quantum field theory

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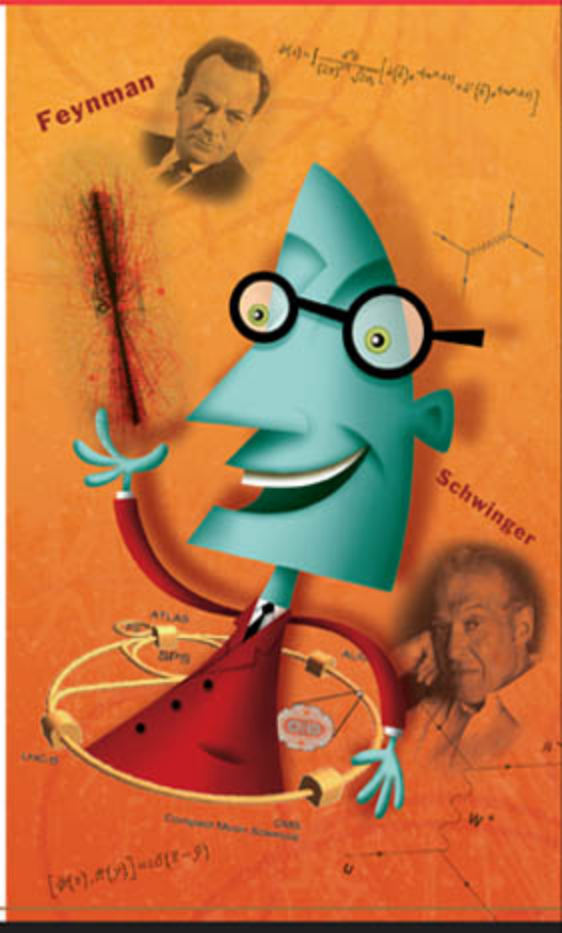

David McMahon

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Quantum Field Theory Demystified

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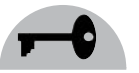

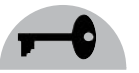

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PREFACE

Quantum field theory is the union of Einstein's special relativity and quantum mechanics. It forms the foundation of what scientists call the standard model, which is a theoretical framework that describes all known particles and interactions with the exception of gravity. There is no time like the present to learn it—the Large Hadron Collider (LHC) being constructed in Europe will test the final pieces of the standard model (the Higgs mechanism) and look for physics beyond the standard model. In addition quantum field theory forms the theoretical underpinnings of string theory, currently the best candidate for unifying all known particles and forces into a single theoretical framework.

Quantum field theory is also one of the most difficult subjects in science. This book aims to open the door to quantum field theory to as many interested people as possible by providing a simplified presentation of the subject. This book is useful as a supplement in the classroom or as a tool for self-study, but be forewarned that the book includes the math that comes along with the subject.

By design, this book is not thorough or complete, and it might even be considered by some "experts" to be shallow or filled with tedious calculations. But this book is not written for the experts or for brilliant graduate students at the top of the class, it is written for those who find the subject difficult or impossible. Certain aspects of quantum field theory have been selected to introduce new people to the subject, or to help refresh those who have been away from physics.

After completing this book, you will find that studying other quantum field theory books will be easier. You can master quantum field theory by tackling the reference list in the back of this book, which includes a list of textbooks used in the development of this one. Frankly, while all of those books are very good and make fine references, most of them are hard to read. In fact many quantum field theory books are impossible to read. My recommendation is to work through this book first, and then tackle *Quantum Field Theory in a Nutshell* by Anthony Zee. Different than all other books on the subject, it's very readable and is packed with great

physical insight. After you've gone through that book, if you are looking for mastery or deep understanding you will be well equipped to tackle the other books on the list.

Unfortunately, learning quantum field theory entails some background in physics and math. The bottom line is, I assume you have it. The background I am expecting includes quantum mechanics, some basic special relativity, some exposure to electromagnetics and Maxwell's equations, calculus, linear algebra, and differential equations. If you lack this background do some studying in these subjects and then give this book a try.

Now let's forge ahead and start learning quantum field theory.

David McMahon

Quantum Field Theory Demystified

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CHAPTER 1

Particle Physics and Special Relativity

Quantum field theory is a theoretical framework that combines *quantum mechanics* and *special relativity*. Generally speaking, quantum mechanics is a theory that describes the behavior of small systems, such as atoms and individual electrons. Special relativity is the study of high energy physics, that is, the motion of particles and systems at velocities near the speed of light (but without gravity). What follows is an introductory discussion to give you a flavor of what quantum field theory is like. We will explore each concept in more detail in the following chapters.

There are three key ideas we want to recall from quantum mechanics, the first being that physical observables are *mathematical operators* in the theory.

For instance, the Hamiltonian (i.e., the energy) of a *simple harmonic oscillator* is the operator

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)$$

where $\hat{a}^{\dagger}$, $\hat{a}$ are the creation and annihilation operators, and $\hbar$ is Planck's constant.

The second key idea you should remember from quantum mechanics is the uncertainty principle. The uncertainty relation between the position operator $\hat{x}$ and the momentum operator $\hat{p}$ is

$$\Delta\hat{x}\,\Delta\hat{p} \geq \frac{\hbar}{2} \tag{1.1}$$

There is also an uncertainty relation between energy and time.

$$\Delta E\,\Delta t \geq \frac{\hbar}{2} \tag{1.2}$$

When considering the uncertainty relation between energy and time, it's important to remember that time is only a parameter in nonrelativistic quantum mechanics, not an operator.

The final key idea to recall from quantum mechanics is the commutation relations. In particular,

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

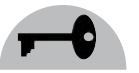
Now let's turn to special relativity. We can jump right to Einstein's famous equation that every lay person knows something about, in order to see how special relativity is going to impact quantum theory. This is the equation that relates energy to mass.

$$E = mc^2 \tag{1.3}$$

What should you take away from this equation? The thing to notice is that if there is enough energy—that is, enough energy proportional to a given particle's mass as described by Eq. (1.3)—then we can "create" the particle. Due to conservation laws, we actually need twice the particle's mass, so that we can create a particle and its antiparticle. So in high energy processes,

- Particle number is not fixed.
- The types of particles present are not fixed.

These two facts are in direct conflict with nonrelativistic quantum mechanics. In nonrelativistic quantum mechanics, we describe the dynamics of a system with the



Schrödinger equation, which for a particle moving in one dimension with a potential V is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V\psi=i\hbar\frac{\partial\psi}{\partial t} \tag{1.4}$$

We can extend this formalism to treat the case when several particles are present. However, the number and types of particles are absolutely fixed. The Schrödinger equation cannot in any shape or form handle changing particle number or new types of particles appearing and disappearing as relativity allows.

In fact, there is no wave equation of the type we are used to from nonrelativistic quantum mechanics that is truly compatible with both relativity and quantum theory. Early attempts to merge quantum mechanics and special relativity focused on generating a relativistic version of the Schrödinger equation. In fact, Schrödinger himself derived a relativistic equation prior to coming up with the wave equation he is now famous for. The equation he derived, which was later discovered independently by Klein and Gordon (and is now known as the Klein-Gordon equation) is

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}=\frac{m^2c^2}{\hbar^2}\varphi$$

We will have more to say about this equation in future chapters. Schrödinger discarded it because it gave the wrong fine structure for the hydrogen atom. It is also plagued by an unwanted feature—it appears to give negative probabilities, something that obviously contradicts the spirit of quantum mechanics. This equation also has a funny feature—it allows negative energy states.

The next attempt at a relativistic quantum mechanics was made by Dirac. His famous equation is

$$i\hbar\frac{\partial\psi}{\partial t}=-i\hbar c\vec{\alpha}\cdot\vec{\nabla}\psi+\beta mc^2\psi$$

Here, $\vec{\alpha}$ and β are actually matrices. This equation, which we will examine in detail in later chapters, resolves some of the problems of the Klein-Gordon equation but also allows for negative energy states.

As we will emphasize later, part of the problem with these relativistic wave equations is in their interpretation. We move forward into a quantum theory of fields by changing how we look at things. In particular, in order to be truly compatible with special relativity we need to discard the notion that φ and ψ in the

Klein-Gordon and Dirac equations, respectively describe single particle states. In their place, we propose the following new ideas:

- The wave functions φ and ψ are not wave functions at all, instead they are *fields*.
- The fields are operators that can create new particles and destroy particles.

Since we have promoted the fields to the status of operators, they must satisfy commutation relations. We will see later that we make a transition of the type

$$[\hat{x}, \hat{p}] \rightarrow \left[\hat{\varphi}(x,t), \hat{\pi}(y,t)\right]$$

Here, $\hat{\pi}(y,t)$ is another field that plays the role of momentum in quantum field theory. Since we are transitioning to the continuum, the commutation relation will be of the form

$$\left[\hat{\varphi}(x,t), \hat{\pi}(y,t)\right] = i\hbar\delta(x-y)$$

where x and y are two points in space. This type of relation holds within it the notion of causality so important in special relativity—if two fields are spatially separated they cannot affect one another.

With fields promoted to operators, you might wonder what happens to the ordinary operators of quantum mechanics. There is one important change you should make sure to keep in mind. In quantum mechanics, position $\hat{x}$ is an operator while time t is just a parameter. In relativity, since time and position are on a similar footing, we might expect that in relativistic quantum mechanics we would also put time and space on a similar footing. This could mean promoting time to an operator $\hat{t}$. This is not what is done in ordinary quantum field theory, where we take the opposite direction—and demote position to a parameter x. So in quantum field theory,

- Fields φ and ψ are operators.
- They are parameterized by spacetime points (x, t).
- Position x and time t are just numbers that fix a point in spacetime—they are not operators.
- Momentum continues to play a role as an operator.

In quantum field theory, we frequently use tools from classical mechanics to deal with fields. Specifically, we often use the Lagrangian

$$L = T - V \tag{1.5}$$

The Lagrangian is important because symmetries (such as rotations) leave the form of the Lagrangian invariant. The classical path taken by a particle is the one which minimizes the action.

$$S = \int L\, dt \tag{1.6}$$

We will see how these methods are applied to fields in Chap. 2.

Special Relativity

The arena in which quantum field theory operates is the high energy domain of special relativity. Therefore, brushing up on some basic concepts in special relativity and familiarizing ourselves with some notation is important to gain some understanding of quantum field theory.

Special relativity is based on two simple postulates. Simply stated, these are:

- The laws of physics are the same for all inertial observers.
- The speed of light c is a constant.

An *inertial frame of reference* is one for which Newton's first law holds. In special relativity, we characterize spacetime by an *event*, which is something that happens at a particular time t and some spatial location (x, y, z). Also notice that the speed of light c can serve in a role as a conversion factor, transforming time into space and vice versa. Space and time therefore form a unified framework and we denote coordinates by (ct, x, y, z).

One consequence of the second postulate is the *invariance of the interval*. In special relativity, we measure distance in space and time together. Imagine a flash of light emitted at the origin at $t = 0$. At some later time t the spherical wavefront of the light can be described by

$$\begin{aligned} c^2t^2 &= x^2 + y^2 + z^2 \\ \Rightarrow\ c^2t^2 - x^2 - y^2 - z^2 &= 0 \end{aligned}$$

Since the speed of light is invariant, this equation must also hold for another observer, who is measuring coordinates with respect to a frame we denote by (ct', x', y', z'). That is,

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = 0$$

It follows that

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

Now, in ordinary space, the differential distance from the origin to some point (x, y, z) is given by

$$dr^2 = dx^2 + dy^2 + dz^2$$

We define an analogous concept in spacetime, called the *interval.* This is denoted by ds^2 and is written as

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{1.7}$$

From Eq. (1.7) it follows that the interval is invariant. Consider two observers in two different inertial frames. Although they measure different spatial coordinates (x, y, z) and (x', y', z') and different time coordinates t and t' to label events, the interval for each observer is the same, that is,

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = ds'^2$$

This is a consequence of the fact that the speed of light is the same for all inertial observers.

It is convenient to introduce an object known as the *metric.* The metric can be used to write down the coefficients of the differentials in the interval, which in this case are just +/–1. The metric of special relativity ("flat space") is given by

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.8}$$

The metric has an inverse, which in this case turns out to be the same matrix. We denote the inverse with lowered indices as

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The symbol $\eta_{\mu\nu}$ is reserved for the metric of special relativity. More generally, the metric is denoted by $g_{\mu\nu}$. This is the convention that we will follow in this book. We have

$$g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu} \tag{1.9}$$

where δ^{ρ}_{μ} is the *Kronecker delta function* defined by

$$\delta^{\rho}_{\mu} = \begin{cases} 1 \text{ if } \mu = \rho \\ 0 \text{ if } \mu \neq \rho \end{cases}$$

Hence Eq. (1.9) is just a statement that

$$gg^{-1} = I$$

where I is the identity matrix.

In relativity, it is convenient to label coordinates by a number called an *index*. We take $ct = x^0$ and $(x, y, z) \rightarrow (x^1, x^2, x^3)$. Then an event in spacetime is labeled by the coordinates of a *contravariant* vector.

$$x^{\mu} = (x^0, x^1, x^2, x^3) \tag{1.10}$$

Contravariant refers to the way the vector transforms under a Lorentz transformation, but just remember that a contravariant vector has raised indices. A *covariant* vector has lowered indices as

$$x_{\mu} = (x_0, x_1, x_2, x_3)$$

An index can be raised or lowered using the metric. Specifically,

$$x_{\alpha} = g_{\alpha\beta}x^{\beta} \qquad x^{\alpha} = g^{\alpha\beta}x_{\beta} \tag{1.11}$$

Looking at the metric, you can see that the components of a covariant vector are related to the components of a contravariant vector by a change in sign as

$$x_0 = x^0 \qquad x_1 = -x^1 \qquad x_2 = -x^2 \qquad x_3 = -x^3$$

We use the *Einstein summation convention* to represent sums. When an index is repeated in an expression once in a lowered position and once in a raised position, this indicates a sum, that is,

$$s_{\alpha}s^{\alpha} \equiv \sum_{\alpha=0}^{3} s_{\alpha}s^{\alpha} = s_0s^0 + s_1s^1 + s_2s^2 + s_3s^3$$

So for example, the index lowering expression in Eq. (1.11) is really shorthand for

$$x_\alpha = g_{\alpha\beta}x^\beta = g_{\alpha 0}x^0 + g_{\alpha 1}x^1 + g_{\alpha 2}x^2 + g_{\alpha 3}x^3$$

Greek letters such as α, β, μ, and ν are taken to range over all spacetime indices, that is, $\mu = 0, 1, 2,$ and 3. If we want to reference spatial indices only, a Latin letter such as *i, j,* and *k* is used. That is, $i = 1, 2,$ and 3.

LORENTZ TRANSFORMATIONS

A *Lorentz transformation* Λ allows us to transform between different inertial reference frames. For simplicity, consider an inertial reference frame x'^μ moving along the x axis with respect to another inertial reference frame x^μ with speed $v < c$. If we define

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{1.12}$$

Then the Lorentz transformation that connects the two frames is given by

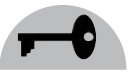
$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma/c & 0 & 0 \\ -\beta\gamma/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.13}$$

Specifically,

$$\begin{aligned} x'^0 &= \gamma\left(x^0 - \frac{\beta}{c}x^1\right) \\ x'^1 &= \gamma\left(x^1 - \frac{\beta}{c}x^0\right) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned} \tag{1.14}$$

We can write a compact expression for a Lorentz transformation relating two sets of coordinates as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.15}$$



The *rapidity* ϕ is defined as

$$\tanh\phi = \beta = \frac{v}{c} \tag{1.16}$$

Using the rapidity, we can view a Lorentz transformation as a kind of rotation (mathematically speaking) that rotates time and spatial coordinates into each other, that is,

$$x'^0 = -x^1 \sinh\phi + x^0 \cosh\phi$$
$$x'^1 = x^1 \cosh\phi - x^0 \sinh\phi$$

Changing velocity to move from one inertial frame to another is done by a Lorentz transformation and we refer to this as a *boost*.

We can extend the shorthand index notation used for coordinates to derivatives. This is done with the following definition:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial x^0} = \partial_0 \qquad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x^1} = \partial_1$$
$$\frac{\partial}{\partial y} \to \frac{\partial}{\partial x^2} = \partial_2 \qquad \frac{\partial}{\partial z} \to \frac{\partial}{\partial x^3} = \partial_3$$

We can raise an index on these expressions so that

$$\partial^\mu = g^{\mu\nu}\partial_\nu$$
$$\partial^0 = \partial_0 \qquad \partial^i = -\partial_i$$

In special relativity many physical vectors have spatial and time components. We call such objects *4-vectors* and denote them with italic font (sometimes with an index) reserving the use of an arrow for the spatial part of the vector. An arbitrary 4-vector A^μ has components

$$A^\mu = (A^0, A^1, A^2, A^3)$$
$$A_\mu = (A_0, -A_1, -A_2, -A_3)$$
$$A^\mu = (A^0, \vec{A})$$
$$A_\mu = (A_0, -\vec{A})$$

We denote the ordinary vector part of a 4-vector as a 3-vector. So the 3-vector part of A^μ is $\vec{A}$. The magnitude of a vector is computed using a generalized dot product, like

$$\begin{aligned} A \cdot A = A^\mu A_\mu &= A^0 A_0 - A^1 A_1 - A^2 A_2 - A^3 A_3 \\ &= g^{\mu\nu} A_\mu A_\nu \end{aligned}$$

This magnitude is a *scalar*, which is invariant under Lorentz transformations. When a quantity is invariant under Lorentz transformations, all inertial observers agree on its value which we call the *scalar product.* A consequence of the fact that the scalar product is invariant, meaning that $x'^\mu x'_\mu = x^\mu x_\mu$, is

$$\Lambda^{\alpha\beta}\Lambda_{\alpha\mu} = \delta^\beta_\mu \tag{1.17}$$

Now let's consider derivatives using relativistic notation. The derivative of a field is written as

$$\frac{\partial\varphi}{\partial x^\mu} = \partial_\mu \varphi \tag{1.18}$$

The index is lowered because as written, the derivative is a covariant 4-vector. The components of the vector are

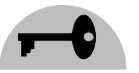
$$\left(\frac{\partial\varphi}{\partial x^0}, \frac{\partial\varphi}{\partial x^1}, \frac{\partial\varphi}{\partial x^2}, \frac{\partial\varphi}{\partial x^3}\right)$$

We also have

$$\frac{\partial\varphi}{\partial x_\mu} = \partial^\mu \varphi$$

which is a contravariant 4-vector. Like any 4-vector, we can compute a scalar product, which is the four-dimensional generalization of the Laplacian called the *D'Alembertian operator* which using ordinary notation is $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box$. Using the relativistic notation for derivatives together with the generalized dot product we have

$$\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box \tag{1.19}$$



One 4-vector that is of particular importance is the *energy-momentum 4-vector* which unifies the energy and momentum into a single object. This is given by

$$\begin{aligned} p_\mu &= (E, -\vec{p}) = (E, -p_1, -p_2, -p_3) \\ \Rightarrow p^\mu &= (E, \vec{p}) = (E, p_1, p_2, p_3) \end{aligned} \tag{1.20}$$

The magnitude of the energy-momentum 4-vector gives us the Einstein relation connecting energy, momentum, and mass.

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 \tag{1.21}$$

We can always choose a Lorentz transformation to boost to a frame in which the 3-momentum of the particle is zero $\vec{p} = 0$ giving Einstein's famous relation between energy and rest mass, like

$$E = mc^2$$

Another important 4-vector is the current 4-vector *J*. The time component of this vector is the charge density ρ while the 3-vector part of *J* is the current density $\vec{J}$. That is,

$$J^\mu = (\rho, J_x, J_y, J_z) \tag{1.22}$$

The current 4-vector is conserved, in the sense that

$$\partial_\mu J^\mu = 0 \tag{1.23}$$

which is nothing other than the familiar relation for conservation of charge as shown here.

$$\begin{aligned} &\partial_\mu J^\mu = \partial_t \rho + \partial_x J^x + \partial_y J^y + \partial_z J^z = 0 \\ &\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \end{aligned}$$

A Quick Overview of Particle Physics

The main application of quantum field theory is to the study of particle physics. This is because quantum field theory describes the fundamental particles and their interactions using what scientists call the *standard model.* In this framework, the standard model is believed to describe all physical phenomena with the exception of gravity. There are three fundamental interactions or forces described in the standard model:

- The electromagnetic interaction
- The weak interaction
- The strong interaction

Each force is mediated by a force-carrying particle called a *gauge boson.* Being a boson, a force-carrying particle has integral spin. The gauge bosons for the electromagnetic, weak, and strong forces are all spin-1 particles. If gravity is quantized, the force-carrying particle (called the *graviton*) is a spin-2 particle.

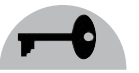
Forces in nature are believed to result from the exchange of the gauge bosons. For each interaction, there is a field, and the gauge bosons are the quanta of that field. The number of gauge bosons that exist for a particular field is given by the number of *generators* of the field. For a particular field, the generators come from the unitary group used to describe the symmetries of the field (this will become clearer later in the book).

THE ELECTROMAGNETIC FORCE

The symmetry group of the electromagnetic field is a unitary transformation, called $U(1)$. Since there is a single generator, the force is mediated by a single particle, which is known to be massless. The electromagnetic force is due to the exchange of photons, which we denote by γ. The photon is spin-1 and has two polarization states. If a particle is massless and spin-1, it can only have two polarization states. Photons do not carry charge.

THE WEAK FORCE

The gauge group of the weak force is $SU(2)$ which has three generators. The three physical gauge bosons that mediate the weak force are W^+, W^-, and Z. As we will see, these particles are superpositions of the generators of the gauge group. The gauge bosons for the weak force are massive.



- W^+ has a mass of 80 GeV/c^2 and carries +1 electric charge.
- W^- has a mass of 80 GeV/c^2 and carries −1 electric charge.
- Z has a mass of 91 GeV/c^2 and is electrically neutral.

The massive gauge bosons of the weak interaction are spin-1 and can have three polarization states.

THE STRONG FORCE

The gauge group of the strong force is $SU(3)$ which has eight generators. The gauge bosons corresponding to these generators are called *gluons*. Gluons mediate interactions between quarks (see below) and are therefore responsible for binding neutrons and protons together in the nucleus. A gluon is a massless spin-1 particle, and like the photon, has two polarization states. Gluons carry the charge of the strong force, called *color*. Since gluons also carry color charge they can interact among themselves, something that is not possible with photons since photons carry no charge. The theory that describes the strong force is called *quantum chromodynamics.*

THE RANGE OF A FORCE

The *range of a force* is dictated primarily by the mass of the gauge boson that mediates this force. We can estimate the range of a force using simple arguments based on the uncertainty principle. The amount of energy required for the exchange of a force mediating particle is found using Einstein's relation for rest mass as

$$\Delta E \approx mc^2$$

Now we use the uncertainty principle to determine how long the particle can exist as shown here.

$$\Delta t \approx \frac{\hbar}{\Delta E} = \frac{\hbar}{mc^2}$$

The special theory of relativity tells us that nothing travels faster than the speed of light c. So, we can use the speed of light to set an upper bound on the velocity of the force-carrying particle, and estimate the range it travels in a time Δt, that is,

$$\text{Velocity} = \frac{\text{distance}}{\text{time}}$$

$$\Rightarrow \quad \Delta x = c\Delta t = \frac{c\hbar}{mc^2} = \frac{\hbar}{mc}$$

This is the range of the force. From this relation, you can see that if $m \to 0$, $\Delta x \to \infty$. So the range of the electromagnetic force is infinite. The range of the weak force, however, is highly constrained because the gauge bosons of the weak force have large masses. Plugging in the mass of the W as $80\ \text{GeV/c}^2$ you can verify that the range is

$$\Delta x \approx 10^{-3}\ \text{fm}$$

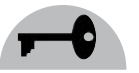
This explains why the weak force is only felt over nuclear distances. This argument does not apply to gluons, which are massless, because the strong force is more complicated and involves a concept known as *confinement.* As stated above, the charge of the strong force is called *color charge*, and gluons carry color. Color charge has a strange property in that it exerts a constant force that binds color-carrying particles together. This can be visualized using the analogy of a rubber band. The stronger you pull on the rubber band, the tighter it feels. If you don't pull on it at all, it hangs loose. The strong force acts like a rubber band. At very short distances, it is relaxed and the particles behave as free particles. As the distance between them increases, the force gets them back in stronger pulling. This limits the range of the strong force, which is believed to be on the order of 10^{-15} m, the dimension of a nuclear particle. As a result of confinement, gluons are involved in mediating interactions between quarks, but are only indirectly responsible for the binding of neutrons and protons, which is accomplished through secondary particles called *mesons*.

Elementary Particles

The elementary particles of quantum field theory are treated as mathematical point-like objects that have no internal structure. The particles that make up matter all carry spin-1/2 and can be divided into two groups, *leptons* and *quarks.* Each group comes in three "families" or "generations." All elementary particles experience the gravitational force.

LEPTONS

Leptons interact via the electromagnetic and weak interaction, but do not participate in the strong interaction. Since they do not carry color charge, they do not participate in the strong interaction. They can carry electric charge e, which we denote as -1



(the charge of the electron), or they can be electrically neutral. The leptons include the following particles:

- The electron e carries charge -1 and has a mass of $0.511\ \text{MeV/c}^2$.
- The muon μ^- carries charge -1 and has a mass of $106\ \text{MeV/c}^2$.
- The tau τ^- carries charge -1 and has a mass of $1777\ \text{MeV/c}^2$.

Each type of lepton described above defines one of the three families that make up the leptons. In short, the muon and tau are just heavy copies of the electron. Physicists are not sure why there are three families of particles. The muon and tau are unstable and decay into electrons and neutrinos.

Corresponding to each particle above, there is a neutrino. It was thought for a long time that neutrinos were massless, but recent evidence indicates this is not the case, although experiment puts small bounds on their masses. Like the electron, muon, and tau, the three types of neutrinos come with masses that increase with each family. They are electrically neutral and are denoted by

- Electron neutrino ν_e
- Muon neutrino ν_μ
- Tau neutrino ν_τ

Since they are electrically neutral, the neutrinos do not participate in the electromagnetic interaction. Since they are leptons, they do not participate in the strong interaction. They interact only via the weak force.

To each lepton there corresponds an antilepton. The antiparticles corresponding to the electron, muon, and tau all carry charge of +1, but they have the same masses. They are denoted as follows:

- The positron e^+ carries charge +1 and has a mass of $0.511\ \text{MeV/c}^2$.
- The antimuon μ^+ carries charge +1 and has a mass of $106\ \text{MeV/c}^2$.
- The antitau τ^+ carries charge +1 and has a mass of $1777\ \text{MeV/c}^2$.

In particle physics, we often indicate an antiparticle (a particle with the same properties but opposite charge) with an overbar; so if p is a given particle, we can indicate its corresponding antiparticle by $\overline{p}$. We will see later that charge is not the only quantum number of interest; a lepton also carries a quantum number called *lepton number*. It is +1 for a particle and -1 for the corresponding antiparticle. The antineutrinos $\overline{\nu}_e, \overline{\nu}_\mu$, and $\overline{\nu}_\tau$, like their corresponding particles, are also electrically neutral, but while the neutrinos ν_e, ν_μ, and ν_τ all have lepton number +1, the antineutrinos $\overline{\nu}_e, \overline{\nu}_\mu$, and $\overline{\nu}_\tau$ have lepton number -1.

In particle interactions, lepton number is always conserved. Particles that are not leptons are assigned a lepton number 0. Lepton number explains why there are antineutrinos, because they are neutral like ordinary neutrinos. Consider the beta decay of a neutron as shown here.

$$n \rightarrow p + e + \overline{\nu}_e$$

A neutron and proton are not leptons, hence they carry lepton number 0. The lepton number must balance on each side of the reaction. On the left we have total lepton number 0. On the right we have

$$0 + n_e + n_{\overline{\nu}_e}$$

Since the electron is a lepton, $n_e = 1$. This tells us that the neutrino emitted in this decay must be an antineutrino, and the lepton number is $n_{\overline{\nu}_e} = -1$ allowing lepton number to be conserved in the reaction.

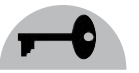
QUARKS

Quarks are fundamental particles that make up the neutron and proton. They carry electrical charge and hence participate in the electromagnetic interaction. They also participate in the weak and strong interactions. *Color charge*, which is the charge of the strong interaction, can come in *red*, *blue*, or *green*. These color designations are just labels, so they should not be taken literally. There is also "anticolor" charge, antired, antiblue, and antigreen. Color charge can only be arranged such that the total color of a particle combination is *white*. There are three ways to get white color charge:

- Put three quarks together, one red, one blue, and one green.
- Put three quarks together, one antired, one antiblue, and one antigreen.
- Put two quarks together, one colored and one anticolored, for example a red quark and an antired quark.

The charge carried by a quark is −1/3 or +2/3 (in units of electric charge e). There are six types or "flavors" of quarks:

- Up quark u with charge +2/3
- Down quark d with charge −1/3
- Strange quark s with charge +2/3



- Charmed quark c with charge −1/3
- Top quark t with charge +2/3
- Bottom quark b with charge −1/3

Like the leptons, the quarks come in three families. One member of a family has charge +2/3 and the other has charge −1/3. The families are (u,d), (s,c), and (t,b). With each family, the mass increases. For example, the mass of the up quark is only

$$m_u \approx 4 \text{ MeV/c}^2$$

while the mass of the top quark is a hefty

$$m_t \approx 172 \text{ GeV/c}^2$$

which is as heavy as a single gold nucleus. Like the leptons, there are antiparticles corresponding to each quark.

Bound states of quarks are called *hadrons.* Bound states of observed quarks consist of two or three quarks only. A *baryon* is a hadron with three quarks or three antiquarks. Two famous baryons are

- The proton, which is the three-quark state uud
- The neutron, which is the three-quark state udd

Bound states consisting of a quark and antiquark are called *mesons.* These include:

The pion $\pi^0 = u\bar{u}$ or $d\bar{d}$

The charged pion $\pi^+ = u\bar{d}$ or $\pi^- = \bar{u}d$

SUMMARY OF PARTICLE GENERATIONS OR FAMILIES

The elementary particles come in three generations:

- The first generation includes the electron, electron neutrino, the up quark, the down quark, and the corresponding antiparticles.
- The second generation includes the muon, muon neutrino, strange quark, and charmed quark, along with the corresponding antiparticles.
- The third generation includes the tau, the tau neutrino, the top quark, and the bottom quark, along with the corresponding antiparticles.

The Higgs Mechanism

As the standard model of particle physics is formulated, the masses of all the particles are 0. An extra field called the *Higgs field* has to be inserted by hand to give the particles mass. The quantum of the Higgs field is a spin-0 particle called the *Higgs boson*. The Higgs boson is electrically neutral.

The Higgs field, if it exists, is believed to fill all of empty space throughout the entire universe. Elementary particles acquire their mass through their interaction with the Higgs field. Mathematically we introduce mass into a theory by adding interaction terms into the Lagrangian that couple the field of the particle in question to the Higgs field. Normally, the lowest energy state of a field would have an expectation value of zero. By symmetry breaking, we introduce a nonzero lowest energy state of the field. This procedure leads to the acquisition of mass by the particles in the theory.

Qualitatively, you might think of the Higgs field by imagining the differences between being on land and being completely submerged in water. On dry land, you can move your arm up and down without any trouble. Under water, moving your arm up and down is harder because the water is resisting your movement. We can imagine the movement of elementary particles being resisted by the Higgs field, with each particle interacting with the Higgs field at a different strength. If the coupling between the Higgs field and the particle is strong, then the mass of the particle is large. If it is weak, then the particle has a smaller mass. A particle like the photon with zero rest mass doesn't interact with the Higgs field at all. If the Higgs field didn't exist at all, then all particles would be massless. It is not certain what the mass of the Higgs boson is, but current estimates place an upper limit of $\approx 140\ \mathrm{GeV/c^2}$. When the Large Hadron Collider begins operation in 2008, it should be able to detect the Higgs, if it exists.

Grand Unification

The standard model, as we have described above, consists of the electromagnetic interaction, the weak force, and quantum chromodynamics. Theorists would like to unify these into a single force or interaction. Many problems remain in theoretical physics, and in the past, many problems have been solved via some kind of unification. In many cases two seemingly different phenomena are actually two sides of the same coin. The quintessential example of this type of reasoning is the discovery by Faraday, Maxwell, and others that light, electricity, and magnetism are all the same physical phenomena that we now group together under electromagnetism.

Electromagnetism and the weak force have been unified into a single theoretical framework called *electroweak theory*. A *grand unified theory* or *GUT* is an attempt to bring quantum chromodynamics (and hence the strong force) into this unified framework.

If such a theory is valid, then there is a *grand unification energy* at which the electromagnetic, weak, and strong forces become unified into a single force. There is some support for this idea since the electromagnetic and weak force are known to become unified at high energies (but at lower energies than where unification with the strong force is imagined to occur).

Supersymmetry

There exists yet another unification scheme beyond that tackled by the GUTs. In particle physics, there are two basic types of particles. These include the spin-1/2 matter particles (fermions) and the spin-1 force-carrying particles (bosons). In elementary quantum mechanics, you no doubt learned that bosons and fermions obey different statistics. While the Pauli exclusion principle prevents two fermions from inhabiting the same state, there is no such limitation for bosons.

One might wonder why there are these two types of particles. In supersymmetry, an attempt is made to apply the reasoning of Maxwell and propose that a symmetry exists between bosons and fermions. For each fermion, supersymmetry proposes that there is a boson with the same mass, and vice versa. The partners of the known particles are called *superpartners*. Unfortunately, at this time there is no evidence that this is the case. The fact that the superpartners do not have the same mass indicates either that the symmetry of the theory is broken, in which case the masses of the superpartners are much larger than expected, or that the theory is not correct at all and supersymmetry does not exist.

String Theory

The ultimate step forward for quantum field theory is a unified theory known as *string theory.* This theory was originally proposed as a theory of the strong interaction, but it fell out of favor when quantum chromodynamics was developed. The basic idea of string theory is that the fundamental objects in the universe are not pointlike elementary particles, but are instead objects spread out in one dimension called strings. Excitations of the string give the different particles we see in the universe.

String theory is popular because it appears to be a completely unified theory. Quantum field theory unifies quantum mechanics and special relativity, and as a result is able to describe interactions involving three of the four known forces.

Gravity, the fourth force, is left out. Currently gravity is best described by Einstein's general theory of relativity, a classical theory that does not take quantum mechanics into account.

Efforts to bring quantum theory into the gravitational realm or vice versa have met with some difficulty. One reason is that interactions at a point cause the theory to "blow up"—in other words you get calculations with infinite results. By proposing that the fundamental objects of the theory are strings rather than point particles, interactions are spread out and the divergences associated with gravitational interactions disappear. In addition, a spin-2 state of the string naturally arises in string theory. It is known that the quantum of the gravitational field, if it exists, will be a massless spin-2 particle. Since this arises naturally in string theory, many people believe it is a strong candidate for a unified theory of all interactions.

Summary

Quantum field theory is a theoretical framework that unifies nonrelativistic quantum mechanics with special relativity. One consequence of this unification is that the types and number of particles can change in an interaction. As a result, the theory cannot be implemented using a single particle wave equation. The fundamental objects of the theory are quantum fields that act as operators, able to create or destroy particles.

Quiz

1. A quantum field
   (a) Is a field with quanta that are operators
   (b) Is a field parameterized by the position operator
   (c) Commutes with the Hamiltonian
   (d) Is an operator that can create or destroy particles
2. The particle generations
   (a) Are in some sense duplicates of each other, with each generation having increasing mass
   (b) Occur in pairs of three particles each
   (c) Have varying electrical charge but the same mass
   (d) Consists of three leptons and three quarks each

3. In relativistic situations
   (a) Particle number and type is not fixed
   (b) Particle number is fixed, but particle types are not
   (c) Particle number can vary, but new particle types cannot appear
   (d) Particle number and types are fixed
4. In quantum field theory
   (a) Time is promoted to an operator
   (b) Time and momentum satisfy a commutation relation
   (c) Position is demoted from being an operator
   (d) Position and momentum continue to satisfy the canonical commutation relation
5. Leptons experience
   (a) The strong force, but not the weak force
   (b) The weak force and electromagnetism
   (c) The weak force only
   (d) The weak force and the strong force
6. The number of force-carrying particles is
   (a) Equivalent to the number of generators for the fields gauge group
   (b) Random
   (c) Proportional to the number of fundamental matter particles involved in the interaction
   (d) Proportional to the number of generators minus one
7. The gauge group of the strong force is:
   (a) $SU(2)$
   (b) $U(1)$
   (c) $SU(3)$
   (d) $SU(1)$
8. Antineutrinos
   (a) Have charge -1 and lepton number 0
   (b) Have lepton number $+1$ and charge 0
   (c) Have lepton number -1 and charge 0
   (d) Are identical to neutrinos, since they carry no charge

9. The lightest family of elementary particles is
   (a) The electron, muon, and neutrino
   (b) The electron, up quark, and down quark
   (c) The electron, electron neutrino, up quark, and down quark
   (d) The electron and its antiparticle
10. The Higgs field
   (a) Couples the W and Z bosons to each other
   (b) Is a zero mass field
   (c) Has zero mass and charge +1
   (d) Gives mass to elementary particles

CHAPTER 2

Lagrangian Field Theory

We begin our study of quantum field theory by building up some fundamental mathematical tools that can be applied to any fundamental physical theory. The main quantity of interest in this chapter is a function known as the *Lagrangian*, which is constructed by taking the difference of the kinetic and potential energies. In classical mechanics the Lagrangian is an equivalent method to Newtonian mechanics that can be used to derive the equations of motion. When Lagrangian methods are applied to fields, we can use the same techniques to derive the field equations.

Basic Lagrangian Mechanics

For now, we work in one spatial dimension x and consider the motion of a single particle. Let T be the kinetic energy of the particle moving in a potential V. The Lagrangian L is defined as

$$L = T - V \tag{2.1}$$

The Lagrangian is a foundational concept which captures all the dynamics of the system and allows us to determine many useful properties such as averages and dynamic behavior.

Given L we can find the equations of motion from the *Euler-Lagrange equations.* For a single particle moving in one dimension, these are given by the single equation.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{2.2}$$

where we have used a dot to denote differentiation with respect to time, that is,

$$\dot{x} = \frac{dx}{dt}$$

EXAMPLE 2.1

Consider a particle of mass m with kinetic energy $T = \frac{1}{2}m\dot{x}^2$ moving in one dimension in a potential $V(x)$. Use the Euler-Lagrange equations to find the equation of motion.

SOLUTION

We follow five basic steps. (1) First we write down the Lagrangian L, then we calculate the derivatives we need for the Euler-Lagrange equation. (2) The first is the derivative of L with respect to $\dot{x}$, $\frac{\partial L}{\partial \dot{x}}$. (3) We take the time derivative of this quantity, $\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}})$. (4) Next is the derivative of L with respect to x, $\frac{\partial L}{\partial x}$. (5) We are finally able to form the equation $\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = \frac{\partial L}{\partial x}$.

Begin by writing the Lagrangian [step (1)] which is

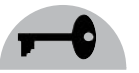
$$L = \frac{1}{2}m\dot{x}^2 - V(x)$$

Next is the derivative of L with respect to $\dot{x}$. When doing calculations involving the Lagrangian, we treat $\dot{x}$ just as though it were an independent variable. For example, $\frac{\partial}{\partial \dot{x}}(\dot{x})^2 = 2\dot{x}$ and $\frac{\partial}{\partial \dot{x}}V(x) = 0$. Applying this [step (2)] we get

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left[\frac{1}{2}m\dot{x}^2 - V(x)\right] = m\dot{x}$$

which is a momentum term: mass times velocity. Now we take the time derivative of this momentum [step (3)] which is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = \dot{m}\dot{x} + m\ddot{x} = m\ddot{x}$$



and we have mass times acceleration. The remaining derivative is now a simple calculation [step (4)], that is,

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}m\dot{x}^2 - V(x)\right] = -\frac{\partial V}{\partial x}$$

Next [step (5)] is writing the equation which describes the dynamical behavior of the system, like

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \Rightarrow m\ddot{x} = -\frac{\partial V}{\partial x}$$

This elegant result is quite familiar. From classical mechanics, we recall that if a force F is conservative then

$$\vec{F} = -\nabla V$$

In one dimension this becomes

$$F = -\frac{\partial V}{\partial x}$$

Therefore, $\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = F$. Using our calculation we have

$$m\ddot{x} = -\frac{\partial V}{\partial x} = F$$

Hence we arrive at Newton's second law,

$$F = m\ddot{x} = ma$$

where the acceleration a is given by $a = d^2x/dt^2 = \ddot{x}$.

EXAMPLE 2.2

Consider a particle of mass m undergoing simple harmonic motion. The force on the particle is given by Hooke's law $F(x) = -kx$. Determine the equation of motion by using the Euler-Lagrange equation.

SOLUTION

Once again, the kinetic energy is given by

$$T = \frac{1}{2}m\dot{x}^2$$

We integrate the force $F(x) = -kx$ to compute the potential and find that

$$V = \frac{1}{2} k x^2$$

Using Eq. (2.1) the Lagrangian is [step (1)]

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2$$

[step (2)] $\frac{\partial}{\partial \dot{x}}(L) = \frac{\partial}{\partial \dot{x}}(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2) = m\dot{x}$. As above, [step (3)] the time derivative of the momentum is a force $\frac{d}{dt}(m\dot{x}) = m\ddot{x}$ and the last derivative [step (4)] is

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2\right] = -kx$$

To obtain the equation of motion [step (5)] for the particle we use Eq. (2.2). We have

$$m\ddot{x} = -kx$$

This is the familiar equation of a simple harmonic oscillator, that is,

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0$$

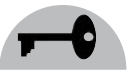
where the natural frequency $\omega_0^2 = k/m$.

The Action and the Equations of Motion

If we integrate the Lagrangian with respect to time, we obtain a new quantity called the *action* which we denote by S

$$S = \int L\, dt \tag{2.3}$$

The action is *functional* because it takes a *function* as argument and returns a *number.* Particles always follow a path of least action. By *varying (minimzing the variance of)* the action, we can determine the path actually followed by a particle. Consider two fixed points $x(t_1)$ and $x(t_2)$. There are an infinite number of paths connecting these points. This means that there are an infinite number of paths for the particle to follow between these two points. The actual path that the particle follows is the path of least action. The path of least action represents a minimum. To find this path we



minimize the variance of the action. We do this by describing the unknown action as minimal term and a variation.

$$S \to S + \delta S$$

then the path followed by the particle is the one for which

$$\delta S = 0 \tag{2.4}$$

This is the path with zero variation. This is the path of least action.

Computing the variation δS leads to the equations of motion for the system. To see how this works, we start with a simple example, returning to the derivation of Newton's second law. We consider a small change in coordinates given by

$$x \to x + \varepsilon$$

where ε is small. This variation is constrained to keep the end points fixed, that is,

$$\varepsilon(t_1) = \varepsilon(t_2) = 0 \tag{2.5}$$

Using a Taylor expansion, the potential can be approximated as

$$V(x+\varepsilon) \approx V(x) + \varepsilon \frac{dV}{dx}$$

The action then becomes

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \frac{1}{2} m\dot{x}^2 - V(x) dt = \int_{t_1}^{t_2} \frac{1}{2} m(\dot{x} + \dot{\varepsilon})^2 - \left(V(x) - \varepsilon \frac{dV}{dx} \right) dt$$

Now we expand out the first term, the kinetic energy term, giving

$$(\dot{x} + \dot{\varepsilon})^2 = \dot{x}^2 + 2\dot{x}\dot{\varepsilon} + \dot{\varepsilon}^2 \approx \dot{x}^2 + 2\dot{x}\dot{\varepsilon}$$

We dropped the $\dot{\varepsilon}^2$ term, since by assumption ε is small, so squaring it gives a term we can neglect. That is, we only keep the leading order terms. So we have

$$S = \int_{t_1}^{t_2} \frac{1}{2} m(\dot{x}^2 + 2\dot{x}\dot{\varepsilon}) - \left(V(x) + \varepsilon \frac{dV}{dx} \right) dt$$

This expression can be written in a more useful fashion with some manipulation. The idea is to isolate the terms containing ε. We can do this by applying integration

by parts to the $2\dot{x}\dot{\varepsilon}$ term, transferring the time derivative from ε to $\dot{x}$. First let's recall the formula for integration by parts

$$\int_{t_1}^{t_2} f(t)\frac{dg}{dt}dt = f(t)g(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} g(t)\frac{df}{dt}dt \tag{2.6}$$

In our case, $f(t)=\dot{x}$ and $\frac{dg}{dt}=\dot{\varepsilon}$. Recalling Eq. (2.5), the fact that the variation vanishes at the end points of the interval, the boundary term vanishes in our case. Hence,

$$\int_{t_1}^{t_2}\frac{1}{2}m(\dot{x}+\dot{\varepsilon})^2dt = \frac{1}{2}m\int_{t_1}^{t_2}(\dot{x}+\dot{\varepsilon})^2dt \approx \frac{1}{2}m\int_{t_1}^{t_2}(\dot{x}^2+2x\ddot{\varepsilon})dt$$

$$\int_{t_1}^{t_2}\frac{1}{2}m(\dot{x}^2+2\dot{x}\dot{\varepsilon})\,dt = \int_{t_1}^{t_2}\frac{1}{2}m\dot{x}^2dt - \int_{t_1}^{t_2}m\ddot{x}\varepsilon dt$$

We can now collect the ε terms

$$\begin{aligned} S &= \int_{t_1}^{t_2}\frac{1}{2}m\dot{x}^2dt - \int_{t_1}^{t_2}m\ddot{x}\varepsilon dt - \int_{t_1}^{t_2}\left(V(x)+\varepsilon\frac{dV}{dx}\right)dt \\ &= \int_{t_1}^{t_2}\frac{1}{2}m\dot{x}^2 - V(x)dt + \int_{t_1}^{t_2}\left(-m\ddot{x}-\frac{dV}{dx}\right)\varepsilon dt \\ &= S+\delta S \end{aligned}$$

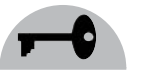
The requirement that $\delta S=0$ can be satisfied only if the integral of the second term is 0. Since the end points are arbitrary, the integrand must be identically 0, that is,

$$m\ddot{x}+\frac{dV}{dx}=0 \Rightarrow m\ddot{x}=-\frac{dV}{dx}$$

Now let's consider a more general situation where there are N generalized coordinates $q_i(t)$ where $i=1,\ldots,N$. Considering a Lagrangian expressed in terms of these coordinates and their first derivatives only, we have an action of the form

$$S=\int_{t_1}^{t_2}L(q,\dot{q})\,dt$$

In this case the system evolves from some initial point $q_1=q(t_1)$ to some final point $q_2=q(t_2)$. To find the trajectory followed by the system, we apply the principle of



least action and solve $\delta S = 0$. Once again, the end points of the trajectory are fixed so that

$$\delta q(t_1) = \delta q(t_2) = 0$$

Also note that

$$\delta \dot{q}(t) = \frac{d}{dt}(\delta q) \tag{2.7}$$

Then,

$$\begin{aligned}
\delta S &= \delta \int_{t_1}^{t_2} L(q, \dot{q})\, dt \\
&= \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right] dt \\
&= \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}(\delta q_i) \right] dt
\end{aligned}$$

To move from the second to the third line, we applied Eq. (2.7). Now we use integration by parts on the second term, giving

$$\delta S = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] dt$$

For this expression to vanish, since δq_i is arbitrary each coordinate index must satisfy

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \tag{2.8}$$

which are of course the Euler-Lagrange equations and $i = 1, \ldots, N$. Therefore, we see that the principle of least action gives rise to the Euler-Lagrange equations. Hence the Lagrangian satisfies the Euler-Lagrange equation independently for each coordinate.

