int_0^4 \int_0^{y/4} f(x,y)\, dx\, dy$$

39.



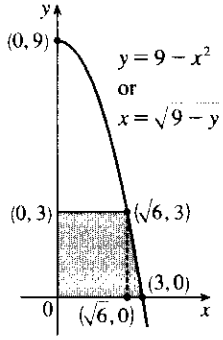
Because the region of integration is

$$D = \left\{(x,y) \mid -\sqrt{9-y^2} \le x \le \sqrt{9-y^2}, 0 \le y \le 3\right\}$$
$$= \{(x,y) \mid 0 \le y \le \sqrt{9-x^2}, -3 \le x \le 3\}$$

we have

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y)\, dx\, dy = \iint_D f(x,y)\, dA$$
$$= \int_{-3}^3 \int_0^{\sqrt{9-x^2}} f(x,y)\, dy\, dx$$

40.



To reverse the order, we must break the region into two separate type I regions. Because the region of integration is

$$D = \{(x,y) \mid 0 \le x \le \sqrt{9-y}, 0 \le y \le 3\}$$
$$= \{(x,y) \mid 0 \le y \le 3, 0 \le x \le \sqrt{6}\} \cup \{(x,y) \mid 0 \le y \le 9 - x^2, \sqrt{6} \le x \le 3\}$$

we have

$$\int_0^3 \int_0^{\sqrt{9-y}} f(x,y)\, dx\, dy = \iint_D f(x,y)\, dA$$
$$= \int_0^{\sqrt{6}} \int_0^3 f(x,y)\, dy\, dx + \int_{\sqrt{6}}^3 \int_0^{9-x^2} f(x,y)\, dy\, dx$$

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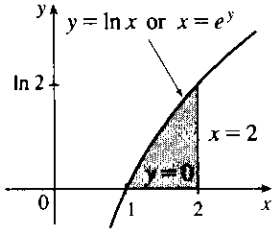
41.



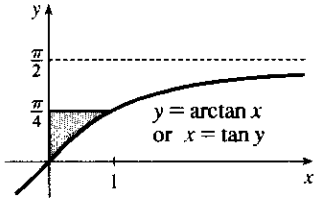
Because the region of integration is

$$D = \{(x, y) \mid 0 \le y \le \ln x, 1 \le x \le 2\}$$
$$= \{(x, y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$$

we have

$$\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA$$
$$= \int_0^{\ln 2} \int_{e^y}^2 f(x, y)\, dx\, dy$$

42.



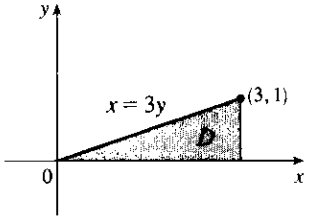
Because the region of integration is

$$D = \{(x, y) \mid \arctan x \le y \le \tfrac{\pi}{4}, 0 \le x \le 1\}$$
$$= \{(x, y) \mid 0 \le x \le \tan y, 0 \le y \le \tfrac{\pi}{4}\}$$

we have

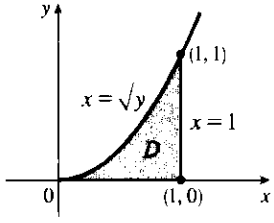
$$\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA$$
$$= \int_0^{\pi/4} \int_0^{\tan y} f(x, y)\, dx\, dy$$

43.



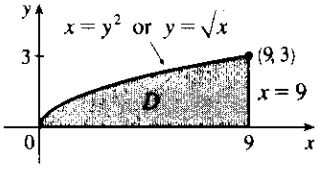
$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx$$
$$= \int_0^3 \left[e^{x^2} y\right]_{y=0}^{y=x/3} dx = \int_0^3 \left(\frac{x}{3}\right) e^{x^2}\, dx$$
$$= \tfrac{1}{6}\, e^{x^2}\Big]_0^3 = \frac{e^9 - 1}{6}$$

44.



$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1}\, dx\, dy = \int_0^1 \int_0^{x^2} \sqrt{x^3 + 1}\, dy\, dx$$
$$= \int_0^1 \left[\sqrt{x^3 + 1}\, y\right]_{y=0}^{y=x^2} dx = \int_0^1 x^2 \sqrt{x^3 + 1}\, dx$$
$$= \tfrac{2}{9}(x^3 + 1)^{3/2}\Big]_0^1 = \tfrac{2}{9}\left(2^{3/2} - 1\right)$$

45.



$$\int_0^3 \int_{y^2}^9 y \cos x^2\, dx\, dy = \int_0^9 \int_0^{\sqrt{x}} y \cos x^2\, dy\, dx$$
$$= \int_0^9 \cos x^2 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^9 \tfrac{1}{2} x \cos x^2\, dx$$
$$= \tfrac{1}{4} \sin x^2\Big]_0^9 = \tfrac{1}{4} \sin 81$$

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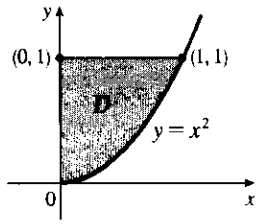
46.



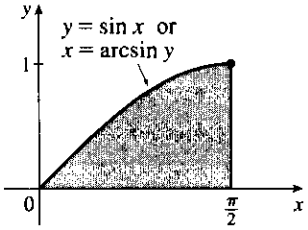
$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^4}{4}\sin(y^3)\right]_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \tfrac{1}{4}y^2 \sin(y^3)\,dy$$

$$= -\tfrac{1}{12}\cos(y^3)\Big]_0^1 = \tfrac{1}{12}(1-\cos 1)$$

47.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\,dy\,dx$$

$$= \int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,[y]_{y=0}^{y=\sin x}\,dx$$

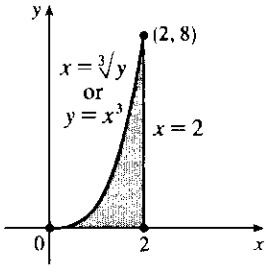
$$= \int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\sin x\,dx$$

[Let $u = \cos x$, $du = -\sin x\,dx$, $dx = du/(-\sin x)$]

$$= \int_1^0 -u\sqrt{1+u^2}\,du = -\tfrac{1}{3}\left(1+u^2\right)^{3/2}\Big]_1^0$$

$$= \tfrac{1}{3}\left(\sqrt{8}-1\right) = \tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

48.



$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$

$$= \int_0^2 e^{x^4}[y]_{y=0}^{y=x^3}\,dx = \int_0^2 x^3 e^{x^4}\,dx$$

$$= \tfrac{1}{4}e^{x^4}\Big]_0^2 = \tfrac{1}{4}(e^{16}-1)$$

49. $D = \{(x,y) \mid 0 \le x \le 1,\ -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0,\ x+1 \le y \le 1\}$

$$\cup \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x-1\},$$

all type I.

$$\iint_D x^2\,dA = \int_0^1\int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0\int_{x+1}^1 x^2\,dy\,dx + \int_0^1\int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0\int_{-1}^{-x-1} x^2\,dy\,dx$$

$$= 4\int_0^1\int_{1-x}^1 x^2\,dy\,dx \qquad \text{[by symmetry of the regions and because } f(x,y) = x^2 \ge 0\text{]}$$

$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

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50. $D = \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le 1+x^2\} \cup \{(x,y) \mid 0 \le x \le 1,\ \sqrt{x} \le y \le 1+x^2\}$

$\cup \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le -\sqrt{x}\}$,

all type I.

$$\begin{aligned}
\iint_D xy\,dA &= \int_{-1}^{0}\int_{-1}^{1+x^2} xy\,dy\,dx + \int_0^1\int_{\sqrt{x}}^{1+x^2} xy\,dy\,dx + \int_0^1\int_{-1}^{-\sqrt{x}} xy\,dy\,dx \\
&= \int_{-1}^{0}\left[\tfrac{1}{2}xy^2\right]_{y=-1}^{y=1+x^2} dx + \int_0^1\left[\tfrac{1}{2}xy^2\right]_{y=\sqrt{x}}^{y=1+x^2} dx + \int_0^1\left[\tfrac{1}{2}xy^2\right]_{y=-1}^{y=-\sqrt{x}} dx \\
&= \int_{-1}^{0}\left(x^3+\tfrac{1}{2}x^5\right) dx + \int_0^1 \tfrac{1}{2}(x^5+2x^3-x^2+x)\,dx + \int_0^1 \tfrac{1}{2}(x^2-x)\,dx \\
&= \left[\tfrac{1}{4}x^4+\tfrac{1}{12}x^6\right]_{-1}^{0} + \tfrac{1}{2}\left[\tfrac{1}{6}x^6+\tfrac{1}{2}x^4-\tfrac{1}{3}x^3+\tfrac{1}{2}x^2\right]_0^1 + \tfrac{1}{2}\left[\tfrac{1}{3}x^3-\tfrac{1}{2}x^2\right]_0^1 \\
&= -\tfrac{1}{3}+\tfrac{5}{12}-\tfrac{1}{12} = 0
\end{aligned}$$

51. For $D = [0,1]\times[0,1]$, $0 \le \sqrt{x^3+y^3} \le \sqrt{2}$ and $A(D) = 1$, so $0 \le \iint_D \sqrt{x^3+y^3}\,dA \le \sqrt{2}$.

52. Since $D = \{(x,y) \mid x^2+y^2 \le \tfrac{1}{4}\}$, $1 = e^0 \le e^{x^2+y^2} \le e^{1/4}$ and $A(D) = \frac{\pi}{4}$, so

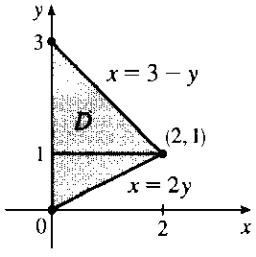
$\frac{\pi}{4} \le \iint_D e^{x^2+y^2}\,dA \le (e^{1/4})\frac{\pi}{4}$.

53. Since $m \le f(x,y) \le M$, $\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$ by (8) $\Rightarrow$

$m\iint_D 1\,dA \le \iint_D f(x,y)\,dA \le M\iint_D 1\,dA$ by (7) $\Rightarrow$

$mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$ by (10).

54.



$$\begin{aligned}
\iint_D f(x,y)\,dA &= \int_0^1\int_0^{2y} f(x,y)\,dx\,dy + \int_1^3\int_0^{3-y} f(x,y)\,dx\,dy \\
&= \int_0^2\int_{x/2}^{3-x} f(x,y)\,dy\,dx
\end{aligned}$$

55. $\iint_D (x^2\tan x + y^3 + 4)\,dA = \iint_D x^2\tan x\,dA + \iint_D y^3\,dA + \iint_D 4\,dA$. But $x^2\tan x$ is an odd function of x and D is symmetric with respect to the y-axis, so $\iint_D x^2\tan x\,dA = 0$. Similarly, y^3 is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D y^3\,dA = 0$. Thus

$$\iint_D (x^2\tan x + y^3 + 4)\,dA = 4\iint_D dA = 4(\text{area of } D) = 4\cdot\pi\left(\sqrt{2}\right)^2 = 8\pi$$

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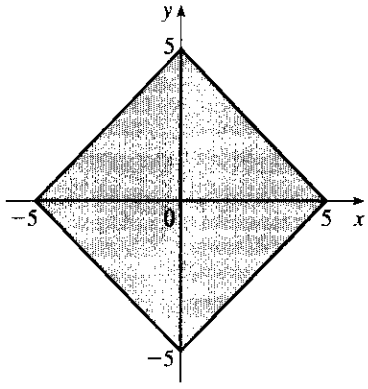
56. First,

$$\iint_D (2 - 3x + 4y)\, dA = \iint_D 2\, dA - \iint_D 3x\, dA + \iint_D 4y\, dA$$

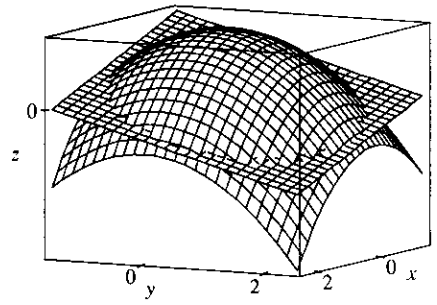


The region D, shown in the figure, is symmetric with respect to the y-axis and $3x$ is an odd function of x, so $\iint_D 3x\, dA = 0$. Similarly, $4y$ is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D 4y\, dA = 0$. Then

$$\begin{aligned}\iint_D (2 - 3x + 4y)\, dA &= \iint_D 2\, dA = 2 \iint_D dA \\ &= 2(\text{area of } D) = 2(50) \\ &= 100\end{aligned}$$

57. Since $\sqrt{1 - x^2 - y^2} \geq 0$, we can interpret $\iint_D \sqrt{1 - x^2 - y^2}\, dA$ as the volume of the solid that lies below the graph of $z = \sqrt{1 - x^2 - y^2}$ and above the region D in the xy-plane. $z = \sqrt{1 - x^2 - y^2}$ is equivalent to $x^2 + y^2 + z^2 = 1$, $z \geq 0$ which meets the xy-plane in the circle $x^2 + y^2 = 1$, the boundary of D. Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2}\left[\frac{4}{3}\pi(1)^3\right] = \frac{2}{3}\pi$.

58. To find the equations of the boundary curves, we require that the z-values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y,y);` and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y,y]`. We find that the curves have equations $y = \dfrac{1 \pm \sqrt{13 + 4x - 4x^2}}{2}$.



To find the two points of intersection of these curves, we use the CAS to solve $13 + 4x - 4x^2 = 0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is

$$V = \int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2} \int_{\left(1-\sqrt{13+4x-4x^2}\right)/2}^{\left(1+\sqrt{13+4x-4x^2}\right)/2} [(4 - x^2 - y^2) - (1 - x - y)]\, dy\, dx = \frac{49\pi}{8}.$$

16.4 Double Integrals in Polar Coordinates ET 15.4

1. The region R is more easily described by polar coordinates: $R = \{(r, \theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$. Thus $\iint_R f(x, y)\, dA = \int_0^{2\pi} \int_0^2 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2 - x\}$. Thus $\iint_R f(x, y)\, dA = \int_0^2 \int_0^{2-x} f(x, y)\, dy\, dx$.

3. The region R is more easily described by rectangular coordinates: $R = \{(x, y) \mid -2 \leq x \leq 2, x \leq y \leq 2\}$. Thus $\iint_R f(x, y)\, dA = \int_{-2}^2 \int_x^2 f(x, y)\, dy\, dx$.

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4. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 1 \le r \le 3, 0 \le \theta \le \frac{\pi}{2}\}$.

Thus $\iint_R f(x,y)\,dA = \int_0^{\pi/2}\int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

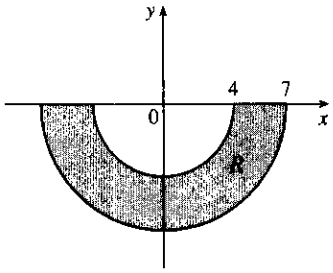
5. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 5, 0 \le \theta \le 2\pi\}$.

Thus $\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

6. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 0 \le r \le 2\sqrt{2}, \frac{\pi}{4} \le \theta \le \frac{5\pi}{4}\}$.

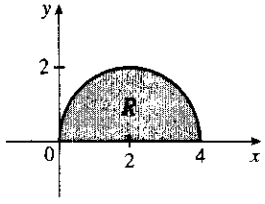
Thus $\iint_R f(x,y)\,dA = \int_{\pi/4}^{5\pi/4}\int_0^{2\sqrt{2}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

7. The integral $\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 4 \le r \le 7, \pi \le \theta \le 2\pi\}$, the lower half of a ring.



$$\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta = \left(\int_\pi^{2\pi} d\theta\right)\left(\int_4^7 r\,dr\right)$$
$$= \left[\theta\right]_\pi^{2\pi}\left[\tfrac{1}{2}r^2\right]_4^7 = \pi\cdot\frac{1}{2}(49-16) = \frac{33\pi}{2}$$

8. The integral $\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 0 \le r \le 4\cos\theta, 0 \le \theta \le \pi/2\}$. Since $r = 4\cos\theta \;\Leftrightarrow\; r^2 = 4r\cos\theta \;\Leftrightarrow\; x^2+y^2 = 4x \;\Leftrightarrow\; (x-2)^2+y^2 = 4$, R is the portion in the first quadrant of a circle of radius 2 with center $(2,0)$.



$$\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=4\cos\theta} d\theta = \int_0^{\pi/2} 8\cos^2\theta\,d\theta$$
$$= \int_0^{\pi/2} 4(1+\cos 2\theta)\,d\theta = 4\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2\pi$$

9. The disk D can be described in polar coordinates as $D = \{(r,\theta) \mid 0 \le r \le 3, 0 \le \theta \le 2\pi\}$. Then

$$\iint_D xy\,dA = \int_0^{2\pi}\int_0^3 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\right)\left(\int_0^3 r^3\,dr\right)$$
$$= \left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^3 = 0$$

10. $\iint_R (x+y)\,dA = \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2}\int_1^2 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$
$$= \left(\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)\,d\theta\right)\left(\int_1^2 r^2\,dr\right) = \left[\sin\theta-\cos\theta\right]_{\pi/2}^{3\pi/2}\left[\tfrac{1}{3}r^3\right]_1^2$$
$$= (-1-0-1+0)\left(\tfrac{8}{3}-\tfrac{1}{3}\right) = -\tfrac{14}{3}$$

11. $\iint_R \cos(x^2+y^2)\,dA = \int_0^\pi\int_0^3 \cos(r^2)\,r\,dr\,d\theta = \left(\int_0^\pi d\theta\right)\left(\int_0^3 r\cos(r^2)\,dr\right)$
$$= \left[\theta\right]_0^\pi\left[\tfrac{1}{2}\sin(r^2)\right]_0^3 = \pi\cdot\tfrac{1}{2}(\sin 9 - \sin 0) = \tfrac{\pi}{2}\sin 9$$

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12. $\iint_R \sqrt{4-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 \sqrt{4-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 r\sqrt{4-r^2}\,dr\right)$

$= \left[\theta\right]_{-\pi/2}^{\pi/2}\left[-\frac{1}{2}\cdot\frac{2}{3}(4-r^2)^{3/2}\right]_0^2 = \left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(-\frac{1}{3}(0-4^{3/2})\right) = \frac{8}{3}\pi$

13. $\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 re^{-r^2}\,dr\right)$

$= \left[\theta\right]_{-\pi/2}^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\frac{1}{2}\right)(e^{-4}-e^0) = \frac{\pi}{2}(1-e^{-4})$

14. $\iint_R ye^x\,dA = \int_0^{\pi/2}\int_0^5 (r\sin\theta)\,e^{r\cos\theta}\,r\,dr\,d\theta = \int_0^5\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta\,dr$. First we integrate $\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta$: Let $u = r\cos\theta \;\Rightarrow\; du = -r\sin\theta\,d\theta$,

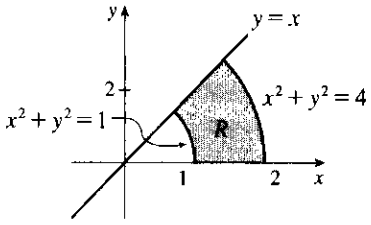
and $\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta = \int_{u=r}^{u=0} -r\,e^u\,du = -r[e^0 - e^r] = re^r - r$. Then

$\int_0^5\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta\,dr = \int_0^5 (re^r - r)\,dr = \left[re^r - e^r - \frac{1}{2}r^2\right]_0^5 = 4e^5 - \frac{23}{2}$, where we integrated by parts in the first term.

15. R is the region shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le \theta \le \pi/4, 1 \le r \le 2\}$. Thus

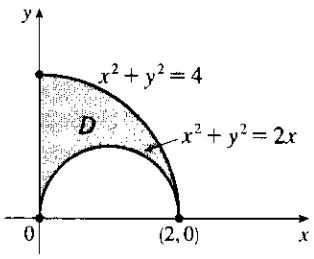
$\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$

since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $0 \le \theta \le \pi/4$, so the integral becomes



$\int_0^{\pi/4}\int_1^2 \theta\,r\,dr\,d\theta = \int_0^{\pi/4}\theta\,d\theta\int_1^2 r\,dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4}\left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32}\cdot\frac{3}{2} = \frac{3}{64}\pi^2$.

16.



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\,y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$

$= \int_0^{\pi/2}\frac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\frac{1}{3}(8\cos^4\theta)\,d\theta$

$= \frac{8}{3} - \frac{8}{12}\left[\cos^3\theta\sin\theta + \frac{3}{2}(\theta+\sin\theta\cos\theta)\right]_0^{\pi/2}$

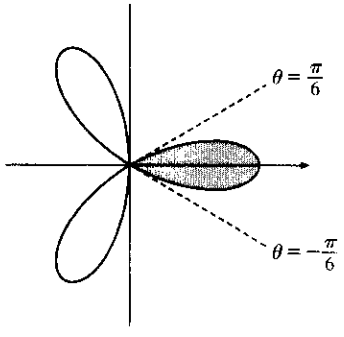
$= \frac{8}{3} - \frac{2}{3}\left[0 + \frac{3}{2}\left(\frac{\pi}{2}\right)\right] = \frac{16-3\pi}{6}$

17. One loop is given by the region $D = \{(r,\theta) \mid -\pi/6 \le \theta \le \pi/6, 0 \le r \le \cos 3\theta\}$, so the area is

$$\iint_D dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\frac{1}{2}\cos^2 3\theta\,d\theta = 2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$



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18. $D = \{(r,\theta) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4 + 3\cos\theta\}$, so

$$A(D) = \iint_D dA = \int_0^{2\pi}\int_0^{4+3\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=4+3\cos\theta} d\theta = \tfrac{1}{2}\int_0^{2\pi}(4+3\cos\theta)^2 d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi}(16 + 24\cos\theta + 9\cos^2\theta)\,d\theta = \tfrac{1}{2}\int_0^{2\pi}\left(16 + 24\cos\theta + 9\cdot\tfrac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \tfrac{1}{2}\left[16\theta + 24\sin\theta + \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{41}{2}\pi$$

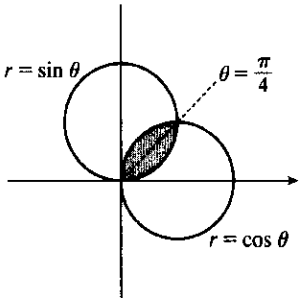
19. By symmetry,

$$A = 2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/4}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=\sin\theta} d\theta$$

$$= \int_0^{\pi/4}\sin^2\theta\,d\theta = \int_0^{\pi/4}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \tfrac{1}{2}\left[\tfrac{\pi}{4} - \tfrac{1}{2}\sin\tfrac{\pi}{2} - 0 + \tfrac{1}{2}\sin 0\right] = \tfrac{1}{8}(\pi - 2)$$



20. $2 = 4\sin\theta$ implies that $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, so

$$A = \int_{\pi/6}^{5\pi/6}\int_2^{4\sin\theta} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6}\left[\tfrac{1}{2}r^2\right]_{r=2}^{r=4\sin\theta} d\theta = \int_{\pi/6}^{5\pi/6}(8\sin^2\theta - 2)\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6}[4(1-\cos 2\theta) - 2]\,d\theta = \left[2\theta - 2\sin 2\theta\right]_{\pi/6}^{5\pi/6} = \tfrac{4\pi}{3} + 2\sqrt{3}.$$

21. $V = \iint\limits_{x^2+y^2\le 9}(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^3 (r^2)\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^3 r^3\,dr = \left[\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^3 = 2\pi\left(\tfrac{81}{4}\right) = \tfrac{81\pi}{2}$

22. The sphere $x^2+y^2+z^2 = 16$ intersects the xy-plane in the circle $x^2+y^2 = 16$, so

$$V = 2\iint\limits_{4\le x^2+y^2\le 16}\sqrt{16-x^2-y^2}\,dA \qquad \text{[by symmetry]}$$

$$= 2\int_0^{2\pi}\int_2^4\sqrt{16-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_2^4 r(16-r^2)^{1/2}dr$$

$$= 2\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}(16-r^2)^{3/2}\right]_2^4 = -\tfrac{2}{3}(2\pi)(0-12^{3/2}) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

23. By symmetry,

$$V = 2\iint\limits_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_0^a r\sqrt{a^2-r^2}\,dr$$

$$= 2\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0+\tfrac{1}{3}a^3\right) = \tfrac{4\pi}{3}a^3$$

24. The paraboloid $z = 10 - 3x^2 - 3y^2$ intersects the plane $z = 4$ when $4 = 10 - 3x^2 - 3y^2$ or $x^2+y^2 = 2$. So

$$V = \iint\limits_{x^2+y^2\le 2}\left[(10-3x^2-3y^2) - 4\right]dA = \int_0^{2\pi}\int_0^{\sqrt{2}}(6-3r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\sqrt{2}}(6r-3r^3)\,dr = \left[\theta\right]_0^{2\pi}\left[3r^2 - \tfrac{3}{4}r^4\right]_0^{\sqrt{2}} = 6\pi$$

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25. The cone $z = \sqrt{x^2 + y^2}$ intersects the sphere $x^2 + y^2 + z^2 = 1$ when $x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2 = 1$ or $x^2 + y^2 = \frac{1}{2}$. So

$$V = \iint_{x^2+y^2 \le 1/2} \left(\sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2}\right) dA = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \left(\sqrt{1 - r^2} - r\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{1/\sqrt{2}} \left(r\sqrt{1 - r^2} - r^2\right) dr = \left[\theta\right]_0^{2\pi} \left[-\tfrac{1}{3}(1 - r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^{1/\sqrt{2}}$$

$$= 2\pi\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{\sqrt{2}} - 1\right) = \tfrac{\pi}{3}\left(2 - \sqrt{2}\right)$$

26. The two paraboloids intersect when $3x^2 + 3y^2 = 4 - x^2 - y^2$ or $x^2 + y^2 = 1$. So

$$V = \iint_{x^2+y^2 \le 1} \left[(4 - x^2 - y^2) - 3(x^2 + y^2)\right] dA = \int_0^{2\pi} \int_0^1 4(1 - r^2)\, r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (4r - 4r^3)\, dr = \left[\theta\right]_0^{2\pi} \left[2r^2 - r^4\right]_0^1 = 2\pi$$

27. The given solid is the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = \sqrt{64 - 4x^2 - 4y^2}$ and $z = -\sqrt{64 - 4x^2 - 4y^2}$. So

$$V = \iint_{x^2+y^2 \le 4} \left[\sqrt{64 - 4x^2 - 4y^2} - \left(-\sqrt{64 - 4x^2 - 4y^2}\right)\right] dA$$

$$= \iint_{x^2+y^2 \le 4} 2\sqrt{64 - 4x^2 - 4y^2}\, dA = 4\int_0^{2\pi} \int_0^2 \sqrt{16 - r^2}\, r \, dr \, d\theta$$

$$= 4\int_0^{2\pi} d\theta \int_0^2 r\sqrt{16 - r^2}\, dr = 4\left[\theta\right]_0^{2\pi} \left[-\tfrac{1}{3}(16 - r^2)^{3/2}\right]_0^2$$

$$= 8\pi\left(-\tfrac{1}{3}\right)(12^{3/2} - 16^{2/3}) = \tfrac{8\pi}{3}\left(64 - 24\sqrt{3}\right)$$

28. (a) Here the region in the xy-plane is the annular region $r_1^2 \le x^2 + y^2 \le r_2^2$ and the desired volume is twice that above the xy-plane. Hence

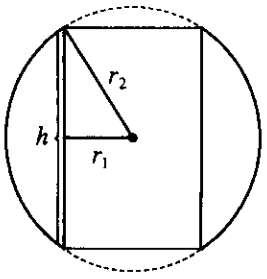
$$V = 2\iint_{r_1^2 \le x^2+y^2 \le r_2^2} \sqrt{r_2^2 - x^2 - y^2}\, dA = 2\int_0^{2\pi} \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\, r \, dr \, d\theta$$

$$= 2\int_0^{2\pi} d\theta \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\, r \, dr = \tfrac{4\pi}{3}\left[-(r_2^2 - r^2)^{3/2}\right]_{r_1}^{r_2} = \tfrac{4\pi}{3}(r_2^2 - r_1^2)^{3/2}$$

(b) A cross-sectional cut is shown in the figure.

So $r_2^2 = \left(\frac{1}{2}h\right)^2 + r_1^2$ or $\frac{1}{4}h^2 = r_2^2 - r_1^2$.

Thus the volume in terms of h is $V = \frac{4\pi}{3}\left(\frac{1}{4}h^2\right)^{3/2} = \frac{\pi}{6}h^3$.



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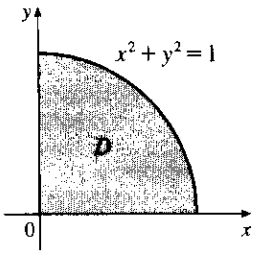
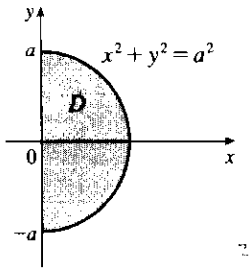
29.



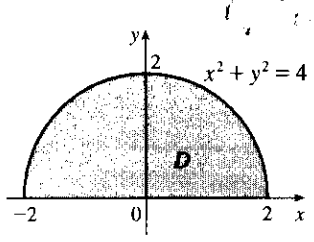
$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx = \int_0^{\pi/2} \int_0^1 e^{r^2} r\,dr\,d\theta$$
$$= \int_0^{\pi/2} d\theta \int_0^1 r e^{r^2}\,dr$$
$$= \left[\theta\right]_0^{\pi/2} \left[\tfrac{1}{2}e^{r^2}\right]_0^1 = \tfrac{1}{4}\pi(e-1)$$

30.



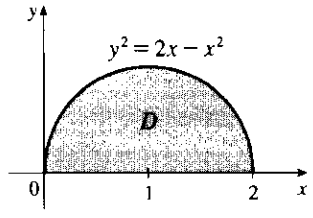
$$\int_{-\pi/2}^{\pi/2} \int_0^a (r^2)^{3/2} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^a r^4\,dr$$
$$= \left[\theta\right]_{-\pi/2}^{\pi/2} \left[\tfrac{1}{5}r^5\right]_0^a$$
$$= \tfrac{1}{5}\pi a^5$$

31.



$$\int_0^{\pi} \int_0^2 (r\cos\theta)^2 (r\sin\theta)^2 r\,dr\,d\theta = \int_0^{\pi} (\sin\theta\cos\theta)^2 d\theta \int_0^2 r^5\,dr$$
$$= \int_0^{\pi} \left(\tfrac{1}{2}\sin 2\theta\right)^2 d\theta \int_0^2 r^5\,dr$$
$$= \tfrac{1}{4}\left[\tfrac{1}{2}\theta - \tfrac{1}{8}\sin 4\theta\right]_0^{\pi} \left[\tfrac{1}{6}r^6\right]_0^2$$
$$= \tfrac{1}{4}\left(\tfrac{\pi}{2}\right)\left(\tfrac{64}{6}\right) = \tfrac{4\pi}{3}$$

32.



$$\int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_0^{\pi/2} \left[\tfrac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta} d\theta$$
$$= \int_0^{\pi/2} \left(\tfrac{8}{3}\cos^3\theta\right) d\theta$$
$$= \tfrac{8}{3}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \tfrac{16}{9}$$

33. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y), then the volume of water in the pool is the volume of the solid that lies above $D = \{(x,y) \mid x^2+y^2 \le 400\}$ and below the graph of $f(x,y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0,-20) = 2$ to $f(0,20) = 7$. The trace in the yz-plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y-20) \quad \Rightarrow \quad z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x,y)$ is independent of x, $f(x,y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r,\theta) \mid 0 \le r \le 20,\ 0 \le \theta \le 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\int_0^{2\pi} \int_0^{20} \left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right) r\,dr\,d\theta = \int_0^{2\pi} \left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta = \int_0^{2\pi} \left(\tfrac{1000}{3}\sin\theta + 900\right) d\theta$$
$$= \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi} = 1800\pi$$

Thus the pool contains $1800\pi \approx 5655$ ft^3 of water.

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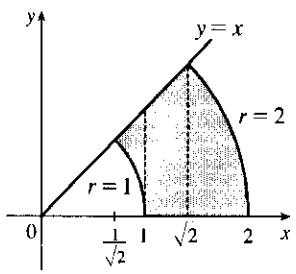
34. (a) The total amount of water supplied each hour to the region within R feet of the sprinkler is

$$V = \int_0^{2\pi}\int_0^R e^{-r} r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^R re^{-r}\,dr = \left[\theta\right]_0^{2\pi}\left[-re^{-r} - e^{-r}\right]_0^R$$
$$= 2\pi[-Re^{-R} - e^{-R} + 0 + 1] = 2\pi(1 - Re^{-R} - e^{-R}) \text{ ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is $\dfrac{V}{\text{area of region}} = \dfrac{V}{\pi R^2} = \dfrac{2(1 - Re^{-R} - e^{-R})}{R^2}$ ft^3 (per hour per square foot). See the definition of the average value of a function on page 1022 [ET 986].

35. $\displaystyle\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$

$$= \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2} d\theta$$
$$= \frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\,d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16}$$



36. (a) $\iint_{D_a} e^{-(x^2+y^2)}dA = \int_0^{2\pi}\int_0^a re^{-r^2}\,dr\,d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi\left(1 - e^{-a^2}\right)$ for each a. Then $\lim_{a\to\infty}\pi\left(1 - e^{-a^2}\right) = \pi$ since $e^{-a^2} \to 0$ as $a \to \infty$. Hence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$.

(b) $\iint_{S_a} e^{-(x^2+y^2)}\,dA = \int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$ for each a.

Then, from (a), $\pi = \iint_{\mathbb{R}^2} -(x^2 + y^2)\,dA$, so

$\pi = \lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$.

To evaluate $\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$, we are using the fact that these integrals are bounded. This is true since on $[-1, 1]$, $0 < e^{-x^2} \le 1$ while on $(-\infty, -1)$, $0 < e^{-x^2} \le e^x$ and on $(1, \infty)$, $0 < e^{-x^2} < e^{-x}$. Hence $0 \le \int_{-\infty}^{\infty} e^{-x^2}\,dx \le \int_{-\infty}^{-1} e^x\,dx + \int_{-1}^{1} dx + \int_1^{\infty} e^{-x}\,dx = 2(e^{-1} + 1)$.

(c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \pi$ and y can be replaced by x, $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$ implies that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \pm\sqrt{\pi}$. But $e^{-x^2} \ge 0$ for all x, so $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

(d) Letting $t = \sqrt{2}x$, $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2}}\left(e^{-t^2/2}\right)dt$, so that $\sqrt{\pi} = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt$ or $\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$.

37. (a) We integrate by parts with $u = x$ and $dv = xe^{-x^2}\,dx$. Then $du = dx$ and $v = -\frac{1}{2}e^{-x^2}$, so

$$\int_0^{\infty} x^2e^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left(-\tfrac{1}{2}xe^{-x^2}\Big]_0^t + \int_0^t \tfrac{1}{2}e^{-x^2}\,dx\right)$$
$$= \lim_{t\to\infty}\left(-\tfrac{1}{2}te^{-t^2}\right) + \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx = 0 + \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx \qquad \text{[by l'Hospital's Rule]}$$
$$= \tfrac{1}{4}\int_{-\infty}^{\infty} e^{-x^2}\,dx \qquad \text{[since } e^{-x^2} \text{ is an even function]}$$
$$= \tfrac{1}{4}\sqrt{\pi} \qquad \text{[by Exercise 36(c)]}$$

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(b) Let $u = \sqrt{x}$. Then $u^2 = x \;\Rightarrow\; dx = 2u\,du \;\Rightarrow$

$$\int_0^\infty \sqrt{x}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du = 2\int_0^\infty u^2e^{-u^2}\,du$$
$$= 2\left(\tfrac{1}{4}\sqrt{\pi}\right) \text{ [by part(a)]} = \tfrac{1}{2}\sqrt{\pi}$$

16.5 Applications of Double Integrals ET 15.5

1. $Q = \iint_D \sigma(x,y)\,dA = \int_1^3\int_0^2 (2xy+y^2)\,dy\,dx = \int_1^3 \left[xy^2+\tfrac{1}{3}y^3\right]_{y=0}^{y=2} dx$
$= \int_1^3 \left(4x+\tfrac{8}{3}\right) dx = \left[2x^2+\tfrac{8}{3}x\right]_1^3 = 16+\tfrac{16}{3} = \tfrac{64}{3}$ C

2. $Q = \iint_D \sigma(x,y)\,dA = \iint_D (x+y+x^2+y^2)\,dA = \int_0^{2\pi}\int_0^2 (r\cos\theta + r\sin\theta + r^2)\,r\,dr\,d\theta$
$= \int_0^{2\pi}\int_0^2 [r^2(\cos\theta+\sin\theta)+r^3]\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{3}r^3(\cos\theta+\sin\theta)+\tfrac{1}{4}r^4\right]_{r=0}^{r=2} d\theta$
$= \int_0^{2\pi}\left[\tfrac{8}{3}(\cos\theta+\sin\theta)+4\right] d\theta = \left[\tfrac{8}{3}(\sin\theta-\cos\theta)+4\theta\right]_0^{2\pi} = 8\pi$ C

3. $m = \iint_D \rho(x,y)\,dA = \int_0^2\int_{-1}^1 xy^2\,dy\,dx = \int_0^2 x\,dx\int_{-1}^1 y^2\,dy = \left[\tfrac{1}{2}x^2\right]_0^2\left[\tfrac{1}{3}y^3\right]_{-1}^1 = 2\cdot\tfrac{2}{3} = \tfrac{4}{3}$,

$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{3}{4}\int_0^2\int_{-1}^1 x^2y^2\,dy\,dx = \frac{3}{4}\int_0^2 x^2\,dx\int_{-1}^1 y^2\,dy = \frac{3}{4}\left[\tfrac{1}{3}x^3\right]_0^2\left[\tfrac{1}{3}y^3\right]_{-1}^1 = \frac{3}{4}\cdot\frac{8}{3}\cdot\frac{2}{3} = \frac{4}{3}$,

$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{3}{4}\int_0^2\int_{-1}^1 xy^3\,dy\,dx = \frac{3}{4}\int_0^2 x\,dx\int_{-1}^1 y^3\,dy = \frac{3}{4}\left[\tfrac{1}{2}x^2\right]_0^2\left[\tfrac{1}{4}y^4\right]_{-1}^1 = \frac{3}{4}\cdot 2\cdot 0 = 0.$

Hence, $(\overline{x},\overline{y}) = \left(\frac{4}{3},0\right)$.

4. $m = \iint_D \rho(x,y)\,dA = \int_0^a\int_0^b cxy\,dy\,dx = c\int_0^a x\,dx\int_0^b y\,dy = c\left[\tfrac{1}{2}x^2\right]_0^a\left[\tfrac{1}{2}y^2\right]_0^b = \tfrac{1}{4}a^2b^2c$,

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^a\int_0^b cx^2y\,dy\,dx = c\int_0^a x^2\,dx\int_0^b y\,dy = c\left[\tfrac{1}{3}x^3\right]_0^a\left[\tfrac{1}{2}y^2\right]_0^b = \tfrac{1}{6}a^3b^2c$, and

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^a\int_0^b cxy^2\,dy\,dx = c\int_0^a x\,dx\int_0^b y^2\,dy = c\left[\tfrac{1}{2}x^2\right]_0^a\left[\tfrac{1}{3}y^3\right]_0^b = \tfrac{1}{6}a^2b^3c$.

Hence, $(\overline{x},\overline{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right) = \left(\frac{2}{3}a,\frac{2}{3}b\right)$.

5. $m = \int_0^2\int_{x/2}^{3-x}(x+y)\,dy\,dx = \int_0^2\left[xy+\tfrac{1}{2}y^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left[x(3-\tfrac{3}{2}x)+\tfrac{1}{2}(3-x)^2-\tfrac{1}{8}x^2\right] dx$
$= \int_0^2\left(-\tfrac{9}{8}x^2+\tfrac{9}{2}\right) dx = \left[-\tfrac{9}{8}\left(\tfrac{1}{3}x^3\right)+\tfrac{9}{2}x\right]_0^2 = 6$,

$M_y = \int_0^2\int_{x/2}^{3-x}(x^2+xy)\,dy\,dx = \int_0^2\left[x^2y+\tfrac{1}{2}xy^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left(\tfrac{9}{2}x-\tfrac{9}{8}x^3\right) dx = \tfrac{9}{2}$, and

$M_x = \int_0^2\int_{x/2}^{3-y}(xy+y^2)\,dy\,dx = \int_0^2\left[\tfrac{1}{2}xy^2+\tfrac{1}{3}y^3\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left(9-\tfrac{9}{2}x\right) dx = 9$.

Hence $m = 6$, $(\overline{x},\overline{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right) = \left(\frac{3}{4},\frac{3}{2}\right)$.

6. $m = \int_0^1\int_y^{4-3y} x\,dx\,dy = \int_0^1\left[\tfrac{1}{2}(4-3y)^2-\tfrac{1}{2}y^2\right] dy = \left[-\tfrac{1}{18}(4-3y)^3-\tfrac{1}{6}y^3\right]_0^1 = \tfrac{10}{3}$,

$M_y = \int_0^1\int_y^{4-3y} x^2\,dx\,dy = \int_0^1\left[\tfrac{1}{3}(4-3y)^3-\tfrac{1}{3}y^3\right] dy = \left[-\tfrac{1}{36}(4-3y)^4-\tfrac{1}{12}y^4\right]_0^1 = 7$,

$M_x = \int_0^1\int_y^{4-3y} xy\,dx\,dy = \int_0^1\left[\tfrac{1}{2}y(4-3y)^2-\tfrac{1}{2}y^3\right] dy = \int_0^1(8y-12y^2+4y^3)\,dy = 1$.

Hence $m = \frac{10}{3}$, $(\overline{x},\overline{y}) = (2.1, 0.3)$.

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7. $m = \int_0^1 \int_0^{e^x} y\,dy\,dx = \int_0^1 \left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x} dx = \frac{1}{2}\int_0^1 e^{2x}\,dx = \frac{1}{4}e^{2x}\Big]_0^1 = \frac{1}{4}(e^2-1)$,

$M_y = \int_0^1 \int_0^{e^x} xy\,dy\,dx = \frac{1}{2}\int_0^1 xe^{2x}\,dx = \frac{1}{2}\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{8}(e^2+1)$, and

$M_x = \int_0^1 \int_0^{e^x} y^2\,dy\,dx = \int_0^1 \left[\frac{1}{3}y^3\right]_{y=0}^{y=e^x} dx = \frac{1}{3}\int_0^1 e^{3x}\,dx = \frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1 = \frac{1}{9}(e^3-1)$.

Hence $m = \frac{1}{4}(e^2-1)$, $(\overline{x},\overline{y}) = \left(\dfrac{\frac{1}{8}(e^2+1)}{\frac{1}{4}(e^2-1)}, \dfrac{\frac{1}{9}(e^3-1)}{\frac{1}{4}(e^2-1)}\right) = \left(\dfrac{e^2+1}{2(e^2-1)}, \dfrac{4(e^3-1)}{9(e^2-1)}\right)$.

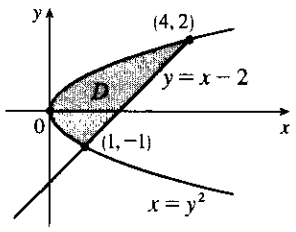
8. $m = \int_0^1 \int_0^{\sqrt{x}} x\,dy\,dx = \int_0^1 x\left[y\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big]_0^1 = \frac{2}{5}$,

$M_y = \int_0^1 \int_0^{\sqrt{x}} x^2\,dy\,dx = \int_0^1 x\left[y\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 x^{5/2}\,dx = \frac{2}{7}x^{7/2}\Big]_0^1 = \frac{2}{7}$, and

$M_x = \int_0^1 \int_0^{\sqrt{x}} yx\,dy\,dx = \int_0^1 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^1 x^2\,dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{6}$.

Hence $m = \frac{2}{5}$, $(\overline{x},\overline{y}) = \left(\frac{2/7}{2/5}, \frac{1/6}{2/5}\right) = \left(\frac{5}{7}, \frac{5}{12}\right)$.

9.



$m = \int_{-1}^2 \int_{y^2}^{y+2} 3\,dx\,dy = \int_{-1}^2 (3y+6-3y^2)\,dy = \frac{27}{2}$,

$M_y = \int_{-1}^2 \int_{y^2}^{y+2} 3x\,dx\,dy = \int_{-1}^2 \frac{3}{2}\left[(y+2)^2 - y^4\right] dy$

$= \left[\frac{1}{2}(y+2)^3 - \frac{3}{10}y^5\right]_{-1}^2 = \frac{108}{5}$

and

$M_x = \int_{-1}^2 \int_{y^2}^{y+2} 3y\,dx\,dy = \int_{-1}^2 (3y^2+6y-3y^3)\,dy$

$= \left[y^3 + 3y^2 - \frac{3}{4}y^4\right]_{-1}^2 = \frac{27}{4}$

Hence $m = \frac{27}{2}$, $(\overline{x},\overline{y}) = \left(\frac{8}{5}, \frac{1}{2}\right)$.

10. $m = \int_0^{\pi/2} \int_0^{\cos x} x\,dy\,dx = \int_0^{\pi/2} x\cos x\,dx = \left[x\sin x + \cos x\right]_0^{\pi/2} = \frac{\pi}{2} - 1$,

$M_y = \int_0^{\pi/2} \int_0^{\cos x} x^2\,dy\,dx = \int_0^{\pi/2} x^2\cos x\,dx = \left[x^2\sin x + 2x\cos x - 2\sin x\right]_0^{\pi/2} = \frac{\pi^2}{4} - 2$, and

$M_x = \int_0^{\pi/2} \int_0^{\cos x} xy\,dy\,dx = \int_0^{\pi/2} \frac{1}{2}x\cos^2 x\,dx = \frac{1}{2}\left[\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x\right]_0^{\pi/2} = \frac{\pi^2}{32} - \frac{1}{8}$.

Hence $m = \dfrac{\pi-2}{2}$, $(\overline{x},\overline{y}) = \left(\dfrac{\pi^2-8}{2(\pi-2)}, \dfrac{\pi+2}{16}\right)$.

11. $\rho(x,y) = ky = kr\sin\theta$, $m = \int_0^{\pi/2}\int_0^1 kr^2\sin\theta\,dr\,d\theta = \frac{1}{3}k\int_0^{\pi/2}\sin\theta\,d\theta = \frac{1}{3}k\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{3}k$,

$M_y = \int_0^{\pi/2}\int_0^1 kr^3\sin\theta\cos\theta\,dr\,d\theta = \frac{1}{4}k\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{1}{8}k\left[-\cos 2\theta\right]_0^{\pi/2} = \frac{1}{8}k$,

$M_x = \int_0^{\pi/2}\int_0^1 kr^3\sin^2\theta\,dr\,d\theta = \frac{1}{4}k\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{8}k\left[\theta + \sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{16}k$.

Hence $(\overline{x},\overline{y}) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

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12. $\rho(x,y) = k(x^2+y^2) = kr^2$, $m = \int_0^{\pi/2}\int_0^1 kr^3\,dr\,d\theta = \frac{\pi}{8}k$,

$M_y = \int_0^{\pi/2}\int_0^1 kr^4\cos\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{5}k\left[\sin\theta\right]_0^{\pi/2} = \frac{1}{5}k$,

$M_x = \int_0^{\pi/2}\int_0^1 kr^4\sin\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2}\sin\theta\,d\theta = \frac{1}{5}k\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{5}k$.

Hence $(\overline{x},\overline{y}) = \left(\frac{8}{5\pi},\frac{8}{5\pi}\right)$.

13. Placing the vertex opposite the hypotenuse at $(0,0)$, $\rho(x,y) = k(x^2+y^2)$. Then

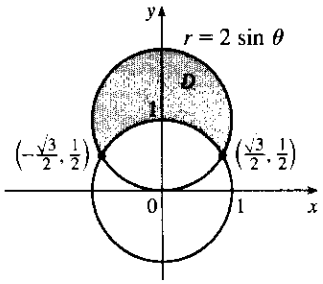
$$m = \int_0^a\int_0^{a-x} k\left(x^2+y^2\right)dy\,dx = k\int_0^a\left[ax^2 - x^3 + \tfrac{1}{3}(a-x)^3\right]dx$$
$$= k\left[\tfrac{1}{3}ax^3 - \tfrac{1}{4}x^4 - \tfrac{1}{12}(a-x)^4\right]_0^a = \tfrac{1}{6}ka^4$$

By symmetry,

$$M_y = M_x = \int_0^a\int_0^{a-x} ky(x^2+y^2)\,dy\,dx = k\int_0^a\left[\tfrac{1}{2}(a-x)^2x^2 + \tfrac{1}{4}(a-x)^4\right]dx$$
$$= k\left[\tfrac{1}{6}a^2x^3 - \tfrac{1}{4}ax^4 + \tfrac{1}{10}x^5 - \tfrac{1}{20}(a-x)^5\right]_0^a = \tfrac{1}{15}ka^5$$

Hence $(\overline{x},\overline{y}) = \left(\frac{2}{5}a,\frac{2}{5}a\right)$.

14.



$\rho(x,y) = k/\sqrt{x^2+y^2} = k/r$,

$$m = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta}\frac{k}{r}\,r\,dr\,d\theta = k\int_{\pi/6}^{5\pi/6}\left[(2\sin\theta)-1\right]d\theta$$
$$= k\left[-2\cos\theta-\theta\right]_{\pi/6}^{5\pi/6} = 2k\left(\sqrt{3}-\tfrac{\pi}{3}\right)$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} kr\sin\theta\,dr\,d\theta = \tfrac{1}{2}k\int_{\pi/6}^{5\pi/6}(4\sin^3\theta - \sin\theta)\,d\theta$$
$$= \tfrac{1}{2}k\left[-3\cos\theta + \tfrac{4}{3}\cos^3\theta\right]_{\pi/6}^{5\pi/6} = \sqrt{3}\,k$$

Hence $(\overline{x},\overline{y}) = \left(0, \frac{3\sqrt{3}}{2\left(3\sqrt{3}-\pi\right)}\right)$.

15. $I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^1\int_0^{e^x} y^2\cdot y\,dy\,dx = \int_0^1\left[\tfrac{1}{4}y^4\right]_{y=0}^{y=e^x}dx = \tfrac{1}{4}\int_0^1 e^{4x}\,dx$

$= \tfrac{1}{4}\left[\tfrac{1}{4}e^{4x}\right]_0^1 = \tfrac{1}{16}(e^4-1)$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^1\int_0^{e^x} x^2y\,dy\,dx = \int_0^1 x^2\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=e^x}dx = \tfrac{1}{2}\int_0^1 x^2e^{2x}\,dx$

$= \tfrac{1}{2}\left[\left(\tfrac{1}{2}x^2 - \tfrac{1}{2}x + \tfrac{1}{4}\right)e^{2x}\right]_0^1$ [integrate by parts twice]

$= \tfrac{1}{8}(e^2-1)$,

and $I_0 = I_x + I_y = \tfrac{1}{16}(e^4-1) + \tfrac{1}{8}(e^2-1) = \tfrac{1}{16}(e^4+2e^2-3)$.

16. $I_x = \int_0^{\pi/2}\int_0^1 (r^2\sin^2\theta)(kr^2)\,r\,dr\,d\theta = \frac{1}{6}k\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{6}k\left[\frac{1}{4}(2\theta-\sin 2\theta)\right]_0^{\pi/2} = \frac{\pi}{24}k$,

$I_y = \int_0^{\pi/2}\int_0^1 (r^2\cos^2\theta)(kr^2)\,r\,dr\,d\theta = \frac{1}{6}k\int_0^{\pi/6}\cos^2\theta\,d\theta = \frac{1}{6}k\left[\frac{1}{4}(2\theta+\sin 2\theta)\right]_0^{\pi/2} = \frac{\pi}{24}k$, and

$I_0 = I_x + I_y = \frac{\pi}{12}k$.

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17. $I_x = \int_{-1}^{2}\int_{y^2}^{y+2} 3y^2\,dx\,dy = \int_{-1}^{2}(3y^3+6y^2-3y^4)\,dy = \left[\frac{3}{4}y^4+2y^3-\frac{3}{5}y^5\right]_{-1}^{2} = \frac{189}{20}$,

$I_y = \int_{-1}^{2}\int_{y^2}^{y+2} 3x^2\,dx\,dy = \int_{-1}^{2}\left[(y+2)^3-y^6\right]dy = \left[\frac{1}{4}(y+2)^4-\frac{1}{7}y^7\right]_{-1}^{2} = \frac{1269}{28}$, and

$I_0 = I_x + I_y = \frac{1917}{35}$.

