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[p. 778] Plenary Session of November 4, 1915

On the General Theory of Relativity

My efforts in recent years were directed toward basing a general theory of relativity,
[1] also for nonuniform motion, upon the supposition of relativity. I believed indeed to have found the only law of gravitation that complies with a reasonably formulated postulate of general relativity; and I tried to demonstrate the truth of precisely this solution in a paper[1] that appeared last year in the *Sitzungsberichte*.

Renewed criticism showed to me that this truth is absolutely impossible to show in the manner suggested. That this seemed to be the case was based upon a misjudgment. The postulate of relativity—*as far as I demanded it there*—is always satisfied if the Hamiltonian principle is chosen as a basis. But in reality, it provides no tool to establish the Hamiltonian function H of the gravitational field. Indeed, equation (77) l.c. which limits the choice of H says only that H has to be an invariant toward linear transformations, a demand that has nothing to do with the relativity of accelerations. Furthermore, the choice determined by equation (78) l.c. does not
[3] determine equation (77) in any way.

For these reasons I lost trust in the field equations I had derived, and instead looked for a way to limit the possibilities in a natural manner. In this pursuit I arrived at the demand of general covariance, a demand from which I parted, though with a heavy heart, three years ago when I worked together with my friend GROSSMANN. As a matter of fact, we were then quite close to that solution of the problem, which will be given in the following.

Just as the special theory of relativity is based upon the postulate that all equations have to be covariant relative to linear orthogonal transformations, so the
[p. 779] theory developed here rests upon the postulate of the *covariance of all systems of equations relative to transformations with the substitution determinant 1*.

Nobody who really grasped it can escape from its charm, because it signifies a real triumph of the general differential calculus as founded by GAUSS, RIEMANN, CHRISTOFFEL, RICCI, and LEVI-CIVITA.

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[1] "Die formale Grundlage der Relativitätstheorie," *Sitzungsberichte* 41 (1914), pp.
[2] 1066–1077. Equations of this paper are quoted in the following with the additional note "l.c." in order to keep them distinct from those in the present paper.

§1. Laws of Forming Covariants

I can be brief on the laws of forming covariants since I gave an elaborate description of the methods of absolute differential calculus in my paper of last year; thus we need only to investigate what will change in the theory of covariants if only substitutions of determinant 1 are permitted. The equation

$$d\tau' = \frac{\partial(x'_1 \ldots x'_4)}{\partial(x_1 \ldots x_4)x} d\tau,$$

which is valid for any substitutions, becomes, due to the premise in our theory, i.e.,

$$\frac{\partial(x'_1 \ldots x'_4)}{\partial(x_1 \ldots x_4)x} = 1 \tag{1}$$

now

$$d\tau' = d\tau \tag{2}$$

and the four-dimensional volume element $d\tau$ is therefore an invariant. Since furthermore (equation (17) l.c.) $\sqrt{-g}\,d\tau$ is an invariant toward arbitrary substitutions, it follows for the group that interests us now

$$\sqrt{-g'} = \sqrt{-g}. \tag{3}$$

The determinant of the $g_{\mu\nu}$ is therefore an invariant. Because of the scalar character of $\sqrt{-g}$ one can simplify the basic formulas of the formation of covariants, as compared to those of general covariance; which in short means, the factors $\sqrt{-g}$ and $1/\sqrt{-g}$ no longer occur in the basic formulas, and the distinction between tensors and *V*-tensors drops out. Specifically one gets:

1. The place of the tensors $G_{iklm} = \sqrt{-g}\ \delta_{iklm}$ and $G^{iklm} = \frac{1}{\sqrt{-g}}\ \delta_{iklm}$ (as in (19) [p. 780]

and (21a) l.c.) is now taken by the tensors

$$G_{iklm} = G^{iklm} = \delta_{iklm} \tag{4}$$

which are of a simpler structure.