Canonical Momentum and the Hamiltonian

In Examples 2.1 and 2.2 we saw that the derivative of the kinetic energy with respect to velocity was the momentum: a completely classical result, that is,

$$\frac{\partial}{\partial \dot{x}}(L) = \frac{\partial}{\partial \dot{x}}(T - V) = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m \dot{x}^2 \right) = m\dot{x}$$

This result can be made more general and more useful. The *canonical* momentum is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2.9}$$

This allows us to define a Hamiltonian function, which is given in terms of the Lagrangian and the canonical momentum as

$$H(p,q) = \sum_i p_i \dot{q}_i - L \tag{2.10}$$

Lagrangian Field Theory

Now that we have reviewed the basics of the Lagrangian formalism, we are prepared to generalize these techniques and apply them to fields, that is, functions on spacetime $\varphi(x,t)$ which we write more compactly as $\varphi(x)$. In the continuous case, we actually work with the Lagrangian density.

$$L = T - V = \int \mathcal{L}\, d^3x \tag{2.11}$$

The action S is then the time integral of this expression.

$$S = \int dt\, L = \int \mathcal{L}\, d^4x \tag{2.12}$$

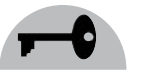
Typically, the Lagrangians encountered in quantum field theory depend only on the fields and their first derivatives.

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(\varphi, \partial_\mu \varphi) \\ L &\to L(\varphi, \partial_\mu \varphi) \end{aligned} \tag{2.13}$$

Moreover we are interested in fields that are *local*, meaning that at a given spacetime point x, the Lagrangian density depends on the fields and their first derivatives evaluated at that point.

Now we apply the principle of least action to Eq. (2.12). We vary the action with respect to the field $\varphi(x)$ and with respect to the first derivative of the field $\partial_\mu \varphi(x)$ as follows:

$$\begin{aligned} 0 &= \delta S \\ &= \delta \int d^4x\, \mathcal{L} \\ &= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial [\partial_\mu \varphi]} \delta(\partial_\mu \varphi) \right\} \end{aligned}$$



Now we use the fact that

$$\delta(\partial_\mu \varphi) = \partial_\mu(\delta\varphi)$$

and apply integration by parts to the second term in this expression. The boundary terms vanish because the end points are fixed and the second term becomes

$$\int d^4x \frac{\partial \mathcal{L}}{\partial[\partial_\mu \varphi]} \delta(\partial_\mu \varphi) = \int d^4x \frac{\partial \mathcal{L}}{\partial[\partial_\mu \varphi]} \partial_\mu(\delta\varphi)$$

$$= -\int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial[\partial_\mu \varphi]} \right) \delta\varphi$$

All together, the variation of the action is then

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial[\partial_\mu \varphi]} \right) \right\} \delta\varphi$$

What does it mean when an integral is 0? There are two cases: Either there is cancellation because the integrand takes positive and negative values or the integrand is 0 over the entire domain of integration. In this case the domain of integration can vary, so we can't rely on cancellation and we know the integrand must be 0. That is, the term inside the braces vanishes. This gives us the Euler-Lagrange equations for a field φ.

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial[\partial_\mu \varphi]} \right) = 0 \tag{2.14}$$

The Einstein summation convention is in force, so there is an implied sum on the second term. That is,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial[\partial_\mu \varphi]} \right) = \partial_t \left(\frac{\partial \mathcal{L}}{\partial[\partial_t \varphi]} \right) - \partial_x \left(\frac{\partial \mathcal{L}}{\partial[\partial_x \varphi]} \right) - \partial_y \left(\frac{\partial \mathcal{L}}{\partial[\partial_y \varphi]} \right) - \partial_z \left(\frac{\partial \mathcal{L}}{\partial[\partial_z \varphi]} \right)$$

The canonical momentum density for the field is given by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \tag{2.15}$$

just as in the classical case. The Hamiltonian density is then

$$\mathcal{H} = \pi(x)\dot{\varphi}(x) - \mathcal{L} \tag{2.16}$$

To obtain the Hamiltonian, we integrate this density over space

$$H = \int \mathcal{H}\, d^3x \tag{2.17}$$

EXAMPLE 2.3
Find the equation of motion and the Hamiltonian corresponding to the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left\{(\partial_\mu\varphi)^2 - m^2\varphi^2\right\}$$

SOLUTION
We can find the field equations with a straightforward application of Eq. (2.14). Let's follow the five-step process from the previous examples. Since we are given the Lagrangian, step (1) is done. The remaining steps require some insight.

The trick in doing the problems is that we treated $\partial_\mu\varphi$ as a variable. If you need a crutch to get used to thinking this way, as an analogy think of φ as x and $\partial_\mu\varphi$ as y. Then $\frac{\partial}{\partial\varphi}[\frac{1}{2}(\partial_\mu\varphi)^2]$ is like calculating $\frac{\partial}{\partial x}[\frac{1}{2}(y)^2] = 0$.

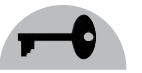
Now we are ready for step (2), compute $\frac{\partial}{\partial(\partial_\mu\varphi)}(L)$.

Begin by expanding the first term in the Lagrangian:

$$(\partial_\mu\varphi)^2 = (\partial_\mu\varphi)(\partial^\mu\varphi) = (\partial_\mu\varphi)g^{\mu\nu}(\partial_\nu\varphi)$$

Hence, we have

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]} &= \frac{\partial}{\partial[\partial_\mu\varphi]}\left(\frac{1}{2}\left\{(\partial_\mu\varphi)^2 - m^2\varphi^2\right\}\right)\\ &= \frac{1}{2}\frac{\partial}{\partial[\partial_\mu\varphi]}(\partial_\mu\varphi)g^{\mu\nu}(\partial_\nu\varphi) - \frac{1}{2}\frac{\partial}{\partial[\partial_\mu\varphi]}(m^2\varphi^2)\end{aligned}$$



Using our observation that we simply act like $\partial_\mu \varphi$ is a variable when computing these derivatives, it is clear that

$$\frac{\partial}{\partial[\partial_\mu \varphi]}(m^2\varphi^2) = 0$$

This means we are left with

$$\frac{\partial \mathcal{L}}{\partial[\partial_\mu \varphi]} = \frac{1}{2}\frac{\partial}{\partial[\partial_\mu \varphi]}(\partial_\mu \varphi) g^{\mu\nu}(\partial_\nu \varphi)$$

But

$$\frac{\partial}{\partial[\partial_\mu \varphi]}(\partial_\mu \varphi) g^{\mu\nu}(\partial_\nu \varphi) = g^{\mu\nu}(\partial_\nu \varphi) + g^{\mu\nu}(\partial_\mu \varphi) = 2\partial^\mu \varphi$$

Meaning that

$$\frac{\partial \mathcal{L}}{\partial[\partial_\mu \varphi]} = \partial^\mu \varphi$$

Step (3) entails taking the derivative of this result.

$$\partial_\mu \left(\frac{\partial L}{\partial[\partial_\mu \varphi]} \right) = \partial_\mu (\partial^\mu \varphi)$$

Next, step (4) is the easiest.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \varphi} &= \frac{\partial}{\partial \varphi}\left[\frac{1}{2}\left\{ (\partial_\mu \varphi)^2 - m^2\varphi^2 \right\} \right] \\
&= \frac{\partial}{\partial \varphi}\left[\frac{1}{2}(\partial_\mu \varphi)^2 \right] - \frac{\partial}{\partial \varphi}\frac{1}{2}(m^2\varphi^2) \\
&= -\frac{\partial}{\partial \varphi}\frac{1}{2}(m^2\varphi^2) = -m^2\varphi
\end{aligned}$$

Finally step (5) is to form the equations of motion.

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial [\partial_\mu \varphi]} \right)$$
$$= -m^2 \varphi - \partial_\mu (\partial^\mu \varphi)$$

Using $\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$, the field equations corresponding to the Lagrangian given in this problem can be written as

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \varphi + m^2 \varphi = 0$$

EXAMPLE 2.4

Derive the *sine-Gordon equation* $\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + \sin \varphi = 0$ from the Lagrangian

$$\mathcal{L} = \frac{1}{2} \left\{ (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right\} + \cos \varphi$$

SOLUTION

First we calculate

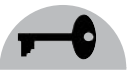
$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial}{\partial \varphi} \left[\frac{1}{2} \left\{ (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right\} + \cos \varphi \right]$$
$$= \frac{\partial}{\partial \varphi} \left[\frac{1}{2} \left\{ (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right\} \right] + \frac{\partial}{\partial \varphi} \cos \varphi$$
$$= \frac{\partial}{\partial \varphi} (\cos \varphi) = -\sin \varphi$$

The Lagrangian only has one spatial coordinate, so

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial [\partial_\mu \varphi]} \right) = \partial_t \left(\frac{\partial \mathcal{L}}{\partial [\partial_t \varphi]} \right) - \partial_x \left(\frac{\partial \mathcal{L}}{\partial [\partial_x \varphi]} \right)$$

Now tackling the time and space derivatives separately leads us to

$$\frac{\partial \mathcal{L}}{\partial [\partial_t \varphi]} = \frac{\partial}{\partial [\partial_t \varphi]} \left[\frac{1}{2} \left\{ (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right\} + \cos \varphi \right] = \partial_t \varphi$$
$$\frac{\partial \mathcal{L}}{\partial [\partial_x \varphi]} = \frac{\partial}{\partial [\partial_x \varphi]} \left[\frac{1}{2} \left\{ (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right\} + \cos \varphi \right] = \partial_x \varphi$$



and so

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\right)=\partial_t\left(\frac{\partial\mathcal{L}}{\partial[\partial_t\varphi]}\right)-\partial_x\left(\frac{\partial\mathcal{L}}{\partial[\partial_x\varphi]}\right)=\partial_t(\partial_t\varphi)-\partial_x(\partial_x\varphi)$$
$$=\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}=\Box\varphi$$

Therefore, the equation of motion in this case is

$$\Box\varphi+\sin\varphi=(\Box+\sin)\varphi=0$$

This equation is called the *sine-Gordon equation*, due to its resemblance to the Klein-Gordon equation.

Symmetries and Conservation Laws

A *symmetry* is a change in perspective that leaves the equations of motion invariant. For example, the change could be a translation in space, a change in time or a rotation. These are external symmetries, that is, they depend upon changes in spacetime. There also exist *internal symmetries*, changes in the fields that do not involve changes with respect to spacetime at all.

A famous theorem in classical mechanics which is very important is known as *Noether's theorem.* This theorem allows us to relate symmetries to conserved quantities like charge, energy, and momentum. Mathematically, a symmetry is some kind of variation to the fields or the Lagrangian that leaves the equations of motion invariant. We will see how to make such a change and then deduce a conserved quantity.

Two of the most fundamental results of physics, conservation of energy and momentum, are due to a symmetry that results from a small displacement in spacetime. That is we let the spacetime coordinates vary according to

$$x^\mu\rightarrow x^\mu+a^\mu \tag{2.18}$$

where a^μ is a small and arbitrary parameter describing a displacement in spacetime. Expanding in Taylor, the field changes according to

$$\varphi(x)\rightarrow\varphi(x+a)=\varphi(x)+a^\mu\partial_\mu\varphi \tag{2.19}$$

Under a small variation (perturbation), the field can be described as

$$\varphi\rightarrow\varphi+\delta\varphi$$

This means we can write the variation in the field explicitly as

$$\delta\varphi = a^{\mu}\partial_{\mu}\varphi \tag{2.20}$$

Now let's reconsider the variation of the Lagrangian. We have, in the case of a Lagrangian depending only on the field and its first derivatives,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\varphi)}\delta(\partial_{\mu}\varphi)$$

From the Euler-Lagrange equations [Eq. (2.14)] we know that

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial[\partial_{\mu}\varphi]}\right)$$

Hence the variation in the Lagrangian can be written as

$$\begin{aligned}\delta\mathcal{L} &= \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial[\partial_{\mu}\varphi]}\right)\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\varphi)}\delta(\partial_{\mu}\varphi)\\ &= \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial[\partial_{\mu}\varphi]}\right)\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\varphi)}\partial_{\mu}(\delta\varphi)\end{aligned}$$

Now we have an expression that can be written as a total derivative. Remember the product rule from ordinary calculus that $(fg)' = f'g + g'f$. We take

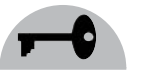
$$f = \frac{\partial\mathcal{L}}{\partial[\partial_{\mu}\varphi]} \qquad g = \delta\varphi$$

Allowing us to write

$$\delta\mathcal{L} = \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial[\partial_{\mu}\varphi]}\delta\varphi\right) = (fg)'$$

Now we can apply Eq. (2.20). Note that the same index can only be used twice in a single expression, so we need to change the label used in Eq. (2.20) to another dummy index, say $\delta\varphi = a^{\mu}\partial_{\mu}\varphi = a^{\nu}\partial_{\nu}\varphi$. The quantities involved are just ordinary scalars, so we can also move them around and write $\delta\varphi = \partial_{\nu}\varphi a^{\nu}$. So, we arrive at the following expression

$$\delta\mathcal{L} = \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial[\partial_{\mu}\varphi]}\partial_{\nu}\varphi\right)a^{\nu}$$



Equivalently, we can also write the variation of the Lagrangian as

$$\delta\mathcal{L} = \partial_\mu(\mathcal{L})a^\mu = \delta^\mu_\nu \partial_\mu(\mathcal{L})a^\nu$$

That is, we consider how it varies directly with respect to the displacement Eq. (2.18). Equating these two results gives

$$\delta\mathcal{L} = \delta^\mu_\nu \partial_\mu(\mathcal{L})a^\nu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\partial_\nu\varphi\right)a^\nu$$

Moving both terms to the same side of the equation gives

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L}\right)a^\nu = 0$$

Now, recall that a^ν is arbitrary. So in order for this expression to vanish, the derivative must be zero. That is,

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L}\right) = 0$$

This expression is so important that we give this quantity its own name. It turns out this is the *energy-momentum tensor.* We write this as

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L} \tag{2.21}$$

Hence the conservation relation expressed by zero total divergence is

$$\partial_\mu T^\mu_\nu = 0 \tag{2.22}$$

Notice that

$$T^0_0 = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\dot{\varphi} - \mathcal{L} = \mathcal{H} \tag{2.23}$$

That is, T^0_0 is nothing other than the Hamiltonian density: it's the energy density and the equation $\partial_0 T^0_0 = 0$ reflects *conservation of energy*. The components of momentum density are given by T^0_i where i runs over the spatial indices. The components of momentum of the field are found by integrating each of these terms over space, that is,

$$P_i = \int d^3x\, T^0_i \tag{2.24}$$

Conserved Currents

Now let's go through the process used in the last section again to see how Noether's theorem can be applied to derive a conserved current and an associated conserved charge. We let the field vary by a small amount

$$\varphi \to \varphi + \delta\varphi \tag{2.25}$$

We then start from the premise that under this variation, the Lagrangian does not change. The variation in the Lagrangian due to Eq. (2.25) will be of the form

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} \tag{2.26}$$

So what we mean is that

$$\delta\mathcal{L} = 0 \tag{2.27}$$

Now following the usual procedure the variation of the Lagrangian due to a variation in the field will be

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(\delta\varphi)$$

Once again, from the Euler-Lagrange equations [Eq. (2.14)] we can write

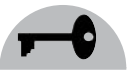
$$\frac{\partial\mathcal{L}}{\partial\varphi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\right)$$

Therefore, we have

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\right)\delta\varphi + \frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\partial_\mu(\delta\varphi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\delta\varphi\right)$$

Since we are operating under the premise that the variation in the field does not change the Lagrangian [Eq. (2.27)], this leads us to the result

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\delta\varphi\right) = 0 \tag{2.28}$$



We call the quantity in the parentheses a *conserved current.* In analogy with electrodynamics we denote it with the letter J and write

$$J^{\mu} = \frac{\partial \mathcal{L}}{\partial[\partial_{\mu}\varphi]}\delta\varphi \tag{2.29}$$

The conservation law Eq. (2.28) then can be written as

$$\partial_{\mu} J^{\mu} = 0 \tag{2.30}$$

This is the central result of Noether's theorem:

- For every continuous symmetry of the Lagrangian—that is, a variation in the field that leaves form of the Lagrangian unchanged—there is a conserved current whose form can be derived from the Lagrangian using Eq. (2.29).

There is a conserved *charge* associated with each conserved current that results from a symmetry of the Lagrangian. This is found by integrating the time component of J:

$$Q = \int d^3x\, J^0 \tag{2.31}$$

We see that the translation symmetry in spacetime worked out in the previous section that led to the energy-momentum tensor is a special case of Noether's theorem, with the conserved "charges" being energy and momentum.

The Electromagnetic Field

The Maxwell or electromagnetic field tensor is given by

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.32}$$

The A^{μ} is the usual vector potential, but this is a 4-vector whose time component is the scalar potential and whose spatial component is the usual vector potential used to write down the magnetic field, that is, $A^{\mu} = (\psi, \vec{\mathrm{A}})$. It can be shown that $F^{\mu\nu}$

satisfies or leads to Maxwell's equations. Note that $F^{\mu\nu}$ is antisymmetric; the sign flips when the indices are interchanged. That is,

$$F^{\mu\nu} = -F^{\nu\mu} \tag{2.33}$$

Without sources, the homogeneous Maxwell's equations can be written in terms of the electromagnetic field tensor as

$$\partial^{\lambda} F^{\mu\nu} + \partial^{\nu} F^{\lambda\mu} + \partial^{\mu} F^{\nu\lambda} \tag{2.34}$$

Meanwhile, the inhomogeneous Maxwell's equations can be written as

$$\partial_{\mu} F^{\mu\nu} - J^{\nu} = 0 \tag{2.35}$$

where J^{ν} are the current densities. It is an instructive exercise to derive Maxwell's equations using a variational procedure so that you can learn how to work with tensors of higher order, that is, vector fields. In the next example we derive Eq. (2.35) from a Lagrangian.

EXAMPLE 2.5

Show that the Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^{\mu} A_{\mu}$ leads to the inhomogeneous Maxwell's equations [Eq. (2.35)] if the potential A_{μ} is varied, leaving the current density constant.

SOLUTION

We begin as usual by writing the action as an integral of the Lagrangian density. The action in this case is

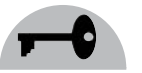
$$S = \int d^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^{\mu} A_{\mu} \right)$$

The variation we will compute is δA_{μ}. We have

$$\delta S = \int d^4 x \left(-\frac{1}{4} (\delta F_{\mu\nu}) F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} (\delta F^{\mu\nu}) - J^{\mu} \delta A_{\mu} \right)$$

Let's consider the first term. Using the definition of the electromagnetic field tensor Eq. (2.32), we can write $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ and so we have

$$-\frac{1}{4} (\delta F_{\mu\nu}) F^{\mu\nu} = -\frac{1}{4} (\partial_{\mu} \delta A_{\nu} - \partial_{\nu} \delta A_{\mu}) F^{\mu\nu}$$



Now we integrate by parts, transferring the derivatives from the δA_ν terms to $F^{\mu\nu}$. This allows us to write

$$-\frac{1}{4}(\delta F_{\mu\nu})F^{\mu\nu} = -\frac{1}{4}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)F^{\mu\nu}$$
$$= \frac{1}{4}(\partial_\mu F^{\mu\nu}\delta A_\nu - \partial_\nu F^{\mu\nu}\delta A_\mu)$$

But, repeated indices are dummy indices. So let's swap μ and ν in the second term, and write this as

$$\frac{1}{4}(\partial_\mu F^{\mu\nu}\delta A_\nu - \partial_\mu F^{\nu\mu}\delta A_\nu)$$

Now we use the antisymmetry of the electromagnetic tensor under interchange of the indices in Eq. (2.23). This will get rid of the minus sign on the second term, giving

$$\frac{1}{4}(\partial_\mu F^{\mu\nu}\delta A_\nu - \partial_\mu F^{\nu\mu}\delta A_\nu) = \frac{1}{4}(\partial_\mu F^{\mu\nu}\delta A_\nu + \partial_\mu F^{\mu\nu}\delta A_\nu)$$
$$= \frac{1}{2}\partial_\mu F^{\mu\nu}\delta A_\nu$$

So we have the result

$$-\frac{1}{4}(\delta F_{\mu\nu})F^{\mu\nu} = \frac{1}{2}\partial_\mu F^{\mu\nu}\delta A_\nu \tag{2.36}$$

Now let's tackle the next term $-\frac{1}{4}F_{\mu\nu}(\delta F^{\mu\nu})$. In this case we have

$$-\frac{1}{4}F_{\mu\nu}(\delta F^{\mu\nu}) = -\frac{1}{4}F_{\mu\nu}\delta(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\frac{1}{4}F_{\mu\nu}(\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu)$$

We're going to have to lower and raise some indices using the metric (see Chap. 1) to get this in the form of Eq. (2.36). The first step is to raise the indices on the field tensor term

$$-\frac{1}{4}F_{\mu\nu}(\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu) = -\frac{1}{4}g_{\mu\rho}g_{\nu\sigma}F^{\rho\sigma}(\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu)$$

Now let's move $F^{\rho\sigma}$ inside the parentheses and integrate by parts to transfer the derivative onto it

$$-\frac{1}{4}g_{\mu\rho}g_{\nu\sigma}F^{\rho\sigma}(\partial^{\mu}\delta A^{\nu}-\partial^{\nu}\delta A^{\mu})=\frac{1}{4}g_{\mu\rho}g_{\nu\sigma}[\partial^{\mu}(F^{\rho\sigma})\delta A^{\nu}-\partial^{\nu}(F^{\rho\sigma})\delta A^{\mu}]$$
$$=\frac{1}{4}g_{\mu\rho}g_{\nu\sigma}\partial^{\mu}(F^{\rho\sigma})\delta A^{\nu}-\frac{1}{4}g_{\mu\rho}g_{\nu\sigma}\partial^{\nu}(F^{\rho\sigma})\delta A^{\mu}$$

Now we lower the indices on the derivative operators to give

$$=\frac{1}{4}g_{\mu\rho}g_{\nu\sigma}\partial^{\mu}(F^{\rho\sigma})\delta A^{\nu}-\frac{1}{4}g_{\mu\rho}g_{\nu\sigma}\partial^{\nu}(F^{\rho\sigma})\delta A^{\mu}$$
$$=\frac{1}{4}g_{\nu\sigma}\partial_{\rho}(F^{\rho\sigma})\delta A^{\nu}-\frac{1}{4}g_{\mu\rho}\partial_{\sigma}(F^{\rho\sigma})\delta A^{\mu}$$

Next, do the same thing to the vector potential terms

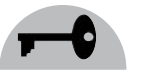
$$\frac{1}{4}g_{\nu\sigma}\partial_{\rho}(F^{\rho\sigma})\delta A^{\nu}-\frac{1}{4}g_{\mu\rho}\partial_{\sigma}(F^{\rho\sigma})\delta A^{\mu}$$
$$=\frac{1}{4}\partial_{\rho}(F^{\rho\sigma})\delta A_{\sigma}-\frac{1}{4}\partial_{\sigma}(F^{\rho\sigma})\delta A_{\rho}$$

Once again, repeated indices are dummy indices so we can change the labels. Focusing on the second term, let's change $\rho\to\nu$, $\sigma\to\mu$. We obtain

$$\frac{1}{4}\partial_{\rho}(F^{\rho\sigma})\delta A_{\sigma}-\frac{1}{4}\partial_{\sigma}(F^{\rho\sigma})\delta A_{\rho}$$
$$=\frac{1}{4}\partial_{\rho}(F^{\rho\sigma})\delta A_{\sigma}-\frac{1}{4}\partial_{\mu}(F^{\nu\mu})\delta A_{\nu}$$

Now apply the antisymmetry of the electromagnetic tensor to the second term, to get rid of the minus sign as shown here.

$$\frac{1}{4}\partial_{\rho}(F^{\rho\sigma})\delta A_{\sigma}-\frac{1}{4}\partial_{\mu}(F^{\nu\mu})\delta A_{\nu}$$
$$=\frac{1}{4}\partial_{\rho}(F^{\rho\sigma})\delta A_{\sigma}+\frac{1}{4}\partial_{\mu}(F^{\nu\mu})\delta A_{\nu}$$



The relabeling procedure can also be applied to the first term. This time we let $\rho \to \mu$, $\sigma \to \nu$ and we get

$$\frac{1}{4}\partial_\rho(F^{\rho\sigma})\delta A_\sigma + \frac{1}{4}\partial_\mu(F^{\mu\nu})\delta A_\nu = \frac{1}{4}\partial_\mu(F^{\mu\nu})\delta A_\nu + \frac{1}{4}\partial_\mu(F^{\mu\nu})\delta A_\nu$$
$$= \frac{1}{2}\partial_\mu(F^{\mu\nu})\delta A_\nu$$

We can combine this result with Eq. (2.36), and we see that the variation in the action becomes

$$\delta S = \int d^4x \left(\frac{1}{2}\partial_\mu F^{\mu\nu}\delta A_\nu + \frac{1}{2}\partial_\mu F^{\mu\nu}\delta A_\nu - J^\mu \delta A_\mu \right)$$
$$= \int d^4x (\partial_\mu F^{\mu\nu} - J^\mu)\delta A_\nu$$

We require that the variation in the action vanish, that is, $\delta S = 0$. Since the variation is arbitrary, δA_ν cannot vanish. Once more we arrive at the conclusion that the integral will be 0 only if the integrand is 0 everywhere in the domain. This means the action will only vanish if Maxwell's equations are satisfied, that is,

$$\partial_\mu F^{\mu\nu} - J^\mu = 0$$

Gauge Transformations

In this section we will consider an extension of the idea of invariance, by introducing what is known as a *gauge transformation.* Here we're only going to provide a brief introduction to these ideas; they will be elaborated as we proceed through the book.

The idea of a gauge transformation follows from studies of electricity and magnetism where we can make changes to the scalar and vector potentials ψ and $\vec{A}$ without changing the field equations and hence the physical fields $\vec{E}$ and $\vec{B}$ themselves. For example, recall that the magnetic field $\vec{B}$ can be defined in terms of the vector potential $\vec{A}$ using the curl relation

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

A rule from vector calculus tells us that $\nabla \bullet (\vec{\nabla} \times \vec{F}) = 0$ for any vector field $\vec{F}$. Hence the Maxwell's equation $\nabla \bullet \vec{B} = 0$ is still satisfied when we make the

definition $\vec{B} = \vec{\nabla} \times \vec{A}$. Now let f be some scalar function and define a new vector potential $\vec{A}'$via

$$\vec{A}' = \vec{A} + \vec{\nabla} f$$

We also know from vector calculus that $\vec{\nabla} \times \vec{\nabla} f = 0$. Hence we can add a term of the form $\vec{\nabla} f$ to the vector potential with impunity if it's *mathematically convenient*. The magnetic field $\vec{B}$ is unchanged since

$$\vec{B} = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla} f) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla} f = \vec{\nabla} \times \vec{A}$$

Therefore the magnetic field, the physical quantity of interest, is unchanged by a transformation of the form $\vec{A}' = \vec{A} + \vec{\nabla} f$. We call this type of transformation in electrodynamics a *gauge transformation*. There are different choices that can be made when implementing a gauge transformation. For example, if we impose the requirement that $\nabla \bullet \vec{A} = 0$, we call this the *Coulomb gauge*. On the other hand, if $\nabla \bullet \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial \psi}{\partial t}$, we have what is known as the *Lorentz gauge.*

In field theory, we arrive at a similar notion by considering transformations to the field that leave the Lagrangian invariant. To see how this works in field theory, let's consider a simple example, the Klein-Gordon Lagrangian with a complex field.

$$\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi \tag{2.37}$$

Let U be a unitary transformation applied to the fields such that U does not, in any way, depend on spacetime. That is, we let

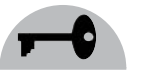
$$\varphi \to U\varphi \tag{2.38}$$

Then

$$\varphi^\dagger \to \varphi^\dagger U^\dagger \tag{2.39}$$

Since the transformation is unitary, we also know that $UU^\dagger = U^\dagger U = 1$. Let's see how this transformation affects the Lagrangian [Eq. (2.37)]. Looking at each term individually, we start with the first term where we have

$$\partial_\mu \varphi^\dagger \partial^\mu \varphi \to \partial_\mu (\varphi^\dagger U^\dagger) \partial^\mu (U\varphi)$$



But U does not depend on spacetime in any way, so the derivative operators do not affect it. Hence

$$\partial_\mu(\varphi^\dagger U^\dagger)\partial^\mu(U\varphi) = \partial_\mu(\varphi^\dagger)(U^\dagger U)\partial^\mu(\varphi) = \partial_\mu\varphi^\dagger\partial^\mu\varphi$$

Similarly for the second term in Eq. (2.37), we have

$$m^2\varphi^\dagger\varphi \to m^2(\varphi^\dagger U^\dagger)(U\varphi) = m^2\varphi^\dagger(U^\dagger U)\varphi = m^2\varphi^\dagger\varphi$$

Therefore we see that under the transformation [Eq. (2.38)], the Lagrangian [Eq. (2.37)] is invariant. Since U is a constant, we can write it in the form

$$U = e^{i\Lambda}$$

where Λ is a constant. However, in certain contexts, Λ can also be a matrix so long as it is Hermitian. Since it is a constant we say that the gauge transformation in this case is *global*, it does not depend on spacetime in any way.

LOCAL GAUGE TRANSFORMATIONS

The gauge transformations that are of interest are local transformations that do depend on spacetime. This type of transformation satisfies the requirements of special relativity—that no signal can travel faster than the speed of light.

Let's return to the transformation $\varphi \to U\varphi$. In the following, we still consider U to be a unitary transformation, however now we let it depend on spacetime so that $U = U(x)$. This means that terms like $\partial_\mu U$ will not vanish. Next consider how the Lagrangian is changed by the transformation $\varphi \to U\varphi$. We use the same Lagrangian we considered in the previous section, namely $\mathcal{L} = \partial_\mu\varphi^\dagger\partial^\mu\varphi - m^2\varphi^\dagger\varphi$. This time the second term in the Lagrangian remains invariant as shown here.

$$m^2\varphi^\dagger\varphi \to m^2\varphi^\dagger U^\dagger(x)U(x)\varphi = m^2\varphi^\dagger\varphi$$

The first term will change due to the spacetime dependence of $U = U(x)$, that is,

$$\partial_\mu\varphi^\dagger \to \partial_\mu(\varphi^\dagger U^\dagger) = (\partial_\mu\varphi^\dagger)U^\dagger + \varphi^\dagger\partial_\mu(U^\dagger)$$

Similarly we find that

$$\partial_\mu\varphi \to \partial_\mu(U\varphi) = (\partial_\mu U)\varphi + U\partial_\mu(\varphi)$$

We can write this in a more useful form by exploiting the fact that U is unitary, like

$$\begin{aligned}\partial_\mu\varphi \to \partial_\mu(U\varphi) &= (\partial_\mu U)\varphi + U\partial_\mu(\varphi) \\ &= UU^\dagger(\partial_\mu U)\varphi + U\partial_\mu(\varphi) \\ &= U[\partial_\mu\varphi + (U^\dagger\partial_\mu U)\varphi]\end{aligned}$$

To maintain invariance, we would like to cancel the extra term that has shown up here. In other words we want to get rid of

$$(U^\dagger\partial_\mu U)\varphi$$

We can do this by introducing a new object, a spacetime dependent field $A_\mu = A_\mu(x)$ called the *gauge potential.* It is given the label A_μ due to the analogy with electrodynamics—we are introducing a hidden field to keep the form of the Lagrangian invariant. As we will see later this has dramatic consequences, and this is one of the most important techniques in quantum field theory.

We introduce a *covariant derivative* that acts on the field as

$$D_\mu\varphi = \partial_\mu\varphi - iA_\mu\varphi \tag{2.40}$$

(for readers of *Relativity Demystified*, notice the similarity to general relativity). Remember what the word *covariant* means—the form of the equations won't change. We introduce this derivative operator to keep the Lagrangian invariant under a local gauge transformation. Recalling the effect of a global gauge transformation $\varphi \to U\varphi$, we found that

$$\partial_\mu\varphi \to U\partial_\mu\varphi$$

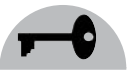
The covariant derivative Eq. (2.40) will allow us to recover this result in the case of the local gauge transformation. That is, $\varphi \to U(x)\varphi$ will lead to $D_\mu\varphi \to U(x)D_\mu\varphi$. This can be accomplished if we define A_μ such that it obeys the similarity transformations.

$$A_\mu \to UA_\mu U^\dagger + iU\partial_\mu U^\dagger \tag{2.41}$$

Note that some authors use the semicolon notation $D_\mu\varphi = \varphi_{;\mu}$ to represent the covariant derivative.

EXAMPLE 2.6

Consider a charged scalar particle of mass m with charge q and describe a suitable modification of the derivative operator $\partial_\mu \to \partial_\mu + qA_\mu$ that will yield a Lorentz invariant, real Lagrangian.



SOLUTION

The field equations for a complex field corresponding to a charged scalar particle are obtained using the covariant derivative in Eq. (2.40), in this case with the form

$$i\partial_\mu \rightarrow i\partial_\mu - qA_\mu$$

The Lagrangian is

$$\mathcal{L} = (i\partial_\mu + qA_\mu)\varphi^*(i\partial_\mu - qA_\mu)\varphi - m^2\varphi^*\varphi$$

Variation of this Lagrangian leads to the equation of motion as shown here.

$$(i\partial_\mu - qA_\mu)^2\varphi - m^2\varphi = 0$$

Summary

The Lagrangian is the difference of kinetic and potential energy given by $L = T - V$. It can be used to obtain the equations of motion for a system by applying variational calculus to the action S which is the integral of the Lagrangian. When extended to continuous systems, these techniques can be applied to fields to obtain the field equations. For problems with expected symmetries or conserved quantities, we require that the Lagrangian remains invariant under corresponding transformations. When a given transformation leaves the form of the Lagrangian invariant, we call that transformation a symmetry. Noether's theorem allows us to derive conservation laws, including conserved charges and currents from symmetries in the Lagrangian. Symmetries can be local, meaning that they are spacetime dependent, or they can be internal symmetries that are intrinsic to the system at hand. To maintain covariance of the equations, a covariant derivative must be introduced which necessitates the use of a gauge potential.

Quiz

1. Find the equation of motion for a forced harmonic oscillator with Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2 + \alpha x$$

Here α is a constant.

2. Consider a Lagrangian given by

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi)$$

   (a) Write down the field equations for this system.

   (b) Find the canonical momentum density $\pi(x)$.

   (c) Write down the Hamiltonian.

3. Consider a free scalar field with Lagrangian $\mathcal{L} = \partial_\mu\varphi\,\partial^\mu\varphi$ and suppose that the field varies according to $\varphi \to \varphi + \alpha$, where α is a constant. Determine the conserved current.

4. Refer to the Lagrangian for a complex scalar field Eq. (2.37). Determine the equations of motion obeyed by the fields φ and $\varphi^\dagger$.

5. Refer to Eq. (2.37) and calculate the conserved charge.

6. Consider the action $S = \frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}d^4x$. Vary the potential according to $A_\mu \to A_\mu + \partial_\mu\varphi$ where φ is a scalar field. Determine the variation in the action.

CHAPTER 3

An Introduction to Group Theory

An abstract branch of mathematics called *group theory* plays a fundamental role in modern particle physics. The reason it does so is because group theory is related to symmetry. For example, an important group in physics is the *rotation group,* which is related to the fact that the laws of physics don't change if you rotate your frame of reference. In general what we are after is a set of equations, or laws of physics, that keep the same mathematical form under various transformations. Group theory is related to this type of symmetry.

Definitions

We begin our discussion of group theory in a somewhat abstract manner, but later we will get to some more concrete material. Let's state what a group is and the four properties it must have.

A *group G* is a set of elements $\{a, b, c, \ldots\}$ which includes a "multiplication" or composition rule such that if $a \in G$ and $b \in G$, then the product is also a member of the group, that is,

$$ab \in G \tag{3.1}$$

We call this the *closure* property if $ab = ba$, we say that the group is *abelian.* On the other hand if $ab \neq ba$, the group is *nonabelian.* The multiplication or composition rule is meant to convey a product in the abstract sense, the actual implementation of this rule can vary from group to group.

A group G must satisfy four axioms. These are

1. Associativity: The multiplication rule is associative, meaning $(ab)c = a(bc)$.
2. Identity element: The group has an identity element e that satisfies $ae = ea = a$. The identity element for the group is unique.
3. Inverse element: For every element $a \in G$, there exists an inverse which we denote by a^{-1} such that $aa^{-1} = a^{-1}a = e$.
4. Order: The order of the group is the number of elements that belong to G.

Representation of the Group

In particle physics we are often interested in what is called a *representation* of the group. Let's denote a representation by F. A representation is a mapping that takes group elements $g \in G$ into linear operators F that preserve the composition rule of the group in the sense that

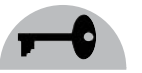
- $F(a)F(b) = F(ab)$.
- The representation also preserves the identity, that is, $F(e) = I$.

Suppose that $a,b \in G$ and $f \in H$ where H is some other group. If the composition rule satisfies

$$f(a)\,f(b) = f(ab)$$

we say that G is *homomorphic* to H. This is a fancy way of saying that the two groups have a similar structure.

EXAMPLE 3.1A
Does the set of all integers form a group under addition?



SOLUTION

The set of all integers forms a group under addition. We can take the composition rule to be addition. Let $z_1 \in Z$ and $z_2 \in Z$. Clearly the sum

$$z_1 + z_2 \in Z$$

is another integer, so it belongs to the group. For the identity, we can take $e = 0$ since

$$z + 0 = 0 + z = z$$

for any $z \in Z$. Addition is commutative, that is,

$$z_1 + z_2 = z_2 + z_1$$

Therefore the group is abelian. The inverse of z is just $-z$, since

$$z + (-z) = e = 0$$

which satisfies $aa^{-1} = a^{-1}a = e$.

EXAMPLE 3.1

Does the set of all integers form a group under multiplication?

SOLUTION

The set of all integers does not form a group under multiplication. We can take the composition rule to be multiplication. Let $z_1 \in Z$ and $z_2 \in Z$. Clearly the product

$$z_1 z_2 \in Z$$

is another integer, so it belongs to the group. For the identity, we can take $e = 1$ since

$$z \times 1 = 1 \times z = z$$

for any $z \in Z$. Multiplication is commutative, that is,

$$z_1 \times z_2 = z_2 \times z_1$$

Therefore the group is abelian. The problem is the inverse: 1/z is not an integer.

$$z \times \frac{1}{z} = e = 1$$

So even though an inverse exists, the inverse is not in the group, and therefore the set of all integers does not form a group under multiplication.

Group Parameters

An ordinary function of position is specified by an input x, that is, we have $y = f(x)$. In an analogous way, a group can also be a function of one or more inputs that we call *parameters.*

Let a group G be such that individual elements $g \in G$ are specified by a finite set of parameters, say n of them. If we denote the set of parameters by

$$\{\theta_1, \theta_2, \ldots, \theta_n\}$$

The group element is then written as

$$g = G(\theta_1, \theta_2, \ldots, \theta_n)$$

The identity is the group element where the parameters are all set to 0.

$$e = G(0, 0, \ldots, 0)$$

Lie Groups

While there are discrete groups with a finite number of elements, most of the groups we will be concerned with have an infinite number of elements. However, they have a finite set of continuously varying parameters.

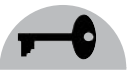
In the expression $g = G(\theta_1, \theta_2, \ldots, \theta_n)$ we have suggestively labeled the parameters as angles, since several important groups in physics are related to rotations. The angles vary continuously over a finite range $0 \ldots 2\pi$. In addition, the group is parameterized by a finite number of parameters, the angles of rotation.

So, if a group G

- Depends on a finite set of continuous parameters θ_i
- Derivatives of the group elements with respect to all the parameters exist

we call the group a *Lie group.* To simplify the discussion we begin with a group with a single parameter θ. We obtain the identity element by setting $\theta = 0$

$$g(\theta)\big|_{\theta=0} = e \tag{3.2}$$



By taking derivatives with respect to the parameters and evaluating the derivative at $\theta = 0$, we obtain the *generators* of the group. Let us denote an abstract generator by X. Then

$$X = \frac{\partial g}{\partial \theta}\Big|_{\theta=0} \tag{3.3}$$

More generally, if there are n parameters of the group, then there will be n generators such that each generator is given by

$$X_i = \frac{\partial g}{\partial \theta_i}\Big|_{\theta_i=0} \tag{3.4}$$

Rotations have a special property, in that they are length preserving (that is, rotate a vector and it maintains the same length). A rotation by $-\theta$ undoes a rotation by θ, hence rotations have an orthogonal or unitary representation. In the case of quantum theory, we seek a unitary representation of the group and choose the generators X_i to be Hermitian. In this case

$$X_i = -i\frac{\partial g}{\partial \theta_i}\Big|_{\theta_i=0} \tag{3.5}$$

For some finite θ, the generators allow us to define a representation of the group. Consider a small real number $\varepsilon > 0$ and use a Taylor expansion to form a representation of the group (which we denote by D)

$$D(\varepsilon\theta) \approx 1 + i\varepsilon\theta X$$

If $\theta = 0$, then clearly the representation gives the identity. You will also recall that the exponential function has a series expansion

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$$

So we can define the representation of the group in terms of the exponential using

$$D(\theta) = \lim_{n\to\infty}\left(1 + i\frac{\theta X}{n}\right)^n = e^{i\theta X} \tag{3.6}$$

Notice that if X is Hermitian, $X = X^\dagger$ and the representation of the group is unitary, since

$$D^\dagger(\theta) = (e^{i\theta X})^\dagger = e^{-i\theta X^\dagger} = e^{-i\theta X}$$
$$\Rightarrow D^\dagger(\theta)D(\theta) = (e^{-i\theta X})\,(e^{i\theta X}) = 1$$

One reason that the generators of a group are important is that they form a vector space. This means we can add two generators of the group together to obtain a third generator, and we can multiply generators by a scalar and still have a generator of the group. A complete vector space can be used as a basis for representing other vector spaces, hence the generators of a group can be used to represent other vector spaces. For example, the Pauli matrices from quantum mechanics can be used to describe any 2×2 matrix.

The character of the group is defined in terms of the generators in the following sense. The generators satisfy a commutation relation we write as

$$[X_i, X_j] = if_{ijk}X_k \tag{3.7}$$

This is called the *Lie algebra* of the group. The quantities f_{ijk} are called the *structure constants* of the group. Looking at the commutation relation for the Lie algebra, Eq. (3.7), you should recognize the fact that you've already been working with group generators in your studies of non-relativistic quantum mechanics. You'll see this explicitly when we discuss the Pauli spin matrices later on.

The Rotation Group

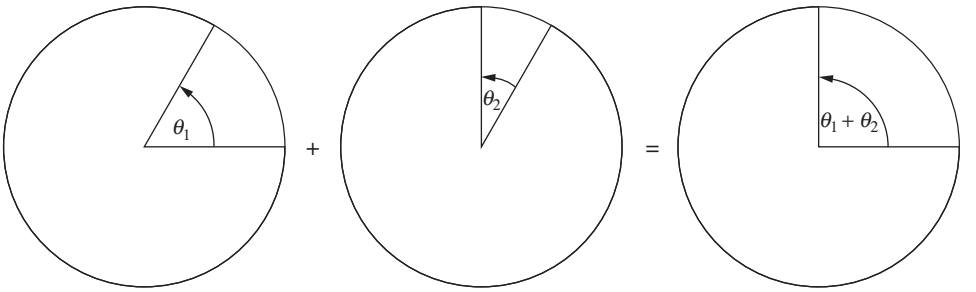
The *rotation group* is the set of all rotations about the origin. A key feature is that rotations preserve the lengths of vectors. This mathematical property is expressed by saying the matrices are unitary. It is easy to see that the set of rotations forms a group. Let's check off each of the basic properties that a group must have.

The first is a group composition rule. Remember that if $a \in G$ and $b \in G$, if G is a group then $ab \in G$ as well. Now let R_1 and R_2 be two rotations. It is clear that the composition of these rotations, say by performing the rotation R_1 first followed by the rotation R_2, is itself just another rotation, as shown here.

$$R_3 = R_2 R_1$$

That is, performing the two rotations as described is the same as doing the single rotation R_3. Hence R_3 is a member of the rotation group. Next, we illustrate how two

rotations in sequence are the same as a single rotation that is the sum of the two angles:



It is not the case that rotations commute. That is, in general

$$R_1 R_2 \neq R_2 R_1$$

To this, put a book on the table in front of you. Then rotate it about two different axes, and then try the experiment again doing the same rotations but in different order. You will see that the end results are not the same. Therefore, the rotation group is nonabelian. However, it's easy to see that rotations are associative, like

$$R_1 (R_2 R_3) = (R_1 R_2) R_3$$

The rotation group has an identity element—this is, simply doing no rotation at all. The inverse of a rotation is simply the rotation carried out in the opposite direction.

Representing Rotations

Let x_i be the coordinates of a two-dimensional vector and let x_i' be the coordinates of the vector rotated by an angle θ in the plane. The components of the two vectors are related by a transformation as

$$x_j' = R_{ij} x_i$$

where R_{ij} is a *rotation matrix*. This is a representation of the rotation group. Specifically, a rotation by an angle θ (in two dimensions) can be represented by the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

So that

$$x_1' = \cos\theta x_1 + \sin\theta x_2$$

$$x_2' = -\sin\theta x_1 + \cos\theta x_2$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{pmatrix}$$

Let's write down the transpose of the rotation matrix as

$$R^T(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Notice that

$$R(\theta)R^T(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$RR^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{pmatrix}$$

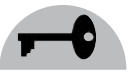
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This tells us that the transpose of the matrix is the inverse group element, since multiplying two matrices together gives the identity. You can see why this is true using basic trigonometry: $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

$$R^T(\theta) = R(-\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So, the inverse of $R(\theta)$ is $R(-\theta)$: $\sin(\theta - \theta) = 0$ and $\cos(\theta - \theta) = 1$.

$$R(\theta)R^T(\theta) = R(\theta)R(-\theta) = R(\theta)R^{-1}(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$



In group theory, various groups are classified according to the determinants of the matrices that represent the groups. Notice that in this case

$$\det R(\theta) = \det \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$
$$= \cos^2\theta + \sin^2\theta = 1$$

In general, the determinant will not be +1. However, when this condition is satisfied, the rotation matrix corresponds to a *proper rotation*.

A more readable notation is to make the angle implicit and to use a subscript. That is,

$$R(\theta_1) \rightarrow R_1$$

This is a better way to discuss problems to multiple angles.