18. If we find the moments of inertia about the x- and y-axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the x-axis is given by

$$I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^2\int_0^2 y^2(1+0.1x)\,dy\,dx = \int_0^2(1+0.1x)\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=2}dx$$

$$= \tfrac{8}{3}\int_0^2(1+0.1x)\,dx = \tfrac{8}{3}\left[x+0.1\cdot\tfrac{1}{2}x^2\right]_0^2 = \tfrac{8}{3}(2.2) \approx 5.87$$

Similarly, the moment of inertia about the y-axis is given by

$$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^2\int_0^2 x^2(1+0.1x)\,dy\,dx = \int_0^2 x^2(1+0.1x)\left[y\right]_{y=0}^{y=2}dx$$

$$= 2\int_0^2(x^2+0.1x^3)\,dx = 2\left[\tfrac{1}{3}x^3+0.1\cdot\tfrac{1}{4}x^4\right]_0^2 = 2\left(\tfrac{8}{3}+0.4\right) \approx 6.13$$

Since $I_y > I_x$, more force is required to rotate the fan blade about the y-axis.

19. Using a CAS, we find $m = \iint_D \rho(x,y)\,dA = \int_0^{\pi}\int_0^{\sin x} xy\,dy\,dx = \dfrac{\pi^2}{8}$. Then

$$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{8}{\pi^2}\int_0^{\pi}\int_0^{\sin x} x^2y\,dy\,dx = \frac{2\pi}{3}-\frac{1}{\pi} \text{ and}$$

$$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{8}{\pi^2}\int_0^{\pi}\int_0^{\sin x} xy^2\,dy\,dx = \frac{16}{9\pi}, \text{ so } (\overline{x},\overline{y}) = \left(\frac{2\pi}{3}-\frac{1}{\pi},\frac{16}{9\pi}\right).$$

The moments of inertia are $I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^{\pi}\int_0^{\sin x} xy^3\,dy\,dx = \dfrac{3\pi^2}{64}$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^{\pi}\int_0^{\sin x} x^3y\,dy\,dx = \dfrac{\pi^2}{16}(\pi^2-3)$, and $I_0 = I_x + I_y = \dfrac{\pi^2}{64}(4\pi^2-9)$.

20. Using a CAS, we find $m = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^2\,dr\,d\theta = \frac{5}{3}\pi$,

$\overline{x} = \frac{1}{m}\iint_D x\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi}\int_0^{2\pi}\int_0^{1+\cos\theta} r^3\cos\theta\,dr\,d\theta = \frac{21}{20}$ and

$\overline{y} = \frac{1}{m}\iint_D y\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi}\int_0^{2\pi}\int_0^{1+\cos\theta} r^3\sin\theta\,dr\,d\theta = 0$, so $(\overline{x},\overline{y}) = (\frac{21}{20},0)$.

The moments of inertia are $I_x = \iint_D y^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^4\sin^2\theta\,dr\,d\theta = \frac{33}{40}\pi$,

$I_y = \iint_D x^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^4\cos^2\theta\,dr\,d\theta = \frac{93}{40}\pi$, and $I_0 = I_x + I_y = \frac{63}{20}\pi$.

21. $I_x = \int_0^a\int_0^a \rho y^2\,dx\,dy = \rho\int_0^a dx\int_0^a y^2\,dy = \rho\left[x\right]_0^a\left[\frac{1}{3}y^3\right]_0^a = \rho a\left(\frac{1}{3}a^3\right) = \frac{1}{3}\rho a^4 = I_y$ by symmetry, and $m = \rho a^2$ since the lamina is homogeneous. Hence $\overline{\overline{x}}^2 = \dfrac{I_y}{m} \quad\Rightarrow\quad \overline{\overline{x}} = \left[\left(\frac{1}{3}\rho a^4\right)/(\rho a^2)\right]^{1/2} = \frac{1}{\sqrt{3}}a$ and

$\overline{\overline{y}}^2 = \dfrac{I_x}{m} \quad\Rightarrow\quad \overline{\overline{y}} = \frac{1}{\sqrt{3}}a.$

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22. $m = \int_0^{\pi}\int_0^{\sin x} \rho\, dy\, dx = \rho\int_0^{\pi} \sin x\, dx = \rho\big[-\cos x\big]_0^{\pi} = 2\rho,$

$I_x = \int_0^{\pi}\int_0^{\sin x} \rho y^2\, dy\, dx = \frac{1}{3}\rho\int_0^{\pi} \sin^3 x\, dx = \frac{1}{3}\rho\int_0^{\pi}(1-\cos^2 x)\sin x\, dx$

$= \frac{1}{3}\rho\big[-\cos\theta + \frac{1}{3}\cos^3\theta\big]_0^{\pi} = \frac{4}{9}\rho,$

$I_y = \int_0^{\pi}\int_0^{\sin x} \rho x^2\, dy\, dx = \rho\int_0^{\pi} x^2 \sin x\, dx$

$= \rho\big[-x^2\cos x + 2x\sin x + 2\cos x\big]_0^{\pi}$ [by integrating by parts twice]

$= \rho(\pi^2 - 4).$

Then $\bar{\bar{y}}^2 = \dfrac{I_x}{m} = \dfrac{2}{9}$, so $\bar{\bar{y}} = \dfrac{\sqrt{2}}{3}$ and $\bar{\bar{x}}^2 = \dfrac{I_y}{m} = \dfrac{\pi^2-4}{2}$, so $\bar{\bar{x}} = \sqrt{\dfrac{\pi^2-4}{2}}$.

23. (a) $f(x,y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,1]\times[0,2]$, we can say

$$\iint_{\mathbb{R}^2} f(x,y)\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = \int_0^1\int_0^2 Cx(1+y)\, dy\, dx$$

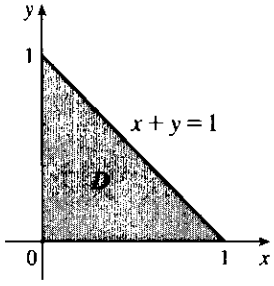
$$= C\int_0^1 x\big[y + \tfrac{1}{2}y^2\big]_{y=0}^{y=2} dx = C\int_0^1 4x\, dx = C\Big[2x^2\Big]_0^1 = 2C$$

Then $2C = 1 \;\Rightarrow\; C = \frac{1}{2}$.

(b) $P(X \le 1, Y \le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y)\, dy\, dx = \int_0^1\int_0^1 \frac{1}{2}x(1+y)\, dy\, dx$

$$= \int_0^1 \tfrac{1}{2}x\big[y + \tfrac{1}{2}y^2\big]_{y=0}^{y=1} dx = \int_0^1 \tfrac{1}{2}x\big(\tfrac{3}{2}\big)\, dx = \tfrac{3}{4}\big[\tfrac{1}{2}x^2\big]_0^1 = \tfrac{3}{8} \text{ or } 0.375$$

(c) $P(X+Y \le 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$P(X+Y \le 1) = \iint_D f(x,y)\, dA = \int_0^1\int_0^{1-x} \tfrac{1}{2}x(1+y)\, dy\, dx$$

$$= \int_0^1 \tfrac{1}{2}x\big[y + \tfrac{1}{2}y^2\big]_{y=0}^{y=1-x} dx = \int_0^1 \tfrac{1}{2}x\big(\tfrac{1}{2}x^2 - 2x + \tfrac{3}{2}\big)\, dx$$

$$= \tfrac{1}{4}\int_0^1 \big(x^3 - 4x^2 + 3x\big)\, dx = \tfrac{1}{4}\Big[\tfrac{x^4}{4} - 4\tfrac{x^3}{3} + 3\tfrac{x^2}{2}\Big]_0^1$$

$$= \tfrac{5}{48} \approx 0.1042$$

24. (a) $f(x,y) \ge 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$. Here, $f(x,y) = 0$ outside the square $[0,1]\times[0,1]$, so $\iint_{\mathbb{R}^2} f(x,y)\, dA = \int_0^1\int_0^1 4xy\, dy\, dx = \int_0^1 \big[2xy^2\big]_{y=0}^{y=1} dx = \int_0^1 2x\, dx = x^2\big]_0^1 = 1$. Thus, $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on Y, so

$$P\big(X \ge \tfrac{1}{2}\big) = \int_{1/2}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = \int_{1/2}^1\int_0^1 4xy\, dy\, dx$$

$$= \int_{1/2}^1 \big[2xy^2\big]_{y=0}^{y=1} dx = \int_{1/2}^1 2x\, dx = x^2\big]_{1/2}^1 = \tfrac{3}{4}$$

(ii) $P\big(X \ge \frac{1}{2}, Y \le \frac{1}{2}\big) = \int_{1/2}^{\infty}\int_{-\infty}^{1/2} f(x,y)\, dy\, dx = \int_{1/2}^1\int_0^{1/2} 4xy\, dy\, dx$

$$= \int_{1/2}^1 \big[2xy^2\big]_{y=0}^{y=1/2} dx = \int_{1/2}^1 \tfrac{1}{2}x\, dx = \tfrac{1}{2}\cdot\tfrac{1}{2}x^2\big]_{1/2}^1 = \tfrac{3}{16}$$

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(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x\, f(x,y)\, dA = \int_0^1 \int_0^1 x(4xy)\, dy\, dx = \int_0^1 2x^2 \left[y^2\right]_{y=0}^{y=1} dx$$
$$= 2\int_0^1 x^2\, dx = 2\left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3}$$

The expected value of Y is

$$\mu_2 = \iint_{\mathbb{R}^2} y\, f(x,y)\, dA = \int_0^1 \int_0^1 y(4xy)\, dy\, dx = \int_0^1 4x\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=1} dx$$
$$= \tfrac{4}{3}\int_0^1 x\, dx = \tfrac{4}{3}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{2}{3}$$

25. (a) $f(x,y) \geq 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$. Here, $f(x,y) = 0$ outside the first quadrant, so

$$\iint_{\mathbb{R}^2} f(x,y)\, dA = \int_0^\infty \int_0^\infty 0.1e^{-(0.5x+0.2y)}\, dy\, dx = 0.1\int_0^\infty \int_0^\infty e^{-0.5x}e^{-0.2y}\, dy\, dx$$
$$= 0.1\int_0^\infty e^{-0.5x}\, dx \int_0^\infty e^{-0.2y}\, dy = 0.1 \lim_{t\to\infty}\int_0^t e^{-0.5x}\, dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\, dy$$
$$= 0.1 \lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t$$
$$= 0.1 \lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right]$$
$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-1) = 1$$

Thus $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on X, so

$$P(Y \geq 1) = \int_{-\infty}^\infty \int_1^\infty f(x,y)\, dy\, dx = \int_0^\infty \int_1^\infty 0.1e^{-(0.5x+0.2y)}\, dy\, dx$$
$$= 0.1\int_0^\infty e^{-0.5x}\, dx \int_1^\infty e^{-0.2y}\, dy = 0.1 \lim_{t\to\infty}\int_0^t e^{-0.5x}\, dx \lim_{t\to\infty}\int_1^t e^{-0.2y}\, dy$$
$$= 0.1 \lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_1^t$$
$$= 0.1 \lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left[-5(e^{-0.2t}-e^{-0.2})\right]$$
$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2}) = e^{-0.2} \approx 0.8187$$

(ii) $P(X \leq 2, Y \leq 4) = \int_{-\infty}^2 \int_{-\infty}^4 f(x,y)\, dy\, dx = \int_0^2 \int_0^4 0.1e^{-(0.5x+0.2y)}\, dy\, dx$
$$= 0.1\int_0^2 e^{-0.5x}\, dx \int_0^4 e^{-0.2y}\, dy = 0.1\left[-2e^{-0.5x}\right]_0^2\left[-5e^{-0.2y}\right]_0^4$$
$$= (0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1)$$
$$= (e^{-1}-1)(e^{-0.8}-1) = 1+e^{-1.8}-e^{-0.8}-e^{-1} \approx 0.3481$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x\, f(x,y)\, dA = \int_0^\infty \int_0^\infty x\left[0.1e^{-(0.5x+0.2y)}\right] dy\, dx$$
$$= 0.1\int_0^\infty xe^{-0.5x}\, dx \int_0^\infty e^{-0.2y}\, dy = 0.1 \lim_{t\to\infty}\int_0^t xe^{-0.5x}\, dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\, dy$$

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To evaluate the first integral, we integrate by parts with $u = x$ and $dv = e^{-0.5x}\,dx$ (or we can use Formula 96 in the Table of Integrals):
$\int xe^{-0.5x}\,dx = -2xe^{-0.5x} - \int -2e^{-0.5x}\,dx = -2xe^{-0.5x} - 4e^{-0.5x} = -2(x+2)e^{-0.5x}$. Thus

$$\begin{aligned}\mu_1 &= 0.1 \lim_{t\to\infty} \left[-2(x+2)e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5e^{-0.2y}\right]_0^t \\ &= 0.1 \lim_{t\to\infty} (-2)\left[(t+2)e^{-0.5t} - 2\right] \lim_{t\to\infty} (-5)\left[e^{-0.2t} - 1\right] \\ &= 0.1(-2)\left(\lim_{t\to\infty} \frac{t+2}{e^{0.5t}} - 2\right)(-5)(-1) = 2 \qquad \text{[by l'Hospital's Rule]}\end{aligned}$$

The expected value of Y is given by

$$\begin{aligned}\mu_2 &= \iint_{\mathbb{R}^2} y\, f(x,y)\,dA = \int_0^\infty \int_0^\infty y\left[0.1e^{-(0.5+0.2y)}\right] dy\,dx \\ &= 0.1 \int_0^\infty e^{-0.5x}\,dx \int_0^\infty ye^{-0.2y}\,dy = 0.1 \lim_{t\to\infty} \int_0^t e^{-0.5x}\,dx \lim_{t\to\infty} \int_0^t ye^{-0.2y}\,dy\end{aligned}$$

To evaluate the second integral, we integrate by parts with $u = y$ and $dv = e^{-0.2y}\,dy$ (or again we can use Formula 96 in the Table of Integrals) which gives
$\int ye^{-0.2y}\,dy = -5ye^{-0.2y} + \int 5e^{-0.2y}\,dy = -5(y+5)e^{-0.2y}$. Then

$$\begin{aligned}\mu_2 &= 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5(y+5)e^{-0.2y}\right]_0^t \\ &= 0.1 \lim_{t\to\infty} \left[-2(e^{-0.5t} - 1)\right] \lim_{t\to\infty} \left(-5\left[(t+5)e^{-0.2t} - 5\right]\right) \\ &= 0.1(-2)(-1)\cdot(-5)\left(\lim_{t\to\infty} \frac{t+5}{e^{0.2t}} - 5\right) = 5 \qquad \text{[by l'Hospital's Rule]}\end{aligned}$$

26. (a) Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x,y) = \begin{cases} 10^{-6}e^{-(x+y)/1000} & \text{if } x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$\begin{aligned}P(X \leq 1000, Y \leq 1000) &= \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x,y)\,dy\,dx \\ &= \int_0^{1000} \int_0^{1000} 10^{-6}e^{-(x+y)/1000}\,dy\,dx \\ &= 10^{-6} \int_0^{1000} e^{-x/1000}\,dx \int_0^{1000} e^{-y/1000}\,dy \\ &= 10^{-6}\left[-1000e^{-x/1000}\right]_0^{1000} \left[-1000e^{-y/1000}\right]_0^{1000} \\ &= \left(e^{-1} - 1\right)^2 \approx 0.3996\end{aligned}$$

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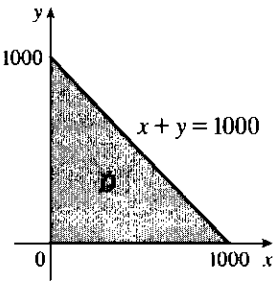
(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X + Y \le 1000)$, or equivalently $P((X, Y) \in D)$ where D is the triangular region shown in the figure.



Then

$$P(X+Y \le 1000) = \iint_D f(x,y)\,dA = \int_0^{1000}\int_0^{1000-x} 10^{-6}e^{-(x+y)/1000}\,dy\,dx$$
$$= 10^{-6}\int_0^{1000}\left[-1000e^{-(x+y)/1000}\right]_{y=0}^{y=1000-x} dx$$
$$= -10^{-3}\int_0^{1000}\left(e^{-1} - e^{-x/1000}\right) dx$$
$$= -10^{-3}\left[e^{-1}x + 1000e^{-x/1000}\right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642$$

27. (a) The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$. The individual density functions for X and Y, then, are $f_1(x) = \dfrac{1}{0.5\sqrt{2\pi}}\,e^{-(x-45)^2/0.5}$ and $f_2(y) = \dfrac{1}{0.1\sqrt{2\pi}}\,e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product

$$f(x,y) = f_1(x)f_2(y) = \frac{1}{0.5\sqrt{2\pi}}\,e^{-(x-45)^2/0.5}\,\frac{1}{0.1\sqrt{2\pi}}\,e^{-(y-20)^2/0.02}$$
$$= \tfrac{10}{\pi}e^{-2(x-45)^2-50(y-20)^2}$$

Then
$$P(40 \le X \le 50,\ 20 \le Y \le 25) = \int_{40}^{50}\int_{20}^{25} f(x,y)\,dy\,dx$$
$$= \tfrac{10}{\pi}\int_{40}^{50}\int_{20}^{25} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx$$

Using a CAS or calculator to evaluate the integral, we get $P(40 \le X \le 50,\ 20 \le Y \le 25) \approx 0.500$.

(b) $P(4(X-45)^2 + 100(Y-20)^2 \le 2) = \iint_D \tfrac{10}{\pi}e^{-2(x-45)^2-50(y-20)^2}\,dA$, where D is the region enclosed by the ellipse $4(x-45)^2 + 100(y-20)^2 = 2$. Solving for y gives $y = 20 \pm \frac{1}{10}\sqrt{2 - 4(x-45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y = 20$ [since the ellipse is centered at $(45, 20)$] $\Rightarrow$ $4(x-45)^2 = 2$ $\Rightarrow$ $x = 45 \pm \frac{1}{\sqrt{2}}$. Thus

$$\iint_D \tfrac{10}{\pi}e^{-2(x-45)^2-50(y-20)^2}\,dA = \tfrac{10}{\pi}\int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}}\ \int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx.$$

Using a CAS or calculator to evaluate the integral, we get $P(4(X-45)^2 + 100(Y-20)^2 \le 2) \approx 0.632$.

28. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50}e^{-x}y & \text{if } x \ge 0,\ 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X \ge Y$. Additionally, Yolanda will wait up to half an

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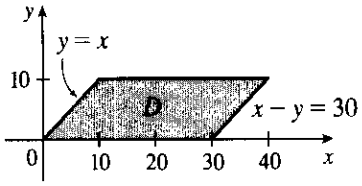
hour but no longer, so they won't meet unless $X - Y \le 30$. Thus the probability that they meet is $P((X, Y) \in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so



$$P((X,Y) \in D) = \iint_D f(x,y)\,dx\,dy = \int_0^{10}\int_y^{y+30} \tfrac{1}{50}e^{-x}y\,dx\,dy = \tfrac{1}{50}\int_0^{10} y\left[-e^{-x}\right]_{x=y}^{x=y+30}\,dy$$

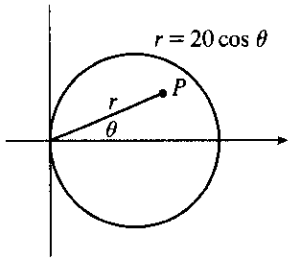
$$= \tfrac{1}{50}\int_0^{10} y(-e^{-(y+30)} + e^{-y})\,dy = \tfrac{1}{50}(1-e^{-30})\int_0^{10} ye^{-y}\,dy$$

By integration by parts (or Formula 96 in the Table of Integrals), this is $\frac{1}{50}(1-e^{-30})\left[-(y+1)e^{-y}\right]_0^{10} = \frac{1}{50}(1-e^{-30})(1-11e^{-10}) \approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

29. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P, and $k\,dA$ is the number of infected individuals in an element of area dA, then $f(P, A)k\,dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA. Integration over D gives the number of infections of the person at A due to all the infected people in D. In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

$$E = \iint_D kf(P,A)\,dA = k\iint_D \frac{20 - d(P,A)}{20}\,dA = k\iint_D \left[1 - \frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{20}\right] dx\,dy.$$

(b) If $A = (0,0)$, then



$$E = k\iint_D \left[1 - \frac{1}{20}\sqrt{x^2+y^2}\right] dx\,dy$$

$$= k\int_0^{2\pi}\int_0^{10} \left(1 - \frac{r}{20}\right) r\,dr\,d\theta = 2\pi k\left[\frac{r^2}{2} - \frac{r^3}{60}\right]_0^{10}$$

$$= 2\pi k\left(50 - \tfrac{50}{3}\right) = \tfrac{200}{3}\pi k \approx 209k$$

For A at the edge of the city, it is convenient to use a polar coordinate system centered at A. Then the polar equation for the circular boundary of the city becomes $r = 20\cos\theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$E = k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta} \left(1 - \frac{r}{20}\right) r\,dr\,d\theta = k\int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2} - \frac{r^3}{60}\right]_{r=0}^{r=20\cos\theta} d\theta$$

$$= k\int_{-\pi/2}^{\pi/2} \left(200\cos^2\theta - \tfrac{400}{3}\cos^3\theta\right) d\theta = 200k\int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta - \tfrac{2}{3}(1-\sin^2\theta)\cos\theta\right] d\theta$$

$$= 200k\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{2}{3}\sin\theta + \tfrac{2}{3}\cdot\tfrac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} = 200k\left[\tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9} + \tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9}\right]$$

$$= 200k\left(\tfrac{\pi}{2} - \tfrac{8}{9}\right) \approx 136k$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

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16.6 Surface Area ET 15.6

1. Here $z = f(x,y) = 2 + 3x + 4y$ and D is the rectangle $[0,5] \times [1,4]$, so by Formula 2 the area of the surface is

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA = \iint_D \sqrt{3^2 + 4^2 + 1}\, dA = \sqrt{26} \iint_D dA$$
$$= \sqrt{26}\, A(D) = \sqrt{26}\,(5)(3) = 15\sqrt{26}$$

2. $z = f(x,y) = 10 - 2x - 5y$ and D is the disk $x^2 + y^2 \le 9$, so by Formula 2

$$A(S) = \iint_D \sqrt{(-2)^2 + (-5)^2 + 1}\, dA = \sqrt{30} \iint_D dA = \sqrt{30}\, A(D)$$
$$= \sqrt{30}\,(\pi \cdot 3^2) = 9\sqrt{30}\,\pi$$

3. $z = f(x,y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\left\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3 - \frac{3}{2}x\right\}$. Thus

$$A(S) = \iint_D \sqrt{(-3)^2 + (-2)^2 + 1}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D)$$
$$= \sqrt{14}\left(\tfrac{1}{2} \cdot 2 \cdot 3\right) = 3\sqrt{14}$$

4. $z = f(x,y) = 1 + 3x + 2y^2$ with $0 \le x \le 2y$, $0 \le y \le 1$. Thus by Formula 2,

$$A(S) = \iint_D \sqrt{(3)^2 + (4y)^2 + 1}\, dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy = \int_0^1 \sqrt{10 + 16y^2}\, \big[x\big]_{x=0}^{x=2y}\, dy$$
$$= \int_0^1 2y\sqrt{10 + 16y^2}\, dy = 2 \cdot \tfrac{1}{32} \cdot \tfrac{2}{3}(10 + 16y^2)^{3/2}\Big]_0^1 = \tfrac{1}{24}(26^{3/2} - 10^{3/2})$$

5. $y^2 + z^2 = 9 \;\Rightarrow\; z = \sqrt{9 - y^2}$. $f_x = 0$, $f_y = -y(9 - y^2)^{-1/2} \;\Rightarrow$

$$A(S) = \int_0^4 \int_0^2 \sqrt{0^2 + [-y(9 - y^2)^{-1/2}]^2 + 1}\, dy\, dx = \int_0^4 \int_0^2 \sqrt{\frac{y^2}{9 - y^2} + 1}\, dy\, dx$$
$$= \int_0^4 \int_0^2 \frac{3}{\sqrt{9 - y^2}}\, dy\, dx = 3\int_0^4 \Big[\sin^{-1}\frac{y}{3}\Big]_{y=0}^{y=2} dx = 3\left[\left(\sin^{-1}\left(\tfrac{2}{3}\right)\right)x\right]_0^4 = 12\sin^{-1}\left(\tfrac{2}{3}\right)$$

6. $z = f(x,y) = 4 - x^2 - y^2$ and D is the projection of the paraboloid $z = 4 - x^2 - y^2$ onto the xy-plane, that is, $D = \left\{(x,y) \mid x^2 + y^2 \le 4\right\}$. So $f_x = -2x$, $f_y = -2y \;\Rightarrow$

$$A(S) = \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\, dA = \iint_D \sqrt{4(x^2 + y^2) + 1}\, dA = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{1}{12}(4r^2 + 1)^{3/2}\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \tfrac{1}{12}(17\sqrt{17} - 1)\, d\theta = \tfrac{\pi}{6}(17\sqrt{17} - 1)$$

7. $z = f(x,y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{1 + 4r^2}\, dr$$
$$= \big[\theta\big]_0^{2\pi} \left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

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8. $z = f(x,y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$A(S) = \iint_D \sqrt{(\sqrt{x})^2 + (\sqrt{y})^2 + 1}\, dA = \int_0^1 \int_0^1 \sqrt{x+y+1}\, dy\, dx$$

$$= \int_0^1 \left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{y=0}^{y=1} dx = \tfrac{2}{3}\int_0^1 \left[(x+2)^{3/2} - (x+1)^{3/2}\right] dx$$

$$= \tfrac{2}{3}\left[\tfrac{2}{5}(x+2)^{5/2} - \tfrac{2}{5}(x+1)^{5/2}\right]_0^1 = \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1)$$

$$= \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1)$$

9. $z = f(x,y) = xy$ with $0 \le x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x$ $\Rightarrow$

$$A(S) = \iint_D \sqrt{y^2 + x^2 + 1}\, dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}(r^2+1)^{3/2}\right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi} \tfrac{1}{3}(2\sqrt{2} - 1)\, d\theta = \tfrac{2\pi}{3}(2\sqrt{2} - 1)$$

10. Given the sphere $x^2 + y^2 + z^2 = 4$, when $z = 1$, we get $x^2 + y^2 = 3$ so $D = \{(x,y) \mid x^2 + y^2 \le 3\}$ and $z = f(x,y) = \sqrt{4 - x^2 - y^2}$. Thus

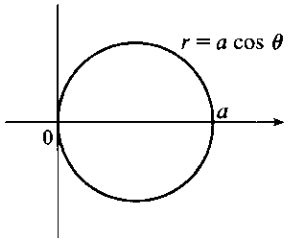
$$A(S) = \iint_D \sqrt{[(-x)(4-x^2-y^2)^{-1/2}]^2 + [(-y)(4-x^2-y^2)^{-1/2}]^2 + 1}\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4-r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}}\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[-2(4-r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta = \int_0^{2\pi}(-2+4)\, d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

11. $z = \sqrt{a^2 - x^2 - y^2}$, $z_x = -x(a^2 - x^2 - y^2)^{-1/2}$, $z_y = -y(a^2 - x^2 - y^2)^{-1/2}$,

$$A(S) = \iint_D \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2} + 1}\, dA$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \sqrt{\frac{r^2}{a^2-r^2} + 1}\, r\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \frac{ar}{\sqrt{a^2-r^2}}\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \left[-a\sqrt{a^2-r^2}\right]_{r=0}^{r=a\cos\theta} d\theta$$
$$= \int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2 - a^2\cos^2\theta} - a\right) d\theta = 2a^2\int_0^{\pi/2} \left(1 - \sqrt{1-\cos^2\theta}\right) d\theta$$
$$= 2a^2\int_0^{\pi/2} d\theta - 2a^2\int_0^{\pi/2} \sqrt{\sin^2\theta}\, d\theta = a^2\pi - 2a^2\int_0^{\pi/2} \sin\theta\, d\theta = a^2(\pi - 2)$$



12. To find the region D: $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z-2)^2 = 4$, so $z = 3$ intersects the upper

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hemisphere. Thus $(z-2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3-2)^2 = 4$, that is, $D = \{(x, y) \mid x^2 + y^2 \le 3\}$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + [(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2}\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + \frac{r^2}{4 - r^2}}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r\, dr}{\sqrt{4 - r^2}}\, d\theta = \int_0^{2\pi} \left[-2(4 - r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta \\ &= \int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta \Big]_0^{2\pi} = 4\pi \end{aligned}$$

13. $z = f(x, y) = e^{-x^2 - y^2}$, $f_x = -2xe^{-x^2-y^2}$, $f_y = -2ye^{-x^2-y^2}$. Then

$$A(S) = \iint_{x^2+y^2 \le 4} \sqrt{(-2xe^{-x^2-y^2})^2 + (-2ye^{-x^2-y^2})^2 + 1}\, dA = \iint_{x^2+y^2 \le 4} \sqrt{4(x^2 + y^2)e^{-2(x^2+y^2)} + 1}\, dA.$$

Converting to polar coordinates we have

$$\begin{aligned} A(S) &= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 e^{-2r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 r \sqrt{4r^2 e^{-2r^2} + 1}\, dr \\ &= 2\pi \int_0^2 r \sqrt{4r^2 e^{-2r^2} + 1}\, dr \approx 13.9783 \text{ using a calculator.} \end{aligned}$$

14. $z = f(x, y) = \cos(x^2 + y^2)$, $f_x = -2x \sin(x^2 + y^2)$, $f_y = -2y \sin(x^2 + y^2)$.

$$\begin{aligned} A(S) &= \iint_{x^2+y^2 \le 1} \sqrt{4x^2 \sin^2(x^2 + y^2) + 4y^2 \sin^2(x^2 + y^2) + 1}\, dA \\ &= \iint_{x^2+y^2 \le 1} \sqrt{4(x^2 + y^2) \sin^2(x^2 + y^2) + 1}\, dA \end{aligned}$$

Converting to polar coordinates gives

$$\begin{aligned} A(S) &= \int_0^{2\pi} \int_0^1 \sqrt{4r^2 \sin^2(r^2) + 1}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 r \sqrt{4r^2 \sin^2(r^2) + 1}\, dr \\ &= 2\pi \int_0^1 r \sqrt{4r^2 \sin^2(r^2) + 1}\, dr \approx 4.1073 \quad \text{using a calculator.} \end{aligned}$$

15. (a) The midpoints of the four squares are $\left(\frac{1}{4}, \frac{1}{4}\right)$, $\left(\frac{1}{4}, \frac{3}{4}\right)$, $\left(\frac{3}{4}, \frac{1}{4}\right)$, and $\left(\frac{3}{4}, \frac{3}{4}\right)$. Here $f(x, y) = x^2 + y^2$, so the Midpoint Rule gives

$$\begin{aligned} A(S) &= \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\, dA \\ &\approx \tfrac{1}{4}\left(\sqrt{\left[2\left(\tfrac{1}{4}\right)\right]^2 + \left[2\left(\tfrac{1}{4}\right)\right]^2 + 1} + \sqrt{\left[2\left(\tfrac{1}{4}\right)\right]^2 + \left[2\left(\tfrac{3}{4}\right)\right]^2 + 1} \right. \\ &\qquad \left. + \sqrt{\left[2\left(\tfrac{3}{4}\right)\right]^2 + \left[2\left(\tfrac{1}{4}\right)\right]^2 + 1} + \sqrt{\left[2\left(\tfrac{3}{4}\right)\right]^2 + \left[2\left(\tfrac{3}{4}\right)\right]^2 + 1} \right) \\ &= \tfrac{1}{4}\left(\sqrt{\tfrac{3}{2}} + 2\sqrt{\tfrac{7}{2}} + \sqrt{\tfrac{11}{2}} \right) \approx 1.8279 \end{aligned}$$

(b) A CAS estimates the integral to be $A(S) = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \int_0^1 \int_0^1 \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx \approx 1.8616$. This agrees with the Midpoint estimate only in the first decimal place.

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16.7 Triple Integrals ET 15.7

1. $\iiint_B xyz^2\,dV = \int_0^1\int_{-1}^2\int_0^3 xyz^2\,dz\,dx\,dy = \int_0^1\int_{-1}^2 xy\left[\frac{1}{3}z^3\right]_{z=0}^{z=3}dx\,dy = \int_0^1\int_{-1}^2 9xy\,dx\,dy$

$= \int_0^1\left[\frac{9}{2}x^2y\right]_{x=-1}^{x=2}dy = \int_0^1 \frac{27}{2}y\,dy = \frac{27}{4}y^2\Big]_0^1 = \frac{27}{4}$

2. There are six different possible orders of integration.

$$\iiint_E (xz-y^3)\,dV = \int_{-1}^1\int_0^2\int_0^1 (xz-y^3)\,dz\,dy\,dx = \int_{-1}^1\int_0^2\left[\tfrac{1}{2}xz^2-y^3z\right]_{z=0}^{z=1}dy\,dx$$
$$= \int_{-1}^1\int_0^2\left(\tfrac{1}{2}x-y^3\right)dy\,dx = \int_{-1}^1\left[\tfrac{1}{2}xy-\tfrac{1}{4}y^4\right]_{y=0}^{y=2}dx$$
$$= \int_{-1}^1 (x-4)\,dx = \left[\tfrac{1}{2}x^2-4x\right]_{-1}^1 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^2\int_{-1}^1\int_0^1 (xz-y^3)\,dz\,dx\,dy = \int_0^2\int_{-1}^1\left[\tfrac{1}{2}xz^2-y^3z\right]_{z=0}^{z=1}dx\,dy$$
$$= \int_0^2\int_{-1}^1\left(\tfrac{1}{2}x-y^3\right)dx\,dy = \int_0^2\left[\tfrac{1}{4}x^2-xy^3\right]_{x=-1}^{x=1}dy$$
$$= \int_0^2 -2y^3\,dy = -\tfrac{1}{2}y^4\Big]_0^2 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_{-1}^1\int_0^1\int_0^2 (xz-y^3)\,dy\,dz\,dx = \int_{-1}^1\int_0^1\left[xyz-\tfrac{1}{4}y^4\right]_{y=0}^{y=2}dz\,dx$$
$$= \int_{-1}^1\int_0^1 (2xz-4)\,dz\,dx = \int_{-1}^1\left[xz^2-4z\right]_{z=0}^{z=1}dx$$
$$= \int_{-1}^1 (x-4)\,dx = \left[\tfrac{1}{2}x^2-4x\right]_{-1}^1 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^1\int_{-1}^1\int_0^2 (xz-y^3)\,dy\,dx\,dz = \int_0^1\int_{-1}^1\left[xyz-\tfrac{1}{4}y^4\right]_{y=0}^{y=2}dx\,dz$$
$$= \int_0^1\int_{-1}^1 (2xz-4)\,dx\,dz = \int_0^1\left[x^2z-4x\right]_{x=-1}^{x=1}dz$$
$$= \int_0^1 -8\,dz = -8z\Big]_0^1 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^2\int_0^1\int_{-1}^1 (xz-y^3)\,dx\,dz\,dy = \int_0^2\int_0^1\left[\tfrac{1}{2}x^2z-xy^3\right]_{x=-1}^{x=1}dz\,dy$$
$$= \int_0^2\int_0^1 -2y^3\,dz\,dy = \int_0^2\left[-2y^3z\right]_{z=0}^{z=1}dy = \int_0^2 -2y^3\,dy = -\tfrac{1}{2}y^4\Big]_0^2 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^1\int_0^2\int_{-1}^1 (xz-y^3)\,dx\,dy\,dz = \int_0^1\int_0^2\left[\tfrac{1}{2}x^2z-xy^3\right]_{x=-1}^{x=1}dy\,dz$$
$$= \int_0^1\int_0^2 -2y^3\,dy\,dz = \int_0^1\left[-\tfrac{1}{2}y^4\right]_{y=0}^{y=2}dz = \int_0^1 -8\,dz = -8z\Big]_0^1 = -8$$

3. $\int_0^1\int_0^z\int_0^{x+z} 6xz\,dy\,dx\,dz = \int_0^1\int_0^z\left[6xyz\right]_{y=0}^{y=x+z}dx\,dz = \int_0^1\int_0^z 6xz(x+z)\,dx\,dz$

$= \int_0^1\left[2x^3z+3x^2z^2\right]_{x=0}^{x=z}dz = \int_0^1(2z^4+3z^4)\,dz = \int_0^1 5z^4\,dz = z^5\Big]_0^1 = 1$

4. $\int_0^1\int_x^{2x}\int_0^y 2xyz\,dz\,dy\,dx = \int_0^1\int_x^{2x}\left[xyz^2\right]_{z=0}^{z=y}dy\,dx = \int_0^1\int_x^{2x} xy^3\,dy\,dx$

$= \int_0^1\left[\tfrac{1}{4}xy^4\right]_{y=x}^{y=2x}dx = \int_0^1\frac{15}{4}x^5\,dx = \frac{5}{8}x^6\Big]_0^1 = \frac{5}{8}$

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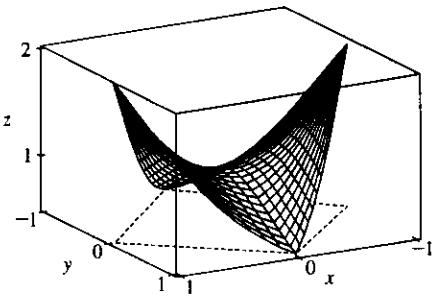
20. Let $f(x,y) = \dfrac{1+x^2}{1+y^2}$. Then $f_x = \dfrac{2x}{1+y^2}$,

$f_y = (1+x^2)\left[-\dfrac{2y}{(1+y^2)^2}\right] = -\dfrac{2y(1+x^2)}{(1+y^2)^2}$. We use a CAS to estimate $\int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|}\sqrt{f_x^2+f_y^2+1}\,dy\,dx \approx 2.6959$. In order to graph only the part of the surface above the square, we use $-(1-|x|) \le y \le 1-|x|$ as the y-range in our plot command.



21. Here $z = f(x,y) = ax+by+c$, $f_x(x,y) = a$, $f_y(x,y) = b$, so

$A(S) = \iint_D \sqrt{a^2+b^2+1}\,dA = \sqrt{a^2+b^2+1}\iint_D dA = \sqrt{a^2+b^2+1}\,A(D)$.

22. Let S be the upper hemisphere. Then $z = f(x,y) = \sqrt{a^2-x^2-y^2}$, so

$$\begin{aligned}A(S) &= \iint_D \sqrt{[-x(a^2-x^2-y^2)^{-1/2}]^2+[-y(a^2-x^2-y^2)^{-1/2}]^2+1}\,dA\\ &= \iint_D \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2}+1}\,dA = \lim_{t\to a^-}\int_0^{2\pi}\int_0^t \sqrt{\frac{r^2}{a^2-r^2}+1}\,r\,dr\,d\theta\\ &= \lim_{t\to a^-}\int_0^{2\pi}\int_0^t \frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta = 2\pi\lim_{t\to a^-}\left[-a\sqrt{a^2-r^2}\right]_0^t = 2\pi\lim_{t\to a^-} -a\left[\sqrt{a^2-t^2}-a\right]\\ &= 2\pi(-a)(-a) = 2\pi a^2.\end{aligned}$$

Thus the surface area of the entire sphere is $4\pi a^2$.

23. If we project the surface onto the xz-plane, then the surface lies "above" the disk $x^2+z^2 \le 25$ in the xz-plane. We have $y = f(x,z) = x^2+z^2$ and, adapting Formula 2, the area of the surface is

$$A(S) = \iint_{x^2+z^2\le 25} \sqrt{[f_x(x,z)]^2+[f_z(x,z)]^2+1}\,dA = \iint_{x^2+z^2\le 25} \sqrt{4x^2+4z^2+1}\,dA$$

Converting to polar coordinates $x = r\cos\theta$, $z = r\sin\theta$ we have

$$\begin{aligned}A(S) &= \int_0^{2\pi}\int_0^5 \sqrt{4r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_0^5 r(4r^2+1)^{1/2}\,dr = \big[\theta\big]_0^{2\pi}\left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_0^5\\ &= \tfrac{\pi}{6}\left(101\sqrt{101}-1\right)\end{aligned}$$

24. First we find the area of the face of the surface that intersects the positive y-axis. As in Exercise 23, we can project the face onto the xz-plane, so the surface lies "above" the disk $x^2+z^2 \le 1$. Then $z = f(x,z) = \sqrt{1-z^2}$ and the area is

$$\begin{aligned}A(S) &= \iint_{x^2+z^2\le 1} \sqrt{[f_x(x,z)]^2+[f_z(x,z)]^2+1}\,dA = \iint_{x^2+z^2\le 1} \sqrt{0+\left(\frac{-z}{\sqrt{1-z^2}}\right)^2+1}\,dA\\ &= \iint_{x^2+z^2\le 1} \sqrt{\frac{z^2}{1-z^2}+1}\,dA = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\,dx\,dz\\ &= 4\int_0^1\int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\,dx\,dz \qquad \text{[by the symmetry of the surface]}\end{aligned}$$

This integral is improper (when $z = 1$), so

$$A(S) = \lim_{t\to 1^-} 4\int_0^t\int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\,dx\,dz = \lim_{t\to 1^-} 4\int_0^t \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}}\,dz = \lim_{t\to 1^-} 4\int_0^t dz = \lim_{t\to 1^-} 4t = 4.$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4) = 16$.

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16. (a) With $m = n = 2$ we have four squares with midpoints $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}\right)$, and $\left(\frac{3}{2}, \frac{3}{2}\right)$. Since $z = xy + x^2 + y^2$, the Midpoint Rule gives

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA = \iint_D \sqrt{(y+2x)^2 + (x+2y)^2 + 1}\, dA$$

$$\approx 1\left(\sqrt{\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{3}{2}\right)^2 + 1} + \sqrt{\left(\tfrac{5}{2}\right)^2 + \left(\tfrac{7}{2}\right)^2 + 1} + \sqrt{\left(\tfrac{7}{2}\right)^2 + \left(\tfrac{5}{2}\right)^2 + 1} + \sqrt{\left(\tfrac{9}{2}\right)^2 + \left(\tfrac{9}{2}\right)^2 + 1}\right)$$

$$= \frac{\sqrt{22}}{2} + \frac{\sqrt{78}}{2} + \frac{\sqrt{78}}{2} + \frac{\sqrt{166}}{2} \approx 17.619$$

(b) Using a CAS, we have

$$A(S) = \iint_D \sqrt{(y+2x)^2 + (x+2y)^2 + 1}\, dA = \int_0^2 \int_0^2 \sqrt{1 + (y+2x)^2 + (x+2y)^2}\, dy\, dx \approx 17.7165.$$

This is within about 0.1 of the Midpoint Rule estimate.

17. $z = 1 + 2x + 3y + 4y^2$, so

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int_1^4 \int_0^1 \sqrt{1 + 4 + (3+8y)^2}\, dy\, dx$$

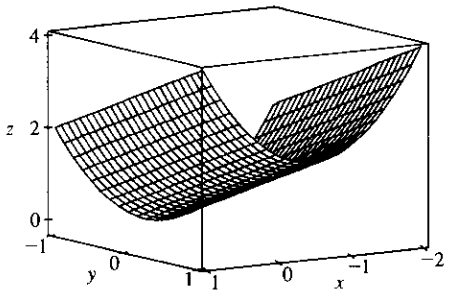
$$= \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\, dy\, dx.$$

Using a CAS, we have

$$\int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\, dy\, dx = \tfrac{45}{8}\sqrt{14} + \tfrac{15}{16}\ln\left(11\sqrt{5} + 3\sqrt{14}\sqrt{5}\right) - \tfrac{15}{16}\ln\left(3\sqrt{5} + \sqrt{14}\sqrt{5}\right)$$

or $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\dfrac{11\sqrt{5} + 3\sqrt{70}}{3\sqrt{5} + \sqrt{70}}$.

18. $f(x,y) = 1 + x + y + x^2 \;\Rightarrow\; f_x = 1 + 2x$, $f_y = 1$. We use a CAS to calculate the integral $A(S) = \int_{-2}^{1}\int_{-1}^{1} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx = \int_{-2}^{1}\int_{-1}^{1} \sqrt{(1+2x)^2 + 2}\, dy\, dx = 2\int_{-2}^{1} \sqrt{4x^2 + 4x + 3}\, dx$ and find that $A(S) = 3\sqrt{11} + 2\sinh^{-1}\left(\frac{3\sqrt{2}}{2}\right)$ or $A(S) = 3\sqrt{11} + \ln\left(10 + 3\sqrt{11}\right)$.



19. $f(x,y) = 1 + x^2y^2 \;\Rightarrow\; f_x = 2xy^2$, $f_y = 2x^2y$. We use a CAS (with precision reduced to five significant digits, to speed up the calculation) to estimate the integral

$$A(S) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2y^4 + 4x^4y^2 + 1}\, dy\, dx, \text{ and find that}$$

$A(S) \approx 3.3213$.

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5. $\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y \, dx \, dz \, dy = \int_0^3 \int_0^1 \left[xze^y\right]_{x=0}^{x=\sqrt{1-z^2}} dz \, dy = \int_0^3 \int_0^1 ze^y\sqrt{1-z^2} \, dz \, dy$

$= \int_0^3 \left[-\frac{1}{3}(1-z^2)^{3/2}e^y\right]_{z=0}^{z=1} dy = \int_0^3 \frac{1}{3}e^y \, dy = \left[\frac{1}{3}e^y\right]_0^3 = \frac{1}{3}(e^3-1)$

6. $\int_0^1 \int_0^z \int_0^y ze^{-y^2} \, dx \, dy \, dz = \int_0^1 \int_0^z \left[xze^{-y^2}\right]_{x=0}^{x=y} dy \, dz = \int_0^1 \int_0^z yze^{-y^2} \, dy \, dz = \int_0^1 \left[-\frac{1}{2}ze^{-y^2}\right]_{y=0}^{y=z} dz$

$= \int_0^1 -\frac{1}{2}z\left(e^{-z^2}-1\right) dz = \frac{1}{2}\int_0^1 \left(z - ze^{-z^2}\right) dz$

$= \frac{1}{2}\left[\frac{1}{2}z^2 + \frac{1}{2}e^{-z^2}\right]_0^1 = \frac{1}{4}(1+e^{-1}-0-1) = \frac{1}{4e}$

7. $\iiint_E 2x \, dV = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^y 2x \, dz \, dx \, dy = \int_0^2 \int_0^{\sqrt{4-y^2}} \left[2xz\right]_{z=0}^{z=y} dx \, dy = \int_0^2 \int_0^{\sqrt{4-y^2}} 2xy \, dx \, dy$

$= \int_0^2 \left[x^2y\right]_{x=0}^{x=\sqrt{4-y^2}} dy = \int_0^2 (4-y^2)y \, dy = \left[2y^2 - \frac{1}{4}y^4\right]_0^2 = 4$

8. $\iiint_E yz\cos(x^5) \, dV = \int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5) \, dz \, dy \, dx = \int_0^1 \int_0^x \left[\frac{1}{2}yz^2\cos(x^5)\right]_{z=x}^{z=2x} dy \, dx$

$= \frac{1}{2}\int_0^1 \int_0^x 3x^2y\cos(x^5) \, dy \, dx = \frac{1}{2}\int_0^1 \left[\frac{3}{2}x^2y^2\cos(x^5)\right]_{y=0}^{y=x} dx$

$= \frac{3}{4}\int_0^1 x^4\cos(x^5) \, dx = \frac{3}{4}\left[\frac{1}{5}\sin(x^5)\right]_0^1 = \frac{3}{20}(\sin 1 - \sin 0) = \frac{3}{20}\sin 1$

9. Here $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1+x+y\}$, so

$\iiint_E 6xy \, dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy \, dz \, dy \, dx = \int_0^1 \int_0^{\sqrt{x}} \left[6xyz\right]_{z=0}^{z=1+x+y} dy \, dx$

$= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y) \, dy \, dx = \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3\right]_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2}) \, dx = \left[x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2}\right]_0^1 = \frac{65}{28}$

10. Here E is the region in the first octant that lies below the plane $2x + 2y + z = 4$ (and above the region in the xy-plane bounded by the lines $x = 0$, $y = 0$, $x + y = 2$). So

$\iiint_E y \, dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y \, dz \, dy \, dx = \int_0^2 \int_0^{2-x} y(4-2x-2y) \, dy \, dx$

$= \int_0^2 \int_0^{2-x} (4y - 2xy - 2y^2) \, dy \, dx = \int_0^2 \left[2y^2 - xy^2 - \frac{2}{3}y^3\right]_{y=0}^{y=2-x} dx$

$= \int_0^2 \left[2(2-x)^2 - x(2-x)^2 - \frac{2}{3}(2-x)^3\right] dx$

$= \int_0^2 \left[(2-x)(2-x)^2 - \frac{2}{3}(2-x)^3\right] dx = \frac{1}{3}\int_0^2 (2-x)^3 \, dx$

$= \frac{1}{3}\left[-\frac{1}{4}(2-x)^4\right]_0^2 = -\frac{1}{12}(0-16) = \frac{4}{3}$

11. Here E is the region that lies below the plane with x-, y-, and z-intercepts 1, 2, and 3 respectively, that is, below the plane $2z + 6x + 3y = 6$ and above the region in the xy-plane bounded by the lines $x = 0$, $y = 0$ and $6x + 3y = 6$. So

$\iiint_E xy \, dV = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} xy \, dz \, dy \, dx = \int_0^1 \int_0^{2-2x} \left(3xy - 3x^2y - \frac{3}{2}xy^2\right) dy \, dx$

$= \int_0^1 \left[\frac{3}{2}xy^2 - \frac{3}{2}x^2y^2 - \frac{1}{2}xy^3\right]_{y=0}^{y=2-2x} dx = \int_0^1 (2x - 6x^2 + 6x^3 - 2x^4) \, dx$

$= \left[x^2 - 2x^3 + \frac{3}{2}x^4 - \frac{2}{5}x^5\right]_0^1 = \frac{1}{10}.$

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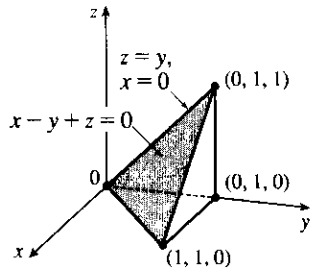
12.

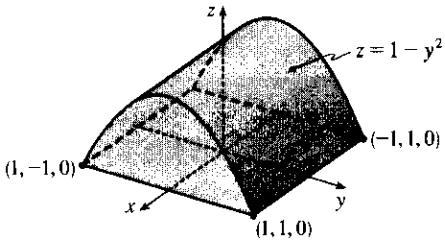


$$\int_0^1 \int_0^y \int_0^{y-z} xz\,dx\,dz\,dy = \int_0^1 \int_0^y \tfrac{1}{2}(y-z)^2 z\,dz\,dy$$

$$= \tfrac{1}{2}\int_0^1 \left[\tfrac{1}{2}y^2z^2 - \tfrac{2}{3}yz^3 + \tfrac{1}{4}z^4\right]_{z=0}^{z=y} dy$$

$$= \tfrac{1}{24}\int_0^1 y^4\,dy = \tfrac{1}{24}\left[\tfrac{1}{5}y^5\right]_0^1 = \tfrac{1}{120}$$

13.



E is the region below the parabolic cylinder $z = 1 - y^2$ and above the square $[-1, 1] \times [-1, 1]$ in the xy-plane.

$$\iiint_E x^2e^y\,dV = \int_{-1}^1 \int_{-1}^1 \int_0^{1-y^2} x^2e^y\,dz\,dy\,dx$$

$$= \int_{-1}^1 \int_{-1}^1 x^2e^y(1-y^2)\,dy\,dx$$

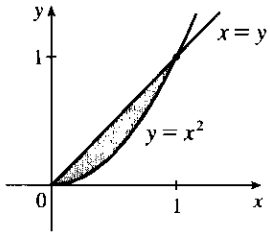
$$= \int_{-1}^1 x^2\,dx \int_{-1}^1 (e^y - y^2e^y)\,dy$$

$$= \left[\tfrac{1}{3}x^3\right]_{-1}^1 \left[e^y - (y^2 - 2y + 2)e^y\right]_{-1}^1$$

[integrate by parts twice]

$$= \tfrac{1}{3}(2)[e - e - e^{-1} + 5e^{-1}] = \tfrac{8}{3e}$$

14.