2. The basic formulas (29) l.c. and (30) l.c. for the extension of tensors can, under our premise, not be replaced by simpler ones, but the equations that define divergence (representing a combination of the equations (30) l.c. and (31) l.c.) can be simplified. This can be written as

$$A^{\alpha_1 \ldots \alpha_l} = \sum_s \frac{\partial A^{\alpha_1 \ldots \alpha_l s}}{\partial x_s} + \sum_{s\tau}\left[\begin{Bmatrix} s\tau \\ \alpha_1 \end{Bmatrix} A^{\tau\alpha_2 \ldots \alpha_l s} + \ldots . \begin{Bmatrix} s\tau \\ \alpha_l \end{Bmatrix} A^{\alpha_1 \cdot \alpha_{l-1}\tau s}\right] + \sum_{s\tau}\begin{Bmatrix} s\tau \\ s \end{Bmatrix} A^{\alpha_1 \ldots \alpha_l \tau}. \tag{5}$$

But according to (24) l.c. and (24a) l.c.

$$\sum_\tau \begin{Bmatrix} s\tau \\ s \end{Bmatrix} = \frac{1}{2}\sum_{\alpha s} g^{s\alpha}\left(\frac{\partial g_{s\alpha}}{\partial x_\tau} + \frac{\partial g_{\tau\alpha}}{\partial x_s} - \frac{\partial g_{s\tau}}{\partial x_\alpha}\right) = \frac{1}{2}\sum g^{s\alpha}\frac{\partial g_{s\alpha}}{\partial x_\tau} = \frac{\partial(lg\sqrt{-g})}{\partial x_\tau}. \quad (6)$$

And this quantity has the characteristics of a vector, due to (3). Consequently, the last term on the right-hand side of (5) is itself a contravariant tensor of rank l. We are therefore entitled to replace (5) by the simple definition of divergence, viz.,

$$A^{\alpha_1 \ldots \alpha_l} = \sum_s \frac{\partial A^{\alpha_1 \ldots \alpha_l s}}{\partial x_s} + \sum_{s\tau}\left[\begin{Bmatrix} s\tau \\ \alpha_1 \end{Bmatrix} A^{\tau\alpha_2 \ldots \alpha_l s} + \ldots . \begin{Bmatrix} s\tau \\ \alpha_l \end{Bmatrix} A^{\alpha_1 \ldots \alpha_{l-1}\tau s}\right], \quad (5a)$$

and we shall do so throughout.

For example, the definition (37) l.c.

$$\Phi = \frac{1}{\sqrt{-g}}\sum_\mu \frac{\partial}{\partial x_\mu}(\sqrt{-g}A^\mu)$$

has to be replaced by the simpler definition

$$\Phi = \sum_\mu \frac{\partial A^\mu}{\partial x_\mu}, \quad (7)$$

and equation (40) l.c. for the divergence of the contravariant six-vector by the simpler

$$A^\mu = \sum_\nu \frac{\partial A^{\mu\nu}}{\partial x_\nu}. \quad (8)$$

In place of (41a) l.c. we have, due to our assumption,

$$A_\sigma = \sum_\nu \frac{\partial A_\sigma^\nu}{\partial x_\nu} - \frac{1}{2}\sum_{\mu\nu\tau} g^{\tau\mu}\frac{\partial g_{\mu\nu}}{\partial x_\sigma} A_\tau^\nu. \quad (9)$$

[p. 781] A comparison with (41b) reveals that under our assumption the law of divergence is the same as that for the divergence of V-tensors in the general differential calculus. This remark applies to any divergence of tensors, as can be derived from (5) and (5a).

3. Our limitation to transformations of determinant 1 brings the farthest-reaching simplification for those covariants which are formed only from the $g_{\mu\nu}$ and their derivatives. It is shown in mathematics that these covariants can all be derived from the RIEMANN-CHRISTOFFEL tensor of rank four, which (in its covariant form) reads:

$$\left.\begin{aligned}(ik,\ lm) = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x_k \partial x_l} + \frac{\partial^2 g_{kl}}{\partial x_i \partial x_m} - \frac{\partial^2 g_{il}}{\partial x_k \partial x_m} - \frac{\partial^2 g_{mk}}{\partial x_l \partial x_i}\right)\\ + \sum_{\rho\sigma} g^{\rho\sigma}\left(\begin{bmatrix} im \\ \rho \end{bmatrix}\begin{bmatrix} kl \\ \sigma \end{bmatrix} - \begin{bmatrix} il \\ \rho \end{bmatrix}\begin{bmatrix} km \\ \sigma \end{bmatrix}\right)\end{aligned}\right\}. \tag{10}$$