Now, if $\det R_1 = 1$ and $\det R_2 = 1$, then the product is unity. We see this from the property of multiplication for determinants.

$$\det(R_1 R_2) = \det R_1 \det R_2 = (1)(1) = 1$$

The product of two rotations also has an inverse, since

$$R_1 R_2 (R_1 R_2)^T = R_1 R_2 R_2^T R_1^T = R_1 R_1^T = I$$

Another way to look at this is to use the closure property of this group. Call the successive rotations R_1 and R_2. Taken together they form the group element R_3:

$$R_1 R_2 = R_3$$

As before, the transpose is the inverse:

$$R_3{}^T = R_3{}^{-1}$$

In terms of the components, we have

$$R_3{}^T = (R_1 R_2)^T = R_2{}^T R_1{}^T \quad \text{and} \quad R_3{}^{-1} = (R_1 R_2)^{-1} = R_2{}^{-1} R_1{}^{-1}$$

We now see

$$R_3 R_3{}^T = I$$

as above.

So the matrix representation preserves the properties of the rotation group. Now that we've introduced the notion of a group, we will explore groups important for particle physics.

SO(N)

The group $SO(N)$ are special orthogonal $N \times N$ matrices. The term *special* is a reference to the fact that these matrices have determinant +1. A larger group, one that contains $SO(N)$ as a subgroup, is the group $O(N)$ which are orthogonal $N \times N$ matrices that can have arbitrary determinant. Generally speaking, rotations can be represented by orthogonal matrices, which themselves form a group. So the group $SO(3)$ is a representation of rotations in three dimensions, and the group consists of 3×3 orthogonal matrices with determinant +1.

A matrix O is called *orthogonal* if the transpose O^T is the inverse, that is,

$$\begin{aligned} &OO^T = O^T O = I \\ &\Rightarrow O \text{ is orthogonal} \end{aligned} \tag{3.8}$$

As we stated above, a special orthogonal matrix is one with positive unit determinant

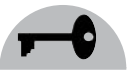
$$\det O = +1 \tag{3.9}$$

Now let's turn to a familiar case, $SO(3)$. This group has three parameters, the three angles defining rotations about the *x, y,* and *z* axes. Let these angles be denoted by ς, ϕ, and θ. Then

$$R_x(\varsigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varsigma & \sin\varsigma \\ 0 & -\sin\varsigma & \cos\varsigma \end{pmatrix} \tag{3.10}$$

$$R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \tag{3.11}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.12}$$



These matrices are the representation of rotations in three dimensions. Rotations in three or more dimensions do not commute, and it is an easy although tedious exercise to show that the rotation matrices written down here do not commute either.

The task now is to find the generators for each group parameter. We do this using Eq. (3.5). Starting with $R_x(\varsigma)$, we have

$$\frac{dR_x}{d\varsigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\varsigma & \cos\varsigma \\ 0 & -\cos\varsigma & \sin\varsigma \end{pmatrix}$$

Now we let $\varsigma \to 0$ to obtain the generator

$$J_x = -i\frac{dR_x}{d\varsigma}\Big|_{\varsigma=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{3.13}$$

Next, we compute the generator for rotations about the y axis evaluated at $\phi = 0$.

$$J_y = -i\frac{dR_y}{d\phi} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \tag{3.14}$$

And finally, for rotations about the z axis we find

$$J_z = -i\frac{dR_z}{d\theta}\Big|_{\theta=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These matrices are, of course, the familiar angular momentum matrices. So we've discovered the famous result that the angular momentum operators are the generators of rotations. We can use the generators to build infinitesimal rotations. For example, an infinitesimal rotation about the z axis by an angle $\varepsilon\theta$, where ε is a small positive parameter, is written as

$$R_z(\varepsilon\theta) = 1 + i J_z \varepsilon\theta$$

From quantum mechanics, you already know the algebra of the group. This is just the commutation relations satisfied by the angular momentum operators.

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \tag{3.15}$$

The structure constants in this case are given by the Levi-Civita tensor, which takes on values of +1, –1, or 0 according to

$$\begin{aligned} \varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = +1 \\ \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1 \end{aligned} \tag{3.16}$$

with all other combinations of the indices giving 0.

EXAMPLE 3.2
Show that the representation of the rotation group is of the form $e^{iJ_y\phi}$.

SOLUTION
What we need to show is that $R_y(\phi) = e^{iJ_y\phi}$. This is easy to do by just writing down the first few terms in the Taylor series expansion of the exponential. Remember Euler's formula which tells us that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

From this we can extract the expansions for the sine and cosine.

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \cdots$$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots$$

Now,

$$e^{iJ_y\phi} = 1 + iJ_y\phi - \frac{1}{2}J_y^2\phi^2 - i\frac{1}{3!}J_y^3\phi^3 + \cdots$$

What are the powers of the J_y matrix? A quick calculation shows that J_y^n takes one of two values depending upon whether n is odd or even.

$$J_y = J_y^3 = J_y^5 = \cdots = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \text{ and } J_y^2 = J_y^4 = J_y^6 = \cdots = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The expansion becomes

$$e^{iJ_y\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + i\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\phi - \frac{1}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}^2 \phi^2 - i\frac{1}{3!}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}^3 \phi^3 + \cdots$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\phi \\ 0 & 0 & 0 \\ \phi & 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \phi^2 \end{pmatrix} - \frac{1}{3!}\begin{pmatrix} 0 & 0 & \phi^3 \\ 0 & 0 & 0 \\ \phi^3 & 0 & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 - \frac{1}{2}\phi^2 + \cdots & 0 & -\left(\phi - \frac{1}{3!}\phi^3 + \cdots\right) \\ 0 & 0 & 0 \\ \phi - \frac{1}{3!}\phi^3 + \cdots & 0 & 1 - \frac{1}{2}\phi^2 + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 0 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$$

So, rotations about the x, y, and z axes are represented by

$$R_x(\varsigma) = e^{iJ_x\varsigma} \qquad R_y(\phi) = e^{iJ_y\phi} \qquad R_z(\theta) = e^{iJ_z\theta} \tag{3.17}$$

A rotation about an arbitrary axis defined by a unit vector $\vec{n}$ is given by

$$R_n(\vec{\theta}) = e^{i\vec{J}\cdot\vec{\theta}} \tag{3.18}$$

As stated earlier, orthogonal transformations (rotations) preserve the lengths of vectors. We say that the length of a vector is *invariant* under rotation. This means that given a vector $\vec{x}$ with length $\vec{x}^2 = x^2 + y^2 + z^2$, when we transform it under a rotation $\vec{x}' = Rx$ we have

$$\begin{aligned} &\vec{x}'^2 = \vec{x}^2 \\ &\Rightarrow x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 \end{aligned} \tag{3.19}$$

The preservation of the lengths of vectors by orthogonal transformations will be important in establishing a correspondence between the unitary transformation $SU(2)$ (see in the next section, Unitary Groups) and rotations in three dimensions represented by $SU(3)$.

Unitary Groups

In particle physics unitary groups play a special role. This is due to the fact that unitary operators play an important role in quantum theory. Specifically, unitary operators preserve inner products—meaning that a unitary transformation leaves the probabilities for different transitions among the states unaffected. That is, quantum physics is invariant under a unitary transformation. As a result unitary groups play a special role in quantum field theory.

When the physical predictions of a theory are invariant under the action of some group, we can represent the group by a unitary operator U. Moreover, this unitary operator commutes with the Hamiltonian as shown here.

$$[U, H] = 0$$

The unitary group $U(N)$ consists of all $N \times N$ unitary matrices. Special unitary groups, denoted by $SU(N)$ are $N \times N$ unitary matrices with positive unit determinant. The dimension of $SU(N)$ and hence the number of generators, is given by $N^2 - 1$. Therefore,

- $SU(2)$ has $2^2 - 1 = 3$ generators.
- $SU(3)$ has $3^2 - 1 = 8$ generators.

The rank of $SU(N)$ is $N - 1$. So

- The rank of $SU(2)$ is $2 - 1 = 1$.
- The rank of $SU(3)$ is $3 - 1 = 2$.

The rank gives the number of operators in the algebra that can be simultaneously diagonalized.

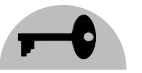
The simplest unitary group is the group $U(1)$. A "1×1" matrix is just a complex number written in polar representation. In another way, we can say a $U(1)$ symmetry has a single parameter θ and is written as

$$U = e^{-i\theta}$$

where θ is a real parameter. It is completely trivial to see that $U(1)$ is abelian, since

$$U_1 U_2 = e^{-i\theta_1} e^{-i\theta_2} = e^{-i\theta_2} e^{-i\theta_1} = U_2 U_1$$

$$U_1 U_2 = e^{-i\theta_1} e^{-i\theta_2} = e^{-i(\theta_1+\theta_2)} = e^{-i(\theta_2+\theta_1)} = e^{-i\theta_2} e^{-i\theta_1} = U_2 U_1$$



We will see that many Lagrangians in field theory are invariant under a $U(1)$ transformation. When you look at the problem in the complex plane this invariance becomes obvious. Consider the arbitrary complex number $z = re^{i\alpha}$. When we multiply z by $e^{i\theta}$ we get

$$e^{i\theta}z = e^{i\theta}re^{i\alpha} = re^{i(\theta+\alpha)}$$

The new complex number has the same length, r, and the angle is increased by θ.

For example, the Lagrangian for a complex scalar field

$$L = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$$

is invariant under the transformation

$$\varphi \to e^{-i\theta}\varphi$$

As described in Chap. 2, when a Lagrangian is invariant under a transformation there is a symmetry, and in this case there is a $U(1)$ symmetry. Force-mediating particles, called *gauge bosons,* will be associated with unitary symmetries like this one. We will see later that when considering electrodynamics, the gauge boson associated with the $U(1)$ symmetry of quantum electrodynamics is the photon.

We will see in later chapters that a $U(1)$ symmetry also manifests itself in terms of the conservation of various quantum numbers. If there is a $U(1)$ symmetry associated with a quantum number a, then

$$U = e^{-ia\theta}$$

The importance of the $U(1)$ symmetry is that the Hamiltonian H is invariant under the transformation $e^{-ia\theta}He^{ia\theta}$, that is,

$$UHU^\dagger = H$$

Again we see that adjoint $U^\dagger$ (the Hermitian conjugate) is also the inverse.

$$U(\theta)U^\dagger(\theta) = U(\theta)U(-\theta) = 1$$

We will see that such symmetries are present in quantum field theory with conservation of lepton and baryon number, for example.

In summary, an element of the group $U(1)$ is a complex number of unit length written as

$$U = e^{-i\theta} \tag{3.20}$$

where θ is a number. This is the familiar unit circle.

Moving right along, the next non-trivial unitary group is $U(2)$, which is the set of all 2×2 unitary matrices. Being unitary these matrices satisfy

$$UU^\dagger = U^\dagger U = I \tag{3.21}$$

For physics, we are interested in a subgroup of $U(2)$, which is the set of all 2×2 unitary matrices with determinant +1. This group is called $SU(2)$. The generators of $SU(2)$ are the Pauli matrices, which we reproduce here for your convenience

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.22}$$

Now we see how the rank of a unitary group comes into play. The rank of $SU(2)$ is 1, and there is one diagonalized operator σ_3 in the basis we have chosen. The generators of $SU(2)$ are actually taken to be $\frac{1}{2}\sigma_i$ and the Lie algebra is the familiar commutation relations that are satisfied by the Pauli matrices

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\varepsilon_{ijk}\frac{\sigma_k}{2} \tag{3.23}$$

The similar algebraic structure between $SU(2)$ as indicated in Eq. (3.23) and $SO(3)$ as indicated by Eq. (3.5) indicates that there will be a correspondence between these two groups. Since the Pauli matrices do not commute, $SU(2)$ is nonabelian, as seen in Eq. (3.23). Recalling that the rank of $SU(2)$ is 1, there is one diagonal generator which we have chosen to be σ_3.

An element of $SU(2)$ can be written as

$$U = e^{i\sigma_j\alpha_j/2} \tag{3.24}$$

where σ_i is one of the Pauli matrices and α_j is a number. To understand the presence of the factor ½ in Eqs. (3.23) and (3.24), let's explore the correspondence between $SU(2)$ and $SO(3)$. We saw in Eq. (3.19) that $SO(3)$ preserves the lengths of vectors. Since it is unitary, $SU(2)$ preserves the lengths of vectors as well. Let $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$. We construct a matrix of the form $\vec{\sigma}\cdot\vec{r}$.

$$\begin{aligned} \vec{\sigma}\cdot\vec{r} &= \sigma_x x + \sigma_y y + \sigma_z z \\ &= \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iy \\ iy & 0 \end{pmatrix} + \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \\ &= \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \end{aligned} \tag{3.25}$$

If we take the determinant of this matrix, we obtain the length of the vector as shown here.

$$\det\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} = -x^2 - y^2 - z^2 = -\vec{x}^2$$

Also note that the matrix $\vec{\sigma}\cdot\vec{r}$ is Hermitian and has zero trace. Now consider a unitary transformation on this matrix. For example, we can take

$$\begin{aligned} U(\vec{\sigma}\cdot\vec{r})U^{\dagger} &= \sigma_x(\vec{\sigma}\cdot\vec{r})\sigma_x \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -z & x+iy \\ x-iy & z \end{pmatrix} \end{aligned}$$

The transformed matrix still has zero trace, and is still Hermitian. Moreover, the determinant is preserved, and it again gives the length of the vector, that is,

$$\det\begin{pmatrix} -z & x+iy \\ x-iy & z \end{pmatrix} = -x^2 - y^2 - z^2 = -\vec{x}^2$$

The conclusion is that like $SO(3)$, $SU(2)$ preserves the lengths of vectors as shown here.

$$\begin{aligned} &\vec{x}'^2 = \vec{x}^2 \\ &\Rightarrow x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 \end{aligned}$$

The correspondence works by considering an $SU(2)$ transformation on a two component spinor, like

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where

$$x = \frac{1}{2}(\beta^2 - \alpha^2) \qquad y = -\frac{i}{2}(\alpha^2 + \beta^2) \qquad z = \alpha\beta$$

Then an $SU(2)$ transformation on $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is equivalent to an $SO(3)$ transformation on $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. As you can see from this equivalence, an $SU(2)$ transformation has three real parameters that correspond to the three angles of an $SO(3)$ transformation. Label the "angles" for the $SU(2)$ transformation by α, β, and γ. *Half* the rotation angle generated by $SU(2)$ corresponds to the rotation generated by $SO(3)$. For arbitrary angle α, a transformation generating a rotation about x in $SU(2)$ is given by

$$U = \begin{pmatrix} \cos\alpha/2 & i\sin\alpha/2 \\ i\sin\alpha/2 & \cos\alpha/2 \end{pmatrix}$$

(see quiz problem 1). This transformation corresponds to the rotation Eq. (3.10), a rotation about the x axis. Next, consider an $SU(2)$ transformation generating a rotation around the y axis. The unitary operator is

$$U = \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix}$$

This corresponds to the $SO(3)$ transformation given in Eq. (3.11). Finally, for a rotation about the z axis we have the $SU(2)$ transformation

$$U = \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix}$$

which corresponds to Eq. (3.12).

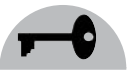
All 2×2 unitary matrices are specified by two parameters, complex numbers a and b where

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

For an element of $SU(2)$, $\det U = +1$ so we require that $|a|^2 + |b|^2 = 1$.

Later we will seek to define Lagrangians that are invariant under an $SU(2)$ symmetry. This symmetry will be of particular importance in the case of electroweak interactions. The gauge bosons corresponding to the $SU(2)$ symmetry will be the W and Z bosons that carry the weak interaction.

Next, we consider the unitary group $SU(3)$, which will be important in the study of quarks and the theory of quantum chromodynamics. Earlier we indicated



that $SU(3)$ has eight generators. These are called the Gell-Mann matrices and are given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Notice two of the matrices are diagonal, λ_3 and λ_8, as we would expect from the rank of the group. The Gell-Mann matrices are traceless and they satisfy the commutation relations

$$\left[\lambda_i, \lambda_j\right] = 2i \sum_{k=1}^{8} f_{ijk} \lambda_k \tag{3.26}$$

This defines the algebraic structure of $SU(3)$. The nonzero structure constants are

$$f_{123} = 1 \qquad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}$$
$$f_{458} = f_{678} = \frac{\sqrt{3}}{2} \tag{3.27}$$

We will see more of $SU(3)$ when we examine the standard model.

Casimir Operators

A *casimir operator* is a nonlinear function of the generators of a group that commutes with all of the generators. The number of casimir operators for a group is given by the rank of the group.

Considering $SU(3)$ as an example, the generators are the angular momentum operators. A casimir operator in this case is

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$$

A casimir operator is an invariant. In this case, the invariance suggests J^2 is a multiple of the group identity element.

Summary

Group theory plays an important role in physics because groups are used to describe symmetries. The structure of a group is defined by the algebra among its generators. If two groups have the same algebra, they are related. Unitary transformations preserve the probabilities of state transitions in quantum theory. As a result, the most important groups in quantum field theory are the unitary groups, specifically $U(1)$, $SU(2)$, and S$U(3)$.

Quiz

1. Consider an element of $SU(2)$ given by $U = e^{i\sigma_x \alpha/2}$. By writing down the power series expansion, write U in terms of trigonometry functions.
2. Consider $SU(3)$ and calculate $tr(\lambda_i \lambda_j)$.
3. How many casimir operators are there for $SU(2)$?
4. Write down the casimir operators for $SU(2)$.

   A Lorentz transformation can be described by boost matrices with rapidity defined by $\tanh\phi = v/c$. A boost in the x direction is represented by the matrix

$$\begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5. Find the generator K_x.

6. Knowing that $[K_x, K_y] = -iJ_z$, where J_z is the angular momentum operator written in four dimensions as

$$J_z = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   find K_y.

7. Do pure Lorentz boosts constitute a group?

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CHAPTER 4

Discrete Symmetries and Quantum Numbers

In Chap. 3 we examined *continuous symmetries,* that is, symmetries with continuously varying parameters such as rotations. Now we consider a different kind of symmetry, a *discrete symmetry.* There are three important discrete symmetries in particle physics: *parity, charge conjugation,* and *time reversal.*

Additive and Multiplicative Quantum Numbers

A *quantum number* is some quantity (a quantized property of a particle) that is conserved in a particle reaction (decay, collision, etc.). An *additive quantum number*

n is one such that if $n_1, n_2, \ldots, n_i, \ldots$ are the quantum numbers before the reaction, and $m_1, m_2, \ldots, m_i, \ldots$ are the quantum numbers after the reaction, then the sum is preserved.

$$\sum_i n_i = \sum_i m_i \tag{4.1}$$

Or, if we have a composite system with quantum numbers $n_1, n_2, \ldots, n_i, \ldots$, and if the quantum number is additive, then the quantum number of the composite system is

$$\sum_i n_i$$

A *multiplicative quantum number* is one such that if $n_1, n_2, \ldots, n_i, \ldots$ are the quantum numbers before the reaction, and $m_1, m_2, \ldots, m_i, \ldots$ are the quantum numbers after the reaction, then the product is preserved.

$$\prod_i n_i = \prod_i m_i \tag{4.2}$$

Or, if we have a composite system, and if a quantum number is multiplicative, then

$$\prod_i n_i$$

is the quantum number for the composite system. If a quantum number is conserved, then it represents a symmetry of the system.

Parity

We begin our discussion of parity by examining nonrelativistic quantum mechanics. Consider a potential V that is symmetric about the origin and therefore $V(-x) = V(x)$. This implies that if $\psi(x)$ is a solution of the Schrödinger equation, then so is $\psi(-x)$, and it solves the equation with the *same eigenvalue*. This is because

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2} + V(-x)\psi(-x) = E\psi(-x)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2} + V(x)\psi(-x) = E\psi(-x)$$

when $V(-x) = V(x)$. If $\psi(x)$ and $\psi(-x)$ both solve the Schrödinger equation with the same eigenvalue E, then they must be related to each other as

$$\psi(x) = \alpha\psi(-x) \tag{4.3}$$

If we let $x \to -x$, then we obtain

$$\psi(-x) = \alpha\psi(x)$$

Inserting this into Eq. (4.3) gives

$$\psi(x) = \alpha\psi(-x) = \alpha\left[\alpha\psi(x)\right] = \alpha^2\psi(x)$$
$$\Rightarrow 1 = \alpha^2$$

This tells us that $\alpha = \pm 1$. Then either

$$\psi(-x) = \psi(x)$$

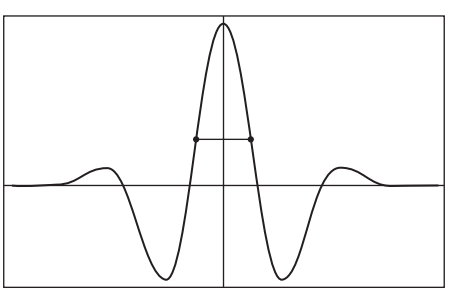
in which case we say that the wave function has *even parity* or

$$\psi(-x) = -\psi(x)$$

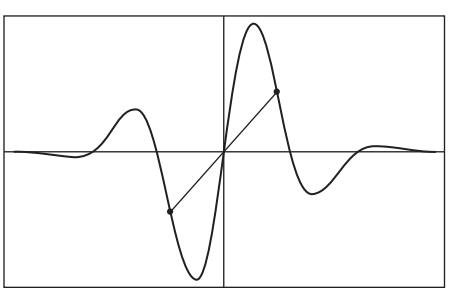
in which case we say the wave function has *odd parity.* This leads us to the concept of the *parity operator P.* The parity operator causes a change in sign when $x \to -x$ in the wave function.

$$P\psi(x) = \psi(-x) \tag{4.4}$$

As seen here in the illustrations, even powers lead to functions with even parity $\left[\psi(-x) = \psi(x)\right]$ while odd powers lead to functions with odd parity $\left[\psi(-x) = -\psi(x)\right]$.



An example of an even function where $\psi(-x) = \psi(x)$. Monomials with even powers (x^2, x^4, x^6, …) are even functions.



An example of an odd function where $\psi(-x) = -\psi(x)$. Monomials with odd powers (x^1, x^3, x^5, …) are odd functions.

Obviously, applying the parity operator twice in succession gives the original wave function back as shown here.

$$P^2\psi(x) = P\psi(-x) = \psi(x)$$

Think of the parity operator as a reflection through the *y* axis. If ψ is even, we see the same function values; if ψ is odd, we see the negative of the function values. In either case, another reflection through the *y* axis brings us back to our initial state. The reflection of the reflection is the image.

It follows that

$$P^2 = I \tag{4.5}$$

The eigenstates of parity are ±1:

$$P|\psi\rangle = \pm|\psi\rangle \tag{4.6}$$

As we have seen, true reflections preserve length.

If $|\psi\rangle$ is an angular momentum state with angular momentum L, that is, $|\psi\rangle = |L, m_z\rangle$, then the parity operator acts as

$$P|L, m_z\rangle = (-1)^L|L, m_z\rangle \tag{4.7}$$

We showed above that $\alpha = \pm 1$ when $\psi(x)$ and $\psi(-x)$ both solve the Schrödinger equation with the same eigenvalue *E*. We can generalize this by saying that if the Parity operator and Hamiltonian commute

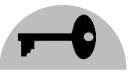
$$[P, H] = 0 = PH - HP$$

then *parity is conserved*. A consequence of this is that a state with parity α cannot evolve into a state with parity $-\alpha$ since the Hamiltonian governs the time evolution of the states. Even parity states remain even during time evolution and odd parity states remain odd. Now if $|\psi\rangle$ is a nondegenerate eigenstate of *H* with eigenvalue *E*, then

$$P\big(H|\psi\rangle\big) = P\big(E|\psi\rangle\big) = EP|\psi\rangle$$

But if $[P, H] = 0 = PH - HP$, then

$$P\big(H|\psi\rangle\big) = H\big(P|\psi\rangle\big) = E\big(P|\psi\rangle\big)$$



So the eigenstates of H are also eigenstates of the parity operator. Also, it follows that the eigenvalues of P are $\alpha = \pm 1$. Notice that this precludes states of mixed, or indefinite parity. This is a powerful constraint mathematically and physically it means that particles are either fermions or bosons.

In quantum field theory, the eigenvalue of parity α is a property of particles called the *intrinsic parity* of the particle. Following our discussion of parity and wave functions at the beginning of this section, if $\alpha = +1$ for a given particle, we say that the particle has even parity. If $\alpha = -1$, the particle has odd parity.

Parity for fermions is assigned as follows:

- Particles with spin-1/2 have positive parity. Hence an electron and a quark both have $\alpha = +1$.
- Antiparticles with spin-1/2 have negative parity. Therefore a positron has $\alpha = -1$.

Bosons have the same intrinsic parity for both particles and antiparticles.

Parity is a multiplicative quantum number. Let $|\psi\rangle = |a\rangle|b\rangle$ define a composite system. If the parities of $|a\rangle$ and $|b\rangle$ are P_a and P_b respectively, then the parity of the composite system is the product of the individual parities, that is,

$$P_\psi = P_a P_b$$

We can construct new parity operators by combining P with one of the conserved charges of the standard model. These are

- The electric charge operator Q
- Lepton number L
- Baryon number B

Earlier we said something about the conservation of parity. Parity *is not always conserved*, and there are specific cases, like

- Parity *is conserved* in the electromagnetic and strong interactions.
- Parity *is not conserved* in the weak interaction.

Particles are often labeled as follows

$$J^P \equiv \text{spin}^{\text{parity}} \tag{4.8}$$

A spin-0 particle with negative parity is called a *pseudoscalar*. Examples of pseudoscalar particles include the π and K mesons. Using the notation in Eq. (4.8) we write 0^- to indicate a pseudoscalar particle.

A spin-0 particle with positive parity is called a *scalar*. We denote a scalar by 0^+. An example of a scalar particle is the *Higgs boson*, the particle corresponding to the

field believed to be responsible for mass generation. The elusive Higgs hasn't been detected at the time of writing, but may be found soon when the Large Hadron Collider (LHC) begins operation.

A *vector boson* has spin-1 and negative parity (1^-). The most famous vector boson is the photon. A pseudovector has unit spin and positive parity 1^+.

Parity is conserved in the electromagnetic and strong interactions, so the total parity of a system before an electromagnetic or strong interaction is the same as the parity after the interaction. In the 1950s, two physicists named Lee and Yang proposed that parity conservation is violated in the weak interaction.

This is called *parity violation.* This was demonstrated experimentally by observing weak decays of Cobalt-60, leading to the Nobel prize for Lee and Yang. Parity violation also became apparent in the weak decays of two particles called the θ and τ mesons. They decay as

$$\theta^+ \rightarrow \pi^+\pi^0$$

$$\tau^+ \rightarrow \pi^+\pi^-\pi^+$$

The final states of these two decays have opposite parity and therefore physicists believed the θ^+ and the τ^+ to be different particles. However, successively refined measurements of the θ^+ and τ^+ mass and lifetimes suggested they were actually the same particle. The discovery of parity violation in the weak interaction resolved this dilemma and today we call this particle the K^+ meson.

Charge Conjugation

We now consider *charge conjugation C*, an operator which converts particles into antiparticles. Let $|\psi\rangle$ represent a particle state and $|\bar{\psi}\rangle$ represent the antiparticle state. Then the charge conjugation operator acts as

$$C|\psi\rangle = |\bar{\psi}\rangle \tag{4.9}$$

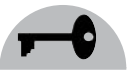
Charge conjugation also acts on antiparticle states, turning them into particle states.

$$C|\bar{\psi}\rangle = |\psi\rangle \tag{4.10}$$

It follows that

$$C^2|\psi\rangle = CC|\psi\rangle = C|\bar{\psi}\rangle = |\psi\rangle$$

We can use this relation to determine the eigenvalues of charge conjugation. It is apparent that like parity, they must be $C = \pm 1$. Charge conjugation is also like parity



in that it is a multiplicative quantum number. Since charge conjugation converts particles into antiparticles and vice versa, it reverses the sign of all quantum numbers (and also changes the sign of magnetic moment). Consider a proton $|p\rangle$. It has positive charge q

$$Q|p\rangle = q|p\rangle$$

and Baryon number $B = +1$. If we operate on the proton with the charge conjugation operator, then $C|p\rangle = |\bar{p}\rangle$ and since

$$Q|\bar{p}\rangle = -q|\bar{p}\rangle$$

the charge has been reversed. The baryon number has also been changed to $B = -1$. Notice that the proton state cannot be an eigenstate of charge conjugation, since the result of $C|p\rangle = |\bar{p}\rangle$ is a state with different quantum numbers—a different quantum state. The eigenstates of the charge conjugation operator are eigenstates with 0 charge, that is, neutral particles. More generally, the eigenstates of C must have all additive quantum numbers equal to 0. An example is a neutral pion π^0. In this case, π^0 is its own antiparticle and so

$$C|\pi^0\rangle = \alpha|\pi^0\rangle$$

for some α. Applying charge conjugation twice

$$C^2|\pi^0\rangle = \alpha C|\pi^0\rangle = \alpha^2|\pi^0\rangle$$
$$\Rightarrow \alpha = \pm 1$$

We can find the charge conjugation properties of the photon in the following way, and hence determine the eigenvalue α for the π^0. First charge conjugation will reverse the sign of the charge density J as shown here.

$$CJC^{-1} = -J$$

Now, the interaction part of the electromagnetic Lagrangian can be used to determine the charge conjugation properties of the photon. We need to find the action of C on $J_\mu A^\mu$. We have

$$CJ_\mu A^\mu C^{-1} = CJ_\mu C^{-1} CA^\mu C^{-1}$$
$$= -J_\mu CA^\mu C^{-1}$$

This can only be invariant if

$$CA^{\mu}C^{-1} = -A^{\mu}$$
$$\Rightarrow CJ_{\mu}A^{\mu}C^{-1} = J_{\mu}A^{\mu}$$

Since A^{μ} is the electromagnetic vector potential, this tells us that the eigenvalue of charge conjugation for the photon is $\alpha = -1$. Therefore if there are *n* photons, the charge conjugation is $(-1)^n$. The π^0 decays into two photons as

$$\pi^0 \rightarrow \gamma + \gamma$$

The two photon state has $\alpha = (-1)^2 = +1$, therefore we conclude that

$$C\left|\pi^0\right\rangle = (+1)\left|\pi^0\right\rangle$$
$$\Rightarrow \alpha = +1 \text{ for the } \pi^0$$

Charge conjugation proceeds in the same way as parity, that is,

- Charge conjugation *C* is conserved in the strong and electromagnetic interactions.
- Charge conjugation is not conserved in the weak interaction.

CP Violation

The fact that charge conjugation and parity are each individually violated in weak interactions led to the hope that the combination of charge conjugation and parity would be conserved. It almost is, but there is a slight violation that can be seen from the decay of neutral K mesons.

The neutral $\left|K^0\right\rangle$ meson is an interesting particle which is observed in a linear combination of states with its antiparticle. This is because the $\left|K^0\right\rangle$ spontaneously transitions into its antiparticle and vice versa as shown here.

$$\left|K^0\right\rangle \leftrightarrow \left|\bar{K}^0\right\rangle$$

The $\left|K^0\right\rangle$ and its antiparticles are pseudoscalars 0^- and so have negative parity, that is

$$P\left|K^0\right\rangle = -\left|K^0\right\rangle$$

$$P\left|\bar{K}^0\right\rangle = -\left|\bar{K}^0\right\rangle$$

Charge conjugation, is of course the operation that transforms the $|K^0\rangle$ into its antiparticle, that is,

$$C|K^0\rangle = |\bar{K}^0\rangle$$

$$C|\bar{K}^0\rangle = |K^0\rangle$$

Taken together, *CP* acts on the states as

$$CP|K^0\rangle = -C|K^0\rangle = -|\bar{K}^0\rangle$$

$$CP|\bar{K}^0\rangle = -C|\bar{K}^0\rangle = -|K^0\rangle$$

We see from this relation that $|K^0\rangle$ and $|\bar{K}^0\rangle$ *are not* eigenstates of *CP*. To see if *CP* is violated, we need to construct states that are eigenstates of *CP* out of $|K^0\rangle$ and $|\bar{K}^0\rangle$. The states that do this are

$$|K_1\rangle = \frac{|K^0\rangle - |\bar{K}^0\rangle}{\sqrt{2}} \qquad |K_2\rangle = \frac{|K^0\rangle + |\bar{K}^0\rangle}{\sqrt{2}}$$

It is helpful to think of particle states in terms of vector spaces. In terms of a rotation by $\pi/4$ we see that

$$\begin{pmatrix} |K_1\rangle \\ |K_2\rangle \end{pmatrix} = R\left(-\frac{\pi}{4}\right)\begin{pmatrix} |K_0\rangle \\ |\bar{K}_0\rangle \end{pmatrix}$$

which demonstrates an advantage of the vector space view of particle.

So we have

$$CP|K_1\rangle = +|K_1\rangle$$

$$CP|K_2\rangle = -|K_2\rangle$$

Fortunately, these states can be created in the laboratory. It turns out that they both decay into π mesons. Now, if *CP* is conserved, then each of these states will decay into a state with the same value of *CP*. That is,

$|K_1\rangle \rightarrow$ decays into a state with $CP = +1$

$|K_2\rangle \rightarrow$ decays into a state with $CP = -1$

A neutral K meson $|K^0\rangle$ can decay into a state with two α mesons or into a state with three π mesons. The charge conjugation and parity eigenvalues of these states are

$$\begin{aligned} &2\pi \text{ mesons} : C = +1,\ P = +1, \\ &\Rightarrow CP = +1 \\ &3\pi \text{ mesons} : C = +1,\ P = -1, \\ &\Rightarrow CP = -1 \end{aligned}$$

If *CP* is conserved then, we would have

$$|K_1\rangle \rightarrow \text{only decays into } 2\pi \text{ mesons}$$

$$|K_1\rangle \rightarrow \text{only decays into } 3\pi \text{ mesons}$$

This is *not* what is observed experimentally. It is found that a small fraction of the time

$$|K_2\rangle \rightarrow \text{decays into } 2\pi \text{ mesons}$$

Hence we have a transition from $CP = -1 \rightarrow CP = +1$. It turns out that the long lived K meson state is

$$|K_L\rangle = \frac{|K_2\rangle + \varepsilon |K_1\rangle}{\sqrt{1 + |\varepsilon|^2}}$$

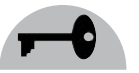
The parameter ε is a measure of the amount of violation of *CP* conservation. Experimental evidence indicates that

$$\varepsilon = 2.3 \times 10^{-3}$$

A small number indeed, but *not zero*. *CP* violation happens because a small fraction of the time, the long lived neutral K meson state, is found in $|K_1\rangle$, giving the unexpected decays.

The CPT Theorem

To restore invariance, we have to bring in one more symmetry, *time reversal*. This is another discrete transformation on states, turning a state $|\psi\rangle$ into a state $|\psi'\rangle$ that evolves with time flowing in the negative direction. Momentums change sign:



linear momentum $p \to -p$ and angular momentum $L \to -L$, but all other quantities maintain the same sign. The time-reversal operator acts to transform the states as

$$T|\psi\rangle = |\psi'\rangle$$

The time-reversal operator is *antiunitary* and *antilinear*. To say that it is antilinear, we mean that

$$T\left(\alpha|\psi\rangle + \beta|\phi\rangle\right) = \alpha^*|\psi'\rangle + \beta^*|\phi'\rangle$$

While a unitary operator U preserves inner products,

$$\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle$$

an antiunitary operator does not, but instead gives the complex conjugate as shown here.

$$\langle A\phi|A\psi\rangle = \langle\phi|\psi\rangle^*$$

The time-reversal operator is antiunitary and can be written as a product of an operator K that converts states into their complex conjugates (note that K does not refer to the K meson of the previous section, but in this context is an operator)

$$K\psi = \psi^*$$

and a unitary operator U

$$T = UK$$

If the time-reversal operator commutes with the Hamiltonian $[T,H] = 0$, and if $|\psi\rangle$ satisfies the Schrödinger equation, $T|\psi\rangle$ will also satisfy the Schrödinger equation when $t \to -t$. Hence the name time-reversal operator. If the laws of physics are unchanged under time reversal, then they are a symmetry of the system.

The *CPT theorem* considers the three symmetries C, P, and T taken together. According to this theorem, if charge conjugation, parity reversal, and time reversal are taken together we have an exact symmetry and so the laws of physics are invariant. More colloquially, the theorem means that if matter is replaced by antimatter (charge conjugation), momentum is reversed with spatial inversion (parity conjugation), and time is reversed, the result would be a universe indistinguishable from the one we live in. For the *CPT* theorem to be valid, all three symmetries must

be valid or if one or more symmetries are violated, another symmetry must be violated to negate the first violation. However we have seen that in the weak interaction, there is *CP* violation. To compensate for the problem of *CP* violation, there must be *T* violation as well.

Now let's look at this (loosely) in the context of quantum field theory. To satisfy special relativity, we need Lorentz invariance. This means that we implement a Lorentz transformation with

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the theory is the same. Quantum theory allows for ϕ to be *complex*. If we take $\phi = i\pi$ and

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then by setting $\phi = i\pi$ we get

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

giving time reversal $t \to -t$ and space inversion $x \to -x$. This is a *PT* invariant theory. If the particles are charged then you recover the entire *CPT* invariance.

Summary

In this chapter we have examined some discrete symmetries, including parity, charge conjugation, and time reversal. Interesting physics arises because these discrete symmetries are not conserved in weak interactions. The symmetry that is always conserved in all interactions is *CPT*, which is stated in the *CPT* theorem.

Quiz

1. Angular momentum states transform under the parity operator as
   (a) $P\left|L,m_z\right\rangle = -\left|L,m_z\right\rangle$
   (b) $P\left|L,m_z\right\rangle = L\left|L,m_z\right\rangle$
   (c) $P\left|L,m_z\right\rangle = (-1)^L\left|L,m_z\right\rangle$
   (d) $P\left|L,m_z\right\rangle = \left|L,m_z\right\rangle$
2. The interaction Lagrangian of electromagnetism is invariant under charge conjugation if
   (a) $CA^\mu C^{-1} = -A^\mu$
   (b) It is not invariant under charge conjugation
   (c) $CJ^\mu C^{-1} = J^\mu$
   (d) $CA^\mu C^{-1} = A^\mu$
3. Parity is
   (a) Conserved in weak and electromagnetic interactions, but is violated in the strong interaction
   (b) Conserved in strong interactions, but is violated in weak and electromagnetic interactions
   (c) Not conserved
   (d) Conserved in the strong and electromagnetic interactions, but is violated in the weak interaction
4. The eigenvalues of charge conjugation are
   (a) $c = \pm 1$
   (b) $c = 0, \pm 1$
   (c) $c = \pm q$
   (d) $c = 0, \pm q$
5. An operator is antiunitary if
   (a) $\langle A\phi | A\psi\rangle = -\langle\phi|\psi\rangle$
   (b) $\langle A\phi | A\psi\rangle = \langle\phi|\psi\rangle$
   (c) $\langle A\phi | A\psi\rangle = \langle\phi|\psi\rangle^*$
   (d) $\langle A\phi | A\psi\rangle = -\langle\phi|\psi\rangle^*$

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CHAPTER 5

The Dirac Equation

In the next chapter we will see that scientists began with an attempt to arrive at a relativistic wave equation that put time and space on an equal footing by "promoting" derivatives with respect to time to second order. This was done because the Schrödinger equation has second-order spatial derivatives. The equation that results is called the *Klein-Gordon equation*. Unfortunately, this leads to several problems as far as quantum theory is concerned. In particular, it gives us a probability density that can be negative and it leads to negative energy states.

While these problems can be resolved by reinterpreting the resulting Klein-Gordon equation, a fruitful line of inquiry results if we try to tackle the first problem head on by taking a different approach. This is exactly what Dirac did in deriving his now famous equation. The Dirac equation applies to spin-1/2 fields, and puts time and space on equal ground in the equation by considering first-order spatial derivatives, rather than increasing the order of the time derivatives.

The Classical Dirac Field

We begin by looking at the Dirac equation in terms of classical field theory. Again, we approach the problem with the goal of satisfying the tenets of special relativity,

hence we want time and space to appear in the equation in a similar fashion. As discussed in Chap. 4, derivatives with respect to time in the Schrödinger equation are first order while derivatives with respect to spatial coordinates are second order. The Klein-Gordon equation attempts to deal with this discrepancy by using second-order derivatives with respect to time. With the Dirac equation, we are going to take the opposite approach and consider using first-order derivatives for the spatial coordinates, while simultaneously keeping the derivatives with respect to time first order as well. The reason for doing this is to avoid the negative probability distributions that we saw arise from the Klein-Gordon equation. In Chap. 4 we saw that this is *due* to the fact that the equation contained second-order derivatives with respect to time. So we will attempt to avoid that problem by keeping time derivatives first order. The result is an equation that beautifully describes spin-1/2 particles.

First let's remind ourselves about the Schrödinger equation.

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{5.1}$$

Let's write it in a more suggestive form using the operator definition $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V$. Then,

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{5.2}$$

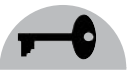
Why is this suggestive? Well, the Dirac equation can be thought of as a type of Schrödinger equation if we just change what the Hamiltonian operator $\hat{H}$ is, that is, applied to the wave function. The form of the Hamiltonian is chosen so that the requirements of special relativity can be satisfied. Assuming that the particle in equation has rest mass m, the form of the Hamiltonian operator used in the Dirac equation is

$$\hat{H} = c\vec{\alpha}\bullet(-i\hbar\nabla) + \beta mc^2 \tag{5.3}$$

We will explain in a bit what $\vec{\alpha}$ and β are. For now, using Eq. (5.3) in Eq. (5.2) gives us the Dirac equation.

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\vec{\alpha}\bullet(-i\hbar\nabla) + \beta mc^2\right]\psi \tag{5.4}$$

This is a relativistically covariant equation. Time and space have been put on the same footing since they both appear in the equation in terms of first-order derivatives. The new terms in the equation, $\vec{\alpha}$ and β, are actually 4×4 *matrices.* Before writing them down, we are going to rewrite Eq. (5.4) using what physicists call the *Dirac*



matrices, or you can call them the gamma matrices if you like. First let's hold on a moment and recall the gradient operator, which is a vector operator

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

or, using the notation of Chap. 1

$$\vec{\nabla} = \frac{\partial}{\partial x^1}\hat{e}_1 + \frac{\partial}{\partial x^2}\hat{e}_2 + \frac{\partial}{\partial x^3}\hat{e}_3$$

Writing this out we see the first-order spatial derivatives that appear in the Dirac equation [Eq. (5.4)] explicitly. We are taking $\vec{\alpha}$ to be a vector as well, whose components are matrices

$$\vec{\alpha} = \alpha_1\hat{e}_1 + \alpha_2\hat{e}_2 + \alpha_3\hat{e}_3$$

Now we define the gamma matrices (or Dirac matrices) in terms of $\vec{\alpha}$ and β in the following way.

$$\gamma^0 = \beta \tag{5.5}$$

$$\gamma^i = \beta\alpha_i \tag{5.6}$$

Adding Quantum Theory

At this point the first hint of quantum theory comes into play. Well of course since they are matrices, we don't necessarily expect the Dirac matrices to commute. In other words it's not necessarily true that $\gamma^1\gamma^2 = \gamma^2\gamma^1$, say. In fact, the Dirac matrices obey an important *anticommutation* rule. The anticommutator of two matrices *A* and *B* is

$$\{A, B\} = AB + BA \tag{5.7}$$

Note that in many texts the anticommutator is denoted by $[A, B]_+$, but we will stick to the notation used in Eq. (5.7). The relationship for the Dirac matrices actually connects them to the spacetime metric (perhaps a connection to quantum gravity here?). It is given by

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{5.8}$$

Using the Dirac matrices, we can write down the Dirac equation in a fancy relativistic notation sure to impress all of your friends. It is

$$i\hbar\gamma^{\mu}\frac{\partial\psi}{\partial x^{\mu}} - mc\psi = 0 \tag{5.9}$$

Better yet, we work in units where we set $\hbar = c = 1$ and use $\frac{\partial}{\partial x^{\mu}} = \partial_{\mu}$ to write the Dirac equation in the compact form

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0 \tag{5.10}$$

The correct way to interpret this equation, when the quantum theory is brought in, is that it applies to the *Dirac field* whose quanta are spin-1/2 particles—electrons.

EXAMPLE 5.1
Show that the Dirac field ψ also satisfies the Klein-Gordon equation.

SOLUTION
That it should ought not to surprise you, the Klein-Gordon equation is nothing other than a restatement of Einstein's relation between energy, mass, and momentum in special relativity derived using the quantum substitutions $E \to i\hbar\frac{\partial}{\partial t}$ and $\vec{p} \to -i\hbar\frac{\partial}{\partial x}$. Since $E^2 = p^2 + m^2$ is an absolutely fundamental relation that applies to everything, all particles and fields, including the Dirac field, must satisfy the Klein-Gordon equation. It turns out that in a sense the Dirac equation is the "square root" of the Klein-Gordon equation. Let's see how to derive the Klein-Gordon equation directly from the Dirac equation. We start with the Dirac equation

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0$$

Now multiply from the left by $i\gamma_{\nu}\partial^{\nu}$. This gives

$$-\gamma_{\nu}\gamma^{\mu}\partial^{\nu}\partial_{\mu}\psi - im\gamma_{\nu}\partial^{\nu}\psi = 0$$

Take a look at the second term. From the Dirac equation itself, we know that

$$i\gamma^{\mu}\partial_{\mu}\psi = m\psi$$

So we can write

$$im\gamma_{\nu}\partial^{\nu}\psi = m^2\psi$$

Using this and moving everything to the other side to get rid of the minus sign

$$-\gamma_{\nu}\gamma^{\mu}\partial^{\nu}\partial_{\mu}\psi - im\gamma_{\nu}\partial^{\nu}\psi = 0$$

becomes

$$\gamma_\nu \gamma^\mu \partial^\nu \partial_\mu \psi + m^2 \psi = 0$$

Now we apply the anticommutation relation obeyed by the Dirac matrices, Eq. (5.8). Restating it here so you don't have to flip through the pages of the book it is

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

This can be applied to write the first term in $\gamma_\nu \gamma^\mu \partial^\nu \partial_\mu \psi + m^2 \psi = 0$ in a symmetric form. This is done as follows:

$$\gamma_\nu \gamma^\mu = g_{\nu\sigma} \gamma^\sigma \gamma^\mu = g_{\nu\sigma} \frac{1}{2} (\gamma^\mu \gamma^\sigma + \gamma^\sigma \gamma^\mu)$$

But we have

$$\begin{aligned} g_{\nu\sigma} \frac{1}{2} \left(\gamma^\mu \gamma^\sigma + \gamma^\sigma \gamma^\mu\right) &= g_{\nu\sigma} \frac{1}{2} \left(2g^{\mu\sigma}\right) \\ &= g_{\nu\sigma} g^{\mu\sigma} \\ &= \delta_\nu^\mu \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= \gamma_\nu \gamma^\mu \partial^\nu \partial_\mu \psi + m^2 \psi \\ &= \delta_\nu^\mu \partial^\nu \partial_\mu \psi + m^2 \psi \\ &= \partial^\mu \partial_\mu \psi + m^2 \psi \end{aligned}$$

This is nothing other than our old friend, the Klein-Gordon equation! This ought to be a really satisfying result—we have *derived* the Klein-Gordon equation directly from the Dirac equation. This shows that Dirac fields (particles . . .) also satisfy the Klein-Gordon equation and hence automatically satisfy the relativistic relation between energy, mass, and momentum.