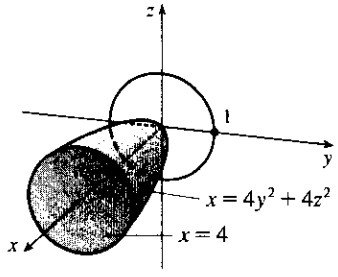
E is the solid above the region shown in the xy-plane and below the plane $z = x$. Thus,

$$\iiint_E (x + 2y)\,dV = \int_0^1 \int_{x^2}^x \int_0^x (x + 2y)\,dz\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^x (x^2 + 2yx)\,dy\,dx = \int_0^1 \left[x^2y + xy^2\right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 (2x^3 - x^4 - x^5)\,dx = \left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6\right]_0^1 = \tfrac{2}{15}$$

15.



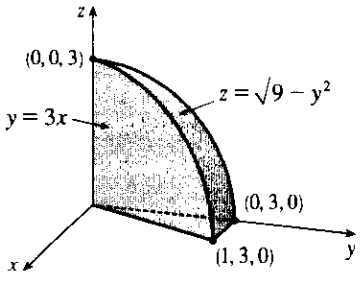
The projection E on the yz-plane is the disk $y^2 + z^2 \le 1$. Using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we get

$$\iiint_E x\,dV = \iint_D \left[\int_{4y^2+4z^2}^4 x\,dx\right] dA$$

$$= \tfrac{1}{2}\iint_D \left[4^2 - (4y^2 + 4z^2)^2\right] dA = 8\int_0^{2\pi}\int_0^1 (1 - r^4)\,r\,dr\,d\theta$$

$$= 8\int_0^{2\pi} d\theta \int_0^1 (r - r^5)\,dr = 8(2\pi)\left[\tfrac{1}{2}r^2 - \tfrac{1}{6}r^6\right]_0^1 = \tfrac{16\pi}{3}$$

16.



$$\int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = \int_0^1 \int_{3x}^3 \tfrac{1}{2}(9 - y^2)\,dy\,dx$$

$$= \int_0^1 \left[\tfrac{9}{2}y - \tfrac{1}{6}y^3\right]_{y=3x}^{y=3} dx$$

$$= \int_0^1 \left[9 - \tfrac{27}{2}x + \tfrac{9}{2}x^3\right] dx$$

$$= \left[9x - \tfrac{27}{4}x^2 + \tfrac{9}{8}x^4\right]_0^1 = \tfrac{27}{8}$$

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17. The plane $2x + y + z = 4$ intersects the xy-plane when

$2x + y + 0 = 4 \quad \Rightarrow \quad y = 4 - 2x$, so

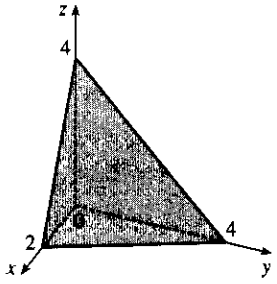
$E = \{(x, y, z) \mid 0 \le x \le 2, 0 \le y \le 4 - 2x, 0 \le z \le 4 - 2x - y\}$ and

$$V = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx = \int_0^2 \int_0^{4-2x} (4 - 2x - y)\, dy\, dx$$

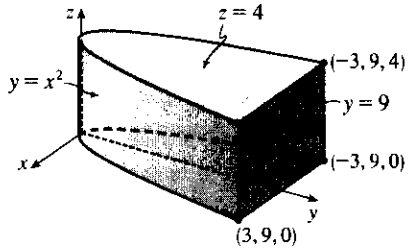
$$= \int_0^2 \left[4y - 2xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx$$

$$= \int_0^2 \left[4(4 - 2x) - 2x(4 - 2x) - \tfrac{1}{2}(4 - 2x)^2\right] dx$$

$$= \int_0^2 (2x^2 - 8x + 8)\, dx = \left[\tfrac{2}{3}x^3 - 4x^2 + 8x\right]_0^2 = \tfrac{16}{3}$$



18.



$$V = \iiint_E dV = \int_{-3}^{3} \int_{x^2}^{9} \int_0^4 dz\, dy\, dx$$

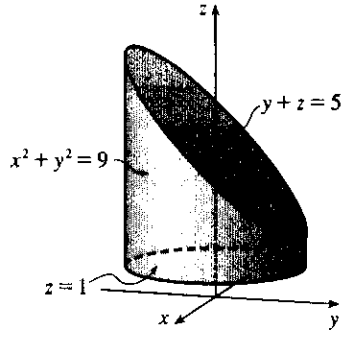
$$= 4 \int_{-3}^{3} \int_{x^2}^{9} dy\, dx = 4 \int_{-3}^{3} \left(9 - x^2\right) dx$$

$$= 4\left[9x - \tfrac{1}{3}x^3\right]_{-3}^{3} = 4(27 - 9 + 27 - 9) = 144$$

19. $$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} dz\, dy\, dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5 - y - 1)\, dy\, dx = \int_{-3}^{3} \left[4y - \tfrac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3} 8\sqrt{9 - x^2}\, dx = 8\left[\tfrac{x}{2}\sqrt{9 - x^2} + \tfrac{9}{2}\sin^{-1}\left(\tfrac{x}{3}\right)\right]_{-3}^{3}$$ [using trigonometric substitution or Formula 30 in the Table of Integrals]

$$= 8\left[\tfrac{9}{2}\sin^{-1}(1) - \tfrac{9}{2}\sin^{-1}(-1)\right] = 36\left(\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right)\right) = 36\pi$$



Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - y)\, dy\, dx = \int_0^{2\pi} \int_0^3 (4 - r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \tfrac{1}{3}r^3\sin\theta\right]_{r=0}^{r=3} d\theta$$

$$= \int_0^{2\pi} (18 - 9\sin\theta)\, d\theta$$

$$= 18\theta + 9\cos\theta\Big]_0^{2\pi} = 36\pi$$

20. The paraboloid $x = y^2 + z^2$ intersects the plane $x = 16$ in the circle $y^2 + z^2 = 16$, $x = 16$.

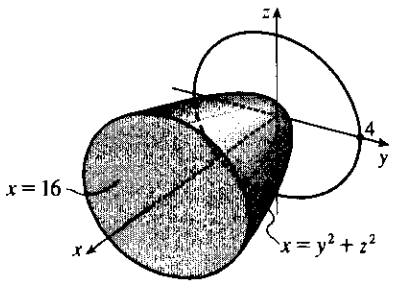
Thus, $E = \left\{(x, y, z) \mid y^2 + z^2 \le x \le 16, y^2 + z^2 \le 16\right\}$.

Let $D = \left\{(y, z) \mid y^2 + z^2 \le 16\right\}$. Then using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we have

$$V = \iint_D \left(\int_{y^2+z^2}^{16} dx\right) dA = \iint_D (16 - (y^2 + z^2))\, dA$$

$$= \int_0^{2\pi} \int_0^4 (16 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^4 (16r - r^3)\, dr$$

$$= \left[\theta\right]_0^{2\pi} \left[8r^2 - \tfrac{1}{4}r^4\right]_0^4 = 2\pi(128 - 64) = 128\pi$$



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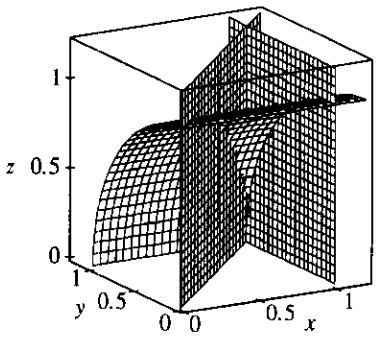
21. (a) The wedge can be described as the region

$$D = \{(x, y, z) \mid y^2 + z^2 \le 1, 0 \le x \le 1, 0 \le y \le x\}$$
$$= \left\{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le \sqrt{1 - y^2}\right\}$$

So the integral expressing the volume of the wedge is

$$\iiint_D dV = \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx.$$



(b) A CAS gives $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx = \frac{\pi}{4} - \frac{1}{3}$.
(Or use Formulas 30 and 87 from the Table of Integrals.)

22. (a) Note that $\Delta V = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, so the Midpoint Rule gives

$$\iiint_B f(x, y, z)\, dV \approx \tfrac{1}{8}\left[f\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right) + f\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{3}{4}\right) + f\left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4}\right) + f\left(\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right)\right.$$
$$\left. + f\left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}\right) + f\left(\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{3}{4}\right) + f\left(\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}\right) + f\left(\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}\right)\right]$$
$$= \tfrac{1}{8}\left[e^{-3(1/4)^2} + 3e^{-2(1/4)^2-(3/4)^2} + 3e^{-(1/4)^2-2(3/4)^2} + e^{-3(3/4)^2}\right] \approx 0.42968$$

(b) A CAS estimates the integral to be $\iiint_B e^{-x^2-y^2-z^2}\, dV \approx 0.42$. The estimate in part (a) is correct to one decimal place, and is larger than the actual value of the integral.

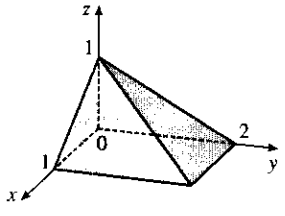
23. Here $f(x, y, z) = \dfrac{1}{\ln(1 + x + y + z)}$ and $\Delta V = 2 \cdot 4 \cdot 2 = 16$, so the Midpoint Rule gives

$$\iiint_B f(x, y, z)\, dV \approx \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\bar{x}_i, \bar{y}_j, \bar{z}_k)\, \Delta V$$
$$= 16\,[f(1, 2, 1) + f(1, 2, 3) + f(1, 6, 1) + f(1, 6, 3)$$
$$+ f(3, 2, 1) + f(3, 2, 3) + f(3, 6, 1) + f(3, 6, 3)]$$
$$= 16\left[\frac{1}{\ln 5} + \frac{1}{\ln 7} + \frac{1}{\ln 9} + \frac{1}{\ln 11} + \frac{1}{\ln 7} + \frac{1}{\ln 9} + \frac{1}{\ln 11} + \frac{1}{\ln 13}\right] \approx 60.533$$

24. Here $f(x, y, z) = \sin(xy^2z^3)$ and $\Delta V = 2 \cdot 1 \cdot \frac{1}{2} = 1$, so the Midpoint Rule gives

$$\iiint_B f(x, y, z)\, dV \approx \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\bar{x}_i, \bar{y}_j, \bar{z}_k)\, \Delta V$$
$$= 1\left[f\left(1, \tfrac{1}{2}, \tfrac{1}{4}\right) + f\left(1, \tfrac{1}{2}, \tfrac{3}{4}\right) + f\left(1, \tfrac{3}{2}, \tfrac{1}{4}\right) + f\left(1, \tfrac{3}{2}, \tfrac{3}{4}\right)\right.$$
$$\left. + f\left(3, \tfrac{1}{2}, \tfrac{1}{4}\right) + f\left(3, \tfrac{1}{2}, \tfrac{3}{4}\right) + f\left(3, \tfrac{3}{2}, \tfrac{1}{4}\right) + f\left(3, \tfrac{3}{2}, \tfrac{3}{4}\right)\right]$$
$$= \sin\tfrac{1}{256} + \sin\tfrac{27}{256} + \sin\tfrac{9}{256} + \sin\tfrac{243}{256} + \sin\tfrac{3}{256} + \sin\tfrac{81}{256} + \sin\tfrac{27}{256} + \sin\tfrac{729}{256} \approx 1.675$$

25. $E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le z \le 1 - x, 0 \le y \le 2 - 2z\}$, the solid bounded by the three coordinate planes and the planes $z = 1 - x$, $y = 2 - 2z$.



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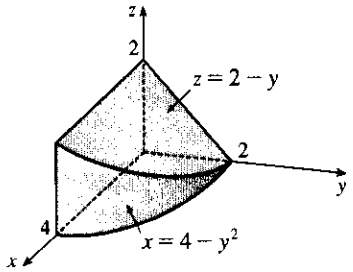
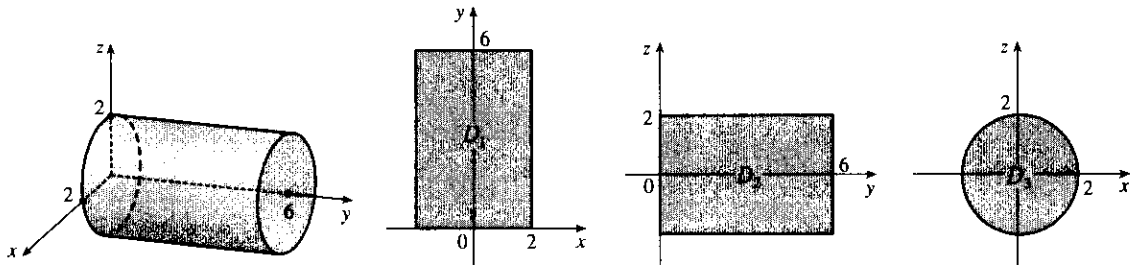
26. $E = \{(x, y, z) \mid 0 \le y \le 2, 0 \le z \le 2 - y, 0 \le x \le 4 - y^2\}$, the solid bounded by the three coordinate planes, the plane $z = 2 - y$, and the cylindrical surface $x = 4 - y^2$.



27.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x, y) \mid -2 \le x \le 2, 0 \le y \le 6\}$$

$$D_2 = \{(y, z) \mid -2 \le z \le 2, 0 \le y \le 6\}$$

$$D_3 = \{(x, z) \mid x^2 + z^2 \le 4\}$$

Therefore

$$E = \left\{(x, y, z) \mid -\sqrt{4 - x^2} \le z \le \sqrt{4 - x^2},\ -2 \le x \le 2, 0 \le y \le 6\right\}$$

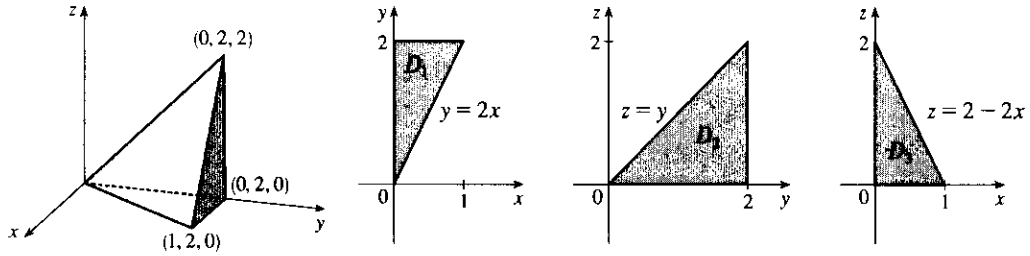
$$= \left\{(x, y, z) \mid -\sqrt{4 - z^2} \le x \le \sqrt{4 - z^2},\ -2 \le z \le 2, 0 \le y \le 6\right\}$$

$$\iiint_E f(x, y, z)\, dV = \int_{-2}^{2} \int_{0}^{6} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y, z)\, dz\, dy\, dx = \int_{0}^{6} \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y, z)\, dz\, dx\, dy$$

$$= \int_{0}^{6} \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x, y, z)\, dx\, dz\, dy = \int_{-2}^{2} \int_{0}^{6} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x, y, z)\, dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{6} f(x, y, z)\, dy\, dz\, dx = \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{0}^{6} f(x, y, z)\, dy\, dx\, dz$$

28.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x, y) \mid 0 \le x \le 1, 2x \le y \le 2\} = \{(x, y) \mid 0 \le y \le 2, 0 \le x \le y/2\},$$

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$$D_2 = \{(y,z) \mid 0 \le y \le 2, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 2, z \le y \le 2\}, \text{ and}$$

$$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le 2-2x\} = \{(x,z) \mid 0 \le z \le 2, 0 \le x \le (2-z)/2\}$$

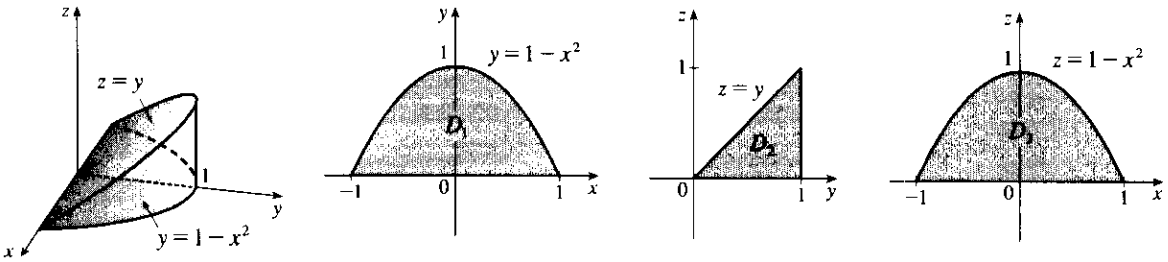
Therefore

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le x \le 1, 2x \le y \le 2, 0 \le z \le y-2x\} \\ &= \{(x,y,z) \mid 0 \le y \le 2, 0 \le x \le y/2, 0 \le z \le y-2x\} \\ &= \{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le y, 0 \le x \le (y-z)/2\} \\ &= \{(x,y,z) \mid 0 \le z \le 2, z \le y \le 2, 0 \le x \le (y-z)/2\} \\ &= \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 2-2x, z+2x \le y \le 2\} \\ &= \{(x,y,z) \mid 0 \le z \le 2, 0 \le x \le (2-z)/2, z+2x \le y \le 2\} \end{aligned}$$

Then

$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \int_0^1 \int_{2x}^2 \int_0^{y-2x} f(x,y,z)\,dz\,dy\,dx \\ &= \int_0^2 \int_0^{y/2} \int_0^{y-2x} f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^2 \int_0^{y} \int_0^{(y-z)/2} f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^2 \int_z^{2} \int_0^{(y-z)/2} f(x,y,z)\,dx\,dy\,dz \\ &= \int_0^1 \int_0^{2-2x} \int_{z+2x}^{2} f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^2 \int_0^{(2-z)/2} \int_{z+2x}^{2} f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

29.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid -1 \le x \le 1, 0 \le y \le 1-x^2\} = \left\{(x,y) \mid 0 \le y \le 1, -\sqrt{1-y} \le x \le \sqrt{1-y}\right\},$$

$$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}, \text{ and}$$

$$D_3 = \{(x,z) \mid -1 \le x \le 1, 0 \le z \le 1-x^2\} = \left\{(x,z) \mid 0 \le z \le 1, -\sqrt{1-z} \le x \le \sqrt{1-z}\right\}$$

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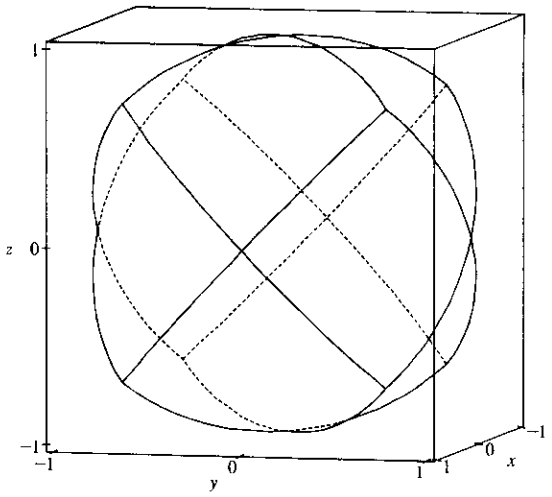
3. To graph the edges of the solid, we use parametrized curves similar to those found in Problem 1 for the intersection of two cylinders. We must restrict the parameter intervals so that each arc extends exactly to the desired vertex. One possible set of parametric equations (with all sign choices allowed) is

$$x = r,\ y = \pm r,\ z = \pm\sqrt{1 - r^2},\ -\tfrac{1}{\sqrt{2}} \le r \le \tfrac{1}{\sqrt{2}};$$

$$x = \pm s,\ y = \pm\sqrt{1 - s^2},\ z = s,\ -\tfrac{1}{\sqrt{2}} \le s \le \tfrac{1}{\sqrt{2}};$$

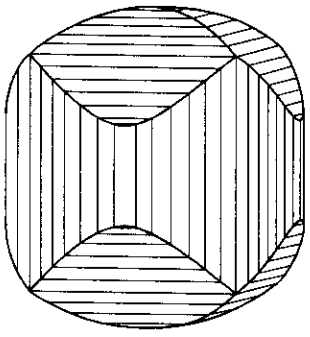
$$x = \pm\sqrt{1 - t^2},\ y = t,\ z = \pm t,\ -\tfrac{1}{\sqrt{2}} \le t \le \tfrac{1}{\sqrt{2}}.$$



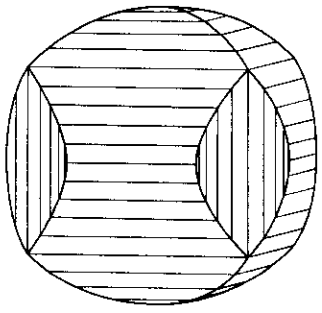
4. Let the three cylinders be $x^2 + y^2 = a^2$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$.

If $a < 1$, then the four faces defined by the cylinder $x^2 + y^2 = 1$ in Problem 1 collapse into a single face, as in the first graph. If $1 < a < \sqrt{2}$, then each pair of vertically opposed faces, defined by one of the other two cylinders, collapse into a single face, as in the second graph. If $a \ge \sqrt{2}$, then the vertical cylinder encloses the solid of intersection of the other two cylinders completely, so the solid of intersection coincides with the solid of intersection of the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, as illustrated in Problem 1.

If we were to vary b or c instead of a, we would get solids with the same shape, but differently oriented.



$a = 0.95, b = c = 1$



$a = 1.1, b = c = 1$

5. If $a < 1$, the solid looks similar to the first graph in Problem 4. As in Problem 2, we split the solid into sixteen congruent pieces, one of which can be described as the solid above the polar region $\{(r, \theta) \mid 0 \le r \le a, 0 \le \theta \le \frac{\pi}{4}\}$ in the xy-plane and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2\cos^2\theta}$. Thus, the total volume is

$$V = 16\int_0^{\pi/4}\int_0^a \sqrt{1 - r^2\cos^2\theta}\, r\, dr\, d\theta$$

If $a > 1$ and $a < \sqrt{2}$, we have a solid similar to the second graph in Problem 4. Its intersection with the xy-plane is graphed below. Again we split the solid into sixteen congruent pieces, one of which is the solid above the region

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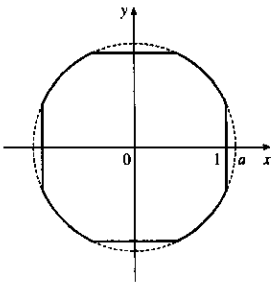
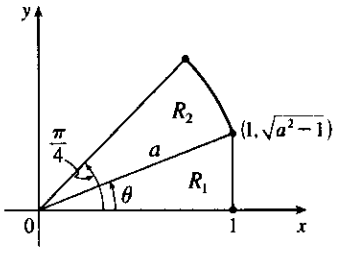
shown in the second figure and below the surface $z = \sqrt{1-x^2} = \sqrt{1-r^2\cos^2\theta}$.





We split the region of integration where the outside boundary changes from the vertical line $x = 1$ to the circle $x^2 + y^2 = a^2$ or $r = 1$. R_1 is a right triangle, so $\cos\theta = \frac{1}{a}$. Thus, the boundary between R_1 and R_2 is $\theta = \cos^{-1}\left(\frac{1}{a}\right)$ in polar coordinates, or $y = \sqrt{a^2-1}\,x$ in rectangular coordinates. Using rectangular coordinates for the region R_1 and polar coordinates for R_2, we find the total volume of the solid to be

$$V = 16\left[\int_0^1 \int_0^{\sqrt{a^2-1}\,x} \sqrt{1-x^2}\,dy\,dx + \int_{\cos^{-1}(1/a)}^{\pi/4} \int_0^a \sqrt{1-r^2\cos^2\theta}\,r\,dr\,d\theta\right]$$

If $a \geq \sqrt{2}$, the cylinder $x^2 + y^2 = 1$ completely encloses the intersection of the other two cylinders, so the solid of intersection of the three cylinders coincides with the intersection of $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as illustrated in Exercise 16.6.24 [ET 15.6.24]. Its volume is

$$V = 16\int_0^1 \int_0^x \sqrt{1-x^2}\,dy\,dx$$

16.9 Change of Variables in Multiple Integrals ET 15.9

1. $x = u + 4v$, $y = 3u - 2v$.

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = 1(-2) - 4(3) = -14.$

2. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 4uv - (-4uv) = 8uv$

3. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{v}{(u+v)^2} & -\dfrac{u}{(u+v)^2} \\ -\dfrac{v}{(u-v)^2} & \dfrac{u}{(u-v)^2} \end{vmatrix} = \dfrac{uv}{(u+v)^2(u-v)^2} - \dfrac{uv}{(u+v)^2(u-v)^2} = 0$

4. $\dfrac{\partial(x,y)}{\partial(\alpha,\beta)} = \begin{vmatrix} \partial x/\partial\alpha & \partial x/\partial\beta \\ \partial y/\partial\alpha & \partial y/\partial\beta \end{vmatrix} = \begin{vmatrix} \sin\beta & \alpha\cos\beta \\ \cos\beta & -\alpha\sin\beta \end{vmatrix} = -\alpha\sin^2\beta - \alpha\cos^2\beta = -\alpha$

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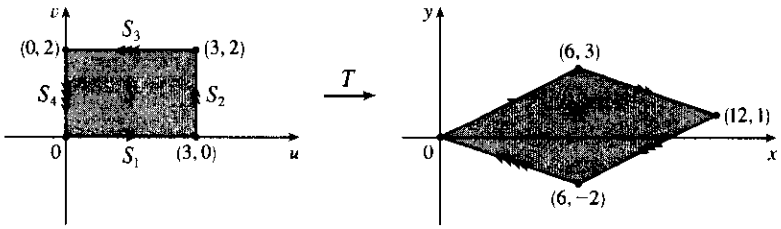
5. $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} v & u & 0 \\ 0 & w & v \\ w & 0 & u \end{vmatrix}$$

$$= v\begin{vmatrix} w & v \\ 0 & u \end{vmatrix} - u\begin{vmatrix} 0 & v \\ w & u \end{vmatrix} + 0\begin{vmatrix} 0 & w \\ w & 0 \end{vmatrix} = v(uw-0) - u(0-vw) = 2uvw$$

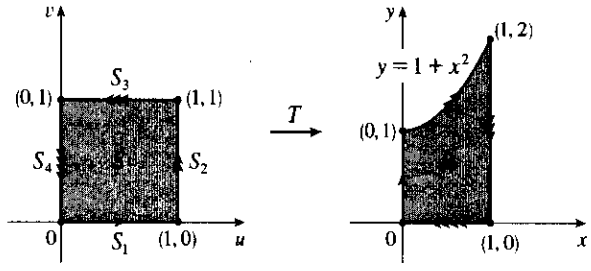
6. $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} e^{u-v} & -e^{u-v} & 0 \\ e^{u+v} & e^{u+v} & 0 \\ e^{u+v+w} & e^{u+v+w} & e^{u+v+w} \end{vmatrix} = e^{u+v+w}\begin{vmatrix} e^{u-v} & -e^{u-v} \\ e^{u+v} & e^{u+v} \end{vmatrix}$$

$$= e^{u+v+w}\left(e^{u-v}e^{u+v} + e^{u-v}e^{u+v}\right) = e^{u+v+w}\left(2e^{2u}\right) = 2e^{3u+v+w}$$

7. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0$ $(0 \le u \le 3)$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u, we have $x = 2y$, $0 \le x \le 6$. S_2 is the line segment $u = 3$, $0 \le v \le 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \quad \Rightarrow \quad x = 6 + 3(3-y) = 15 - 3y$, $6 \le x \le 12$. S_3 is the line segment $v = 2$, $0 \le u \le 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \quad \Rightarrow \quad x = 2y + 10$, $6 \le x \le 12$. Finally, S_4 is the segment $u = 0$, $0 \le v \le 2$, so $x = 3v$ and $y = -v \quad \Rightarrow \quad x = -3y$, $0 \le x \le 6$. The image of set S is the region R shown in the xy-plane, a parallelogram bounded by these four segments.



8. S_1 is the line segment $v = 0$, $0 \le u \le 1$, so $x = v = 0$ and $y = u(1 + v^2) = u$. Since $0 \le u \le 1$, the image is the line segment $x = 0$, $0 \le y \le 1$. S_2 is the segment $u = 1$, $0 \le v \le 1$, so $x = v$ and $y = u(1 + v^2) = 1 + x^2$. Thus the image is the portion of the parabola $y = 1 + x^2$ for $0 \le x \le 1$. S_3 is the segment $v = 1$, $0 \le u \le 1$, so $x = 1$ and $y = 2u$. The image is the segment $x = 1$, $0 \le y \le 2$. S_4 is described by $u = 0$, $0 \le v \le 1$, so $0 \le x = v \le 1$ and $y = u(1 + v^2) = 0$. The image is the line segment $y = 0$, $0 \le x \le 1$. Thus, the image of S is the region R bounded by the parabola $y = 1 + x^2$, the x-axis, and the lines $x = 0$, $x = 1$.



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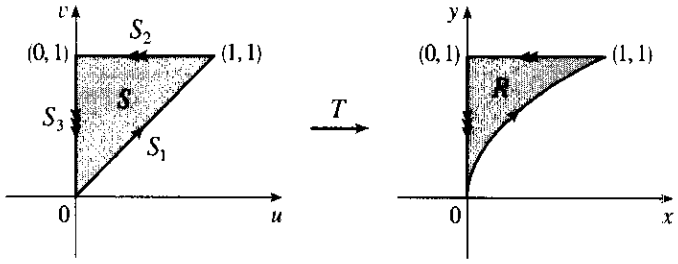
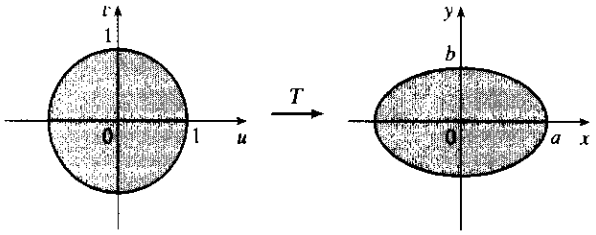
9. S_1 is the line segment $u = v$, $0 \le u \le 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \le u \le 1$, the image is the portion of the parabola $x = y^2$, $0 \le y \le 1$. S_2 is the segment $v = 1$, $0 \le u \le 1$, thus $y = v = 1$ and $x = u^2$, so $0 \le x \le 1$. The image is the line segment $y = 1$, $0 \le x \le 1$. S_3 is the segment $u = 0$, $0 \le v \le 1$, so $x = u^2 = 0$ and $y = v \;\Rightarrow\; 0 \le y \le 1$. The image is the segment $x = 0$, $0 \le y \le 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x = y^2$, the y-axis, and the line $y = 1$.



10. Substituting $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$ into $u^2 + v^2 \le 1$ gives $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$, so the image of $u^2 + v^2 \le 1$ is the elliptical region $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$.



11. $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ and $x - 3y = (2u + v) - 3(u + 2v) = -u - 5v$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0, 0)$ and $(2, 1)$ is $y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u + v) \;\Rightarrow\; v = 0$; the line through $(2, 1)$ and $(1, 2)$ is $x + y = 3$ which is the image of $(2u + v) + (u + 2v) = 3 \;\Rightarrow\; u + v = 1$; the line through $(0, 0)$ and $(1, 2)$ is $y = 2x$ which is the image of $u + 2v = 2(2u + v) \;\Rightarrow\; u = 0$. Thus S is the triangle $0 \le v \le 1 - u$, $0 \le u \le 1$ in the uv-plane and

$$\begin{aligned} \iint_R (x - 3y)\, dA &= \int_0^1 \int_0^{1-u} (-u - 5v)\, |3|\, dv\, du \\ &= -3 \int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_{v=0}^{v=1-u} du = -3 \int_0^1 \left(u - u^2 + \tfrac{5}{2}(1 - u)^2\right) du \\ &= -3\left[\tfrac{1}{2}u^2 - \tfrac{1}{3}u^3 - \tfrac{5}{6}(1 - u)^3\right]_0^1 = -3\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{5}{6}\right) = -3 \end{aligned}$$

12. $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = \dfrac{1}{4}$, $4x + 8y = 4 \cdot \frac{1}{4}(u + v) + 8 \cdot \frac{1}{4}(v - 3u) = 3v - 5u$. R is a parallelogram bounded by the lines $x - y = -4$, $x - y = 4$, $3x + y = 0$, $3x + y = 8$. Since $u = x - y$ and $v = 3x + y$, R is the image of

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the rectangle enclosed by the lines $u=-4$, $u=4$, $v=0$, and $v=8$. Thus

$$\iint_R (4x+8y)\,dA = \int_{-4}^{4}\int_{0}^{8}(3v-5u)\left|\tfrac{1}{4}\right|\,dv\,du = \tfrac{1}{4}\int_{-4}^{4}\left[\tfrac{3}{2}v^2-5uv\right]_{v=0}^{v=8}du$$
$$= \tfrac{1}{4}\int_{-4}^{4}(96-40u)\,du = \tfrac{1}{4}\left[96u-20u^2\right]_{-4}^{4} = 192$$

13. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$, $x^2 = 4u^2$ and the planar ellipse $9x^2+4y^2 \le 36$ is the image of the disk $u^2+v^2 \le 1$. Thus

$$\iint_R x^2\,dA = \iint_{u^2+v^2\le 1}(4u^2)(6)\,du\,dv = \int_0^{2\pi}\int_0^1 (24r^2\cos^2\theta)\,r\,dr\,d\theta$$
$$= 24\int_0^{2\pi}\cos^2\theta\,d\theta\int_0^1 r^3\,dr = 24\left[\tfrac{1}{2}x+\tfrac{1}{4}\sin 2x\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1$$
$$= 24(\pi)\left(\tfrac{1}{4}\right) = 6\pi$$

14. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \dfrac{4}{\sqrt{3}}$, $x^2-xy+y^2 = 2u^2+2v^2$ and the planar ellipse $x^2-xy+y^2 \le 2$ is the image of the disk $u^2+v^2 \le 1$. Thus

$$\iint_R (x^2-xy+y^2)\,dA = \iint_{u^2+v^2\le 1}(2u^2+2v^2)\left(\tfrac{4}{\sqrt{3}}\,du\,dv\right) = \int_0^{2\pi}\int_0^1 \tfrac{8}{\sqrt{3}}r^3\,dr\,d\theta = \tfrac{4\pi}{\sqrt{3}}$$

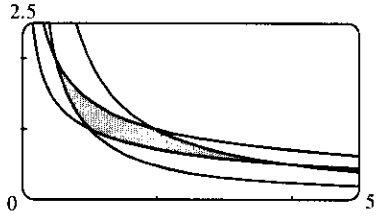
15. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, $xy=u$, $y=x$ is the image of the parabola $v^2=u$, $y=3x$ is the image of the parabola $v^2=3u$, and the hyperbolas $xy=1$, $xy=3$ are the images of the lines $u=1$ and $u=3$ respectively. Thus

$$\iint_R xy\,dA = \int_1^3\int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right)dv\,du = \int_1^3 u\left(\ln\sqrt{3u}-\ln\sqrt{u}\right)du$$
$$= \int_1^3 u\ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3$$

16. Here $y=\dfrac{v}{u}$, $x=\dfrac{u^2}{v}$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{vmatrix} = \dfrac{1}{v}$

and R is the image of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$. So

$$\iint_R y^2\,dA = \int_1^2\int_1^2 \frac{v^2}{u^2}\left(\frac{1}{v}\right)du\,dv = \int_1^2 \frac{v}{2}\,dv = \frac{3}{4}$$



17. (a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u=\dfrac{x}{a}$, $v=\dfrac{y}{b}$, $w=\dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2+v^2+w^2 \le 1$. So

$$\iiint_E dV = \iiint_{u^2+v^2+w^2\le 1} abc\,du\,dv\,dw = (abc)(\text{volume of the ball}) = \tfrac{4}{3}\pi abc$$

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(b) If we approximate the surface of Earth by the ellipsoid $\frac{x^2}{6378^2}+\frac{y^2}{6378^2}+\frac{z^2}{6356^2}=1$, then we can estimate the volume of Earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV=\frac{4}{3}\pi(6378)(6378)(6356)\approx 1.083\times 10^{12}\ \text{km}^3$.

18. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=abc$ and the solid enclosed by the ellipsoid is the image of the ball $u^2+v^2+w^2\le 1$.

Now $x^2y=(a^2u^2)(bv)$, so

$$\begin{aligned}\iiint_E x^2y\,dV &= \iiint_{u^2+v^2+w^2\le 1}(a^2bu^2v)(abc)\,du\,dv\,dw\\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (a^3b^2c)(\rho^2\sin^2\phi\cos^2\theta)(\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\ &= a^3b^2c\int_0^{2\pi}\int_0^{\pi}\int_0^1(\rho^5\sin^4\phi\cos^2\theta\sin\theta)\,d\rho\,d\phi\,d\theta\\ &= a^3b^2c\int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta\int_0^{\pi}\sin^4\phi\,d\phi\int_0^1\rho^5 d\rho\\ &= 0 \quad \text{since } \int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta=0\end{aligned}$$

19. Letting $u=x-2y$ and $v=3x-y$, we have $x=\frac{1}{5}(2v-u)$ and $y=\frac{1}{5}(v-3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}-1/5 & 2/5\\ -3/5 & 1/5\end{vmatrix}=\frac{1}{5}$ and R is the image of the rectangle enclosed by the lines $u=0$, $u=4$, $v=1$, and $v=8$. Thus $\iint_R \frac{x-2y}{3x-y}\,dA=\int_0^4\int_1^8 \frac{u}{v}\left|\frac{1}{5}\right|dv\,du=\frac{1}{5}\int_0^4 u\,du\int_1^8\frac{1}{v}\,dv=\frac{1}{5}\left[\frac{1}{2}u^2\right]_0^4\left[\ln|v|\right]_1^8=\frac{8}{5}\ln 8$.

20. Letting $u=x+y$ and $v=x-y$, we have $x=\frac{1}{2}(u+v)$ and $y=\frac{1}{2}(u-v)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{vmatrix}=-\frac{1}{2}$ and R is the image of the rectangle enclosed by the lines $u=0$, $u=3$, $v=0$, and $v=2$. Thus

$$\begin{aligned}\iint_R (x+y)\,e^{x^2-y^2}\,dA &= \int_0^3\int_0^2 ue^{uv}\left|-\tfrac{1}{2}\right|dv\,du=\tfrac{1}{2}\int_0^3\left[e^{uv}\right]_{v=0}^{v=2}du=\tfrac{1}{2}\int_0^3(e^{2u}-1)\,du\\ &= \tfrac{1}{2}\left[\tfrac{1}{2}e^{2u}-u\right]_0^3=\tfrac{1}{2}\left(\tfrac{1}{2}e^6-3-\tfrac{1}{2}\right)=\tfrac{1}{4}(e^6-7)\end{aligned}$$

21. Letting $u=y-x$, $v=y+x$, we have $y=\frac{1}{2}(u+v)$, $x=\frac{1}{2}(v-u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}-1/2 & 1/2\\ 1/2 & 1/2\end{vmatrix}=-\frac{1}{2}$ and R is the image of the trapezoidal region with vertices $(-1,1)$, $(-2,2)$, $(2,2)$, and $(1,1)$. Thus

$$\begin{aligned}\iint_R \cos\frac{y-x}{y+x}\,dA &= \int_1^2\int_{-v}^{v}\cos\frac{u}{v}\left|-\frac{1}{2}\right|du\,dv=\frac{1}{2}\int_1^2\left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v}dv\\ &= \tfrac{1}{2}\int_1^2 2v\sin(1)\,dv=\tfrac{3}{2}\sin 1\end{aligned}$$

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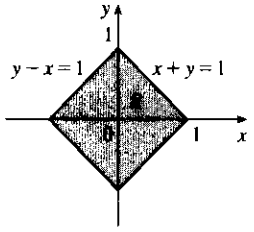
22. Letting $u = 3x$, $v = 2y$, we have $9x^2 + 4y^2 = u^2 + v^2$, $x = \frac{1}{3}u$, and $y = \frac{1}{2}v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{6}$ and R is the image of the quarter-disk D given by $u^2 + v^2 \le 1$, $u \ge 0$, $v \ge 0$. Thus

$$\iint_R \sin(9x^2 + 4y^2)\,dA = \iint_D \tfrac{1}{6}\sin(u^2 + v^2)\,du\,dv = \int_0^{\pi/2}\int_0^1 \tfrac{1}{6}\sin(r^2)\,r\,dr\,d\theta$$

$$= \tfrac{\pi}{12}\left[-\tfrac{1}{2}\cos r^2\right]_0^1 = \tfrac{\pi}{24}(1 - \cos 1)$$

23. Let $u = x + y$ and $v = -x + y$. Then $u + v = 2y \;\Rightarrow\; y = \frac{1}{2}(u + v)$ and

$u - v = 2x \;\Rightarrow\; x = \frac{1}{2}(u - v)$. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$.



Now $|u| = |x + y| \le |x| + |y| \le 1 \;\Rightarrow\; -1 \le u \le 1$, and

$|v| = |-x + y| \le |x| + |y| \le 1 \Rightarrow\; -1 \le v \le 1$. R is the image of the square region with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$.

So $\iint_R e^{x+y}\,dA = \frac{1}{2}\int_{-1}^1\int_{-1}^1 e^u\,du\,dv = \frac{1}{2}\left[e^u\right]_{-1}^1\left[v\right]_{-1}^1 = e - e^{-1}$.

24. Let $u = x + y$ and $v = y$, then $x = u - v$, $y = v$, $\dfrac{\partial(x,y)}{\partial(u,v)} = 1$ and R is the image under T of the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Thus

$$\iint_R f(x+y)\,dA = \int_0^1\int_0^u (1)\,f(u)\,dv\,du = \int_0^1 f(u)\left[v\right]_{v=0}^{v=u}du = \int_0^1 uf(u)\,du \quad \text{as desired.}$$

16 Review ET 15

CONCEPT CHECK

1. (a) A double Riemann sum of f is $\sum_{i=1}^m\sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)\,\Delta A$, where ΔA is the area of each subrectangle and (x_{ij}^*, y_{ij}^*) is a sample point in each subrectangle. If $f(x,y) \ge 0$, this sum represents an approximation to the volume of the solid that lies above the rectangle R and below the graph of f.

(b) $\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^m\sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)\,\Delta A$

(c) If $f(x,y) \ge 0$, $\iint_R f(x,y)\,dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x,y)$. If f takes on both positive and negative values, $\iint_R f(x,y)\,dA$ is the difference of the volume above R but below the surface $z = f(x,y)$ and the volume below R but above the surface $z = f(x,y)$.

(d) We usually evaluate $\iint_R f(x,y)\,dA$ as an iterated integral according to Fubini's Theorem (see Theorem 16.2.4 [ET 15.2.4]).

(e) The Midpoint Rule for Double Integrals says that we approximate the double integral $\iint_R f(x,y)\,dA$ by the double Riemann sum $\sum_{i=1}^m\sum_{j=1}^n f(\bar{x}_i, \bar{y}_j)\,\Delta A$ where the sample points $(\bar{x}_i, \bar{y}_j)$ are the centers of the subrectangles.

(f) $f_{\text{ave}} = \dfrac{1}{A(R)}\iint_R f(x,y)\,dA$ where $A(R)$ is the area of R.

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2. (a) See (1) and (2) and the accompanying discussion in Section 16.3 [ET 15.3].

(b) See (3) and the accompanying discussion in Section 16.3 [ET 15.3].

(c) See (5) and the preceding discussion in Section 16.3 [ET 15.3].

(d) See (6)–(11) in Section 16.3 [ET 15.3].

3. We may want to change from rectangular to polar coordinates in a double integral if the region R of integration is more easily described in polar coordinates. To accomplish this, we use $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ where R is given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$.

4. (a) $m = \iint_D \rho(x,y)\,dA$

(b) $M_x = \iint_D y\rho(x,y)\,dA$, $M_y = \iint_D x\rho(x,y)\,dA$

(c) The center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$.

(d) $I_x = \iint_D y^2\rho(x,y)\,dA$, $I_y = \iint_D x^2\rho(x,y)\,dA$, $I_0 = \iint_D (x^2+y^2)\rho(x,y)\,dA$

5. (a) $P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$

(b) $f(x,y) \ge 0$ and $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$.

(c) The expected value of X is $\mu_1 = \iint_{\mathbb{R}^2} xf(x,y)\,dA$; the expected value of Y is $\mu_2 = \iint_{\mathbb{R}^2} yf(x,y)\,dA$.

6. $A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\,dA$

7. (a) $\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$

(b) We usually evaluate $\iiint_B f(x,y,z)\,dV$ as an iterated integral according to Fubini's Theorem for Triple Integrals (see Theorem 16.7.4 [ET 15.7.4]).

(c) See the paragraph following Example 16.7.1 [ET 15.7.1].

(d) See (5) and (6) and the accompanying discussion in Section 16.7 [ET 15.7].

(e) See (10) and the accompanying discussion in Section 16.7 [ET 15.7].

(f) See (11) and the preceding discussion in Section 16.7 [ET 15.7].

8. (a) $m = \iiint_E \rho(x,y,z)\,dV$

(b) $M_{yz} = \iiint_E x\rho(x,y,z)\,dV$, $M_{xz} = \iiint_E y\rho(x,y,z)\,dV$, $M_{xy} = \iiint_E z\rho(x,y,z)\,dV$.

(c) The center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \dfrac{M_{yz}}{m}$, $\overline{y} = \dfrac{M_{xz}}{m}$, and $\overline{z} = \dfrac{M_{xy}}{m}$.

(d) $I_x = \iiint_E (y^2+z^2)\rho(x,y,z)\,dV$, $I_y = \iiint_E (x^2+z^2)\rho(x,y,z)\,dV$, $I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV$.

9. (a) See Formula 16.8.2 [ET 15.8.2] and the accompanying discussion.

(b) See Formula 16.8.4 [ET 15.8.4] and the accompanying discussion.

(c) We may want to change from rectangular to cylindrical or spherical coordinates in a triple integral if the region E of integration is more easily described in cylindrical or spherical coordinates or if the triple integral is easier to evaluate using cylindrical or spherical coordinates.

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10. (a) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$

(b) See (9) and the accompanying discussion in Section 16.9 [ET 15.9].

(c) See (13) and the accompanying discussion in Section 16.9 [ET 15.9].

TRUE-FALSE QUIZ

1. This is true by Fubini's Theorem.

2. False. $\int_0^1 \int_0^x \sqrt{x+y^2}\,dy\,dx$ describes the region of integration as a Type I region. To reverse the order of integration, we must consider the region as a Type II region: $\int_0^1 \int_y^1 \sqrt{x+y^2}\,dx\,dy$.

3. True. See the discussion following Example 4 on page 1029 [ET page 993].

4. $\int_{-1}^1 \int_0^1 e^{x^2+y^2} \sin y\,dx\,dy = \left(\int_0^1 e^{x^2}\,dx\right)\left(\int_{-1}^1 e^{y^2}\sin y\,dy\right) = \left(\int_0^1 e^{x^2}\,dx\right)(0) = 0$, since $e^{y^2}\sin y$ is an odd function. Therefore the statement is true.

5. True:
$\iint_D \sqrt{4-x^2-y^2}\,dA =$ the volume under the surface $x^2+y^2+z^2=4$ and above the xy-plane
$= \frac{1}{2}$ (the volume of the sphere $x^2+y^2+z^2=4$) $= \frac{1}{2}\cdot\frac{4}{3}\pi(2)^3 = \frac{16}{3}\pi$

6. This statement is true because in the given region, $\left(x^2+\sqrt{y}\right)\sin(x^2y^2) \le (1+2)(1) = 3$, so $\int_1^4\int_0^1 \left(x^2+\sqrt{y}\right)\sin(x^2y^2)\,dx\,dy \le \int_1^4\int_0^1 3\,dA = 3A(D) = 3(3) = 9$.

7. The volume enclosed by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=2$ is, in cylindrical coordinates, $V = \int_0^{2\pi}\int_0^2\int_r^2 r\,dz\,dr\,d\theta \ne \int_0^{2\pi}\int_0^2\int_r^2 dz\,dr\,d\theta$, so the assertion is false.

8. True. The moment of inertia about the z-axis of a solid E with constant density k is $I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iiint_E (kr^2)\,r\,dz\,dr\,d\theta = \iiint_E kr^3\,dz\,dr\,d\theta$.

EXERCISES

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x,y)\,dA$ by a Riemann sum with $m=n=3$ and the sample points the upper right corners of each square, so

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i,y_j)\,\Delta A \\ &= \Delta A\,[f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2) \\ &\qquad + f(2,3)+f(3,1)+f(3,2)+f(3,3)]\end{aligned}$$

Using the contour lines to estimate the function values, we have

$$\iint_R f(x,y)\,dA \approx 1[2.7+4.7+8.0+4.7+6.7+10.0+6.7+8.6+11.9] \approx 64.0$$

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2. As in Exercise 1, we have $m = n = 3$ and $\Delta A = 1$. Using the contour map to estimate the value of f at the center of each subsquare, we have

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(\bar{x}_i,\bar{y}_j)\,\Delta A \\ &= \Delta A\,[f(0.5,0.5)+(0.5,1.5)+(0.5,2.5)+(1.5,0.5)+f(1.5,1.5) \\ &\qquad + f(1.5,2.5)+(2.5,0.5)+f(2.5,1.5)+f(2.5,2.5)] \\ &\approx 1[1.2+2.5+5.0+3.2+4.5+7.1+5.2+6.5+9.0] = 44.2\end{aligned}$$

3. $\int_1^2\int_0^2 (y+2xe^y)\,dx\,dy = \int_1^2 \left[xy+x^2e^y\right]_{x=0}^{x=2} dy = \int_1^2 (2y+4e^y)\,dy = \left[y^2+4e^y\right]_1^2$
$= 4+4e^2-1-4e = 4e^2-4e+3$

4. $\int_0^1\int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1} dy = \int_0^1 (e^y-1)\,dy = \left[e^y-y\right]_0^1 = e-2$

5. $\int_0^1\int_0^x \cos(x^2)\,dy\,dx = \int_0^1 \left[\cos(x^2)y\right]_{y=0}^{y=x} dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big]_0^1 = \frac{1}{2}\sin 1$

6. $\int_0^1\int_x^{e^x} 3xy^2\,dy\,dx = \int_0^1 \left[xy^3\right]_{y=x}^{y=e^x} dx = \int_0^1 (xe^{3x}-x^4)\,dx$
$= \frac{1}{3}xe^{3x}\Big]_0^1 - \int_0^1 \frac{1}{3}e^{3x}\,dx - \left[\frac{1}{5}x^5\right]_0^1$ [integrating by parts in the first term]
$= \frac{1}{3}e^3 - \left[\frac{1}{9}e^{3x}\right]_0^1 - \frac{1}{5} = \frac{2}{9}e^3 - \frac{4}{45}$

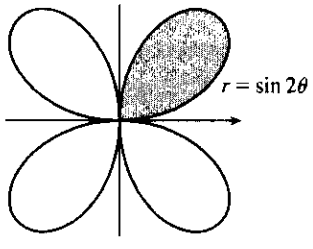
7. $\int_0^\pi\int_0^1\int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx = \int_0^\pi\int_0^1 \left[(y\sin x)z\right]_{z=0}^{z=\sqrt{1-y^2}} dy\,dx = \int_0^\pi\int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$
$= \int_0^\pi \left[-\frac{1}{3}(1-y^2)^{3/2}\sin x\right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\Big]_0^\pi = \frac{2}{3}$

8. $\int_0^1\int_0^y\int_x^1 6xyz\,dz\,dx\,dy = \int_0^1\int_0^y \left[3xyz^2\right]_{z=x}^{z=1} dx\,dy = \int_0^1\int_0^y (3xy-3x^3y)\,dx\,dy$
$= \int_0^1 \left[\frac{3}{2}x^2y-\frac{3}{4}x^4y\right]_{x=0}^{x=y} dy = \int_0^1 \left(\frac{3}{2}y^3-\frac{3}{4}y^5\right) dy = \left[\frac{3}{8}y^4-\frac{1}{8}y^6\right]_0^1 = \frac{1}{4}$

9. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 4, 0 \le \theta \le \pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^\pi\int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

10. The region R is a type II region that can be described as the region enclosed by the lines $y = 4-x$, $y = 4+x$, and the x-axis. So using rectangular coordinates, we can say $R = \{(x,y) \mid y-4 \le x \le 4-y, 0 \le y \le 4\}$ and $\iint_R f(x,y)\,dA = \int_0^4\int_{y-4}^{4-y} f(x,y)\,dx\,dy$.