It is in the nature of gravitation that we are most interested in tensors of rank two, which can be formed by inner multiplication of this tensor of rank four with the $g_{\mu\nu}$. Due to the symmetry properties of the RIEMANNIAN tensor, apparent from (10), viz.,

$$\begin{aligned}(ik,\ lm) &= (lm,\ ik)\\ (ik,\ lm) &= -(ki,\ lm),\end{aligned} \tag{11}$$

this multiplication can be formed only in *one* way; whereby one obtains the tensor

$$G_{im} = \sum_{kl} g^{kl}(ik,\ lm). \tag{12}$$

It is more advantageous for our purposes to derive this tensor from a different form of (10) which CHRISTOFFEL has given,[2] i.e.,

$$\{ik,\ lm\} = \sum_{\rho} g^{k\rho}(i\rho,\ lm) = \frac{\partial \begin{Bmatrix} il \\ k \end{Bmatrix}}{\partial x_m} - \frac{\partial \begin{Bmatrix} im \\ k \end{Bmatrix}}{\partial x_l} + \sum_{\rho}\left[\begin{Bmatrix} il \\ \rho \end{Bmatrix}\begin{Bmatrix} \rho m \\ k \end{Bmatrix} - \begin{Bmatrix} im \\ \rho \end{Bmatrix}\begin{Bmatrix} \rho l \\ k \end{Bmatrix}\right]. \tag{13}$$

When this tensor is multiplied (inner multiplication) with the tensor

$$\delta_k^l = \sum_{\alpha} g_{k\alpha} g^{\alpha l}$$

one obtains G_{im}, viz.,

$$G_{im} = \{il,\ lm\} = R_{im} + S_{im} \tag{13}$$ [p. 782] {1}

$$R_{im} = -\frac{\partial \begin{Bmatrix} im \\ l \end{Bmatrix}}{\partial x_l} + \sum_{\rho}\begin{Bmatrix} il \\ \rho \end{Bmatrix}\begin{Bmatrix} \rho m \\ l \end{Bmatrix} \tag{13a}$$ [4]

[2]A simple proof of the tensorial character of this expression can be found on page 1053 of my repeatedly quoted paper.

$$S_{im} = \frac{\partial \left\{ {il \atop l} \right\}}{\partial x_m} - \left\{ {im \atop \rho} \right\} \left\{ {\rho l \atop l} \right\}. \tag{13b}$$

Under the constraint to transformations with determinants 1, not only (G_{im}) is a tensor, but (R_{im}) and (S_{im}) also have tensorial character. It follows indeed from the fact that $\sqrt{-g}$ is a scalar, and because of (6), that $\left\{ {il \atop l} \right\}$ is a covariant four-vector. (S_{im}), however, is, due to (29) l.c., nothing other than the extension of this four-vector, which means it is also a tensor. From the tensorial character of (G_{im}) and (S_{im}) follows the same for (R_{im}), from (13). The tensor (R_{im}) is of utmost importance for the theory of gravitation.

§2. Notes on the Differential Laws of "Material" Processes

1. The energy-momentum theorem for matter (including electromagnetic processes in a vacuum.

According to the general considerations of the previous paragraph, equation (42a) l.c. has to be replaced by

$$\sum_{\nu} \frac{\partial T_{\sigma}^{\nu}}{\partial x_{\nu}} = \frac{1}{2} \sum_{\mu\tau\nu} g^{\tau\mu} \frac{\partial g_{\mu\nu}}{\partial x_{\sigma}} T_{\tau}^{\nu} + K_{\sigma}, \tag{14}$$

where T_{σ}^{ν} is an ordinary tensor, K_{σ} an ordinary four-vector (not a V-tensor, V-vector, resp.). We have to attach a remark to this equation, because it is important for the following. The equations of conservation led me in the past to view the quantities

$$\frac{1}{2} \sum_{\mu} g^{\tau\mu} \frac{\partial g_{\mu\nu}}{\partial x_{\sigma}}$$