The Form of the Dirac Matrices

It will be hard to get anywhere if we don't know how to explicitly write down the Dirac matrices. There are actually a couple of different ways to do it. One way, which we introduce first, is the Dirac-Pauli representation, and it's pretty straightforward. Keep in mind that these matrices are 4×4 matrices. The first Dirac

matrix is an extension of the unassuming identity matrix, which has 1s all along the diagonal. In the 2×2 case,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In the Dirac-Pauli representation, we can write the first gamma matrix as

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{5.11}$$

where I is the 2×2 identity matrix shown above. The 0s are 2×2 blocks of 0s. So this matrix is actually

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now, an aside, note that since the Dirac matrices are 4×4 matrices (think operators), the Dirac field ψ must be a four component vector that they can act on, since they appear in the Dirac equation. We call this vector a *spinor* and write

$$\psi(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \\ \psi^3(x) \\ \psi^4(x) \end{pmatrix} \tag{5.12}$$

Now that the aside is over, let's turn to the other Dirac matrices. They are written in terms of the Pauli matrices that you should be familiar with ordinary nonrelativistic quantum mechanics. To review, they are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.13}$$

In the Dirac-Pauli representation, then

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{5.14}$$

so that

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

Let's write down one example explicitly as shown here.

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

The Dirac matrices can also be written down using what is known as the *chiral representation.* In this case,

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{5.15}$$

Some Tedious Properties of the Dirac Matrices

Regardless of representation used, the Dirac matrices satisfy several tedious but important relations that are useful when doing calculations. First we define yet another matrix, the mysterious gamma five matrix.

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{5.16}$$

It is Hermitian

$$\left(\gamma^5\right)^\dagger = \gamma^5 \tag{5.17}$$

and it squares to the identity

$$\left(\gamma^5\right)^2 = I \tag{5.18}$$

In the chiral representation we write it as

$$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.19}$$

but in the Dirac representation we write it as

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{5.20}$$

(notice that these matrices are traceless). Now we also have $(\gamma^\mu)^2 = I$. Hence,

$$\gamma^\mu \gamma_\mu = 4I \tag{5.21}$$

We can show this explicitly:

$$\gamma^\mu \gamma_\mu = \gamma^1\gamma_1 + \gamma^2\gamma_2 + \gamma^3\gamma_3 + \gamma^4\gamma_4 = 4\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \tag{5.22}$$

Again, we see this by writing out the implied summation:

$$\gamma^\mu \gamma^\nu \gamma_\mu = \gamma^1\gamma^\nu\gamma_1 + \gamma^2\gamma^\nu\gamma_2 + \gamma^3\gamma^\nu\gamma_3 + \gamma^4\gamma^\nu\gamma_4 = -2\gamma^\nu$$

You should repeat these computations yourself to ensure you understand how to work with these matrices.

EXAMPLE 5.2
Find the anticommutator $\{\gamma^5, \gamma^0\}$.

SOLUTION
We are helped by the fact that $\{\gamma^i, \gamma^0\} = 0$. This is because we apply Eq. (5.8) and use the fact that $g^{i0} = 0$. For instance,

$$\{\gamma^3, \gamma^0\} = \gamma^3\gamma^0 + \gamma^0\gamma^3 = 2g^{03} = 0$$

Using this fact, that is $\{\gamma^i, \gamma^0\} = 0$, we can write

$$\gamma^i\gamma^0 = -\gamma^0\gamma^i$$

Therefore,

$$\begin{aligned}\{\gamma^5,\gamma^0\} &= \gamma^5\gamma^0+\gamma^0\gamma^5\\ &= (i\gamma^0\gamma^1\gamma^2\gamma^3)\gamma^0+\gamma^0(i\gamma^0\gamma^1\gamma^2\gamma^3)\\ &= -(i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3)+\gamma^0(i\gamma^0\gamma^1\gamma^2\gamma^3)\\ &= -(i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3)+(\gamma^0)^2(i\gamma^1\gamma^2\gamma^3)\\ &= -(i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3)+(i\gamma^1\gamma^2\gamma^3)\\ &= (i\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3)+(i\gamma^1\gamma^2\gamma^3)\\ &= -(\gamma^0)^2(i\gamma^1\gamma^2\gamma^3)+(i\gamma^1\gamma^2\gamma^3)\\ &= -(i\gamma^1\gamma^2\gamma^3)+(i\gamma^1\gamma^2\gamma^3)=0\end{aligned}$$

We conclude that $\{\gamma^5,\gamma^0\}=0$.

EXAMPLE 5.3
Find $tr(\gamma^5)$.

SOLUTION
We use $(\gamma^\mu)^2 = I$ together with the result derived in Example 5.2, which means that $\gamma^5\gamma^0 = -\gamma^0\gamma^5$. We have

$$\begin{aligned}tr(\gamma^5) &= tr(I\gamma^5)\\ &= tr(\gamma^0\gamma^0\gamma^5)\\ &= tr(-\gamma^0\gamma^5\gamma^0)\end{aligned}$$

Now let's recall a basic property of the trace. Remember that the trace operation is cyclic, meaning that

$$tr(ABC) = tr(CBA) = tr(BCA) = tr(BAC)$$

We can also pull scalars (numbers) right outside the trace, that is, $tr(\alpha A) = \alpha tr(A)$
Hence,

$$\begin{aligned}tr(\gamma^5) &= tr(-\gamma^0\gamma^5\gamma^0)\\ &= -tr(\gamma^0\gamma^5\gamma^0)\\ &= -tr(\gamma^0\gamma^0\gamma^5)\\ &= -tr(\gamma^5)\end{aligned}$$

We have found that

$$tr(\gamma^5) = -tr(\gamma^5)$$

This can only be true if

$$tr(\gamma^5) = 0$$

which is easily verified by looking at the explicit representation of this matrix in Eq. (5.19).

Adjoint Spinors and Transformation Properties

The adjoint spinor of ψ is not simply $\psi^\dagger$, but turns out to be

$$\bar{\psi} = \psi^\dagger \gamma^0 \tag{5.23}$$

We can form composite objects among the fields $\psi, \bar{\psi}$, and the Dirac matrices. Each of these objects transforms in a different way, so we can construct vectors, tensors, and pseudovectors, for example. We can form the following Lorentz scalar

$$\bar{\psi}\psi \tag{5.24}$$

Using the gamma matrices, we can construct a pseudoscalar which means it is a quantity that changes sign under either parity or space inversion. This pseudoscalar is

$$\bar{\psi}\gamma_5\psi \tag{5.25}$$

Taking an arbitrary gamma matrix, we get an object that transforms as a four vector.

$$\bar{\psi}\gamma^\mu\psi \tag{5.26}$$

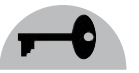
Another scalar that can be constructed is

$$\bar{\psi}\gamma^\mu\partial_\mu\psi \tag{5.27}$$

Since $\bar{\psi}\psi$ is a scalar and the mass m is a scalar, we can use this to write down the Lagrangian that can be used to derive the Dirac equation using the usual methods. Remember that the Lagrangian must transform as a scalar. The Lagrangian that works is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \tag{5.28}$$

Notice that both terms in this Lagrangian are scalars. By varying $\bar{\psi}$ we obtain the Dirac equation as the equation of motion for this to Lagrangian.



Slash Notation

In quantum field theory texts and papers you will often see a shorthand notation developed by Feynman called *slash notation.* Slash notation is used to indicate a contraction between a 4-vector and a gamma matrix. Let a_μ be some 4-vector. Then,

$$\not{a} = \gamma^\mu a_\mu = \gamma^0 a_0 + \gamma^1 a_1 + \gamma^2 a_2 + \gamma^3 a_3 \tag{5.29}$$

So for momentum, we have

$$\not{p} = \gamma^\mu p_\mu \tag{5.30}$$

In fact, this can be written as the 4×4 matrix.

$$\not{p} = \begin{pmatrix} E & -\vec{\sigma} \bullet \vec{p} \\ \vec{\sigma} \bullet \vec{p} & -E \end{pmatrix} \tag{5.31}$$

Solutions of the Dirac Equation

Let's introduce two new components of the Dirac field so that we can write down a spinor as a two-component object. We call these components *u* and *v* where

$$\psi(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \\ \psi^3(x) \\ \psi^4(x) \end{pmatrix} = \begin{pmatrix} u^1(x) \\ u^2(x) \\ v^1(x) \\ v^2(x) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{5.32}$$

It will be easy to find solutions of the Dirac equation if we consider momentum space, since the single spatial derivative will be converted into momentum and we will arrive at an algebraic relationship. So consider the Fourier expansion of the Dirac field as

$$\psi(x) = \int \frac{d^4k}{(2\pi)^4} \psi(k) e^{-ik_\mu x^\mu} \tag{5.33}$$

Now we return to the Dirac equation, which for your convenience we reproduce here as

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

Using the Fourier expansion of the field Eq. (5.33), we have

$$\begin{aligned} i\gamma^{\nu}\partial_{\nu}\psi &= i\gamma^{\mu}\partial_{\mu}\int \frac{d^4k}{(2\pi)^4}\,\psi(k)\,e^{-ik_{\mu}x^{\mu}} \\ &= \int \frac{d^4k}{(2\pi)^4}\,i\gamma^{\nu}\psi(k)\,\partial_{\nu}e^{-ik_{\mu}x^{\mu}} \\ &= \int \frac{d^4k}{(2\pi)^4}\,i\gamma^{\nu}(-ik_{\nu})\psi(k)\,e^{-ik_{\mu}x^{\mu}} \\ &= \int \frac{d^4k}{(2\pi)^4}\,[\gamma^{\nu}k_{\nu}\psi(k)]\,e^{-ik_{\mu}x^{\mu}} \end{aligned}$$

The other piece of the equation is

$$\begin{aligned} m\psi(x) &= m\int \frac{d^4k}{(2\pi)^4}\,\psi(k)\,e^{-ik_{\mu}x^{\mu}} \\ &= \int \frac{d^4k}{(2\pi)^4}\,m\,\psi(k)\,e^{-ik_{\mu}x^{\mu}} \end{aligned}$$

Putting these terms together gives

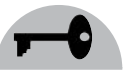
$$\begin{aligned} 0 &= \int \frac{d^4k}{(2\pi)^4}\,[\gamma^{\nu}k_{\nu}\psi(k)]\,e^{-ik_{\mu}x^{\mu}} - \int \frac{d^4k}{(2\pi)^4}\,m\,\psi(k)\,e^{-ik_{\mu}x^{\mu}} \\ &= \int \frac{d^4k}{(2\pi)^4}\,[\gamma^{\nu}k_{\nu}\psi(k) - m\,\psi(k)]\,e^{-ik_{\mu}x^{\mu}} \end{aligned}$$

Once more, the only way for this integral to be 0 is for the integrand to be 0. So it must be true that

$$\gamma^{\nu}k_{\nu}\psi(k) - m\,\psi(k) = 0$$

Well of course you could have arrived at this, which is just the momentum space equivalent of the Dirac equation, using the usual representation of momentum as a derivative in position space. To see how to proceed toward a solution, let's write down the Dirac representation of the gamma matrices again. We have

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$



together with

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

Let's take a look at the individual terms in the equation $\gamma^v k_v \psi(k) - m\psi(k) = 0$. First we have

$$\gamma^0 k_0 \psi = k_0 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} k_0 u \\ -k_0 v \end{pmatrix}$$

and

$$m\psi = m \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} mu \\ mv \end{pmatrix}$$

Next we have

$$\gamma^1 k_1 \psi = k_1 \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = k_1 \begin{pmatrix} \sigma_1 v \\ -\sigma_1 u \end{pmatrix}$$

Similarly,

$$\gamma^2 k_2 \psi = k_2 \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = k_2 \begin{pmatrix} \sigma_2 v \\ -\sigma_2 u \end{pmatrix}$$

and

$$\gamma^3 k_3 \psi = k_3 \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = k_3 \begin{pmatrix} \sigma_3 v \\ -\sigma_3 u \end{pmatrix}$$

Putting everything together, these relations and the Dirac equation in momentum space can be written in matrix form as

$$\begin{pmatrix} k_0 - m & -\vec{k} \cdot \vec{\sigma} \\ \vec{k} \bullet \vec{\sigma} & -(k_0 + m) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{5.34}$$

So there are two coupled equations

$$\begin{aligned} (k_0 - m)u - (\vec{k} \bullet \vec{\sigma})v = 0 \\ (k_0 + m)v - (\vec{k} \bullet \vec{\sigma})u = 0 \end{aligned} \tag{5.35}$$

For the sake of simplicity, let's identify the matrix in Eq. (5.34) as

$$K = \begin{pmatrix} k_0 - m & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\bullet\vec{\sigma} & -(k_0 + m) \end{pmatrix}$$

For a solution to this system to exist, it must be true that K has vanishing determinant. That is,

$$\det\begin{pmatrix} k_0 - m & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\bullet\vec{\sigma} & -(k_0 + m) \end{pmatrix} = 0$$

The determinant works out to be

$$\begin{aligned} \det K &= \det\begin{pmatrix} k_0 - m & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\bullet\vec{\sigma} & -(k_0 + m) \end{pmatrix} \\ &= -(k_0 - m)(k_0 + m) + (\vec{k}\cdot\vec{\sigma})^2 \end{aligned}$$

To work out this result, we'll need to calculate $(\vec{k}\cdot\vec{\sigma})^2$. From ordinary quantum mechanics recall that the Pauli matrices satisfy $\sigma_j^2 = I$. In addition, they satisfy the anticommutation relation.

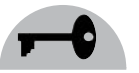
$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{5.36}$$

This greatly simplifies the calculation of $(\vec{k}\cdot\vec{\sigma})^2$. Writing out the terms, it's going to be

$$(\vec{k}\cdot\vec{\sigma})^2 = (k_1\sigma_1 + k_2\sigma_2 + k_3\sigma_3)(k_1\sigma_1 + k_2\sigma_2 + k_3\sigma_3)$$

Since the Pauli matrices satisfy Eq. (5.36), mixed terms in this expression will vanish. As a specific example consider

$$\begin{aligned} k_1k_2\sigma_1\sigma_2 + k_2k_1\sigma_2\sigma_1 &= k_1k_2(\sigma_1\sigma_2 + \sigma_2\sigma_1) \\ &= k_1k_2 2\delta_{12} \\ &= 0 \end{aligned}$$



So we are left with

$$\begin{aligned}(\vec{k}\cdot\vec{\sigma})^2 &= (k_1\sigma_1 + k_2\sigma_2 + k_3\sigma_3)(k_1\sigma_1 + k_2\sigma_2 + k_3\sigma_3)\\ &= {k_1}^2{\sigma_1}^2 + {k_2}^2{\sigma_2}^2 + {k_3}^2{\sigma_3}^2\\ &= {k_1}^2 + {k_2}^2 + {k_3}^2\\ &= \vec{k}^2\end{aligned}$$

Hence,

$$\begin{aligned}\det K &= -(k_0 - m)(k_0 + m) + (\vec{k}\cdot\vec{\sigma})^2\\ &= -k_0^2 + m^2 + \vec{k}^2\end{aligned}$$

This is just another way to write down our old friend, the relativistic relation between energy, mass, and momentum. Since we are working in units where $\hbar = 1$, $\vec{k} = \vec{p}$, and $k_0 = p_0 = E$, we can also write the energy in terms of a frequency since as you recall $E = \hbar\omega$.

Remember for a solution to exist, this quantity must vanish $(\det K = 0)$. That is,

$$k_0^2 = \vec{k}^2 + m^2$$

Taking the square root, we see that the possible energies are

$$\omega_k = k_0 = \pm\sqrt{\vec{k}^2 + m^2}$$

This means that the Dirac equation is still plagued by negative energies. If we take the positive solution, so that $E = \omega_k > 0$, we call the solution a *positive frequency* solution. The solution with $E = \omega_k < 0$ is the *negative energy* solution. We get out of this problem the same way we did with the Klein-Gordon equation—we interpret the positive energy solutions as corresponding to particles with positive energy, and the negative energy solutions correspond to antiparticles with positive energy. Dirac predicted their existence with the notion of a "sea" of negative energy states. However his logic used to predict the existence of antiparticles is wrong, so we will not discuss it. There is no Dirac sea. You can read about this in most quantum field theory texts.

Free Space Solutions

If a particle is at rest (or let's rephrase that and say you are observing it in its rest frame), it has no motion and hence no momentum. This means we can disregard the spatial derivatives in the Dirac equation. Let's use this case to develop the free

space solution for a particle at rest. Once you have that, you can always do a Lorentz boost to find the solution for a particle with arbitrary momentum.

In the case where the particle is at rest, the Dirac equation reduces to

$$i\gamma^0 \frac{\partial \psi}{\partial t} - m\psi = 0 \tag{5.37}$$

Once again we take $\psi = \binom{u}{v}$ where u and v each have two components and we work with the Dirac-Pauli representation and take

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

The first term works out to be

$$\begin{aligned} i\gamma^0 \frac{\partial \psi}{\partial t} &= i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} \\ &= i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \\ &= i\begin{pmatrix} \dot{u} \\ -\dot{v} \end{pmatrix} \end{aligned}$$

where $\dot{u} = \frac{\partial u}{\partial t}$ and similarly for v. The minus sign that appears here will once again lead to negative energies. The other term in the Dirac equation is

$$m\psi = m\begin{pmatrix} u \\ v \end{pmatrix}$$

So we have the system

$$\begin{aligned} 0 &= i\gamma^0 \frac{\partial \psi}{\partial t} - m\psi \\ &= i\begin{pmatrix} \dot{u} \\ -\dot{v} \end{pmatrix} - m\begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

This leads to the two elementary differential equations

$$i\frac{\partial u}{\partial t} = mu$$

$$i\frac{\partial v}{\partial t} = -mv$$

with the solutions

$$u(t) = u(0)e^{-imt}$$
$$v(t) = v(0)e^{imt}$$

Thinking back to nonrelativistic quantum mechanics, a free space solution has a time dependence of the form e^{-iEt}. So we make a comparison in each case to determine the energy, which in this case is the rest energy. For u we have the correspondence

$$e^{-imt} \sim e^{-iEt}$$

So for u we have the pleasing relationship that $E = m$. Since we're working in units where $c = 1$, this is just the statement that the rest-mass energy is $E = mc^2$. Making the same comparison for v, we see that

$$e^{imt} \sim e^{-iEt}$$

This time we have $E = -m$. Yet again negative-energy states have reared their ugly head. Therefore we conclude that u is a two-component spinor representing a *particle*, while v is a two-component spinor representing an *antiparticle.* These are two-component objects because this equation describes spin-1/2 particles. Recall that the column vector representation of a spin-1/2 particle in quantum mechanics is

$$\phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where α and β are the probability amplitudes to find the particle spin-up or spin-down, respectively.

Summarizing, we've learned that using the two component spinors u and v, the Dirac spinor can be thought of as

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \text{particle} \\ \text{antiparticle} \end{pmatrix} \tag{5.38}$$

Now let's consider a particle moving with arbitrary momentum p. We will use the scalar product

$$p \cdot x = Et - \vec{p} \cdot \vec{x} \tag{5.39}$$

Free space solutions are going to be plane waves, and we can immediately use the rest frame solutions to guess at a solution for a particle with momentum p. We use Eq. (5.39) and take

$$u \propto e^{-ip\cdot x}$$
$$v \propto e^{ip\cdot x}$$

Notice that if $u \propto e^{-ip\cdot x}$, then

$$\begin{aligned} i\gamma^\mu \partial_\mu u &= i\gamma^\mu \partial_\mu e^{-ip\cdot x} \\ &= \gamma^\mu p_\mu e^{-ip\cdot x} \\ &= \not{p}\psi \end{aligned}$$

For $v \propto e^{ip\cdot x}$ we find

$$\begin{aligned} i\gamma^\mu \partial_\mu v &= i\gamma^\mu \partial_\mu e^{ip\cdot x} \\ &= -\gamma^\mu p_\mu e^{ip\cdot x} \\ &= -\not{p}\psi \end{aligned}$$

The Dirac equation, as you should have ingrained in your mind by now, is $i\gamma^\mu \partial_\mu \psi - m\psi = 0$. Using the above results we have two algebraic relations for u and v.

$$(\not{p} - m)u = 0 \tag{5.40}$$
$$(\not{p} + m)v = 0 \tag{5.41}$$

Notice that

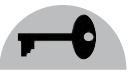
$$\begin{aligned} (\not{p} - m)(\not{p} + m) &= (\gamma^\mu p_\mu - m)(\gamma^\nu p_\nu + m) \\ &= \gamma^\mu \gamma^\nu p_\mu p_\nu + \gamma^\nu \gamma^\mu p_\mu p_\nu + m\gamma^\mu p_\mu - m\gamma^\nu p_\nu - m^2 \end{aligned}$$

When an index in an expression is repeated, it is a dummy index so we can relabel it. This allows us to get rid of two of the terms in this expression, because we have

$$m\gamma^\mu p_\mu - m\gamma^\nu p_\nu = m\gamma^\mu p_\mu - m\gamma^\mu p_\mu = 0$$

This leaves

$$\begin{aligned} (\not{p} - m)(\not{p} + m) &= \gamma^\mu \gamma^\nu p_\mu p_\nu + \gamma^\nu \gamma^\mu p_\mu p_\nu - m^2 \\ &= (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) p_\mu p_\nu - m^2 \end{aligned}$$



Using a now familiar friend, the anticommutation relation for the gamma matrices Eq. (5.8), this simplifies to

$$\begin{aligned}(p-m)(p+m) &= (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)p_\mu p_\nu - m^2 \\ &= p_\mu p^\mu - m^2 \\ &= p^2 - m^2 \\ &= E^2 - \vec{p}^2 - m^2 = m^2 - m^2 = 0\end{aligned}$$

We take the solutions to be of the form

$$u(p) = (\not{p} + m)u(0) \tag{5.42}$$

$$v(p) = (\not{p} - m)v(0) \tag{5.43}$$

Notice that

$$\begin{aligned}(p-m)u(p) &= (p-m)(p+m)u(0) \\ &= (p^2 - m^2)u(0) \\ &= 0\end{aligned}$$

The Dirac equation, written as Eq. (5.40), is satisfied. The initial states are given by ordinary spin-up and spin-down states, that we are free to choose. For example, we can take

$$u(0) = \begin{pmatrix}1\\0\end{pmatrix} \qquad v(0) = \begin{pmatrix}0\\1\end{pmatrix}$$

Boosts, Rotations, and Helicity

We have already written down the Dirac matrices and their defining anticommutation relation as

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Now we need to figure out how to use them to generate boosts and rotations for the spin-1/2 case. The so-called Lorentz algebra requires that we seek operators $J^{\mu\nu}$ that satisfy

$$[J^{\mu\nu}, J^{\alpha\beta}] = i(g^{\nu\alpha}J^{\mu\beta} - g^{\mu\alpha}J^{\nu\beta} - g^{\nu\beta}J^{\mu\alpha} + g^{\mu\beta}J^{\nu\alpha})$$

This will work if we define the tensor

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{5.44}$$

For a spin-1/2 particle, the generator of a Lorentz boost in the *j*th direction is

$$S^{0j} = \frac{i}{4}[\gamma^0, \gamma^j] \tag{5.45}$$

The generator of a rotation for a spin-1/2 particle is

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{5.46}$$

Now let

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{5.47}$$

so that, for example

$$\Sigma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

This can be used to define the *helicity operator*. Helicity tells us how closely aligned the spin of a particle is with its direction of motion. The direction of a particle's motion is given by its spatial momentum vector $\vec{p}$, so we can write the helicity operator as

$$h = \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|} \tag{5.48}$$

Looking at Eq. (5.47) you can see that a simple way to write the helicity operator is $\vec{\sigma} \cdot \vec{p}$.

Weyl Spinors

Let's return to the chiral representation, which is useful when considering a special type of spinor known as a *Weyl spinor*. In the chiral representation, the Dirac matrices are represented by

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

In the chiral representation, we have

$$\begin{aligned}\not{p} &= \gamma^\mu p_\mu = \gamma^0 p_0 - \gamma^1 p_1 - \gamma^2 p_2 - \gamma^3 p_3 \\ &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} p_0 - \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} p_1 - \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} p_2 - \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} p_3 \\ &= \begin{pmatrix} 0 & E - \vec{p}\cdot\vec{\sigma} \\ E + \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix}\end{aligned}$$

where $p_0 = E$. Considering the simple massless case, we write the Dirac spinor as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{5.49}$$

We call Eq. (5.49) a Weyl spinor. The components ψ_L and ψ_R will be left-handed and right-handed spinors. Like *u* and *v*, these are two component spinors. We also use the term chiral representation because these components are eigenstates of the helicity operator.

When $m = 0$, the Dirac equation reduces to the pleasingly simple form like

$$\not{p}\psi = 0$$

or in matrix form

$$\begin{pmatrix} 0 & E - \vec{p}\cdot\vec{\sigma} \\ E + \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

This produces two equations.

$$\left(E + \vec{p}\cdot\vec{\sigma}\right)\psi_L = 0 \tag{5.50}$$

$$\left(E - \vec{p}\cdot\vec{\sigma}\right)\psi_R = 0 \tag{5.51}$$

Let's take a look at the first equation. Using $(\vec{p}\cdot\vec{\sigma})^2 = |\vec{p}|^2$, we have

$$\begin{aligned} 0 &= (E - \vec{p}\cdot\vec{\sigma})(E + \vec{p}\cdot\vec{\sigma})\psi_L \\ &= (E^2 - |\vec{p}|^2)\psi_L \\ \Rightarrow E^2 - |\vec{p}|^2 &= 0,\ E = |\vec{p}| \end{aligned}$$

A similar relation with a sign change can be found for the right-handed spinor. Writing the helicity operator as $\vec{\sigma}\cdot\vec{p}$, the left- and right-handed spinors satisfy the eigenvalue equations.

$$(\vec{\sigma}\cdot\vec{p})\psi_L = -E\psi_L = -|\vec{p}|\psi_L \tag{5.52}$$

$$(\vec{\sigma}\cdot\vec{p})\psi_R = E\psi_R = |\vec{p}|\psi_R \tag{5.53}$$

The Weyl spinors are also eigenstates of γ_5, as we show in Example 5.4.

EXAMPLE 5.4
Consider massless Weyl spinors and show that they are eigenstates of the γ_5 matrix if we take $\psi_L = \frac{1}{2}(I-\gamma_5)\psi$, $\psi_R = \frac{1}{2}(I+\gamma_5)\psi$.

SOLUTION
In the chiral representation we have

$$\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

Applying this to the Weyl spinor, we get

$$\gamma_5\psi = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -\psi_L \\ \psi_R \end{pmatrix}$$

Now of course

$$I\psi = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Therefore,

$$(I-\gamma_5)\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} - \begin{pmatrix} -\psi_L \\ \psi_R \end{pmatrix} = 2\begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$$

This leads to the relationship

$$\psi_L = \frac{1}{2}(I-\gamma_5)\psi$$

Similarly, we have

$$\psi_R = \frac{1}{2}(I+\gamma_5)\psi$$

Notice that

$$\gamma_5^{\,2} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I$$

So we have

$$\begin{aligned} \gamma_5\psi_L &= \frac{1}{2}(\gamma_5 I - \gamma_5^{\,2})\psi \\ &= \frac{1}{2}(\gamma_5 - I)\psi \\ &= -\frac{1}{2}(I - \gamma_5)\psi \\ &= -\psi_L \end{aligned}$$

This shows that the left-handed Weyl spinor is an eigenstate of γ_5 with eigenvalue −1. Similarly,

$$\begin{aligned} \gamma_5\psi_R &= \frac{1}{2}(\gamma_5 I + \gamma_5^{\,2})\psi \\ &= \frac{1}{2}(\gamma_5 + I)\psi \\ &= \frac{1}{2}(I + \gamma_5)\psi \\ &= \psi_R \end{aligned}$$

That is, ψ_R is an eigenstate of γ_5 with eigenvalue +1.

Summary

In this chapter we have introduced the Dirac equation. This equation was derived by Dirac in an attempt to get a relativistic equation with time and space on the same footing, while avoiding the negative probability densities associated with the Klein-Gordon equation. The equation still has solutions with negative energy, which are the result of the fact that it describes antiparticles, as well as particles.

Quiz

1. Given the Lagrangian

$$\mathcal{L} = \bar{\psi}(x)[i\gamma^{\mu}\partial_{\mu} - m]\psi$$

vary $\psi(x)$ to find the equation of motion obeyed by $\bar{\psi}(x)$.

2. Calculate $\{\gamma_5, \gamma^{\mu}\}$.
3. Consider the solution of the Dirac equation with $E = \omega_k > 0$. Find a relationship between the *u* and *v* components of the Dirac field.
4. Find the normalization of the free space solutions of the Dirac equation using the density $\bar{\psi}\gamma^0\psi$.
5. Find S^{01}, the generator of a boost in the *x* direction.
6. We can introduce an electromagnetic field with a vector potential A_{μ}. Let the source charge be *q*. Using the substitution $p_{\mu} \to p_{\mu} - qA_{\mu}$, determine the form of the Dirac equation in the presence of an electromagnetic field.

CHAPTER 6

Scalar Fields

The first attempts to merge the theory of relativity with quantum mechanics involved what you might think of as relativistic generalizations of the Schrödinger equation that were imagined to apply to a single particle. In fact, Schrödinger himself derived a relativistic equation—one that we first learned in Chap. 2—the Klein-Gordon equation, before coming up with his famous nonrelativistic wave equation. He ended up discarding the Klein-Gordon equation as the correct one for quantum mechanics for three main reasons:

- It appeared to have solutions with negative energy.
- It appeared to lead to a negative probability distribution.
- It gave an incorrect spectrum for the hydrogen atom.

Looking at these factors he ended up discarding what we now know as the Klein-Gordon equation in favor of what is now known as the Schrödinger equation. But as we'll see later, the main problem with the Klein-Gordon equation is a problem in interpretation.

We can begin our path to a relativistic wave equation by thinking about what we learned in Chap. 1. Relativity treats time and space in a similar fashion. In a wave

equation, this implies that the derivatives applied to the time and spatial coordinates must be of the same order. In the nonrelativistic Schrödinger equation, there is a first-order derivative with respect to time but derivatives with respect to the spatial coordinates are of second order. Let's write down the Schrödinger equation in the case of one spatial dimension to remind ourselves of this explicitly.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi \tag{6.1}$$

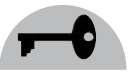
This equation cannot be relativistic since we have a first-order derivative with respect to time $\frac{\partial \psi}{\partial t}$ on the left hand side, while we have a second-order derivative with respect to the spatial coordinate $\frac{\partial^2 \psi}{\partial x^2}$ on the right-hand side. To incorporate special relativity into quantum theory, we would expect symmetry. This situation is rectified in the Klein-Gordon equation where second-order derivatives are applied to both the time and spatial coordinates. In contrast, Dirac, when deriving his famous equation that applies to spin-1/2 particles, insisted that first-order derivatives be applied to both the spatial and time coordinates. We will see later why Dirac decided to "demote" the spatial derivative to first order to get the symmetry in time and space we are looking for when we see how a second-order time derivative causes problems in the Klein-Gordon equation. In this chapter we will discuss the Klein-Gordon equation, which applies to scalar fields.

Arriving at the Klein-Gordon Equation

We begin our examination of relativistic wave equations by returning to an equation we learned briefly in the second chapter, the *Klein-Gordon equation.* In Chap. 2 we saw how it could be derived from a Lagrangian which was given to us, but the ultimate origin of this equation may have seemed mysterious. What we are going to see in a moment is that the Klein-Gordon equation follows from the application of two fundamental principles—one taken from special relativity and the other taken from quantum mechanics. These are

- The relativistic relation between energy, mass, and momentum derived by Einstein
- The promotion of measurable quantities ("observables") to mathematical operators in quantum mechanics

Now let's go forward and see how Schrödinger, Klein, and Gordon (my apologies to any other dead people who derived this equation I am leaving out) derived the equation. The Klein-Gordon equation is very easy to derive in two steps. We start



by writing down the fundamental relation between energy, momentum, and mass used in special relativity.

$$E^2 = p^2c^2 + m^2c^4 \tag{6.2}$$

Now we turn immediately to quantum mechanics. In quantum theory, observables turn into mathematical operators using a specific prescription you are no doubt very familiar with. We can see how this is done looking at the nonrelativistic Schrödinger equation [Eq. (6.1)]. You remember that the time-independent version of this equation is given by

$$E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi \tag{6.3}$$

So it might occur to you that the Schrödinger equation can be thought of as a statement of the nonrelativistic definition of energy. Hence we make the following substitution for energy, promoting it to an operator that takes the derivative with respect to time.

$$E \to i\hbar\frac{\partial}{\partial t} \tag{6.4}$$

We also recall that in ordinary quantum mechanics, momentum p is given by a spatial derivative, that is,

$$p \to -i\hbar\frac{\partial}{\partial x} \tag{6.5}$$

Generalizing to three dimensions, the relation is

$$\vec{p} \to -i\hbar\nabla \tag{6.6}$$

To derive the Klein-Gordon equation, all we do is put the substitutions [Eqs. (6.4) and (6.6)] into the Einstein relation for energy, momentum, and mass Eq. (6.2) and apply it to a wave function φ. Using Eq. (6.4) we see that

$$E^2 \to -\hbar^2\frac{\partial^2}{\partial t^2}$$

Now, using Eq. (6.6) we have

$$p^2 = -\hbar^2 \nabla^2$$

Therefore, in terms of operators, the Einstein relation between energy, momentum, and mass Eq. (6.2) can be written as

$$-\hbar^2 \frac{\partial^2}{\partial t^2} = -\hbar^2 c^2 \nabla^2 + m^2 c^4$$

This isn't going to be of much use or make any sense unless we *do something* with it. So we'll apply this operator to a function of space and time $\varphi = \varphi(\vec{x}, t)$. Doing this and rearranging terms a little gives us the Klein-Gordon equation.

$$\hbar^2 \frac{\partial^2 \varphi}{\partial t^2} - \hbar^2 c^2 \nabla^2 \varphi + m^2 c^4 \varphi = 0 \tag{6.7}$$

As discussed in Chap. 1, in particle physics we typically work in units where $\hbar = c = 1$ (*natural units*) so this becomes

$$\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi + m^2 \varphi = 0 \tag{6.8}$$

We can simplify the appearance of the equation a little further by using different notation. In fact we'll write it in two different ways. The first is to recall the D'Alembertian operator in Minkowski space as

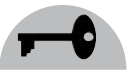
$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2$$

This allows us to write Eq. (6.8) in the following simplified way:

$$(\Box + m^2)\varphi = 0$$

This is a nice way to write the equation for the following reason. We know that $\Box$ is a relativistic invariant, that is, it is the same in all inertial reference frames because it transforms as a scalar. The mass m is of course a scalar so the operator given by

$$\Box + m^2$$



is also a scalar. What this tells us is that the Klein-Gordon equation will be covariant provided that the function φ—which we will interpret later as a field—also transforms as a scalar. In Chap. 1 we learned that the coordinates x^μ transform as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{6.9}$$

under a Lorentz transformation. If a field $\varphi(x)$ is a *scalar* field, then it transforms as

$$\varphi'(x) = \varphi(\Lambda^{-1}x) \tag{6.10}$$

We are led to the first important characteristics of the Klein-Gordon equation.

- It applies to scalar particles (actually scalar fields).
- These particles are spin-0 particles.

We can also write Eq. (6.7) in a nice, compact style using the notation developed in Chap. 1. Using $\partial_\mu\partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$ it becomes

$$(\partial_\mu\partial^\mu + m^2)\varphi = 0 \tag{6.11}$$

As it is written, Eq. (6.11) describes a free particle. The free particle solution is given by

$$\varphi(\vec{x},t) = e^{-ip\cdot x}$$

Remember, we are applying special relativity here so p and x are 4-vectors given by $p = (E,\vec{p})$ and $x = (t,\vec{x})$, respectively. The scalar product in the exponent is

$$p\cdot x = p_\mu x^\mu = Et - \vec{p}\cdot\vec{x} \tag{6.12}$$

The free particle solution implies the relativistic relation between energy, mass, and momentum. This is very easy to show, so let's do it. For simplicity, we consider one spatial dimension only. Since

$$\frac{\partial\varphi}{\partial t} = \frac{\partial}{\partial t}e^{-i(Et-px)} = -iEe^{-i(Et-px)} = -iE\varphi$$

and

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}e^{-i(Et-px)} = ipe^{-i(Et-px)} = ip\varphi$$

Therefore, we have

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = -E^2\varphi + p^2\varphi$$

Hence, applying the full Klein-Gordon equation [Eq. (6.8)] we have

$$(E^2 - p^2)\varphi = m^2\varphi$$

Canceling the wave function and rearranging terms gives $E^2 = p^2 + m^2$, the desired result. Solving for the energy, we take the square root, being careful to include both positive and negative square roots.

$$E = \pm\sqrt{p^2 + m^2} \tag{6.13}$$

This is a dramatic result which is one reason Schrödinger discarded the Klein-Gordon equation. The solution for the energy of the particle tells us that it is possible to have both positive and negative energy states—a nonphysical result. How do we get around this?

Finding negative energy states was the first indication that the *interpretation* of the Klein-Gordon equation as a single particle wave equation is incorrect. It turns out we deal with the negative energy states in the following way. Solutions for particles with negative energy are actually solutions describing *antiparticles,* that is, particles with the same mass but opposite charge, with positive energy.

However as we mentioned in the introduction, there are more problems with the Klein-Gordon equation. The second problem we will see is the fact that the time derivatives are second order leads to the problem of *negative probability densities*, which is nonsense. At least if you are constrained by the ideas of nonrelativistic quantum mechanics. The way around this is that we will decide the equation is not describing a single particle wave function the way the nonrelativistic Schrödinger equation does. Let's see how a negative probability arises from the free particle solution in Example 6.1.

EXAMPLE 6.1

Show that the Klein-Gordon equation leads to a negative probability density in the free particle case. For simplicity, consider one spatial dimension.

SOLUTION

We begin by assuming that the *probability current* assumes the same form as it does in ordinary quantum mechanics. Keeping $\hbar = 1$ we define the probability current as

$$J = -i\varphi^* \frac{\partial\varphi}{\partial x} + i\varphi\frac{\partial\varphi^*}{\partial x} \tag{6.14}$$

Now,

$$\frac{\partial J}{\partial x} = -i\frac{\partial \varphi^*}{\partial x}\frac{\partial \varphi}{\partial x} - i\varphi^*\frac{\partial^2 \varphi}{\partial x^2} + i\frac{\partial \varphi}{\partial x}\frac{\partial \varphi^*}{\partial x} + i\varphi\frac{\partial^2 \varphi^*}{\partial x^2}$$

$$= -i\varphi^*\frac{\partial^2 \varphi}{\partial x^2} + i\varphi\frac{\partial^2 \varphi^*}{\partial x^2}$$

We can transform the derivatives with respect to the spatial coordinate into derivatives with respect to the time coordinate using the Klein-Gordon equation [Eq. (6.8)]. Sticking to one spatial dimension, we rearrange terms a bit and find that

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial t^2} + m^2\varphi \tag{6.15}$$

So we see that

$$\frac{\partial J}{\partial x} = -i\varphi^*\frac{\partial^2 \varphi}{\partial x^2} + i\varphi\frac{\partial^2 \varphi^*}{\partial x^2}$$

$$= -i\varphi^*\left(\frac{\partial^2 \varphi}{\partial t^2} + m^2\varphi\right) + i\varphi\left(\frac{\partial^2 \varphi^*}{\partial t^2} + m^2\varphi^*\right)$$

$$\Rightarrow \frac{\partial J}{\partial x} = -i\left(\varphi^*\frac{\partial^2 \varphi}{\partial t^2} - \varphi\frac{\partial^2 \varphi^*}{\partial t^2}\right)$$

Now we recall another fundamental result from ordinary quantum mechanics. The probability current and probability density ρ satisfy a conservation equation, called the *conservation of probability,* which becomes, when working in one spatial dimension

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0 \tag{6.16}$$

Hence we find that

$$\frac{\partial \rho}{\partial t} = i\left(\varphi^*\frac{\partial^2 \varphi}{\partial t^2} - \varphi\frac{\partial^2 \varphi^*}{\partial t^2}\right)$$

This equation will be satisfied if

$$\rho = i\left(\varphi^*\frac{\partial \varphi}{\partial t} - \varphi\frac{\partial \varphi^*}{\partial t}\right) \tag{6.17}$$

Let's summarize what we have found. The fact that the Klein-Gordon equation includes second-order derivatives with respect to time leads to a probability density Eq. (6.17) that has a very different form than it takes in ordinary quantum mechanics. In fact you probably remember the probability density is defined in terms of the wave function ψ as

$$\rho = |\psi|^2 = \psi^* \psi$$

This is a direct result of the expression we found for the probability current, namely

$$\frac{\partial J}{\partial x} = -i\left(\varphi^* \frac{\partial^2 \varphi}{\partial t^2} - \varphi \frac{\partial^2 \varphi^*}{\partial t^2}\right) \tag{6.18}$$

together with the Klein-Gordon equation. Now let's see what happens when we consider the free particle solution. The presence of the first-order time derivatives in the probability density Eq. (6.17) together with the solutions for the energy Eq. (6.13) leads to problems. Remembering the free particle solution

$$\varphi(\vec{x}, t) = e^{-ip \cdot x} = e^{-i(Et - px)}$$

the time derivatives are

$$\frac{\partial \varphi}{\partial t} = -iEe^{-i(Et-px)} \qquad \frac{\partial \varphi^*}{\partial t} = iEe^{i(Et-px)}$$

We have

$$\varphi^* \frac{\partial \varphi}{\partial t} = e^{i(Et-px)}[-iEe^{-i(Et-px)}] = -iE$$

$$\varphi \frac{\partial \varphi^*}{\partial t} = e^{-i(Et-px)}[iEe^{i(Et-px)}] = iE$$

So the probability density Eq. (6.17) is

$$\rho = i\left(\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}\right) = i(-iE - iE) = 2E$$

Looks good so far—except for those pesky negative energy solutions. Remember that

$$E = \pm\sqrt{p^2 + m^2}$$

In the case of the negative energy solution

$$\rho = 2E = -2\sqrt{p^2 + m^2} < 0$$

which is a negative probability density, something which simply does not make sense. Why doesn't it make sense? A probability of 1 means the particle is in hand. Probability of ½ means you might find it. A probability of 0 means you can't find the particle. A negative probability, say −1, has no consistent interpretation.

Reinterpreting the Field

The solution to the problem of negative probability density with the Klein-Gordon equation involves reinterpreting what the equation represents. Instead of imagining that the equation governs the wave function of a scalar particle, we instead imagine that φ is a field. We promote φ to an operator $\varphi \to \hat{\varphi}$ that includes creation and annihilation operators that create and destroy quanta of the field (the particles), and we will force φ to obey the usual/respected/canonical commutation relations.

Field Quantization of Scalar Fields

We now turn to the task of quantizing a given field $\varphi(x)$. The process of quantization which basically means creating a quantum theory from a classical one is based on imposing commutation relations. *Canonical quantization* refers to the process of imposing the fundamental commutation relation on the position and momentum operators.

$$[\hat{x}, \hat{p}] = i \tag{6.19}$$

In total, the quantization procedure is to

- Promote position and momentum functions to operators
- Impose the commutation relation Eq. (6.19)

We will follow a similar procedure for quantizing a classical field theory. In this case, the procedure is called *second quantization.*

SECOND QUANTIZATION

In quantum field theory, we quantize the fields themselves rather than quantizing dynamical variables like position. Once again we are faced with the problem of having to put space and time on an equal footing. In nonrelativistic quantum mechanics, position and momentum are operators. The position operator acts on a wave function according to

$$\hat{X}\psi(x) = x\psi(x)$$

The momentum operator acts as

$$\hat{p}\,\psi(x) = -i\hbar\frac{\partial\psi}{\partial x}$$

On the other hand, time t is nothing but a parameter in nonrelativistic quantum mechanics. Clearly it is treated differently than position, as there is no operator that acts as

$$\hat{T}\psi(x,t) = t\psi(x,t)$$

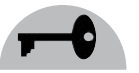
Maybe you could try to construct a theory based on promoting time to such an operator, but that is not what is done in quantum field theory. What happens in quantum field theory is that we actually take the opposite approach, and demote position and momentum from their lofty status as operators. In quantum field theory, time t and position x are just parameters that label a position in spacetime for a field as shown here.

$$\varphi(x,t)$$

To quantize the theory, we are going to take a different approach and treat the fields themselves as operators. The procedure of second quantization is therefore to

- Promote the fields to operators, and
- Impose *equal time* commutation relations on the fields and their conjugate momenta

Since we are quantizing the fields rather than the position and momentum, we call this procedure second quantization—the type of quantization used in ordinary quantum mechanics is first quantization.



This is important so let's summarize. In quantum field theory,

- Position x and momentum p are not operators—they are just numbers like in classical physics.
- The fields $\varphi(x,t)$ and their conjugate momentum fields $\pi(x,t)$ are operators.
- Canonical commutation relations are imposed on the fields.

The fields are operators in the following sense. We have quantum states as we do in quantum mechanics, but these are states of the field. The field operators act on these states to destroy or create particles. This is important because in special relativity,

- Particle number is not fixed. Particles can be created and destroyed.
- To create a particle, we need at least twice the rest-mass energy $E = mc^2$.

The mathematics that describe a quantum theory with changing particle number has its roots in the simple harmonic oscillator, one of the few exactly solvable models. We will briefly review this now.

The Simple Harmonic Oscillator

The Hamiltonian for a simple harmonic oscillator in nonrelativistic quantum mechanics is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \tag{6.20}$$

We define two non-Hermitian operators, which are known as the *annihilation* and *creation* operators, respectively.

$$\hat{a} = \sqrt{\frac{m\omega}{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \tag{6.21}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) \tag{6.22}$$

It is straightforward to show, using $[\hat{x}, \hat{p}] = i$, that

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{6.23}$$

The Hamiltonian can be written in terms of these operators. It is given by

$$\hat{H} = \omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) \tag{6.24}$$

If we define the number operator as

$$\hat{N} = \hat{a}^{\dagger}\hat{a} \tag{6.25}$$

then the Hamiltonian takes on the wonderfully simple form

$$\hat{H} = \omega\left(\hat{N} + \frac{1}{2}\right) \tag{6.26}$$

The eigenstates of the Hamiltonian satisfy

$$\hat{H}|n\rangle = \omega\left(n + \frac{1}{2}\right)|n\rangle \tag{6.27}$$

This tells us that the energy of the state $|n\rangle$ is

$$E_n = \omega\left(n + \frac{1}{2}\right) \tag{6.28}$$

We call the states $|n\rangle$ the *number states.* They are eigenstates of the number operator

$$\hat{N}|n\rangle = n|n\rangle \tag{6.29}$$

The number n is an integer. The number operator obeys the following commutation relations with the annihilation and creation operators.

$$[\hat{N}, \hat{a}] = -\hat{a} \tag{6.30}$$

$$[\hat{N}, \hat{a}^{\dagger}] = \hat{a}^{\dagger} \tag{6.31}$$

The annihilation operator drops n by one unit.

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \tag{6.32}$$

The creation operator increases n by one unit.

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{6.33}$$

There is a lowest lying state, otherwise the system would be able to degenerate into negative energy states. We call the lowest energy state the *ground state* and denote it by $|0\rangle$. In quantum field theory, we will often refer to this state as the *vacuum state.* The vacuum state is annihilated by the annihilation operator, rather it's destroyed by it.

$$\hat{a}|0\rangle = 0 \tag{6.34}$$

Meanwhile, $\hat{a}^\dagger$ raises the energy of the system so that $|n\rangle \to |n+1\rangle$ without limit. The state $|n\rangle$ is obtained from the ground state through repeated applications of $\hat{a}^\dagger$.