11.



The region whose area is given by $\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta$ is $\{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le \sin 2\theta\}$, which is the region contained in the loop in the first quadrant of the four-leaved rose $r = \sin 2\theta$.

12. The solid is $\{(\rho,\theta,\phi) \mid 1 \le \rho \le 2, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\}$ which is the region in the first octant on or between the two spheres $\rho = 1$ and $\rho = 2$.

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7. (a) Since $|xyz| < 1$ except at $(1, 1, 1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xyz} = \sum_{n=0}^{\infty} (xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)^3} = \frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

(b) Since $|-xyz| < 1$, except at $(1, 1, 1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1+xyz} = \sum_{n=0}^{\infty}(-xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1+xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(-xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (-xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} = \frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}$$

To evaluate this sum, we first write out a few terms: $s = 1 - \dfrac{1}{2^3} + \dfrac{1}{3^3} - \dfrac{1}{4^3} + \dfrac{1}{5^3} - \dfrac{1}{6^3} \approx 0.8998$. Notice that $a_7 = \dfrac{1}{7^3} < 0.003$. By the Alternating Series Estimation Theorem from Section 12.5 [ET 11.5], we have $|s - s_6| \le a_7 < 0.003$. This error of 0.003 will not affect the second decimal place, so we have $s \approx 0.90$.

8.
$$\int_0^{\infty}\frac{\arctan\pi x - \arctan x}{x}\,dx = \int_0^{\infty}\left[\frac{\arctan yx}{x}\right]_{y=1}^{y=\pi}dx = \int_0^{\infty}\int_1^{\pi}\frac{1}{1+y^2x^2}\,dy\,dx$$

$$= \int_1^{\pi}\int_0^{\infty}\frac{1}{1+y^2x^2}\,dx\,dy = \int_1^{\pi}\lim_{t\to\infty}\left[\frac{\arctan yx}{y}\right]_{x=0}^{x=t}dy$$

$$= \int_1^{\pi}\frac{\pi}{2y}\,dy = \tfrac{\pi}{2}\left[\ln y\right]_1^{\pi} = \tfrac{\pi}{2}\ln\pi$$

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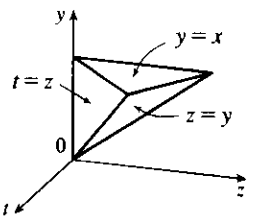
9. $\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy = \iiint_E f(t)\,dV$, where

$E = \{(t,z,y) \mid 0 \le t \le z, 0 \le z \le y, 0 \le y \le x\}$.



If we let D be the projection of E on the yt-plane then

$D = \{(y,t) \mid 0 \le t \le x, t \le y \le x\}$. And we see from the diagram that $E = \{(t,z,y) \mid t \le z \le y, t \le y \le x, 0 \le t \le x\}$. So

$$\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy = \int_0^x \int_t^x \int_t^y f(t)\,dz\,dy\,dt = \int_0^x \left[\int_t^x (y-t)\,f(t)\,dy\right] dt$$

$$= \int_0^x \left[\left(\tfrac{1}{2}y^2 - ty\right) f(t)\right]_{y=t}^{y=x} dt = \int_0^x \left[\tfrac{1}{2}x^2 - tx - \tfrac{1}{2}t^2 + t^2\right] f(t)\,dt$$

$$= \int_0^x \left[\tfrac{1}{2}x^2 - tx + \tfrac{1}{2}t^2\right] f(t)\,dt = \int_0^x \left(\tfrac{1}{2}x^2 - 2tx + t^2\right) f(t)\,dt$$

$$= \tfrac{1}{2}\int_0^x (x-t)^2 f(t)\,dt$$

10. (a) Consider a polar division of the disk, similar to that in Figure 16.4.4 [ET 15.4.4], where $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi$, $0 = r_1 < r_2 < \cdots < r_m = R$, and where the polar subrectangle R_{ij}, as well as r_i^*, θ_j^*, Δr and $\Delta\theta$ are the same as in that figure. Thus $\Delta A_i = r_i^*\,\Delta r\,\Delta\theta$. The mass of R_{ij} is $\rho\,\Delta A_i$, and its distance from m is $s_{ij} \approx \sqrt{(r_i^*)^2 + d^2}$. According to Newton's Law of Gravitation, the force of attraction experienced by m due to this polar subrectangle is in the direction from m towards R_{ij} and has magnitude $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}$. The symmetry of the lamina with respect to the x- and y-axes and the position of m are such that all horizontal components of the gravitational force cancel, so that the total force is simply in the z-direction. Thus, we need only be concerned with the components of this vertical force; that is, $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}\sin\alpha$, where α is the angle between the origin, r_i^* and the mass m. Thus $\sin\alpha = \dfrac{d}{s_{ij}}$ and the previous result becomes $\dfrac{Gm\rho d\,\Delta A_i}{s_{ij}^3}$. The total attractive force is just the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{Gm\rho d\,\Delta A_i}{s_{ij}^3} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{Gm\rho d(r_i^*)\,\Delta r\,\Delta\theta}{[(r_i^*)^2 + d^2]^{3/2}} \text{ which becomes } \int_0^R \int_0^{2\pi} \frac{Gm\rho d}{(r^2+d^2)^{3/2}}\,r\,d\theta\,dr$$

as $m \to \infty$ and $n \to \infty$. Therefore,

$$F = 2\pi Gm\rho d \int_0^R \frac{r}{(r^2+d^2)^{3/2}}\,dr = 2\pi Gm\rho d\left[-\frac{1}{\sqrt{r^2+d^2}}\right]_0^R = 2\pi Gm\rho d\left(\frac{1}{d} - \frac{1}{\sqrt{R^2+d^2}}\right)$$

(b) This is just the result of part (a) in the limit as $R \to \infty$. In this case $\dfrac{1}{\sqrt{R^2+d^2}} \to 0$, and we are left with

$$F = 2\pi Gm\rho d\left(\frac{1}{d} - 0\right) = 2\pi Gm\rho.$$

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Chapter 17

Adapted from the

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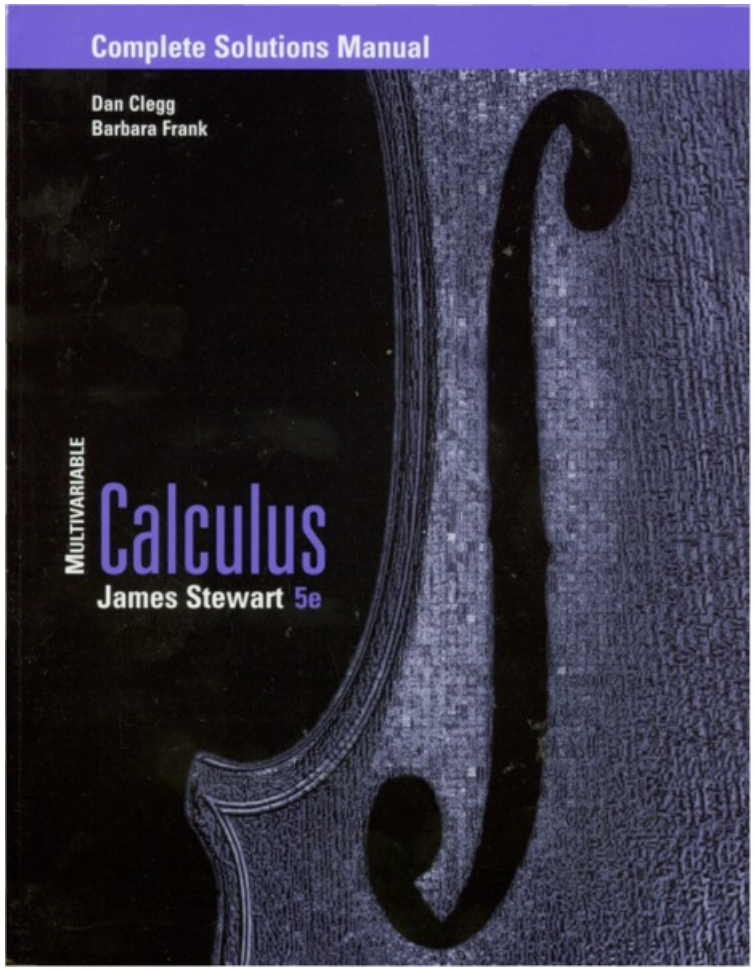

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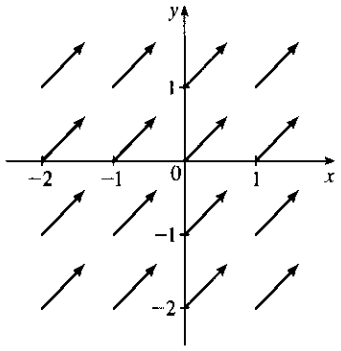
17.1 Vector Fields ET 16.1

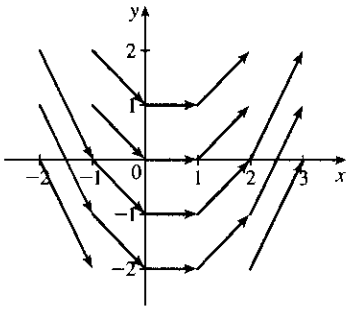
1. $\mathbf{F}(x, y) = \frac{1}{2}(\mathbf{i} + \mathbf{j})$

All vectors in this field are identical, with length $\frac{1}{\sqrt{2}}$ and direction parallel to the line $y = x$.



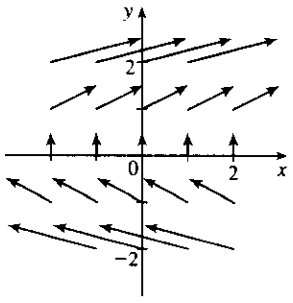
2. $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$

The length of the vector $\mathbf{i} + x\,\mathbf{j}$ is $\sqrt{1 + x^2}$. Vectors are tangent to parabolas opening about the y-axis.



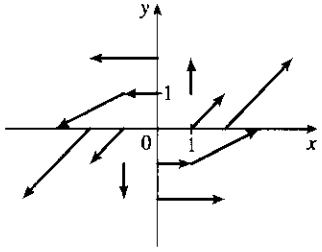
3. $\mathbf{F}(x, y) = y\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$.

The length of the vector $y\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$ is $\sqrt{y^2 + \frac{1}{4}}$. Vectors are tangent to parabolas opening about the x-axis.



4. $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$

The length of the vector $(x - y)\,\mathbf{i} + x\,\mathbf{j}$ is $\sqrt{(x - y)^2 + x^2}$. Vectors along the line $y = x$ are vertical.



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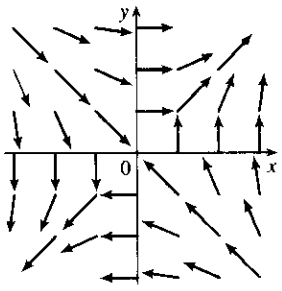
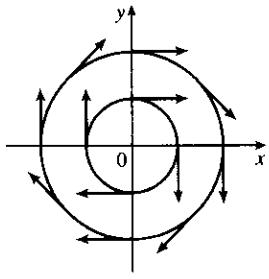
5. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

The length of the vector $\dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$ is 1.



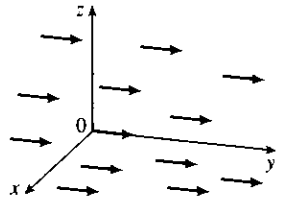
6. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

All the vectors $\mathbf{F}(x, y)$ are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2 + y^2}$.



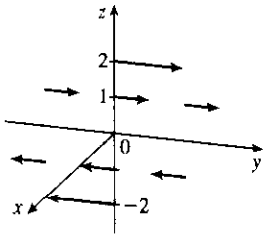
7. $\mathbf{F}(x, y, z) = \mathbf{j}$

All vectors in this field are parallel to the y-axis and have length 1.



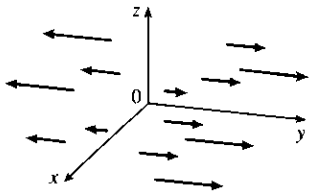
8. $\mathbf{F}(x, y, z) = z\,\mathbf{j}$

At each point (x, y, z), $\mathbf{F}(x, y, z)$ is a vector of length $|z|$. For $z > 0$, all point in the direction of the positive y-axis while for $z < 0$, all are in the direction of the negative y-axis.



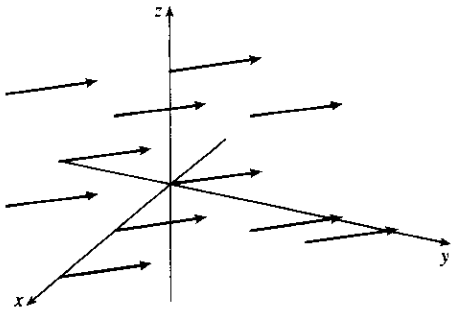
9. $\mathbf{F}(x, y, z) = y\,\mathbf{j}$

The length of $\mathbf{F}(x, y, z)$ is $|y|$. No vectors emanate from the xz-plane since $y = 0$ there. In each plane $y = b$, all the vectors are identical.



10. $\mathbf{F}(x, y, z) = \mathbf{j} - \mathbf{i}$

All vectors in this field have length $\sqrt{2}$ and point in the same direction, parallel to the xy-plane.



11. $\mathbf{F}(x, y) = \langle y, x \rangle$ corresponds to graph II. In the first quadrant all the vectors have positive x- and y-components, in the second quadrant all vectors have positive x-components and negative y-components, in the third quadrant all vectors have negative x- and y-components, and in the fourth quadrant all vectors have negative x-components and positive y-components. In addition, the vectors get shorter as we approach the origin.

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12. $\mathbf{F}(x, y) = \langle 1, \sin y \rangle$ corresponds to graph IV since the x-component of each vector is constant, the vectors are independent of x (vectors along horizontal lines are identical), and the vector field appears to repeat the same pattern vertically.

13. $\mathbf{F}(x, y) = \langle x - 2, x + 1 \rangle$ corresponds to graph I since the vectors are independent of y (vectors along vertical lines are identical) and, as we move to the right, both the x- and the y-components get larger.

14. $\mathbf{F}(x, y) = \langle y, 1/x \rangle$ corresponds to graph III. As in Exercise 11, all the vectors in the first quadrant have positive x- and y-components, in the second quadrant all vectors have positive x-components and negative y-components, in the third quadrant all vectors have negative x- and y-components, and in the fourth quadrant all vectors have negative x-components and positive y-components. Also, the vectors become longer as we approach the y-axis.

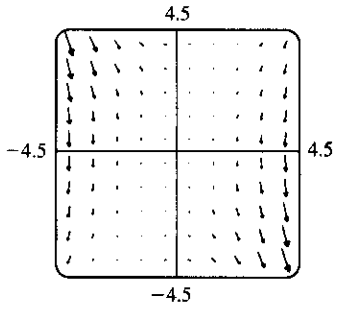
15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph IV, since all vectors have identical length and direction.

16. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph I, since the horizontal vector components remain constant, but the vectors above the xy-plane point generally upward while the vectors below the xy-plane point generally downward.

17. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph III; the projection of each vector onto the xy-plane is $x\,\mathbf{i} + y\,\mathbf{j}$, which points away from the origin, and the vectors point generally upward because their z-components are all 3.

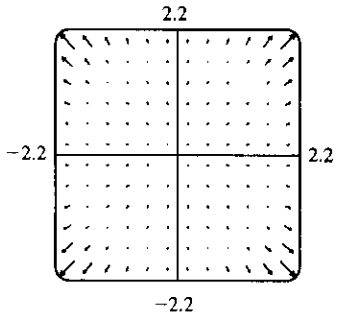
18. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph II; each vector $\mathbf{F}(x, y, z)$ has the same length and direction as the position vector of the point (x, y, z), and therefore the vectors all point directly away from the origin.

19.



The vector field seems to have very short vectors near the line $y = 2x$. For $\mathbf{F}(x, y) = \langle 0, 0 \rangle$ we must have $y^2 - 2xy = 0$ and $3xy - 6x^2 = 0$. The first equation holds if $y = 0$ or $y = 2x$, and the second holds if $x = 0$ or $y = 2x$. So both equations hold [and thus $\mathbf{F}(x, y) = \mathbf{0}$] along the line $y = 2x$.

20.



From the graph, it appears that all of the vectors in the field lie on lines through the origin, and that the vectors have very small magnitudes near the circle $|\mathbf{x}| = 2$ and near the origin. Note that $\mathbf{F}(\mathbf{x}) = \mathbf{0} \Leftrightarrow r(r - 2) = 0 \Leftrightarrow r = 0$ or 2, so as we suspected, $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ for $|\mathbf{x}| = 2$ and for $|\mathbf{x}| = 0$. Note that where $r^2 - r < 0$, the vectors point towards the origin, and where $r^2 - r > 0$, they point away from the origin.

21. $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\,\mathbf{j} = \dfrac{1}{x + 2y}\,\mathbf{i} + \dfrac{2}{x + 2y}\,\mathbf{j}$

22. $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \left[x^{\alpha}\left(-\beta e^{-\beta x}\right) + \alpha x^{\alpha - 1} e^{-\beta x}\right]\mathbf{i} + 0\,\mathbf{j} = (\alpha - \beta x)x^{\alpha - 1}e^{-\beta x}\,\mathbf{i}$

23. $\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{k}$$

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24. $\nabla f(x,y,z) = f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}$

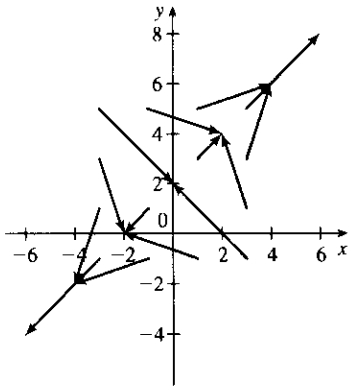
$$= \left(\cos\frac{y}{z}\right)\mathbf{i} - x\left(\sin\frac{y}{z}\right)\left(\frac{1}{z}\right)\mathbf{j} - x\left(\sin\frac{y}{z}\right)\left(-\frac{y}{z^2}\right)\mathbf{k}$$

$$= \left(\cos\frac{y}{z}\right)\mathbf{i} - \frac{x}{z}\left(\sin\frac{y}{z}\right)\mathbf{j} + \frac{xy}{z^2}\left(\sin\frac{y}{z}\right)\mathbf{k}$$

25. $f(x,y) = xy - 2x \;\Rightarrow$

$\nabla f(x,y) = (y-2)\,\mathbf{i} + x\,\mathbf{j}$.

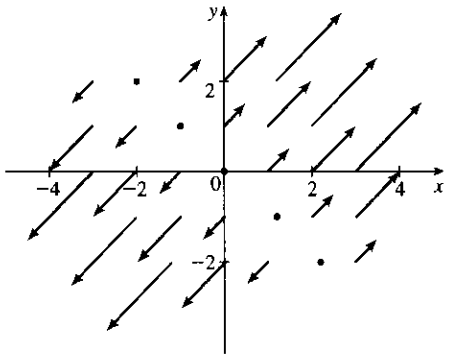
The length of $\nabla f(x,y)$ is $\sqrt{(y-2)^2 + x^2}$ and $\nabla f(x,y)$ terminates on the line $y = x + 2$ at the point $(x + y - 2, x + y)$.



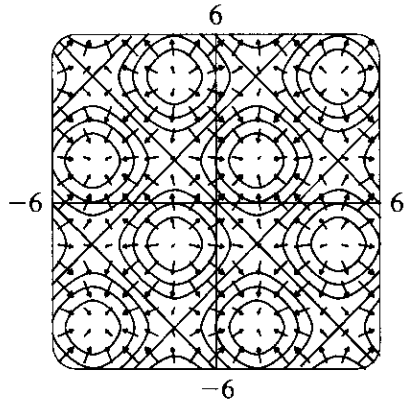
26. $f(x,y) = \frac{1}{4}(x+y)^2 \;\Rightarrow$

$\nabla f(x,y) = \frac{1}{2}(x+y)\,\mathbf{i} + \frac{1}{2}(x+y)\,\mathbf{j}$.

The length of $\nabla f(x,y)$ is $\sqrt{\frac{1}{2}(x+y)^2} = \frac{1}{\sqrt{2}}\,|x+y|$. The vectors are perpendicular to the line $y = -x$ and point away from the line, with length that increases as the distance from the line $y = -x$ increases.

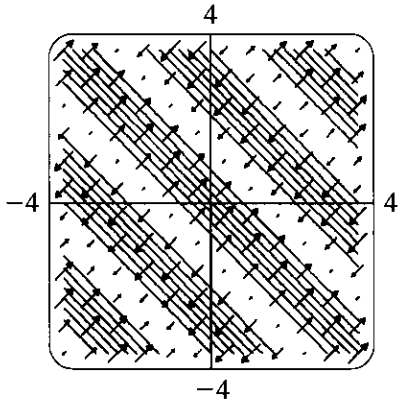


27. We graph ∇f along with a contour map of f.



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

28. We graph ∇f along with a contour map of f.



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

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29. $f(x,y) = xy \;\Rightarrow\; \nabla f(x,y) = y\,\mathbf{i} + x\,\mathbf{j}$. In the first quadrant, both components of each vector are positive, while in the third quadrant both components are negative. However, in the second quadrant each vector's x-component is positive while its y-component is negative (and vice versa in the fourth quadrant). Thus, ∇f is graph IV.

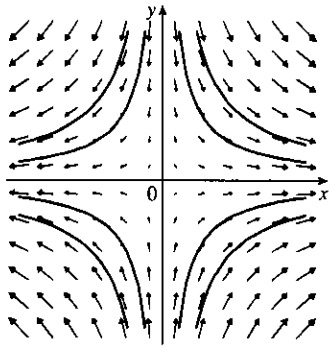
30. $f(x,y) = x^2 - y^2 \;\Rightarrow\; \nabla f(x,y) = 2x\,\mathbf{i} - 2y\,\mathbf{j}$. In the first quadrant, the x-component of each vector is positive while the y-component is negative. The other three quadrants are similar, where the x-component of each vector has the same sign as the x-value of its initial point, and the y-component has sign opposite that of the y-value of the initial point. Thus, ∇f is graph III.

31. $f(x,y) = x^2 + y^2 \;\Rightarrow\; \nabla f(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Thus, each vector $\nabla f(x,y)$ has the same direction and twice the length of the position vector of the point (x,y), so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph II.

32. $f(x,y) = \sqrt{x^2+y^2} \;\Rightarrow\; \nabla f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\mathbf{j}$. Then

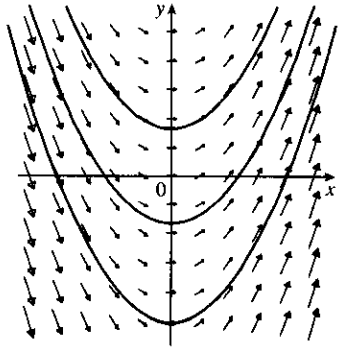
$|\nabla f(x,y)| = \dfrac{1}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2} = 1$, so all vectors are unit vectors. In addition, each vector $\nabla f(x,y)$ has the same direction as the position vector of the point (x,y), so the vectors all point directly away from the origin. Hence, ∇f is graph I.

33. (a) We sketch the vector field $\mathbf{F}(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be hyperbolas with shape similar to the graph of $y = \pm 1/x$, so we might guess that the flow lines have equations $y = C/x$.



(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x,y) is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = x\,\mathbf{i} - y\,\mathbf{j} \;\Rightarrow\; dx/dt = x,\ dy/dt = -y$. To solve these differential equations, we know $dx/dt = x \;\Rightarrow\; dx/x = dt \;\Rightarrow\; \ln|x| = t + C \;\Rightarrow\; x = \pm e^{t+C} = Ae^t$ for some constant A, and $dy/dt = -y \;\Rightarrow\; dy/y = -dt \;\Rightarrow\; \ln|y| = -t + K \;\Rightarrow\; y = \pm e^{-t+K} = Be^{-t}$ for some constant B. Therefore $xy = Ae^t Be^{-t} = AB =$ constant. If the flow line passes through $(1,1)$ then $(1)\,(1) =$ constant $= 1 \;\Rightarrow\; xy = 1 \;\Rightarrow\; y = 1/x,\ x > 0$.

34. (a) We sketch the vector field $\mathbf{F}(x,y) = \mathbf{i} + x\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be parabolas.



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(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have

$$x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = \mathbf{i} + x\,\mathbf{j} \quad \Rightarrow \quad \frac{dx}{dt} = 1,\ \frac{dy}{dt} = x. \text{ Thus } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x}{1} = x.$$

(c) From part (b), $dy/dx = x$. Integrating, we have $y = \frac{1}{2}x^2 + c$. Since the particle starts at the origin, we know $(0, 0)$ is on the curve, so $0 = 0 + c \quad \Rightarrow \quad c = 0$ and the path the particle follows is $y = \frac{1}{2}x^2$.

17.2 Line Integrals ET 16.2

1. $x = t^2$ and $y = t$, $0 \le t \le 2$, so by Formula 3

$$\int_C y\,ds = \int_0^2 t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^2 t\,\sqrt{(2t)^2 + (1)^2}\,dt$$

$$= \int_0^2 t\,\sqrt{4t^2+1}\,dt = \tfrac{1}{12}\left(4t^2+1\right)^{3/2}\Big]_0^2 = \tfrac{1}{12}\left(17\sqrt{17} - 1\right)$$

2. $\int_C \frac{y}{x}\,ds = \int_{1/2}^1 \frac{t^3}{t^4}\sqrt{(4t^3)^2 + (3t^2)^2}\,dt = \int_{1/2}^1 \frac{1}{t}\sqrt{16t^6 + 9t^4}\,dt = \int_{1/2}^1 t\,\sqrt{16t^2+9}\,dt$

$$= \tfrac{1}{48}(16t^2+9)^{3/2}\Big]_{1/2}^1 = \tfrac{1}{48}(25^{3/2} - 13^{3/2}) = \tfrac{1}{48}\left(125 - 13\sqrt{13}\right)$$

3. Parametric equations for C are $x = 4\cos t$, $y = 4\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2}(4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt$$

$$= \int_{-\pi/2}^{\pi/2} 4^5\cos t\sin^4 t\sqrt{16\left(\sin^2 t + \cos^2 t\right)}\,dt$$

$$= 4^5\int_{-\pi/2}^{\pi/2}\left(\sin^4 t\cos t\right)(4)\,dt = (4)^6\left[\tfrac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \tfrac{2\cdot 4^6}{5} = 1638.4$$

4. Parametric equations for C are $x = 1 + 3t$, $y = 2 + 5t$, $0 \le t \le 1$. Then

$$\int_C ye^x\,ds = \int_0^1 (2+5t)\,e^{1+3t}\sqrt{3^2+5^2}\,dt = \sqrt{34}\int_0^1 (2+5t)\,e^{1+3t}\,dt$$

Integrating by parts with $u = 2 + 5t \quad \Rightarrow \quad du = 5\,dt$, $dv = e^{1+3t} \quad \Rightarrow \quad v = \frac{1}{3}e^{1+3t}\,dt$ gives

$$\int_C ye^x\,ds = \sqrt{34}\left[\tfrac{1}{3}(2+5t)\,e^{1+3t} - \tfrac{5}{9}e^{1+3t}\right]_0^1$$

$$= \sqrt{34}\left[\left(\tfrac{7}{3} - \tfrac{5}{9}\right)e^4 - \left(\tfrac{2}{3} - \tfrac{5}{9}\right)e\right] = \tfrac{\sqrt{34}}{9}\left(16e^4 - e\right)$$

5. If we choose x as the parameter, parametric equations for C are $x = x$, $y = x^2$ for $1 \le x \le 3$ and

$$\int_C (xy + \ln x)\,dy = \int_1^3 \left(x\cdot x^2 + \ln x\right)2x\,dx = \int_1^3 2\left(x^4 + x\ln x\right)dx$$

$$= 2\left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\right]_1^3 \quad \text{(by integrating by parts in the second term)}$$

$$= 2\left(\tfrac{243}{5} + \tfrac{9}{2}\ln 3 - \tfrac{9}{4} - \tfrac{1}{5} + \tfrac{1}{4}\right) = \tfrac{464}{5} + 9\ln 3$$

6. Choosing y as the parameter, we have $x = e^y$, $y = y$, $0 \le y \le 1$. Then
$\int_c xe^y\,dx = \int_0^1 e^y(e^y)e^y\,dy = \int_0^1 e^{3y}\,dy = \tfrac{1}{3}e^{3y}\Big]_0^1 = \tfrac{1}{3}(e^3 - 1)$.

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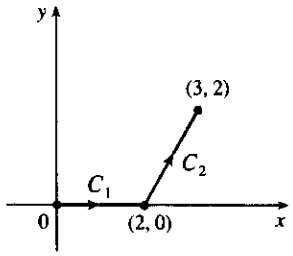
7.



$C = C_1 + C_2$

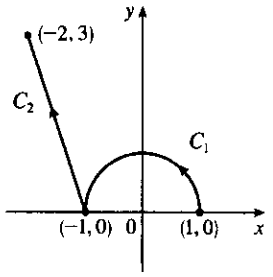
On C_1: $x = x, y = 0 \Rightarrow dy = 0\,dx, 0 \le x \le 2.$

On C_2: $x = x, y = 2x - 4 \Rightarrow dy = 2\,dx, 2 \le x \le 3.$

Then

$$\begin{aligned}\int_C xy\,dx + (x-y)\,dy &= \int_{C_1} xy\,dx + (x-y)\,dy + \int_{C_2} xy\,dx + (x-y)\,dy \\ &= \int_0^2 (0+0)\,dx + \int_2^3 [(2x^2 - 4x) + (-x+4)(2)]\,dx \\ &= \int_2^3 (2x^2 - 6x + 8)\,dx = \tfrac{17}{3}\end{aligned}$$

8.



$C = C_1 + C_2$

On C_1: $x = \cos t \Rightarrow dx = -\sin t\,dt, y = \sin t \Rightarrow$
$dy = \cos t\,dt, 0 \le t \le \pi.$

On C_2: $x = -1 - t \Rightarrow dx = -dt, y = 3t \Rightarrow$
$dy = 3\,dt, 0 \le t \le 1.$

Then
$$\begin{aligned}\int_C \sin x\,dx + \cos y\,dy &= \int_{C_1} \sin x\,dx + \cos y\,dy + \int_{C_2} \sin x\,dx + \cos y\,dy \\ &= \int_0^\pi \sin(\cos t)(-\sin t\,dt) + \cos(\sin t)\cos t\,dt \\ &\qquad + \int_0^1 \sin(-1-t)(-dt) + \cos(3t)(3\,dt) \\ &= \left[-\cos(\cos t) + \sin(\sin t)\right]_0^\pi + \left[-\cos(-1-t) + \sin(3t)\right]_0^1 \\ &= -\cos(\cos\pi) + \sin(\sin\pi) + \cos(\cos 0) - \sin(\sin 0) \\ &\quad -\cos(-2) + \sin(3) + \cos(-1) - \sin(0) \\ &= -\cos(-1) + \sin 0 + \cos(1) - \sin 0 - \cos(-2) + \sin 3 + \cos(-1) \\ &= -\cos 1 + \cos 1 - \cos 2 + \sin 3 + \cos 1 = \cos 1 - \cos 2 + \sin 3\end{aligned}$$

where we have used the identity $\cos(-\theta) = \cos\theta$.

9. $x = 4\sin t, y = 4\cos t, z = 3t, 0 \le t \le \frac{\pi}{2}$. Then by Formula 9,

$$\begin{aligned}\int_C xy^3\,ds &= \int_0^{\pi/2} (4\sin t)(4\cos t)^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\ &= \int_0^{\pi/2} 4^4 \cos^3 t \sin t \sqrt{(4\cos t)^2 + (-4\sin t)^2 + (3)^2}\,dt \\ &= \int_0^{\pi/2} 256\cos^3 t \sin t \sqrt{16(\cos^2 t + \sin^2 t) + 9}\,dt \\ &= 1280 \int_0^{\pi/2} \cos^3 t \sin t\,dt = -320\cos^4 t\Big]_0^{\pi/2} = 320\end{aligned}$$

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10. Parametric equations for C are $x = 4t$, $y = 6 - 5t$, $z = -1 + 6t$, $0 \le t \le 1$. Then

$$\int_C x^2 z\,ds = \int_0^1 (4t)^2(6t-1)\sqrt{4^2+(-5)^2+6^2}\,dt = \sqrt{77}\int_0^1 (96t^3 - 16t^2)\,dt$$
$$= \sqrt{77}\left[96\cdot\frac{t^4}{4} - 16\cdot\frac{t^3}{3}\right]_0^1 = \tfrac{56}{3}\sqrt{77}$$

11. Parametric equations for C are $x = t$, $y = 2t$, $z = 3t$, $0 \le t \le 1$. Then

$$\int_C xe^{yz}\,ds = \int_0^1 te^{(2t)(3t)}\sqrt{1^2+2^2+3^2}\,dt = \sqrt{14}\int_0^1 te^{6t^2}\,dt$$
$$= \sqrt{14}\left[\tfrac{1}{12}e^{6t^2}\right]_0^1 = \tfrac{\sqrt{14}}{12}(e^6 - 1)$$

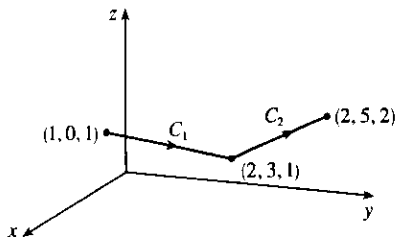
12. $\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2} = \sqrt{1^2+(2t)^2+(3t^2)^2} = \sqrt{1+4t^2+9t^4}$. Then

$$\int_C (2x+9z)\,ds = \int_0^1 (2t+9t^3)\sqrt{1+4t^2+9t^4}\,dt \quad [\text{let } u = 1+4t^2+9t^4 \Rightarrow \tfrac{1}{4}\,du = (2t+9t^3)\,dt]$$
$$= \int_1^{14} \tfrac{1}{4}\sqrt{u}\,du = \tfrac{1}{6}u^{3/2}\Big]_1^{14} = \tfrac{1}{6}(14^{3/2} - 1)$$

13. $\int_C x^2 y\sqrt{z}\,dz = \int_0^1 (t^3)^2(t)\sqrt{t^2}\cdot 2t\,dt = \int_0^1 2t^9\,dt = \tfrac{1}{5}t^{10}\Big]_0^1 = \tfrac{1}{5}$

14. $\int_C z\,dx + x\,dy + y\,dz = \int_0^1 t^2\cdot 2t\,dt + t^2\cdot 3t^2\,dt + t^3\cdot 2t\,dt = \int_0^1 (2t^3+5t^4)\,dt$
$$= \left[\tfrac{1}{2}t^4 + t^5\right]_0^1 = \tfrac{1}{2} + 1 = \tfrac{3}{2}$$

15.

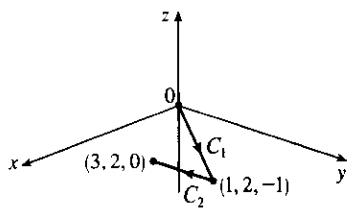


On C_1: $x = 1+t \Rightarrow dx = dt$, $y = 3t \Rightarrow dy = 3\,dt$, $z = 1$
$\Rightarrow dz = 0\,dt$, $0 \le t \le 1$.

On C_2: $x = 2 \Rightarrow dx = 0\,dt$, $y = 3+2t \Rightarrow$
$dy = 2\,dt$, $z = 1+t \Rightarrow dz = dt$, $0 \le t \le 1$.

Then $\int_C (x+yz)\,dx + 2x\,dy + xyz\,dz$
$$= \int_{C_1}(x+yz)\,dx + 2x\,dy + xyz\,dz + \int_{C_2}(x+yz)\,dx + 2x\,dy + xyz\,dz$$
$$= \int_0^1 (1+t+(3t)(1))\,dt + 2(1+t)\cdot 3\,dt + (1+t)(3t)(1)\cdot 0\,dt$$
$$+ \int_0^1 (2+(3+2t)(1+t))\cdot 0\,dt + 2(2)\cdot 2\,dt + (2)(3+2t)(1+t)dt$$
$$= \int_0^1 (10t+7)\,dt + \int_0^1 (4t^2+10t+14)\,dt$$
$$= \left[5t^2+7t\right]_0^1 + \left[\tfrac{4}{3}t^3 + 5t^2 + 14t\right]_0^1 = 12 + \tfrac{61}{3} = \tfrac{97}{3}$$

16.



On C_1: $x = t \Rightarrow dx = dt$, $y = 2t \Rightarrow dy = 2\,dt$, $z = -t$
$\Rightarrow dz = -dt$, $0 \le t \le 1$.

On C_2: $x = 1+2t \Rightarrow dx = 2\,dt$, $y = 2 \Rightarrow$
$dy = 0\,dt$, $z = -1+t \Rightarrow dz = dt$, $0 \le t \le 1$.

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(b) $\oint_C xy^2\,dx + x^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(xy^2)\right] dA = \int_0^2 \int_0^3 (3x^2 - 2xy)\,dy\,dx$

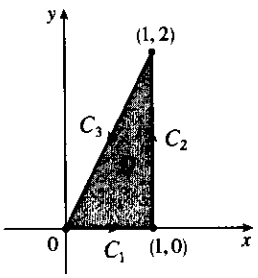
$= \int_0^2 (9x^2 - 9x)\,dx = 24 - 18 = 6$

2. (a) $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Then

$\oint_C y\,dx - x\,dy = \int_0^{2\pi} [\sin t(-\sin t) - \cos t(\cos t)]\,dt = -\int_0^{2\pi} dt = -2\pi.$

(b) $\oint_C y\,dx - x\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right] dA = -2\iint_D dA = -2A(D) = -2\pi(1)^2 = -2\pi$

3. (a)



C_1: $x = t \Rightarrow dx = dt$, $y = 0 \Rightarrow dy = 0\,dt$, $0 \le t \le 1$.

C_2: $x = 1 \Rightarrow dx = 0\,dt$, $y = t \Rightarrow dy = dt$, $0 \le t \le 2$.

C_3: $x = 1 - t \Rightarrow dx = -dt$, $y = 2 - 2t \Rightarrow dy = -2\,dt$, $0 \le t \le 1$.

Thus $\oint_C xy\,dx + x^2y^3\,dy = \oint_{C_1+C_2+C_3} xy\,dx + x^2y^3\,dy$

$= \int_0^1 0\,dt + \int_0^2 t^3\,dt + \int_0^1 \left[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3\right] dt$

$= 0 + \left[\frac{1}{4}t^4\right]_0^2 + \left[\frac{2}{3}(1-t)^3 + \frac{8}{3}(1-t)^6\right]_0^1 = 4 - \frac{10}{3} = \frac{2}{3}$

(b) $\oint_C xy\,dx + x^2y^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$

$= \int_0^1 \left[\frac{1}{2}xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$

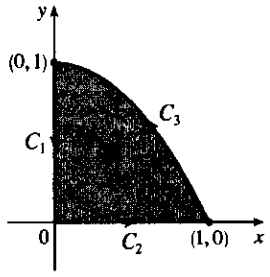
4. (a) $C_1 : x = 0 \Rightarrow dx = 0\,dt$, $y = 1 - t \Rightarrow$

$dy = -dt$, $0 \le t \le 1$

$C_2 : x = t \Rightarrow dx = dt$, $y = 0 \Rightarrow dy = 0\,dt$, $0 \le t \le 1$

$C_3 : x = 1 - t \Rightarrow dx = -dt$, $y = 1 - (1-t)^2 = 2t - t^2 \Rightarrow$

$dy = (2 - 2t)\,dt$, $0 \le t \le 1$



Thus

$\oint_C x\,dx + y\,dy = \oint_{C_1+C_2+C_3} x\,dx + y\,dy$

$= \int_0^1 (0\,dt + (1-t)(-dt)) + \int_0^1 (t\,dt + 0\,dt) + \int_0^1 ((1-t)(-dt) + (2t - t^2)(2 - 2t)\,dt)$

$= \left[\frac{1}{2}t^2 - t\right]_0^1 + \left[\frac{1}{2}t^2\right]_0^1 + \left[\frac{1}{2}t^4 - 2t^3 + \frac{5}{2}t^2 - t\right]_0^1$

$= -\frac{1}{2} + \frac{1}{2} + \left(\frac{1}{2} - 2 + \frac{5}{2} - 1\right) = 0$

(b) $\oint_C x\,dx + y\,dy = \iint_D \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right] dA = \iint_D 0\,dA = 0$

5. We can parametrize C as $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$. Then the line integral is $\oint_C P\,dx + Q\,dy = \int_0^{2\pi} \cos^4\theta \sin^5\theta\,(-\sin\theta)\,d\theta + \int_0^{2\pi} (-\cos^7\theta \sin^6\theta)\cos\theta\,d\theta = -\frac{29\pi}{1024}$, according to a CAS. The double integral is

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-7x^6y^6 - 5x^4y^4)\,dy\,dx = -\frac{29\pi}{1024},$$

verifying Green's Theorem in this case.

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6. Since $y = x^2$ along the first part of C and $y = x$ along the second part, the line integral is

$$\begin{aligned}\oint_C P\,dx + Q\,dy &= \int_0^1 \left[x^4 \sin x + x^2 \sin(x^2)(2x)\right] dx + \int_1^0 (x^2 \sin x + x^2 \sin x)\,dx \\ &= -16\cos 1 - 23\sin 1 + 28\end{aligned}$$

according to a CAS. The double integral is

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_{x^2}^{x} (2x \sin y - 2y \sin x)\,dy\,dx = -16\cos 1 - 23\sin 1 + 28$$

7. The region D enclosed by C is $[0,1] \times [0,1]$, so

$$\begin{aligned}\int_C e^y\,dx + 2xe^y\,dy &= \iint_D \left[\frac{\partial}{\partial x}\left(2xe^y\right) - \frac{\partial}{\partial y}\left(e^y\right)\right] dA = \int_0^1 \int_0^1 (2e^y - e^y)\,dy\,dx \\ &= \int_0^1 dx \int_0^1 e^y\,dy = (1)\left(e^1 - e^0\right) = e - 1\end{aligned}$$

8. The region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 1, 3x \le y \le 3\}$, so

$$\begin{aligned}\int_C x^2y^2\,dx + 4xy^3\,dy &= \iint_D \left[\frac{\partial}{\partial x}\left(4xy^3\right) - \frac{\partial}{\partial y}\left(x^2y^2\right)\right] dA = \int_0^1 \int_{3x}^3 (4y^3 - 2x^2y)\,dy\,dx \\ &= \int_0^1 \left[y^4 - x^2y^2\right]_{y=3x}^{y=3} dx = \int_0^1 (81 - 9x^2 - 72x^4)\,dx = 81 - 3 - \tfrac{72}{5} = \tfrac{318}{5}\end{aligned}$$

9. $$\begin{aligned}\int_C \left(y + e^{\sqrt{x}}\right) dx + \left(2x + \cos y^2\right) dy &= \iint_D \left[\frac{\partial}{\partial x}\left(2x + \cos y^2\right) - \frac{\partial}{\partial y}\left(y + e^{\sqrt{x}}\right)\right] dA \\ &= \int_0^1 \int_{y^2}^{\sqrt{y}} (2 - 1)\,dx\,dy = \int_0^1 (y^{1/2} - y^2)\,dy = \tfrac{1}{3}\end{aligned}$$

10. $$\begin{aligned}\int_C xe^{-2x}\,dx + \left(x^4 + 2x^2y^2\right) dy &= \iint_D \left[\frac{\partial}{\partial x}\left(x^4 + 2x^2y^2\right) - \frac{\partial}{\partial y}\left(xe^{-2x}\right)\right] dA = \iint_D (4x^3 + 4xy^2 - 0)\,dA \\ &= 4\iint_D x(x^2 + y^2)\,dA = 4\int_0^{2\pi} \int_1^2 (r\cos\theta)(r^2)\,r\,dr\,d\theta \\ &= 4\int_0^{2\pi} \cos\theta\,d\theta \int_1^2 r^4\,dr = 4\left[\sin\theta\right]_0^{2\pi} \left[\tfrac{1}{5}r^5\right]_1^2 = 0\end{aligned}$$

11. $$\begin{aligned}\int_C y^3\,dx - x^3\,dy &= \iint_D \left[\frac{\partial}{\partial x}\left(-x^3\right) - \frac{\partial}{\partial y}\left(y^3\right)\right] dA = \iint_D (-3x^2 - 3y^2)\,dA = \int_0^{2\pi} \int_0^2 (-3r^2)\,r\,dr\,d\theta \\ &= -3\int_0^{2\pi} d\theta \int_0^2 r^3\,dr = -3(2\pi)(4) = -24\pi\end{aligned}$$

12. $$\int_C \sin y\,dx + x\cos y\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(x\cos y\right) - \frac{\partial}{\partial y}\left(\sin y\right)\right] dA = \iint_D (\cos y - \cos y)\,dA = \iint_D 0\,dA = 0$$

13. $\mathbf{F}(x,y) = \left\langle \sqrt{x} + y^3, x^2 + \sqrt{y} \right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le \pi, 0 \le y \le \sin x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= -\int_{-C} \left(\sqrt{x} + y^3\right) dx + \left(x^2 + \sqrt{y}\right) dy = -\iint_D \left[\frac{\partial}{\partial x}\left(x^2 + \sqrt{y}\right) - \frac{\partial}{\partial y}\left(\sqrt{x} + y^3\right)\right] dA \\ &= -\int_0^{\pi} \int_0^{\sin x} (2x - 3y^2)\,dy\,dx = -\int_0^{\pi} \left[2xy - y^3\right]_{y=0}^{y=\sin x} dx \\ &= -\int_0^{\pi} (2x\sin x - \sin^3 x)\,dx = -\int_0^{\pi} (2x\sin x - (1 - \cos^2 x)\sin x)\,dx \\ &= -\left[2\sin x - 2x\cos x + \cos x - \tfrac{1}{3}\cos^3 x\right]_0^{\pi} \qquad \text{[integrate by parts in the first term]} \\ &= -\left(2\pi - 2 + \tfrac{2}{3}\right) = \tfrac{4}{3} - 2\pi\end{aligned}$$

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14. $\mathbf{F}(x,y) = \langle y^2\cos x, x^2 + 2y\sin x\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C}(y^2\cos x)\,dx + (x^2+2y\sin x)\,dy\\ &= -\iint_D \left[\tfrac{\partial}{\partial x}(x^2+2y\sin x) - \tfrac{\partial}{\partial y}(y^2\cos x)\right]dA\\ &= -\iint_D (2x + 2y\cos x - 2y\cos x)\,dA = -\int_0^2\int_0^{3x} 2x\,dy\,dx\\ &= -\int_0^2 2x\,[y]_{y=0}^{y=3x}\,dx = -\int_0^2 6x^2\,dx = -2x^3\big]_0^2 = -16\end{aligned}$$

15. $\mathbf{F}(x,y) = \langle e^x + x^2y, e^y - xy^2\rangle$ and the region D enclosed by C is the disk $x^2+y^2 \le 25$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C}(e^x + x^2y)\,dx + (e^y - xy^2)\,dy\\ &= -\iint_D \left[\tfrac{\partial}{\partial x}(e^y - xy^2) - \tfrac{\partial}{\partial y}(e^x + x^2y)\right]dA = -\iint_D(-y^2 - x^2)\,dA\\ &= \iint_D (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^5 (r^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_0^5 r^3\,dr = 2\pi\left[\tfrac{1}{4}r^4\right]_0^5 = \tfrac{625}{2}\pi\end{aligned}$$

16. $\mathbf{F}(x,y) = \left\langle y - \ln(x^2+y^2), 2\tan^{-1}\left(\frac{y}{x}\right)\right\rangle$ and the region D enclosed by C is the disk with radius 1 centered at $(2,3)$. C is oriented positively, so

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (y - \ln(x^2+y^2))\,dx + \left(2\tan^{-1}\left(\frac{y}{x}\right)\right)dy\\ &= \iint_D \left[\frac{\partial}{\partial x}\left(2\tan^{-1}\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial y}(y - \ln(x^2+y^2))\right]dA\\ &= \iint_D \left[2\left(\frac{-yx^{-2}}{1+(y/x)^2}\right) - \left(1 - \frac{2y}{x^2+y^2}\right)\right]dA = \iint_D \left[-\frac{2y}{x^2+y^2} - 1 + \frac{2y}{x^2+y^2}\right]dA\\ &= -\iint_D dA = -(\text{area of } D) = -\pi\end{aligned}$$

17. By Green's Theorem, $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C x(x+y)\,dx + xy^2\,dy = \iint_D (y^2 - x)\,dy\,dx$ where C is the path described in the question and D is the triangle bounded by C. So

$$\begin{aligned}W &= \int_0^1\int_0^{1-x}(y^2 - x)\,dy\,dx = \int_0^1 \left[\tfrac{1}{3}y^3 - xy\right]_{y=0}^{y=1-x}dx = \int_0^1 \left(\tfrac{1}{3}(1-x)^3 - x(1-x)\right)dx\\ &= \left[-\tfrac{1}{12}(1-x)^4 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right]_0^1 = \left(-\tfrac{1}{2} + \tfrac{1}{3}\right) - \left(-\tfrac{1}{12}\right) = -\tfrac{1}{12}\end{aligned}$$

18. By Green's Theorem, $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C x\,dx + (x^3 + 3xy^2)\,dy = \iint_D (3x^2 + 3y^2 - 0)\,dA$, where D is the semicircular region bounded by C. Converting to polar coordinates, we have $W = 3\int_0^2\int_0^\pi r^2\cdot r\,d\theta\,dr = 3\pi\left[\tfrac{1}{4}r^4\right]_0^2 = 12\pi$.

19. Let C_1 be the arch of the cycloid from $(0,0)$ to $(2\pi,0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi,0)$ to $(0,0)$, so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \le t \le 2\pi$. Then $C = C_1 \cup C_2$ is traversed

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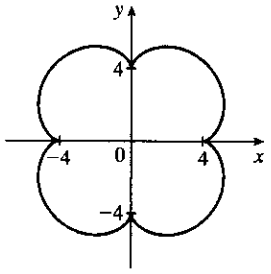
clockwise, so $-C$ is oriented positively. Thus $-C$ encloses the area under one arch of the cycloid and from (5) we have

$$A = -\oint_{-C} y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi}(1-\cos t)(1-\cos t)\,dt + \int_0^{2\pi} 0\,(-dt)$$
$$= \int_0^{2\pi}(1 - 2\cos t + \cos^2 t)\,dt + 0 = \left[t - 2\sin t + \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi$$

20.



$$A = \oint_C x\,dy = \int_0^{2\pi}(5\cos t - \cos 5t)(5\cos t - 5\cos 5t)\,dt$$
$$= \int_0^{2\pi}(25\cos^2 t - 30\cos t\cos 5t + 5\cos^2 5t)\,dt$$
$$= \left[25\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) - 30\left(\tfrac{1}{8}\sin 4t + \tfrac{1}{12}\sin 6t\right) + 5\left(\tfrac{1}{2}t + \tfrac{1}{20}\sin 10t\right)\right]_0^{2\pi}$$

[Use Formula 80 in the Table of Integrals]

$$= 30\pi$$

21. (a) Using Equation 17.2.8 [ET 16.2.8], we write parametric equations of the line segment as $x = (1-t)x_1 + tx_2$, $y = (1-t)y_1 + ty_2$, $0 \le t \le 1$. Then $dx = (x_2 - x_1)\,dt$ and $dy = (y_2 - y_1)\,dt$, so

$$\int_C x\,dy - y\,dx = \int_0^1 [(1-t)x_1 + tx_2](y_2 - y_1)\,dt + [(1-t)y_1 + ty_2](x_2 - x_1)\,dt$$
$$= \int_0^1 \left(x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[(y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)(y_2 - y_1)]\right)dt$$
$$= \int_0^1 (x_1y_2 - x_2y_1)\,dt = x_1y_2 - x_2y_1$$

(b) We apply Green's Theorem to the path $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, \ldots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1). From (5), $\frac{1}{2}\int_C x\,dy - y\,dx = \iint_D dA$, where D is the polygon bounded by C. Therefore

$$\text{area of polygon} = A(D) = \iint_D dA = \tfrac{1}{2}\int_C x\,dy - y\,dx$$
$$= \tfrac{1}{2}\left(\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \cdots + \int_{C_{n-1}} x\,dy - y\,dx + \int_{C_n} x\,dy - y\,dx\right)$$

To evaluate these integrals we use the formula from (a) to get
$A(D) = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$.