as the natural expressions of the components of the gravitational field, even though the formulas of the absolute differential calculus seem to suggest the CHRISTOFFEL symbols $\left\{ {\nu\sigma \atop \tau} \right\}$ instead, as being the more natural quantities. The former view was a fateful prejudice. The preference for the CHRISTOFFEL symbols justifies itself
[p. 783] especially because of the symmetry in their covariant indices (here ν and σ) and, furthermore, because they occur in the fundamentally important equations of the geodesic line (23b) l.c.; and these latter are—from a physical point of view—the equations of motion of a material point in a gravitational field. Equation (14) cannot

serve as a counterargument because the first term on its right-hand side can be brought into the form

$$\sum_{\nu\tau} \begin{Bmatrix} \sigma\nu \\ \tau \end{Bmatrix} T_\tau^\nu .$$

Therefore, from now on we shall call the quantities

$$\Gamma_{\mu\nu}^\sigma = -\begin{Bmatrix} \mu\nu \\ \sigma \end{Bmatrix} = -\sum_\alpha g^{\sigma\alpha} \begin{bmatrix} \mu\nu \\ \alpha \end{bmatrix} = -\frac{1}{2}\sum_\alpha g^{\sigma\alpha}\left(\frac{\partial g_{\mu\alpha}}{\partial x_\nu} + \frac{\partial g_{\nu\alpha}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right) \quad (15)$$

the components of the gravitational field. K_ν vanishes when T_σ^ν denotes the energy tensor of all "material" processes, and the conservation theorem (14) takes the form

$$\sum_\alpha \frac{\partial T_\sigma^\alpha}{\partial x_\alpha} = -\sum_{\alpha\beta} \Gamma_{\sigma\beta}^\alpha T_\alpha^\beta. \quad (14a)$$

We note that the equations of motion (23b) l.c. of a material point in a gravitational field take the form

$$\frac{d^2 x_\tau}{ds^2} = \sum_{\mu\nu} \Gamma_{\mu\nu}^\tau \frac{dx_\mu}{ds}\frac{dx_\nu}{ds}. \quad (15)$$ {2}

2. The considerations in paragraphs 10 and 11 of the quoted paper remain unchanged, except that the structures which were there called *V*-scalars and *V*-tensors are now ordinary scalars and tensors, respectively.

§3. The Field Equations of Gravitation

From what has been said, it seems appropriate to write the field equations of gravitation in the form

$$R_{\mu\nu} = -\kappa T_{\mu\nu} \quad (16)$$

since we already know that these equations are covariant under any transformation of a determinant equal to 1. Indeed, these equations satisfy all conditions we can demand. Written out in more detail, and according to (13a) and (15), they are

$$\sum_\alpha \frac{\partial \Gamma_{\mu\nu}^\alpha}{\partial x_\alpha} + \sum_{\alpha\beta} \Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta = -\kappa T_{\mu\nu}. \quad (16a)$$

We wish to show now that these field equations can be brought into the HAMILTONIAN form [p. 784]

$$\left.\begin{aligned}\delta\left\{\int\left(\mathfrak{L} - \kappa\sum_{\mu\nu} g^{\mu\nu} T_{\mu\nu}\right)d\tau\right\}\\ \mathfrak{L} = \sum_{\sigma\tau\alpha\beta} g^{\sigma\tau}\Gamma^{\alpha}_{\sigma\beta}\Gamma^{\beta}_{\tau\alpha}\end{aligned}\right\}, \tag{17}$$

where the $g^{\mu\nu}$ have to be varied while the $T_{\mu\nu}$ are to be treated as constants. The reason is that (17) is equivalent to the equations

$$\sum_{\alpha}\frac{\partial}{\partial x_{\alpha}}\left(\frac{\partial\mathfrak{L}}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial\mathfrak{L}}{\partial g^{\mu\nu}} = -\kappa T_{\mu\nu}, \tag{18}$$

where $\mathfrak{L}$ has to be thought of as a function of the $g^{\mu\nu}$ and the $\dfrac{\partial g^{\mu\nu}}{\partial x_{\sigma}}(= g^{\mu\nu}_{\sigma})$. On the other hand, a lengthy but uncomplicated calculation yields the relations

$$\frac{\partial\mathfrak{L}}{\partial g^{\mu\nu}} = -\sum_{\alpha\beta}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \tag{19}$$

$$\frac{\partial\mathfrak{L}}{\partial g^{\mu\nu}_{\alpha}} = \Gamma^{\alpha}_{\mu\nu}. \tag{19a}$$

These together with (18) provide the field equations (16a).