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{6.35}$$

These ideas carry over to quantum field theory, but with a different interpretation. In quantum mechanics we are talking about a single particle with state $|n\rangle$ and energy levels $E_n = \omega(n+\frac{1}{2})$. The creation and annihilation operators move the state of the particle up and down in energy from the ground.

In quantum field theory, we take the notion of "number operator" literally. The state $|n\rangle$ is not a state of a single particle, rather it is a state of the field with n particles present. The ground state which is also the lowest energy state is a state of the field with 0 particles (but the field is still there). The creation operator $\hat{a}^\dagger$ adds a single quantum (a particle) to the field, while the annihilation operator $\hat{a}$ destroys a single quantum (removes a single particle) from the field. As we will see, in general there will be creation operators and annihilation operators for particles as well as for antiparticles.

These operators will be functions of momentum. The fields will become operators which will be written as sums over annihilation and creation operators.

SCALAR FIELD QUANTIZATION

The best way to learn about quantizing fields is to consider the simplest case first, the real scalar field that satisfies the Klein-Gordon equation as shown here.

$$\partial_\mu \partial^\mu \varphi + m^2\varphi = \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2\varphi + m^2\varphi = 0$$

We saw that the free field solution of the Klein-Gordon equation is of the form

$$\varphi(x,t) \sim e^{-i(Et-\vec{p}\cdot\vec{x})}$$

Let's write this using the wave number k and let $E \to k_0 = \omega_k, \vec{p} \to \vec{k}$

$$\varphi(x) \sim e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})}$$

where we have also adopted the relativistic notation for position. The reason we are doing this is so that we can write down the general solution of the Klein-Gordon equation in terms of a Fourier expansion.

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\varphi(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \varphi^*(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \right] \tag{6.36}$$

Now we apply step one of the quantization process—we promote the field $\varphi(x)$ to an operator. This is done by replacing the Fourier transforms of the field $\varphi(\vec{k})$ and $\varphi^*(\vec{k})$ by annihilation and creation operators, associated with each mode. That is,

$$\varphi(\vec{k}) \to \hat{a}(\vec{k})$$

$$\varphi^*(\vec{k}) \to \hat{a}^\dagger(\vec{k})$$

Now that the field is an operator, we will add a caret to it and write it as $\hat{\varphi}(x)$ to remind us of that fact. In terms of the creation and annihilation operators, the field is written as

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \right] \tag{6.37}$$

To have a quantum theory, we need to have a conjugate momentum to the field so that we can impose commutation relations. Reminding ourselves of what that is, we repeat the definition starting with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$$

We showed that the conjugate momentum to the field is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} = \partial_0 \varphi$$

Now

$$\partial_0\hat{\varphi}(x) = \partial_0 \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left[\hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right]$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left[\hat{a}(\vec{k})\partial_0(e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})}) + \hat{a}^\dagger(\vec{k})\partial_0(e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})})\right]$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left[\hat{a}(\vec{k})(-i\omega_k)e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger(\vec{k})(+i\omega_k)e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right]$$

$$= -i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}}\left[\hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} - \hat{a}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right]$$

Therefore, the conjugate momentum to the field in Eq. (6.37) is

$$\hat{\pi}(x) = -i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}}\left[\hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} - \hat{a}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right] \tag{6.38}$$

The commutation relations we impose follow from the canonical commutation relations used in ordinary quantum mechanics. For Cartesian coordinates x_i we have

$$[x_i, p_j] = i\delta_{ij}$$

$$[x_i, x_j] = [p_i, p_j] = 0$$

where δ_{ij} is the Kronecker delta function. Going to the continuum, with spatial locations $\vec{x}$ and $\vec{y}$ we let

$$\delta_{ij} \to \delta(\vec{x} - \vec{y})$$

Now we consider commutators between the fields evaluated at *the same time.* We say that the fields obey *equal time commutation relations.* The fields are evaluated at different spatial locations $\vec{x}$ and $\vec{y}$, but $x^0 = y^0$. Then we have

$$[\hat{\varphi}(x), \hat{\pi}(y)] = i\delta(\vec{x} - \vec{y}) \tag{6.39}$$

$$[\hat{\varphi}(x), \hat{\varphi}(y)] = 0 \tag{6.40}$$

$$[\hat{\pi}(x), \hat{\pi}(y)] = 0 \tag{6.41}$$

EXAMPLE 6.2
Suppose that a real scalar field is given by

$$\varphi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(\vec{p}) e^{ipx} + a^\dagger(\vec{p}) e^{-ipx} \right]$$

Compute the equal time commutator

$$[\varphi(x), \pi(y)]$$

where $x^0 = y^0$.

SOLUTION
The momentum is

$$\begin{aligned} \pi(x) &= \frac{\partial \varphi}{\partial x^0} \\ &= \frac{\partial}{\partial x^0} \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(\vec{p}) e^{ipx} + a^\dagger(\vec{p}) e^{-ipx} \right] \\ &= i \int \frac{d^3p}{\sqrt{(2\pi)^3}} \sqrt{\frac{p^0}{2}} \left[a(\vec{p}) e^{ipx} - a^\dagger(\vec{p}) e^{-ipx} \right] \end{aligned}$$

Now the commutator is

$$[\varphi(x), \pi(y)] = \varphi(x)\pi(y) - \pi(y)\varphi(x)$$

where $x^0 = y^0$ (equal time commutation relation). Looking at the first term, we have

$$\varphi(x)\pi(y) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(\vec{p}) e^{ipx} + a^\dagger(\vec{p}) e^{-ipx} \right] i \int \frac{d^3p'}{\sqrt{(2\pi)^3}} \sqrt{\frac{p'^0}{2}} \left[a(\vec{p}') e^{ip'y} - a^\dagger(\vec{p}') e^{-ip'y} \right]$$

$$= \underbrace{i \int \frac{d^3p}{\sqrt{(2\pi)^3}} \frac{d^3p'}{\sqrt{(2\pi)^3}} \frac{1}{2} \sqrt{\frac{p'^0}{p^0}}}_{\text{Phase space factors}} \underbrace{\left[a(\vec{p}) e^{ipx} + a^\dagger(\vec{p}) e^{-ipx} \right]}_{\text{x terms}} \underbrace{\left[a(\vec{p}') e^{ip'y} - a^\dagger(\vec{p}') e^{-ip'y} \right]}_{\text{y terms}}$$

Unfortunately in this case, there is only one way to proceed in order to complete the calculation—and that is to use brute force. Multiplying out term by term we get

$$\begin{aligned}\varphi(x)\pi(y) = i\int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \frac{d^3p'}{\sqrt{(2\pi)^3}} \frac{1}{2}\sqrt{\frac{p'^0}{p^0}} \{a(p)a(p')e^{ipx}e^{ip'y} \\ - a(p)a^\dagger(p')e^{ipx}e^{-ip'y} + a^\dagger(p)a(p')e^{-ipx}e^{ip'y} \\ \times a^\dagger(p)a^\dagger(p')e^{-ipx}e^{-ip'y}\}\end{aligned}$$

Now we compute the other term in the commutator, which turns out to be

$$\begin{aligned}\pi(y)\varphi(x) = i\int \frac{d^3p'}{\sqrt{(2\pi)^3}} \frac{d^3p}{\sqrt{(2\pi)^3}} \frac{1}{2}\sqrt{\frac{p'^0}{p^0}} \{a(p')a(p)e^{ipx}e^{ip'y} \\ - a^\dagger(p')a(p)e^{ipx}e^{-ip'y} + a(p')a^\dagger(p)e^{-ipx}e^{ip'y} \\ \times a^\dagger(p')a^\dagger(p)e^{-ipx}e^{-ip'y}\}\end{aligned}$$

The next step is to take the difference and to collect terms using the creation and annihilation operators. Not surprisingly, they obey similar commutation relations to the creation and annihilation operators used with the simple harmonic oscillator—we simply generalize to the continuous case. The relevant relations are

$$\left[a(\vec{p}), a^\dagger(\vec{p}')\right] = \delta(\vec{p} - \vec{p}') \tag{6.42}$$

$$\left[a(\vec{p}), a(\vec{p}')\right] = 0 \tag{6.43}$$

$$\left[a^\dagger(\vec{p}), a^\dagger(\vec{p}')\right] = 0 \tag{6.44}$$

Taking the difference of the first term we computed for $\varphi(x)\pi(y)$ with the first term we computed for $\pi(y)\varphi(x)$, we get

$$\begin{aligned}a(p)a(p')e^{ipx}e^{ip'y} - a(p')a(p)e^{ipx}e^{ip'y} &= \left[a(p), a(p')\right]e^{ipx}e^{ip'y} \\ &= 0\end{aligned}$$

using Eq. (6.43).

In a similar fashion, if we take the difference of the last terms in each expression, we also get 0 since $\left[a^\dagger(\vec{p}), a^\dagger(\vec{p}')\right] = 0$. Now let's look at the second term in each

expression. Taking the difference of the second term in $\varphi(x)\pi(y)$ and the second term in $\pi(y)\varphi(x)$, we get

$$\begin{aligned}-a(p)a^{\dagger}(p')e^{ipx}e^{-ip'y}+a^{\dagger}(p')a(p)e^{ipx}e^{-ip'y} &= -\left(a(p)a^{\dagger}(p')-a^{\dagger}(p')a(p)\right)e^{ipx}e^{-ip'y}\\ &= -\left[a(p),a^{\dagger}(p')\right]e^{ipx}e^{-ip'y}\\ &= -\delta(\vec{p}-\vec{p}')e^{ipx}e^{-ip'y}\\ &= -\delta(\vec{p}-\vec{p}')e^{i\vec{p}(\vec{x}-\vec{y})}\end{aligned}$$

To get to the last step, we used the fact that

$$\delta(p-p')f(p)=\delta(p-p')f(p')$$

together with the fact that $x^0=y^0$ to get rid of the time component. Now we apply the same procedure to the difference of the third terms in each expression. The result is

$$-\delta(\vec{p}-\vec{p}')e^{i\vec{p}(\vec{x}-\vec{y})}$$

Putting everything together, we obtain

$$\begin{aligned}\left[\varphi(x),\pi(y)\right] &= \varphi(x)\pi(y)-\pi(y)\varphi(x)\\ &= i\int\frac{d^3p}{\sqrt{(2\pi)^3}}\frac{d^3p'}{\sqrt{(2\pi)^3}}\sqrt{\frac{p'^0}{p^0}}\frac{1}{2}\left[-\delta(\vec{p}-\vec{p}')e^{i\vec{p}(\vec{x}-\vec{y})}-\delta(\vec{p}-\vec{p}')e^{-i\vec{p}(\vec{x}-\vec{y})}\right]\\ &= -i\int\frac{d^3p}{\sqrt{(2\pi)^3}}\frac{1}{2}\left[e^{i\vec{p}(\vec{x}-\vec{y})}+e^{-i\vec{p}(\vec{x}-\vec{y})}\right]\end{aligned}$$

But one definition of the Dirac delta function is

$$\delta(\vec{x}-\vec{y})=\int\frac{d^3p}{(2\pi)^3}e^{i(\vec{x}-\vec{y})p} \tag{6.45}$$

and from the symmetry of the delta functions we know $\delta(\vec{x}-\vec{y})=\delta(\vec{y}-\vec{x})$, so

$$\begin{aligned}\left[\varphi(x),\pi(y)\right] = \varphi(x)\pi(y)-\pi(y)\varphi(x) &= -i\int\frac{d^3p}{\sqrt{(2\pi)^3}}\frac{1}{2}\left[e^{i\vec{p}(\vec{x}-\vec{y})}+e^{-i\vec{p}(\vec{x}-\vec{y})}\right]\\ &= -i\frac{1}{2}\left[\delta(\vec{x}-\vec{y})+\delta(\vec{y}-\vec{x})\right]\\ &= -i\delta(\vec{x}-\vec{y})\end{aligned}$$

States in Quantum Field Theory

Now that we know how to write down the scalar field in terms of creation and annihilation operators, we are ready to see how the operators act on the states of the field. We already have some idea of how they act by the analogy with the simple harmonic oscillator. As always let's start with the simplest case, the state of lowest energy or the ground state, which is commonly referred to as the *vacuum (or vacuum state)* in quantum field theory. The vacuum, represented by $|0\rangle$, is destroyed by the annihilation operator.

$$\hat{a}(\vec{k})|0\rangle = 0 \tag{6.46}$$

Now notice that the creation and annihilation operators entered the field via the Fourier expansion. Therefore, we have been labeling them by the momentum $\vec{p}$ or wave number $\vec{k}$. States can be denoted by momentum, so we can step up from the vacuum to a state $|\vec{k}\rangle$ with an application of the creation operator.

$$|\vec{k}\rangle = \hat{a}^{\dagger}(\vec{k})|0\rangle \tag{6.47}$$

This describes a one-particle state. We can apply multiple creation operators of different modes $\vec{k}_1, \vec{k}_2, \ldots$ and so on. For example, the two-particle state $|\vec{k}_1, \vec{k}_2\rangle$ is created by

$$|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^{\dagger}(\vec{k}_1)\hat{a}^{\dagger}(\vec{k}_2)|0\rangle$$

By extension, we can create an *n*-particle state using

$$|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\rangle = \hat{a}^{\dagger}(\vec{k}_1)\hat{a}^{\dagger}(\vec{k}_2) \ldots \hat{a}^{\dagger}(\vec{k}_n)|0\rangle \tag{6.48}$$

Each creation operator $\hat{a}^{\dagger}(\vec{k}_i)$ creates a single particle with momentum $\hbar\vec{k}_i$ and energy $\hbar\omega_{k_i}$ (we are restoring the $\hbar$'s for the moment, for clarity) where

$$\omega_{k_i} = \sqrt{\vec{k}_i^{\,2} + m^2}$$

An annihilation operator $\hat{a}(\vec{k}_i)$ destroys a particle with the said momentum and energy.

Positive and Negative Frequency Decomposition

We can decompose the field into two parts, a positive frequency part and a negative frequency part. The positive frequency part consists of annihilation operators and is written as

$$\hat{\varphi}^{+}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \tag{6.49}$$

The negative frequency part of the field is composed of creation operators.

$$\hat{\varphi}^{-}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}^{\dagger}(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \tag{6.50}$$

Therefore, since $\hat{a}(\vec{k})|0\rangle = 0$, the positive frequency part of the field annihilates the vacuum.

$$\hat{\varphi}^{+}(x)|0\rangle = 0 \tag{6.51}$$

And the negative frequency part creates particles.

$$\begin{aligned} \hat{\varphi}^{-}(x)|0\rangle &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \hat{a}^{\dagger}(\vec{k})|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \left|\vec{k}\right\rangle \end{aligned} \tag{6.52}$$

Number Operators

We can construct a number operator from the creation and annihilation operators.

$$\hat{N}(\vec{k}) = \hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k}) \tag{6.53}$$

The eigenvalues of the number operator are called *occupation numbers.* These are integers

$$n(\vec{k}) = 0,1,2,\ldots \tag{6.54}$$

which tell us how many particles there are of momentum $\vec{k}$ for a given state. The state

$$\left|\vec{k}_1,\vec{k}_2,\ldots,\vec{k}_n\right\rangle = \hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_2)\ldots\hat{a}^\dagger(\vec{k}_n)\left|0\right\rangle$$

consists of n particles, with a single particle with momentum $\vec{k}_1$, a single particle with momentum $\vec{k}_2$, a single particle with momentum $\vec{k}_3$, and so on. However, we can have states where there are multiple particles with the same momentum. Suppose that we have two particles with momentum $\vec{k}_1$ and a single particle with momentum $\vec{k}_2$. We can write the state as

$$\left|\vec{k}_1,\vec{k}_1,\vec{k}_2\right\rangle = \frac{\hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_1)}{\sqrt{2}}\hat{a}^\dagger(\vec{k}_2)\left|0\right\rangle$$

We can also write this state as

$$\left|\vec{k}_1,\vec{k}_1,\vec{k}_2\right\rangle = \left|n(\vec{k}_1)n(\vec{k}_2)\right\rangle$$

where $n(\vec{k}_1) = 2$, $n(\vec{k}_2) = 1$. From the vacuum state, we have

$$\left|n(\vec{k}_1)n(\vec{k}_2)\right\rangle = \frac{\hat{a}^\dagger(\vec{k}_1)^{n(\vec{k}_1)}}{\sqrt{n(k_1)!}}\frac{\hat{a}^\dagger(\vec{k}_2)^{n(\vec{k}_2)}}{\sqrt{n(k_2)!}}\left|0\right\rangle$$

In general,

$$\left|n(\vec{k}_1)n(\vec{k}_2)\ldots n(\vec{k}_m)\right\rangle = \prod_j \frac{\hat{a}^\dagger(\vec{k}_j)^{n(\vec{k}_j)}}{\sqrt{n(k_j)!}}\left|0\right\rangle$$

As it is written, the number operator Eq. (6.53) is actually a density. It tells us the number density of particles in a given state, so to get the total number of particles we have to integrate over all of the states in momentum space. Doing so one obtains

$$\hat{N} = \int d^3k\, \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$$

EXAMPLE 6.3
Find $\hat{N}\left|\vec{k}'\right\rangle$.

SOLUTION
Since

$$\left|\vec{k}'\right\rangle = \hat{a}^{\dagger}(\vec{k}')|0\rangle$$

and

$$\left[\hat{a}(\vec{k}),\hat{a}^{\dagger}(\vec{k}')\right] = \hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k}') - \hat{a}^{\dagger}(\vec{k}')\hat{a}(\vec{k}) = \delta(\vec{k}-\vec{k}')$$

we have

$$\begin{aligned}\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})\left|\vec{k}'\right\rangle &= \hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k}')|0\rangle \\ &= \hat{a}^{\dagger}(\vec{k})\left[\hat{a}^{\dagger}(\vec{k}')\hat{a}(\vec{k}) + \delta(\vec{k}-\vec{k}')\right]|0\rangle \\ &= \hat{a}^{\dagger}(\vec{k})\delta(\vec{k}-\vec{k}')|0\rangle \\ &= \delta(\vec{k}-\vec{k}')\hat{a}^{\dagger}(\vec{k}')|0\rangle = \delta(\vec{k}-\vec{k}')\left|\vec{k}'\right\rangle\end{aligned}$$

To get from the second to the third line, remember that $\hat{a}|0\rangle = 0$. So we find that

$$\begin{aligned}\hat{N}\left|\vec{k}'\right\rangle &= \int d^3k\,\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})\left|\vec{k}'\right\rangle \\ &= \left\{\int d^3k\,\delta(\vec{k}-\vec{k}')\right\}\left|\vec{k}'\right\rangle \\ &= \left|\vec{k}'\right\rangle\end{aligned}$$

Hence we've found that the single particle state $\left|\vec{k}'\right\rangle$ has $n(\vec{k}') = 1$.

Normalization of the States

An important question that always comes up in quantum theory is the normalization of a given state. How do we tackle it here? First we start with the premise that the vacuum is normalized to unity.

$$\langle 0|0\rangle = 1 \tag{6.55}$$

Then, to compute the normalization of an arbitrary state $|\vec{k}\rangle$, we proceed by using the commutation relation Eq. (6.42). This is shown in the Example 6.4.

EXAMPLE 6.4
Compute the normalization of the state $|\vec{k}\rangle$ by considering the inner product $\langle\vec{k}|\vec{k}'\rangle$.

SOLUTION
We proceed using the fact that $\hat{a}^\dagger(\vec{k})|0\rangle = |\vec{k}\rangle$ and that the adjoint of this expression is $\langle 0| = \langle\vec{k}|\hat{a}(\vec{k})$. Then,

$$\begin{aligned}
\langle\vec{k}|\vec{k}'\rangle &= \langle 0|\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}')|0\rangle \\
&= \langle 0|\hat{a}^\dagger(\vec{k}')\hat{a}(\vec{k}) + \delta(\vec{k}-\vec{k}')|0\rangle \\
&= \langle 0|\hat{a}^\dagger(\vec{k}')\hat{a}(\vec{k})|0\rangle + \langle 0|\delta(\vec{k}-\vec{k}')|0\rangle \\
&= \delta(\vec{k}-\vec{k}')\langle 0|0\rangle \\
&= \delta(\vec{k}-\vec{k}') \\
\Rightarrow \langle\vec{k}|\vec{k}'\rangle &= \delta(\vec{k}-\vec{k}')
\end{aligned}$$

Bose-Einstein Statistics

The theory being developed in this chapter applies to *bosons,* which are indistinguishable particles of integral spin (or spin-0 in this case). To see this, we note that we can interchange the order of creation operators as applied to a state. So

$$|\vec{k}_1,\vec{k}_2\rangle = \hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_2)|0\rangle$$

but

$$\begin{aligned}
|\vec{k}_1,\vec{k}_2\rangle &= \hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_2)|0\rangle \\
&= \hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0\rangle \\
&= |\vec{k}_2,\vec{k}_1\rangle \\
\Rightarrow |\vec{k}_1,\vec{k}_2\rangle &= |\vec{k}_2,\vec{k}_1\rangle
\end{aligned}$$

This tells us we are dealing with a theory that describes bosons. If we had fermions, there would have been a sign change in this calculation.

ENERGY AND MOMENTUM

Next we turn to the question of computing the energy and momentum of the field. Starting with the operator expansion of the field

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \right]$$

and using the number operator $\hat{N} = \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})$, it can be shown that the Hamiltonian operator is

$$\hat{H} = \int d^3k \, \omega_k \left[\hat{N}(\vec{k}) + \frac{1}{2} \right] \tag{6.56}$$

The momentum in the field is

$$\hat{P} = \int d^3k \, \vec{k} \left[\hat{N}(\vec{k}) + \frac{1}{2} \right] \tag{6.57}$$

EXAMPLE 6.5
For the real scalar field, find the energy of the vacuum.

SOLUTION
The solution to this example is the famous infinite energy of the vacuum, which may or may not be a problem depending on your point of view. To find the energy of the vacuum, we need to compute

$$\langle 0 | \hat{H} | 0 \rangle \tag{6.58}$$

We have

$$\begin{aligned} \langle 0 | \hat{H} | 0 \rangle &= \langle 0 | \int d^3k \, \omega_k \left(\hat{N}(\vec{k}) + \frac{1}{2} \right) | 0 \rangle \\ &= \langle 0 | \int d^3k \, \omega_k \left(\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \frac{1}{2} \right) | 0 \rangle \end{aligned}$$

$$= \langle 0| \int d^3k\, \omega_k \left(\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k})\right)|0\rangle + \langle 0| \int d^3k\, \omega_k \left(\frac{1}{2}\right)|0\rangle$$

$$= \frac{\omega_k}{2} \int d^3k \langle 0|0\rangle$$

$$= \frac{\omega_k}{2} \int d^3k$$

This solution is reminiscent of the energy of the harmonic oscillator in ordinary quantum mechanics. In that case the energy of the ground state is $\frac{1}{2}\hbar\omega$. We have found a similar term here, but the integral blows up, since we're integrating over all momentum space as shown here.

$$\int d^3k \to \infty$$

This result can be ignored, or swept under the rug depending on your point of view. The usual explanation is that we only measure energy *differences*, so energy is measured relative to the ground state and this term falls out. The end result is we just throw it in the trash and say that the energy is 0. We simply subtract the infinity and say we are "renormalizing" the theory. This trick works but you have to think about the fact that we have to resort to a mathematical sleight of hand to make the theory work—perhaps it's an indicator that things are not quite right.

The *renormalized Hamiltonian* is constructed by subtracting off the term that gives rise to the infinite energy. Thus,

$$\hat{H}_R = \hat{H} - \int d^3k$$

$$= \int d^3k\, \omega_k \hat{N}(\vec{k}) = \int d^3k\, \omega_k \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) \tag{6.59}$$

EXAMPLE 6.6
Find the energy of the state $|\vec{k}\rangle$ using the renormalized Hamiltonian.

SOLUTION
We have

$$\langle \vec{k}|\hat{H}_R|\vec{k}\rangle = \langle \vec{k}| \int d^3k'\omega_{k'}\, \hat{a}^\dagger(\vec{k}')\hat{a}(\vec{k}')|\vec{k}\rangle$$

$$= \langle \vec{k}| \int d^3k'\omega_{k'}\, \delta(\vec{k}-\vec{k}')|\vec{k}'\rangle$$

$$= \langle \vec{k}|\omega_k|\vec{k}\rangle$$

$$= \omega_k$$

Normal and Time-Ordered Products

In quantum field theory we often find it desirable to write expressions in a way such that all creation operators are to *the left* of all annihilation operators. When an expression is written in this way, we say that we are using *normal ordering*. When normal ordering is applied to an expression, we denote this by enclosing it in two colons, so the normal ordering of ψ is denoted by writing $:\psi:$. Since normal ordering means move all creation operators to the left of all annihilation operators, then

$$:\hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k}):=\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k}) \tag{6.60}$$

The normal ordering of a scalar field can be written down using the positive and negative frequency parts. Recall that

$$\hat{\varphi}^{+}(x)=\int\frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\hat{a}(\vec{k})e^{-i(\omega_k x^0-\vec{k}\cdot\vec{x})}$$

while

$$\hat{\varphi}^{-}(x)=\int\frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\hat{a}^{\dagger}(\vec{k})e^{i(\omega_k x^0-\vec{k}\cdot\vec{x})}$$

Normal ordering puts creation operators to the left, so we expect the normal ordered field to have negative frequency parts to the left of positive frequency components. Explicitly

$$:\varphi(x)\varphi(y):=\varphi^{+}(x)\varphi^{+}(y)+\varphi^{-}(x)\varphi^{+}(y)+\varphi^{-}(y)\varphi^{+}(x)+\varphi^{-}(x)\varphi^{-}(y)$$

A *time-ordered product* is a mathematical representation of the physical fact that a particle has to be created before it gets destroyed. Time ordering is accomplished using the time-ordering operator which acts on the product $\varphi(t_1)\psi(t_2)$ as

$$T[\varphi(t_1)\psi(t_2)]=\begin{cases}\varphi(t_1)\psi(t_2) & \text{if } t_1>t_2\\ \psi(t_2)\varphi(t_1) & \text{if } t_2>t_1\end{cases} \tag{6.61}$$

Remember that the fields are operators. Operators act in a right to left order. So a product of operators $\hat{A}\hat{B}$ acts on a state $|\psi\rangle$ in such a way that $\hat{B}$ acts on the state first, and then $\hat{A}$ acts on the result. Therefore if $t_1>t_2$ which means that t_1 is later in time, $\psi(t_2)$ acts on the state first, followed by the action of $\varphi(t_1)$. The order is reversed if $t_2>t_1$.

The Complex Scalar Field

Now let's quantize the complex scalar field. This is a good step forward because the complex scalar field represents particles with charge q and antiparticles with charge $-q$, so we will be able to tackle a relatively simple case and see how antiparticles can be represented in quantum field theory.

When we are dealing with antiparticles, the field is expanded in terms of positive frequency modes (annihilation operators) for particles and negative frequency modes (creation operators) for antiparticles. It is common to use the creation and annihilation operators $\hat{a}^\dagger$, $\hat{a}$ for particles

$$\hat{a}^\dagger(\vec{k}) \qquad \hat{a}(\vec{k}) \quad \text{(particles)}$$

We use $\hat{b}^\dagger, \hat{b}$ to represent the creation and annihilation operators for antiparticles.

$$\hat{b}^\dagger(\vec{k}) \qquad \hat{b}(\vec{k}) \quad \text{(antiparticles)}$$

Hence $\hat{a}^\dagger(\vec{k})$ creates a *particle* of momentum $\hbar k$ and energy $\hbar\omega_k$, while $\hat{b}^\dagger(\vec{k})$ creates an *antiparticle* of momentum $\hbar k$ and energy $\hbar\omega_k$. To write the field operator, we sum up positive frequency parts for particles together with negative frequency parts for antiparticles to get

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{b}^\dagger(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \tag{6.62}$$

There is an adjoint field (not surprising since it's a *complex* field) given by

$$\hat{\varphi}^\dagger(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \hat{a}^\dagger(\vec{k}) e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{b}(\vec{k}) e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \tag{6.63}$$

We still require that $\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')\right] = \delta(\vec{k} - \vec{k}')$, and similarly for the creation and annihilation operators for antiparticles

$$\left[\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k}')\right] = \delta(\vec{k} - \vec{k}') \tag{6.64}$$

There are two conjugate momenta corresponding to the field and its adjoint. For example,

$$\begin{aligned}\hat{\pi}(x) &= \partial_0 \hat{\varphi}(x) \\ &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}(-i\omega_k)\hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + (i\omega_k)\hat{b}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})} \\ &= -i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}}\left[\hat{a}(\vec{k})e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \hat{b}^\dagger(\vec{k})e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\right] \end{aligned} \tag{6.65}$$

In the case of the charged complex field, we have two number operators. The first is the familiar number operator that corresponds to the number of particles.

$$\hat{N}_{\hat{a}} = \int d^3k\, \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) \tag{6.66}$$

The second is the number operator that represents the number of antiparticles.

$$\hat{N}_{\hat{b}} = \int d^3k\, \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}) \tag{6.67}$$

The total energy in the field is expressed as the energy of the particles added to the energy of the antiparticles.

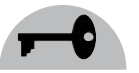
$$\hat{H} = \int d^3k\, \omega_k \left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right] \tag{6.68}$$

Notice that the energy density is the number density of particles added to the number density of antiparticles multiplied by the energy ω_k. Then, to get the total energy we integrate over all modes of the field. Next, the total momentum is the momentum due to particles added to the momentum due to antiparticles.

$$\hat{P} = \int d^3k\, \vec{k}\left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right] \tag{6.69}$$

A complex field corresponds to a charged field. Particles and antiparticles have opposite charge. The total charge is found by subtracting the charge due to antiparticles from the charge due to particles. The charge operator is

$$\begin{aligned}\hat{Q} &= \int d^3k \left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right] \\ &= \hat{N}_{\hat{a}} - \hat{N}_{\hat{b}} \end{aligned} \tag{6.70}$$



Finally, the fields and the conjugate momenta satisfy a series of commutation relations. Once again we consider equal time commutation relations such that $x^0 = y^0$. Then,

$$\left[\hat{\varphi}(x), \hat{\pi}(y)\right] = \left[\hat{\varphi}^\dagger(x), \hat{\pi}^\dagger(y)\right] = i\delta(\vec{x} - \vec{y}) \tag{6.71}$$

All the equal time commutators vanish, leaving the following, which is *not* an equal time commutator.

$$\left[\hat{\varphi}(x), \hat{\varphi}^\dagger(y)\right] = i\Delta(x - y) \tag{6.72}$$

At equal times,

$$\left[\hat{\varphi}(x), \hat{\varphi}^\dagger(y)\right] = \left[\hat{\varphi}(x), \hat{\varphi}(y)\right] = \left[\hat{\varphi}^\dagger(x), \hat{\varphi}^\dagger(y)\right] = 0$$

The commutator $\left[\hat{\varphi}(x), \hat{\varphi}^\dagger(y)\right] = i\Delta(x - y)\partial$ represents a new function called the *propagator,* which we will explore in detail in the next chapter.

Summary

The Klein-Gordon equation results from a straightforward substitution of the quantum mechanical operators for energy and momentum into the Einstein relation for energy, momentum, and mass from special relativity. This leads to inconsistencies such as negative probabilities and negative energy states. We can get around the inconsistencies by reinterpreting the equation. Rather than viewing it as a single particle wave equation, we instead apply it to a field that includes creation and annihilation operators similar to the harmonic oscillator of quantum mechanics. There is one difference, however, in that the creation and annihilation operators now create and destroy particles, rather than changing the energy level of an individual particle.

Quiz

1. Compute $\left[\hat{N}(\vec{k}), \hat{N}^\dagger(\vec{k}')\right]$ for the real scalar field.
2. Find $\hat{N}(\vec{k})\hat{a}^\dagger(\vec{k})\left|n(\vec{k})\right\rangle$.

3. Find $\hat{N}|0\rangle$.
4. Consider the complex scalar field. Determine if charge is conserved by examining the Heisenberg equation of motion for the charge operator $\hat{Q}$.

$$\dot{Q} = [H, Q]$$

Do this computation by writing out the operators using Eqs. (6.68) and (6.70) and using the commutation relations Eq. (6.64).

CHAPTER 7

The Feynman Rules

The tricky mathematics of quantum field theory have been distilled into a series of operations called the Feynman rules. The rules can be thought of as prescriptions to describe all manner of processes in quantum field theory and are embodied in a pictorial form, the famous *Feynman diagrams*. In this chapter we will develop the Feynman rules and show how to construct Feynman diagrams. The ultimate goal is to compute physical parameters for various particle interactions. We discuss these here.

In quantum theory we make experimental predictions by calculating the probability amplitude that a process will occur. This remains true in quantum field theory, where we calculate amplitudes for particle interactions such as decays and scattering events. The primary tool used to do such calculations is known as the *S matrix*. Any given physical process can be considered as a transition from an initial state $|i\rangle = |\alpha(t_0)\rangle$ to a final output state we denote by $|f\rangle = |\alpha(t)\rangle$, that is,

$$|i\rangle \rightarrow |f\rangle$$

This transition occurs via the action of a unitary operator, the S matrix, where S stands for scattering, in the following way.

$$|f\rangle = S|i\rangle$$

Since the S matrix is unitary, it satisfies

$$S^{\dagger}S = SS^{\dagger} = I$$

From ordinary quantum mechanics, we know that the time evolution of states is described using the unitary time evolution operator $U(t,t_0)$. Then the amplitude to evolve from $|\alpha_I(t_0)\rangle$ at time t_0 to a final state $|\alpha_F(t)\rangle$ at a later time t is

$$\langle\alpha_F(t)|U(t,t_0)|\alpha_I(t_0)\rangle \tag{7.1}$$

The initial and final states involve *free* particles that come in from $t = -\infty$, interact, and then move off as different free particle states at $t = +\infty$. An element of the S matrix is the limit of Eq. (7.1) where

$$S_{FI} = \lim_{\substack{t_0\to-\infty \\ t\to+\infty}} \langle\alpha_F(t)|U(t,t_0)|\alpha_I(t_0)\rangle \tag{7.2}$$

In momentum space, the S matrix is proportional to the amplitude M_{FI} for a given process to occur, as follows:

$$S_{FI} \propto -i(2\pi)^4\delta^4(p_F - p_I)M_{FI} \tag{7.3}$$

where p_F is the total four momentum of the outgoing states and similarly for the incoming momenta. The Dirac delta function enforces the conservation of momentum in the process. The *Feynman rules* allow us to calculate the amplitude M_{FI} for the process rather easily, using a graphical representation known as a Feynman diagram of each physical process that can occur. The amplitude M_{FI} is calculated using a perturbative process. Imagine taking the probability amplitude for a process, and expanding it in a series. There is one Feynman diagram for each term in the perturbative expansion, and we add them up to get the total amplitude. Suppose that M is the amplitude for a given event. The same initial and final particle states might result from a set of processes, each with amplitude M_i. The total amplitude is

$$M_{\text{total}} = \sum_{i=0}^{n} g^{k_i} M_i \tag{7.4}$$

where the g^{k_i} are coupling constants for each M_i. There will be a Feynman diagram for each term M_i in the sum. Each amplitude scales by a coupling constant denoted by g which describes the strength of the interaction, and k describes the order of the interaction. For a first-order process, $k = 1$; second-order $k = 2$ and so on. Higher order terms in Eq. (7.4) will have more factors of g. Therefore if g is small, as i gets larger, that is as we take more terms in the sum Eq. (7.4), the higher order terms will begin to become negligible and we can cut the sum off at some n to get a reasonable estimate of the amplitude, like

$$M = \sum_{i=0}^{n} g^{k_i} M_i$$

For example, in quantum electrodynamics (QED) the coupling constant is proportional to $\alpha = 1/137 \approx 0.0073$ which is a small number. Second-order probabilities are proportional to $\alpha^2 \approx 0.0000053$. So as factors of the coupling constant appear as products, the terms become small enough that we can ignore them in our calculations. This will make more sense as we do explicit derivations of amplitudes. In this chapter we will illustrate the procedure with simple and easy to understand scattering and decay events that are abstract yet instructive. In the next chapter we will begin to study physical processes when we examine QED.

In quantum field theory, it is helpful to examine the evolution of a system using the interaction picture which yields the amplitudes in Eq. (7.4). So we begin by reviewing interactions in quantum mechanics.

The Interaction Picture

In this section we will be considering quantum mechanics in three different pictures. The first two will be the Schrödinger picture, and the second is the interaction picture. In between is Heisenberg picture. To keep the states and operators in the two pictures separate, we will use an S subscript for the Schrödinger picture and an I subscript for the interaction picture.

We move to the interaction picture in quantum theory by splitting the Hamiltonian into two parts, a *free field Hamiltonian* H_0 which is time independent, and a time-dependent interaction Hamiltonian H_I.

$$H = H_0 + H_I \tag{7.5}$$

Let us denote the states and operators in the Schrödinger picture with a subscript S. For example, in the Schrödinger picture a state vector is written as $|\alpha\rangle_S$ and an

operator is denoted by A_S. In this picture, operators are fixed and states evolve in time according to

$$i\frac{\partial}{\partial t}\left|\alpha(t)\right\rangle_S = H\left|\alpha(t)\right\rangle_S \tag{7.6}$$

Here H is the full Hamiltonian. A state vector in the interaction picture $\left|\alpha(t)\right\rangle_I$ is related to the state vector in the Schrödinger picture via the action of the free part of the Hamiltonian.

$$\left|\alpha(t)\right\rangle_I = e^{iH_0 t}\left|\alpha(t)\right\rangle_S \tag{7.7}$$

Now, using the interaction picture, we take an intermediate view between the Schrödinger picture and the Heisenberg picture, which moves the time evolution from the states to the operators. That is, in the interaction picture the operators also evolve in time. An interaction picture operator A_I is related to a Schrödinger picture operator A_S in the following way.

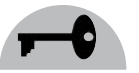
$$A_I = e^{iH_0 t}A_S e^{-iH_0 t} \tag{7.8}$$

Now we differentiate Eq. (7.7) and use Eq. (7.6) to arrive at a dynamical equation for the state as shown here.

$$\begin{aligned}
\frac{\partial}{\partial t}\left|\alpha(t)\right\rangle_I &= \frac{\partial}{\partial t}\left(e^{iH_0 t}\left|\alpha(t)\right\rangle_S\right) \\
&= iH_0 e^{iH_0 t}\left|\alpha(t)\right\rangle_S + e^{iH_0 t}\frac{\partial}{\partial t}\left|\alpha(t)\right\rangle_S \\
&= iH_0 e^{iH_0 t}\left|\alpha(t)\right\rangle_S + e^{iH_0 t}\left(-iH\left|\alpha(t)\right\rangle_S\right) \\
&= iH_0 e^{iH_0 t}\left|\alpha(t)\right\rangle_S + e^{iH_0 t}(-iH_0 - iH_I)\left|\alpha(t)\right\rangle_S \\
&= -ie^{iH_0 t}H_I\left|\alpha(t)\right\rangle_S \\
&= -iH_I\left|\alpha(t)\right\rangle_I
\end{aligned}$$

Therefore, we conclude that the time evolution of the states in the interaction picture is

$$i\frac{\partial}{\partial t}\left|\alpha(t)\right\rangle_I = H_I\left|\alpha(t)\right\rangle_I \tag{7.9}$$



This tells us that the time evolution of the states is determined by the *interaction part of the Hamiltonian.* Now let's take a look at the interaction picture operators and see how they evolve with time. We do this by differentiating Eq. (7.8).

$$\begin{aligned}\frac{\partial}{\partial t}A_I &= \frac{\partial}{\partial t}(e^{iH_0t}A_S e^{-iH_0t}) \\ &= iH_0 e^{iH_0t}A_S e^{-iH_0t} + e^{iH_0t}\left(\frac{\partial A_S}{\partial t}\right)e^{-iH_0t} - ie^{iH_0t}A_S H_0 e^{-iH_0t} \\ &= iH_0 e^{iH_0t}A_S e^{-iH_0t} - ie^{iH_0t}A_S H_0 e^{-iH_0t} \\ &= iH_0 A_I - iA_I H_0 \\ &= i[H_0, A_I]\end{aligned}$$

That is, the time evolution of operators in the interaction picture is determined by *the free part of the Hamiltonian.*

$$\frac{\partial}{\partial t}A_I = i[H_0, A_I] \tag{7.10}$$

Let's summarize how the Hamiltonian affects time evolution in the interaction picture:

H_0 (free)	affects time evolution of operators
H_I (interaction)	affects time evolution of states

In quantum field theory, we have seen that the fields themselves are operators. Equation (7.10) implies that the time evolution of the fields will be characterized by the free field Hamiltonian.

Perturbation Theory

Now we know that the time evolution in quantum mechanics can also be described by a unitary operator $U(t,t_0)$.

$$|\alpha(t)\rangle_I = U(t,t_0)|\alpha(t_0)\rangle_I \tag{7.11}$$

Let's differentiate both sides of this equation. On the left, we have

$$\frac{\partial}{\partial t}|\alpha(t)\rangle_I = -iH_I|\alpha(t)\rangle_I = -iH_I U(t,t_0)|\alpha(t_0)\rangle_I$$

On the right hand side, we obtain

$$\frac{\partial}{\partial t}U(t,t_0)\left|\alpha(t_0)\right\rangle_I \tag{7.12}$$

Hence the time evolution of $U(t,t_0)$ is described by the equation

$$i\frac{\partial U}{\partial t} = H_I U \tag{7.13}$$

Now, if we set $t = t_0$ in Eq. (7.11) expect to see now change in the system. Let's compute the form of the *U* operator in this instant. We have

$$\left|\alpha(t_0)\right\rangle_I = U(t_0,t_0)\left|\alpha(t_0)\right\rangle_I$$

So it must be the case that

$$U(t_0,t_0) = 1$$

The identity says there has been no change in the system in zero time. We take this as the initial condition in Eq. (7.13) and integrate to obtain

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t')U(t',t_0)\,dt' \tag{7.14}$$

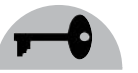
This is an integral equation which describes how the *U* operator changes as we go from time t_0 to time *t*. This immediately suggests an iterative refinement where we improve the computation of the *U* operator in a series of small steps.

As an example, let's start with a rough guess for which we will call U_0. Our first refinement will be U_1 and is given by

$$U_1(t,t_0) = 1 - i\int_{t_0}^{t} H_I(\tau)U_0(\tau,t_0)\,d\tau$$

However, we saw a large change as we went from U_0 to U_1 so we know we need to keep refining our calculation. So we will use the output U_1 to calculate U_2.

$$U_2(t,t_0) = 1 - i\int_{t_0}^{t} H_I(\tau)U_1(\tau,t_0)\,d\tau$$



These refinements produce a rather hideous expression as shown here.

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t')\left(1 - i\int_{t_0}^{t'} H_I(t'')U(t'',t_0)dt''\right)dt'$$

$$= 1 - i\int_{t_0}^{t} dt' H_I(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t')H_I(t'') + \cdots$$

$$+ (-i)^n \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \cdots \int_{t_0}^{t^{(n-1)}} dt^{(n)} H_I(t')H_I(t'') \cdots H_I(t^{(n)})$$

We assume that the times satisfy $t > t' > t'' > \cdots > t^{(n)}$ so we can write the *nth* term as

$$U_n(t,t_0) = (-i)^n \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \cdots \int_{t_0}^{t^{(n-1)}} dt^{(n)} T\left\{H_I(t')\boldsymbol{H}_I(t'') \cdots H_I(t^{(n)})\right\}$$

$$= \frac{(-i)^n}{n!} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \cdots \int_{t_0}^{t} dt^{(n)} T\left\{H_I(t')H_I(t'') \cdots H_I(t^{(n)})\right\}$$

where T is the time-ordering operator. Then we sum up these terms to obtain a *Dyson series*. We can find an approximate solution by cutting off the expansion at a suitable number of terms.

Now, returning to the states, we need to solve

$$i\frac{\partial}{\partial t}\left|\alpha(t)\right\rangle = H_I\left|\alpha(t)\right\rangle$$

where the Hamiltonian H_I is time dependent. Note the form of this equation. If the interaction H_I goes to 0, then the states are constant in time telling us that any *transitions in the state are due to interactions*. To simplify notation let's denote the initial state of the system at time t_0 as

$$\left|\alpha(t_0)\right\rangle = \left|i\right\rangle$$

The initial state of the system $\left|i\right\rangle$ is the state of the system before a scattering event, in other words the state of the system as $t \to -\infty$. This is the noninteracting state of the particles prior to the scattering event. The final state, long after the scattering event, is taken as $\lim_{t\to\infty}\left|\alpha(t)\right\rangle = S\left|i\right\rangle$. We wish to calculate the amplitude for a system in this state to end up in some specific final state $\left|f\right\rangle$

$$\left\langle f\right|S\left|i\right\rangle = S_{fi}$$

which is a component of the S matrix. Therefore the probability $\mathbf{P}_{fi}$ for a system that starts in the state $|i\rangle$ and ends up in the final state $|f\rangle$ after undergoing an interaction described by S is

$$\mathbf{P}_{fi} = \left|\langle f|S|i\rangle\right|^2 = S_{fi}S_{fi}^*$$

$$\left|\langle f|S|i\rangle\right|^2$$

The state of the system at time t can be written in an iterative expansion we used for the Dyson series. With an initial state $|i\rangle$, we have for the first two terms

$$|\alpha(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} H_I(t')|\alpha(t')\rangle dt'$$

Since $S_{FI} = \lim_{\substack{t_0\to-\infty \\ t\to+\infty}} \langle\alpha_F(t)|U(t,t_0)|\alpha_I(t_0)\rangle$, and we can write U in terms of a Dyson series, we can cast the S matrix as a series.

$$S = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \cdots \int_{-\infty}^{t^{(n-1)}} dt^n H_I(t_1)H_I(t_2)\cdots H_I(t_n) \tag{7.15}$$

To summarize, to calculate the amplitude $\langle f|S|i\rangle$ we use perturbation theory and compute terms to a suitable order (to acceptable level of error). In quantum field theory we must describe processes like the creation of matter and antimatter:

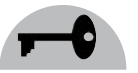
$$_i e^- + e^- \to e^- + e^- + e^+ + e^-$$

You can see how we have a different set of particles in the before and after states (think special relativity). Because the sets of particles change, we use terms that annihilate particles in the initial state and create particles in the final state.

Confused? Who wouldn't be. Luckily Fenyman understood all this stuff well enough to distill it down to a simple recipe. We will now forget everything we've done so far and use the Feynman rules to calculate amplitudes.

Basics of the Feynman Rules

The crux of the perturbative expansion is this: we refine our calculation using corrections that are becoming smaller and smaller. The reduction in importance is quantified by the power of the perturbation parameter, which in this case is the



coupling strength. At some point we are adding refinements that are too small to be measured and we know we can stop adding refinements.

In such a perturbation expansion, as we have seen earlier, the amplitude M for a given process can be computed using an expansion of the type

$$M = \sum_n g^{k_n} M_n$$

Each individual amplitude M_n is a specific particle reaction (scattering, decay) that can be drawn as a Feynman diagram. The higher the order of term M_n, the less likely it is to occur and the less it contributes to the overall amplitude. The terms M_n have the same incoming and outgoing particles, but represent different intermediate states. The intermediate states correspond to terms in the Dyson series. Since each term M_n is scaled by a coupling constant g, which represents the strength of a given interaction, when the relative strength of a given interaction is small then it can be analyzed using perturbation theory.