(c) $A = \frac{1}{2}[(0\cdot 1 - 2\cdot 0) + (2\cdot 3 - 1\cdot 1) + (1\cdot 2 - 0\cdot 3) + (0\cdot 1 - (-1)\cdot 2) + (-1\cdot 0 - 0\cdot 1)]$
$= \frac{1}{2}(0 + 5 + 2 + 2) = \frac{9}{2}$

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and
$-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D(-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

23. Here $A = \frac{1}{2}(1)(1) = \frac{1}{2}$ and $C = C_1 + C_2 + C_3$, where C_1: $x = x$, $y = 0$, $0 \le x \le 1$;
C_2: $x = x$, $y = 1 - x$, $x = 1$ to $x = 0$; and C_3: $x = 0$, $y = 1$ to $y = 0$. Then
$\overline{x} = \frac{1}{2A}\int_C x^2\,dy = \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = 0 + \int_1^0 (x^2)(-dx) + 0 = \frac{1}{3}$. Similarly,
$\overline{y} = -\frac{1}{2A}\int_C y^2\,dx = \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = 0 + \int_1^0 (1-x)^2(-dx) + 0 = \frac{1}{3}$.
Therefore $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \frac{1}{3}\right)$.

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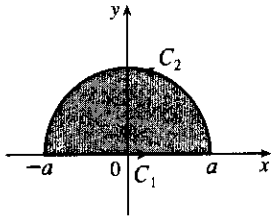
24. $A = \frac{\pi a^2}{2}$ so $\overline{x} = \frac{1}{\pi a^2}\oint_C x^2\,dy$ and $\overline{y} = -\frac{1}{\pi a^2}\oint_C y^2\,dx$.



Orienting the semicircular region as in the figure,

$$\overline{x} = \frac{1}{\pi a^2}\oint_{C_1+C_2} x^2\,dy$$
$$= \frac{1}{\pi a^2}\left[0 + \int_0^{\pi}(a^2\cos^2 t)(a\cos t)\,dt\right] = 0$$

and

$$\overline{y} = -\frac{1}{\pi a^2}\left[\int_{-a}^{a} 0\,dx + \int_0^{\pi}(a^2\sin^2 t)(-a\sin t)\,dt\right] = \frac{a}{\pi}\int_0^{\pi}\sin^3 t\,dt = \frac{a}{\pi}\left[-\cos t + \tfrac{1}{3}(\cos^3 t)\right]_0^{\pi} = \frac{4a}{3\pi}.$$

Thus $(\overline{x},\overline{y}) = \left(0, \frac{4a}{3\pi}\right)$.

25. By Green's Theorem, $-\frac{1}{3}\rho\oint_C y^3\,dx = -\frac{1}{3}\rho\iint_D(-3y^2)\,dA = \iint_D y^2\rho\,dA = I_x$ and

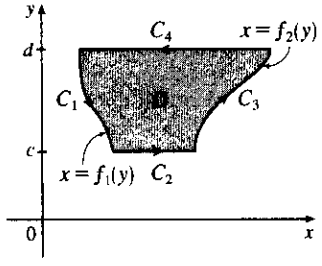
$\frac{1}{3}\rho\oint_C x^3\,dy = \frac{1}{3}\rho\iint_D(3x^2)\,dA = \iint_D x^2\rho\,dA = I_y$.

26. By symmetry the moments of inertia about any two diameters are equal. Centering the disk at the origin, the moment of inertia about a diameter equals

$$I_y = \tfrac{1}{3}\rho\oint_C x^3\,dy = \tfrac{1}{3}\rho\int_0^{2\pi}(a^4\cos^4 t)\,dt = \tfrac{1}{3}a^4\rho\int_0^{2\pi}\left[\tfrac{3}{8} + \tfrac{1}{2}\cos 2t + \tfrac{1}{8}\cos 4t\right]dt$$
$$= \tfrac{1}{3}a^4\rho\cdot\tfrac{3(2\pi)}{8} = \tfrac{1}{4}\pi a^4\rho$$

27. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D. Thus $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 17.3.33(a) [ET 16.3.33(a)], $\partial P/\partial y = \partial Q/\partial x$, so $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D 0\,dA = 0$.

28. We express D as a type II region: $D = \{(x,y) \mid f_1(y) \le x \le f_2(y),\ c \le y \le d\}$ where f_1 and f_2 are continuous functions. Then $\iint_D \frac{\partial Q}{\partial x}\,dA = \int_c^d\int_{f_1(y)}^{f_2(y)}\frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d [Q(f_2(y),y) - Q(f_1(y),y)]\,dy$ by the Fundamental Theorem of Calculus. But referring to the figure,



$\oint_C Q\,dy = \oint_{C_1+C_2+C_3+C_4} Q\,dy$. Then $\int_{C_1} Q\,dy = \int_d^c Q(f_1(y),y)\,dy$,

$\int_{C_2} Q\,dy = \int_{C_4} Q\,dy = 0$, and $\int_{C_3} Q\,dy = \int_c^d Q(f_2(y),y)\,dy$. Hence

$\oint_C Q\,dy = \int_c^d [Q(f_2(y),y) - Q(f_1(y),y)]\,dy = \iint_D(\partial Q/\partial x)\,dA$.

29. Using the first part of (5), we have that $\iint_R dx\,dy = A(R) = \int_{\partial R} x\,dy$. But $x = g(u,v)$, and

$dy = \dfrac{\partial h}{\partial u}\,du + \dfrac{\partial h}{\partial v}\,dv$, and we orient ∂S by taking the positive direction to be that which corresponds, under the

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mapping, to the positive direction along ∂R, so

$$\int_{\partial R} x\,dy = \int_{\partial S} g(u,v)\left(\frac{\partial h}{\partial u}\,du + \frac{\partial h}{\partial v}\,dv\right) = \int_{\partial S} g(u,v)\,\frac{\partial h}{\partial u}\,du + g(u,v)\,\frac{\partial h}{\partial v}\,dv$$

$$= \pm\iint_S \left[\frac{\partial}{\partial u}\left(g(u,v)\,\frac{\partial h}{\partial v}\right) - \frac{\partial}{\partial v}\left(g(u,v)\,\frac{\partial h}{\partial u}\right)\right] dA \quad \text{[using Green's Theorem in the } uv\text{-plane]}$$

$$= \pm\iint_S \left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} + g(u,v)\,\frac{\partial^2 h}{\partial u\,\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} - g(u,v)\,\frac{\partial^2 h}{\partial v\,\partial u}\right) dA \qquad \text{[using the Chain Rule]}$$

$$= \pm\iint_S \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) dA \text{ [by the equality of mixed partials]} = \pm\iint_S \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

The sign is chosen to be positive if the orientation that we gave to ∂S corresponds to the usual positive orientation, and it is negative otherwise. In either case, since $A(R)$ is positive, the sign chosen must be the same as the sign of $\dfrac{\partial(x,y)}{\partial(u,v)}$. Therefore $A(R) = \displaystyle\iint_R dx\,dy = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$.

17.5 Curl and Divergence — ET 16.5

1. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 0 & -x^2y \end{vmatrix} = (-x^2-0)\,\mathbf{i} - (-2xy-xy)\,\mathbf{j} + (0-xz)\,\mathbf{k}$

$= -x^2\,\mathbf{i} + 3xy\,\mathbf{j} - xz\,\mathbf{k}$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(xyz) + \dfrac{\partial}{\partial y}(0) + \dfrac{\partial}{\partial z}(-x^2y) = yz + 0 + 0 = yz$

2. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} = (xz^2 - xy^2)\,\mathbf{i} - (yz^2 - x^2y)\,\mathbf{j} + (y^2z - x^2z)\,\mathbf{k}$

$= x(z^2-y^2)\,\mathbf{i} + y(x^2-z^2)\,\mathbf{j} + z(y^2-x^2)\,\mathbf{k}$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(x^2yz) + \dfrac{\partial}{\partial y}(xy^2z) + \dfrac{\partial}{\partial z}(xyz^2) = 2xyz + 2xyz + 2xyz = 6xyz$

3. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix} = (x-y)\,\mathbf{i} - (y-0)\,\mathbf{j} + (1-0)\,\mathbf{k}$

$= (x-y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(1) + \dfrac{\partial}{\partial y}(x+yz) + \dfrac{\partial}{\partial z}(xy-\sqrt{z}) = z - \dfrac{1}{2\sqrt{z}}$

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4. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \cos xz & -\sin xy \end{vmatrix}$

$= (-x\cos xy + x\sin xz)\,\mathbf{i} - (-y\cos xy - 0)\,\mathbf{j} + (-z\sin xz - 0)\,\mathbf{k}$

$= x(\sin xz - \cos xy)\,\mathbf{i} + y\cos xy\,\mathbf{j} - z\sin xz\,\mathbf{k}$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(0) + \dfrac{\partial}{\partial y}(\cos xz) + \dfrac{\partial}{\partial z}(-\sin xy) = 0 + 0 + 0 = 0$

5. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x\sin y & e^x\cos y & z \end{vmatrix} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (e^x\cos y - e^x\cos y)\,\mathbf{k} = \mathbf{0}$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(e^x\sin y) + \dfrac{\partial}{\partial y}(e^x\cos y) + \dfrac{\partial}{\partial z}(z) = e^x\sin y - e^x\sin y + 1 = 1$

6. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{x^2+y^2+z^2} & \dfrac{y}{x^2+y^2+z^2} & \dfrac{z}{x^2+y^2+z^2} \end{vmatrix}$

$= \dfrac{1}{(x^2+y^2+z^2)^2}\left[(-2yz+2yz)\,\mathbf{i} - (-2xz+2xz)\,\mathbf{j} + (-2xy+2xy)\,\mathbf{k}\right] = \mathbf{0}$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2+z^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2+z^2}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{x^2+y^2+z^2}\right)$

$= \dfrac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2} + \dfrac{x^2+y^2+z^2-2y^2}{(x^2+y^2+z^2)^2} + \dfrac{x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2}$

$= \dfrac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \dfrac{1}{x^2+y^2+z^2}$

7. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln x & \ln(xy) & \ln(xyz) \end{vmatrix} = \left(\dfrac{xz}{xyz} - 0\right)\mathbf{i} - \left(\dfrac{yz}{xyz} - 0\right)\mathbf{j} + \left(\dfrac{y}{xy} - 0\right)\mathbf{k}$

$= \left\langle \dfrac{1}{y}, -\dfrac{1}{x}, \dfrac{1}{x} \right\rangle$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(\ln x) + \dfrac{\partial}{\partial y}(\ln(xy)) + \dfrac{\partial}{\partial z}(\ln(xyz)) = \dfrac{1}{x} + \dfrac{x}{xy} + \dfrac{xy}{xyz} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$

8. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^{-y} & xz & ze^y \end{vmatrix} = (ze^y - x)\,\mathbf{i} - (0-0)\,\mathbf{j} + (z - xe^{-y}(-1))\,\mathbf{k}$

$= \langle ze^y - x, 0, z + xe^{-y}\rangle$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(xe^{-y}) + \dfrac{\partial}{\partial y}(xz) + \dfrac{\partial}{\partial z}(ze^y) = e^{-y} + 0 + e^y = e^y + e^{-y}$

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9. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the x-component of each vector of $\mathbf{F}$ is 0, so $P = 0$, hence $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial y} = \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. Q decreases as y increases, so $\dfrac{\partial Q}{\partial y} < 0$, but Q doesn't change in the x- or z-directions, so $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial z} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + \dfrac{\partial Q}{\partial y} + 0 < 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$

10. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, P and Q don't vary in the z-direction, so $\dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = \dfrac{\partial P}{\partial z} = \dfrac{\partial Q}{\partial z} = 0$. As x increases, the x-component of each vector of $\mathbf{F}$ increases while the y-component remains constant, so $\dfrac{\partial P}{\partial x} > 0$ and $\dfrac{\partial Q}{\partial x} = 0$. Similarly, as y increases, the y-component of each vector increases while the x-component remains constant, so $\dfrac{\partial Q}{\partial y} > 0$ and $\dfrac{\partial P}{\partial y} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + 0 > 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$

11. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the y-component of each vector of $\mathbf{F}$ is 0, so $Q = 0$, hence $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial y} = \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. P increases as y increases, so $\dfrac{\partial P}{\partial y} > 0$, but P doesn't change in the x- or z-directions, so $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial z} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + 0 + 0 = 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k}$

$= (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + \left(0 - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = -\dfrac{\partial P}{\partial y}\mathbf{k}$

Since $\dfrac{\partial P}{\partial y} > 0$, $-\dfrac{\partial P}{\partial y}\mathbf{k}$ is a vector pointing in the negative z-direction.

12. (a) $\operatorname{curl} f = \nabla \times f$ is meaningless because f is a scalar field.

(b) $\operatorname{grad} f$ is a vector field.

(c) $\operatorname{div}\mathbf{F}$ is a scalar field.

(d) $\operatorname{curl}(\operatorname{grad} f)$ is a vector field.

(e) $\operatorname{grad}\mathbf{F}$ is meaningless because $\mathbf{F}$ is not a scalar field.

(f) $\operatorname{grad}(\operatorname{div}\mathbf{F})$ is a vector field.

(g) $\operatorname{div}(\operatorname{grad} f)$ is a scalar field.

(h) $\operatorname{grad}(\operatorname{div} f)$ is meaningless because f is a scalar field.

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(i) curl (curl $\mathbf{F}$) is a vector field.

(j) div(div $\mathbf{F}$) is meaningless because div $\mathbf{F}$ is a scalar field.

(k) (grad f) $\times$ (div $\mathbf{F}$) is meaningless because div $\mathbf{F}$ is a scalar field.

(l) div(curl (grad f)) is a scalar field.

13. $$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = (x-x)\,\mathbf{i} - (y-y)\,\mathbf{j} + (z-z)\,\mathbf{k} = \mathbf{0}$$

and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ with component functions which have continuous partial derivatives, so by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_x(x,y,z) = yz$ implies $f(x,y,z) = xyz + g(y,z)$ and $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz$, so $g(y,z) = h(z)$ and $f(x,y,z) = xyz + h(z)$. Thus $f_z(x,y,z) = xy + h'(z)$ but $f_z(x,y,z) = xy$ so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x,y,z) = xyz + K$.

14. $$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3z^2 & \cos y & 2xz \end{vmatrix} = (0-0)\,\mathbf{i} - (2z-6z)\,\mathbf{j} + (0-0)\,\mathbf{k} = 4z\,\mathbf{j} \neq \mathbf{0},$$

so $\mathbf{F}$ is not conservative.

15. $$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2+2yz & y^2 \end{vmatrix} = (2y-2y)\,\mathbf{i} - (0-0)\,\mathbf{j} + (2x-2x)\,\mathbf{k} = \mathbf{0},$$ $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = 2xy$ implies $f(x,y,z) = x^2y + g(y,z)$ and $f_y(x,y,z) = x^2 + g_y(y,z)$. But $f_y(x,y,z) = x^2 + 2yz$, so $g(y,z) = y^2z + h(z)$ and $f(x,y,z) = x^2y + y^2z + h(z)$. Thus $f_z(x,y,z) = y^2 + h'(z)$ but $f_z(x,y,z) = y^2$ so $h(z) = K$ and $f(x,y,z) = x^2y + y^2z + K$.

16. $$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^z & 1 & xe^z \end{vmatrix} = (0-0)\,\mathbf{i} - (e^z-e^z)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$$ and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ with component functions that have continuous partial deriatives, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = e^z$ implies $f(x,y,z) = xe^z + g(y,z) \Rightarrow$ $f_y(x,y,z) = g_y(y,z)$. But $f_y(x,y,z) = 1$, so $g(y,z) = y + h(z)$ and $f(x,y,z) = xe^z + y + h(z)$. Thus $f_z(x,y,z) = xe^z + h'(z)$ but $f_z(x,y,z) = xe^z$, so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x,y,z) = xe^z + y + K$.

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17. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^{-x} & e^{-x} & 2z \end{vmatrix} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (-e^{-x} - e^{-x})\,\mathbf{k} = -2e^{-x}\,\mathbf{k} \neq \mathbf{0},$

so $\mathbf{F}$ is not conservative.

18. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\cos xy & x\cos xy & -\sin z \end{vmatrix}$

$= (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + [(-xy\sin xy + \cos xy) - (-xy\sin xy + \cos xy)]\,\mathbf{k} = \mathbf{0}$

$\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = y\cos xy$ implies $f(x,y,z) = \sin xy + g(y,z) \quad \Rightarrow \quad f_y(x,y,z) = x\cos xy + g_y(y,z)$. But $f_y(x,y,z) = x\cos xy$, so $g(y,z) = h(z)$ and $f(x,y,z) = \sin xy + h(z)$. Thus $f_z(x,y,z) = h'(z)$ but $f_z(x,y,z) = -\sin z$ so $h(z) = \cos z + K$ and a potential function for $\mathbf{F}$ is $f(x,y,z) = \sin xy + \cos z + K$.

19. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = y^2 + z^2 + x^2 \neq 0$, which contradicts Theorem 11.

20. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = xz \neq 0$ which contradicts Theorem 11.

21. $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}.$

Hence $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$ is irrotational.

22. $\operatorname{div}\mathbf{F} = \dfrac{\partial(f(y,z))}{\partial x} + \dfrac{\partial(g(x,z))}{\partial y} + \dfrac{\partial(h(x,y))}{\partial z} = 0$ so $\mathbf{F}$ is incompressible.

For Exercises 23–29, let $\mathbf{F}(x,y,z) = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G}(x,y,z) = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$.

23. $\operatorname{div}(\mathbf{F}+\mathbf{G}) = \dfrac{\partial(P_1+P_2)}{\partial x} + \dfrac{\partial(Q_1+Q_2)}{\partial y} + \dfrac{\partial(R_1+R_2)}{\partial z}$

$= \left(\dfrac{\partial P_1}{\partial x} + \dfrac{\partial Q_1}{\partial y} + \dfrac{\partial R_1}{\partial z}\right) + \left(\dfrac{\partial P_2}{\partial x} + \dfrac{\partial Q_2}{\partial y} + \dfrac{\partial R_3}{\partial z}\right) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

24. $\operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G} = \left[\left(\dfrac{\partial R_1}{\partial y} - \dfrac{\partial Q_1}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P_1}{\partial z} - \dfrac{\partial R_1}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q_1}{\partial x} - \dfrac{\partial P_1}{\partial y}\right)\mathbf{k}\right]$

$+ \left[\left(\dfrac{\partial R_2}{\partial y} - \dfrac{\partial Q_2}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P_2}{\partial z} - \dfrac{\partial R_2}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q_2}{\partial x} - \dfrac{\partial P_2}{\partial y}\right)\mathbf{k}\right]$

$= \left[\dfrac{\partial(R_1+R_2)}{\partial y} - \dfrac{\partial(Q_1+Q_2)}{\partial z}\right]\mathbf{i} + \left[\dfrac{\partial(P_1+P_2)}{\partial z} - \dfrac{\partial(R_1+R_2)}{\partial x}\right]\mathbf{j}$

$+ \left[\dfrac{\partial(Q_1+Q_2)}{\partial x} - \dfrac{\partial(P_1+P_2)}{\partial y}\right]\mathbf{k} = \operatorname{curl}(\mathbf{F}+\mathbf{G})$

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25. $\operatorname{div}(f\mathbf{F}) = \dfrac{\partial(fP_1)}{\partial x} + \dfrac{\partial(fQ_1)}{\partial y} + \dfrac{\partial(fR_1)}{\partial z}$

$$= \left(f\frac{\partial P_1}{\partial x} + P_1\frac{\partial f}{\partial x}\right) + \left(f\frac{\partial Q_1}{\partial y} + Q_1\frac{\partial f}{\partial y}\right) + \left(f\frac{\partial R_1}{\partial z} + R_1\frac{\partial f}{\partial z}\right)$$

$$= f\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \langle P_1, Q_1, R_1\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$$

26. $\operatorname{curl}(f\mathbf{F}) = \left[\dfrac{\partial(fR_1)}{\partial y} - \dfrac{\partial(fQ_1)}{\partial z}\right]\mathbf{i} + \left[\dfrac{\partial(fP_1)}{\partial z} - \dfrac{\partial(fR_1)}{\partial x}\right]\mathbf{j} + \left[\dfrac{\partial(fQ_1)}{\partial x} - \dfrac{\partial(fP_1)}{\partial y}\right]\mathbf{k}$

$$= \left[f\frac{\partial R_1}{\partial y} + R_1\frac{\partial f}{\partial y} - f\frac{\partial Q_1}{\partial z} - Q_1\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[f\frac{\partial P_1}{\partial z} + P_1\frac{\partial f}{\partial z} - f\frac{\partial R_1}{\partial x} - R_1\frac{\partial f}{\partial x}\right]\mathbf{j}$$

$$+ \left[f\frac{\partial Q_1}{\partial x} + Q_1\frac{\partial f}{\partial x} - f\frac{\partial P_1}{\partial y} - P_1\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$= f\left[\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right]\mathbf{i} + f\left[\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right]\mathbf{j} + f\left[\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right]\mathbf{k}$$

$$+ \left[R_1\frac{\partial f}{\partial y} - Q_1\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[P_1\frac{\partial f}{\partial z} - R_1\frac{\partial f}{\partial x}\right]\mathbf{j} + \left[Q_1\frac{\partial f}{\partial x} - P_1\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$= f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$$

27. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \nabla\cdot(\mathbf{F}\times\mathbf{G}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix} = \dfrac{\partial}{\partial x}\begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} - \dfrac{\partial}{\partial y}\begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix} + \dfrac{\partial}{\partial z}\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}$

$$= \left[Q_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial Q_1}{\partial x} - Q_2\frac{\partial R_1}{\partial x} - R_1\frac{\partial Q_2}{\partial x}\right]$$

$$- \left[P_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial P_1}{\partial y} - P_2\frac{\partial R_1}{\partial y} - R_1\frac{\partial P_2}{\partial y}\right]$$

$$+ \left[P_1\frac{\partial Q_2}{\partial z} + Q_2\frac{\partial P_1}{\partial z} - P_2\frac{\partial Q_1}{\partial z} - Q_1\frac{\partial P_2}{\partial z}\right]$$

$$= \left[P_2\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2\left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2\left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\right]$$

$$- \left[P_1\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right) + Q_1\left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right) + R_1\left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\right]$$

$$= \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$$

28. $\operatorname{div}(\nabla f\times\nabla g) = \nabla g\cdot\operatorname{curl}(\nabla f) - \nabla f\cdot\operatorname{curl}(\nabla g)$ [by Exercise 27] $= 0$ (by Theorem 3)

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29. $\operatorname{curl}\operatorname{curl}\mathbf{F} = \nabla\times(\nabla\times\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial R_1/\partial y - \partial Q_1/\partial z & \partial P_1/\partial z - \partial R_1/\partial x & \partial Q_1/\partial x - \partial P_1/\partial y \end{vmatrix}$

$$= \left(\frac{\partial^2 Q_1}{\partial y\,\partial x} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial z\,\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 R_1}{\partial z\,\partial y} - \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial x\,\partial y}\right)\mathbf{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial x\,\partial z} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial y\,\partial z}\right)\mathbf{k}$$

Now let's consider $\operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$ and compare with the above.
(Note that $\nabla^2\mathbf{F}$ is defined on page 1130 [ET 1094].)

$$\operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F} = \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial x\,\partial y} + \frac{\partial^2 R_1}{\partial x\,\partial z}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y\,\partial x} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial y\,\partial z}\right)\mathbf{j}\right.$$
$$\left. + \left(\frac{\partial^2 P_1}{\partial z\,\partial x} + \frac{\partial^2 Q_1}{\partial z\,\partial y} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$
$$- \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} + \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}\right.$$
$$\left. + \left(\frac{\partial^2 R_1}{\partial x^2} + \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$
$$= \left(\frac{\partial^2 Q_1}{\partial x\,\partial y} + \frac{\partial^2 R_1}{\partial x\,\partial z} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y\,\partial x} + \frac{\partial^2 R_1}{\partial y\,\partial z} - \frac{\partial^2 Q_1}{\partial x^2} - \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial z\,\partial x} + \frac{\partial^2 Q_1}{\partial z\,\partial y} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_2}{\partial y^2}\right)\mathbf{k}$$

Then applying Clairaut's Theorem to reverse the order of differentiation in the second partial derivatives as needed and comparing, we have $\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$ as desired.

30. (a) $\nabla\cdot\mathbf{r} = \left(\dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} + \dfrac{\partial}{\partial z}\mathbf{k}\right)\cdot(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = 1 + 1 + 1 = 3$

(b) $\nabla\cdot(r\mathbf{r}) = \nabla\cdot\sqrt{x^2+y^2+z^2}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

$$= \left(\frac{x^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right) + \left(\frac{y^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right)$$
$$+ \left(\frac{z^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right)$$
$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\left(4x^2+4y^2+4z^2\right) = 4\sqrt{x^2+y^2+z^2} = 4r$$

Another method:

By Exercise 25, $\nabla\cdot(r\mathbf{r}) = \operatorname{div}(r\mathbf{r}) = r\operatorname{div}\mathbf{r} + \mathbf{r}\cdot\nabla r = 3r + \mathbf{r}\cdot\dfrac{\mathbf{r}}{r}$ [see Exercise 31(a) below] $= 4r$.

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(c) $\nabla^2 r^3 = \nabla^2 \left(x^2+y^2+z^2\right)^{3/2}$

$$= \tfrac{\partial}{\partial x}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2x)\right] + \tfrac{\partial}{\partial y}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2y)\right] + \tfrac{\partial}{\partial z}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2z)\right]$$

$$= 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)(x) + (x^2+y^2+z^2)^{1/2}\right] + 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)(y) + (x^2+y^2+z^2)^{1/2}\right] + 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)(z) + (x^2+y^2+z^2)^{1/2}\right]$$

$$= 3(x^2+y^2+z^2)^{-1/2}(4x^2+4y^2+4z^2) = 12(x^2+y^2+z^2)^{1/2}$$

$$= 12r$$

Another method: $\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{3/2} = 3x\sqrt{x^2+y^2+z^2} \;\Rightarrow\; \nabla r^3 = 3r(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}) = 3r\,\mathbf{r}$, so $\nabla^2 r^3 = \nabla\cdot\nabla r^3 = \nabla\cdot(3r\,\mathbf{r}) = 3(4r) = 12r$ by part (b).

31. (a) $\nabla r = \nabla\sqrt{x^2+y^2+z^2} = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$

$$= \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{r}}{r}$$

(b) $$\nabla\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]\mathbf{k} = \mathbf{0}$$

(c) $$\nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{-\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2x)}{x^2+y^2+z^2}\,\mathbf{i} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2y)}{x^2+y^2+z^2}\,\mathbf{j} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2z)}{x^2+y^2+z^2}\,\mathbf{k}$$

$$= -\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

(d) $\nabla\ln r = \nabla\ln\left(x^2+y^2+z^2\right)^{1/2} = \frac{1}{2}\nabla\ln\left(x^2+y^2+z^2\right)$

$$= \frac{x}{x^2+y^2+z^2}\,\mathbf{i} + \frac{y}{x^2+y^2+z^2}\,\mathbf{j} + \frac{z}{x^2+y^2+z^2}\,\mathbf{k} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

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32. $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \;\Rightarrow\; r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$, so

$$\mathbf{F} = \frac{\mathbf{r}}{r^p} = \frac{x}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{k}$$

Then $\dfrac{\partial}{\partial x}\dfrac{x}{(x^2+y^2+z^2)^{p/2}} = \dfrac{(x^2+y^2+z^2)-px^2}{(x^2+y^2+z^2)^{1+p/2}} = \dfrac{r^2-px^2}{r^{p+2}}$. Similarly,

$\dfrac{\partial}{\partial y}\dfrac{y}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2-py^2}{r^{p+2}}$ and $\dfrac{\partial}{\partial z}\dfrac{z}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2-pz^2}{r^{p+2}}$. Thus

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{r^2-px^2}{r^{p+2}} + \frac{r^2-py^2}{r^{p+2}} + \frac{r^2-pz^2}{r^{p+2}} = \frac{3r^2-px^2-py^2-pz^2}{r^{p+2}}$$
$$= \frac{3r^2-p(x^2+y^2+z^2)}{r^{p+2}} = \frac{3r^2-pr^2}{r^{p+2}} = \frac{3-p}{r^p}$$

Consequently, if $p = 3$ we have $\operatorname{div}\mathbf{F} = 0$.

33. By (13), $\oint_C f(\nabla g)\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}(f\nabla g)\,dA = \iint_D [f\operatorname{div}(\nabla g) + \nabla g\cdot\nabla f]\,dA$ by Exercise 25. But $\operatorname{div}(\nabla g) = \nabla^2 g$. Hence $\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$.

34. By Exercise 33, $\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$ and $\iint_D g\nabla^2 f\,dA = \oint_C g(\nabla f)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$. Hence

$$\iint_D (f\nabla^2 g - g\nabla^2 f)\,dA = \oint_C [f(\nabla g)\cdot\mathbf{n} - g(\nabla f)\cdot\mathbf{n}]\,ds + \iint_D (\nabla f\cdot\nabla g - \nabla g\cdot\nabla f)\,dA$$
$$= \oint_C [f\nabla g - g\nabla f]\cdot\mathbf{n}\,ds$$

35. (a) We know that $\omega = v/d$, and from the diagram $\sin\theta = d/r \;\Rightarrow\; v = d\omega = (\sin\theta)r\omega = |\mathbf{w}\times\mathbf{r}|$. But $\mathbf{v}$ is perpendicular to both $\mathbf{w}$ and $\mathbf{r}$, so that $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.

(b) From (a), $\mathbf{v} = \mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (0\cdot z - \omega y)\,\mathbf{i} + (\omega x - 0\cdot z)\,\mathbf{j} + (0\cdot y - x\cdot 0)\,\mathbf{k} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$

(c) $\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(\omega x)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(-\omega y) - \frac{\partial}{\partial x}(0)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right]\mathbf{k}$$
$$= [\omega - (-\omega)]\,\mathbf{k} = 2\omega\,\mathbf{k} = 2\mathbf{w}$$

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36. Let $\mathbf{H} = \langle h_1, h_2, h_3 \rangle$ and $\mathbf{E} = \langle E_1, E_2, E_3 \rangle$.

(a) $\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (\operatorname{curl} \mathbf{E}) = \nabla \times \left(-\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{1}{c} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial h_1/\partial t & \partial h_2/\partial t & \partial h_3/\partial t \end{vmatrix}$

$$= -\frac{1}{c} \left[\left(\frac{\partial^2 h_3}{\partial y \, \partial t} - \frac{\partial^2 h_2}{\partial z \, \partial t} \right) \mathbf{i} + \left(\frac{\partial^2 h_1}{\partial z \, \partial t} - \frac{\partial^2 h_3}{\partial x \, \partial t} \right) \mathbf{j} + \left(\frac{\partial^2 h_2}{\partial x \, \partial t} - \frac{\partial^2 h_1}{\partial y \, \partial t} \right) \mathbf{k} \right]$$

$$= -\frac{1}{c} \frac{\partial}{\partial t} \left[\left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \right) \mathbf{k} \right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$= -\frac{1}{c} \frac{\partial}{\partial t} \operatorname{curl} \mathbf{H} = -\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

(b) $\nabla \times (\nabla \times \mathbf{H}) = \nabla \times (\operatorname{curl} \mathbf{H}) = \nabla \times \left(\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right) = \frac{1}{c} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial E_1/\partial t & \partial E_2/\partial t & \partial E_3/\partial t \end{vmatrix}$

$$= \frac{1}{c} \left[\left(\frac{\partial^2 E_3}{\partial y \, \partial t} - \frac{\partial^2 E_2}{\partial z \, \partial t} \right) \mathbf{i} + \left(\frac{\partial^2 E_1}{\partial z \, \partial t} - \frac{\partial^2 E_3}{\partial x \, \partial t} \right) \mathbf{j} + \left(\frac{\partial^2 E_2}{\partial x \, \partial t} - \frac{\partial^2 E_1}{\partial y \, \partial t} \right) \mathbf{k} \right]$$

$$= \frac{1}{c} \frac{\partial}{\partial t} \left[\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} \right) \mathbf{k} \right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$= \frac{1}{c} \frac{\partial}{\partial t} \operatorname{curl} \mathbf{E} = \frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

(c) Using Exercise 29, we have that $\operatorname{curl} \operatorname{curl} \mathbf{E} = \operatorname{grad} \operatorname{div} \mathbf{E} - \nabla^2 \mathbf{E} \Rightarrow$

$$\nabla^2 \mathbf{E} = \operatorname{grad} \operatorname{div} \mathbf{E} - \operatorname{curl} \operatorname{curl} \mathbf{E} = \operatorname{grad} 0 + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \text{ [from part (a)]} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

(d) As in part (c), $\nabla^2 \mathbf{H} = \operatorname{grad} \operatorname{div} \mathbf{H} - \operatorname{curl} \operatorname{curl} \mathbf{H} = \operatorname{grad} 0 + \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}$ [using part (b)] $= \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}$.

37. For any continuous function f on $\mathbb{R}^3$, define a vector field $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0 \rangle$ where $g(x, y, z) = \int_0^x f(t, y, z) \, dt$. Then

$\operatorname{div} \mathbf{G} = \frac{\partial}{\partial x} (g(x, y, z)) + \frac{\partial}{\partial y} (0) + \frac{\partial}{\partial z} (0) = \frac{\partial}{\partial x} \int_0^x f(t, y, z) \, dt = f(x, y, z)$ by the Fundamental Theorem of Calculus. Thus every continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.

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17.6 Parametric Surfaces and Their Areas ET 16.6

1. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u^2\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x = u\cos v, y = u\sin v, z = u^2$. For any point (x,y,z) on the surface, we have $x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = z$. Since no restrictions are placed on the parameters, the surface is $z = x^2 + y^2$, which we recognize as a circular paraboloid opening upward whose axis is the z-axis.

2. $\mathbf{r}(u,v) = (1+2u)\,\mathbf{i} + (-u+3v)\,\mathbf{j} + (2+4u+5v)\,\mathbf{k} = \langle 1,0,2\rangle + u\langle 2,-1,4\rangle + v\langle 0,3,5\rangle$. From Example 3, we recognize this as a vector equation of a plane through the point $(1,0,2)$ and containing vectors $\mathbf{a} = \langle 2,-1,4\rangle$ and $\mathbf{b} = \langle 0,3,5\rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{vmatrix} = -17\,\mathbf{i} - 10\,\mathbf{j} + 6\,\mathbf{k}$$

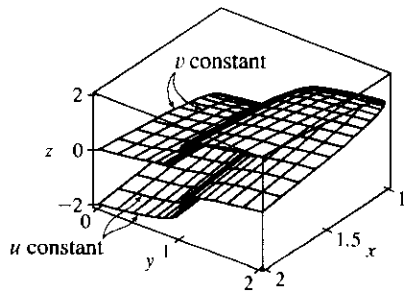
and an equation of the plane is $-17(x-1) - 10(y-0) + 6(z-2) = 0$ or $-17x - 10y + 6z = -5$.

3. $\mathbf{r}(x,\theta) = \langle x, \cos\theta, \sin\theta\rangle$, so the corresponding parametric equations for the surface are $x = x, y = \cos\theta, z = \sin\theta$. For any point (x,y,z) on the surface, we have $y^2 + z^2 = \cos^2\theta + \sin^2\theta = 1$, so any vertical trace in $x = k$ is the circle $y^2 + z^2 = 1, x = k$. Since $x = x$ with no restriction, the surface is a circular cylinder with radius 1 whose axis is the x-axis.

4. $\mathbf{r}(x,\theta) = \langle x, x\cos\theta, x\sin\theta\rangle$, so the corresponding parametric equations for the surface are $x = x, y = x\cos\theta$, $z = x\sin\theta$. For any point (x,y,z) on the surface, we have $y^2 + z^2 = x^2\cos^2\theta + x^2\sin^2\theta = x^2$. With $x = x$ and no restrictions on the parameters, the surface is $x^2 = y^2 + z^2$, which we recognize as a circular cone whose axis is the x-axis.

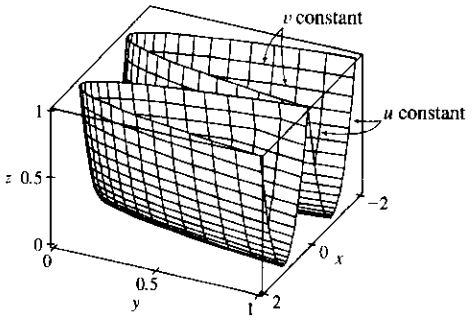
5. $\mathbf{r}(u,v) = \langle u^2+1, v^3+1, u+v\rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.

The surface has parametric equations $x = u^2 + 1$, $y = v^3 + 1$, $z = u + v$, $-1 \le u \le 1$, $-1 \le v \le 1$. If we keep u constant at u_0, $x = u_0^2 + 1$, a constant, so the corresponding grid curves must be the curves parallel to the yz-plane. If v is constant, we have $y = v_0^3 + 1$, a constant, so these grid curves are the curves parallel to the xz-plane.



6. $\mathbf{r}(u,v) = \langle u+v, u^2, v^2\rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.

The surface has parametric equations $x = u + v$, $y = u^2$, $z = v^2$, $-1 \le u \le 1$, $-1 \le v \le 1$. If $u = u_0$ is constant, $y = u_0^2 =$ constant, so the corresponding grid curves are the curves parallel to the xz-plane. If $v = v_0$ is constant, $z = v_0^2 =$ constant, so the corresponding grid curves are the curves parallel to the xy-plane.



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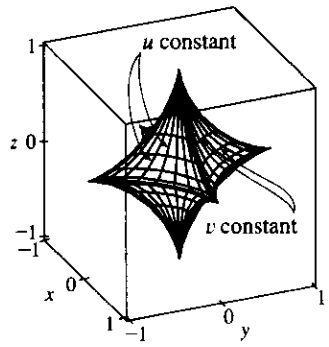
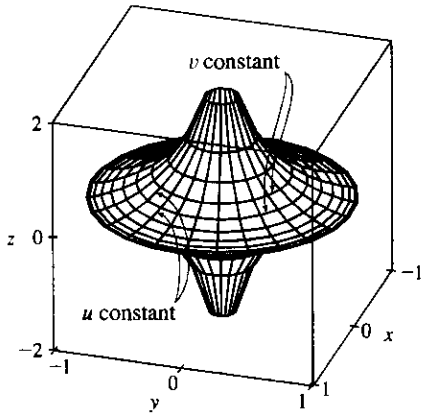
7. $\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$.

The surface has parametric equations $x = \cos^3 u \cos^3 v$, $y = \sin^3 u \cos^3 v$, $z = \sin^3 v$, $0 \le u \le \pi$, $0 \le v \le 2\pi$. Note that if $v = v_0$ is constant then $z = \sin^3 v_0$ is constant, so the corresponding grid curves must be the curves parallel to the xy-plane. The vertically oriented grid curves, then, correspond to $u = u_0$ being held constant, giving $x = \cos^3 u_0 \cos^3 v$, $y = \sin^3 u_0 \cos^3 v$, $z = \sin^3 v$. These curves lie in vertical planes that contain the z-axis.



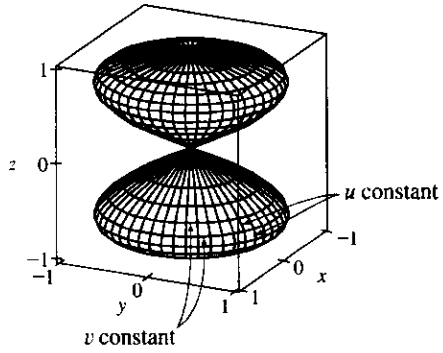
8. $\mathbf{r}(u, v) = \langle \cos u \sin v, \sin u \sin v, \cos v + \ln \tan(v/2) \rangle$.

The surface has parametric equations $x = \cos u \sin v$, $y = \sin u \sin v$, $z = \cos v + \ln \tan(v/2)$, $0 \le u \le 2\pi$, $0.1 \le v \le 6.2$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin v_0$, $y = \sin u \sin v_0$, $z = \cos v_0 + \ln \tan(v_0/2)$ which represent a circle of radius $\sin v_0$ in the plane $z = \cos v_0 + \ln \tan(v_0/2)$. So the circular grid curves we see lying horizontally are the grid curves with v constant. The vertically oriented grid curves correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin v$, $y = \sin u_0 \sin v$, $z = \cos v + \ln \tan(v/2)$. These curves lie in vertical planes that contain the z-axis.



9. $x = \cos u \sin 2v$, $y = \sin u \sin 2v$, $z = \sin v$.

The complete graph of the surface is given by the parametric domain $0 \le u \le \pi, 0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin 2v_0$, $y = \sin u \sin 2v_0$, $z = \sin v_0$ which represent a circle of radius $\sin 2v_0$ in the plane $z = \sin v_0$. So the circular grid curves we see lying horizontally are the grid curves which have v constant. The vertical grid curves, then, correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin 2v$ and $y = \sin u_0 \sin 2v$ with $z = \sin v$ which has a "figure-eight" shape.



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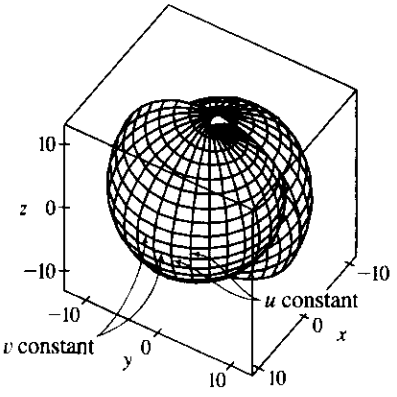
10. $x = u \sin u \cos v$, $y = u \cos u \cos v$, $z = u \sin v$.

We graph the portion of the surface with parametric domain $0 \le u \le 4\pi, 0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = u \sin u \cos v_0$, $y = u \cos u \cos v_0$, $z = u \sin v_0$. The equations for x and y show that the projections onto the xy-plane give a spiral shape, so the corresponding grid curves are the almost-horizontal spiral curves we see. The vertical grid curves, which look approximately circular, correspond to $u = u_0$ being held constant, giving $x = u_0 \sin u_0 \cos v$, $y = u_0 \cos u_0 \cos v$, $z = u_0 \sin v$.



11. $\mathbf{r}(u, v) = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + u\,\mathbf{k}$. The parametric equations for the surface are $x = \cos v$, $y = \sin v$, $z = u$. Then $x^2 + y^2 = \cos^2 v + \sin^2 v = 1$ and $z = u$ with no restriction on u, so we have a circular cylinder, graph IV. The grid curves with u constant are the horizontal circles we see in the plane $z = u$. If v is constant, both x and y are constant with z free to vary, so the corresponding grid curves are the lines on the cylinder parallel to the z-axis.

12. $\mathbf{r}(u, v) = u \cos v\,\mathbf{i} + u \sin v\,\mathbf{j} + u\,\mathbf{k}$. The parametric equations for the surface are $x = u \cos v$, $y = u \sin v$, $z = u$. Then $x^2 + y^2 = u^2 \cos^2 v + u^2 \sin^2 v = u^2 = z^2$, which represents the equation of a cone with axis the z-axis, graph V. The grid curves with u constant are the horizontal circles we see, corresponding to the equations $x^2 + y^2 = u^2$ in the plane $z = u$. If v is constant, x, y, z are each scalar multiples of u, corresponding to the straight line grid curves through the origin.

13. $\mathbf{r}(u, v) = u \cos v\,\mathbf{i} + u \sin v\,\mathbf{j} + v\,\mathbf{k}$. The parametric equations for the surface are $x = u \cos v$, $y = u \sin v$, $z = v$. We look at the grid curves first; if we fix v, then x and y parametrize a straight line in the plane $z = v$ which intersects the z-axis. If u is held constant, the projection onto the xy-plane is circular; with $z = v$, each grid curve is a helix. The surface is a spiraling ramp, graph I.

14. $x = u^3$, $y = u \sin v$, $z = u \cos v$. Then $y^2 + z^2 = u^2 \sin v^2 + u^2 \cos v^2 = u^2$, so if u is held constant, each grid curve is a circle of radius u in the plane $x = u^3$. The graph then must be graph III. If v is held constant, so $v = v_0$, we have $y = u \sin v_0$ and $z = u \cos v_0$. Then $y = (\tan v_0)z$, so the grid curves we see running lengthwise along the surface in the planes $y = kz$ correspond to keeping v constant.

15. $x = (u - \sin u) \cos v$, $y = (1 - \cos u) \sin v$, $z = u$. If u is held constant, x and y give an equation of an ellipse in the plane $z = u$, thus the grid curves are horizontally oriented ellipses. Note that when $u = 0$, the "ellipse" is the single point $(0, 0, 0)$, and when $u = \pi$, we have $y = 0$ while x ranges from $-\pi$ to π, a line segment parallel to the x-axis in the plane $z = \pi$. This is the upper "seam" we see in graph II. When v is held constant, $z = u$ is free to vary, so the corresponding grid curves are the curves we see running up and down along the surface.

16. $x = (1 - u)(3 + \cos v) \cos 4\pi u$, $y = (1 - u)(3 + \cos v) \sin 4\pi u$, $z = 3u + (1 - u) \sin v$. These equations correspond to graph VI: when $u = 0$, then $x = 3 + \cos v$, $y = 0$, and $z = \sin v$, which are equations of a circle with radius 1 in the xz-plane centered at $(3, 0, 0)$. When $u = \frac{1}{2}$, then $x = \frac{3}{2} + \frac{1}{2} \cos v$, $y = 0$, and $z = \frac{3}{2} + \frac{1}{2} \sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz-plane centered at $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$. When $u = 1$, then $x = y = 0$ and $z = 3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

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17. From Example 3, parametric equations for the plane through the point $(1, 2, -3)$ that contains the vectors $\mathbf{a} = \langle 1, 1, -1\rangle$ and $\mathbf{b} = \langle 1, -1, 1\rangle$ are $x = 1 + u(1) + v(1) = 1 + u + v$, $y = 2 + u(1) + v(-1) = 2 + u - v$, $z = -3 + u(-1) + v(1) = -3 - u + v$.

18. Solving the equation for z gives $z^2 = 1 - 2x^2 - 4y^2 \quad \Rightarrow \quad z = -\sqrt{1 - 2x^2 - 4y^2}$ (since we want the lower half of the ellipsoid). If we let x and y be the parameters, parametric equations are $x = x$, $y = y$, $z = -\sqrt{1 - 2x^2 - 4y^2}$.

Alternate solution: The equation can be rewritten as $\dfrac{x^2}{\left(1/\sqrt{2}\right)^2} + \dfrac{y^2}{(1/2)^2} + z^2 = 1$, and if we let $x = \dfrac{1}{\sqrt{2}}\, u \cos v$ and $y = \frac{1}{2} u \sin v$, then $z = -\sqrt{1 - 2x^2 - 4y^2} = -\sqrt{1 - u^2\cos^2 v - u^2 \sin^2 v} = -\sqrt{1 - u^2}$, where $0 \le u \le 1$ and $0 \le v \le 2\pi$.

19. Solving the equation for y gives $y^2 = 1 - x^2 + z^2 \quad \Rightarrow \quad y = \sqrt{1 - x^2 + z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $y \ge 0$.) If we let x and z be the parameters, parametric equations are $x = x$, $z = z$, $y = \sqrt{1 - x^2 + z^2}$.

20. $x = 4 - y^2 - 2z^2$, $y = y$, $z = z$ where $y^2 + 2z^2 \le 4$ since $x \ge 0$. Then the associated vector equation is $\mathbf{r}(y, z) = (4 - y^2 - 2z^2)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

21. Since the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \le 2$.

Alternate solution: Using spherical coordinates, $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$ where $0 \le \phi \le \frac{\pi}{4}$ and $0 \le \theta \le 2\pi$.

22. In spherical coordinates, parametric equations are $x = 4\sin\phi\cos\theta$, $y = 4\sin\phi\sin\theta$, $z = 4\cos\phi$. The intersection of the sphere with the plane $z = 2$ corresponds to $z = 4\cos\phi = 2 \quad \Rightarrow \quad \cos\phi = \frac{1}{2} \quad \Rightarrow \quad \phi = \frac{\pi}{3}$. By symmetry, the intersection of the sphere with the plane $z = -2$ corresponds to $\phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Thus the surface is described by $0 \le \theta \le 2\pi$, $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$.

23. Parametric equations are $x = x$, $y = 4\cos\theta$, $z = 4\sin\theta$, $0 \le x \le 5$, $0 \le \theta \le 2\pi$.

24. Using x and y as the parameters, $x = x$, $y = y$, $z = x + 3$ where $0 \le x^2 + y^2 \le 1$. Also, since the plane intersects the cylinder in an ellipse, the surface is a planar ellipse in the plane $z = x + 3$. Thus, parametrizing with respect to s and θ, we have $x = s\cos\theta$, $y = s\sin\theta$, $z = 3 + s\cos\theta$ where $0 \le s \le 1$ and $0 \le \theta \le 2\pi$.

25. The surface appears to be a portion of a circular cylinder of radius 3 with axis the x-axis. An equation of the cylinder is $y^2 + z^2 = 9$, and we can impose the restrictions $0 \le x \le 5$, $y \le 0$ to obtain the portion shown.
To graph the surface on a CAS, we can use parametric equations $x = u$, $y = 3\cos v$, $z = 3\sin v$ with the parameter

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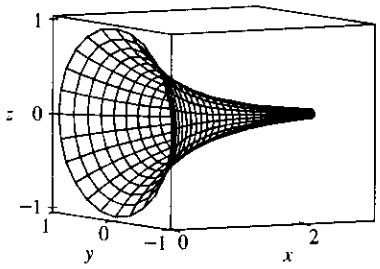
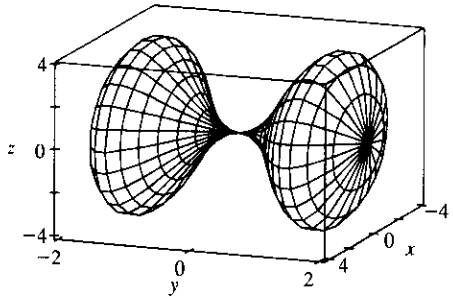
domain $0 \le u \le 5$, $\frac{\pi}{2} \le v \le \frac{3\pi}{2}$. Alternatively, we can regard x and z as parameters. Then parametric equations are $x = x$, $z = z$, $y = -\sqrt{9 - z^2}$, where $0 \le x \le 5$ and $-3 \le z \le 3$.

26. The surface appears to be a portion of a sphere of radius 1 centered at the origin. In spherical coordinates, the sphere has equation $\rho = 1$, and imposing the restrictions $\frac{\pi}{2} \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \pi$ will give only the portion of the sphere shown. Thus, to graph the surface on a CAS we can either use spherical coordinates with the stated restrictions, or we can use parametric equations: $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, $\frac{\pi}{2} \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \pi$.

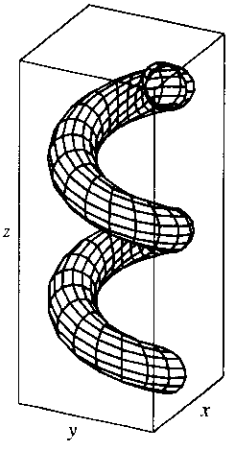
27. Using Equations 3, we have the parametrization $x = x$, $y = e^{-x}\cos\theta$, $z = e^{-x}\sin\theta$, $0 \le x \le 3$, $0 \le \theta \le 2\pi$.



28. Letting θ be the angle of rotation about the y-axis, we have the parametrization $x = (4y^2 - y^4)\cos\theta$, $y = y$, $z = (4y^2 - y^4)\sin\theta$, $-2 \le y \le 2$, $0 \le \theta \le 2\pi$.



29. (a) Replacing $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$ gives parametric equations $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = u + \cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = 0$, draws a circle in the clockwise direction for each value of v. The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.