It can now also be easily shown that the principle of the conservation of energy and momentum is satisfied. Multiplying (18) by $g^{\mu\nu}_{\sigma}$ with summation over the indices μ and ν, one obtains after customary rearrangement

$$\sum_{\alpha\mu\nu}\frac{\partial}{\partial x_{\alpha}}\left(g^{\mu\nu}_{\sigma}\frac{\partial\mathfrak{L}}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial\mathfrak{L}}{\partial x_{\sigma}} = -\kappa\sum_{\mu\nu} T_{\mu\nu}g^{\mu\nu}_{\sigma}.$$

According to (14), on the other hand, for the *total* energy tensor of matter one has

$$\sum_{\lambda}\frac{\partial T^{\lambda}_{\sigma}}{\partial x_{\lambda}} = -\frac{1}{2}\sum_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x_{\sigma}}T_{\mu\nu}.$$

From the last two equations follows

$$\sum_{\lambda}\frac{\partial}{\partial x_{\lambda}}(T^{\lambda}_{\sigma} + t^{\lambda}_{\sigma}) = 0, \tag{20}$$

where

$$t^{\lambda}_{\sigma} = \frac{1}{2\kappa}\left(\mathfrak{L}\,\delta^{\lambda}_{\sigma} - \sum_{\mu\nu} g^{\mu\nu}_{\sigma}\frac{\partial\mathfrak{L}}{\partial g^{\mu\nu}_{\lambda}}\right) \tag{20a}$$

[p. 785] denotes the "energy tensor" of the gravitational field which, by the way, has tensorial character only under linear transformations. After a simple rearrangement, one gets from (20a) and (19a)

$$t_\sigma^\lambda = \frac{1}{2}\delta_\sigma^\lambda \sum_{\mu\nu\alpha\beta} g^{\mu\nu}\Gamma_{\mu\beta}^\alpha\Gamma_{\nu\alpha}^\beta - \sum_{\mu\nu\alpha} g^{\mu\nu}\Gamma_{\mu\sigma}^\alpha\Gamma_{\nu\alpha}^\lambda. \qquad (20b)$$

Finally, it is of interest to derive two scalar equations that result from the field equations. After multiplying (16a) by $g^{\mu\nu}$ with summation over μ and ν, we get after simple rearranging

$$\sum_{\alpha\beta}\frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta} - \sum_{\sigma\tau\alpha\beta} g^{\sigma\tau}\Gamma_{\sigma\beta}^\alpha\Gamma_{\tau\alpha}^\beta + \sum_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\left(g^{\alpha\beta}\frac{\partial lg\sqrt{-g}}{\partial x_\beta}\right) = -\kappa\sum_\sigma T_\sigma^\sigma. \qquad (21)$$

On the other hand, multiplying (16a) by $g^{\nu\lambda}$ and summing over ν, we get

$$\sum_{\alpha\nu}\frac{\partial}{\partial x_\alpha}(g^{\nu\lambda}\Gamma_{\mu\nu}^\alpha) - \sum_{\alpha\beta\nu} g^{\nu\beta}\Gamma_{\nu\mu}^\alpha\Gamma_{\beta\alpha}^\lambda = -\kappa T_\mu^\tau,$$

or, also considering (20b),

$$\sum_{\alpha\nu}\frac{\partial}{\partial x_\alpha}(g^{\nu\lambda}\Gamma_{\mu\nu}^\alpha) - \frac{1}{2}\delta_\mu^\lambda\sum_{\mu\nu\alpha\beta} g^{\mu\nu}\Gamma_{\mu\beta}^\alpha\Gamma_{\nu\alpha}^\beta = -\kappa(T_\mu^\lambda + t_\mu^\lambda).$$

Taking (20) into account, and after simple rearranging, this yields

$$\frac{\partial}{\partial x_\mu}\left[\sum_{\alpha\beta}\frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha\partial x_\beta} - \sum_{\sigma\tau\alpha\beta} g^{\sigma\tau}\Gamma_{\sigma\beta}^\alpha\Gamma_{\tau\alpha}^\beta\right] = 0. \qquad (22)$$