A Feynman diagram consists of one or more *external* lines that represent the incoming and outgoing particles, connected by a vertex. Time can be taken to flow from the bottom to the top of the diagram, or from the left to the right. For example, imagine a particle decay process involving particles *A, B,* and *C* that proceeds with *A* breaking up into particles *B* and *C*.

$$A \rightarrow B + C$$

If we draw time going from the bottom to the top, we obtain the Feynman diagram shown in Fig. 7.1.

If we draw a diagram such that time is moving from the left to the right, then we obtain the diagram shown in Fig. 7.2.

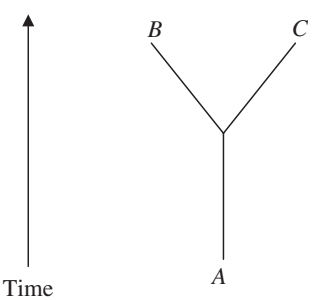


Figure 7.1 A Feynman diagram for the decay process $A \rightarrow B + C$. Time flows from the bottom of the diagram to the top.

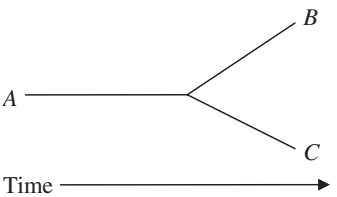


Figure 7.2 A Feynman diagram for the decay $A \rightarrow B + C$, with time flowing from left to right.

Feynman diagrams are a qualitative, symbolic representation of some particle interaction. So don't think of the flow of time as an actual time axis. Normally, the direction of time flow is not explicitly indicated on the diagram, rather it is understood from the context.

Scattering events will involve an intermediate state or particle that is drawn in the diagram as an *internal* line. Suppose that particles A and B scatter with particles C and D leaving the process, and let's suppose that the scattering process involves in intermediary I. The scattering event is

$$A + B \rightarrow C + D$$

It is represented by the Feynman diagram in Fig. 7.3, which includes the internal line with the intermediate state I. The correct way to interpret the intermediate state is that it is a force carrying particle that transmits the given force between the particles A and B. For example, if this were an electromagnetic interaction, say the scattering of an electron and a positron, the internal line would be a photon. The way the reaction is drawn in Fig. 7.3, particles A and

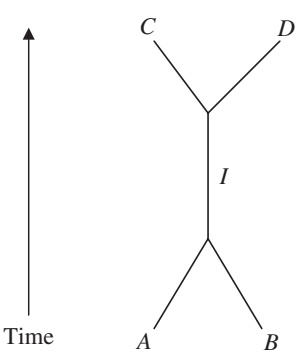


Figure 7.3 A Feynman diagram for the reaction $A + B \rightarrow C + D$.

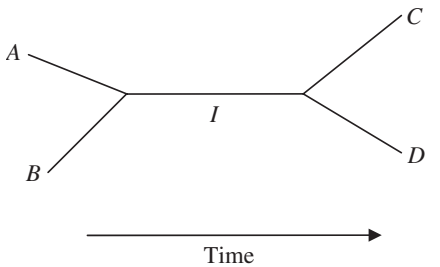


Figure 7.4 The process $A + B \rightarrow C + D$ with time flowing from left to right.

B meet and annihilate, producing the state I, which decays at a later time into the particles C and D.

Now let's draw the same reaction with time flowing from left to right. This is shown in Fig. 7.4.

Particles can scatter via the exchange of a boson (a force-carrying particle). Let's represent the scattering event

$$C + D \rightarrow C + D$$

where particles C and D exchange a boson B during a scattering event. This is shown in Fig. 7.5. Time flows from bottom to the top of the diagram. Particles C and D move in, scatter via the exchange of the boson B, and then move off.

Now we know that in quantum field theory, antiparticles, in addition to particles, take part in many processes. Let us indicate the labels for antiparticles by a tick mark so that A is the particle and A' is the antiparticle. The lines for particles in a Feynman diagram are indicated by an arrow that flows with the direction of time. The lines for antiparticles are indicated in a Feynman diagram

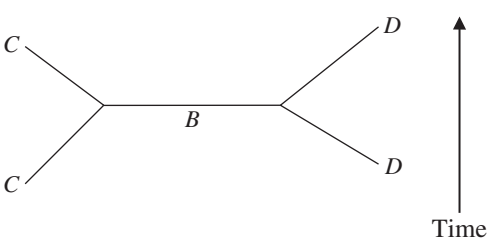


Figure 7.5 A scattering event $C + D \rightarrow C + D$ with the exchange of a boson B.

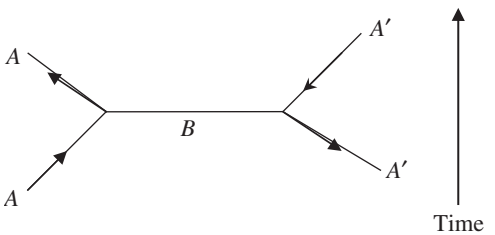


Figure 7.6 The scattering event $A + A' \rightarrow A + A'$, using arrows flowing with time to indicate a particle and arrows flowing against time to indicate an antiparticle.

by an arrow that flows in the opposite direction to the flow of time. Consider a reaction

$$A + A' \rightarrow A + A'$$

where A scatters with the antiparticle A' via the exchange of a boson B. The reaction is drawn in Fig. 7.6.

One of Feynman's brilliant observations was that a particle traveling *forward* in time is equivalent to an antiparticle traveling *backward* in time (refer to Chap. 3). This is why we arrows indicating a particle is traveling backward in time.

Now consider the annihilation reaction, where A and A' meet and annihilate, producing a boson B, which then decays into A and A'. This version of the reaction $A + A' \rightarrow A + A'$ is shown in Fig. 7.7.

Each line in a Feynman diagram is characterized by a four momentum. Suppose once again that $A + A' \rightarrow A + A'$ occurs with the exchange of a boson B as shown

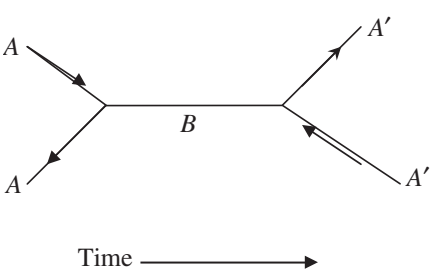


Figure 7.7 Another representation of the reaction $A + A' \rightarrow A + A'$.

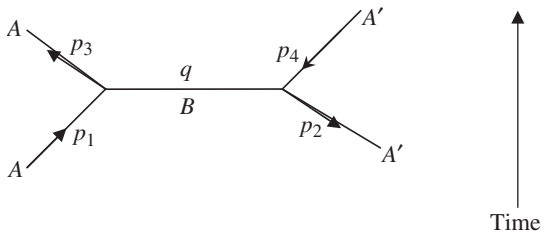


Figure 7.8 A Feynman diagram showing momenta.

in Fig. 7.6. We indicate the momenta of each incoming and outgoing particle (or external line) with a *p*. Internal momenta are indicated with a *q*. This is shown in Fig. 7.8.

Calculating Amplitudes

To actually calculate an amplitude *M*, we integrate over all of the internal momenta. Fortunately, all the integrals involve Dirac delta functions so they can be done by inspection. This is due to the sampling property of the delta function, that is,

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x')\,dx = f(x')$$

As mentioned earlier, the reason the delta functions are in the amplitudes is to enforce conservation of energy and momentum. At each vertex, we assign a positive sign for a momentum entering a vertex and a negative sign for a momentum leaving a vertex. For instance, consider the vertex shown in Fig. 7.9.

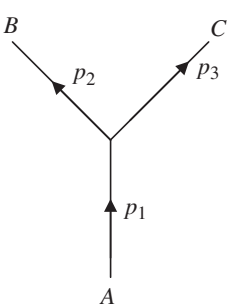


Figure 7.9 A particle decay showing momenta entering and leaving a vertex.

Momentum and energy are conserved and does not change because of the interaction. This conservation is described as $p_1 = p_2 + p_3$, and we include this using the Dirac delta function (recall that $\delta(x_1 - x_2) = 0$ unless $x_1 = x_2$). The delta function that enforces conservation of energy and momentum at the vertex shown in Fig. 7.9 is

$$(2\pi)^4 \delta(p_1 - p_2 - p_3)$$

The value of the delta function is 1 when $p_1 = p_2 + p_3$ and 0 otherwise.

The direction of the arrows indicates whether the given line is for a particle or antiparticle and does not have anything to do with the direction of momentum. If a line goes into a vertex, the momentum is entering the vertex. Following the direction of time (which can be up from the vertex or right from the vertex), lines leaving the vertex should be assigned momenta with minus signs.

Conservation of energy and momentum is enforced at a vertex involving an internal line as well. Consider the vertex shown in Fig. 7.10, where the boson B carries away momentum q.

The delta function that will enforce conservation of energy and momentum at the vertex shown in Fig. 7.10 is

$$(2\pi)^4 \delta(p_1 - p_3 - q)$$

We need to assign a direction to the internal momentum q. If we do so as shown in Fig. 7.11, then the delta function which must be of the form

$$\delta\left(\sum \text{incoming momenta} - \sum \text{outgoing momenta}\right)$$

is written as

$$(2\pi)^4 \delta(p_2 - p_4 + q)$$

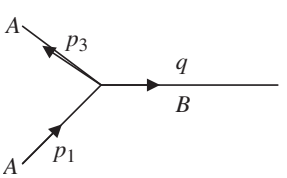


Figure 7.10 Conservation of energy and momentum is also enforced at a vertex with an internal line using a delta function.

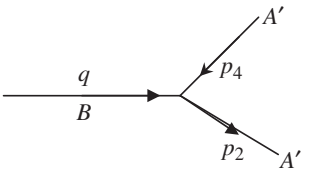


Figure 7.11 Conservation of momentum in this figure is enforced by $(2\pi)^4 \delta(p_2 - p_4 + q)$.

Steps to Construct an Amplitude

Constructing an amplitude from a Feynman diagram involves the following steps:

- Write down a delta function to conserve energy and momentum at each vertex. Multiply these terms together.
- Write down the one coupling constant for each vertex in the figure.
- Write down a propagator for each internal line.
- Multiply all the factors together.
- Integrate over internal momenta.

The total amplitude M is the sum of all the amplitudes M_i that can occur for a specific process with the same incoming and outgoing particles. Each M_i corresponds to a Feynman diagram. So the total amplitude for the process $A + A' \rightarrow A + A'$ is the sum of the two amplitudes represented by Figs. 7.6 and 7.7. In fact, higher-order diagrams for each of those processes can be drawn with different intermediate states. Now let's discuss the two remaining pieces of the process, the coupling constants and propagators.

COUPLING CONSTANTS

Every force has some fundamental strength, and the force manifests itself in the Feynman calculus as a coupling constant g. For quantum electrodynamics, for example, the coupling constant g_e is related to the fine structure constant α as

$$g_e = \sqrt{4\pi\alpha} \tag{7.16}$$

The fine structure constant is a dimensionless number that contains fundamental constants that appear in electromagnetic theory, the electric charge e, the speed of light c, and Planck's constant $\hbar$.

$$\alpha = \frac{e^2}{4\pi\hbar c} \approx 1/137$$

In a good theory like QED, or any interaction where the coupling constant is small (<<1), higher order diagrams contribute less and less because they include higher and higher powers of g. This means we can terminate the series at a point where we have the accuracy we need to describe a given process.

For each vertex in a Feynman diagram, we include one copy of the coupling constant as follows:

$$-ig$$

PROPAGATORS

We associate a *propagator* with each internal line in a Feynman diagram. A propagator is a factor that represents the transfer or propagation of momentum from one particle to another. Right now we will introduce propagators for three types of particles; you will come across others later in your studies.

The simplest case we can consider is an internal line for a spin-0 boson. In this case, the propagator is

$$\frac{i}{q^2 - m^2} \tag{7.17}$$

The mass in this term is the mass of the particle that corresponds to the internal line. In a Feynman diagram, an internal line for a spin-0 boson can be shown as a dashed line, as shown in Fig. 7.12.

For a spin-1/2 particle, we indicate the internal line the same way we would for an external line, as a solid line with an arrow pointing in the direction of momentum for a particle and against the direction of momentum for an antiparticle. In this case the propagator is

$$i\frac{\not{q} + m}{q^2 - m^2} = \frac{i}{\not{q} - m} \tag{7.18}$$

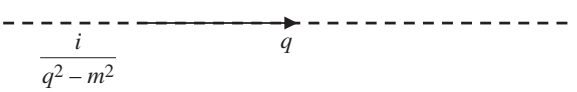


Figure 7.12 An internal line for a spin-0 boson.

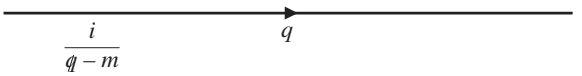


Figure 7.13 An internal line for a Fermion.

where

$$\not q = \gamma^\mu q_\mu$$

An internal line for a Fermion is shown in Fig. 7.13.

The photon propagator is

$$\frac{i}{k^2}\left(-g^{\mu\nu} + (1-\zeta)\frac{k^\mu k^\nu}{k^2}\right) \tag{7.19}$$

In the Feynman gauge $\zeta = 1$, and the photon propagator is just

$$-\frac{i}{k^2} g^{\mu\nu} \tag{7.20}$$

In the examples in this chapter, we will use spin-0 bosons as the force-carrying particles in all of our calculations because they are simpler to deal with.

Now let's review again the steps used to build up the amplitude from a Feynman diagram. We take each factor and multiply them together as a product.

- Write down one factor of $-ig$ for each vertex in the diagram where g is the coupling constant for the interaction depicted.
- Write down a delta function $(2\pi)^4 \delta\left(\sum \text{incoming momenta} - \sum \text{outgoing momenta}\right)$ for each vertex to conserve momentum.
- Add a propagator for each internal line.

The next step is to integrate over all internal momenta. For each internal momentum q we add an integration measure to enforce normalization in phase space:

$$\frac{1}{(2\pi)^4} d^4 q \tag{7.21}$$

Then we integrate for each internal momentum q. In the end, there will be a final delta function left over that enforces the conservation of energy and momentum for

the *external lines*. We simply discard this factor, and the result left over is the amplitude for the given process.

EXAMPLE 7.1

A particle A annihilates with its antiparticle A', producing a spin-0 scalar boson B, which subsequently decays into A and A'. The mass of the scalar boson is m_B. Calculate the amplitude and probability for this process shown in Fig. 7.14.

SOLUTION

Following the rules, we start by writing down a factor of $-ig$ for each vertex. There are two vertices in Fig. 7.14, hence we get two factors of $-ig$.

$$(-ig)(-ig) = -g^2 \tag{7.22}$$

Next, we multiply this by the Dirac delta functions that will conserve 4-momentum at each vertex. For the first vertex at the bottom of Fig. 7.14, we have incoming momenta p_1 and p_2 and outgoing momentum q. This is represented by the delta function

$$(2\pi)^4 \delta(p_1 + p_2 - q)$$

We multiply this by Eq. (7.22), giving

$$-g^2(2\pi)^4 \delta(p_1 + p_2 - q) \tag{7.23}$$

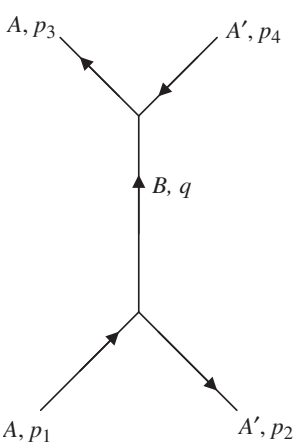
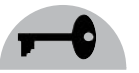


Figure 7.14 The annihilation-creation process with spin-0 boson.



At the top vertex, we have incoming momentum q, and outgoing momenta p_3 and p_4. We indicate this with the delta function

$$(2\pi)^4 \delta(q - p_3 + p_4)$$

Adding this to the product in Eq. (7.23) gives

$$-g^2 (2\pi)^4 \delta(p_1 + p_2 - q)(2\pi)^4 \delta(q - p_3 + p_4) \tag{7.24}$$

The next step is to add the propagator for the internal line. Since this is a spin-0 boson, we use Eq. (7.17), which we multiply by Eq. (7.24) to give

$$\frac{-ig^2}{q^2 - m_B{}^2} (2\pi)^4 \delta(p_1 + p_2 - q)(2\pi)^4 \delta(q - p_3 + p_4)$$

Now we integrate over q, using the measure

$$\frac{1}{(2\pi)^4} d^4 q$$

So we have

$$M = \int \frac{-ig^2}{q^2 - m_B{}^2} (2\pi)^4 \delta(p_1 + p_2 - q) \delta(q - p_3 + p_4)\, dq$$

This integral can be done by inspection. Using the second delta function, we set

$$q = p_3 + p_4$$

Therefore, the amplitude for this process is

$$\begin{aligned} M &= \int \frac{-ig^2}{q^2 - m_B{}^2} (2\pi)^4 \delta(p_1 + p_2 - q) \delta(q - p_3 + p_4)\, dq \\ &= \frac{-ig^2}{(p_3 + p_4)^2 - m_B{}^2} \end{aligned}$$

where we have discarded the remaining delta function which is

$$(2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)$$

which enforces overall conservation of energy and momentum (at the external lines). The probability for the process to occur is the modulus squared of M, that is,

$$|M|^2 = \frac{g^4}{\left((p_3 + p_4)^2 - m_B{}^2\right)^2}$$

EXAMPLE 7.2
A particle decays as follows

$$u \to w + v$$

with a coupling strength g_w. The particle w then decays into

$$w \to v' + e$$

with a strength given by the same coupling constant. Draw the Feynman diagram for this process and calculate the amplitude for it to occur. The w particle is a spin-0 boson with mass m_w.

SOLUTION
The Feynman diagram for this process is shown in Fig. 7.15.

We include a factor of $-ig_w$ for each vertex. There are two vertices shown in Fig. 7.15, so we have

$$(-ig_w)^2 = -g_w^2$$

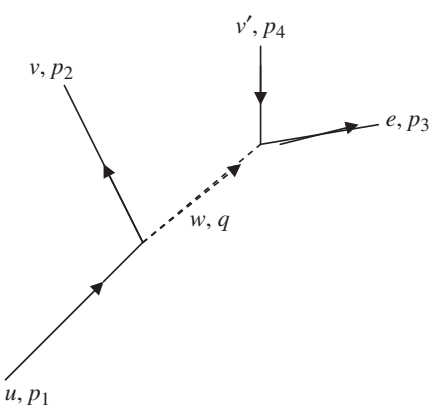


Figure 7.15 The process described in Example 7.2. Note that v' is an antiparticle, so the arrow for its external line is pointing in the opposite direction.

At vertex 1 (at the first vertex), there is incoming momentum p_1 and outgoing momenta p_2 and q. Therefore, we include a delta function giving

$$-g_w^2(2\pi)^4\delta(p_1-p_2-q)$$

The particle w decays into the final products, so it's represented by an internal line. The propagator for this particle is given by

$$\frac{i}{q^2-m_w^2}$$

At the second vertex, we have incoming momentum q and outgoing momenta p_3 and p_4, so we multiply by a delta function

$$(2\pi)^4\delta(q-p_3-p_4)$$

Putting everything together, we get

$$-g_w^2(2\pi)^4\delta(p_1-p_2-q)\left(\frac{i}{q^2-m_w^2}\right)(2\pi)^4\delta(q-p_3-p_4)$$

Now we integrate to obtain

$$\int -g_w^2(2\pi)^4\delta(p_1-p_2-q)\left(\frac{i}{q^2-m_w^2}\right)(2\pi)^4\delta(q-p_3-p_4)\frac{d^4q}{(2\pi)^4}$$

$$=-\frac{ig_w^2(2\pi)^4\delta(p_1-p_2-p_3-p_4)}{(p_3+p_4)^2-m_w^2}$$

We have used $q=p_3+p_4$ from the second delta function in the integral. To get the amplitude for the process to occur, we drop the $(2\pi)^4\delta(p_1-p_2-p_3-p_4)$ term

$$M=-\frac{ig_w^2(2\pi)^4\delta(p_1-p_2-p_3-p_4)}{(p_3+p_4)^2-m_w^2}$$

Rates of Decay and Lifetimes

Decay processes are very important in nuclear and particle physics since many nuclei and particles are unstable—they will eventually decay into something else. In fact, very few particles are fundamental and immune to decay. So decay rates and lifetimes are an essential quantities of interest.

The rate of decay for a process is proportional to the squared amplitude.

$$\Gamma \propto |M|^2 \tag{7.25}$$

The lifetime of a particle that decays is the inverse of the amplitude squared.

$$\tau \propto \frac{1}{|M|^2} \tag{7.26}$$

Summary

Feynman diagrams allow us to represent the amplitude for a process to occur with a picture. External lines represent incoming and outgoing particle states. At each vertex, conservation of energy and momentum is enforced with a delta function, and the strength of the interaction is included with a coupling constant. Internal lines can represent force-carrying particles or particles that spontaneously decay into the end products. Each internal line is accompanied by a propagator that represents transfer of momentum to the final states.

Quiz

1. What is the amplitude for the process shown in Fig. 7.16?

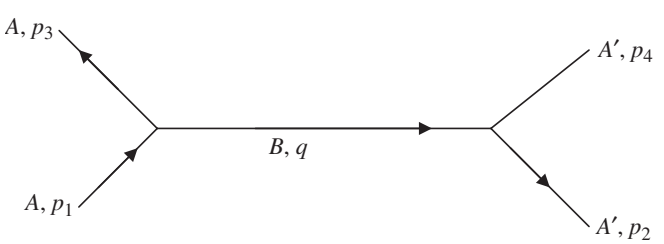


Figure 7.16 Feynman diagram for Question 1.

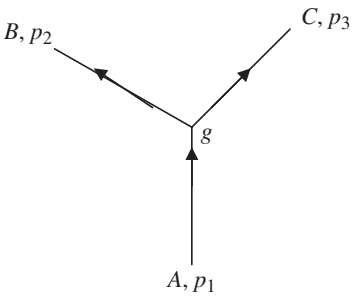


Figure 7.17 Feynman diagram for Question 1.

2. Find the lifetime for the decay as shown in Fig. 7.17.
3. An internal line corresponds to a spin-0 boson of mass m. The propagator is
   (a) $i\dfrac{q+m}{q^2-m^2}$
   (b) $\dfrac{i}{q^2-m^2}$
   (c) $\dfrac{i}{q-m}$
   (d) $\delta(q^2-m^2)$
4. In the interaction picture,
   (a) The time evolution of states is governed by the free Hamiltonian
   (b) States are stationary, operators evolve according to the interaction part of the Lagrangian
   (c) States evolve according to the interaction part of the Hamiltonian, fields evolve according to the free part of the Hamiltonian
   (d) States obey the Heisenberg equation of motion
5. Each vertex in a Feynman diagram requires the addition of
   (a) One factor of the coupling constant $-ig$
   (b) One factor of the coupling constant $-g$
   (c) One factor of the coupling constant $-ig^2$
   (d) One factor of the coupling constant $-i\sqrt{g}$
6. What number is the coupling constant for quantum electrodynamics related to?

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CHAPTER 8

Quantum Electrodynamics

Quantum electrodynamics or *QED* was the first true quantum field theory that was developed. As the name implies, it is a quantum field theory that describes electromagnetic interactions. It is sometimes called the prototype quantum field theory. In a sense, physicists would like all physical interactions described by a theory like quantum electrodynamics.

The development of quantum electrodynamics brought the notion of describing forces with particle exchange to the forefront. In quantum electrodynamics, electromagnetic forces are the result of the exchange of *virtual photons*. We say the photons are virtual because they are not observed directly, rather they are exchanged between two charged particles. The momentum carried by the photons causes a recoil between the two electrons giving rise to a repulsive force. We can illustrate a process like this with a Feynman diagram. The photon, which is the exchanged particle, is represented with a wavy line as shown in Fig. 8.1. A γ can also be used to indicate a photon.

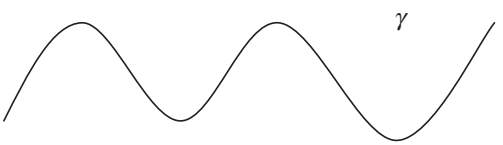


Figure 8.1 A schematic representation of a photon that we will use in a Feynman diagram for QED processes is a wavy line. We can include a γ for clarity if desired.

In Fig. 8.2, we show a basic QED process. This is the repulsion of two electrons mentioned in the introductory paragraph. In the diagram, time is moving in a direction from bottom to top. Two electrons enter and scatter with the exchange of a photon.

Recall that the strength of a given interaction is described by its coupling constant. The coupling constant for QED processes is the *fine structure constant* which is denoted by α. It has a precisely known numerical value given by

$$\alpha = 1/137$$

In terms of fundamental constants, it can be written as

$$\alpha = \frac{e^2}{\hbar c} \tag{8.1}$$

The fact that $\alpha \ll 1$ is very helpful. This means that if we expand some quantity in a series in terms of α, higher order terms will contribute less and less because $\alpha^n \to 0$ as n gets large. This fact makes QED calculations using perturbation theory, and in particular Feynman diagrams possible.

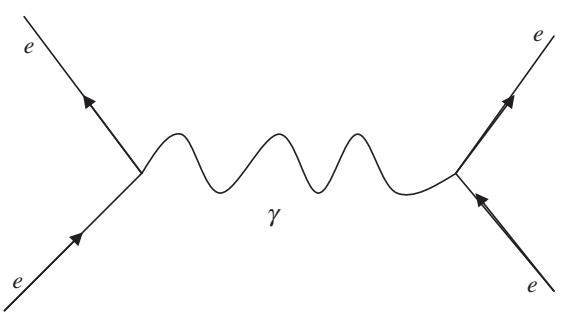
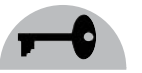


Figure 8.2 A basic QED process: repulsion between electrons. Two electrons enter from the bottom, exchange a photon, then move off.



Reviewing Classical Electrodynamics Again

In an earlier chapter we touched on a relativistic description of the electromagnetic field. We are going to review that here, and incorporate photon polarization into the picture. Once we have done that we can unify the electromagnetic field with the description of the electron using the Dirac equation to develop quantum electrodynamics.

We return to Maxwell's equations

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho \qquad & \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} &= 0 \qquad & \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} &= \vec{J} \end{aligned} \tag{8.2}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, ρ is the charge density, and $\vec{j}$ is the current density. In field theory, we work with the 4-vector potential

$$A^{\mu} = (\phi, \vec{A}) \tag{8.3}$$

which allows us to define the electric and magnetic fields as

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla}\times\vec{A} \tag{8.4}$$

The electromagnetic field tensor is defined as

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} \tag{8.5}$$

which turns out to be the matrix given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{8.6}$$

Using this formulation, Maxwell's equations can be written in the compact form

$$\begin{aligned} \partial^{\alpha}F^{\mu\nu} + \partial^{\beta}F^{\alpha\mu} + \partial^{\mu}F^{\nu\alpha} &= 0 \\ \partial_{\mu}F^{\mu\nu} &= J^{\nu} \end{aligned}$$

where $J^\mu = (\rho, \vec{J})$. The Lagrangian for the electromagnetic field is

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu$$

From this Lagrangian, the field playing the role of canonical momentum to $F^{\mu\nu}(x)$ is

$$\pi^\mu(x) = \frac{\partial L}{\partial(\partial_0 A_\mu)} = -F^{\mu 0}(x) \tag{8.7}$$

The continuity equation expressing conservation of electric charge can be derived from this Lagrangian using the usual techniques and is

$$\partial_\mu J^\mu = 0 \tag{8.8}$$

A gauge transformation can be applied to the 4-vector potential A^μ by adding the derivative of a scalar field χ.

$$A'_\mu = A_\mu + \partial_\mu \chi \tag{8.9}$$

The field equations remain unchanged by such a mathematical transformation. This allows us to pick a form of A^μ that is convenient in some way. For example, we can require that the divergence of the four potential vanishes, something called the *Lorentz condition.*

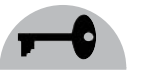
$$\partial_\mu A^\mu = 0 \tag{8.10}$$

This equation allows us to approach the electromagnetic field in a similar way that the Klein-Gordon equation can be dealt with. While the vector potential plays an ancillary role as a mathematical tool in classical electrodynamics, in QED we treat A^μ itself as the photon field. In free space, the electromagnetic field will have plane wave solutions that we write as

$$A^\mu \propto e^{-ip\cdot x} \varepsilon^\mu(p)$$

As usual

$$p \cdot x = Et - \vec{p} \cdot \vec{x}$$



However, the photon is a massless particle with $E = |\vec{p}|$ and hence we have

$$p^\mu p_\mu = 0$$

The quantity $\varepsilon^\mu(p)$ is called the *polarization vector.* This vector plays the role of the spin part of the wave function for the photon. The Lorentz condition $\partial_\mu A^\mu = 0$ results in a constraint on the polarization vector.

$$\begin{aligned} 0 = \partial_\mu A^\mu &= \partial_\mu e^{-ip\cdot x}\varepsilon^\mu(p) \\ &= \partial_\mu e^{-ip_\mu x^\mu}\varepsilon^\mu(p) \\ &= -ip_\mu e^{-ip_\mu x^\mu}\varepsilon^\mu(p) + e^{-ip_\mu x^\mu}\partial_\mu\varepsilon^\mu(p) \\ &= -ip_\mu e^{-ip_\mu x^\mu}\varepsilon^\mu(p) \\ \Rightarrow p_\mu\varepsilon^\mu &= 0 \end{aligned}$$

Starting with the Lorentz gauge condition

$$\partial_\mu A^\mu = 0$$

we substitute the free-space form of the potential A and get the differential equation as shown here.

$$\partial_\mu\left[e^{-ip_\mu x^\mu}\varepsilon^\mu(p)\right] = 0$$

Now we apply the product rule for derivatives

$$\partial_\mu\left[e^{-ip_\mu x^\mu}\varepsilon^\mu(p)\right] = (-ip_\mu)e^{-ip_\mu x^\mu}\varepsilon^\mu(p) + e^{-ip_\mu x^\mu}\partial_\mu\varepsilon^\mu(p)$$

and make use of the fact that $\partial_\mu\varepsilon^\mu(p) = 0$ because $\varepsilon^\mu(p)$ is a function of momentum and does not depend on x^μ. This leads to

$$\partial_\mu\left[e^{-ip_\mu x^\mu}\varepsilon^\mu(p)\right] = (-ip_\mu)e^{-ip_\mu x^\mu}\varepsilon^\mu(p) = 0$$

Since the spacetime position x^μ is completely arbitrary, the exponential $e^{-ip_\mu x^\mu}$ can be nonzero and the only way to ensure the equality is to have

$$p_\mu\varepsilon^\mu(p) = 0$$

The form of the polarization vector is fixed by the choice of reference frame. It is common to take $\vec{p}$ in the z direction.

In the Coulomb gauge, where the gradient of the 3-vector potential is 0, that is,

$$\vec{\nabla} \cdot \vec{A} = 0$$

the polarization vector is perpendicular to the spatial component of momentum

$$\varepsilon \cdot \vec{p} = 0$$

This just says that the polarization is transverse—the polarization vector lies in a plane perpendicular to the direction of motion of the field. A massless spin-s particle has two possible spin states. Let's compute the spin states in the Coulomb gauge for the photon, which has $s = 1$. In this gauge, we take the time component of the polarization vector to be 0, that is,

$$\varepsilon^0 = 0$$

Then the two polarization states for a photon are

$$\varepsilon_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \varepsilon_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

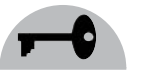
Normalization of the polarization vector is expressed as

$$\varepsilon^\mu \cdot (\varepsilon^\nu)^* = g^{\mu\nu}$$

The Quantized Electromagnetic Field

The process of quantization takes us from classical electrodynamics to field theory. We quantize the electromagnetic field by imposing commutation relations and writing the field in terms of creation and annihilation operators. The canonical equal-time commutation rule is

$$\left[A_\mu(\vec{x},t), \pi^\nu(\vec{y},t) \right] = i g_\mu^\nu \delta^3(\vec{x} - \vec{y}) \tag{8.11}$$



Furthermore we have

$$\left[A_\mu(\vec{x},t),A^\nu(\vec{y},t)\right]=\left[\pi_\mu(\vec{x},t),\pi^\nu(\vec{y},t)\right]=0 \tag{8.12}$$

We can quantize the electromagnetic field rather easily by looking at the classical free-space solution written in Fourier modes. The solution is summed over momentum $\vec{p}$ and polarization $\lambda=1,2$ with complex expansion coefficients $a_{k,\lambda}$ as

$$\vec{A}=\frac{1}{\sqrt{V}}\sum_{\vec{p}}\sum_{\lambda}\frac{\varepsilon_\lambda(\vec{p})}{\sqrt{2\omega_p}}\left[a_{p,\lambda}e^{i(\vec{p}\cdot\vec{r}-\omega t)}+a^*{}_{p,\lambda}e^{-i(\vec{p}\cdot\vec{r}-\omega t)}\right]$$

To quantize the field, we promote the expansion coefficients to creation and annihilation operators as shown here.

$$a_{p,\lambda}\rightarrow\hat{a}_{p,\lambda}$$
$$a^*{}_{p,\lambda}\rightarrow a^\dagger{}_{p,\lambda}$$

The creation operator $a^\dagger{}_{p,\lambda}$ creates a photon of momentum $\vec{p}$ and polarization λ, while the annihilation operator $\hat{a}_{p,\lambda}$ destroys such a photon. These operators obey the usual commutation relations, where we now also take into account polarization so that

$$\left[a_{\lambda,p},a^\dagger{}_{\lambda',p'}\right]=g^{\lambda\lambda'}2p^0(2\pi)^3\delta^3(\vec{p}-\vec{p}')=\delta_{\lambda,\lambda'}2p^0(2\pi)^3\delta^3(\vec{p}-\vec{p}')$$

and the field operator is

$$\begin{aligned}A_\mu&=\int\frac{d^3p}{2p_0(2\pi)^3}\sum_\lambda\left[a_{\lambda,p}\varepsilon_\mu^{(\lambda)}e^{ip_\mu x^\mu}+a^\dagger{}_{\lambda,p}\left(\varepsilon_\mu^{(\lambda)}\right)^*e^{-ip_\mu x^\mu}\right]\\&=A_\mu^+ + A_\mu^-\end{aligned}$$

where

$$A_\mu^+=\int\frac{d^3p}{2p_0(2\pi)^3}\sum_\lambda\left[a_{\lambda,p}\varepsilon_\mu^{(\lambda)}e^{ip_\mu x^\mu}\right]$$
$$A_\mu^-=\int\frac{d^3p}{2p_0(2\pi)^3}\sum_\lambda\left[a^\dagger{}_{\lambda,p}\left(\varepsilon_\mu^{(\lambda)}\right)^*e^{-ip_\mu x^\mu}\right]$$

(where we have dropped the hats from the operators and we have broken up the field into positive and negative frequency components).

Gauge Invariance and QED

Since we are talking about QED, let's review gauge invariance for this theory. The gauge invariance we are going to have to satisfy is local and involves three terms: a Lagrangian term for the electromagnetic field, the Dirac Lagrangian, which will involve two terms, a kinetic energy term and a mass term; and an interaction term that couples the Dirac and electromagnetic fields. The kinetic energy part of the Lagrangian for the electromagnetic field is of the form

$$L_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

From the Dirac equation, we have the Lagrangian

$$L_{\text{Dirac}} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi$$

The Lagrangian for interaction of a particle with charge q and the electromagnetic field is given by

$$L_{\text{int}} = -q\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

We can construct the total Lagrangian describing the electromagnetic field and interactions with a Dirac field like the electron by putting all of these terms together.

$$\begin{aligned} L &= L_{\text{EM}} + L_{\text{Dirac}} + L_{\text{int}} \\ &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi - q\bar{\psi}\gamma^{\mu}\psi A_{\mu} \end{aligned}$$

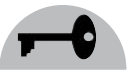
Now, the Dirac portion of the Lagrangian is invariant under a global $U(1)$ symmetry; that is, the Lagrangian does not change when we change the field

$$\psi(x) \to e^{i\theta}\psi(x)$$

which of course implies

$$\bar{\psi}(x) \to e^{-i\theta}\bar{\psi}(x)$$

Recall that for a global symmetry, θ is just a parameter, a complex number—it is not a function of spacetime. This means that $\partial_{\mu} e^{i\theta} = 0$.



It is trivial to see that the mass term in the Dirac part of the Lagrangian is invariant under this transformation.

$$m\bar{\psi}\psi \rightarrow m[e^{-i\theta}\bar{\psi}(x)]\,[e^{i\theta}\psi(x)] = m\bar{\psi}\psi$$

Since the transformation is global, the kinetic energy term of the Dirac Lagrangian is invariant as well since the derivative ∂_μ doesn't affect it.

$$\begin{aligned} i\bar{\psi}\gamma^\mu\partial_\mu\psi \rightarrow ie^{-i\theta}\bar{\psi}(x)\gamma^\mu\partial_\mu[e^{i\theta}\psi(x)] &= ie^{-i\theta}\bar{\psi}(x)\gamma^\mu e^{i\theta}\partial_\mu\psi(x) \\ &= i\bar{\psi}\gamma^\mu\partial_\mu\psi \end{aligned}$$

We have shown that the Dirac Lagrangian is invariant under a global $U(1)$ symmetry. This is all very nice, but what we are interested in for quantum field theory to maintain the spirit of relativity is invariance under a local transformation. Keep that in mind—the standard model of particle physics requires invariance under local transformations. Recall that this requires us to make the parameter θ space-time dependent

$$\theta \rightarrow \theta(x)$$

This would give us a local $U(1)$ symmetry. In this case the mass term of the Dirac Lagrangian is unchanged.

$$m\bar{\psi}\psi \rightarrow m[e^{-i\theta(x)}\bar{\psi}(x)][e^{i\theta(x)}\psi(x)] = m\bar{\psi}\psi$$

But for the kinetic energy term we have a problem since the derivative of the transformation term is no longer 0. That is, $\partial_\mu e^{i\theta(x)} = i[\partial_\mu\theta(x)]e^{i\theta(x)} \neq 0$. Here is the problem:

$$\begin{aligned} &i\bar{\psi}\gamma^\mu\partial_\mu\psi \rightarrow ie^{-i\theta(x)}\,\bar{\psi}(x)\gamma^\mu\partial_\mu\left[e^{i\theta(x)}\psi(x)\right] \\ &= ie^{-i\theta(x)}\bar{\psi}(x)\,\gamma^\mu e^{i\theta(x)}\partial_\mu\psi(x) - \bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\theta \\ &\neq i\bar{\psi}\gamma^\mu\partial_\mu\psi \end{aligned}$$

Physically (and experimentally) we find invariance in nature and so we will insist our theory also has invariance. So the challenge becomes how do we recover invariance under a local gauge transformation? One way is to create a transformation of the electromagnetic field of the form, like

$$A^\mu \rightarrow A^\mu - \frac{1}{q}\partial^\mu\theta$$

This will cancel the errant terms in the kinetic energy term. To see this, examine the interaction portion of the Lagrangian as follows:

$$\begin{aligned} L_{\text{int}} &= -q\bar{\psi}\gamma^{\mu}\psi A_{\mu} \rightarrow -q\bar{\psi}\gamma^{\mu}\psi\left(A_{\mu} - \frac{1}{q}\partial_{\mu}\theta\right) \\ &= -q\bar{\psi}\gamma^{\mu}\psi A_{\mu} + \bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\theta \\ &= L_{\text{int}} + \bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\theta \end{aligned}$$

Now let's look at the results for both transformations together. We have the local $U(1)$ gauge transformation

$$\psi(x) \rightarrow e^{i\theta(x)}\psi(x)$$

and the new transformation which restores invariance

$$A^{\mu} \rightarrow A^{\mu} - \frac{1}{q}\partial^{\mu}\theta$$

The Dirac and interaction parts of the Lagrangian transform as

$$\begin{aligned} &L_{\text{Dirac}} + L_{\text{int}} \rightarrow ie^{-i\theta(x)}\bar{\psi}\gamma^{\mu}e^{i\theta(x)}\partial_{\mu}\psi - \bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\theta - m\bar{\psi}\psi - q\bar{\psi}\gamma^{\mu}\psi A_{\mu} + \bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\theta \\ &= L_{\text{Dirac}} + L_{\text{int}} \end{aligned}$$

and we have restored the luster of the theory—the invariance.

The invariance is restored because we forced $A^{\mu} \rightarrow A^{\mu} - \frac{1}{q}\partial^{\mu}\theta$. That is, we have introduced the covariant derivative

$$D_{\mu} = \partial_{\mu} + iqA_{\mu}$$

This adjustment of the derivative operator is called the *minimal coupling prescription.* Hence the term

$$\bar{\psi}\gamma^{\mu}D_{\mu}\psi$$

is invariant under a local $U(1)$ transformation. We can understand the origin of the covariant derivative by considering how A^{μ} transforms under a Lorentz gauge transformation. It can be shown that the similarity transformation

$$U(\Lambda)A^{\mu}U^{-1}(\Lambda) = \Lambda^{\mu}{}_{\nu}A^{\nu} + \partial_{\mu}\theta(x)$$

This is why we require invariance under a transformation of the form $A^{\mu} \rightarrow A^{\mu} - \frac{1}{q}\partial^{\mu}\theta$. This gives us a Lagrangian that is invariant under a Lorentz gauge transformation.

Feynman Rules for QED

The Feynman rules for QED apply to any lepton–photon interactions, but for our purposes we will just discuss electrons and positrons. We use the Dirac states from Chap. 5 to represent electrons and positrons, so $u(p,s)$ is a particle state of momentum p and spin s and $v(p,s)$ is the antiparticle state. An incoming electron is a spinor state $u(p,s)$ with an arrow pointing in the direction of positive time flow. This is indicated schematically in Fig. 8.3.

An outgoing electron state has the arrow flowing with the direction of time, but we replace the spinor state $u(p,s)$ by its adjoint $\overline{u}(p,s)$. This is illustrated in Fig. 8.4.

Next we will need to represent incoming and outgoing positrons, which are the antiparticles of electrons. We use the spinors $v(p,s)$ for positrons. An incoming positron is represented by the adjoint spinor $\overline{v}(p,s)$. Since the positron is an antiparticle, the arrow used to represent it points *opposite* the direction of time flow. This is shown in Fig. 8.5.

Outgoing positron states are represented by the spinor $v(p,s)$. An outgoing positron state is shown in Fig. 8.6.

At each vertex of a Feynman diagram for a QED process we need to include a coupling constant, g_e. If we define it in terms of the fine structure constant as

$$g_e = \sqrt{4\pi\alpha} \tag{8.13}$$

then we need a term at each vertex of the form

$$ig_e\gamma^{\mu} \tag{8.14}$$

Next we consider an internal photon line. Since it is an internal line, it will be characterized by a propagator. The form that propagates a field from a spacetime point x

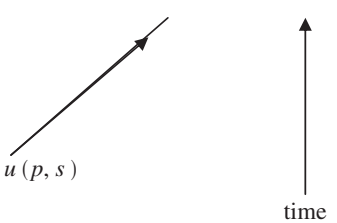


Figure 8.3 An electron entering an interaction.

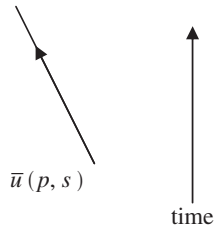


Figure 8.4 An electron leaving an interaction.

to a spacetime point y is determined by calculating the time ordered vacuum expectation value of the fields as shown here.

$$-i\Delta_{\mu\nu}(x-y)=\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle$$

This can be done by breaking up the field into positive and negative frequency components, like

$$\begin{aligned}-i\Delta_{\mu\nu}(x-y)&=\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle\\&=\theta(x-y)\int\frac{d^3pd^3p'}{\sqrt{2p_0 2p^{0'}}(2\pi)^3}\varepsilon_\mu^{(\lambda)}(p)e^{ip_\mu x^\mu}\left[\varepsilon_\nu^{(\lambda')}(p')\right]^* e^{-ip'_\mu y^\mu}\delta_{\lambda,\lambda'}\delta^3(\vec{p}-\vec{p}')\\&+\theta(y-x)\int\frac{d^3pd^3p'}{\sqrt{2p_0 2p^{0'}}(2\pi)^3}\varepsilon_\nu^{(\lambda)}(p')e^{ip'_\mu y^\mu}\left[\varepsilon_\mu^{(\lambda')}(p')\right]^* e^{-ip_\mu x^\mu}\delta_{\lambda,\lambda'}\delta^3(\vec{p}-\vec{p}')\end{aligned}$$

The Kronecker delta terms ensure that the polarization states are the same, that is, $\lambda'=\lambda$. We set $\lambda'\to\lambda$ and then readily do the integrals. The Dirac delta terms $\delta^3(\vec{p}-\vec{p}')$ enforce momentum conservation and the propagator simplifies to

$$\begin{aligned}-i\Delta_{\mu\nu}(x-y)&=\int\frac{d^3p}{2|\vec{p}|(2\pi)^3}e^{ip\cdot(x-y)}\theta(x-y)\sum_\lambda\varepsilon_\mu^{(\lambda)}(p)\left[\varepsilon_\nu^{(\lambda')}(p)\right]^*\\&+\int\frac{d^3p}{2|\vec{p}|(2\pi)^3}e^{-ip\cdot(x-y)}\theta(y-x)\sum_\lambda\varepsilon_\nu^{(\lambda)}(p)\left[\varepsilon_\mu^{(\lambda')}(p)\right]^*\end{aligned}$$

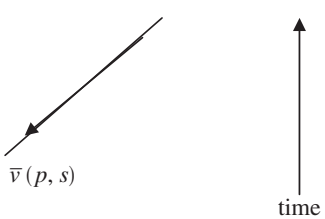


Figure 8.5 An incoming positron state.

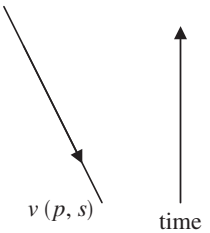


Figure 8.6 An outgoing positron state.

But it can be shown that

$$\sum_{\lambda} \varepsilon_{\lambda}^{i}(p)\left[\varepsilon_{\lambda}^{j}(p)\right]^{*} = \delta_{ij} - \frac{p_i p_j}{\left|\vec{p}\right|^2} \tag{8.15}$$

We are also taking the time component of the polarization vectors to be 0, that is, $\varepsilon_{\lambda}^{0}(p) = 0$. This just means that the polarization is fixed and does not vary with time. Then we define

$$P_{\mu\nu}(p) = \sum_{\lambda} \varepsilon_{\lambda}^{\mu}(p)\left[\varepsilon_{\lambda}^{\nu}(p)\right]^{*}$$

and use the following definition of the unit step or Heaviside function

$$\theta(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-isx}}{sx}\, ds \tag{8.16}$$

Then we can write the photon propagator as (note that q is momentum in this context, and not charge)

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4 q}{(2\pi)^4} P_{\mu\nu}(q) \frac{e^{iq\cdot(x-y)}}{q^2 - i\varepsilon}$$

We have made the notational change $p \to q$ because this is momentum for an internal line in the Feynman diagram. It can be shown that this reduces the propagator to

$$\Delta_{\mu\nu} = -\frac{i g_{\mu\nu}}{q^2} \tag{8.17}$$

In QED we can also have internal lines for electrons and positrons and the propagator is more complicated. For each internal line involving an electron or positron we include a factor

$$\frac{i(\gamma^{\mu}q_{\mu}+m)}{q^2-m^2} \tag{8.18}$$

The procedure is similar to that outlined in the last chapter, where conservation of momentum is enforced at each vertex using a Dirac delta function. However, in the case of QED spin must be taken into account. Let's see how to set up some basic calculations with a couple of examples.