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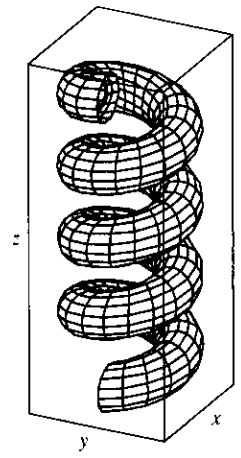
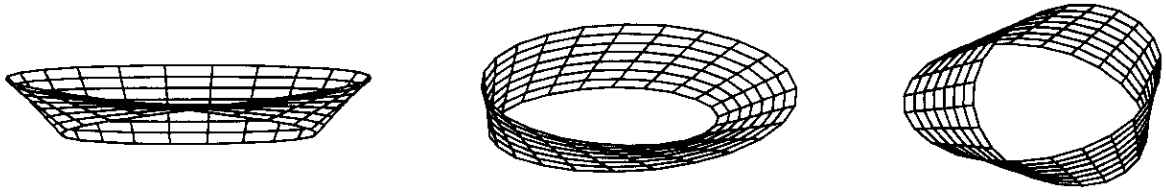
(b) Replacing $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$ gives parametric equations $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = u + \cos v$. From the graph, it appears that the number of coils in the surface doubles within the same parametric domain. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = 0$ (where v is constant), complete circular revolutions for $0 \le u \le \pi$ while the original surface requires $0 \le u \le 2\pi$ for a complete revolution. Thus, the new surface winds around twice as fast as the original surface, and since the equation for z is identical in both surfaces, we observe twice as many circular coils in the same z-interval.

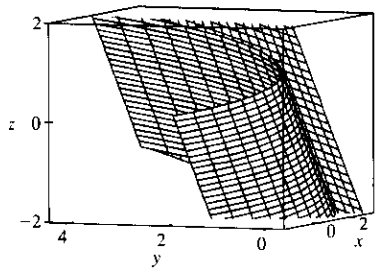


30. First we graph the surface as viewed from the front, then from two additional viewpoints.

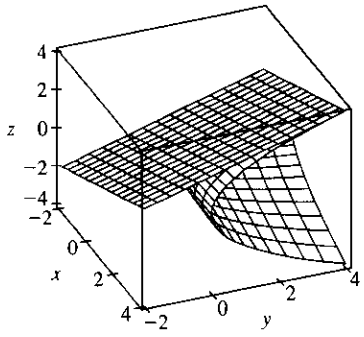


The surface appears as a twisted sheet, and is unusual because it has only one side. (The Möbius strip is discussed in more detail in Section 17.7 [ET 16.7].)

31. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + 3u^2\,\mathbf{j} + (u - v)\,\mathbf{k}$.
$\mathbf{r}_u = \mathbf{i} + 6u\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_v = \mathbf{i} - \mathbf{k}$, so
$\mathbf{r}_u \times \mathbf{r}_v = -6u\,\mathbf{i} + 2\,\mathbf{j} - 6u\,\mathbf{k}$. Since the point $(2, 3, 0)$ corresponds to $u = 1$, $v = 1$, a normal vector to the surface at $(2, 3, 0)$ is $-6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.



32. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + v^2\,\mathbf{j} + uv\,\mathbf{k} \;\Rightarrow\; \mathbf{r}(1, 1) = (1, 1, 1)$.
$\mathbf{r}_u = 2u\,\mathbf{i} + v\,\mathbf{k}$ and $\mathbf{r}_v = 2v\,\mathbf{j} + u\,\mathbf{k}$, so a normal vector to the surface at the point $(1, 1, 1)$ is
$\mathbf{r}_u(1, 1) \times \mathbf{r}_v(1, 1) = (2\,\mathbf{i} + \mathbf{k}) \times (2\,\mathbf{j} + \mathbf{k}) = -2\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k}$.
Thus an equation of the tangent plane at the point $(1, 1, 1)$ is $-2(x - 1) - 2(y - 1) + 4(z - 1) = 0$ or $x + y - 2z = 0$.



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(c) As in part (a),

$$\begin{aligned}\operatorname{curl}(f\nabla g + g\nabla f) &= \operatorname{curl}(f\nabla g) + \operatorname{curl}(g\nabla f) \qquad \text{(by Exercise 17.5.24 [ET 16.5.24])}\\ &= (\nabla f \times \nabla g) + (\nabla g \times \nabla f) = \mathbf{0} \qquad [\text{since } \mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})]\end{aligned}$$

Hence by Stokes' Theorem, $\int_C (f\nabla g + g\nabla f)\cdot d\mathbf{r} = \iint_S \operatorname{curl}(f\nabla g + g\nabla f)\cdot d\mathbf{S} = 0$.

17.9 The Divergence Theorem ET 16.9

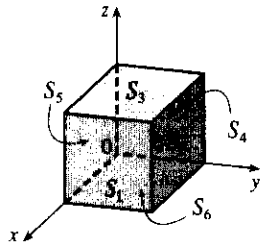
1. The vectors that end near P_1 are longer than the vectors that start near P_1, so the net flow is inward near P_1 and $\operatorname{div}\mathbf{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2, so the net flow is outward near P_2 and $\operatorname{div}\mathbf{F}(P_2)$ is positive.

2. (a) The vectors that end near P_1 are shorter than the vectors that start near P_1, so the net flow is outward and P_1 is a source. The vectors that end near P_2 are longer than the vectors that start near P_2, so the net flow is inward and P_2 is a sink.

(b) $\mathbf{F}(x,y) = \langle x, y^2\rangle \;\Rightarrow\; \operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = 1 + 2y$. The y-value at P_1 is positive, so $\operatorname{div}\mathbf{F} = 1+2y$ is positive, thus P_1 is a source. At P_2, $y < -1$, so $\operatorname{div}\mathbf{F} = 1+2y$ is negative, and P_2 is a sink.

3. $\operatorname{div}\mathbf{F} = 3 + x + 2x = 3 + 3x$, so

$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^1 (3x+3)\,dx\,dy\,dz = \frac{9}{2}$ (notice the triple integral is three times the volume of the cube plus three times $\overline{x}$).



To compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$, on S_1: $\mathbf{n} = \mathbf{i}$, $\mathbf{F} = 3\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$, and

$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} 3\,dS = 3$;

S_2: $\mathbf{F} = 3x\,\mathbf{i} + x\,\mathbf{j} + 2xz\,\mathbf{k}$, $\mathbf{n} = \mathbf{j}$ and $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} x\,dS = \frac{1}{2}$;

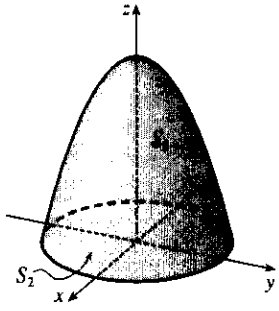
S_3: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2x\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$ and $\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_3} 2x\,dS = 1$;

S_4: $\mathbf{F} = \mathbf{0}$, $\iint_{S_4}\mathbf{F}\cdot d\mathbf{S} = 0$; S_5: $\mathbf{F} = 3x\,\mathbf{i} + 2x\,\mathbf{k}$, $\mathbf{n} = -\mathbf{j}$ and $\iint_{S_5}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_5} 0\,dS = 0$;

S_6: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_6}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_6} 0\,dS = 0$. Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{9}{2}$.

4. $\operatorname{div}\mathbf{F} = 2x + x + 1 = 3x + 1$ so

$$\begin{aligned}\iiint_E \operatorname{div}\mathbf{F}\,dV &= \iiint_E (3x+1)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3r\cos\theta + 1)\,r\,dz\,dr\,d\theta\\ &= \int_0^2\int_0^{2\pi} r(3r\cos\theta+1)(4-r^2)\,d\theta\,dr\\ &= \int_0^{2\pi} r(4-r^2)\left[3r\sin\theta+\theta\right]_{\theta=0}^{\theta=2\pi}\,dr\\ &= 2\pi\int_0^2 (4r - r^3)\,dr = 2\pi\left[2r^2 - \tfrac{1}{4}r^4\right]_0^2\\ &= 2\pi(8-4) = 8\pi\end{aligned}$$



On S_1: The surface is $z = 4 - x^2 - y^2$, $x^2 + y^2 \le 4$, with upward orientation, and $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j} + (4 - x^2 - y^2)\,\mathbf{k}$. Then

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_D [-(x^2)(-2x) - (xy)(-2y) + (4 - x^2 - y^2)]\,dA\\ &= \iint_D \left[2x(x^2+y^2) + 4 - (x^2+y^2)\right]\,dA = \int_0^{2\pi}\int_0^2 (2r\cos\theta\cdot r^2 + 4 - r^2)\,r\,dr\,d\theta\\ &= \int_0^{2\pi}\left[\tfrac{2}{5}r^5\cos\theta + 2r^2 - \tfrac{1}{4}r^4\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi}\left(\tfrac{64}{5}\cos\theta + 4\right)\,d\theta\\ &= \left[\tfrac{64}{5}\sin\theta + 4\theta\right]_0^{2\pi} = 8\pi\end{aligned}$$

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On S_2: The surface is $z = 0$ with downward orientation, so $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} 0\, dS = 0$.

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 8\pi$.

5. $\operatorname{div} \mathbf{F} = x + y + z$, so

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \int_0^{2\pi} \int_0^1 \int_0^1 (r\cos\theta + r\sin\theta + z)\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 \left(r^2\cos\theta + r^2\sin\theta + \tfrac{1}{2}r\right) dr\, d\theta$$
$$= \int_0^{2\pi} \left(\tfrac{1}{3}\cos\theta + \tfrac{1}{3}\sin\theta + \tfrac{1}{4}\right) d\theta = \tfrac{1}{4}(2\pi) = \tfrac{\pi}{2}$$

Let S_1 be the top of the cylinder, S_2 the bottom, and S_3 the vertical edge. On S_1, $z = 1$, $\mathbf{n} = \mathbf{k}$, and $\mathbf{F} = xy\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$, so $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} x\, dS = \int_0^{2\pi} \int_0^1 (r\cos\theta)\, r\, dr\, d\theta = \left[\sin\theta\right]_0^{2\pi} \left[\tfrac{1}{3}r^3\right]_0^1 = 0$. On S_2, $z = 0$, $\mathbf{n} = -\mathbf{k}$, and $\mathbf{F} = xy\,\mathbf{i}$ so $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} 0\, dS = 0$. S_3 is given by $\mathbf{r}(\theta, z) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 1$. Then $\mathbf{r}_\theta \times \mathbf{r}_z = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z)\, dA = \int_0^{2\pi} \int_0^1 (\cos^2\theta\sin\theta + z\sin^2\theta)\, dz\, d\theta$$
$$= \int_0^{2\pi} \left(\cos^2\theta\sin\theta + \tfrac{1}{2}\sin^2\theta\right) d\theta = \left[-\tfrac{1}{3}\cos^3\theta + \tfrac{1}{4}\left(\theta - \tfrac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = \tfrac{\pi}{2}$$

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0 + 0 + \frac{\pi}{2} = \frac{\pi}{2}$.

6. $\operatorname{div} \mathbf{F} = 1 + 1 + 1 = 3$, so $\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 3\, dV = 3\,(\text{volume of ball}) = 3\left(\frac{4}{3}\pi\right) = 4\pi$. To find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ we use spherical coordinates. S is the unit sphere, represented by $\mathbf{r}(\phi, \theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\cos\theta\,\mathbf{i} + \sin^2\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$ (see Example 17.6.10 [ET 16.6.10]) and $\mathbf{F}(\mathbf{r}(\phi, \theta)) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$. Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA = \int_0^{2\pi} \int_0^\pi \left(\sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi\right) d\phi\, d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^\pi \sin\phi\, d\phi = (2\pi)(2) = 4\pi$$

7. $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(e^x\sin y) + \frac{\partial}{\partial y}(e^x\cos y) + \frac{\partial}{\partial z}(yz^2) = e^x\sin y - e^x\sin y + 2yz = 2yz$, so by the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV = \int_0^1 \int_0^1 \int_0^2 2yz\, dz\, dy\, dx = 2\int_0^1 dx \int_0^1 y\, dy \int_0^1 z\, dz$$
$$= 2\left[x\right]_0^1 \left[\tfrac{1}{2}y^2\right]_0^1 \left[\tfrac{1}{2}z^2\right]_0^2 = 2$$

8. $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2z^3) + \frac{\partial}{\partial y}(2xyz^3) + \frac{\partial}{\partial z}(xz^4) = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$, so by the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV = \int_{-1}^1 \int_{-2}^2 \int_{-3}^3 8xz^3\, dz\, dy\, dx = 8\int_{-1}^1 x\, dx \int_{-2}^2 dy \int_{-3}^3 z^3\, dz$$
$$= 8\left[\tfrac{1}{2}x^2\right]_{-1}^1 \left[y\right]_{-2}^2 \left[\tfrac{1}{4}z^4\right]_{-3}^3 = 0$$

9. $\operatorname{div} \mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (3y^2 + 3z^2)\, dV = \int_0^{2\pi} \int_0^1 \int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta)\, r\, dx\, dr\, d\theta$$
$$= 3\int_0^{2\pi} d\theta \int_0^1 r^3\, dr \int_{-1}^2 dx = 3(2\pi)\left(\tfrac{1}{4}\right)(3) = \tfrac{9\pi}{2}$$

10. $\operatorname{div} \mathbf{F} = 3x^2y - 2x^2y - x^2y = 0$, so $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 0\, dV = 0$.

11. $\operatorname{div} \mathbf{F} = y\sin z + 0 - y\sin z = 0$, so by the Divergence Theorem, $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 0\, dV = 0$.

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12. $\operatorname{div} \mathbf{F} = 2xy + 2xy + 2xy = 6xy$, so

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 6xy\,dV \\ &= \int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} 6xy\,dz\,dx\,dy = \int_0^1 \int_0^{2-2y} 6xy(2-x-2y)\,dx\,dy \\ &= \int_0^1 \int_0^{2-2y} (12xy - 6x^2y - 12xy^2)\,dx\,dy = \int_0^1 \left[6x^2y - 2x^3y - 6x^2y^2\right]_{x=0}^{x=2-2y} dy \\ &= \int_0^1 y(2-2y)^3\,dy = \left[-\tfrac{8}{5}y^5 + 6y^4 - 8y^3 + 4y^2\right]_0^1 = \tfrac{2}{5}\end{aligned}$$

13. $\operatorname{div} \mathbf{F} = y^2 + 0 + x^2 = x^2 + y^2$ so

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E (x^2+y^2)\,dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \cdot r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^2 r^3(4-r^2)\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta \int_0^2 (4r^3 - r^5)\,dr = 2\pi\left[r^4 - \tfrac{1}{6}r^6\right]_0^2 = \tfrac{32}{3}\pi\end{aligned}$$

14. $\operatorname{div} \mathbf{F} = 4x^3 + 4xy^2$ so

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 4x(x^2+y^2)\,dV = \int_0^{2\pi} \int_0^1 \int_0^{r\cos\theta+2} (4r^3\cos\theta)\,r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^1 (4r^5\cos^2\theta + 8r^4\cos\theta)\,dr\,d\theta = \int_0^{2\pi} \left(\tfrac{2}{3}\cos^2\theta + \tfrac{8}{5}\cos\theta\right) d\theta = \tfrac{2}{3}\pi\end{aligned}$$

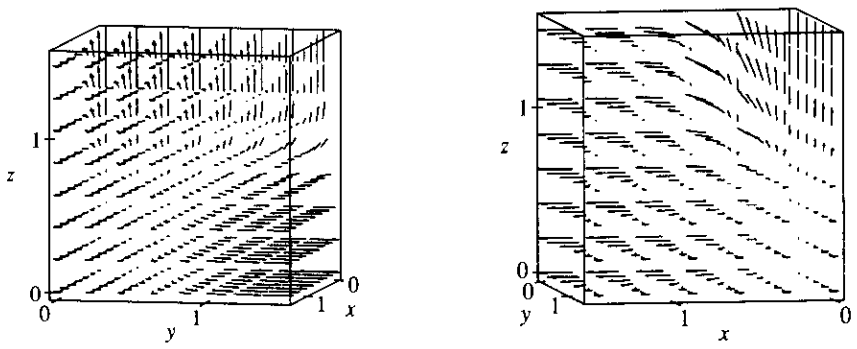
15. $\operatorname{div} \mathbf{F} = 12x^2z + 12y^2z + 12z^3$ so

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 12z(x^2+y^2+z^2)\,dV = \int_0^{2\pi} \int_0^{\pi} \int_0^R 12(\rho\cos\phi)(\rho^2)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 12\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\cos\phi\,d\phi \int_0^R \rho^5\,d\rho = 12(2\pi)\left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi}\left[\tfrac{1}{6}\rho^6\right]_0^R = 0\end{aligned}$$

16. $$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 3(x^2+y^2+1)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 3(\rho^2\sin^2\phi + 1)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 2\pi\int_0^{\pi/2}\left[\tfrac{93}{5}\sin^3\phi + 7\sin\phi\right] d\phi = 2\pi\left[\tfrac{93}{5}\left(-\cos\phi + \tfrac{1}{3}\cos^3\phi\right) - 7\cos\phi\right]_0^{\pi/2} = \tfrac{194}{5}\pi\end{aligned}$$

17. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \sqrt{3-x^2}\,dV = \int_{-1}^1 \int_{-1}^1 \int_0^{2-x^4-y^4} \sqrt{3-x^2}\,dz\,dy\,dx = \frac{341}{60}\sqrt{2} + \frac{81}{20}\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

18.



By the Divergence Theorem, the flux of $\mathbf{F}$ across the surface of the cube is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2} \left[\cos x\cos^2 y + 3\sin^2 y\cos y\cos^4 z + 5\sin^4 z\cos z\cos^6 x\right] dz\,dy\,dx = \tfrac{19}{64}\pi^2.$$

19. For S_1 we have $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot(-\mathbf{k}) = -x^2z - y^2 = -y^2$ (since $z = 0$ on S_1). So if D is the unit disk, we get $\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D (-y^2)\,dA = -\int_0^{2\pi}\int_0^1 r^2(\sin^2\theta)\,r\,dr\,d\theta = -\frac{1}{4}\pi$. Now since S_2 is closed, we can use the Divergence Theorem. Since $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3 + \tan z\right) + \frac{\partial}{\partial z}(x^2z + y^2) = z^2 + y^2 + x^2$, we use spherical coordinates to get

$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi$. Finally

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} - \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \frac{2}{5}\pi - \left(-\frac{1}{4}\pi\right) = \frac{13}{20}\pi$.

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20. As in the hint to Exercise 19, we create a closed surface $S_2 = S \cup S_1$, where S is the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$, and S_1 is the disk $x^2 + y^2 = 1$ on the plane $z = 1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F} \cdot (-\mathbf{k}) = -z = -1$ on S_1. So $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} (-1)\, dS = -A(S_1) = -\pi$. Let E be the region bounded by S_2. Then

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 1\, dV = \int_0^1 \int_0^{2\pi} \int_1^{2-r^2} r\, dz\, d\theta\, dr = \int_0^1 \int_0^{2\pi} (r - r^3)\, d\theta\, dr$$
$$= (2\pi)\tfrac{1}{4} = \tfrac{\pi}{2}$$

Thus the flux of $\mathbf{F}$ across S is $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

21. Since $\dfrac{\mathbf{x}}{|\mathbf{x}|^3} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2 + y^2 + z^2)^{3/2}}\right) = \dfrac{(x^2 + y^2 + z^2) - 3x^2}{(x^2 + y^2 + z^2)^{5/2}}$ with similar expressions for $\dfrac{\partial}{\partial y}\left(\dfrac{y}{(x^2 + y^2 + z^2)^{3/2}}\right)$ and $\dfrac{\partial}{\partial z}\left(\dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}\right)$, we have

$\operatorname{div}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|^3}\right) = \dfrac{3(x^2 + y^2 + z^2) - 3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} = 0$, except at $(0, 0, 0)$ where it is undefined.

22. We first need to find $\mathbf{F}$ so that $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (2x + 2y + z^2)\, dS$, so $\mathbf{F} \cdot \mathbf{n} = 2x + 2y + z^2$.

But for S, $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Thus $\mathbf{F} = 2\,\mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ and $\operatorname{div} \mathbf{F} = 1$. If $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$, then $\iint_S (2x + 2y + z^2)\, dS = \iiint_B dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$.

23. $\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{a}\, dV = 0$ since $\operatorname{div} \mathbf{a} = 0$.

24. $\frac{1}{3} \iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \iiint_E \operatorname{div} \mathbf{F}\, dV = \frac{1}{3} \iiint_E 3\, dV = V(E)$

25. $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\, dV = 0$ by Theorem 17.5.11 [ET 16.5.11].

26. $\iint_S D_{\mathbf{n}} f\, dS = \iint_S (\nabla f \cdot \mathbf{n})\, dS = \iiint_E \operatorname{div}(\nabla f)\, dV = \iiint_E \nabla^2 f\, dV$

27. $\iint_S (f \nabla g) \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div}(f \nabla g)\, dV = \iiint_E (f \nabla^2 g + \nabla g \cdot \nabla f)\, dV$ by Exercise 17.5.25 [ET 16.5.25].

28. $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n}\, dS = \iiint_E \left[(f \nabla^2 g + \nabla g \cdot \nabla f) - (g \nabla^2 f + \nabla g \cdot \nabla f)\right] dV$ [by Exercise 27].
But $\nabla g \cdot \nabla f = \nabla f \cdot \nabla g$, so that $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n}\, dS = \iiint_E (f \nabla^2 g - g \nabla^2 f)\, dV$.

29. If $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$ is an arbitrary constant vector, we define $\mathbf{F} = f\mathbf{c} = fc_1\,\mathbf{i} + fc_2\,\mathbf{j} + fc_3\,\mathbf{k}$. Then $\operatorname{div} \mathbf{F} = \operatorname{div} f\mathbf{c} = \dfrac{\partial f}{\partial x} c_1 + \dfrac{\partial f}{\partial y} c_2 + \dfrac{\partial f}{\partial z} c_3 = \nabla f \cdot \mathbf{c}$ and the Divergence Theorem says

$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV \quad \Rightarrow \quad \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \nabla f \cdot \mathbf{c}\, dV$. In particular, if $\mathbf{c} = \mathbf{i}$ then

$\iint_S f\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_E \nabla f \cdot \mathbf{i}\, dV \quad \Rightarrow \quad \displaystyle\iint_S f n_1\, dS = \iiint_E \frac{\partial f}{\partial x}\, dV$ (where $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$).

Similarly, if $\mathbf{c} = \mathbf{j}$ we have $\displaystyle\iint_S f n_2\, dS = \iiint_E \frac{\partial f}{\partial y}\, dV$, and $\mathbf{c} = \mathbf{k}$ gives $\displaystyle\iint_S f n_3\, dS = \iiint_E \frac{\partial f}{\partial z}\, dV$. Then

$$\iint_S f\mathbf{n}\, dS = \left(\iint_S f n_1\, dS\right)\mathbf{i} + \left(\iint_S f n_2\, dS\right)\mathbf{j} + \left(\iint_S f n_3\, dS\right)\mathbf{k}$$
$$= \left(\iiint_E \frac{\partial f}{\partial x}\, dV\right)\mathbf{i} + \left(\iiint_E \frac{\partial f}{\partial y}\, dV\right)\mathbf{j} + \left(\iiint_E \frac{\partial f}{\partial z}\, dV\right)\mathbf{k}$$
$$= \iiint_E \left(\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}\right) dV = \iiint_E \nabla f\, dV$$

as desired.

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30. By Exercise 29, $\iint_S p\mathbf{n}\,dS = \iiint_E \nabla p\,dV$, so

$$\mathbf{F} = -\iint_S p\mathbf{n}\,dS = -\iiint_E \nabla p\,dV = -\iiint_E \nabla(\rho g z)\,dV = -\iiint_E (\rho g\,\mathbf{k})\,dV$$
$$= -\rho g\left(\iiint_E dV\right)\mathbf{k} = -\rho g V(E)\,\mathbf{k}$$

But the weight of the displaced liquid is volume × density × $g = \rho g V(E)$, thus $\mathbf{F} = -W\mathbf{k}$ as desired.

17 Review ET 16

CONCEPT CHECK

1. See Definitions 1 and 2 in Section 17.1 [ET 16.1]. A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vectors of Earth's gravitational field at a location in space.

2. (a) A conservative vector field $\mathbf{F}$ is a vector field which is the gradient of some scalar function f.

(b) The function f in part (a) is called a potential function for $\mathbf{F}$, that is, $\mathbf{F} = \nabla f$.

3. (a) See Definition 17.2.2 [ET 16.2.2].

(b) We normally evaluate the line integral using Formula 17.2.3 [ET 16.2.3].

(c) The mass is $m = \int_C \rho(x,y)\,ds$, and the center of mass is $(\overline{x},\overline{y})$ where $\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds$, $\overline{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds$.

(d) See (5) and (6) in Section 17.2 [ET 16.2] for plane curves; we have similar definitions when C is a space curve (see the equation preceding (10) in Section 17.2 [ET 16.2]).

(e) For plane curves, see Equations 17.2.7 [ET 16.2.7]. We have similar results for space curves (see the equation preceding (10) in Section 17.2 [ET 16.2]).

4. (a) See Definition 17.2.13 [ET 16.2.13].

(b) If $\mathbf{F}$ is a force field, $\int_C \mathbf{F}\cdot d\mathbf{r}$ represents the work done by $\mathbf{F}$ in moving a particle along the curve C.

(c) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$

5. See Theorem 17.3.2 [ET 16.3.2].

6. (a) $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial and terminal points.

(b) See Theorem 17.3.4 [ET 16.3.4].

7. See the statement of Green's Theorem on page 1119 [ET 1083].

8. See Equations 17.4.5 [ET 16.4.5].

9. (a) $\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \nabla\times\mathbf{F}$

(b) $\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla\cdot\mathbf{F}$

(c) For $\operatorname{curl}\mathbf{F}$, see the discussion accompanying Figure 1 on page 1129 [ET 1093] as well as Figure 6 and the accompanying discussion on page 1160 [ET 1124]. For $\operatorname{div}\mathbf{F}$, see the discussion following Example 5 on page 1130 [ET 1094] as well as the discussion preceding (8) on page 1167 [ET 1131].

10. See Theorem 17.3.6 [ET 16.3.6]; see Theorem 17.5.4 [ET 16.5.4].

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11. (a) See (1) and (2) and the accompanying discussion in Section 17.6 [ET 16.6]; See Figure 4 and the accompanying discussion on page 1135 [ET 1099].

(b) See Definition 17.6.6 [ET 16.6.6].

(c) See Equation 17.6.9 [ET 16.6.9].

12. (a) See (1) in Section 17.7 [ET 16.7].

(b) We normally evaluate the surface integral using Formula 17.7.3 [ET 16.7.3].

(c) See Formula 17.7.2 [ET 16.7.2].

(d) The mass is $m = \iint_S \rho(x, y, z)\, dS$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \frac{1}{m} \iint_S x\rho(x, y, z)\, dS$, $\overline{y} = \frac{1}{m} \iint_S y\rho(x, y, z)\, dS$, $\overline{z} = \frac{1}{m} \iint_S z\rho(x, y, z)\, dS$.

13. (a) See Figures 7 and 8 and the accompanying discussion in Section 17.7 [ET 16.7]. A Möbius strip is a nonorientable surface; see Figures 5 and 6 and the accompanying discussion on page 1149 [ET 1113].

(b) See Definition 17.7.7 [ET 16.7.7].

(c) See Formula 17.7.9 [ET 16.7.9].

(d) See Formula 17.7.8 [ET 16.7.8].

14. See the statement of Stokes' Theorem on page 1157 [ET 1121].

15. See the statement of the Divergence Theorem on page 1163 [ET 1127].

16. In each theorem, we have an integral of a "derivative" over a region on the left side, while the right side involves the values of the original function only on the boundary of the region.

TRUE-FALSE QUIZ

1. False; div $\mathbf{F}$ is a scalar field.

2. True. (See Definition 17.5.1 [ET 16.5.1].)

3. True, by Theorem 17.5.3 [ET 16.5.3] and the fact that div $\mathbf{0} = 0$.

4. True, by Theorem 17.3.2 [ET 16.3.2].

5. False. See Exercise 17.3.33 [ET 16.3.33]. (But the assertion is true if D is simply-connected; see Theorem 17.3.6 [ET 16.3.6].)

6. False. See the discussion accompanying Figure 8 on page 1103 [ET 1067].

7. True. Apply the Divergence Theorem and use the fact that div $\mathbf{F} = 0$.

8. False by Theorem 17.5.11 [ET 16.5.11], because if it were true, then div curl $\mathbf{F} = 3 \neq 0$.

EXERCISES

1. (a) Vectors starting on C point in roughly the direction opposite to C, so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is negative. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is negative.

(b) The vectors that end near P are shorter than the vectors that start near P, so the net flow is outward near P and div $\mathbf{F}\,(P)$ is positive.

2. We can parametrize C by $x = x$, $y = x^2$, $0 \le x \le 1$ so

$$\int_C x\, ds = \int_0^1 x\sqrt{1 + (2x)^2}\, dx = \tfrac{1}{12}(1 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(5\sqrt{5} - 1\right).$$

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3. $\int_C x^3 z\,ds = \int_0^{\pi/2} (2\sin t)^3 (2\cos t)\sqrt{(2\cos t)^2 + (1)^2 + (-2\sin t)^2}\,dt = \int_0^{\pi/2} (16\sin^3 t\cos t)\sqrt{5}\,dt$
$= 4\sqrt{5}\sin^4 t\big]_0^{\pi/2} = 4\sqrt{5}$

4. $\int_C xy\,dx + y\,dy = \int_0^{\pi/2} (x\sin x + \sin x\cos x)\,dx = -x\cos x + \sin x - \frac{1}{4}\cos 2x\big]_0^{\pi/2} = \frac{3}{2}$

5. $x = \cos t \Rightarrow dx = -\sin t\,dt$, $y = \sin t \Rightarrow dy = \cos t\,dt$, $0 \le t \le 2\pi$ and

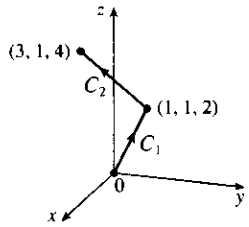
$\int_C x^3 y\,dx - x\,dy = \int_0^{2\pi} (-\cos^3 t\sin^2 t - \cos^2 t)\,dt = \int_0^{2\pi} (-\cos^3 t\sin^2 t - \cos^2 t)\,dt = -\pi$

Or: Since C is a simple closed curve, apply Green's Theorem giving

$$\iint\limits_{x^2+y^2\le 1} (-1 - x^3)\,dA = \int_0^1 \int_0^{2\pi} (-r - r^4\cos^3\theta)\,d\theta = -\pi.$$

6. $\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz = \int_0^1 \left(\sqrt{t^4\cdot t^2}\cdot 4t^3 + e^{t^2}\cdot 2t + t^4\cdot t^3\cdot 3t^2\right) dt$
$= \int_0^1 \left(4t^6 + 2te^{t^2} + 3t^9\right)\,dt = \left[\frac{4}{7}t^7 + e^{t^2} + \frac{3}{10}t^{10}\right]_0^1$
$= e - \frac{9}{70}$

7.



C_1: $x = t$, $y = t$, $z = 2t$, $0 \le t \le 1$;

C_2: $x = 1 + 2t$, $y = 1$, $z = 2 + 2t$, $0 \le t \le 1$.

Then

$$\int_C y\,dx + z\,dy + x\,dz = \int_0^1 5t\,dt + \int_0^1 (4 + 4t)\,dt = \frac{17}{2}$$

8. $\mathbf{F}(\mathbf{r}(t)) = (\sin t)(1+t)\,\mathbf{i} + (\sin^2 t)\,\mathbf{j}$, $\mathbf{r}'(t) = \cos t\,\mathbf{i} + \mathbf{j}$ and

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} ((1+t)\sin t\cos t + \sin^2 t)\,dt = \int_0^{\pi} \left(\frac{1}{2}(1+t)\sin 2t + \sin^2 t\right) dt$
$= \left[\frac{1}{2}\left((1+t)\left(-\frac{1}{2}\cos 2t\right) + \frac{1}{4}\sin 2t\right) + \frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{\pi} = \frac{\pi}{4}.$

9. $\mathbf{F}(\mathbf{r}(t)) = e^{-t}\,\mathbf{i} + t^2(-t)\,\mathbf{j} + (t^2 + t^3)\,\mathbf{k}$, $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} - \mathbf{k}$ and

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (2te^{-t} - 3t^5 - (t^2 + t^3))\,dt = \left[-2te^{-t} - 2e^{-t} - \frac{1}{2}t^6 - \frac{1}{3}t^3 - \frac{1}{4}t^4\right]_0^1 = \frac{11}{12} - \frac{4}{e}.$

10. (a) C: $x = 3 - 3t$, $y = \frac{\pi}{2}t$, $z = 3t$, $0 \le t \le 1$. Then

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left[3t\,\mathbf{i} + (3-3t)\,\mathbf{j} + \tfrac{\pi}{2}t\,\mathbf{k}\right]\cdot\left[-3\,\mathbf{i} + \tfrac{\pi}{2}\,\mathbf{j} + 3\,\mathbf{k}\right] dt = \int_0^1 \left[-9t + \tfrac{3\pi}{2}\right] dt$$
$$= \tfrac{1}{2}(3\pi - 9)$$

(b) $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} (3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + t\,\mathbf{k})\cdot(-3\sin t\,\mathbf{i} + \mathbf{j} + 3\cos t\,\mathbf{k})\,dt$

$= \int_0^{\pi/2} (-9\sin^2 t + 3\cos t + 3t\cos t)\,dt$

$= \left[-\frac{9}{2}(t - \sin t\cos t) + 3\sin t + 3(t\sin t + \cos t)\right]_0^{\pi/2} = -\frac{9\pi}{4} + 3 + \frac{3\pi}{2} - 3 = -\frac{3\pi}{4}$

11. $\frac{\partial}{\partial y}\left[(1+xy)e^{xy}\right] = 2xe^{xy} + x^2 ye^{xy} = \frac{\partial}{\partial x}\left[e^y + x^2 e^{xy}\right]$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_y(x,y) = e^y + x^2 e^{xy}$ implies $f(x,y) = e^y + xe^{xy} + g(x)$ and then $f_x(x,y) = xye^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. But $f_x(x,y) = (1+xy)e^{xy}$, so $g'(x) = 0 \Rightarrow g(x) = K$. Thus $f(x,y) = e^y + xe^{xy} + K$ is a potential function for $\mathbf{F}$.

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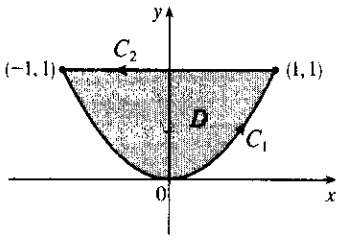
12. $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, its components have continuous partial derivatives, and $\operatorname{curl}\mathbf{F} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative by Theorem 17.5.4 [ET 16.5.4]. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = \sin y$ implies $f(x,y,z) = x\sin y + g(y,z)$ and then $f_y(x,y,z) = x\cos y + g_y(y,z)$. But $f_y(x,y,z) = x\cos y$, so $g_y(y,z) = 0 \;\Rightarrow\; g(y,z) = h(z)$. Then $f(x,y,z) = x\sin y + h(z)$ implies $f_z(x,y,z) = h'(z)$. But $f_z(x,y,z) = -\sin z$, so $h(z) = \cos z + K$. Thus a potential function for $\mathbf{F}$ is $f(x,y,z) = x\sin y + \cos z + K$.

13. Since $\frac{\partial}{\partial y}\left(4x^3y^2 - 2xy^3\right) = 8x^3y - 6xy^2 = \frac{\partial}{\partial x}\left(2x^4y - 3x^2y^2 + 4y^3\right)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, $\mathbf{F}$ is conservative. Furthermore $f(x,y) = x^4y^2 - x^2y^3 + y^4$ is a potential function for $\mathbf{F}$. $t = 0$ corresponds to the point $(0,1)$ and $t = 1$ corresponds to $(1,1)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(0,1) = 1 - 1 = 0$.

14. Here $\operatorname{curl}\mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^3$, and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative. Furthermore $f(x,y,z) = xe^y + ye^z$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4,0,3) - f(0,2,0) = 4 - 2 = 2$.

15.



C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $-1 \le t \le 1$;

C_2: $\mathbf{r}(t) = -t\,\mathbf{i} + \mathbf{j}$, $-1 \le t \le 1$.

Then

$$\int_C xy^2\,dx - x^2y\,dy = \int_{-1}^{1}(t^5 - 2t^5)\,dt + \int_{-1}^{1} t\,dt$$

$$= \left[-\tfrac{1}{6}t^6\right]_{-1}^{1} + \left[\tfrac{1}{2}t^2\right]_{-1}^{1} = 0$$

Using Green's Theorem, we have

$$\int_C xy^2\,dx - x^2y\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(-x^2y\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right] dA = \iint_D (-2xy - 2xy)\,dA$$

$$= \int_{-1}^{1}\int_{x^2}^{1} -4xy\,dy\,dx = \int_{-1}^{1}\left[-2xy^2\right]_{y=x^2}^{y=1} dx$$

$$= \int_{-1}^{1}\left(2x^5 - 2x\right) dx = \left[\tfrac{1}{3}x^6 - x^2\right]_{-1}^{1} = 0$$

16. $\int_C \sqrt{1+x^3}\,dx + 2xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}\left(\sqrt{1+x^3}\right)\right] dA = \int_0^1\int_0^{3x}(2y - 0)\,dy\,dx$

$$= \int_0^1 9x^2\,dx = 3x^3\Big]_0^1 = 3$$

17. $\int_C x^2y\,dx - xy^2\,dy = \iint\limits_{x^2+y^2\le 4}\left[\frac{\partial}{\partial x}\left(-xy^2\right) - \frac{\partial}{\partial y}\left(x^2y\right)\right] dA$

$$= \iint\limits_{x^2+y^2\le 4}\left(-y^2 - x^2\right) dA = -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -8\pi$$

18. $\operatorname{curl}\mathbf{F} = \left(0 - e^{-y}\cos z\right)\mathbf{i} - \left(e^{-z}\cos x - 0\right)\mathbf{j} + \left(0 - e^{-x}\cos y\right)\mathbf{k} = -e^{-y}\cos z\,\mathbf{i} - e^{-z}\cos x\,\mathbf{j} - e^{-x}\cos y\,\mathbf{k}$, $\operatorname{div}\mathbf{F} = -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$

19. If we assume there is such a vector field $\mathbf{G}$, then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 2 + 3z - 2xz$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for all vector fields $\mathbf{F}$. Thus such a $\mathbf{G}$ cannot exist.

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20. Let $\mathbf{F} = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G} = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$$\begin{aligned}
&\mathbf{F}\,\mathrm{div}\,\mathbf{G} - \mathbf{G}\,\mathrm{div}\,\mathbf{F} \\
&= \left[P_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{i} + Q_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{j} + R_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{k}\right] \\
&\quad - \left[P_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{i} + Q_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{j} + R_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]
\end{aligned}$$

and

$$\begin{aligned}
(\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G} &= \left[\left(P_2\frac{\partial P_1}{\partial x} + Q_2\frac{\partial P_1}{\partial y} + R_2\frac{\partial P_1}{\partial z}\right)\mathbf{i} + \left(P_2\frac{\partial Q_1}{\partial x} + Q_2\frac{\partial Q_1}{\partial y} + R_2\frac{\partial Q_1}{\partial z}\right)\mathbf{j}\right. \\
&\qquad \left. + \left(P_2\frac{\partial R_1}{\partial x} + Q_2\frac{\partial R_1}{\partial y} + R_2\frac{\partial R_1}{\partial z}\right)\mathbf{k}\right] \\
&\quad - \left[\left(P_1\frac{\partial P_2}{\partial x} + Q_1\frac{\partial P_2}{\partial y} + R_1\frac{\partial P_2}{\partial z}\right)\mathbf{i} + \left(P_1\frac{\partial Q_2}{\partial x} + Q_1\frac{\partial Q_2}{\partial y} + R_1\frac{\partial Q_2}{\partial z}\right)\mathbf{j}\right. \\
&\qquad \left. + \left(P_1\frac{\partial R_2}{\partial x} + Q_1\frac{\partial R_2}{\partial y} + R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbf{F}\,\mathrm{div}\,\mathbf{G} - \mathbf{G}\,\mathrm{div}\,\mathbf{F} + (\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G} \\
&= \left[\left(P_1\frac{\partial Q_2}{\partial y} + Q_2\frac{\partial P_1}{\partial x}\right) - \left(P_2\frac{\partial Q_1}{\partial y} + Q_1\frac{\partial P_2}{\partial y}\right)\right. \\
&\qquad \left. - \left(P_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial P_2}{\partial z}\right) + \left(P_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial P_1}{\partial z}\right)\right]\mathbf{i} \\
&\quad + \left[\left(Q_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial Q_1}{\partial z}\right) - \left(Q_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial Q_2}{\partial z}\right)\right. \\
&\qquad \left. - \left(P_1\frac{\partial Q_2}{\partial x} + Q_2\frac{\partial P_1}{\partial x}\right) + \left(P_2\frac{\partial Q_1}{\partial x} + Q_1\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j} \\
&\quad + \left[\left(P_2\frac{\partial R_1}{\partial x} + R_1\frac{\partial P_2}{\partial x}\right) - \left(P_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial P_1}{\partial x}\right)\right. \\
&\qquad \left. - \left(Q_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial Q_1}{\partial y}\right) + \left(Q_2\frac{\partial R_1}{\partial y} + R_1\frac{\partial Q_2}{\partial y}\right)\right]\mathbf{k} \\
&= \left[\frac{\partial}{\partial y}(P_1Q_2 - P_2Q_1) - \frac{\partial}{\partial z}(P_2R_1 - P_1R_2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(Q_1R_2 - Q_2R_1) - \frac{\partial}{\partial x}(P_1Q_2 - P_2Q_1)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}(P_2R_1 - P_1R_2) - \frac{\partial}{\partial y}(Q_1R_2 - Q_2R_1)\right]\mathbf{k} \\
&= \mathrm{curl}\,(\mathbf{F}\times\mathbf{G})
\end{aligned}$$

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21. For any piecewise-smooth simple closed plane curve C bounding a region D, we can apply Green's Theorem to $\mathbf{F}(x, y) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j}$ to get $\int_C f(x)\,dx + g(y)\,dy = \iint_D \left[\frac{\partial}{\partial x}\,g(y) - \frac{\partial}{\partial y}\,f(x)\right] dA = \iint_D 0\,dA = 0$.

22.
$$\nabla^2(fg) = \frac{\partial^2(fg)}{\partial x^2} + \frac{\partial^2(fg)}{\partial y^2} + \frac{\partial^2(fg)}{\partial z^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\,g + f\,\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\,g + f\,\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\,g + f\,\frac{\partial g}{\partial z}\right) \quad \text{[Product Rule]}$$

$$= \frac{\partial^2 f}{\partial x^2}\,g + 2\,\frac{\partial f}{\partial x}\,\frac{\partial g}{\partial x} + f\,\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\,g + 2\,\frac{\partial f}{\partial y}\,\frac{\partial g}{\partial y}$$

$$+ f\,\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,g + 2\,\frac{\partial f}{\partial z}\,\frac{\partial g}{\partial z} + f\,\frac{\partial^2 g}{\partial z^2} \quad \text{[Product Rule]}$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right\rangle$$

$$= f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

Another method: Using the rules in Exercises 15.6.37(b) [ET 14.6.37(b)] and 17.5.25 [ET 16.5.25], we have

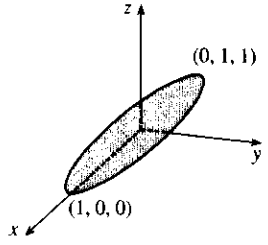
$$\nabla^2(fg) = \nabla \cdot \nabla(fg) = \nabla \cdot (g\,\nabla f + f\,\nabla g) = \nabla g \cdot \nabla f + g\nabla \cdot \nabla f + \nabla f \cdot \nabla g + f\nabla \cdot \nabla g$$
$$= g\,\nabla^2 f + f\,\nabla^2 g + 2\nabla f \cdot \nabla g$$

23. $\nabla^2 f = 0$ means that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f_y\,dx - f_x\,dy = \iint_D \left[\frac{\partial}{\partial x}\,(-f_x) - \frac{\partial}{\partial y}\,(f_y)\right] dA = -\iint_D (f_{xx} + f_{yy})\,dA$$
$$= -\iint_D 0\,dA = 0$$

Therefore the line integral is independent of path, by Theorem 17.3.3 [ET 16.3.3].

24. (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2 + y^2 = 1$. But also $y = z$, so C lies on the plane $y = z$. Thus C is the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$.



(b) Apply Stokes' Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$:

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z \end{vmatrix}$$

$$= \left\langle -2y\csc^2 z - (-2y\csc^2 z), 0, 4xe^{2y} - 4xe^{2y}\right\rangle = \mathbf{0}$$

Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$.

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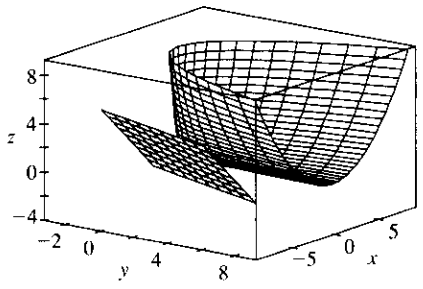
25. $z = f(x,y) = x^2 + 2y$ with $0 \le x \le 1$, $0 \le y \le 2x$. Thus

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4}\, dA = \int_0^1 \int_0^{2x} \sqrt{5 + 4x^2}\, dy\, dx = \int_0^1 2x\sqrt{5 + 4x^2}\, dx$$
$$= \tfrac{1}{6}(5 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{6}\left(27 - 5\sqrt{5}\right).$$

26. (a) $\mathbf{r}_u = -v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = 2v\,\mathbf{i} - u\,\mathbf{j}$ and $\mathbf{r}_u \times \mathbf{r}_v = 2u^2\,\mathbf{i} + 4uv\,\mathbf{j} + 2v^2\,\mathbf{k}$. Since the point $(4,-2,1)$ corresponds to $u = 1, v = 2$ (or $u = -1, v = -2$ but $\mathbf{r}_u \times \mathbf{r}_v$ is the same for both), a normal vector to the surface at $(4,-2,1)$ is $2\,\mathbf{i} + 8\,\mathbf{j} + 8\,\mathbf{k}$ and an equation of the tangent plane is $2x + 8y + 8z = 0$ or $x + 4y + 4z = 0$.

(b)



(c) By Definition 17.6.6 [ET 16.6.6], the area of S is given by
$A(S) = \int_0^3 \int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2}\, dv\, du = 2\int_0^3 \int_{-3}^3 \sqrt{u^4 + 4u^2v^2 + v^4}\, dv\, du.$

(d) By Equation 17.7.9 [ET 16.7.9], the surface integral is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^3 \int_{-3}^3 \left\langle \frac{(u^2)^2}{1 + (v^2)^2}, \frac{(v^2)^2}{1 + (-uv)^2}, \frac{(-uv)^2}{1 + (u^2)^2} \right\rangle \cdot \left\langle 2u^2, 4uv, 2v^2 \right\rangle dv\, du$$
$$= \int_0^3 \int_{-3}^3 \left(\frac{2u^6}{1 + v^4} + \frac{4uv^5}{1 + u^2v^2} + \frac{2u^2v^4}{1 + u^4} \right) dv\, du \approx 1524.0190$$

27. $z = f(x,y) = x^2 + y^2$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (using upward orientation). Then

$$\iint_S z\, dS = \iint_{x^2 + y^2 \le 4} (x^2 + y^2)\sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi} \int_0^2 r^3 s\sqrt{1 + 4r^2}\, dr\, d\theta$$
$$= \tfrac{1}{60}\pi\left(391\sqrt{17} + 1\right)$$

(Substitute $u = 1 + 4r^2$ and use tables.)

28. $z = f(x,y) = 4 + x + y$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\iint_S (x^2 z + y^2 z)\, dS = \iint_{x^2 + y^2 \le 4} (x^2 + y^2)(4 + x + y)\sqrt{3}\, dA$$
$$= \int_0^2 \int_0^{2\pi} \sqrt{3}\, r^3 (4 + r\cos\theta + r\sin\theta)\, d\theta\, dr = \int_0^2 8\pi\sqrt{3}\, r^3\, dr = 32\pi\sqrt{3}$$

29. Since the sphere bounds a simple solid region, the Divergence Theorem applies and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (z - 2)\, dV = \iiint_E z\, dV - 2\iiint_E dV = m\bar{z} - 2\left(\tfrac{4}{3}\pi 2^3\right) = -\tfrac{64}{3}\pi.$$

Alternate solution: $\mathbf{F}(\mathbf{r}(\phi,\theta)) = 4\sin\phi\cos\theta\cos\phi\,\mathbf{i} - 4\sin\phi\sin\theta\,\mathbf{j} + 6\sin\phi\cos\theta\,\mathbf{k}$,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\cos\theta\,\mathbf{i} + 4\sin^2\phi\sin\theta\,\mathbf{j} + 4\sin\phi\cos\phi\,\mathbf{k}$, and

$\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16\sin^3\phi\cos^2\theta\cos\phi - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\phi\cos\theta$. Then

$$\iint_S F \cdot dS = \int_0^{2\pi} \int_0^{\pi} (16\sin^3\phi\cos\phi\cos^2\theta - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\phi\cos\theta)\, d\phi\, d\theta$$
$$= \int_0^{2\pi} \tfrac{4}{3}(-16\sin^2\theta)\, d\theta = -\tfrac{64}{3}\pi$$

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30. $z = f(x,y) = x^2 + y^2$, $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (because of upward orientation) and $\mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -2x^3 - 2xy^2 + x^2 + y^2$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2 \le 1} (-2x^3 - 2xy^2 + x^2 + y^2)\,dA$$

$$= \int_0^1 \int_0^{2\pi} (-2r^3\cos^3\theta - 2r^3\cos\theta\sin^2\theta + r^2)\,r\,dr\,d\theta = \int_0^1 r^3(2\pi)\,dr = \tfrac{\pi}{2}$$

31. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\iint_S (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} = 0$. We parametrize C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$ and $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos^2 t \sin t + \sin^2 t \cos t)\,dt = \frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t\Big]_0^{2\pi} = 0$.

32. $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$ where C: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t \sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + e^{4\cos t \sin t}\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\cos^2 t \sin^2 t + 4\sin t\cos t$. Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-16\cos^2 t \sin^2 t + 4\sin t\cos t)\,dt$$

$$= \left[-16\left(-\tfrac{1}{4}\sin t\cos^3 t + \tfrac{1}{16}\sin 2t + \tfrac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi$$

33. The surface is given by $x + y + z = 1$ or $z = 1 - x - y$, $0 \le x \le 1$, $0 \le y \le 1 - x$ and $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D (-y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dA$$

$$= \iint_D (-1)\,dA = -(\text{area of } D) = -\tfrac{1}{2}$$

34. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^2 (3r^2 + 3z^2)\,r\,dz\,dr\,d\theta = 2\pi\int_0^1 (6r^3 + 8r)\,dr = 11\pi$

35. $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_{x^2+y^2+z^2 \le 1} 3\,dV = 3(\text{volume of sphere}) = 4\pi$. Then $\mathbf{F}(\mathbf{r}(\phi,\theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi = \sin\phi$ and $\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} \sin\phi\,d\phi\,d\theta = (2\pi)(2) = 4\pi$.

36. Here we must use Equation 17.9.6 [ET 16.9.6] since $\mathbf{F}$ is not defined at the origin. Let S_1 be the sphere of radius 1 with center at the origin and outer unit normal $\mathbf{n}_1$. Let S_2 be the surface of the ellipsoid with outer unit normal $\mathbf{n}_2$ and let E be the solid region between S_1 and S_2. Then the outward flux of $\mathbf{F}$ through the ellipsoid is given by $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\,dS = -\iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\,dS + \iiint_E \operatorname{div}\mathbf{F}\,dV$. But $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$, so $\operatorname{div}\mathbf{F} = \nabla \cdot \left(|\mathbf{r}|^{-3}\,\mathbf{r}\right) = |\mathbf{r}|^{-3}(\nabla \cdot \mathbf{r}) + \mathbf{r} \cdot \left(\nabla |\mathbf{r}|^{-3}\right) = |\mathbf{r}|^{-3}(3) + \mathbf{r} \cdot \left(-3|\mathbf{r}|^{-4}\right)\left(\mathbf{r}|\mathbf{r}|^{-1}\right) = 0$. (Here we have used Exercises 17.5.30(a) [ET 16.5.30(a)] and 17.5.31(a) [ET 16.5.31(a)].) And $\mathbf{F} \cdot \mathbf{n}_1 = \dfrac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|} = |\mathbf{r}|^{-2} = 1$ on S_1. Thus $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\,dS = \iint_{S_1} dS + \iiint_E 0\,dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi$.