However, we demand somewhat beyond that:

$$\sum_{\alpha\beta}\frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha\partial x_\beta} - \sum_{\sigma\tau\alpha\beta} g^{\sigma\tau}\Gamma_{\sigma\beta}^\alpha\Gamma_{\tau\alpha}^\beta = 0, \qquad (22a)$$

whereupon (21) becomes

$$\sum_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\left(g^{\alpha\beta}\frac{\partial lg\sqrt{-g}}{\partial x_\beta}\right) = -\kappa\sum_\sigma T_\sigma^\sigma. \qquad (21a)$$

Equation (21a) shows the impossibility to choose the coordinate system such that $\sqrt{-g}$ equals 1, because the scalar of the energy tensor cannot be set to zero. [5]

Equation (22a) is a relation of the $g_{\mu\nu}$ alone; it would not be valid in a new coordinate system which would result from the original one by a forbidden transformation. The equation therefore shows how the coordinate system has to be adapted to the manifold.

[p. 786]

§4. Some Remarks on the Physical Qualities of the Theory

A first approximation of the equations (22a) is

$$\sum_{\alpha\beta} \frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta} = 0.$$

This does not yet fix the coordinate system, because this would require 4 equations. We are therefore entitled to put for a first approximation arbitrarily

[3]

$$\sum_{\beta} \frac{\partial g^{\alpha\beta}}{\partial x_\beta} = 0. \tag{22}$$

For further simplification we want to introduce the imaginary time as a fourth variable. The field equations (16a) then take, as a first approximation, the form

$$\frac{1}{2}\sum_{\alpha} \frac{\partial^2 g_{\mu\nu}}{\partial x_\alpha^2} = \kappa T_{\mu\nu}, \tag{16b}$$

from which one sees immediately that it contains NEWTON's law as an approximation.—

That the new theory complies with the relativity of motion follows from the fact that among the permissible transformations are those that correspond to a rotation of
[6] the new relative to the old system (with arbitrarily variable angular velocity), and also those where the origin of the new system performs an arbitrarily prescribed motion relative to that of the old one.

Indeed, the substitutions

$$\begin{aligned} x' &= x \cos\tau + y \sin\tau \\ y' &= -x \sin\tau + y \cos\tau \\ z' &= z \\ t' &= t \end{aligned}$$

and

$$\begin{aligned} x' &= x - \tau_1 \\ y' &= y - \tau_2 \\ z' &= z - \tau_3 \\ t' &= t, \end{aligned}$$

where τ and τ_1, τ_2, τ_3 respectively are arbitrary functions of t and substitutions with the determinant 1.

Additional notes by translator

{1} The number (13) has already been assigned to an equation above.
{2} The number (15) has already been assigned to an equation above.
{3} The number (22) has already been assigned to an equation above.

Doc. 22

[p. 799]

On the General Theory of Relativity (Addendum)

by A. Einstein

In a recent investigation[1] I have shown how RIEMANN's theory of covariants in multidimensional manifolds can be utilized as a basis for a theory of the gravitational field. I now want to show here that an even more concise and logical structure of the theory can be achieved by introducing an admittedly bold additional hypothesis on the structure of matter.

The hypothesis whose justification we want to consider relates to the following topic. The energy tensor of "matter" T_μ^λ has a scalar $\sum_\mu T_\mu{}^\mu$, whose vanishing for the electromagnetic field is well known. In contrast, it seems to differ from zero for matter *proper*. Because, if we consider the most simple special case, that of an "incoherent" continuous fluid (with pressure neglected), then we are used to writing

$$T^{\mu\nu} = \sqrt{-g}\,\rho_0 \frac{dx_\mu}{ds}\frac{dx_\nu}{ds},$$

and we have

$$\sum_\mu T_\mu^\mu = \sum_{\mu\nu} g_{\mu\nu} T^{\mu\nu} = \rho_0\sqrt{-g}.$$

The scalar of the energy tensor does not vanish in this approach.