Consider electron-electron scattering, sometimes known as Møller scattering, which is shown in Fig. 8.7.

To begin, we pick up a factor of $ig_e\gamma^{\mu}$ at each vertex. However, we need to be careful of the order of the factors because we are dealing with particles with spin and summations. We start at the left of the diagram, but move "backward in time" writing down factors from left to right. The exiting state on the left is the electron state $\overline{u}(p_3,s_3)$ and the input state is $u(p_1,s_1)$. So we get

$$\overline{u}(p_3,s_3)ig_e\gamma^{\mu}u(p_1,s_1)$$

Now we stir in a Dirac delta function to enforce conservation of momentum at the left vertex. We have an incoming momentum p_1 associated with $u(p_1,s_1)$ and an outgoing momentum p_3 associated with $\overline{u}(p_3,s_3)$. We also have a momentum q that is carried away from this vertex by the photon. The appropriate delta function is

$$(2\pi)^4\delta(p_1-p_3-q)$$

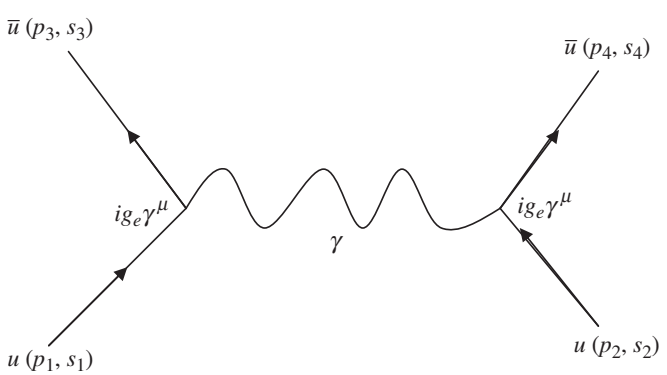
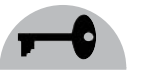


Figure 8.7 The simplest representation of Møller scattering.



So far we have

$$\bar{u}(p_3,s_3)ig_e\gamma^\mu u(p_1,s_1)(2\pi)^4\delta(p_1-p_3-q)$$

The next step is to add the photon propagator Eq. (8.17). Since there is one internal photon line, we only need one factor. Our expression for the Feynman diagram becomes

$$-\bar{u}(p_3,s_3)ig_e\gamma^\mu u(p_1,s_1)(2\pi)^4\delta(p_1-p_3-q)\left(\frac{ig_{\mu\nu}}{q^2}\right)$$

Now we multiply another factor of $ig_e\gamma^\mu$ for the vertex on the right side, along with terms for the electron states entering and leaving at that vertex, which is $\bar{u}(p_4,s_4)ig_e\gamma^\mu u(p_2,s_2)$. All together we have

$$-\bar{u}(p_3,s_3)ig_e\gamma^\mu u(p_1,s_1)(2\pi)^4\delta(p_1-p_3-q)\left(\frac{ig_{\mu\nu}}{q^2}\right)\bar{u}(p_4,s_4)ig_e\gamma^\mu u(p_2,s_2)$$

which we can rearrange as

$$-i(2\pi)^4 g_e{}^2\bar{u}(p_3,s_3)\gamma^\mu u(p_1,s_1)\delta(p_1-p_3-q)\left(\frac{g_{\mu\nu}}{q^2}\right)\bar{u}(p_4,s_4)\gamma^\mu u(p_2,s_2)$$

Integrating overall internal momenta q gives

$$\begin{aligned}
-iM &= \int -i(2\pi)^4 g_e{}^2\bar{u}(p_3,s_3)\gamma^\mu u(p_1,s_1)\delta(p_1-p_3-q)\left(\frac{g_{\mu\nu}}{q^2}\right)\bar{u}(p_4,s_4)\gamma^\mu u(p_2,s_2)d^4q \\
&= -ig_e{}^2\bar{u}(p_3,s_3)\gamma^\mu u(p_1,s_1)\left(\frac{g_{\mu\nu}}{(p_1-p_3)^2}\right)\bar{u}(p_4,s_4)\gamma^\mu u(p_2,s_2) \\
\Rightarrow M &= -g_e{}^2\bar{u}(p_3,s_3)\gamma^\mu u(p_1,s_1)\left(\frac{g_{\mu\nu}}{(p_1-p_3)^2}\right)\bar{u}(p_4,s_4)\gamma^\mu u(p_2,s_2)
\end{aligned}$$

Overall conservation of momentum is enforced by adding a delta function of the form $(2\pi)^4\delta(p_1+p_2-p_3-p_4)$ which can be ignored when writing down the amplitude.

It turns out we aren't done. Møller scattering includes one more lowest-order diagram, which is shown in Fig. 8.8.

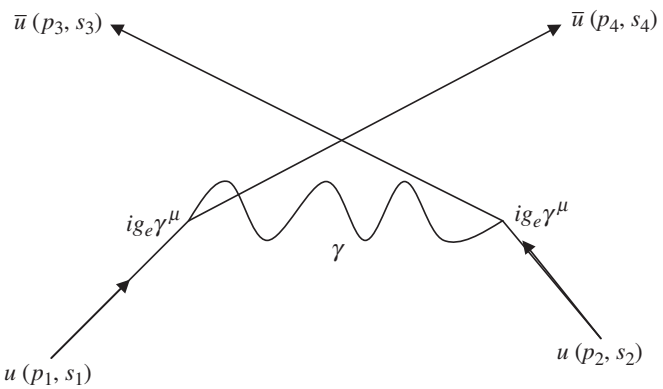


Figure 8.8 To complete the calculation for Møller scattering we need to include this diagram.

This process isn't exactly identical to the one in Fig. 8.7. This time $u(p_1, s_1)$ enters but then crosses over and exits as $\bar{u}(p_4, s_4)$, while $u(p_2, s_2)$ enters on the right and exits on the left as $\bar{u}(p_3, s_3)$. Conservation of momentum is enforced by the Dirac delta function on the left side of the diagram where the photon is emitted with momentum q.

$$(2\pi)^4 \delta(p_1 - p_4 - q)$$

Aside from this, the form of amplitude for this process is similar to the last one we wrote down, but we are swapping output states as shown here.

$$\bar{u}(p_3, s_3) \rightleftarrows \bar{u}(p_4, s_4)$$

Since we are exchanging two identical fermions, we must make a sign change. In total, the amplitude for the process illustrated in Fig. 8.8 is

$$M' = g_e{}^2 \bar{u}(p_4, s_4)\gamma^\mu u(p_1, s_1)\left(\frac{g_{\mu\nu}}{(p_1 - p_4)^2}\right)\bar{u}(p_3, s_3)\gamma^\mu u(p_2, s_2)$$

The total amplitude for the process is the sum of the two amplitudes we have written down, that is,

$$\begin{aligned} M_{\text{Møller}} &= M + M' \\ &= -g_e{}^2 \bar{u}(p_3, s_3)\gamma^\mu u(p_1, s_1)\left(\frac{g_{\mu\nu}}{(p_1 - p_3)^2}\right)\bar{u}(p_4, s_4)\gamma^\mu u(p_2, s_2) \\ &\quad + g_e{}^2 \bar{u}(p_4, s_4)\gamma^\mu u(p_1, s_1)\left(\frac{g_{\mu\nu}}{(p_1 - p_4)^2}\right)\bar{u}(p_3, s_3)\gamma^\mu u(p_2, s_2) \end{aligned}$$

This is a useful theoretical result. To calculate a measurable quantity we have to

- Pick a reference frame, either the lab frame or the center of mass frame
- Assign helicities to the particles or average/sum over all possible spin states

Let's consider the process in Fig. 8.7. It is easy to choose the center of mass frame. Let's suppose that all particles have a helicity of +1. This means that if the particle is moving along the positive z axis, its wave function will be of the form

$$u = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \dfrac{p}{E+m} \\ 0 \end{pmatrix}$$

modulo a normalization factor. If the particle is moving toward negative z, the wave function will be of the form

$$u = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{p}{E+m} \end{pmatrix}$$

where m is the mass of the electron. When the particles are incoming, we have the situation illustrated in Fig. 8.9.

Now, since $u\,(p_1, s_1)$ moves in the direction of positive z, the state is

$$u(p_1, s_1) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \dfrac{p}{E+m} \\ 0 \end{pmatrix}$$

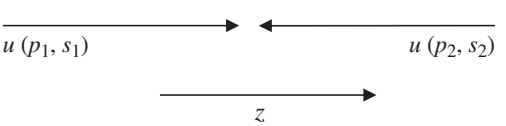


Figure 8.9 The incoming particle states. The state $u(p_1, s_1)$ moves in the direction of positive z.

All states have momentum p since we are using the center of mass frame. Now, $u(p_2, s_2)$ is moving in the direction of negative z so the state is

$$u(p_2, s_2) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{p}{E+m} \end{pmatrix}$$

Now let's look at the outgoing states, shown in Fig. 8.10. The direction of motion is reversed for each state, so the form of the states is different. Also recall that $\bar{\psi} = \psi^\dagger \gamma^0$. We have

$$\bar{u}(p_3, s_3) = u^\dagger(p_3, s_3)\gamma^0 = \sqrt{E+m}\begin{pmatrix} 0 & 1 & 0 & \dfrac{p}{E+m} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \sqrt{E+m}\begin{pmatrix} 0 & 1 & 0 & -\dfrac{p}{E+m} \end{pmatrix}$$

In the case of $\bar{u}(p_4, s_4)$, the state is moving in the positive z direction and so can be written as

$$\bar{u}(p_4, s_4) = u^\dagger(p_4, s_4)\gamma^0 = \sqrt{E+m}\begin{pmatrix} 1 & 0 & \dfrac{p}{E+m} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \sqrt{E+m}\begin{pmatrix} 1 & 0 & -\dfrac{p}{E+m} & 0 \end{pmatrix}$$

Now we can use these results to do some explicit calculations. Recall that the amplitude we found for the process shown in Fig. 8.7 was given by

$$M = -g_e{}^2 \bar{u}(p_3, s_3)\gamma^\mu u(p_1, s_1)\left(\frac{g_{\mu\nu}}{(p_1 - p_3)^2}\right)\bar{u}(p_4, s_4)\gamma^\mu u(p_2, s_2) \qquad (8.19)$$

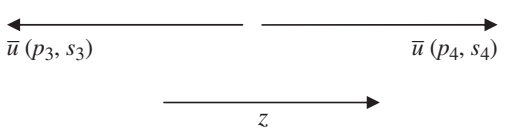
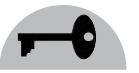


Figure 8.10 The outgoing states reverse direction.



So we need to calculate

$$\bar{u}(p_4,s_4)\gamma^0 u(p_2,s_2), \bar{u}(p_4,s_4)\gamma^1 u(p_2,s_2), \bar{u}(p_4,s_4)\gamma^2 u(p_2,s_2)$$

and

$$\bar{u}(p_4,s_4)\gamma^3 u(p_2,s_2)$$

There is only one way to do it, using brute force. The first term is

$$\bar{u}(p_4,s_4)\gamma^0 u(p_2,s_2) =$$

$$\sqrt{E+m}\begin{pmatrix}1 & 0 & -\dfrac{p}{E+m} & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\sqrt{E+m}\begin{pmatrix}0\\ 1\\ 0\\ \dfrac{p}{E+m}\end{pmatrix}$$

$$= (E+m)\begin{pmatrix}1 & 0 & -\dfrac{p}{E+m} & 0\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\\ -\dfrac{p}{E+m}\end{pmatrix}$$

$$= 0$$

Next we find

$$\bar{u}(p_4,s_4)\gamma^1 u(p_2,s_2) =$$

$$\sqrt{E+m}\begin{pmatrix}1 & 0 & -\dfrac{p}{E+m} & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}\sqrt{E+m}\begin{pmatrix}0\\ 1\\ 0\\ \dfrac{p}{E+m}\end{pmatrix}$$

$$= (E+m)\begin{pmatrix}1 & 0 & -\dfrac{p}{E+m} & 0\end{pmatrix}\begin{pmatrix}\dfrac{p}{E+m}\\ 0\\ -1\\ 0\end{pmatrix}$$

$$= 2p$$

Continuing we find

$$\bar{u}(p_4,s_4)\gamma^2 u(p_2,s_2) =$$

$$\sqrt{E+m}\begin{pmatrix}1 & 0 & -\dfrac{p}{E+m} & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & i & 0 & 0\\ -i & 0 & 0 & 0\end{pmatrix}\sqrt{E+m}\begin{pmatrix}0\\ 1\\ 0\\ \dfrac{p}{E+m}\end{pmatrix}$$

$$=(E+m)\begin{pmatrix}1 & 0 & -\dfrac{p}{E+m} & 0\end{pmatrix}\begin{pmatrix}\dfrac{-ip}{E+m}\\ 0\\ i\\ 0\end{pmatrix}$$

$$=-2ip$$

and finally

$$\bar{u}(p_4,s_4)\gamma^3 u(p_2,s_2) =$$

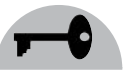
$$\sqrt{E+m}\begin{pmatrix}1 & 0 & -\dfrac{p}{E+m} & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\sqrt{E+m}\begin{pmatrix}0\\ 1\\ 0\\ \dfrac{p}{E+m}\end{pmatrix}$$

$$=(E+m)\begin{pmatrix}1 & 0 & -\dfrac{p}{E+m} & 0\end{pmatrix}\begin{pmatrix}0\\ -\dfrac{p}{E+m}\\ 0\\ 1\end{pmatrix}$$

$$=0$$

Using

$$u(p_1,s_1)=\sqrt{E+m}\begin{pmatrix}1\\ 0\\ \dfrac{p}{E+m}\\ 0\end{pmatrix},\quad \bar{u}(p_3,s_3)=\sqrt{E+m}\begin{pmatrix}0 & 1 & 0 & -\dfrac{p}{E+m}\end{pmatrix}$$



We also obtain

$$\bar{u}(p_3,s_3)\gamma^0 u(p_1,s_1) = 0$$
$$\bar{u}(p_3,s_3)\gamma^1 u(p_1,s_1) = 2p$$
$$\bar{u}(p_3,s_3)\gamma^2 u(p_1,s_1) = 2ip$$
$$\bar{u}(p_3,s_3)\gamma^3 u(p_1,s_1) = 0$$

which the reader should verify.

Now we apply the summation convention to each term in the amplitude [Eq. (8.19)]. For the left term we have

$$\begin{aligned} g_{\mu\nu}\bar{u}(p_4,s_4)\gamma^\mu u(p_2,s_2) &= g_{00}\bar{u}(p_4,s_4)\gamma^0 u(p_2,s_2) + g_{11}\bar{u}(p_4,s_4)\gamma^1 u(p_2,s_2) \\ &\quad + g_{22}\bar{u}(p_4,s_4)\gamma^2 u(p_2,s_2) + g_{33}\bar{u}(p_4,s_4)\gamma^3 u(p_2,s_2) \\ &= \bar{u}(p_4,s_4)\gamma^1 u(p_2,s_2) + \bar{u}(p_4,s_4)\gamma^2 u(p_2,s_2) \\ &= 2p(1-i) \end{aligned}$$

For $\bar{u}(p_3,s_3)\gamma^\mu u(p_1,s_1)$ we get $2p(1+i)$ and so Eq. (8.19) becomes

$$\begin{aligned} M &= -g_e{}^2\bar{u}(p_3,s_3)\gamma^\mu u(p_1,s_1)\left(\frac{g_{\mu\nu}}{(p_1-p_3)^2}\right)\bar{u}(p_4,s_4)\gamma^\mu u(p_2,s_2) \\ &= -g_e{}^2\frac{2p(1+i)2p(1-i)}{(p_1-p_3)^2} = -g_e{}^2\frac{8p^2}{(p_1-p_3)^2} \end{aligned}$$

Using $E^2 = m^2 + p^2$ you can show that $(p_1 - p_3)^2 = 2m^2 - 2(E^2 + p^2) = -4p^2$ and so the amplitude becomes

$$M = -g_e{}^2\frac{8p^2}{-4p^2} = 2g_e{}^2$$

Often, the helicities of the particles are not known. When this is the case, we say that the cross section is *unpolarized.* In that case it is necessary to average and sum over the spins. We compute the average over all spins for incoming particles and then sum over all possible spin states for outgoing particles. A useful tool for doing so is

$$\sum_s u(p,s)\bar{u}(p,s) = \frac{\not{p}+m}{2m} \tag{8.20}$$

(consider deriving this relation). For example, let's look at the lowest order Feynman diagram for electron–muon scattering. This is Fig. 8.7, but we replace the incoming and outgoing particle on the right with a muon. The amplitude is

$$M = -g_e{}^2 \bar{u}(p_3)\gamma^\mu u(p_1)\frac{g_{\mu\nu}}{q^2}\bar{u}(p_4)\gamma^\mu u(p_2)$$

where $\bar{u}(p_3)\gamma^\mu u(p_1)$ is due to the electron states and $\bar{u}(p_4)\gamma^\mu u(p_2)$ is due to the muon states. The final amplitude, when summing and averaging over all outgoing and incoming spins is

$$\left|\bar{M}\right|^2 = \frac{g_e^4}{q^4} L_e^{\mu\nu} L_{\mu\nu}{}^{\text{muon}}$$

We will focus on the electron term only, the muon term is similar. The electron term is

$$L_e^{\mu\nu} = \frac{1}{2}\sum_s \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]^*$$

Using Eq. (8.20) we find

$$L_e^{\mu\nu} = \frac{1}{2} Tr(\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu \tag{8.21}$$

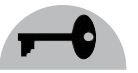
These terms can be evaluated using the so-called trace theorems. These include

$$\begin{aligned}
Tr(I) &= 4 \\
Tr(\not{a}\not{b}) &= 4a\cdot b \\
Tr(\not{a}\not{b}\not{c}\not{d}) &= 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)] \\
\gamma_\mu \not{a}\gamma^\mu &= -2\not{a} \\
Tr(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu}
\end{aligned} \tag{8.22}$$

These theorems are based only on basic linear algebra and the properties of the Dirac matrices.

Multiplying out the terms in Eq. (8.21) and applying trace theorems gives the result

$$\begin{aligned}
L_e^{\mu\nu} &= \frac{1}{2} Tr(\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu \\
&= \frac{1}{2} Tr(\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu + m\not{p}_3\gamma^\mu\gamma^\nu + m\gamma^\mu\not{p}_1\gamma^\nu + m^2\gamma^\mu\gamma^\nu) \\
&= 2p_3^\mu p_1^\nu + 2p_3^\nu p_1^\mu - 2(p_3\cdot p_1 - m^2)g^{\mu\nu}
\end{aligned}$$



At last we are done. The amplitude for the electron–muon scattering is

$$\left|\bar{M}\right|^2 = \frac{g_e^4}{q^4} L_e^{\mu\nu} L_{\mu\nu}^{\text{muon}} = \frac{g_e^4}{q^4} \left[Tr(p_3 + m)\gamma^\mu (p_1 + m)\gamma^\nu \right] \times \left[p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - (p_3 \cdot p_1 - m^2) g^{\mu\nu} \right]$$

Summary

In this first treatment of quantum electrodynamics, we have introduced the basic concept of electromagnetic forces as being due to the exchange of photons. Quantum electrodynamics ties together the Dirac theory of the electron (and other leptons, the principles are the same) with electromagnetics, which is a description of the photon field. It does this by considering interactions which are mediated by the photon, the force-carrying particle for the electromagnetic interaction. Electrons and other charged particles interact electromagnetically by exchanging photons. In this initial treatment, we have ignored higher order processes which include internal "loops" that lead to divergences.

The gauge symmetry for quantum electrodynamics is a local $U(1)$ symmetry. The requirement that the Lagrangian be invariant under this symmetry led to the minimal coupling prescription of the covariant derivative. We then extended this concept to compute measurable quantities like scattering amplitudes.

Quiz

1. Compute $[D_\mu, D_\nu]$.
2. The Lagrangian of quantum electrodynamics can be best described as
   (a) Admitting a local $U(1)$ symmetry
   (b) Admitting a global $U(1)$ symmetry
   (c) Admitting a local $SU(2)$ symmetry
   (d) Admitting a local $SU(1)$ symmetry
3. Write down the amplitude for electron-positron scattering as shown in Fig. 8.11.

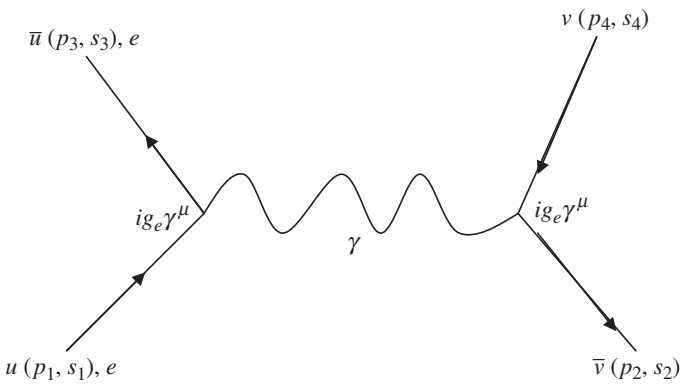


Figure 8.11 Electron-muon scattering to lowest order.

4. The minimal coupling prescription for the QED Lagrangian is
   (a) $D_\mu = \partial_\mu + ig_e A_\mu$
   (b) $D_\mu = \partial_\mu - ig_e A_\mu$
   (c) $D_\mu = \partial_\mu + iqA_\mu$
   (d) $D_\mu = \partial_\mu + iq\gamma^\mu A_\mu$
5. In a QED process an incoming antiparticle state is written as
   (a) $\overline{v}\,(p,s)$
   (b) $\overline{u}\,(p,s)$
   (c) $u\,(p,s)$
   (d) $v\,(p,s)$

CHAPTER 9

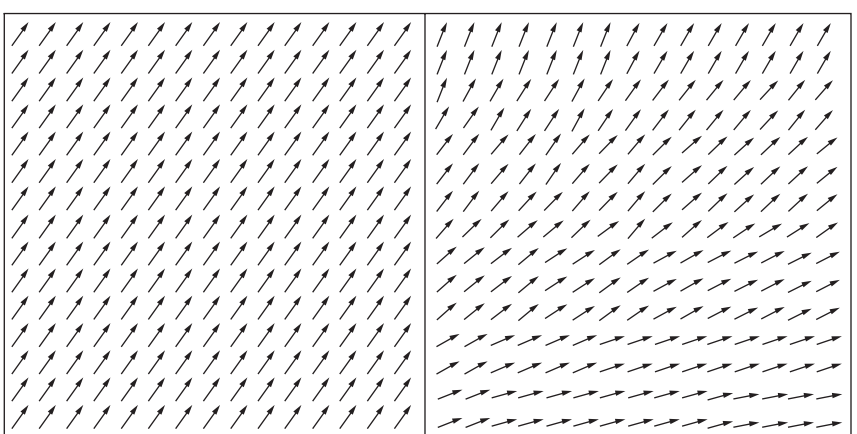
Spontaneous Symmetry Breaking and the Higgs Mechanism

Let's review some important concepts. Noether's theorem relates conservation laws to symmetries in the Lagrangian. When quantum theory is invoked, these symmetries can take the form of invariance under a unitary transformation. For example, a $U(1)$ symmetry means that a Lagrangian $L = L(\varphi, \partial_\mu \varphi)$ is invariant under a transformation of the form

$$\varphi(x) \rightarrow \varphi'(x) = e^{-i\theta}\varphi(x) \tag{9.1}$$

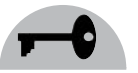
When θ does not depend on the spacetime coordinate x, then we say that Eq. (9.1) is a global symmetry. In quantum field theory global symmetries represent something that cannot be measured, like the phase of a wave function in quantum mechanics. The wave function $\psi(x,t)$ and $e^{-i\alpha}\psi(x,t)$ give the same physical predictions. On the other hand, if θ does depend on the spacetime coordinate so that $\theta = \theta(x)$, then Eq. (9.1) depends on where you are and hence represents a *local* symmetry. Local symmetries are very important in relativistic physics because they represent the physical fact that quantities that are conserved like charge and lepton number are conserved *locally*. Charge would not be conserved locally if you could have a current on earth disappear and suddenly reappear on the moon. The charge must travel across the intervening space to appear on the moon, and the way it moves from the earth to the moon is dictated by the fact that nothing travels faster than the speed of light. Said another way, a local symmetry preserves *causality* as required by special relativity.

Two pictures help distinguish between local and global $U(1)$ transformations. We know from complex variables that the exponential $e^{i\theta}$ represents a point on the unit circle. We can think of this point as a vector—an arrow from the origin to the point on the unit circle. In a global transformation, let θ assume a fixed value, so that $e^{i\theta}$ has a constant value throughout spacetime. Hence the name global. On the other hand, $e^{i\theta(x)}$ where x is a point in spacetime, has a different value depending upon location because $\theta(x)$ is now a function on spacetime. So we talk of a value at a specific point. The figure below shows these two cases. On the left, notice that each vector is the same, but on the right, the direction of the vector depends on it's location. So the left side illustrates a global transformation, while the right side illustrates a local transformation.



A global transformation: The value of $e^{i\theta}$ is the same everywhere.

A local transformation: The value of $e^{i\theta}$ depends upon the space-time location.



We saw in Chap. 3 that more complicated unitary transformations such as $SU(2)$ appear in quantum field theory. It is also possible to have Lagrangians that are invariant under other types of transformations, such as $\varphi \rightarrow -\varphi$. We will use this type of symmetry to introduce the concept of *spontaneous symmetry breaking*. An example of a Lagrangian that is invariant under $\varphi \rightarrow -\varphi$ is the Lagrangian of the so-called (φ^4) theory.

$$L = \partial_\mu \varphi \, \partial^\mu \varphi - m^2 \varphi^2 + \lambda \varphi^4 \tag{9.2}$$

Clearly, since the field φ only appears in the Lagrangian in terms of even powers and even derivatives, the Lagrangian is unchanged under the transformation $\varphi \rightarrow -\varphi$. We will see, however, this Lagrangian is even more interesting than it appears at first sight.

It turns out that in many cases, a system that has some symmetry that exists in the Lagrangian may have a ground state (i.e., a vacuum state) that *does not* satisfy the same symmetry. This is the case for the Lagrangian given in Eq. (9.2). When a situation like this exists, we say that the system has undergone *spontaneous symmetry breaking*. Before jumping into the mathematics of the situation, let's describe the concept with a simple physical example. Imagine an upside down steel bowl placed on flat ground. We place a marble on top of the bowl right in the center. This system is symmetric—from the point of view of the marble every direction from the top of the bowl to the ground is equivalent. However, the system is *unstable*. The marble starts out at rest, but the slightest perturbation will send it rolling down the bowl to the ground. In analogy with quantum field theory, think of the marble sitting on top of the bowl as a ground state that is unstable.

Now suppose that the marble is perturbed and rolls off the bowl. It will roll in one particular direction and come to rest below on the flat ground. In short, the perturbation has *spontaneously broken* the symmetry that existed before. Moreover, the marble has now arrived at a state of minimum potential energy. In short, the marble was not really in the ground state when it was resting on top of the bowl—the true ground state of potential energy exists when the symmetry is broken and the marble finds itself resting on the ground below.

Symmetry Breaking in Field Theory

In quantum field theory, we often have Lagrangians that exhibit similar properties to an upside down bowl. We will see a vacuum state that is an apparent ground state, but in fact there will be a *true* ground state or vacuum state of lower energy that leads to symmetry breaking. What is vacuum? Vacuum is the state with no fields, that is, $\varphi = 0$. In our calculations by applying perturbation theory, we expand

about $\varphi = 0$, the fields are then viewed as fluctuations about the ground state. You can think of $\varphi = 0$ as the minimum of potential energy.

However, when considering different Lagrangians, it turns out that the state with $\varphi = 0$ is not always the minimum. Remember that the Lagrangian is the difference between kinetic energy T and potential energy V.

$$L = T - V$$

In field theory, remember that kinetic energy terms will be of the form $\partial_\mu \varphi \partial^\mu \varphi$. The potential V will be some function of the fields φ so that $V = V(\varphi)$. Therefore, to find the minimum we use ordinary calculus, that is, we seek to find the minimum of the potential by computing its derivative. That is, we will find a φ that allows us to satisfy

$$\frac{\partial V}{\partial \varphi} = 0 \tag{9.3}$$

This procedure will give us the true ground state of the system, which may not be $\varphi = 0$.

EXAMPLE 9.1

Consider the Lagrangian for φ^4 theory when

$$L = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4}\lambda\varphi^4 \tag{9.4}$$

where φ is a real scalar field. Describe the minimum of potential energy when $m^2 > 0$ and $m^2 < 0$.

SOLUTION

The kinetic energy term in the Lagrangian is

$$\frac{1}{2}(\partial_\mu \varphi)^2$$

The potential is

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda\varphi^4$$

What force does this potential create? We compute the derivative of V with respect to the field φ:

$$\begin{aligned}\frac{\partial V}{\partial \varphi} &= m^2\varphi + \lambda\varphi^3 \\ &= \varphi(m^2 + \lambda\varphi^2)\end{aligned}$$

We obtain the minima by setting this expression equal to 0. One extremum jumps out immediately, it is the one we naively expect to find

$$\varphi = 0$$

This case corresponds to the case $m^2 > 0$, which represents a scalar field of mass *m*. The φ^4 term represents self-interactions of the field with a coupling strength given by λ. The e potential in this case is shown in Fig. 9.1.

When the ground state is at $\varphi = 0$, it obviously satisfies the symmetry present in the Lagrangian, $\varphi \to -\varphi$, and does so trivially.

Now let's consider the other alternative minimum that result from our calculation of $\partial V/\partial\varphi = 0$. In this case we have

$$m^2 + \lambda\varphi^2 = 0$$

Since φ appears as a square, this leads to two possible minima given by

$$\varphi = \pm\sqrt{\frac{-m^2}{\lambda}} = \pm v \tag{9.5}$$

In this case, in order for the field φ to be real, it must be the case that $m^2 < 0$. This is a situation which corresponds to the ball sitting on top of the bowl. The potential in this case is shown in Fig. 9.2. Notice that $\varphi = 0$ corresponds to the unstable point where the marble is resting on top of the bowl. We can go to one or the other minimum, where $\varphi = +v$ or $\varphi = -v$, giving the true ground state. But choosing one of the other breaks the symmetry. This is analogous to the marble rolling off the bowl and coming to rest at some particular point on the ground.

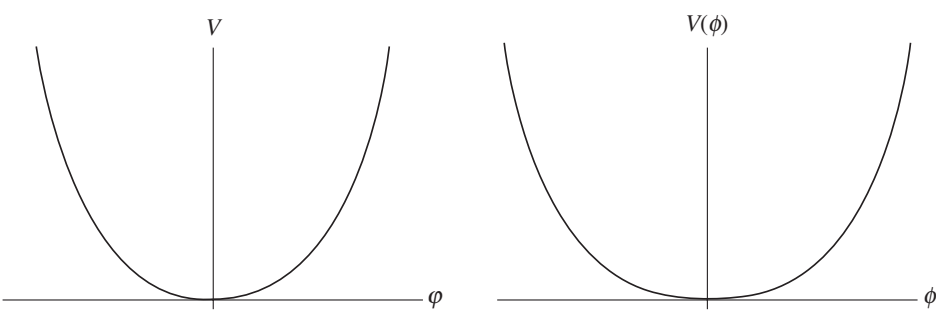


Figure 9.1 The potential for the Lagrangian given in Eq. (9.4), for the case of $m^2 > 0$. The minimum is at $\varphi = 0$.

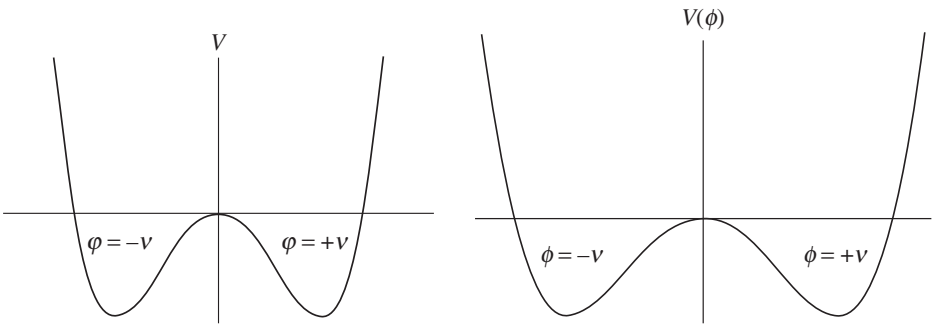


Figure 9.2 The potential for the Lagrangian given in Eq. (9.4) when $m^2 < 0$.

In the case of the Lagrangian, we have found the true minimum at $\varphi = \pm v$. The point $\varphi = 0$ is an unstable point, therefore a perturbative expansion about this point will not converge. In contrast, a perturbative expansion about one of $\varphi = \pm v$ will converge, allowing us to do calculations using the Feynman rules.

However, the symmetry has been broken. There are two ground states, the minima at $\varphi = +v$ and $\varphi = -v$. In the next section we will see that the Lagrangian is no longer invariant under $\varphi \to -\varphi$. While we loose this benefit, we will gain knowledge, specifically the true mass of the particle associated with the field φ.

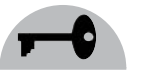
Mass Terms in the Lagrangian

A key task in exercises involving spontaneous symmetry breaking is the ability to recognize mass terms in the Lagrangian. Doing so is usually pretty simple; to see this let's go back to square one—which means digging up the Klein-Gordon equation. The Lagrangian in this case is

$$L = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2$$

We already know that in the case of the Klein-Gordon equation, the field quanta φ are particles of mass m. Looking at the Lagrangian, we therefore recognize the mass term as

$$-\frac{1}{2}m^2\varphi^2$$



where $m^2 > 0$ and m is the mass of the associated particle. So this is a straightforward exercise. We conclude that

> A mass term in the Lagrangian is one that is quadratic in the fields, which is a term of the form $\alpha^2\varphi^2$ for some α.

However, it turns out that identifying the mass terms in a Lagrangian by inspection is not always possible. Many Lagrangians have mass terms that are hidden in one way or another. To see this, consider a fictitious Lagrangian given by

$$L = \frac{1}{2}(\partial_\mu\varphi)^2 + \ln(1-\alpha\varphi) \tag{9.6}$$

Is there a mass term in this Lagrangian? By inspection, we don't see any terms that are quadratic in the field, so we might jump to the conclusion that $m = 0$ in this case. The Lagrangian appears to describe a massless field like a photon, say. But a closer look at Eq. (9.6) will reveal otherwise. Once again we call upon our skills we learned in freshman calculus. To expand the Lagrangian in a series we need to expand the logarithmic term. The trick is to start with the geometric series as shown here.

$$\frac{1}{1-\alpha x} = 1 + \alpha x + (\alpha x)^2 + (\alpha x)^3 + \cdots \text{ for } |\alpha x| < 1$$

(If you have forgotten this expansion, just do the division manually.) To introduce a logarithm, we need to integrate this expression to get

$$\ln(1-\alpha x) = -\alpha\left[\int 1 + \alpha x + (\alpha x)^2 + \cdots dx\right]$$

which is just

$$\ln(1-\alpha x) = -\alpha x - \frac{1}{2}\alpha^2 x^2 - \frac{1}{3}\alpha^3 x^3 - O(x^4)$$

Writing Eq. (9.6) using this expansion, we see that the Lagrangian does in fact contain a term that is quadratic in the field.

$$\begin{aligned} L &= \frac{1}{2}(\partial_\mu\varphi)^2 + \ln(1-\alpha\varphi) \\ &= \frac{1}{2}(\partial_\mu\varphi)^2 - \alpha\varphi - \frac{1}{2}\alpha^2\varphi^2 - \frac{1}{3}\alpha^3\varphi^3 - O(\varphi^4) \end{aligned}$$

Provided that, $\alpha^2 > 0$, this Lagrangian describes particles with mass $\alpha = m$. The mass term has been disguised by the original representation of the Lagrangian given in Eq. (9.6). So, when it's not obvious as to whether or not a given Lagrangian contains a mass term

- Expand the potential in a series.
- Look for terms that are quadratic in the fields.

EXAMPLE 9.2
Do the Lagrangians

$$L_1 = \frac{1}{2}(\partial_\mu \varphi)^2 - e^{\alpha^3 \varphi^3}$$
$$L_2 = \frac{1}{2}(\partial_\mu \varphi)^2 - e^{\alpha \varphi}$$

represent massive or massless fields?

SOLUTION
We apply our little recipe and follow the guidance provided by the Klein-Gordon equation. That is, we use the expansion of the exponential function and look for terms that are quadratic in the field. Recall that

$$e^{\alpha x} = 1 + \alpha x + \frac{1}{2!}(\alpha x)^2 + \frac{1}{3!}(\alpha x)^3 + O(x^4)$$

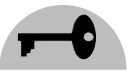
In the case of L_1, we have

$$\begin{aligned} L_1 &= \frac{1}{2}(\partial_\mu \varphi)^2 - e^{\alpha^3 \varphi^3} \\ &= \frac{1}{2}(\partial_\mu \varphi)^2 - 1 - \alpha^3 \varphi^3 - \frac{1}{2}\alpha^6 \varphi^6 \end{aligned}$$

plus higher order terms. There is not a term involving φ^2, hence we conclude that L_1 is a Lagrangian for a massless field.

Now let's consider L_2. Expanding the exponential in this case gives us

$$\begin{aligned} L_2 &= \frac{1}{2}(\partial_\mu \varphi)^2 - e^{\alpha \varphi} \\ &= \frac{1}{2}(\partial_\mu \varphi)^2 - 1 - \alpha \varphi - \frac{1}{2}\alpha^2 \varphi^2 - \frac{1}{6}\alpha^3 \varphi^3 - \cdots \end{aligned}$$



The presence of the term

$$\frac{1}{2}\alpha^2\varphi^2$$

tells us that L_2 is a Lagrangian for a massive field. The mass of the particle is given by comparison with the Klein-Gordon equation, hence the mass is $m = \alpha$.

Aside on Units

Now a short note on units and scaling with mass terms. When we put all the $\hbar$'s and c's back in the Klein-Gordon equation, it is written as

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi - \frac{m^2c^2}{\hbar^2}\varphi = 0$$

So, if we have a term in the Lagrangian we are considering that looks like

$$\frac{1}{2}\alpha^2\varphi^2$$

Then the mass of the particle is related to the constant α in the following way.

$$\alpha = \frac{mc}{\hbar} \tag{9.7}$$

That is, the mass of the particle is

$$m = \frac{\hbar\alpha}{c} \tag{9.8}$$

where α is a unitless number and m will have dimensions of mass inherited from the values used for $\hbar$ and c.

If the quadratic term in the Lagrangian is missing the ½ scale factor, that is, if it contains a term

$$\alpha^2\varphi^2$$

you need to account for the missing ½ when comparing to the Klein-Gordon equation to get the mass. In this case we have a relationship given by

$$\alpha^2 = \frac{1}{2}\frac{m^2c^2}{\hbar^2}$$

Hence, the mass of the particle is

$$m = \sqrt{2}\,\frac{\alpha\hbar}{c}$$

Spontaneous Symmetry Breaking and Mass

Now that we know how to recognize mass terms in the Lagrangian, let's go back to the φ^4 theory and reconsider the situation. To keep you from having to flip back and forth through the pages, remember that the Lagrangian we used in Example 9.1 was given by

$$L = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4}\lambda\varphi^4$$

We found that the true ground state or minima of the potential was spontaneously broken and given by

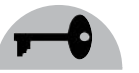
$$\varphi = \pm\sqrt{\frac{-m^2}{\lambda}} = \pm v$$

Now what to do with this information? The minimum is not at $\varphi = 0$, instead it's located at $\varphi = \pm v$. We consider the case where $\varphi = v$ and rescale the field to represent this fact

$$\varphi(x) = v + \eta(x) \tag{9.9}$$

We've written the field as fluctuations described by $\eta(x)$ about the right-hand minimum v. Next we will rewrite the Lagrangian using the new form [Eq. (9.9)]. The kinetic energy terms are easy to write down because v is just a number, so

$$\partial_\mu\varphi(x) = \partial_\mu\left[v + \eta(x)\right] = \partial_\mu\eta(x)$$



Now we square Eq. (9.9) to give

$$\varphi^2 = (v+\eta)^2 = v^2 + 2v\eta + \eta^2$$

and the fourth power of the field becomes

$$\varphi^4 = (v+\eta)^4 = v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4$$

Putting these terms together, the Lagrangian becomes

$$\begin{aligned} L &= \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4}\lambda\varphi^4 \\ &= \frac{1}{2}(\partial_\mu\eta)^2 - \frac{1}{2}m^2(v^2 + 2v\eta + \eta^2) - \frac{1}{4}\lambda(v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4) \end{aligned}$$

Again, we remember that v is just a number. We can drop all terms that are constant from the Lagrangian since a constant does not contribute to the field equations for the system. The first term in the potential can then be written as

$$\begin{aligned} \frac{1}{2}m^2(v^2 + 2v\eta + \eta^2) &= -\frac{1}{2}\lambda v^2(v^2 + 2v\eta + \eta^2) \\ &= -\lambda v^3\eta - \frac{1}{2}\lambda v^2\eta^2 \end{aligned}$$

where we used Eq. (9.5) to write $m^2 = -\lambda v^2$ and we dropped constant terms. Dropping constant terms from the last term of the potential gives

$$\begin{aligned} L &= \frac{1}{2}(\partial_\mu\eta)^2 - \frac{1}{2}m^2(v^2 + 2v\eta + \eta^2) - \frac{1}{4}\lambda(v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4) \\ &= \frac{1}{2}(\partial_\mu\eta)^2 + \lambda v^3\eta + \frac{1}{2}\lambda v^2\eta^2 - \lambda v^3\eta - \frac{3}{2}\lambda v^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4 \end{aligned}$$

Finally, we arrive at the new Lagrangian

$$L = \frac{1}{2}(\partial_\mu\eta)^2 - \lambda v^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4 \tag{9.10}$$

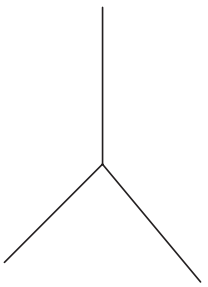


Figure 9.3 A Feynman diagram representation of the self-interaction term in the Lagrangian $\lambda v \eta^3$.

Now we apply our rule. Look for terms that are quadratic in the fields η; they should have negative signs in front of them. The mass term for Eq. (9.10) is

$$\lambda v^2 \eta^2$$

Comparing to a mass term in the Klein-Gordon–type Lagrangian

$$\frac{1}{2} m^2 \phi^2$$

we see that the mass of the particle in the case of Eq. (9.10) is

$$m = \sqrt{2\lambda v^2} = \sqrt{2\lambda}\, v$$

Notice that we have taken into account the missing ½ factor. What about the other terms in the Lagrangian? These represent self-interaction terms of the field $\eta(x)$. In particular, the cubic term η^3 is a vertex in a Feynman diagram with three legs and a coupling given by λv. This is illustrated in Fig. 9.3.

The last term, $\frac{1}{4}\lambda\eta^4$, is another self-interaction term that will have four legs in a Feynman diagram. This is illustrated in Fig. 9.4.

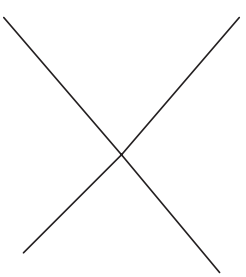


Figure 9.4 The $\frac{1}{4}\lambda\eta^4$ term in the Lagrangian given in Eq. (9.10) is represented by a four legged vertex in a Feynman diagram.

We have now accounted for every term in the Lagrangian:

$$L = \frac{1}{2}(\partial_\mu \eta)^2 - \lambda v^2 \eta^3 - \lambda v \eta^3 - \frac{1}{4}\lambda \eta^4$$

Kinetic energy — Mass term — Self interaction, three legs — Self interaction, four legs

Lagrangians with Multiple Particles

In most, if not all, real cases of physical interest, the spontaneous symmetry breaking of a given Lagrangian will result in the appearance of more than one particle. It may be that these particles have different masses; perhaps some will have mass and some will not. Let's illustrate this with a complex field and a Lagrangian that gives rise to one massive and one massless particle. First let's define the field in terms of two real fields φ_1 and φ_2.