37. Because $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative, and if $f(x,y,z) = x^3yz - 3xy + z^2$, then $\nabla f = \mathbf{F}$. Hence $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0,3,0) - f(0,0,2) = 0 - 4 = -4$.

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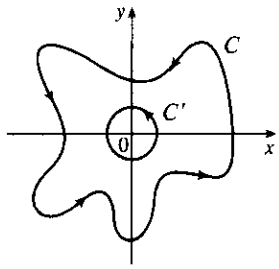
38. Let C' be the circle with center at the origin and radius a as in the figure.



Let D be the region bounded by C and C'. Then D's positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0, \text{ so}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot dr = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left[\frac{2a^3 \cos^3 t + 2a^3 \cos t \sin^2 t - 2a \sin t}{a^2} (-a \sin t) \right.$$

$$\left. + \frac{2a^3 \sin^3 t + 2a^3 \cos^2 t \sin t + 2a \cos t}{a^2} (a \cos t) \right] dt$$

$$= \int_0^{2\pi} \frac{2a^2}{a^2}\, dt = 4\pi$$

39. By the Divergence Theorem, $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV = 3(\text{volume of } E) = 3(8 - 1) = 21$.

40. The stated conditions allow us to use the Divergence Theorem. Hence $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\, dV = 0$ since $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

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PROBLEMS PLUS

1. Let S_1 be the portion of $\Omega(S)$ between $S(a)$ and S, and let ∂S_1 be its boundary. Also let S_L be the lateral surface of S_1 [that is, the surface of S_1 except S and $S(a)$]. Applying the Divergence Theorem we have $\iint_{\partial S_1} \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} \nabla\cdot\dfrac{\mathbf{r}}{r^3}\,dV$. But

$$\nabla\cdot\frac{\mathbf{r}}{r^3} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$= \frac{(x^2+y^2+z^2-3x^2)+(x^2+y^2+z^2-3y^2)+(x^2+y^2+z^2-3z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$$

$\Rightarrow \iint_{\partial S_1} \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} 0\,dV = 0$. On the other hand, notice that for the surfaces of ∂S_1 other than $S(a)$ and S, $\mathbf{r}\cdot\mathbf{n} = 0 \Rightarrow$

$$0 = \iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S_L} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$$

$$= \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$$

$\Rightarrow \iint_S \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = -\iint_{S(a)} \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$. Notice that on $S(a)$, $r = a \Rightarrow \mathbf{n} = -\dfrac{\mathbf{r}}{r} = -\dfrac{\mathbf{r}}{a}$ and $\mathbf{r}\cdot\mathbf{r} = r^2 = a^2$, so that $-\iint_{S(a)} \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iint_{S(a)} \dfrac{\mathbf{r}\cdot\mathbf{r}}{a^4}\,dS = \iint_{S(a)} \dfrac{a^2}{a^4}\,dS = \dfrac{1}{a^2}\iint_{S(a)} dS = \dfrac{\text{area of } S(a)}{a^2} = |\Omega(S)|$.

Therefore $|\Omega(S)| = \iint_S \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$.

2. By Green's Theorem

$$\int_C (y^3 - y)\,dx - 2x^3\,dy = \iint_D \left[\frac{\partial(-2x^3)}{\partial x} - \frac{\partial(y^3-y)}{\partial y}\right] dA = \iint_D (1 - 6x^2 - 3y^2)\,dA$$

Notice that for $6x^2 + 3y^2 > 1$, the integrand is negative. The integral has maximum value if it is evaluated only in the region where the integrand is positive, which is within the ellipse $6x^2 + 3y^2 = 1$. So the simple closed curve that gives a maximum value for the line integral is the ellipse $6x^2 + 3y^2 = 1$.

3. The given line integral $\frac{1}{2}\int_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$ can be expressed as $\int_C \mathbf{F}\cdot d\mathbf{r}$ if we define the vector field $\mathbf{F}$ by $\mathbf{F}(x,y,z) = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k} = \frac{1}{2}(bz - cy)\,\mathbf{i} + \frac{1}{2}(cx - az)\,\mathbf{j} + \frac{1}{2}(ay - bx)\,\mathbf{k}$. Then define S to be the planar interior of C, so S is an oriented, smooth surface. Stokes' Theorem says $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$. Now

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \left(\tfrac{1}{2}a + \tfrac{1}{2}a\right)\mathbf{i} + \left(\tfrac{1}{2}b + \tfrac{1}{2}b\right)\mathbf{j} + \left(\tfrac{1}{2}c + \tfrac{1}{2}c\right)\mathbf{k} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k} = \mathbf{n}$$

so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{n} = |\mathbf{n}|^2 = 1$, hence $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_S dS$ which is simply the surface area of S. Thus, $\int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}\int_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$ is the plane area enclosed by C.

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4. (a) First we place the piston on coordinate axes so the top of the cylinder is at the origin and $x(t) \geq 0$ is the distance from the top of the cylinder to the piston at time t. Let C_1 be the curve traced out by the piston during one four-stroke cycle, so C_1 is given by $\mathbf{r}(t) = x(t)\,\mathbf{i}$, $a \leq t \leq b$. (Thus, the curve lies on the positive x-axis and reverses direction several times.) The force on the piston is $AP(t)\,\mathbf{i}$, where A is the area of the top of the piston and $P(t)$ is the pressure in the cylinder at time t. As in Section 17.2 [ET 16.2], the work done on the piston is $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_a^b AP(t)\,\mathbf{i} \cdot x'(t)\,\mathbf{i}\,dt = \int_a^b AP(t)\,x'(t)\,dt$. Here, the volume of the cylinder at time t is $V(t) = Ax(t) \;\Rightarrow\; V'(t) = Ax'(t) \;\Rightarrow\; \int_a^b AP(t)\,x'(t)\,dt = \int_a^b P(t)\,V'(t)\,dt$. Since the curve C in the PV-plane corresponds to the values of P and V at time t, $a \leq t \leq b$, we have

$$W = \int_a^b AP(t)\,x'(t)\,dt = \int_a^b P(t)\,V'(t)\,dt = \int_C P\,dV$$

Another method: If we divide the time interval $[a, b]$ into n subintervals of equal length Δt, the amount of work done on the piston in the ith time interval is approximately $AP(t_i)[x(t_i) - x(t_{i-1})]$. Thus we estimate the total work done during one cycle to be $\sum_{i=1}^{n} AP(t_i)[x(t_i) - x(t_{i-1})]$. If we allow $n \to \infty$, we have

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} AP(t_i)[x(t_i) - x(t_{i-1})] = \lim_{n\to\infty} \sum_{i=1}^{n} P(t_i)[Ax(t_i) - Ax(t_{i-1})]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(t_i)[V(t_i) - V(t_{i-1})] = \int_C P\,dV$$

(b) Let C_L be the lower loop of the curve C and C_U the upper loop. Then $C = C_L \cup C_U$. C_L is positively oriented, so from Formula 17.4.5 [ET 16.4.5] we know the area of the lower loop in the PV-plane is given by $-\oint_{C_L} P\,dV$. C_U is negatively oriented, so the area of the upper loop is given by $-\left(-\oint_{C_U} P\,dV\right) = \oint_{C_U} P\,dV$. From part (a),

$$W = \int_C P\,dV = \int_{C_L \cup C_U} P\,dV = \oint_{C_L} P\,dV + \oint_{C_U} P\,dV = \oint_{C_U} P\,dV - \left(-\oint_{C_L} P\,dV\right),$$

the difference of the areas enclosed by the two loops of C.

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Chapter 18

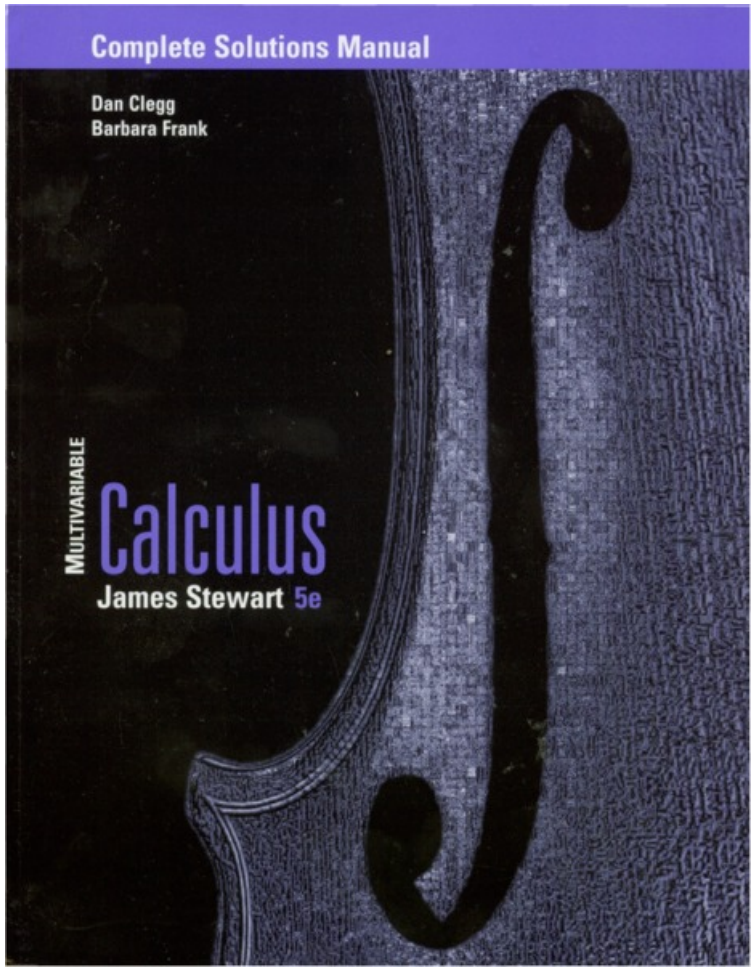
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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18 ☐ SECOND-ORDER DIFFERENTIAL EQUATIONS ☐ ET 17

18.1 Second-Order Linear Equations ET 17.1

1. The auxiliary equation is $r^2 - 6r + 8 = 0 \Rightarrow (r-4)(r-2) = 0 \Rightarrow r = 4, r = 2$. Then by (8) the general solution is $y = c_1 e^{4x} + c_2 e^{2x}$.

2. The auxiliary equation is $r^2 - 4r + 8 = 0 \Rightarrow r = 2 \pm 2i$. Then by (11) the general solution is $y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$.

3. The auxiliary equation is $r^2 + 8r + 41 = 0 \Rightarrow r = -4 \pm 5i$. Then by (11) the general solution is $y = e^{-4x}(c_1 \cos 5x + c_2 \sin 5x)$.

4. The auxiliary equation is $2r^2 - r - 1 = (2r+1)(r-1) = 0 \Rightarrow r = 1, r = -\frac{1}{2}$. Then the general solution is $y = c_1 e^{x} + c_2 e^{-x/2}$.

5. The auxiliary equation is $r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$. Then by (10), the general solution is $y = c_1 e^{x} + c_2 x e^{x}$.

6. The auxiliary equation is $3r^2 - 5r = r(3r-5) = 0 \Rightarrow r = 0, r = \frac{5}{3}$, so $y = c_1 + c_2 e^{5x/3}$.

7. The auxiliary equation is $4r^2 + 1 = 0 \Rightarrow r = \pm\frac{1}{2}i$, so $y = c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right)$.

8. The auxiliary equation is $16r^2 + 24r + 9 = (4r+3)^2 = 0 \Rightarrow r = -\frac{3}{4}$, so $y = c_1 e^{-3x/4} + c_2 x e^{-3x/4}$.

9. The auxiliary equation is $4r^2 + r = r(4r+1) = 0 \Rightarrow r = 0, r = -\frac{1}{4}$, so $y = c_1 + c_2 e^{-x/4}$.

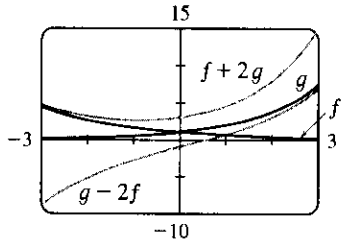
10. The auxiliary equation is $9r^2 + 4 = 0 \Rightarrow r = \pm\frac{2}{3}i$, so $y = c_1 \cos\left(\frac{2}{3}x\right) + c_2 \sin\left(\frac{2}{3}x\right)$.

11. The auxiliary equation is $r^2 - 2r - 1 = 0 \Rightarrow r = 1 \pm \sqrt{2}$, so $y = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t}$.

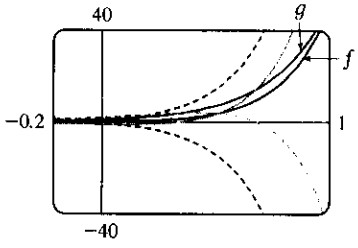
12. The auxiliary equation is $r^2 - 6r + 4 = 0 \Rightarrow r = 3 \pm \sqrt{5}$, so $y = c_1 e^{(3+\sqrt{5})t} + c_2 e^{(3-\sqrt{5})t}$.

13. The auxiliary equation is $r^2 + r + 1 = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, so $y = e^{-t/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right]$.

14. $6r^2 - r - 2 = (2r+1)(3r-2) = 0$ so $y = c_1 e^{-x/2} + c_2 e^{2x/3}$. The solutions $(c_1, c_2) = (0,1)$, $(1,0)$, $(1,2)$, $(-2,1)$ are shown. Each solution consists of a single continuous curve that approaches either 0 or $\pm\infty$ as $x \to \pm\infty$.



15. $r^2 - 8r + 16 = (r-4)^2 = 0$ so $y = c_1 e^{4x} + c_2 x e^{4x}$. The graphs are all asymptotic to the x-axis as $x \to -\infty$, and as $x \to \infty$ the solutions tend to $\pm\infty$.



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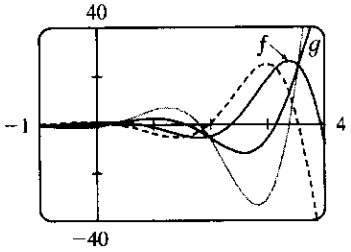
16. $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i$ and the solution is $y = e^x(c_1 \cos 2x + c_2 \sin 2x)$. Graphs for $(c_1, c_2) = (1, 0), (0, 1), (1, -1), (-1, 2)$ are shown. The solutions are all asymptotic to the x-axis as $x \to -\infty$ and they all oscillate. The amplitudes of the oscillations become arbitrarily large as $x \to \infty$ and arbitrarily small as $x \to -\infty$.



17. $2r^2 + 5r + 3 = (2r + 3)(r + 1) = 0$, so $r = -\frac{3}{2}$, $r = -1$ and the general solution is $y = c_1 e^{-3x/2} + c_2 e^{-x}$. Then $y(0) = 3 \Rightarrow c_1 + c_2 = 3$ and $y'(0) = -4 \Rightarrow -\frac{3}{2}c_1 - c_2 = -4$, so $c_1 = 2$ and $c_2 = 1$. Thus the solution to the initial-value problem is $y = 2e^{-3x/2} + e^{-x}$.

18. $r^2 + 3 = 0 \Rightarrow r = \pm\sqrt{3}i$ and the general solution is $y = e^{0x}\big(c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x)\big) = c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x)$. Then $y(0) = 1 \Rightarrow c_1 = 1$ and $y'(0) = 3 \Rightarrow c_2 = \sqrt{3}$, so the solution to the initial-value problem is $y = \cos(\sqrt{3}\,x) + \sqrt{3}\sin(\sqrt{3}\,x)$.

19. $4r^2 - 4r + 1 = (2r - 1)^2 = 0 \Rightarrow r = \frac{1}{2}$ and the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. Then $y(0) = 1 \Rightarrow c_1 = 1$ and $y'(0) = -1.5 \Rightarrow \frac{1}{2}c_1 + c_2 = -1.5$, so $c_2 = -2$ and the solution to the initial-value problem is $y = e^{x/2} - 2xe^{x/2}$.

20. $2r^2 + 5r - 3 = (2r - 1)(r + 3) = 0 \Rightarrow r = \frac{1}{2}, r = -3$ and the general solution is $y = c_1 e^{x/2} + c_2 e^{-3x}$. Then $1 = y(0) = c_1 + c_2$ and $4 = y'(0) = \frac{1}{2}c_1 - 3c_2$ so $c_1 = 2$, $c_2 = -1$ and the solution to the initial-value problem is $y = 2e^{x/2} - e^{-3x}$.

21. $r^2 + 16 = 0 \Rightarrow r = \pm 4i$ and the general solution is $y = e^{0x}(c_1 \cos 4x + c_2 \sin 4x) = c_1 \cos 4x + c_2 \sin 4x$. Then $y\left(\frac{\pi}{4}\right) = -3 \Rightarrow -c_1 = -3 \Rightarrow c_1 = 3$ and $y'\left(\frac{\pi}{4}\right) = 4 \Rightarrow -4c_2 = 4 \Rightarrow c_2 = -1$, so the solution to the initial-value problem is $y = 3\cos 4x - \sin 4x$.

22. $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i$ and the general solution is $y = e^x(c_1 \cos 2x + c_2 \sin 2x)$. Then $0 = y(\pi) = e^\pi(c_1 + 0) \Rightarrow c_1 = 0$ and $2 = y'(\pi) = (c_1 + 2c_2)e^\pi \Rightarrow c_2 = 1/e^\pi$ and the solution to the initial-value problem is $y = \dfrac{e^x}{e^\pi} \sin 2x = e^{x-\pi} \sin 2x$.

23. $r^2 + 2r + 2 = 0 \Rightarrow r = -1 \pm i$ and the general solution is $y = e^{-x}(c_1 \cos x + c_2 \sin x)$. Then $2 = y(0) = c_1$ and $1 = y'(0) = c_2 - c_1 \Rightarrow c_2 = 3$ and the solution to the initial-value problem is $y = e^{-x}(2\cos x + 3\sin x)$.

24. $r^2 + 12r + 36 = (r + 6)^2 = 0 \Rightarrow r = -6$ and the general solution is $y = c_1 e^{-6x} + c_2 x e^{-6x}$. Then $0 = y(1) = c_1 e^{-6} + c_2 e^{-6} \Rightarrow c_1 + c_2 = 0$ and $1 = y'(1) = -6c_1 e^{-6} - 5c_2 e^{-6} \Rightarrow 6c_1 + 5c_2 = -e^6$, so $c_1 = -e^6$ and $c_2 = e^6$. The solution to the initial-value problem is $y = -e^6 e^{-6x} + e^6 x e^{-6x} = (x - 1)e^{6-6x}$.

25. $4r^2 + 1 = 0 \Rightarrow r = \pm\frac{1}{2}i$ and the general solution is $y = c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right)$. Then $3 = y(0) = c_1$ and $-4 = y(\pi) = c_2$, so the solution of the boundary-value problem is $y = 3\cos\left(\frac{1}{2}x\right) - 4\sin\left(\frac{1}{2}x\right)$.

26. $r^2 + 2r = r(2 + r) = 0 \Rightarrow r = 0, r = -2$ and the general solution is $y = c_1 + c_2 e^{-2x}$. Then $1 = y(0) = c_1 + c_2$ and $2 = y(1) = c_1 + c_2 e^{-2}$ so $c_2 = \dfrac{e^2}{1 - e^2}$, $c_1 = \dfrac{1 - 2e^2}{1 - e^2}$. The solution of the boundary-value problem is $y = \dfrac{1 - 2e^2}{1 - e^2} + \dfrac{e^2}{1 - e^2} \cdot e^{-2x}$.

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27. $r^2 - 3r + 2 = (r-2)(r-1) = 0 \Rightarrow r = 1, r = 2$ and the general solution is $y = c_1e^x + c_2e^{2x}$. Then $1 = y(0) = c_1 + c_2$ and $0 = y(3) = c_1e^3 + c_2e^6$ so $c_2 = 1/(1-e^3)$ and $c_1 = e^3/(e^3-1)$. The solution of the boundary-value problem is $y = \dfrac{e^{x+3}}{e^3-1} + \dfrac{e^{2x}}{1-e^3}$.

28. $r^2 + 100 = 0 \Rightarrow r = \pm 10i$ and the general solution is $y = c_1 \cos 10x + c_2 \sin 10x$. But $2 = y(0) = c_1$ and $5 = y(\pi) = c_1$, so there is no solution.

29. $r^2 - 6r + 25 = 0 \Rightarrow r = 3 \pm 4i$ and the general solution is $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $1 = y(0) = c_1$ and $2 = y(\pi) = c_1e^{3\pi} \Rightarrow c_1 = 2/e^{3\pi}$, so there is no solution.

30. $r^2 - 6r + 9 = (r-3)^2 = 0 \Rightarrow r = 3$ and the general solution is $y = c_1e^{3x} + c_2xe^{3x}$. Then $1 = y(0) = c_1$ and $0 = y(1) = c_1e^3 + c_2e^3 \Rightarrow c_2 = -1$. The solution of the boundary-value problem is $y = e^{3x} - xe^{3x}$.

31. $r^2 + 4r + 13 = 0 \Rightarrow r = -2 \pm 3i$ and the general solution is $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$. But $2 = y(0) = c_1$ and $1 = y\left(\frac{\pi}{2}\right) = e^{-\pi}(-c_2)$, so the solution to the boundary-value problem is $y = e^{-2x}(2\cos 3x - e^{\pi} \sin 3x)$.

32. $9r^2 - 18r + 10 = 0 \Rightarrow r = 1 \pm \frac{1}{3}i$ and the general solution is $y = e^x\left(c_1 \cos \frac{x}{3} + c_2 \sin \frac{x}{3}\right)$. Then $0 = y(0) = c_1$ and $1 = y(\pi) = e^{\pi}\left(\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2\right) \Rightarrow c_2 = \dfrac{2}{\sqrt{3}\,e^{\pi}}$. The solution of the boundary-value problem is $y = \dfrac{2e^x}{\sqrt{3}\,e^{\pi}} \sin\left(\dfrac{x}{3}\right) = \dfrac{2}{\sqrt{3}}\, e^{x-\pi} \sin\left(\dfrac{x}{3}\right)$.

33. (a) *Case 1* ($\lambda = 0$): $y'' + \lambda y = 0 \Rightarrow y'' = 0$ which has an auxiliary equation $r^2 = 0 \Rightarrow r = 0 \Rightarrow y = c_1 + c_2x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2L \Rightarrow c_1 = c_2 = 0$. Thus, $y = 0$.

Case 2 ($\lambda < 0$): $y'' + \lambda y = 0$ has auxiliary equation $r^2 = -\lambda \Rightarrow r = \pm\sqrt{-\lambda}$ (distinct and real since $\lambda < 0$) $\Rightarrow y = c_1e^{\sqrt{-\lambda}x} + c_2e^{-\sqrt{-\lambda}x}$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1 + c_2$ $(*)$ and $0 = y(L) = c_1e^{\sqrt{-\lambda}L} + c_2e^{-\sqrt{-\lambda}L}$ $(\dagger)$.

Multiplying $(*)$ by $e^{\sqrt{-\lambda}L}$ and subtracting $(\dagger)$ gives $c_2\left(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right) = 0 \Rightarrow c_2 = 0$ and thus $c_1 = 0$ from $(*)$. Thus, $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) $y'' + \lambda y = 0$ has an auxiliary equation $r^2 + \lambda = 0 \Rightarrow r = \pm i\sqrt{\lambda} \Rightarrow y = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 \sin \sqrt{\lambda}L$ since $c_1 = 0$. Since we cannot have a trivial solution, $c_2 \neq 0$ and thus $\sin \sqrt{\lambda}\,L = 0 \Rightarrow \sqrt{\lambda}\,L = n\pi$ where n is an integer $\Rightarrow \lambda = n^2\pi^2/L^2$ and $y = c_2 \sin(n\pi x/L)$ where n is an integer.

34. The auxiliary equation is $ar^2 + br + c = 0$. If $b^2 - 4ac > 0$, then any solution is of the form $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$ where $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. But a, b, and c are all positive so both r_1 and r_2 are negative and $\lim_{x\to\infty} y(x) = 0$. If $b^2 - 4ac = 0$, then any solution is of the form $y(x) = c_1e^{rx} + c_2xe^{rx}$ where $r = -b/(2a) < 0$ since a, b are positive. Hence $\lim_{x\to\infty} y(x) = 0$. Finally if $b^2 - 4ac < 0$, then any solution is of the form $y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ where $\alpha = -b/(2a) < 0$ since a and b are positive. Thus $\lim_{x\to\infty} y(x) = 0$.

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18.2 Nonhomogeneous Linear Equations ET 17.2

1. The auxiliary equation is $r^2 + 3r + 2 = (r + 2)(r + 1) = 0$, so the complementary solution is $y_c(x) = c_1e^{-2x} + c_2e^{-x}$. We try the particular solution $y_p(x) = Ax^2 + Bx + C$, so $y_p' = 2Ax + B$ and $y_p'' = 2A$. Substituting into the differential equation, we have $(2A) + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$ or $2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2$. Comparing coefficients gives $2A = 1$, $6A + 2B = 0$, and $2A + 3B + 2C = 0$, so $A = \frac{1}{2}$, $B = -\frac{3}{2}$, and $C = \frac{7}{4}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1e^{-2x} + c_2e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$.

2. The auxiliary equation is $r^2 + 9 = 0$ with roots $r = \pm 3i$, so the complementary solution is $y_c(x) = c_1\cos(3x) + c_2\sin(3x)$. Try the particular solution $y_p(x) = Ae^{3x}$, so $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substitution into the differential equation gives $9Ae^{3x} + 9(Ae^{3x}) = e^{3x}$ or $18Ae^{3x} = e^{3x}$. Thus $A = \frac{1}{18}$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1\cos(3x) + c_2\sin(3x) + \frac{1}{18}e^{3x}$.

3. The auxiliary equation is $r^2 - 2r = r(r - 2) = 0$, so the complementary solution is $y_c(x) = c_1 + c_2e^{2x}$. Try the particular solution $y_p(x) = A\cos 4x + B\sin 4x$, so $y_p' = -4A\sin 4x + 4B\cos 4x$ and $y_p'' = -16A\cos 4x - 16B\sin 4x$. Substitution into the differential equation gives $(-16A\cos 4x - 16B\sin 4x) - 2(-4A\sin 4x + 4B\cos 4x) = \sin 4x \Rightarrow$ $(-16A - 8B)\cos 4x + (8A - 16B)\sin 4x = \sin 4x$. Then $-16A - 8B = 0$ and $8A - 16B = 1 \Rightarrow A = \frac{1}{40}$ and $B = -\frac{1}{20}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1 + c_2e^{2x} + \frac{1}{40}\cos 4x - \frac{1}{20}\sin 4x$.

4. The auxiliary equation is $r^2 + 6r + 9 = (r + 3)^2 = 0$, so the complementary solution is $y_c(x) = c_1e^{-3x} + c_2xe^{-3x}$. Try the particular solution $y_p(x) = Ax + B$, so $y_p' = A$ and $y_p'' = 0$. Substitution into the differential equation gives $0 + 6A + 9(Ax + B) = 1 + x$ or $(9A)x + (6A + 9B) = 1 + x$. Comparing coefficients, we have $9A = 1$ and $6A + 9B = 1$, so $A = \frac{1}{9}$ and $B = \frac{1}{27}$. Thus the general solution is $y(x) = c_1e^{-3x} + c_2xe^{-3x} + \frac{1}{9}x + \frac{1}{27}$.

5. The auxiliary equation is $r^2 - 4r + 5 = 0$ with roots $r = 2 \pm i$, so the complementary solution is $y_c(x) = e^{2x}(c_1\cos x + c_2\sin x)$. Try $y_p(x) = Ae^{-x}$, so $y_p' = -Ae^{-x}$ and $y_p'' = Ae^{-x}$. Substitution gives $Ae^{-x} - 4(-Ae^{-x}) + 5(Ae^{-x}) = e^{-x} \Rightarrow 10Ae^{-x} = e^{-x} \Rightarrow A = \frac{1}{10}$. Thus the general solution is $y(x) = e^{2x}(c_1\cos x + c_2\sin x) + \frac{1}{10}e^{-x}$.

6. $y_c(x) = e^{-x}(c_1x + c_2)$. Try $y_p(x) = x^2(Ax + B)e^{-x}$ so that no term in y_p is a solution of the complementary equation. Then $y_p' = [-Ax^3 + (3A - B)x^2 + 2Bx]e^{-x}$, $y_p'' = [Ax^3 + (B - 6A)x^2 + (6A - 4B)x + 2B]e^{-x}$ and substitution gives $[Ax^3 + (B - 6A)x^2 + (6A - 4B)x + 2B] + 2[-Ax^3 + (3A - B)x^2 + 2Bx] + (Ax^3 + Bx^2) = x \Rightarrow$ $6Ax + 2B = x$. So $y_p(x) = x^2\left(\frac{1}{6}x\right)e^{-x}$ and the general solution is $y(x) = e^{-x}(c_1x + c_2) + \frac{1}{6}x^3e^{-x}$.

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7. The auxiliary equation is $r^2 + 1 = 0$ with roots $r = \pm i$, so the complementary solution is $y_c(x) = c_1 \cos x + c_2 \sin x$. For $y'' + y = e^x$ try $y_{p_1}(x) = Ae^x$. Then $y'_{p_1} = y''_{p_1} = Ae^x$ and substitution gives $Ae^x + Ae^x = e^x \Rightarrow A = \frac{1}{2}$, so $y_{p_1}(x) = \frac{1}{2}e^x$. For $y'' + y = x^3$ try $y_{p_2}(x) = Ax^3 + Bx^2 + Cx + D$. Then $y'_{p_2} = 3Ax^2 + 2Bx + C$ and $y''_{p_2} = 6Ax + 2B$. Substituting, we have $6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3$, so $A = 1$, $B = 0$, $6A + C = 0 \Rightarrow C = -6$, and $2B + D = 0 \Rightarrow D = 0$. Thus $y_{p_2}(x) = x^3 - 6x$ and the general solution is $y(x) = y_c(x) + y_{p_1}(x) + y_{p_2}(x) = c_1 \cos x + c_2 \sin x + \frac{1}{2}e^x + x^3 - 6x$. But $2 = y(0) = c_1 + \frac{1}{2} \Rightarrow c_1 = \frac{3}{2}$ and $0 = y'(0) = c_2 + \frac{1}{2} - 6 \Rightarrow c_2 = \frac{11}{2}$. Thus the solution to the initial-value problem is $y(x) = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{1}{2}e^x + x^3 - 6x$.

8. The auxiliary equation is $r^2 - 4 = 0$ with roots $r = \pm 2$, so the complementary solution is $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$. Try $y_p(x) = e^x(A\cos x + B\sin x)$, so $y'_p = e^x(A\cos x + B\sin x + B\cos x - A\sin x)$ and $y''_p = e^x(2B\cos x - 2A\sin x)$. Substitution gives $e^x(2B\cos x - 2A\sin x) - 4e^x(A\cos x + B\sin x) = e^x\cos x \Rightarrow$ $(2B - 4A)e^x\cos x + (-2A - 4B)e^x\sin x = e^x\cos x \Rightarrow A = -\frac{1}{5}$, $B = \frac{1}{10}$. Thus the general solution is $y(x) = c_1e^{2x} + c_2e^{-2x} + e^x\left(-\frac{1}{5}\cos x + \frac{1}{10}\sin x\right)$. But $1 = y(0) = c_1 + c_2 - \frac{1}{5}$ and $2 = y'(0) = 2c_1 - 2c_2 - \frac{1}{10}$. Then $c_1 = \frac{9}{8}$, $c_2 = \frac{3}{40}$, and the solution to the initial-value problem is $y(x) = \frac{9}{8}e^{2x} + \frac{3}{40}e^{-2x} + e^x\left(-\frac{1}{5}\cos x + \frac{1}{10}\sin x\right)$.

9. The auxiliary equation is $r^2 - r = 0$ with roots $r = 0$, $r = 1$ so the complementary solution is $y_c(x) = c_1 + c_2e^x$. Try $y_p(x) = x(Ax + B)e^x$ so that no term in y_p is a solution of the complementary equation. Then $y'_p = (Ax^2 + (2A + B)x + B)e^x$ and $y''_p = (Ax^2 + (4A + B)x + (2A + 2B))e^x$. Substitution into the differential equation gives $(Ax^2 + (4A + B)x + (2A + 2B))e^x - (Ax^2 + (2A + B)x + B)e^x = xe^x \Rightarrow$ $(2Ax + (2A + B))e^x = xe^x \Rightarrow A = \frac{1}{2}$, $B = -1$. Thus $y_p(x) = \left(\frac{1}{2}x^2 - x\right)e^x$ and the general solution is $y(x) = c_1 + c_2e^x + \left(\frac{1}{2}x^2 - x\right)e^x$. But $2 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_2 - 1$, so $c_2 = 2$ and $c_1 = 0$. The solution to the initial-value problem is $y(x) = 2e^x + \left(\frac{1}{2}x^2 - x\right)e^x = e^x\left(\frac{1}{2}x^2 - x + 2\right)$.

10. $y_c(x) = c_1e^x + c_2e^{-2x}$. For $y'' + y' - 2y = x$ try $y_{p_1}(x) = Ax + B$. Then $y'_{p_1} = A$, $y''_{p_1} = 0$, and substitution gives $0 + A - 2(Ax + B) = x \Rightarrow A = -\frac{1}{2}$, $B = -\frac{1}{4}$, so $y_{p_1}(x) = -\frac{1}{2}x - \frac{1}{4}$. For $y'' + y' - 2y = \sin 2x$ try $y_{p_2}(x) = A\cos 2x + B\sin 2x$. Then $y'_{p_2} = -2A\sin 2x + 2B\cos 2x$, $y''_{p_2} = -4A\cos 2x - 4B\sin 2x$, and substitution gives $(-4A\cos 2x - 4B\sin 2x) + (-2A\sin 2x + 2B\cos 2x) - 2(A\cos 2x + B\sin 2x) = \sin 2x \Rightarrow A = -\frac{1}{20}$, $B = -\frac{3}{20}$. Thus $y_{p_2}(x) = -\frac{1}{20}\cos 2x + -\frac{3}{20}\sin 2x$ and the general solution is $y(x) = c_1e^x + c_2e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$. But $1 = y(0) = c_1 + c_2 - \frac{1}{4} - \frac{1}{20}$ and $0 = y'(0) = c_1 - 2c_2 - \frac{1}{2} - \frac{3}{10} \Rightarrow c_1 = \frac{17}{15}$ and $c_2 = \frac{1}{6}$. Thus the solution to the initial-value problem is $y(x) = \frac{17}{15}e^x + \frac{1}{6}e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$.

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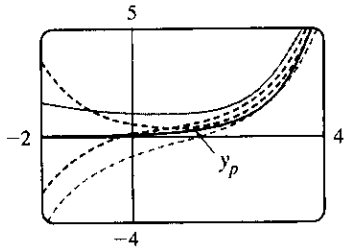
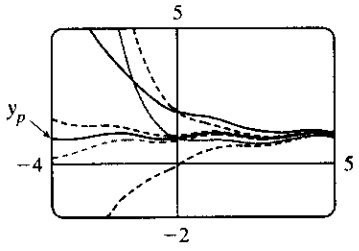
11. $y_c(x) = c_1e^{-x/4} + c_2e^{-x}$. Try $y_p(x) = Ae^x$. Then $10Ae^x = e^x$, so $A = \frac{1}{10}$ and the general solution is $y(x) = c_1e^{-x/4} + c_2e^{-x} + \frac{1}{10}e^x$. The solutions are all composed of exponential curves and with the exception of the particular solution (which approaches 0 as $x \to -\infty$), they all approach either ∞ or $-\infty$ as $x \to -\infty$. As $x \to \infty$, all solutions are asymptotic to $y_p = \frac{1}{10}e^x$.



12. The auxiliary equation is $(2r + 1)(r + 1) = 0$, so $r = -1, -\frac{1}{2}$ and $y_c(x) = c_1e^{-x} + c_2e^{-x/2}$. For $2y'' + 3y' + y = 1$, try $y_{p_1}(x) = A$; substituting gives $y_{p_1}(x) = 1$. For $2y'' + 3y' + y = \cos 2x$ try $y_{p_2} = A\cos 2x + B\sin 2x \quad \Rightarrow \quad y'_{p_2} = -2A\sin 2x + 2B\cos 2x$, $y''_{p_2} = -4A\cos 2x - 4B\sin 2x$. Substituting into the differential equation gives $\cos 2x = (6B - 7A)\cos 2x + (-7B - 6A)\sin 2x$. Then solving the equations $6B - 7A = 1$ and $-7B - 6A = 0$ gives $A = -\frac{7}{85}$, $B = \frac{6}{85}$. Thus, $y_{p_2}(x) = -\frac{7}{85}\cos 2x + \frac{6}{85}\sin 2x$ and the general solution is $y(x) = c_1e^{-x} + c_2e^{-x/2} + 1 - \frac{7}{85}\cos 2x + \frac{6}{85}\sin 2x$.

The graph shows $y_p = y_{p_1} + y_{p_2}$ and several other solutions. Notice that all solutions are asymptotic to y_p as $x \to \infty$.



13. Here $y_c(x) = c_1\cos 3x + c_2\sin 3x$. For $y'' + 9y = e^{2x}$ try $y_{p_1}(x) = Ae^{2x}$ and for $y'' + 9y = x^2\sin x$ try $y_{p_2}(x) = (Bx^2 + Cx + D)\cos x + (Ex^2 + Fx + G)\sin x$. Thus a trial solution is $y_p(x) = y_{p_1}(x) + y_{p_2}(x) = Ae^{2x} + (Bx^2 + Cx + D)\cos x + (Ex^2 + Fx + G)\sin x$.

14. Since $y_c(x) = c_1 + c_2e^{-9x}$, try $y_p(x) = (Ax + B)e^{-x}\cos \pi x + (Cx + D)e^{-x}\sin \pi x$.

15. Here $y_c(x) = c_1 + c_2e^{-9x}$. For $y'' + 9y' = 1$ try $y_{p_1}(x) = Ax$ (since $y = A$ is a solution to the complementary equation) and for $y'' + 9y' = xe^{9x}$ try $y_{p_2}(x) = (Bx + C)e^{9x}$.

16. Since $y_c(x) = c_1e^x + c_2e^{-4x}$ try $y_p(x) = x(Ax^3 + Bx^2 + Cx + D)e^x$ so that no term of $y_p(x)$ satisfies the complementary equation.

17. Since $y_c(x) = e^{-x}(c_1\cos 3x + c_2\sin 3x)$ we try $y_p(x) = x(Ax^2 + Bx + C)e^{-x}\cos 3x + x(Dx^2 + Ex + F)e^{-x}\sin 3x$ (so that no term of y_p is a solution of the complementary equation).

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18. Here $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. For $y'' + 4y = e^{3x}$ try $y_{p_1}(x) = Ae^{3x}$ and for $y'' + 4y = x \sin 2x$ try $y_{p_2}(x) = x(Bx + C)\cos 2x + x(Dx + E)\sin 2x$ (so that no term of y_{p_2} is a solution of the complementary equation).

Note: Solving Equations (7) and (9) in The Method of Variation of Parameters gives

$$u_1' = -\frac{Gy_2}{a\,(y_1y_2' - y_2y_1')} \qquad \text{and} \qquad u_2' = \frac{Gy_1}{a\,(y_1y_2' - y_2y_1')}$$

We will use these equations rather than resolving the system in each of the remaining exercises in this section.

19. (a) The complementary solution is $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. A particular solution is of the form $y_p(x) = Ax + B$. Thus, $4Ax + 4B = x \;\Rightarrow\; A = \frac{1}{4}$ and $B = 0 \;\Rightarrow\; y_p(x) = \frac{1}{4}x$. Thus, the general solution is $y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x$.

(b) In (a), $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$, so set $y_1 = \cos 2x$, $y_2 = \sin 2x$. Then $y_1y_2' - y_2y_1' = 2\cos^2 2x + 2\sin^2 2x = 2$ so $u_1' = -\frac{1}{2}x \sin 2x \;\Rightarrow$ $u_1(x) = -\frac{1}{2}\int x \sin 2x\,dx = -\frac{1}{4}\left(-x\cos 2x + \frac{1}{2}\sin 2x\right)$ [by parts] and $u_2' = \frac{1}{2}x\cos 2x$ $\Rightarrow \quad u_2(x) = \frac{1}{2}\int x\cos 2x dx = \frac{1}{4}\left(x\sin 2x + \frac{1}{2}\cos 2x\right)$ [by parts]. Hence $y_p(x) = -\frac{1}{4}\left(-x\cos 2x + \frac{1}{2}\sin 2x\right)\cos 2x + \frac{1}{4}\left(x\sin 2x + \frac{1}{2}\cos 2x\right)\sin 2x = \frac{1}{4}x$. Thus $y(x) = y_c(x) + y_p(x) = c_1\cos 2x + c_2 \sin 2x + \frac{1}{4}x$.

20. (a) Here $r^2 - 3r + 2 = 0 \;\Rightarrow\; r = 1$ or 2 and $y_c(x) = c_1e^{2x} + c_2e^x$. We try a particular solution of the form $y_p(x) = A\cos x + B\sin x \;\Rightarrow\; y_p' = -A\sin x + B\cos x$ and $y_p'' = -A\cos x - B\sin x$. Then the equation $y'' - 3y' + 2y = \sin x$ becomes $(A - 3B)\cos x + (B + 3A)\sin x = \sin x \;\Rightarrow\; A - 3B = 0$ and $B + 3A = 1 \;\Rightarrow\; A = \frac{3}{10}$ and $B = \frac{1}{10}$. Thus, $y_p(x) = \frac{3}{10}\cos x + \frac{1}{10}\sin x$. Therefore, the general solution is $y(x) = y_c(x) + y_p(x) = c_1e^{2x} + c_2e^x + \frac{3}{10}\cos x + \frac{1}{10}\sin x$.

(b) From (a) we know that $y_c(x) = c_1e^{2x} + c_2e^x$. Setting $y_1 = e^{2x}$, $y_2 = e^x$, we have $y_1y_2' - y_2y_1' = e^{3x} - 2e^{3x} = -e^{3x}$. Thus $u_1' = -\dfrac{\sin x e^x}{-e^{3x}} = \sin x e^{-2x}$ and $u_2' = \dfrac{\sin x e^{2x}}{-e^{3x}} = -\sin x e^{-x}$. Then $u_1(x) = \int e^{-2x}\sin x\,dx = \frac{1}{5}e^{-2x}(-2\sin x - \cos x)$ [by parts] and $u_2(x) = -\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(-\sin x - \cos x)$. Thus $y_p(x) = \frac{1}{5}(-2\sin x - \cos x) + \frac{1}{2}(\sin x + \cos x) = \frac{1}{10}\sin x + \frac{3}{10}\cos x$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1e^{2x} + c_2e^x + \frac{1}{10}\sin x + \frac{3}{10}\cos x$.

21. (a) $r^2 - r = r(r-1) = 0 \;\Rightarrow\; r = 0, 1$, so the complementary solution is $y_c(x) = c_1e^x + c_2xe^x$. A particular solution is of the form $y_p(x) = Ae^{2x}$. Thus $4Ae^{2x} - 4Ae^{2x} + Ae^{2x} = e^{2x} \;\Rightarrow\; Ae^{2x} = e^{2x} \;\Rightarrow\; A = 1$ $\Rightarrow\; y_p(x) = e^{2x}$. So a general solution is $y(x) = y_c(x) + y_p(x) = c_1e^x + c_2xe^x + e^{2x}$.

(b) From (a), $y_c(x) = c_1e^x + c_2xe^x$, so set $y_1 = e^x$, $y_2 = xe^x$. Then, $y_1y_2' - y_2y_1' = e^{2x}(1 + x) - xe^{2x} = e^{2x}$ and so $u_1' = -xe^x \;\Rightarrow\; u_1(x) = -\int xe^x\,dx = -(x-1)e^x$ [by parts] and $u_2' = e^x \;\Rightarrow$ $u_2(x) = \int e^x\,dx = e^x$. Hence $y_p(x) = (1-x)e^{2x} + xe^{2x} = e^{2x}$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1e^x + c_2xe^x + e^{2x}$.

22. (a) Here $r^2 - 2r + 1 = (r+1)^2 = 0 \;\Rightarrow\; r = -1$ and $y_c(x) = c_1 + c_2e^x$ and so we try a particular solution of the form $y_p(x) = Axe^x$. Thus, after calculating the necessary derivatives, we get $y'' - y' = e^x \;\Rightarrow$ $Ae^x(2 + x) - Ae^x(1 + x) = e^x \;\Rightarrow\; A = 1$. Thus $y_p(x) = xe^x$ and the general solution is $y(x) = c_1 + c_2e^x + xe^x$.

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(b) From (a) we know that $y_c(x) = c_1 + c_2e^x$, so setting $y_1 = 1$, $y_2 = e^x$, then $y_1y_2' - y_2y_1' = e^x - 0 = e^x$. Thus $u_1' = -e^{2x}/e^x = -e^x$ and $u_2' = e^x/e^x = 1$. Then $u_1(x) = -\int e^x dx = -e^x$ and $u_2(x) = x$. Thus $y_p(x) = -e^x + xe^x$ and the general solution is $y(x) = c_1 + c_2e^x - e^x + xe^x = c_1 + c_3e^x + xe^x$.

23. As in Example 6, $y_c(x) = c_1 \sin x + c_2 \cos x$, so set $y_1 = \sin x$, $y_2 = \cos x$. Then

$y_1y_2' - y_2y_1' = -\sin^2 x - \cos^2 x = -1$, so $u_1' = -\dfrac{\sec x \cos x}{-1} = 1 \quad \Rightarrow \quad u_1(x) = x$ and

$u_2' = \dfrac{\sec x \sin x}{-1} = -\tan x \quad \Rightarrow \quad u_2(x) = -\int \tan x dx = \ln|\cos x| = \ln(\cos x)$ on $0 < x < \frac{\pi}{2}$. Hence

$y_p(x) = x \sin x + \cos x \ln(\cos x)$ and the general solution is $y(x) = (c_1 + x)\sin x + [c_2 + \ln(\cos x)]\cos x$.

24. Setting $y_1 = \sin x$, $y_2 = \cos x$, then $y_1y_2' - y_2y_1' = -\sin^2 x - \cos^2 x = -1$. Thus $u_1' = -\dfrac{\cot x \cos x}{-1} = \dfrac{\cos^2 x}{\sin x}$

and $u_2' = \dfrac{\cot x \sin x}{-1} = -\cos x$. Then $u_1(x) = \displaystyle\int \frac{\cos^2 x}{\sin x}\, dx = \int (\csc x - \sin x)\, dx = \ln(\csc x - \cot x) + \cos x$

and $u_2(x) = -\sin x$. Thus $y_p(x) = [\cos x + \ln(\csc x - \cot x)]\sin x + (-\sin x)(\cos x)$ and the general solution is $y(x) = c_1 \sin x + c_2 \cos x + \sin x \ln(\csc x - \cot x)$.

25. $y_1 = e^x$, $y_2 = e^{2x}$ and $y_1y_2' - y_2y_1' = e^{3x}$. So $u_1' = \dfrac{-e^{2x}}{(1 + e^{-x})e^{3x}} = -\dfrac{e^{-x}}{1 + e^{-x}}$ and

$u_1(x) = \displaystyle\int -\frac{e^{-x}}{1 + e^{-x}}\, dx = \ln(1 + e^{-x})$. $u_2' = \dfrac{e^x}{(1 + e^{-x})e^{3x}} = \dfrac{e^x}{e^{3x} + e^{2x}}$ so

$u_2(x) = \displaystyle\int \frac{e^x}{e^{3x} + e^{2x}}\, dx = \ln\left(\frac{e^x + 1}{e^x}\right) - e^{-x} = \ln(1 + e^{-x}) - e^{-x}$. Hence

$y_p(x) = e^x \ln(1 + e^{-x}) + e^{2x}[\ln(1 + e^{-x}) - e^{-x}]$ and the general solution is

$y(x) = [c_1 + \ln(1 + e^{-x})]e^x + [c_2 - e^{-x} + \ln(1 + e^{-x})]e^{2x}$.

26. $y_1 = e^{-x}$, $y_2 = e^{-2x}$ and $y_1y_2' - y_2y_1' = -e^{-3x}$. So $u_1' = -\dfrac{(\sin e^x)e^{-2x}}{-e^{-3x}} = e^x \sin e^x$ and

$u_2' = \dfrac{(\sin e^x)e^{-x}}{-e^{-3x}} = -e^{2x} \sin e^x$. Hence $u_1(x) = \int e^x \sin e^x dx = -\cos e^x$ and

$u_2(x) = \int -e^{2x} \sin e^x dx = e^x \cos e^x - \sin e^x$. Then $y_p(x) = -e^{-x} \cos e^x - e^{-2x}[\sin e^x - e^x \cos e^x]$ and the general solution is $y(x) = (c_1 - \cos e^x)e^{-x} + [c_2 - \sin e^x + e^x \cos e^x]e^{-2x}$.

27. $y_1 = e^{-x}$, $y_2 = e^x$ and $y_1y_2' - y_2y_1' = 2$. So $u_1' = -\dfrac{e^x}{2x}$, $u_2' = \dfrac{e^{-x}}{2x}$ and

$y_p(x) = -e^{-x} \displaystyle\int \frac{e^x}{2x}\, dx + e^x \int \frac{e^{-x}}{2x}\, dx$. Hence the general solution is

$y(x) = \left(c_1 - \displaystyle\int \frac{e^x}{2x}\, dx\right)e^{-x} + \left(c_2 + \int \frac{e^{-x}}{2x}\, dx\right)e^x$.

28. $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$ and $y_1y_2' - y_2y_1' = e^{-4x}$. Then $u_1' = \dfrac{-e^{-2x}xe^{-2x}}{x^3e^{-4x}} = -\dfrac{1}{x^2}$ so $u_1(x) = x^{-1}$ and

$u_2' = \dfrac{e^{-2x}e^{-2x}}{x^3e^{-4x}} = \dfrac{1}{x^3}$ so $u_2(x) = -\dfrac{1}{2x^2}$. Thus $y_p(x) = \dfrac{e^{-2x}}{x} - \dfrac{xe^{-2x}}{2x^2} = \dfrac{e^{-2x}}{2x}$ and the general solution is

$y(x) = e^{-2x}[c_1 + c_2x + 1/(2x)]$.

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18.3 Applications of Second-Order Differential Equations ET 17.3

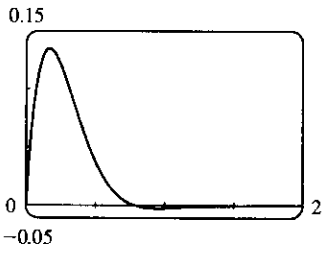
1. By Hooke's Law $k(0.6) = 20$ so $k = \frac{100}{3}$ is the spring constant and the differential equation is $3x'' + \frac{100}{3}x = 0$. The general solution is $x(t) = c_1 \cos\left(\frac{10}{3}t\right) + c_2 \sin\left(\frac{10}{3}t\right)$. But $0 = x(0) = c_1$ and $1.2 = x'(0) = \frac{10}{3}c_2$, so the position of the mass after t seconds is $x(t) = 0.36 \sin\left(\frac{10}{3}t\right)$.

2. $k(0.3) = 24.3$ or $k = 81$ is the spring constant and the resulting initial-value problem is $4x'' + 81x = 0$, $x(0) = -0.5$ (since compressed), $x'(0) = 0$. The general solution is $x(t) = c_1 \cos\left(\frac{9}{2}t\right) + c_2 \sin\left(\frac{9}{2}t\right)$. But $-0.2 = x(0) = c_1$ and $0 = x'(0) = \frac{9}{2}c_2$. Thus the position is given by $x(t) = -0.2\cos(4.5t)$.

3. $k(0.5) = 6$ or $k = 12$ is the spring constant, so the initial-value problem is $2x'' + 14x' + 12x = 0$, $x(0) = 1$, $x'(0) = 0$. The general solution is $x(t) = c_1 e^{-6t} + c_2 e^{-t}$. But $1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -6c_1 - c_2$. Thus the position is given by $x(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$.

4. (a) The differential equation is $3x'' + 30x' + 123x = 0$ with general solution $x(t) = e^{-5t}(c_1 \cos 4t + c_2 \sin 4t)$. Then $0 = x(0) = c_1$ and $2 = x'(0) = 4c_2$, so the position is given by $x(t) = \frac{1}{2}e^{-5t} \sin 4t$.