One now has to remember that by our knowledge "matter" is not to be perceived as something primitively given or physically plain. There even are those, and not just a few, who hope to reduce matter to purely electrodynamic processes, which of course would have to be done in a theory more completed than MAXWELL's electrodynamics. Now let us just assume that in such completed electrodynamics the scalar of the energy tensor also would vanish! Would the result, shown above, prove that matter cannot be constructed in this theory? I think I can answer this question
[p. 800] in the negative, because it might very well be that in "matter," to which the previous expression relates, gravitational fields do form an important constituent. In that case, $\sum T_\mu^\mu$ can appear positive for the entire structure while in reality only $\sum (T_\mu^\mu + t_\mu^\mu)$ is positive and $\sum T_\mu^\mu$ vanishes everywhere. *In the following we assume the conditions* $\sum T_\mu^\mu = 0$ *really to be generally true.*

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[1]These *Sitzungsberichte*, p. 778.

Whoever does not categorically reject the possibility that gravitational fields could constitute an *essential* part of matter will find powerful support for this conception in the following.[2]

Derivation of the Field Equations

Our hypothesis allows us to take the last step that the idea of general relativity may consider as desirable. It allows us, namely, also to write the field equations of gravitation in a *general* covariant form. I have shown in the previous paper (equation (13)) that

$$G_{im} = \sum_l \{il,\ lm\} = R_{im} + S_{im} \tag{13}$$

is a covariant tensor. And we had

$$R_{im} = -\sum_l \frac{\partial \begin{Bmatrix} im \\ l \end{Bmatrix}}{\partial x_l} + \sum_{\rho l} \begin{Bmatrix} il \\ \rho \end{Bmatrix} \begin{Bmatrix} \rho m \\ l \end{Bmatrix} \tag{13a}$$

$$S_{im} = \sum_l \frac{\partial \begin{Bmatrix} il \\ l \end{Bmatrix}}{\partial x_m} - \sum_{\rho l} \begin{Bmatrix} im \\ \rho \end{Bmatrix} \begin{Bmatrix} \rho l \\ l \end{Bmatrix}. \tag{13b}$$

This tensor G_{im} is the only tensor available for the establishment of generally covariant equations of gravitation.

We have won generally covariant field equations if we agree that the field equations of gravitation should be

$$G_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{16b}$$

These, together with the generally covariant laws, provided for by the absolute differential calculus, express the causal nexus for "material" processes in nature; and they express it in a form that emphasizes the fact that any special choice of coordinate system—which logically has nothing to do with nature's law anyway—is [p. 801] not used in the formulation of these laws.

[2]In writing this paper I was not yet aware that the hypothesis $\sum T_\mu^\mu = 0$ is, in principle, admissible.

Based upon this system one can—by retroactive choice of coordinates—return to those laws which I established in my recent paper, and without any actual change in these laws, because it is clear that we can introduce a new coordinate system such that relative to it

$$\sqrt{-g} = 1$$

holds everywhere. S_{im} then vanishes and one returns to the system of field equations

$$R_{\mu\nu} = -\kappa T_{\mu\nu} \tag{16}$$

of the recent paper. The formulas of absolute differential calculus degenerate exactly in the manner shown in said paper. And our choice of coordinates still allows only transformations of determinant 1.

The only difference in content between the field equations derived from general covariance and those of the recent paper is that the value of $\sqrt{-g}$ could not be prescribed in the latter. This value was rather determined by the equation

{1}

$$\sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha}\left(g^{\alpha\beta}\frac{\partial lg\sqrt{-g}}{\partial x_\beta}\right) = -\kappa\sum_\sigma T_\sigma^\sigma. \tag{21a}$$

This equation shows that here $\sqrt{-g}$ can only be constant if the scalar of the energy tensor vanishes.

Under our present derivation $\sqrt{-g} = 1$ due to our arbitrary choice of coordinates. The vanishing of the scalar of the energy tensor of "matter" follows now from our field equations instead of from equation (21a). *The generally covariant field equations (16b), which form our starting point, do not lead to a contradiction only when the hypothesis, which we explained in the introduction, applies*. Then, however, we are also entitled to add to our previous field equation the limiting condition:

$$\sqrt{-g} = 1. \tag{21b}$$

Additional note by translator

{1} The "∂x_α" inside of the parentheses has been corrected to "∂x_β."