(b)



5. For critical damping we need $c^2 - 4mk = 0$ or $m = c^2/(4k) = 14^2/(4 \cdot 12) = \frac{49}{12}$ kg.

6. For critical damping we need $c^2 = 4mk$ or $c = 2\sqrt{mk} = 2\sqrt{3 \cdot 123} = 6\sqrt{41}$.

7. We are given $m = 1$, $k = 100$, $x(0) = -0.1$ and $x'(0) = 0$. From (3), the differential equation is $\frac{d^2x}{dt^2} + c\frac{dx}{dt} + 100x = 0$ with auxiliary equation $r^2 + cr + 100 = 0$. If $c = 10$, we have two complex roots $r = -5 \pm 5\sqrt{3}i$, so the motion is underdamped and the solution is $x = e^{-5t}\left[c_1 \cos\left(5\sqrt{3}\,t\right) + c_2 \sin\left(5\sqrt{3}\,t\right)\right]$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = 5\sqrt{3}\,c_2 - 5c_1 \quad \Rightarrow \quad c_2 = -\frac{1}{10\sqrt{3}}$, so $x = e^{-5t}\left[-0.1\cos\left(5\sqrt{3}\,t\right) - \frac{1}{10\sqrt{3}}\sin\left(5\sqrt{3}\,t\right)\right]$. If $c = 15$, we again have underdamping since the auxiliary equation has roots $r = -\frac{15}{2} \pm \frac{5\sqrt{7}}{2}i$. The general solution is $x = e^{-15t/2}\left[c_1 \cos\left(\frac{5\sqrt{7}}{2}t\right) + c_2 \sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$, so $-0.1 = x(0) = c_1$ and $0 = x'(0) = \frac{5\sqrt{7}}{2}c_2 - \frac{15}{2}c_1 \quad \Rightarrow \quad c_2 = -\frac{3}{10\sqrt{7}}$. Thus $x = e^{-15t/2}\left[-0.1\cos\left(\frac{5\sqrt{7}}{2}t\right) - \frac{3}{10\sqrt{7}}\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$. For $c = 20$, we have equal roots $r_1 = r_2 = -10$, so the oscillation is critically damped and the solution is $x = (c_1 + c_2 t)e^{-10t}$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = -10c_1 + c_2 \quad \Rightarrow \quad c_2 = -1$, so $x = (-0.1 - t)e^{-10t}$. If $c = 25$ the auxiliary equation has roots $r_1 = -5$, $r_2 = -20$, so we have overdamping and the solution is $x = c_1 e^{-5t} + c_2 e^{-20t}$. Then $-0.1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -5c_1 - 20c_2 \quad \Rightarrow \quad c_1 = -\frac{2}{15}$ and $c_2 = \frac{1}{30}$,

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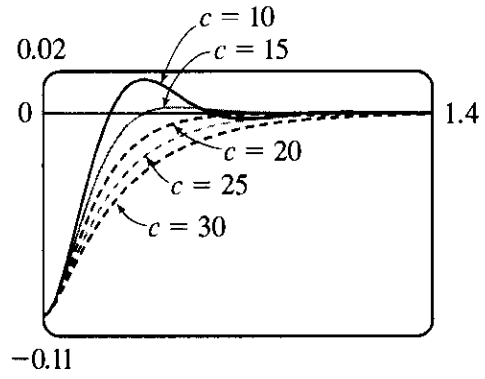
so $x = -\frac{2}{15}e^{-5t} + \frac{1}{30}e^{-20t}$. If $c = 30$ we have roots $r = -15 \pm 5\sqrt{5}$, so the motion is overdamped and the solution is $x = c_1 e^{(-15+5\sqrt{5})t} + c_2 e^{(-15-5\sqrt{5})t}$. Then $-0.1 = x(0) = c_1 + c_2$ and

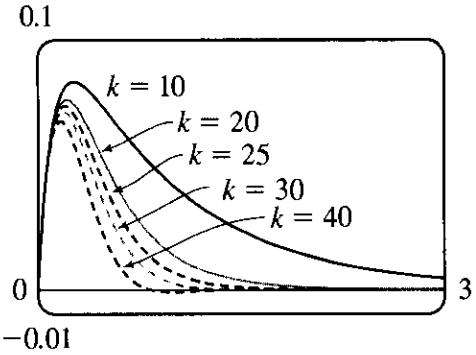
$0 = x'(0) = (-15 + 5\sqrt{5})\,c_1 + (-15 - 5\sqrt{5})\,c_2 \quad \Rightarrow$

$c_1 = \frac{-5-3\sqrt{5}}{100}$ and $c_2 = \frac{-5+3\sqrt{5}}{100}$, so

$x = \left(\frac{-5-3\sqrt{5}}{100}\right) e^{(-15+5\sqrt{5})t} + \left(\frac{-5+3\sqrt{5}}{100}\right) e^{(-15-5\sqrt{5})t}$.



8. We are given $m = 1$, $c = 10$, $x(0) = 0$ and $x'(0) = 1$. The differential equation is $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + kx = 0$ with auxiliary equation $r^2 + 10r + k = 0$. $k = 10$: the auxiliary equation has roots $r = -5 \pm \sqrt{15}$ so we have overdamping and the solution is $x = c_1 e^{(-5+\sqrt{15})t} + c_2 e^{(-5-\sqrt{15})t}$. Entering the initial conditions gives $c_1 = \frac{1}{2\sqrt{15}}$ and $c_2 = -\frac{1}{2\sqrt{15}}$, so $x = \frac{1}{2\sqrt{15}}\,e^{(-5+\sqrt{15})t} - \frac{1}{2\sqrt{15}}e^{(-5-\sqrt{15})t}$. $k = 20$: $r = -5 \pm \sqrt{5}$ and the solution is $x = c_1 e^{(-5+\sqrt{5})t} + c_2 e^{(-5-\sqrt{5})t}$ so again the motion is overdamped. The initial conditions give $c_1 = \frac{1}{2\sqrt{5}}$ and $c_2 = -\frac{1}{2\sqrt{5}}$, so $x = \frac{1}{2\sqrt{5}}e^{(-5+\sqrt{5})t} - \frac{1}{2\sqrt{5}}e^{(-5-\sqrt{5})t}$. $k = 25$: we have equal roots $r_1 = r_2 = -5$, so the motion is critically damped and the solution is $x = (c_1 + c_2 t)e^{-5t}$. The initial conditions give $c_1 = 0$ and $c_2 = 1$, so $x = te^{-5t}$. $k = 30$: $r = -5 \pm \sqrt{5}\,i$ so the motion is underdamped and the solution is $x = e^{-5t}\left[c_1 \cos\left(\sqrt{5}\,t\right) + c_2 \sin\left(\sqrt{5}\,t\right)\right]$. The initial conditions give $c_1 = 0$ and $c_2 = \frac{1}{\sqrt{5}}$, so $x = \frac{1}{\sqrt{5}}\,e^{-5t}\sin\left(\sqrt{5}\,t\right)$. $k = 40$: $r = -5 \pm \sqrt{15}\,i$ so we again have underdamping. The solution is $x = e^{-5t}\left[c_1 \cos\left(\sqrt{15}\,t\right) + c_2 \sin\left(\sqrt{15}\,t\right)\right]$, and the initial conditions give $c_1 = 0$ and $c_2 = \frac{1}{\sqrt{15}}$. Thus $x = \frac{1}{\sqrt{15}}e^{-5t}\sin\left(\sqrt{15}\,t\right)$.



9. The differential equation is $mx'' + kx = F_0 \cos\omega_0 t$ and $\omega_0 \neq \omega = \sqrt{k/m}$. Here the auxiliary equation is $mr^2 + k = 0$ with roots $\pm\sqrt{k/m}\,i = \pm\omega i$ so $x_c(t) = c_1 \cos\omega t + c_2 \sin\omega t$. Since $\omega_0 \neq \omega$, try

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$x_p(t) = A\cos\omega_0 t + B\sin\omega_0 t$. Then we need

$(m)\left(-\omega_0^2\right)(A\cos\omega_0 t + B\sin\omega_0 t) + k(A\cos\omega_0 t + B\sin\omega_0 t) = F_0\cos\omega_0 t$ or $A\left(k - m\omega_0^2\right) = F_0$ and $B\left(k - m\omega_0^2\right) = 0$. Hence $B = 0$ and $A = \dfrac{F_0}{k - m\omega_0^2} = \dfrac{F_0}{m(\omega^2 - \omega_0^2)}$ since $\omega^2 = \dfrac{k}{m}$. Thus the motion of the mass is given by $x(t) = c_1\cos\omega t + c_2\sin\omega t + \dfrac{F_0}{m(\omega^2 - \omega_0^2)}\cos\omega_0 t$.

10. As in Exercise 9, $x_c(t) = c_1\cos\omega t + c_2\sin\omega t$. But the natural frequency of the system equals the frequency of the external force, so try $x_p(t) = t(A\cos\omega t + B\sin\omega t)$. Then we need $m(2\omega B - \omega^2 At)\cos\omega t - m(2\omega A + \omega^2 Bt)\sin\omega t + kAt\cos\omega t + kBt\sin\omega t = F_0\cos\omega t$ or $2m\omega B = F_0$ and $-2m\omega A = 0$ (noting $-m\omega^2 A + kA = 0$ and $-m\omega^2 B + kB = 0$ since $\omega^2 = k/m$). Hence the general solution is $x(t) = c_1\cos\omega t + c_2\sin\omega t + [F_0 t/(2m\omega)]\sin\omega t$.

11. From Equation 6, $x(t) = f(t) + g(t)$ where $f(t) = c_1\cos\omega t + c_2\sin\omega t$ and $g(t) = \dfrac{F_0}{m(\omega^2 - \omega_0^2)}\cos\omega_0 t$. Then f is periodic, with period $\frac{2\pi}{\omega}$, and if $\omega \neq \omega_0$, g is periodic with period $\frac{2\pi}{\omega_0}$. If $\frac{\omega}{\omega_0}$ is a rational number, then we can say $\frac{\omega}{\omega_0} = \frac{a}{b} \;\Rightarrow\; a = \frac{b\omega}{\omega_0}$ where a and b are non-zero integers. Then

$$x\left(t + a\cdot\tfrac{2\pi}{\omega}\right) = f\left(t + a\cdot\tfrac{2\pi}{\omega}\right) + g\left(t + a\cdot\tfrac{2\pi}{\omega}\right) = f(t) + g\left(t + \tfrac{b\omega}{\omega_0}\cdot\tfrac{2\pi}{\omega}\right)$$
$$= f(t) + g\left(t + b\cdot\tfrac{2\pi}{\omega_0}\right) = f(t) + g(t) = x(t)$$

so $x(t)$ is periodic.

12. (a) The graph of $x = c_1 e^{rt} + c_2 te^{rt}$ has a t-intercept when $c_1 e^{rt} + c_2 te^{rt} = 0 \;\Leftrightarrow\; e^{rt}(c_1 + c_2 t) = 0 \;\Leftrightarrow\; c_1 = -c_2 t$. Since $t > 0$, x has a t-intercept if and only if c_1 and c_2 have opposite signs.

(b) For $t > 0$, the graph of x crosses the t-axis when $c_1 e^{r_1 t} + c_2 e^{r_2 t} = 0 \;\Leftrightarrow\; c_2 e^{r_2 t} = -c_1 e^{r_1 t} \;\Leftrightarrow\; c_2 = -c_1\dfrac{e^{r_1 t}}{e^{r_2 t}} = -c_1 e^{(r_1 - r_2)t}$. But $r_1 > r_2 \;\Rightarrow\; r_1 - r_2 > 0$ and since $t > 0$, $e^{(r_1 - r_2)t} > 1$. Thus $|c_2| = |c_1|\,e^{(r_1 - r_2)t} > |c_1|$, and the graph of x can cross the t-axis only if $|c_2| > |c_1|$.

13. Here the initial-value problem for the charge is $Q'' + 20Q' + 500Q = 12$, $Q(0) = Q'(0) = 0$. Then $Q_c(t) = e^{-10t}(c_1\cos 20t + c_2\sin 20t)$ and try $Q_p(t) = A \;\Rightarrow\; 500A = 12$ or $A = \frac{3}{125}$. The general solution is $Q(t) = e^{-10t}(c_1\cos 20t + c_2\sin 20t) + \frac{3}{125}$. But $0 = Q(0) = c_1 + \frac{3}{125}$ and $Q'(t) = I(t) = e^{-10t}[(-10c_1 + 20c_2)\cos 20t + (-10c_2 - 20c_1)\sin 20t]$ but $0 = Q'(0) = -10c_1 + 20c_2$. Thus the charge is $Q(t) = -\frac{1}{250}e^{-10t}(6\cos 20t + 3\sin 20t) + \frac{3}{125}$ and the current is $I(t) = e^{-10t}\left(\frac{3}{5}\right)\sin 20t$.

14. (a) Here the initial-value problem for the charge is $2Q'' + 24Q' + 200Q = 12$ with $Q(0) = 0.001$ and $Q'(0) = 0$. Then $Q_c(t) = e^{-6t}(c_1\cos 8t + c_2\sin 8t)$ and try $Q_p(t) = A \;\Rightarrow\; A = \frac{3}{50}$ and the general solution is $Q(t) = e^{-6t}(c_1\cos 8t + c_2\sin 8t) + \frac{3}{50}$. But $0.001 = Q(0) = c + \frac{3}{50}$ so $c_1 = -0.059$. Also $Q'(t) = I(t) = e^{-6t}[(-6c_1 + 8c_2)\cos 8t + (-6c_2 - 8c_1)\sin 8t]$ and $0 = Q'(0) = -6c_1 + 8c_2$ so

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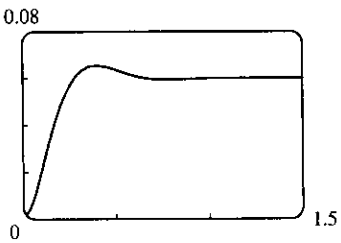
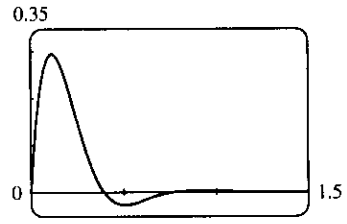
$c_2 = -0.04425$. Hence the charge is $Q(t) = -e^{-6t}(0.059\cos 8t + 0.04425\sin 8t) + \frac{3}{50}$ and the current is $I(t) = e^{-6t}(0.7375)\sin 8t$.

(b)



charge, $Q(t)$

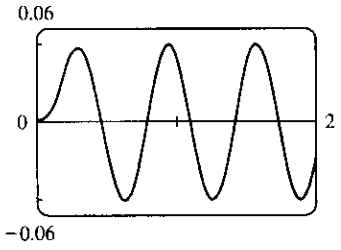


current, $I(t) = Q'(t)$

15. As in Exercise 13, $Q_c(t) = e^{-10t}(c_1\cos 20t + c_2\sin 20t)$ but $E(t) = 12\sin 10t$ so try $Q_p(t) = A\cos 10t + B\sin 10t$. Substituting into the differential equation gives $(-100A + 200B + 500A)\cos 10t + (-100B - 200A + 500B)\sin 10t = 12\sin 10t \quad \Rightarrow \quad 400A + 200B = 0$ and $400B - 200A = 12$. Thus $A = -\frac{3}{250}$, $B = \frac{3}{125}$ and the general solution is $Q(t) = e^{-10t}(c_1\cos 20t + c_2\sin 20t) - \frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t$. But $0 = Q(0) = c_1 - \frac{3}{250}$ so $c_1 = \frac{3}{250}$. Also $Q'(t) = \frac{3}{25}\sin 10t + \frac{6}{25}\cos 10t + e^{-10t}[(-10c_1 + 20c_2)\cos 20t + (-10c_2 - 20c_1)\sin 20t]$ and $0 = Q'(0) = \frac{6}{25} - 10c_1 + 20c_2$ so $c_2 = -\frac{3}{500}$. Hence the charge is given by $Q(t) = e^{-10t}\left[\frac{3}{250}\cos 20t - \frac{3}{500}\sin 20t\right] - \frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t$.

16. (a) As in Exercise 14, $Q_c(t) = e^{-6t}(c_1\cos 8t + c_2\sin 8t)$ but try $Q_p(t) = A\cos 10t + B\sin 10t$. Substituting into the differential equation gives $(-200A + 240B + 200A)\cos 10t + (-200B - 240A + 200B)\sin 10t = 12\sin 10t$, so $B = 0$ and $A = -\frac{1}{20}$. Hence, the general solution is $Q(t) = e^{-6t}(c_1\cos 8t + c_2\sin 8t) - \frac{1}{20}\cos 10t$. But $0.001 = Q(0) = c_1 - \frac{1}{20}$, $Q'(t) = e^{-6t}[(-6c_1 + 8c_2)\cos 8t + (-6c_2 - 8c_1)\sin 8t] - \frac{1}{2}\sin 10t$ and $0 = Q'(0) = -6c_1 + 8c_2$, so $c_1 = 0.051$ and $c_2 = 0.03825$. Thus the charge is given by $Q(t) = e^{-6t}(0.051\cos 8t + 0.03825\sin 8t) - \frac{1}{20}\cos 10t$.

(b)



17. $x(t) = A\cos(\omega t + \delta) \quad \Leftrightarrow \quad x(t) = A[\cos\omega t\cos\delta - \sin\omega t\sin\delta] \quad \Leftrightarrow \quad x(t) = A\left(\frac{c_1}{A}\cos\omega t + \frac{c_2}{A}\sin\omega t\right)$ where $\cos\delta = c_1/A$ and $\sin\delta = -c_2/A \quad \Leftrightarrow \quad x(t) = c_1\cos\omega t + c_2\sin\omega t$. (Note that $\cos^2\delta + \sin^2\delta = 1 \quad \Rightarrow \quad c_1^2 + c_2^2 = A^2$.)

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18. (a) We approximate $\sin\theta$ by θ and, with $L = 1$ and $g = 9.8$, the differential equation becomes $\dfrac{d^2\theta}{dt^2} + 9.8\theta = 0$.

The auxiliary equation is $r^2 + 9.8 = 0 \;\Rightarrow\; r = \pm\sqrt{9.8}\,i$, so the general solution is $\theta(t) = c_1\cos\left(\sqrt{9.8}\,t\right) + c_2\sin\left(\sqrt{9.8}\,t\right)$. Then $0.2 = \theta(0) = c_1$ and $1 = \theta'(0) = \sqrt{9.8}\,c_2 \;\Rightarrow\; c_2 = \frac{1}{\sqrt{9.8}}$, so the equation is $\theta(t) = 0.2\cos\left(\sqrt{9.8}\,t\right) + \frac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}\,t\right)$.

(b) $\theta'(t) = -0.2\sqrt{9.8}\sin\left(\sqrt{9.8}\,t\right) + \cos\left(\sqrt{9.8}\,t\right) = 0$ or $\tan\left(\sqrt{9.8}\,t\right) = \frac{5}{\sqrt{9.8}}$, so the critical numbers are $t = \frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right) + \frac{n}{\sqrt{9.8}}\pi$ (n any integer). The maximum angle from the vertical is $\theta\left(\frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right)\right) \approx 0.377$ radians (or about $21.7°$).

(c) From part (b), the critical numbers of $\theta(t)$ are spaced $\frac{\pi}{\sqrt{9.8}}$ apart, and the time between successive maximum values is $2\left(\frac{\pi}{\sqrt{9.8}}\right)$. Thus the period of the pendulum is $\frac{2\pi}{\sqrt{9.8}} \approx 2.007$ seconds.

(d) $\theta(t) = 0 \;\Rightarrow\; 0.2\cos\left(\sqrt{9.8}\,t\right) + \frac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}\,t\right) = 0 \;\Rightarrow\; \tan\left(\sqrt{9.8}\,t\right) = -0.2\sqrt{9.8} \;\Rightarrow\; t = \frac{1}{\sqrt{9.8}}\left[\tan^{-1}\left(-0.2\sqrt{9.8}\right) + \pi\right] \approx 0.825$ seconds.

(e) $\theta'(0.825) \approx -1.180$ rad/s.

18.4 Series Solutions — ET 17.4

1. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and the given equation, $y' - y = 0$, becomes $\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$. Replacing n by $n+1$ in the first sum gives $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - \sum_{n=0}^{\infty} c_n x^n = 0$, so $\sum_{n=0}^{\infty}[(n+1)c_{n+1} - c_n]x^n = 0$. Equating coefficients gives $(n+1)c_{n+1} - c_n = 0$, so the recursion relation is $c_{n+1} = \dfrac{c_n}{n+1}$, $n = 0, 1, 2, \ldots$. Then $c_1 = c_0$, $c_2 = \frac{1}{2}c_1 = \frac{c_0}{2}$, $c_3 = \frac{1}{3}c_2 = \frac{1}{3}\cdot\frac{1}{2}c_0 = \frac{c_0}{3!}$, $c_4 = \frac{1}{4}c_3 = \frac{c_0}{4!}$, and in general, $c_n = \dfrac{c_0}{n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{c_0}{n!}\,x^n = c_0\sum_{n=0}^{\infty}\frac{x^n}{n!} = c_0 e^x$$

2. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = xy \;\Rightarrow\; y' - xy = 0 \;\Rightarrow\; \sum_{n=1}^{\infty} n c_n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$. Replacing n with $n+1$ in the first sum and n with $n-1$ in the second gives $\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$ or $c_1 + \sum_{n=1}^{\infty}(n+1)\,c_{n+1}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$. Thus,

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$c_1 + \sum_{n=1}^{\infty} [(n+1)c_{n+1} - c_{n-1}]\, x^n = 0$. Equating coefficients gives $c_1 = 0$ and $(n+1)\, c_{n+1} - c_{n-1} = 0$. Thus, the recursion relation is $c_{n+1} = \dfrac{c_{n-1}}{n+1}$, $n = 1, 2, \ldots$. But $c_1 = 0$, so $c_3 = 0$ and $c_5 = 0$ and in general $c_{2n+1} = 0$. Also, $c_2 = \dfrac{c_0}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{c_0}{4 \cdot 2} = \dfrac{c_0}{2^2 \cdot 2!}$, $c_6 = \dfrac{c_4}{6} = \dfrac{c_0}{6 \cdot 4 \cdot 2} = \dfrac{c_0}{2^3 \cdot 3!}$ and in general $c_{2n} = \dfrac{c_0}{2^n \cdot n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{c_0}{2^n \cdot n!}\, x^{2n} = c_0 \sum_{n=0}^{\infty} \frac{\left(x^2/2\right)^n}{n!} = c_0 e^{x^2/2}$$

3. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$ and $-x^2 y = -\sum_{n=0}^{\infty} c_n x^{n+2} = -\sum_{n=2}^{\infty} c_{n-2} x^n$. Hence, the equation $y' = x^2 y$ becomes $\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=2}^{\infty} c_{n-2} x^n = 0$ or $c_1 + 2c_2 x + \sum_{n=2}^{\infty} [(n+1)c_{n+1} - c_{n-2}]\, x^n = 0$. Equating coefficients gives $c_1 = c_2 = 0$ and $c_{n+1} = \dfrac{c_{n-2}}{n+1}$ for $n = 2, 3, \ldots$. But $c_1 = 0$, so $c_4 = 0$ and $c_7 = 0$ and in general $c_{3n+1} = 0$. Similarly $c_2 = 0$ so $c_{3n+2} = 0$. Finally $c_3 = \dfrac{c_0}{3}$, $c_6 = \dfrac{c_3}{6} = \dfrac{c_0}{6 \cdot 3} = \dfrac{c_0}{3^2 \cdot 2!}$, $c_9 = \dfrac{c_6}{9} = \dfrac{c_0}{9 \cdot 6 \cdot 3} = \dfrac{c_0}{3^3 \cdot 3!}$, $\ldots$, and $c_{3n} = \dfrac{c_0}{3^n \cdot n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{3n} x^{3n} = \sum_{n=0}^{\infty} \frac{c_0}{3^n \cdot n!} x^{3n} = c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 \sum_{n=0}^{\infty} \frac{\left(x^3/3\right)^n}{n!} = c_0 e^{x^3/3}$$

4. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \;\Rightarrow\; y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$. Then the differential equation becomes $(x-3)\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0 \;\Rightarrow$

$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^{n+1} - 3\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0 \;\Rightarrow$

$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 3(n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0 \;\Rightarrow\; \sum_{n=0}^{\infty} [(n+2)c_n - 3(n+1)c_{n+1}]\, x^n = 0$ $\left(\text{since } \sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n\right)$. Equating coefficients gives $(n+2)c_n - 3(n+1)c_{n+1} = 0$, thus the recursion relation is $c_{n+1} = \dfrac{(n+2)c_n}{3(n+1)}$, $n = 0, 1, 2, \ldots$. Then $c_1 = \dfrac{2c_0}{3}$, $c_2 = \dfrac{3c_1}{3(2)} = \dfrac{3c_0}{3^2}$, $c_3 = \dfrac{4c_2}{3(3)} = \dfrac{4c_0}{3^3}$, $c_4 = \dfrac{5c_3}{3(4)} = \dfrac{5c_0}{3^4}$, and in general, $c_n = \dfrac{(n+1)c_0}{3^n}$. Thus the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n}\, x^n. \left[\text{Note that } c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n}\, x^n = \frac{9c_0}{(3-x)^2} \text{ for } |x| < 3.\right]$$

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5. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. The differential equation becomes $\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + n c_n + c_n] x^n$ (since $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$). Equating coefficients gives $(n+2)(n+1) c_{n+2} + (n+1) c_n = 0$, thus the recursion relation is $c_{n+2} = \dfrac{-(n+1) c_n}{(n+2)(n+1)} = -\dfrac{c_n}{n+2}$, $n = 0, 1, 2, \ldots$. Then the even coefficients are given by $c_2 = -\dfrac{c_0}{2}$, $c_4 = -\dfrac{c_2}{4} = \dfrac{c_0}{2 \cdot 4}$, $c_6 = -\dfrac{c_4}{6} = -\dfrac{c_0}{2 \cdot 4 \cdot 6}$, and in general, $c_{2n} = (-1)^n \dfrac{c_0}{2 \cdot 4 \cdot \cdots \cdot 2n} = \dfrac{(-1)^n c_0}{2^n\, n!}$. The odd coefficients are $c_3 = -\dfrac{c_1}{3}$, $c_5 = -\dfrac{c_3}{5} = \dfrac{c_1}{3 \cdot 5}$, $c_7 = -\dfrac{c_5}{7} = -\dfrac{c_1}{3 \cdot 5 \cdot 7}$, and in general, $c_{2n+1} = (-1)^n \dfrac{c_1}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)} = \dfrac{(-2)^n\, n!\, c_1}{(2n+1)!}$. The solution is

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\, n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n\, n!}{(2n+1)!} x^{2n+1}.$$

6. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum\limits_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum\limits_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. Hence, the equation $y'' = y$ becomes $\sum\limits_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum\limits_{n=0}^{\infty} c_n x^n = 0$ or $\sum\limits_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} - c_n] x^n = 0$. So the recursion relation is $c_{n+2} = \dfrac{c_n}{(n+2)(n+1)}$, $n = 0, 1, \ldots$. Given c_0 and c_1, $c_2 = \dfrac{c_0}{2 \cdot 1}$, $c_4 = \dfrac{c_2}{4 \cdot 3} = \dfrac{c_0}{4!}$, $c_6 = \dfrac{c_4}{6 \cdot 5} = \dfrac{c_0}{6!}, \ldots, c_{2n} = \dfrac{c_0}{(2n)!}$ and $c_3 = \dfrac{c_1}{3 \cdot 2}$, $c_5 = \dfrac{c_3}{5 \cdot 4} = \dfrac{c_1}{5 \cdot 4 \cdot 3 \cdot 2} = \dfrac{c_1}{5!}$, $c_7 = \dfrac{c_5}{7 \cdot 6} = \dfrac{c_1}{7!}, \ldots$, $c_{2n+1} = \dfrac{c_1}{(2n+1)!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1} = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

The solution can be written as $y(x) = c_0 \cosh x + c_1 \sinh x$

$$\left[\text{or } y(x) = c_0 \frac{e^x + e^{-x}}{2} + c_1 \frac{e^x - e^{-x}}{2} = \frac{c_0 + c_1}{2} e^x + \frac{c_0 - c_1}{2} e^{-x}\right].$$

7. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$, $xy' = \sum_{n=0}^{\infty} n c_n x^n$ and $(x^2+1) y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. The differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + [n(n-1) + n - 1] c_n] x^n = 0$. The recursion relation is $c_{n+2} = -\dfrac{(n-1) c_n}{n+2}$, $n = 0, 1, 2, \ldots$. Given c_0 and c_1, $c_2 = \dfrac{c_0}{2}$, $c_4 = -\dfrac{c_2}{4} = -\dfrac{c_0}{2^2 \cdot 2!}$, $c_6 = -\dfrac{3c_4}{6} = (-1)^2 \dfrac{3c_0}{2^3 \cdot 3!}, \ldots$, $c_{2n} = (-1)^{n-1} \dfrac{1 \cdot 3 \cdot \cdots \cdot (2n-3)\, c_0}{2^n\, n!} = (-1)^{n-1} \dfrac{(2n-3)!\, c_0}{2^n 2^{n-2}\, n!\, (n-2)!} = (-1)^{n-1} \dfrac{(2n-3)!\, c_0}{2^{2n-2}\, n!\, (n-2)!}$ for $n = 2, 3, \ldots$. $c_3 = \dfrac{0 \cdot c_1}{3} = 0 \quad \Rightarrow \quad c_{2n+1} = 0$ for $n = 1, 2, \ldots$. Thus the solution is

$$y(x) = c_0 + c_1 x + c_0 \frac{x^2}{2} + c_0 \sum_{n=2}^{\infty} \frac{(-1)^{n-1} (2n-3)!}{2^{2n-2}\, n!\, (n-2)!} x^{2n}.$$

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8. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, $y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and

$-xy(x) = -\sum_{n=0}^{\infty} c_n x^{n+1} = -\sum_{n=1}^{\infty} c_{n-1}x^n$. The equation $y'' = xy$ becomes

$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$ or $2c_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - c_{n-1}]x^n = 0$. Equating

coefficients gives $c_2 = 0$ and $c_{n+2} = \dfrac{c_{n-1}}{(n+2)(n+1)}$ for $n = 1, 2, \ldots$. Since $c_2 = 0$,

$c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$. Given c_0, $c_3 = \dfrac{c_0}{3 \cdot 2}$, $c_6 = \dfrac{c_3}{6 \cdot 5} = \dfrac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}, \ldots,$

$c_{3n} = \dfrac{c_0}{3n(3n-1)(3n-3)(3n-4) \cdot \cdots \cdot 6 \cdot 5 \cdot 3 \cdot 2}$. Given c_1, $c_4 = \dfrac{c_1}{4 \cdot 3}$, $c_7 = \dfrac{c_4}{7 \cdot 6} = \dfrac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}, \ldots,$

$c_{3n+1} = \dfrac{c_1}{(3n+1)3n(3n-2)(3n-3)\ldots 7 \cdot 6 \cdot 4 \cdot 3}$. The solution can be written as

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(3n-2)(3n-5) \cdot \cdots \cdot 7 \cdot 4 \cdot 1}{(3n)!} x^{3n} + c_1 \sum_{n=0}^{\infty} \frac{(3n-1)(3n-4) \cdot \cdots \cdot 8 \cdot 5 \cdot 2}{(3n+1)!} x^{3n+1}$$

9. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $-xy'(x) = -x \sum_{n=1}^{\infty} nc_n x^{n-1} = -\sum_{n=1}^{\infty} nc_n x^n = -\sum_{n=0}^{\infty} nc_n x^n$,

$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$, and the equation $y'' - xy' - y = 0$ becomes

$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - nc_n - c_n]x^n = 0$. Thus, the recursion relation is

$c_{n+2} = \dfrac{nc_n + c_n}{(n+2)(n+1)} = \dfrac{c_n(n+1)}{(n+2)(n+1)} = \dfrac{c_n}{n+2}$ for $n = 0, 1, 2, \ldots$. One of the given conditions is

$y(0) = 1$. But $y(0) = \sum_{n=0}^{\infty} c_n(0)^n = c_0 + 0 + 0 + \cdots = c_0$, so $c_0 = 1$. Hence, $c_2 = \dfrac{c_0}{2} = \dfrac{1}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{1}{2 \cdot 4}$,

$c_6 = \dfrac{c_4}{6} = \dfrac{1}{2 \cdot 4 \cdot 6}, \ldots, c_{2n} = \dfrac{1}{2^n n!}$. The other given condition is $y'(0) = 0$. But

$y'(0) = \sum_{n=1}^{\infty} nc_n(0)^{n-1} = c_1 + 0 + 0 + \cdots = c_1$, so $c_1 = 0$. By the recursion relation, $c_3 = \dfrac{c_1}{3} = 0$, $c_5 = 0, \ldots,$

$c_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. Thus, the solution to the initial-value problem is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = e^{x^2/2}$$

10. Assuming that $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $x^2 y = \sum_{n=0}^{\infty} c_n x^{n+2}$ and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=-2}^{\infty} (n+4)(n+3)c_{n+4}x^{n+2}$$

$$= 2c_2 + 6c_3 x + \sum_{n=0}^{\infty} (n+4)(n+3)c_{n+4}x^{n+2}$$

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Thus, the equation $y'' + x^2y = 0$ becomes $2c_2 + 6c_3x + \sum_{n=0}^{\infty} [(n+4)(n+3)c_{n+4} + c_n]\, x^{n+2} = 0$. So

$c_2 = c_3 = 0$ and the recursion relation is $c_{n+4} = -\dfrac{c_n}{(n+4)(n+3)}$, $n = 0, 1, 2, \ldots$.

But $c_1 = y'(0) = 0 = c_2 = c_3$ and by the recursion relation, $c_{4n+1} = c_{4n+2} = c_{4n+3} = 0$ for $n = 0, 1, 2, \ldots$.

Also, $c_0 = y(0) = 1$, so

$$c_4 = -\frac{c_0}{4 \cdot 3} = -\frac{1}{4 \cdot 3},\ c_8 = -\frac{c_4}{8 \cdot 7} = \frac{(-1)^2}{8 \cdot 7 \cdot 4 \cdot 3},\ \ldots,\ c_{4n} = \frac{(-1)^n}{4n(4n-1)(4n-4)(4n-5) \cdot \cdots \cdot 4 \cdot 3}.$$

Thus, the solution to the initial-value problem is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=0}^{\infty} c_{4n} x^{4n} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{4n}}{4n(4n-1)(4n-4)(4n-5) \cdot \cdots \cdot 4 \cdot 3}$$

11. Assuming that $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $xy = x \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$,

$$x^2y' = x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n c_n x^{n+1},$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1} \quad \text{[replace } n \text{ with } n+3\text{]}$$

$$= 2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1},$$

and the equation $y'' + x^2y' + xy = 0$ becomes $2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} + nc_n + c_n]\, x^{n+1} = 0$.

So $c_2 = 0$ and the recursion relation is $c_{n+3} = \dfrac{-nc_n - c_n}{(n+3)(n+2)} = -\dfrac{(n+1)c_n}{(n+3)(n+2)}$, $n = 0, 1, 2, \ldots$.

But $c_0 = y(0) = 0 = c_2$ and by the recursion relation, $c_{3n} = c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$.

Also, $c_1 = y'(0) = 1$, so

$$c_4 = -\frac{2c_1}{4 \cdot 3} = -\frac{2}{4 \cdot 3},\ c_7 = -\frac{5c_4}{7 \cdot 6} = (-1)^2 \frac{2 \cdot 5}{7 \cdot 6 \cdot 4 \cdot 3} = (-1)^2 \frac{2^2 5^2}{7!},\ \ldots,$$

$c_{3n+1} = (-1)^n \dfrac{2^2 5^2 \cdot \cdots \cdot (3n-1)^2}{(3n+1)!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = x + \sum_{n=1}^{\infty} \left[(-1)^n \frac{2^2 5^2 \cdot \cdots \cdot (3n-1)^2 x^{3n+1}}{(3n+1)!} \right]$$

12. (a) Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $x^2y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^{n+2}$,

$xy'(x) = \sum_{n=1}^{\infty} n c_n x^n = \sum_{n=-1}^{\infty} (n+2) c_{n+2} x^{n+2} = c_1 x + \sum_{n=0}^{\infty} (n+2) c_{n+2} x^{n+2}$, and the equation

$x^2y'' + xy' + x^2y = 0$ becomes $c_1 x + \sum_{n=0}^{\infty} \{[(n+2)(n+1) + (n+2)]c_{n+2} + c_n\} x^{n+2} = 0$. So $c_1 = 0$

and the recursion relation is $c_{n+2} = -\dfrac{c_n}{(n+2)^2}$, $n = 0, 1, 2, \ldots$. But $c_1 = y'(0) = 0$ so $c_{2n+1} = 0$ for

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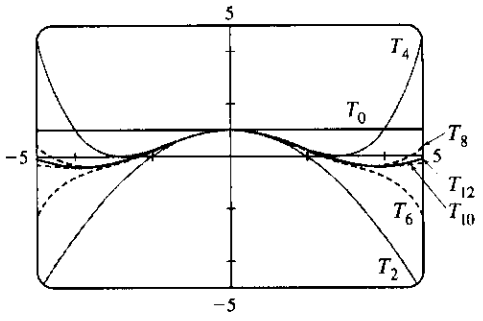
$n = 0, 1, 2, \ldots$. Also, $c_0 = y(0) = 1$, so $c_2 = -\dfrac{1}{2^2}$, $c_4 = -\dfrac{c_2}{4^2} = (-1)^2 \dfrac{1}{4^2 2^2} = (-1)^2 \dfrac{1}{2^4 (2!)^2}$,

$c_6 = -\dfrac{c_4}{6^2} = (-1)^3 \dfrac{1}{2^6 (3!)^2}, \ldots, c_{2n} = (-1)^n \dfrac{1}{2^{2n} (n!)^2}$. The solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} (n!)^2}$$

(b) The Taylor polynomials T_0 to T_{12} are shown in the graph. Because T_{10} and T_{12} are close together throughout the interval $[-5, 5]$, it is reasonable to assume that T_{12} is a good approximation to the Bessel function on that interval.



18 Review ET 17

CONCEPT CHECK

1. (a) $ay'' + by' + cy = 0$ where a, b, and c are constants.

(b) $ar^2 + br + c = 0$

(c) If the auxiliary equation has two distinct real roots r_1 and r_2, the solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. If the roots are real and equal, the solution is $y = c_1 e^{rx} + c_2 x e^{rx}$ where r is the common root. If the roots are complex, we can write $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, and the solution is $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$.

2. (a) An initial-value problem consists of finding a solution y of a second-order differential equation that also satisfies given conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$, where y_0 and y_1 are constants.

(b) A boundary-value problem consists of finding a solution y of a second-order differential equation that also satisfies given boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.

3. (a) $ay'' + by' + cy = G(x)$ where a, b, and c are constants and G is a continuous function.

(b) The complementary equation is the related homogeneous equation $ay'' + by' + cy = 0$. If we find the general solution y_c of the complementary equation and y_p is any particular solution of the original differential equation, then the general solution of the original differential equation is $y(x) = y_p(x) + y_c(x)$.

(c) See Examples 1–5 and the associated discussion in Section 18.2 [ET 17.2].

(d) See the discussion on pages 1188–1190 [ET 1152-1154].

4. Second-order linear differential equations can be used to describe the motion of a vibrating spring or to analyze an electric circuit; see the discussion in Section 18.3 [ET 17.3].

5. See Example 1 and the preceding discussion in Section 18.4 [ET 17.4].

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TRUE-FALSE QUIZ

1. True. See Theorem 18.1.3 [ET 17.1.3].
2. False. The differential equation is not homogeneous.
3. True. $\cosh x$ and $\sinh x$ are linearly independent solutions of this linear homogeneous equation.
4. False. $y = Ae^x$ is a solution of the complementary equation, so we have to take $y_p(x) = Axe^x$.

EXERCISES

1. The auxiliary equation is $r^2 - 2r - 15 = 0 \Rightarrow (r-5)(r+3) = 0 \Rightarrow r = 5, r = -3$. Then the general solution is $y = c_1e^{5x} + c_2e^{-3x}$.

2. The auxiliary equation is $r^2 + 4r + 13 = 0 \Rightarrow r = -2 \pm 3i$, so $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$.

3. The auxiliary equation is $r^2 + 3 = 0 \Rightarrow r = \pm\sqrt{3}\,i$. Then the general solution is $y = c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x)$.

4. The auxiliary equation is $4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)^2 = 0 \Rightarrow r = -\frac{1}{2}$, so the general solution is $y = c_1e^{-x/2} + c_2xe^{-x/2}$.

5. $r^2 - 4r + 5 = 0 \Rightarrow r = 2 \pm i$, so $y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Try $y_p(x) = Ae^{2x} \Rightarrow y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$. Substitution into the differential equation gives $4Ae^{2x} - 8Ae^{2x} + 5Ae^{2x} = e^{2x} \Rightarrow A = 1$ and the general solution is $y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + e^{2x}$.

6. $r^2 + r - 2 = 0 \Rightarrow r = 1, r = -2$ and $y_c(x) = c_1e^x + c_2e^{-2x}$. Try $y_p(x) = Ax^2 + Bx + C \Rightarrow y_p' = 2Ax + B$ and $y_p'' = 2A$. Substitution gives $2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2 \Rightarrow A = B = -\frac{1}{2}, C = -\frac{3}{4}$ so the general solution is $y(x) = c_1e^x + c_2e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$.

7. $r^2 - 2r + 1 = 0 \Rightarrow r = 1$ and $y_c(x) = c_1e^x + c_2xe^x$. Try $y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x \Rightarrow y_p' = (C - Ax - B)\sin x + (A + Cx + D)\cos x$ and $y_p'' = (2C - B - Ax)\cos x + (-2A - D - Cx)\sin x$. Substitution gives $(-2Cx + 2C - 2A - 2D)\cos x + (2Ax - 2A + 2B - 2C)\sin x = x\cos x \Rightarrow A = 0$, $B = C = D = -\frac{1}{2}$. The general solution is $y(x) = c_1e^x + c_2xe^x - \frac{1}{2}\cos x - \frac{1}{2}(x+1)\sin x$.

8. $r^2 + 4 = 0 \Rightarrow r = \pm 2i$ and $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. Try $y_p(x) = Ax \cos 2x + Bx \sin 2x$ so that no term of y_p is a solution of the complementary equation. Then $y_p' = (A + 2Bx)\cos 2x + (B - 2Ax)\sin 2x$ and $y_p'' = (4B - 4Ax)\cos 2x + (-4A - 4Bx)\sin 2x$. Substitution gives $4B \cos 2x - 4A \sin 2x = \sin 2x \Rightarrow A = -\frac{1}{4}$ and $B = 0$. The general solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}x \cos 2x$.

9. $r^2 - r - 6 = 0 \Rightarrow r = -2, r = 3$ and $y_c(x) = c_1e^{-2x} + c_2e^{3x}$. For $y'' - y' - 6y = 1$, try $y_{p_1}(x) = A$. Then $y_{p_1}'(x) = y_{p_1}''(x) = 0$ and substitution into the differential equation gives $A = -\frac{1}{6}$. For $y'' - y' - 6y = e^{-2x}$ try $y_{p_2}(x) = Bxe^{-2x}$ (since $y = Be^{-2x}$ satisfies the complementary equation). Then $y_{p_2}' = (B - 2Bx)e^{-2x}$ and $y_{p_2}'' = (4Bx - 4B)e^{-2x}$, and substitution gives $-5Be^{-2x} = e^{-2x} \Rightarrow B = -\frac{1}{5}$. The general solution then is $y(x) = c_1e^{-2x} + c_2e^{3x} + y_{p_1}(x) + y_{p_2}(x) = c_1e^{-2x} + c_2e^{3x} - \frac{1}{6} - \frac{1}{5}xe^{-2x}$.

10. Using variation of parameters, $y_c(x) = c_1 \cos x + c_2 \sin x$, $u_1'(x) = -\csc x \sin x = -1 \Rightarrow u_1(x) = -x$, and $u_2'(x) = \dfrac{\csc x \cos x}{x} = \cot x \Rightarrow u_2(x) = \ln|\sin x| \Rightarrow y_p = -x \cos x + \sin x \ln|\sin x|$. The solution is $y(x) = (c_1 - x)\cos x + (c_2 + \ln|\sin x|)\sin x$.

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11. The auxiliary equation is $r^2 + 6r = 0$ and the general solution is $y(x) = c_1 + c_2 e^{-6x} = k_1 + k_2 e^{-6(x-1)}$. But $3 = y(1) = k_1 + k_2$ and $12 = y'(1) = -6k^2$. Thus $k_2 = -2$, $k_1 = 5$ and the solution is $y(x) = 5 - 2e^{-6(x-1)}$.

12. The auxiliary equation is $r^2 - 6r + 25 = 0$ and the general solution is $y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $2 = y(0) = c_1$ and $1 = y'(0) = 3c_1 + 4c_2$. Thus the solution is $y(x) = e^{3x}\left(2\cos 4x - \frac{5}{4}\sin 4x\right)$.

13. The auxiliary equation is $r^2 - 5r + 4 = 0$ and the general solution is $y(x) = c_1 e^x + c_2 e^{4x}$. But $0 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_1 + 4c_2$, so the solution is $y(x) = \frac{1}{3}(e^{4x} - e^x)$.

14. $y_c(x) = c_1 \cos(x/3) + c_2 \sin(x/3)$. For $9y'' + y = 3x$, try $y_{p_1}(x) = Ax + B$. Then $y_{p_1}(x) = 3x$. For $9y'' + y = e^{-x}$, try $y_{p_2}(x) = Ae^{-x}$. Then $9Ae^{-x} + Ae^{-x} = e^{-x}$ or $y_{p_2}(x) = \frac{1}{10}e^{-x}$. Thus the general solution is $y(x) = c_1 \cos(x/3) + c_2 \sin(x/3) + 3x + \frac{1}{10}e^{-x}$. But $1 = y(0) = c_1 + \frac{1}{10}$ and $2 = y'(0) = \frac{1}{3}c_2 + 3 - \frac{1}{10}$, so $c_1 = \frac{9}{10}$ and $c_2 = -\frac{27}{10}$. Hence the solution is $y(x) = \frac{1}{10}[9\cos(x/3) - 27\sin(x/3)] + 3x + \frac{1}{10}e^{-x}$.

15. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$ and the differential equation becomes $\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + (n+1)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = -c_n/(n+2)$ for $n = 0, 1, 2, \ldots$. But $c_0 = y(0) = 0$, so $c_{2n} = 0$ for $n = 0, 1, 2, \ldots$. Also $c_1 = y'(0) = 1$, so $c_3 = -\dfrac{1}{3}$, $c_5 = \dfrac{(-1)^2}{3\cdot 5}$, $c_7 = \dfrac{(-1)^3}{3\cdot 5\cdot 7} = \dfrac{(-1)^3 2^3 3!}{7!}$, $\ldots$, $c_{2n+1} = \dfrac{(-1)^n\, 2^n\, n!}{(2n+1)!}$ for $n = 0, 1, 2, \ldots$. Thus the solution to the initial-value problem is $y(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, n!}{(2n+1)!}\, x^{2n+1}$.

16. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n\,(n-1)\,c_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$ and the differential equation becomes $\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} - (n+2)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = \dfrac{c_n}{n+1}$ for $n = 0, 1, 2, \ldots$. Given c_0 and c_1, we have $c_2 = \dfrac{c_0}{1}$, $c_4 = \dfrac{c_2}{3} = \dfrac{c_0}{1\cdot 3}$, $c_6 = \dfrac{c_4}{5} = \dfrac{c_0}{1\cdot 3\cdot 5}, \ldots, c_{2n} = \dfrac{c_0}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} = c_0\dfrac{2^{n-1}(n-1)!}{(2n-1)!}$. Similarly $c_3 = \dfrac{c_1}{2}$, $c_5 = \dfrac{c_3}{4} = \dfrac{c_1}{2\cdot 4}$, $c_7 = \dfrac{c_5}{6} = \dfrac{c_1}{2\cdot 4\cdot 6}, \ldots, c_{2n+1} = \dfrac{c_1}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n} = \dfrac{c_1}{2^n\, n!}$. Thus the general solution is $y(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n = c_0 + c_0\sum_{n=1}^{\infty}\frac{2^{n-1}(n-1)!\,x^{2n}}{(2n-1)!} + c\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^n\, n!}$. But $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^n\, n!} = x\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}x^2\right)^n}{n!} = xe^{x^2/2}$, so $y(x) = c_1 x e^{x^2/2} + c_0 + c_0\displaystyle\sum_{n=1}^{\infty}\frac{2^{n-1}(n-1)!\,x^{2n}}{(2n-1)!}$.

17. Here the initial-value problem is $2Q'' + 40Q' + 400Q = 12$, $Q(0) = 0.01$, $Q'(0) = 0$!. Then $Q_c(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t)$ and we try $Q_p(t) = A$!. Thus the general solution is $Q(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + \frac{3}{100}$!. But $0.01 = Q'(0) = c_1 + 0.03$ and $0 = Q''(0) = -10c_1 + 10c_2$, so $c_1 = -0.02 = c_2$!. Hence the charge is given by $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$.

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18. By Hooke's Law the spring constant is $k = 64$ and the initial-value problem is $2x'' + 16x' + 64x = 0, x(0) = 0, x'(0) = 2.4$. Thus the general solution is $x(t) = e^{-4t}(c_1 \cos 4t + c_2 \sin 4t)$. But $0 = x(0) = c_1$ and $2.4 = x'(0) = -4c_1 + 4c_2 \Rightarrow c_1 = 0, c_2 = 0.6$. Thus the position of the mass is given by $x(t) = 0.6e^{-4t} \sin 4t$.

19. (a) Since we are assuming that the earth is a solid sphere of uniform density, we can calculate the density ρ as follows: $\rho = \dfrac{\text{mass of earth}}{\text{volume of earth}} = \dfrac{M}{\frac{4}{3}\pi R^3}$. If V_r is the volume of the portion of the earth which lies within a distance r of the center, then $V_r = \frac{4}{3}\pi r^3$ and $M_r = \rho V_r = \dfrac{Mr^3}{R^3}$. Thus $F_r = -\dfrac{GM_r m}{r^2} = -\dfrac{GMm}{R^3} r$.

(b) The particle is acted upon by a varying gravitational force during its motion. By Newton's Second Law of Motion, $m\dfrac{d^2y}{dt^2} = F_y = -\dfrac{GMm}{R^3} y$, so $y''(t) = -k^2 y(t)$ where $k^2 = \dfrac{GM}{R^3}$. At the surface, $-mg = F_R = -\dfrac{GMm}{R^2}$, so $g = \dfrac{GM}{R^2}$. Therefore $k^2 = \dfrac{g}{R}$.

(c) The differential equation $y'' + k^2 y = 0$ has auxiliary equation $r^2 + k^2 = 0$. (This is the r of Section 18.1 [ET 17.1], not the r measuring distance from the earth's center.) The roots of the auxiliary equation are $\pm ik$, so by (11) in Section 18.1 [ET 17.1], the general solution of our differential equation for t is $y(t) = c_1 \cos kt + c_2 \sin kt$. It follows that $y'(t) = -c_1 k \sin kt + c_2 k \cos kt$. Now $y(0) = R$ and $y'(0) = 0$, so $c_1 = R$ and $c_2 k = 0$. Thus $y(t) = R \cos kt$ and $y'(t) = -kR \sin kt$. This is simple harmonic motion (see Section 18.3 [ET 17.3]) with amplitude R, frequency k, and phase angle 0. The period is $T = 2\pi/k$. $R \approx 3960$ mi $= 3960 \cdot 5280$ ft and $g = 32$ ft/s^2, so $k = \sqrt{g/R} \approx 1.24 \times 10^{-3} s^{-1}$ and $T = 2\pi/k \approx 5079$ s ≈ 85 min.

(d) $y(t) = 0 \Leftrightarrow \cos kt = 0 \Leftrightarrow kt = \frac{\pi}{2} + \pi n$ for some integer $n \Rightarrow$ $y'(t) = -kR \sin\left(\frac{\pi}{2} + \pi n\right) = \pm kR$. Thus the particle passes through the center of the earth with speed $kR \approx 4.899$ mi/s $\approx 17{,}600$ mi/h.

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