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \tag{9.11}$$

The Lagrangian we consider is

$$L = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi + \lambda(\varphi^\dagger \varphi)^2 \tag{9.12}$$

Now

$$\varphi^\dagger \varphi = \left(\frac{\varphi_1 - i\varphi_2}{\sqrt{2}}\right)\left(\frac{\varphi_1 + i\varphi_2}{\sqrt{2}}\right) = \frac{1}{2}(\varphi_1^2 + \varphi_2^2)$$

Using this the Lagrangian in Eq. (9.12) becomes

$$L = \frac{1}{2}(\partial_\mu \varphi_1)^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - \frac{1}{2}m^2(\varphi_1^2 + \varphi_2^2) + \frac{1}{4}\lambda(\varphi_1^4 + \varphi_2^4) + \frac{1}{2}\lambda\varphi_1^2\varphi_2^2 \tag{9.13}$$

The potential is

$$V = \frac{1}{2}m^2(\varphi_1^2 + \varphi_2^2) - \frac{1}{4}\lambda(\varphi_1^4 + \varphi_2^4) - \frac{1}{2}\lambda\varphi_1^2\varphi_2^2$$

The Lagrangian has a (φ_1, φ_2) symmetry that can be described by rotations in (φ_1, φ_2) space. These can be written in the matrix form

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

That is,

$$\varphi_1' = \cos\alpha\,\varphi_1 + \sin\alpha\,\varphi_2$$
$$\varphi_2' = -\sin\alpha\,\varphi_1 + \cos\alpha\,\varphi_2$$

The minima of the potential lie on a circle that is described by

$$\varphi_1^2 + \varphi_2^2 = \frac{m^2}{\lambda}$$

To break the $U(1)$ symmetry, we think back to the original example of a marble sitting on top of the bowl. We pick out a specific direction. Following the notation of the last example, we denote the minima by v using a subscript to indicate the minima of φ_1 and φ_2. In this case, we pick the minimum at

$$v_1 = \frac{m}{\sqrt{\lambda}} \qquad v_2 = 0 \tag{9.14}$$

Now we rewrite the field. This time we need two fields χ and ψ that fluctuate about the minimum given by Eq. (9.14). We have

$$\varphi = \frac{m}{\sqrt{\lambda}} + \frac{\chi + i\psi}{\sqrt{2}}$$

We have taken

$$\chi = \varphi_1 - \frac{m}{\sqrt{\lambda}} \qquad \psi = \varphi_2 \tag{9.15}$$

The change in the coordinate systems in going from $\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$ to $\begin{pmatrix}\chi\\ \psi\end{pmatrix}$ amounts to shifting the coordinate system to the right by an amount $\frac{m}{\sqrt{\lambda}}$. In other words, we have shifted the origin to the actual minimum of the potential. Then,

$$\varphi_1^2 = \left(\chi + \frac{m}{\sqrt{\lambda}}\right)^2 = \chi^2 + 2\frac{m}{\sqrt{\lambda}}\chi + \frac{m^2}{\lambda}$$
$$\varphi_2^2 = \psi^2$$

and for the quadratic terms

$$\varphi_1^4 = \left(\chi^2 + 2\frac{m}{\sqrt{\lambda}}\chi + \frac{m^2}{\lambda}\right)^2$$
$$= \chi^4 + 4\frac{m}{\sqrt{\lambda}}\chi^3 + 6\frac{m^2}{\lambda}\chi^2 + 4\frac{m^3}{\sqrt{\lambda^3}}\chi + \frac{m^4}{\lambda^2}$$

and

$$\varphi_2^4 = \psi^4$$

Finally,

$$\frac{1}{2}\lambda\varphi_1^2\varphi_2^2 = \frac{1}{2}\lambda\left(\chi^2\psi^2 + \frac{2m}{\sqrt{\lambda}}\chi\psi^2 + \frac{m^2}{\lambda}\psi^2\right)$$

Remember that terms of the form φ^n where $n > 2$ represent interaction terms. To get the mass terms, we need to ignore those and look at the *free Lagrangian.* Also remember we can drop constants because they do not contribute to the field equations that would be derived from the Lagrangian. Now, dropping everything except the quadratic terms, the free parts of the potential are

$$\frac{1}{2}m^2\left(\varphi_1^2 + \varphi_2^2\right)_{\text{free}} = \frac{1}{2}m^2(\chi^2 + \psi^2)$$
$$-\frac{1}{4}\lambda\left(\varphi_1^4 + \varphi_2^4\right)_{\text{free}} = -\frac{1}{4}\lambda\left(6\frac{m^2}{\lambda}\chi^2\right)$$
$$-\frac{1}{2}\lambda\left(\varphi_1^2\varphi_2^2\right)_{\text{free}} = -\frac{1}{2}\lambda\left(\frac{m^2}{\lambda}\psi^2\right)$$

Notice that

$$\frac{1}{2}m^2\left(\varphi_1^2+\varphi_2^2\right)_{\text{free}}-\frac{1}{2}\lambda\left(\varphi_1^2\varphi_2^2\right)_{\text{free}}=\frac{1}{2}m^2(\chi^2+\psi^2)-\frac{1}{2}\lambda\left(\frac{m^2}{\lambda}\psi^2\right)$$
$$=\frac{1}{2}m^2\chi^2$$

Putting all this together, the free or noninteraction part of the Lagrangian is

$$L_{\text{free}}=\frac{1}{2}(\partial_\mu\chi)^2+\frac{1}{2}(\partial_\mu\psi)^2-\frac{1}{2}m^2\chi^2$$

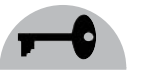
Hence, spontaneous symmetry breaking of the $U(1)$ symmetry given by a rotation in (φ_1,φ_2) space for the complex field in Eq. (9.11) together with the Lagrangian in Eq. (9.12) gives us a field χ with mass m and a field χ that is *massless*. The mass m is defined by the minima on the circle that we choose to be

$$v_1=\frac{m}{\sqrt{\lambda}}$$

Hence $m=\sqrt{\lambda}v_1$. This example involved the use of scalar fields, so the particles associated with these fields are spin-0 particles. When a massless spin-0 particle appears in a theory due to symmetry breaking, it is called a *Goldstone boson.*

The Higgs Mechanism

In the previous section, we examined spontaneous symmetry breaking by considering the case of a complex field with two real components, and a $U(1)$ symmetry. For simplicity, we've considered global gauge invariance. Now we wish to extend this idea to a more complicated situation in which we include a gauge field A_μ and require *local* gauge invariance under a $U(1)$ transformation. We will start with a massless gauge field A_μ, and show that symmetry breaking results in a massive vector field. We'll see later that in electroweak theory this type of procedure gives rise to the massive vector bosons, the $W^\pm$ and the Z^0. The mechanism involving spontaneous symmetry breaking involving a gauge field and local $U(1)$ invariance is known as the *Higgs mechanism*, named for its founder Peter Higgs who discovered the effect in 1964. A major task ahead for experimentalists when the Large Hadron Collider begins operation in the summer of 2008 will be to find the quanta of the Higgs field $h(x)$.



Once again, recall that $U(1)$ invariance implies that the Lagrangian is invariant under a transformation of the form

$$\varphi \rightarrow \varphi' = e^{-i\theta}\varphi$$

Previously, we considered the case of a global gauge transformation, wherein θ is a scalar, just a number, and not a function of spacetime. Now, however, we extend this concept and let $\theta \rightarrow \theta(x)$, giving us a local gauge transformation. We want the Lagrangian to be invariant under the transformation that is,

$$\varphi(x) \rightarrow \varphi'(x) = e^{-iq\theta(x)}\varphi(x) \tag{9.16}$$

Now q is a number, but $\theta \rightarrow \theta(x)$ is a function of spacetime, meaning that the transformation varies from point to point. Invariance with a local gauge transformation will require the introduction of a *gauge field.* In analogy with electrodynamics, we will see that the requirement of local gauge invariance forces us to use a covariant derivative in order to restore the invariance of the Lagrangian. Define the gauge field as a vector potential A_μ in analogy to electromagnetics. The gauge field must also be invariant under a $U(1)$ gauge transformation, with the transformation assuming the form

$$A_\mu \rightarrow A_\mu{}' = A_\mu + \partial_\mu\theta \tag{9.17}$$

This is the same θ present in Eq. (9.16), and we note that A_μ is a function of spacetime as well, that is, $A_\mu = A_\mu(x)$.

Up until this point, in this chapter we have been using ordinary derivatives in the Lagrangian. That is, the kinetic terms in the Lagrangian have been of the form

$$\partial_\mu\varphi^\dagger\partial^\mu\varphi$$

In order to have a gauge invariant Lagrangian, considering Eqs. (9.16) and (9.17), we will need to use a covariant derivative. A suitable covariant derivative in this case is

$$D_\mu = \partial_\mu + iqA_\mu \tag{9.18}$$

With this definition, the Lagrangian assumes the form

$$L = D_\mu\varphi^\dagger D^\mu\varphi - V(\varphi^\dagger\varphi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.19}$$

Summarizing, this Lagrangian describes a theory that includes a complex scalar field φ and a *massless* gauge field A_μ. With a gauge field in the theory, terms like $F_{\mu\nu}F^{\mu\nu}$ represent kinetic energy terms. In analogy with electrodynamics, the following definition is used.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{9.20}$$

If a mass term was present, we would see a contraction on the gauge field $A_\mu A^\mu$ that is analogous to the quadratic mass terms of the scalar field. Since the Lagrangian given in Eq. (9.19) has no terms of this form, the gauge field is massless. What we will see in a moment is that a certain type of symmetry breaking will cause the gauge field A_μ to acquire mass. This is the essence of the Higgs mechanism.

The potential in Eq. (9.19) is given by

$$V(\varphi^\dagger\varphi) = \frac{m^2}{2v^2}(\varphi^\dagger\varphi - v^2)^2 \tag{9.21}$$

where we have included a constant v in anticipation of the search for a minimum. In this case, v is the minimum for the theory when the symmetry is *unbroken.*

Now we proceed as in the previous examples. This time, we seek the minimum potential energy when both the gauge field A_μ and potential V vanish. When the symmetry is unbroken, the minimum of the potential, the vacuum state, is at

$$|\varphi|^2 = v$$

where v is some real number. But we can do a gauge transformation since

$$\varphi'^\dagger\varphi' = (\varphi^\dagger e^{iq\theta(x)})(e^{-iq\theta(x)}\varphi) = \varphi^\dagger\varphi = |\varphi|^2 = v$$

So a local gauge transformation gives us the same minimum. And if the field is complex, there are infinity vacuum states. Since the gauge transformation gives the same minimum, we have a symmetry.

How can we break the symmetry? There is a hint in the fact that the minimum is obtained with the squared amplitude of a complex field. We can break the symmetry by requiring that the field be *real.* So the situation we have is

- The value v is the minimum of the potential when the symmetry is unbroken.
- We seek a gauge transformation that gives the field in terms of fluctuations about v.

This can be done when the vacuum v is perturbed by a real field $h(x)$.

$$\varphi \to \varphi' = v + \frac{h(x)}{\sqrt{2}} \tag{9.22}$$

The field $h(x)$ is the Higgs field. Since the field is real, $\varphi^\dagger = \varphi$ and the potential becomes

$$\begin{aligned} V &= \frac{m^2}{2v^2}\left[\sqrt{2}vh(x) + \frac{h^2(x)}{2}\right]^2 \\ &= m^2h^2 + \frac{m^2h^2}{2v^2}\left(\sqrt{2}vh + \frac{h^2}{4}\right) \end{aligned} \tag{9.23}$$

Using Eq. (9.22) together with the definition of the covariant derivative in Eq. (9.18), we get

$$\begin{aligned} D^\mu\varphi' &= (\partial^\mu + iqA'^\mu)\left(v + \frac{h}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}}\partial^\mu h + iqvA'^\mu + \frac{iqh}{\sqrt{2}}A'^\mu \end{aligned}$$

Similarly, we find that

$$\begin{aligned} D_\mu\varphi' &= (\partial_\mu - iqA')\left(v + \frac{h}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}}\partial_\mu h - iqvA'_\mu - \frac{iqh}{\sqrt{2}}A'_\mu \end{aligned}$$

Therefore,

$$\begin{aligned} D_\mu\varphi' D^\mu\phi' &= \left(\frac{1}{\sqrt{2}}\partial_\mu h - iqvA'_\mu - \frac{iqh}{\sqrt{2}}A'_\mu\right)\left(\frac{1}{\sqrt{2}}\partial^\mu h + iqvA'^\mu + \frac{iqh}{\sqrt{2}}A'^\mu\right) \\ &= \frac{1}{2}\partial_\mu h\,\partial^\mu h + q^2v^2A'_\mu A'^\mu + \sqrt{2}q^2vhA'_\mu A'^\mu + \frac{q^2h}{2}A'_\mu A'^\mu \end{aligned}$$

We can put this together with the expression we obtained for the potential to obtain the full Lagrangian in the case of the gauge transformation in Eq. (9.22)

where we chose the field to be real. Let's drop the primes on the vector potential terms to simplify writing. We find that

$$\begin{aligned} L &= D_\mu \varphi^\dagger D^\mu \varphi - V(\varphi^\dagger \varphi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= \frac{1}{2} \partial_\mu h \partial^\mu h + q^2 v^2 A'_\mu A'^\mu + \sqrt{2} q^2 v h A'_\mu A'^\mu + \frac{q^2 h}{2} A'_\mu A'^\mu \\ &\quad - m^2 h^2 - \frac{m^2 h^2}{2v^2} \left(\sqrt{2} v h + \frac{h^2}{4} \right) \end{aligned}$$

This Lagrangian has several components. The first part we will take a look at is the free part of the Lagrangian involving the Higgs field $h(x)$. This is

$$L^h_{\text{free}} = \frac{1}{2} \partial_\mu h \partial^\mu h - m^2 h^2 \tag{9.24}$$

By now, this should look very familiar. It's a Klein-Gordon equation type Lagrangian for a scalar field $h(x)$ with mass

$$\sqrt{2}\, m \tag{9.25}$$

So in the example we've done here, the Higgs field is a scalar field, a spin-0 boson with mass $\sqrt{2}m$. Now let's look at some of the other terms. Next we have a free Lagrangian for the gauge field. This is given by

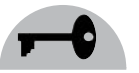
$$L^B_{\text{free}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + q^2 v^2 A'_\mu A'^\mu$$

This is a remarkable result. The kinetic term $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ was present in the original Lagrangian in Eq. (9.19), but before symmetry breaking the gauge field was massless. By choosing a real field, that is, a perturbation about the unbroken vacuum v, we have picked up a mass term which is given by

$$q^2 v^2 A'_\mu A'^\mu$$

We can determine the mass by comparison with a mass term that would appear in a Klein-Gordon type Lagrangian

$$\frac{1}{2} M^2 \phi^2$$



for some field ϕ. Hence, comparing the two terms, we see that symmetry breaking has given rise to a vector boson with mass

$$M = \sqrt{2}\, q\, v \tag{9.26}$$

The remaining terms in the Lagrangian are interaction terms. The first term we can write down includes the self-interaction terms for the Higgs field as shown here.

$$L_{\text{int}}^{h} = -\frac{m^2 h^2}{2v^2}\left(\sqrt{2}vh + \frac{h^2}{4}\right)$$

And finally, there is an interaction Lagrangian representing coupling between the Higgs field h and the gauge field A_μ. This is

$$L_{\text{int}}^{\text{coup}} = q^2 A'_\mu A'^{\mu}\left(\sqrt{2}vh + \frac{1}{2}h^2\right)$$

Summary

In this chapter we learned about spontaneous symmetry breaking, a process which leads to the appearance of massive particles in the Lagrangian. The procedure works by considering a Lagrangian with some vacuum state. The system is then reconsidered by breaking the symmetry, leading us to a new vacuum state. Gauge invariance leads to the appearance of new particles. For a scalar theory, a massive Goldstone boson appears. When we combine a complex scalar theory with a massless gauge field, breaking the symmetry by forcing the field to be a real fluctuation about the unbroken vacuum v leads to the appearance of a massive scalar field called the Higgs field, and the gauge field acquires mass. The Higgs field and the gauge field are coupled through an interaction Lagrangian.

Quiz

1. Suppose that $L = \frac{1}{2}(\partial_\mu \varphi)^2 + \cosh(b\varphi)$. Does this describe a massive or massless particle?
2. A mass term appears in the Lagrangian
   (a) As a squared scalar multiplying the field
   (b) As a scalar term multiplying the field squared

(c) As a scalar term multiplying the field to the fourth power

(d) Must be put in by hand

Consider a Lagrangian with a potential given by

$$V = \frac{\lambda}{4}\left(\varphi^*\varphi - \frac{\mu^2}{\lambda}\right)^2$$

3. Let $\varphi \to \psi(x)e^{i\theta(x)}$. Write down the form of the Lagrangian.
4. Identify the mass term in the resulting Lagrangian.
5. Is there a self-interaction term?

CHAPTER 10

Electroweak Theory

In this chapter we will explore the electroweak part of the standard model of particle physics, which unifies the electromagnetic and weak interactions. The gauge group that does this is

$$SU(2) \otimes U(1)$$

The weak interactions are mediated by the $SU(2)$ gauge bosons, which includes the charged $W^{\pm}$ and the neutral Z^0. The $U(1)$ sector of the interaction is the electromagnetic interaction, which is mediated by the massless photon. The theory that describes the electroweak interaction is known as the *Weinberg-Salam* model, after the two codiscoverers of the theory. They shared the Nobel Prize with Sheldon Glashow in 1979 for the development of this theory and their prediction of the $W^{\pm}$ and Z^0 masses.

The Higgs field is introduced into the model causing spontaneous symmetry breaking. This leads the electron and its heavy partners, the muon (m) and the tau (τ),

to acquire mass. In addition, the gauge bosons $W^{\pm}$ and Z^0 acquire mass, but the photon remains massless. So far so good—the results are in good agreement with experiment. However, the Weinberg-Salam model also predicts that neutrinos are massless. Recent experimental evidence indicates that while their mass may be small (<1 *eV*), neutrinos probably *do* have mass. This problem is currently one of the great outstanding problems[1] in particle physics, and solving the neutrino mass problem may lead to new physics beyond the standard model.

In this chapter, we will focus on the electroweak interactions of leptons, and will leave out hadron interactions.

Right- and Left-Handed Spinors

Let's briefly review the concept of left- and right-handed spinors. We write a Dirac spinor as a two-component object, with the top component being the right-handed spinor and the lower component being the left-handed spinor.

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{10.1}$$

Each component, ψ_R and ψ_L, is itself a two-component object. We can pick out the left- and right-handed components of a Dirac field ψ by using an operator composed of the identity and the γ_5 matrix. Refreshing our memory, the γ_5 matrix is a 4×4 matrix given by

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{10.2}$$

Now let's see how we can pick out the left- and right-handed components of ψ. First we write

$$\frac{1}{2}(1-\gamma_5) = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

[1]Neutrinos with mass could solve the long-standing solar neutrino deficit. The nuclear physics of our sun is well-understood and in agreement with measurement except for the number of neutrinos. Terrestrial measurements by different laboratories agree that the solar neutrino flux is one-third of what is expected, a problem that has been solved by neutrino oscillations.

Hence,

$$\frac{1}{2}(1-\gamma_5)\psi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_L \end{pmatrix} = \psi_L$$

Similarly, we have

$$\frac{1}{2}(1+\gamma_5)\psi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} \psi_R \\ 0 \end{pmatrix} = \psi_R$$

Also notice that we can write the Dirac field as

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_L \end{pmatrix} + \begin{pmatrix} \psi_R \\ 0 \end{pmatrix} = \psi_L + \psi_R$$

A Massless Dirac Lagrangian

We begin with the standard Dirac Lagrangian, setting the mass term to 0. This gives

$$L = i\bar{\psi}\gamma^\mu \partial_\mu \psi \tag{10.3}$$

where as usual

$$\bar{\psi} = \psi^\dagger \gamma^0$$

We wish to split up the Lagrangian into two parts, one for the left-handed spinor and one for the right-handed spinor. This is actually very straightforward. Proceeding we have

$$\begin{aligned} L &= i\bar{\psi}\gamma^\mu \partial_\mu \psi \\ &= i(\bar{\psi}_L + \bar{\psi}_R)\gamma^\mu \partial_\mu (\psi_L + \psi_R) \\ &= i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R + i(\bar{\psi}_L \gamma^\mu \partial_\mu \psi_R + \bar{\psi}_R \gamma^\mu \partial_\mu \psi_L) \end{aligned}$$

The last term actually vanishes. This is because

$$\begin{aligned}\bar{\psi}_L \gamma^\mu \partial_\mu \psi_R &= \frac{1}{2}\left(\frac{1-\gamma_5}{2}\right)\bar{\psi}\gamma^\mu \partial_\mu \left(\frac{1+\gamma_5}{2}\right)\psi \\ &= \frac{1}{4}\left(1-\gamma_5+\gamma_5-\gamma_5^2\right)\bar{\psi}\gamma^\mu \partial_\mu \psi \\ &= \frac{1}{4}\left(1-\gamma_5^2\right)\bar{\psi}\gamma^\mu \partial_\mu \psi\end{aligned}$$

But since

$$\gamma_5^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = I$$

the mixed terms vanish. So we are left with

$$L = i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R \tag{10.4}$$

And the Lagrangian separates nicely into left- and right-handed parts. In the case of electroweak theory, there is an asymmetry between left- and right-handed weak interactions. As a result, the actual Lagrangian used will reflect this.

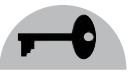
Leptonic Fields of the Electroweak Interactions

For reasons as of yet unclear, fundamental particles belong to one of three families. The distinction is one of mass; otherwise particles within a family behave in a similar fashion (have the same charge and spin, for example). When considering just leptons, the fields of the electroweak interaction consists of the electron (e), muon (μ), and tau (τ), together with their corresponding neutrinos. These are the three families of leptons. In short we can write

$$L = L_e + L_\mu + L_\tau$$

However, the muon and tau are just heavier duplicates of the electron, so we can learn everything we need to about electroweak theory by just focusing on the electron.[2]

[2]So what are the two other families? Mysteries like this are driving physicists to look at options like string theory.



The electron field and its associated neutrino field are combined together into a two-component object. Considering left-handed components only, we have the left-handed spinor

$$\psi_L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \tag{10.5}$$

where ν_e is the electron neutrino and e_L is a left-handed electron field. The left- and right-handed electron fields are related to the electron e in the usual way.

$$e_L = \left(\frac{1-\gamma_5}{2}\right)e \qquad e_R = \left(\frac{1+\gamma_5}{2}\right)e \tag{10.6}$$

If we take the mass of the neutrino to be 0, that is, $m_{\nu_e} = 0$, then there is only a left-handed component of the neutrino field. Since the field is entirely left-handed, it satisfies the equation

$$\left(\frac{1-\gamma_5}{2}\right)\nu_e = \nu_e \tag{10.7}$$

With no right-handed component of the neutrino field, we can define

$$\psi_R = \begin{pmatrix} 0 \\ e_R \end{pmatrix} \tag{10.8}$$

When considering only electrons, the Lagrangian describing the Dirac fields of the electroweak interaction can therefore be written in the form stated above, that is,

$$L = i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R$$

If we wanted to consider the full theory for leptons, we would simply add terms for the muon and tau, which would be identical in form to that of the electron.

Charges of the Electroweak Interaction

Charged current interactions work as follows. Charged currents couple to left-handed particles and to right-handed antiparticles. In electroweak theory, there exist three types of charge.

Description	Label
Electric Charge	Q
Weak Isospin	I
Weak Hypercharge	Y

These charges are related by the *Gell-Mann-Nishijima relation*

$$Q = I^3 + \frac{Y}{2} \tag{10.9}$$

where I^3 is the third component of weak isospin. The neutrino is assigned an isospin of

$$I_v^3 = +\frac{1}{2}$$

while a left-handed electron has

$$I_e^3 = -\frac{1}{2}$$

The right-handed electron field has

$$I_R = 0$$

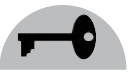
For a left-handed spinor, $Y = -1$, while for a right-handed spinor, $Y = -2$. Hence overall the charges for left- and right-handed spinors in electroweak theory area are

$$\begin{aligned} &I_L^3 = +\frac{1}{2} \qquad Y_L = -1 \qquad Q_L = 0 \text{ (neutrino)} \\ &I_L^3 = -\frac{1}{2} \qquad Y_L = -1 \qquad Q_L = -1 \text{ (electron—left handed)} \\ &I_R^3 = 0 \qquad Y_L = -2 \qquad Q_R = -1 \text{ (electron—right handed)} \end{aligned} \tag{10.10}$$

The weak hypercharge Y and weak isospin charge I are independent, hence

$$[Y, \vec{I}] = 0$$

In the next section we will see that there are four gauge fields denoted by W_μ^1, W_μ^2, W_μ^3, and B_μ that correspond to the weak isospin charge I and weak hypercharge Y, respectively. If a particle has a given type of charge, then it can interact with the



field associated with that charge, and the value of the charge determines the strength of that interaction. Since the neutrino participates in weak interactions but does not interact with the photon, it has quantum numbers $I_L^3 = +\frac{1}{2}, Y_L = -1$, and $Q_L = 0$. A left-handed electron participates in the weak interaction and interacts with the photon and accordingly has all nonzero charges.

A right-handed electron is a little different. It has $I_R^3 = 0$, $Y_R = -2$, and $Q_R = -1$. As we will see below, the isospin charge allows interactions with the gauge bosons W_μ. A right-handed electron does not interact with the gauge field. It will, however, interact with the B_μ field, and in fact does so with twice the strength of a left-handed electron. The right-handed electron does interact with the photon since $Q_R = -1$.

Unitary Transformations and the Gauge Fields of the Theory

Next we consider the possible symmetries of the theory and proceed to introduce the gauge bosons. As mentioned above, electroweak theory has two independent symmetries $SU(2)$ and $U(1)$; we call the combination

$$SU(2) \otimes U(1)$$

The $SU(2)$ symmetry leads to three gauge bosons as mentioned here.

$$SU(2): W_\mu^1, W_\mu^2, W_\mu^3$$

The conservation of the weak hypercharge Y actually arises from invariance under a $U(1)$ transformation. Hence, there is an additional gauge field associated with $U(1)$ invariance. We denote this field as B_μ. Summarizing:

$$U(1): B_\mu$$

After we introduce the gauge fields, the Lagrangian will be expanded as

$$L = L_{\text{leptons}} + L_{\text{gauge}}$$

Let's take a look at the $U(1)$ transformation first. It is clear by looking at the Lagrangian that it is invariant under a standard $U(1)$ transformation on the right-handed spinor

$$\psi_R \rightarrow \psi_R' = e^{i\beta}\psi_R$$

where β is a scalar. This transformation does not change the Lagrangian

$$L \to L' = \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R e^{-i\beta} \gamma^\mu \partial_\mu e^{i\beta} \psi_R + i\bar{L}\gamma^\mu \partial_\mu L$$
$$= i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R = L$$

So clearly the Lagrangian is invariant under $\psi_R \to \psi_R' = e^{i\beta}\psi_R$. Naively, one would expect the $U(1)$ transformation to be the same for the left-handed field, $\psi_L \to \psi_L' = e^{i\beta}\psi_L$, but it is not. We already know that the left- and right-handed fields have different weak hypercharge, so we expect them to transform differently. The correct transformation for the left-handed field is of the form

$$\psi_L \to \psi_L' = e^{in\beta}\psi_L$$

Since $Y_R = -2$, but $Y_L = -1$, the left-handed field interacts at half the strength, so $n = \frac{1}{2}$ and the correct transformation is

$$\psi_L \to \psi_L' = e^{i\beta/2}\psi_L$$

We can arrange the neutrino, left- and right-handed electrons into a single object as

$$\begin{pmatrix} \nu_e \\ e_L \\ e_R \end{pmatrix}$$

Then the $U(1)$ transformation can be written in nice matrix form as

$$\begin{pmatrix} \nu_e \\ e_L \\ e_R \end{pmatrix} \to \begin{pmatrix} \nu_e' \\ e_L' \\ e_R' \end{pmatrix} = \begin{pmatrix} e^{i\beta/2} & 0 & 0 \\ 0 & e^{i\beta/2} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} \nu_e \\ e_L \\ e_R \end{pmatrix} \tag{10.11}$$

Now, given a gauge field B_μ, we define the field strength tensor $f_{\mu\nu}$ where

$$f_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{10.12}$$

Hence, the gauge field B_μ is included in the Lagrangian with the addition of the term

$$L_B = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} \tag{10.13}$$

We want to introduce this term but preserve of the action under variation $\delta S = 0$. Once more the gauge field B_μ forces us to introduce extra terms into the derivative in order to maintain covariance. This is possible if we take

$$\partial_\mu \to \partial_\mu + \frac{ig_B}{2} B_\mu$$

where g_B is a coupling constant associated with the gauge field B_μ. In a similar fashion, we will define a field strength tensor associated with the gauge fields W_μ^1, W_μ^2, and W_μ^3. First let's consider $SU(2)$ transformations.

We now reintroduce the Pauli matrices in anticipation of including $SU(2)$ as

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.14}$$

Since these are just the Pauli matrices, we have the $SU(2)$ algebra.

$$[\tau_i, \tau_j] = 2i\varepsilon_{ijk}\tau_k \tag{10.15}$$

The τ_i generators define weak isospin space. We now use them to consider a $SU(2)$ transformation of the form

$$U(\alpha) = \exp(i\alpha_j \tau_j / 2) \tag{10.16}$$

The right-handed electron spinor e_R is invariant under the $SU(2)$ transformation.

$$\psi_R \to \psi'_R = U(\alpha)\psi_R = \psi_R$$

However, the left-handed spinor transforms in the usual way as

$$\psi_L \to \psi'_L = e^{-i(\tau\cdot\alpha)/2}\psi_L$$

The reason for these transformation properties is that the right-handed electron e_R does not carry any weak isospin charge ($\vec{I}_R = I_R^3 = 0$) which is associated with the $SU(2)$ transformation—hence it does not couple to the W_μ^1, W_μ^2, and W_μ^3 fields.

The electron neutrino and left-handed electron state do carry isospin charge, so we need to apply the $SU(2)$ transformation in that case. In matrix form, the $SU(2)$ transformation can be written as

$$\begin{pmatrix} \nu_e \\ e_L \\ e_R \end{pmatrix} \rightarrow \begin{pmatrix} \nu_e' \\ e_L' \\ e_R' \end{pmatrix} = \begin{pmatrix} e^{-i(\tau\cdot\alpha)/2} & 0 & 0 \\ 0 & e^{-i(\tau\cdot\alpha)/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_e \\ e_L \\ e_R \end{pmatrix} \tag{10.17}$$

Now let's consider the field strength tensor corresponding to the gauge fields W_μ^1, W_μ^2, and W_μ^3. It assumes the form

$$F_{\mu\nu}^{\ell} = \partial_\mu W_\nu^{\ell} - \partial_\nu W_\mu^{\ell} - g_W \varepsilon^{\ell mn} W_\mu^m W_\nu^n \tag{10.18}$$

We add the field tensor to the Lagrangian by summing up $F_{\mu\nu}^{\ell} F^{\ell,\mu\nu}$ over $\ell = 1,2,3$ to include each of the gauge fields W_μ^1, W_μ^2, and W_μ^3. That is, we take the trace and can write the contribution to the Lagrangian as

$$L_W = -\frac{1}{8} Tr(F_{\mu\nu} F^{\mu\nu}) \tag{10.19}$$

To keep the derivative covariant, now we need to add an extra term to account for the presence of Eq. (10.18) in the Lagrangian. This is done by adding the following term to the derivative:

$$ig_W \frac{\vec{\tau}}{2} \cdot \vec{W}_\mu = \frac{ig_W}{2} \left(\tau_1 W_\mu^1 + \tau_2 W_\mu^2 + \tau_3 W_\mu^3 \right)$$

Since the right-handed lepton field does not participate in the interaction involving weak isospin, this term is not added to the derivative in that case. We only add the term to account for the presence of the gauge field B_μ. Hence,

$$i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R \rightarrow i\bar{\psi}_R \gamma^\mu \left(\partial_\mu + \frac{ig_B}{2} B_\mu \right) \psi_R$$

For the left-handed field, we have

$$i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L \rightarrow i\bar{\psi}_L \gamma^\mu \left(\partial_\mu + \frac{ig_B}{2} B_\mu + ig_W \frac{\vec{\tau}}{2} \cdot \vec{W}_\mu \right) \psi_L$$

So the total leptonic portion of the Lagrangian is

$$L_{\text{Lepton}} = i\bar{\psi}_R\gamma^\mu\left(\partial_\mu + \frac{ig_B}{2}B_\mu\right)\psi_R + i\bar{\psi}_L\gamma^\mu\left(\partial_\mu + \frac{ig_B}{2}B_\mu + ig_W\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu\right)\psi_L$$

The total Lagrangian includes the gauge field Lagrangians as

$$\begin{aligned} L &= L_{\text{lepton}} + L_{\text{gauge}} \\ &= i\bar{\psi}_R\gamma^\mu\left(\partial_\mu + \frac{ig_B}{2}B_\mu\right)\psi_R + i\bar{\psi}_L\gamma^\mu\left(\partial_\mu + \frac{ig_B}{2}B_\mu + ig_W\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu\right)\psi_L \\ &\quad -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{8}Tr\left(F_{\mu\nu}F^{\mu\nu}\right) \end{aligned} \tag{10.20}$$

When you step back and look at the Lagrangian, the asymmetry between the chiral fields strikes a note of discord. While the Standard Model has been a great success, it now leaves theorists with many questions that will need to be solved with physics beyond the Standard Model.

Weak Mixing or Weinberg Angle

In the following sections we will see that it is convenient to relate the coupling constants g_B and g_W in the following way:

$$\tan\theta_W = \frac{g_W}{g_B} \tag{10.21}$$

The angle θ_W is often called the *Weinberg angle.* It is also sometimes called the *weak mixing angle.* The reason is that it mixes the gauge fields to give

$$\begin{aligned} A_\mu &= B_\mu\cos\theta_W + W_\mu^3\sin\theta_W \\ Z_\mu &= -B_\mu\sin\theta_W + W_\mu^3\cos\theta_W \end{aligned} \tag{10.22}$$

We can view this as a rotation, writing the relationship in matrix form:

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = R(\theta_W)\begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}$$

The gauge field A_μ is nothing other than the electromagnetic vector potential which couples the photon to the theory. We will explore this in more detail after we introduce the Higgs mechanism.

Symmetry Breaking

At this point we have put together a theory describing two leptons, the electron and its corresponding neutrino, and four gauge bosons. All the particles described so far are massless. It's time to introduce mass into the theory and this will be done using the symmetry breaking methods of Chap. 9. We will introduce a Higgs field that will force the gauge bosons to acquire mass. In this section we will also introduce the photon field explicitly (besides talking about charge) so that we can present a unified picture of the electroweak interaction.

Following the example of Chap. 9, we introduce the Higgs field as a scalar (spin-0) field. However this time it is a two-component object as shown here.

$$\varphi = \begin{pmatrix} \varphi^A \\ \varphi^B \end{pmatrix} \tag{10.23}$$

Each component is a complex scalar field and is written as

$$\begin{aligned} \varphi^A &= \frac{\varphi_3 + i\varphi_4}{\sqrt{2}} \\ \varphi^B &= \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \end{aligned} \tag{10.24}$$

So the Higgs field is really composed of four real, scalar fields. We see that

$$\begin{aligned} \varphi^\dagger\varphi &= (\varphi^A)^\dagger\varphi^A + (\varphi^B)^\dagger\varphi^B \\ &= \frac{1}{2}\left(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2\right) \end{aligned}$$

The Higgs field carries charges of the weak interaction, specifically

$$Y_\varphi = +1 \qquad I_\varphi = 1/2 \tag{10.25}$$

The $SU(2)$ transformation is given in Eq. (10.16), which we restate here.

$$U(\alpha) = \exp\left(i\alpha_j\tau_j/2\right)$$

We have a gauge freedom that can be exploited to simplify the form of the Higgs field. This can be done by insisting that each of the "angles" are functions of space-time $\alpha_j = \alpha_j(x)$; this is more than whimsy. It is because we want a local symmetry.

Here is how the gauge freedom helps us. We can choose the $\alpha_j = \alpha_j(x)$ in such a way that

$$\varphi^A = 0$$

and

$$\varphi^B = \varphi_0 + \frac{h(x)}{\sqrt{2}}$$

This leaves us with a particularly simple and useful form of the Higgs field as

$$\varphi = \begin{pmatrix} 0 \\ \varphi_0 + \dfrac{h(x)}{\sqrt{2}} \end{pmatrix} \tag{10.26}$$

The parameter φ_0 and field $h(x)$ are both real. The φ_0 parameter allows us to break the symmetry. Instead of taking the ground state to be $\varphi \to 0$, we take it to be

$$\varphi_G = \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix}$$

The Higgs field will make its appearance in the Lagrangian Eq. (10.20) in several ways. It will do so through a potential $V = V(\varphi^\dagger \varphi)$, kinetic energy terms $D_\mu \varphi D^\mu \varphi$, and via interaction terms that couple the Higgs field to the electron and gauge bosons, giving them mass. Let's use Eq. (10.26) and write the form of the potential. First

$$\varphi^\dagger \varphi = \begin{pmatrix} 0 & \varphi_0 + \dfrac{h}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ \dfrac{h}{\sqrt{2}} \end{pmatrix}$$

$$= \varphi_0^2 + \sqrt{2}\varphi_0 h + \frac{1}{2}h^2$$

The potential is

$$V(\varphi^\dagger \varphi) = \mu^2 \varphi^\dagger \varphi + \lambda(\varphi^\dagger \varphi)^2 \tag{10.27}$$

If $\mu^2 > 0$, then the minimum of the potential is located at $\varphi = 0$. Following the procedure of Chap. 9, we take $\mu^2 < 0$ to break the symmetry, since this causes the Higgs field to attain a vacuum expectation value.

What is the new minimum? As usual, we apply elementary calculus to get

$$\frac{\partial V}{\partial \varphi} = \mu^2 \varphi^\dagger + 2\lambda(\varphi^\dagger \varphi) = 0$$

That is,

$$\varphi_{\min} = -\frac{\mu^2}{2\lambda}$$

Defining

$$\varphi_0{}^2 = -\frac{\mu^2}{2\lambda}$$

We arrive at the result $\varphi_G = \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix}$. The Higgs field has a mass given by

$$m_h = \sqrt{-2\mu^2} \tag{10.28}$$

A review of Chap. 9 will help clarify this result.

Giving Mass to the Lepton Fields

A Dirac Lagrangian consisting of left- and right-handed fields with mass m is written as

$$L = i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R - m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \tag{10.29}$$

Hence a mass term in the Lagrangian is of the form

$$-m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \tag{10.30}$$

where m is a scalar (a number, not a function dependent on spacetime). In Weinberg-Salam theory, an interaction term (known as the *Yukawa* term) is introduced that couples the matter fields to the Higgs field. The Yukawa coupling G_e

which defines the strength of the interaction between the Higgs field and the electron-lepton fields. The interaction Lagrangian is

$$L_{\text{int}} = -G_e(\bar{\psi}_L \varphi \psi_R + \bar{\psi}_R \varphi^{\dagger} \psi_L) \tag{10.31}$$

Let's look at each term.

$$\bar{\psi}_L \varphi = (\bar{\nu}_e \quad \bar{e}_L)\begin{pmatrix} \varphi^A \\ \varphi^B \end{pmatrix} = \bar{\nu}_e \varphi^A + \bar{e}_L \varphi^B$$

Now, using the gauge choice that led to the form of the Higgs field in Eq. (10.26), the neutrino term drops out—a key step—and we have

$$\bar{\psi}_L \varphi = (\bar{\nu}_e \quad \bar{e}_L)\begin{pmatrix} \varphi^A \\ \varphi^B \end{pmatrix} = \bar{\nu}_e \varphi^A + \bar{e}_L \varphi^B = \bar{e}_L \left(\varphi_0 + \frac{h(x)}{\sqrt{2}} \right)$$

Using $\psi_R = \begin{pmatrix} 0 \\ e_R \end{pmatrix}$ we have

$$\bar{\psi}_L \varphi \psi_R = \bar{e}_L \left(\varphi_0 + \frac{h(x)}{\sqrt{2}} \right) e_R \tag{10.32}$$

For the next term in Eq. (10.31), we have

$$\begin{aligned} \bar{\psi}_R \varphi^{\dagger} \psi_L &= \begin{pmatrix} 0 & \bar{e}_R \end{pmatrix} \begin{pmatrix} 0 & \varphi_0 + \dfrac{h(x)}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \\ &= (0 \quad \bar{e}_R) \left(\varphi_0 + \frac{h(x)}{\sqrt{2}} \right) e_L \\ &= \bar{e}_R \left(\varphi_0 + \frac{h(x)}{\sqrt{2}} \right) e_L \end{aligned}$$

Once again, the neutrino term drops out. Using these results the interaction Lagrangian becomes

$$\begin{aligned} L_{\text{int}} &= -G_e(\bar{\psi}_L \varphi \psi_R + \bar{\psi}_R \varphi^{\dagger} \psi_L) \\ &= -G_e \left[\bar{e}_L \left(\varphi_0 + \frac{h(x)}{\sqrt{2}} \right) e_R + \bar{e}_R \left(\varphi_0 + \frac{h(x)}{\sqrt{2}} \right) e_L \right] \\ &= -G_e \varphi_0 \left(\bar{e}_L e_R + \bar{e}_R e_L \right) - G_e \frac{h(x)}{\sqrt{2}} \left(\bar{e}_L e_R + \bar{e}_R e_L \right) \end{aligned}$$

Looking for a mass, we ignore the second term because a mass term in the Lagrangian is multiplied by an overall scalar (a number). The second term is multiplied by the Higgs field $h(x)$. So the mass term in the Lagrangian is

$$L_{\text{mass}} = -G_e\varphi_0(\overline{e}_L e_R + \overline{e}_R e_L) \tag{10.33}$$

Comparison with Eq. (10.30) leads us to the mass of the electron

$$m = G_e\varphi_0 \tag{10.34}$$

The conclusion is that the interaction of the Higgs field with the electron field leaves the neutrino massless and gives the electron the mass defined in Eq. (10.34). So while these are instructive results, they do not seem to completely reflect what we see in nature as the neutrinos described here have no mass. As noted earlier experimental results indicate that neutrinos have a small, but nonzero mass. This indicates that the theory is incomplete.

Gauge Masses

Now we will see how the Higgs mechanism gives mass to the gauge bosons. The gauge bosons will acquire a mass through the action of the covariant derivative on the Higgs field. First let's set up some notation that is frequently used. The fields W_μ^1 and W_μ^2 are electrically charged, and can be combined into the physical fields as shown here.

$$W_\mu^+ = \frac{W_\mu^1 - iW_\mu^2}{\sqrt{2}} \tag{10.35}$$

$$W_\mu^- = \frac{W_\mu^1 + iW_\mu^2}{\sqrt{2}} \tag{10.36}$$

We have a covariant derivative given by

$$D_\mu = \partial_\mu + i\frac{g_B}{2} + i\frac{g_W}{2}\vec{\tau}\cdot\vec{W}_\mu \tag{10.37}$$

Now we apply the covariant derivative to the Higgs field

$$D_\mu\varphi = \partial_\mu\varphi + i\frac{g_B}{2}\varphi + i\frac{g_W}{2}\vec{\tau}\cdot\vec{W}_\mu\varphi \tag{10.38}$$

Notice that

$$\begin{aligned}\vec{\tau}\cdot\vec{W}_\mu &= \tau_1 W_\mu^1 + \tau_2 W_\mu^2 + \tau_3 W_\mu^3 \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W_\mu^1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} W_\mu^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_\mu^3 \\ &= \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\end{aligned}$$

So

$$i\frac{g_W}{2}\vec{\tau}\cdot\vec{W}_\mu\varphi = i\frac{g_W}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\begin{pmatrix} 0 \\ \varphi_0 + \dfrac{h(x)}{\sqrt{2}} \end{pmatrix}$$

$$= i\frac{g_W}{2}\begin{pmatrix} \sqrt{2}W_\mu^+\left(\varphi_0 + \dfrac{h(x)}{\sqrt{2}}\right) \\ -W_\mu^3\left(\varphi_0 + \dfrac{h(x)}{\sqrt{2}}\right) \end{pmatrix}$$

The first term in Eq. (10.37) just gives the ordinary derivative of the Higgs field.

$$\partial_\mu\begin{pmatrix} 0 \\ \left(\varphi_0 + \dfrac{h(x)}{\sqrt{2}}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{1}{\sqrt{2}}\partial_\mu h \end{pmatrix}$$

This term is the kinetic energy term and does not contribute to the generation of the boson masses, so we won't worry about it. But putting everything together we have

$$D_\mu\varphi = \begin{pmatrix} 0 \\ \dfrac{1}{\sqrt{2}}\partial_\mu h \end{pmatrix} + i\frac{g_W}{2}\begin{pmatrix} \sqrt{2}W_\mu^+\left(\varphi_0 + \dfrac{h(x)}{\sqrt{2}}\right) \\ -W_\mu^3\left(\varphi_0 + \dfrac{h(x)}{\sqrt{2}}\right) \end{pmatrix} + i\frac{g_B}{2}B_\mu\begin{pmatrix} 0 \\ \varphi_0 + \dfrac{h(x)}{\sqrt{2}} \end{pmatrix}$$

To find the mass terms, we calculate $(D_\mu\varphi)^\dagger D_\mu\varphi$ and only keep terms of quadratic order. Now

$$(D_\mu\varphi)^\dagger = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\partial_\mu h \end{pmatrix} - i\frac{g_W}{2}\left(\sqrt{2}W_\mu^-\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right) - W_\mu^3\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)\right)$$
$$- i\frac{g_B}{2}B_\mu\begin{pmatrix} 0 & \varphi_0+\frac{h(x)}{\sqrt{2}} \end{pmatrix}$$

The first term in the product $(D_\mu\varphi)^\dagger D_\mu\varphi$ is

$$\frac{1}{2}(\partial_\mu h)^2$$

The next term we get is

$$-i\frac{g_W}{2}\left(\sqrt{2}W_\mu^-\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right) - W_\mu^3\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)\right) i\frac{g_W}{2}\begin{pmatrix} \sqrt{2}W^{+,\mu}\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right) \\ -W^{3,\mu}\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)\end{pmatrix}$$
$$= \frac{{g_W}^2}{2}W_\mu^- W^{+,\mu}\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)^2 + \frac{{g_W}^2}{4}W_\mu^3 W^{3,\mu}\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)^2$$

And then we have a cross term

$$-i\frac{g_W}{2}\left(\sqrt{2}W_\mu^-\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right) - W_\mu^3\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)\right) i\frac{g_B}{2}B^\mu\begin{pmatrix} 0 \\ \varphi_0+\frac{h(x)}{\sqrt{2}}\end{pmatrix}$$
$$= -\frac{g_W g_B}{4}W_\mu^3 B^\mu\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)^2$$

The second cross term is

$$-i\frac{g_B}{2}B_\mu\begin{pmatrix} 0 & \varphi_0+\frac{h(x)}{\sqrt{2}}\end{pmatrix} i\frac{g_W}{2}\begin{pmatrix} \sqrt{2}W^{+,\mu}\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right) \\ -W^{3,\mu}\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)\end{pmatrix}$$
$$= -\frac{g_B g_W}{4}B_\mu W^{3,\mu}\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)^2$$

The final quadratic term of interest is

$$-i\frac{g_B}{2}B_\mu\begin{pmatrix}0 & \varphi_0+\frac{h(x)}{\sqrt{2}}\end{pmatrix}i\frac{g_B}{2}B^\mu\begin{pmatrix}0\\ \varphi_0+\frac{h(x)}{\sqrt{2}}\end{pmatrix}$$

$$=\frac{{g_B}^2}{4}B_\mu B^\mu\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)^2$$

In the last section, we noted that mass terms will be multiplied by numbers. So we can ignore any terms that include $h(x)$. For the boson fields we are looking for terms of the form $m^2 A_\mu A^\mu$.

We wish to write these expressions in terms of the physical fields. Using the Weinberg angle, we can invert Eq. (10.22) to give

$$\begin{pmatrix}B_\mu\\ W_\mu^3\end{pmatrix}=R(-\theta_W)\begin{pmatrix}A_\mu\\ Z_\mu\end{pmatrix}=\begin{pmatrix}\cos\theta_W & -\sin\theta_W\\ \sin\theta_W & \cos\theta_W\end{pmatrix}\begin{pmatrix}A_\mu\\ Z_\mu\end{pmatrix}$$

$$B_\mu = A_\mu\cos\theta_W - Z_\mu\sin\theta_W \tag{10.39}$$

$$W_\mu^3 = A_\mu\sin\theta_W + Z_\mu\cos\theta_W \tag{10.40}$$

Keeping only the leading term in $\left(\varphi_0+\frac{h(x)}{\sqrt{2}}\right)^2$, which is the only term that gives a scalar (number), we find

$$\begin{aligned}&\frac{{g_W}^2}{2}W_\mu^- W^{+,\mu}{\varphi_0}^2+\frac{{g_W}^2}{4}W_\mu^3 W^{3,\mu}{\varphi_0}^2\\ &=\frac{{g_W}^2}{2}W_\mu^- W^{+,\mu}{\varphi_0}^2+\frac{{g_W}^2}{4}(A_\mu\sin\theta_W+Z_\mu\cos\theta_W)\\ &\quad\times(A^\mu\sin\theta_W+Z^\mu\cos\theta_W){\varphi_0}^2\\ &=\frac{{g_W}^2}{2}W_\mu^- W^{+,\mu}{\varphi_0}^2+\frac{{g_W}^2}{4}(A_\mu A^\mu\sin^2\theta_W+A_\mu Z^\mu\sin\theta_W\cos\theta_W\\ &\quad+Z_\mu A^\mu\sin\theta_W\cos\theta_W+Z_\mu Z^\mu\cos^2\theta_W){\varphi_0}^2\end{aligned} \tag{10.41}$$

The next term becomes

$$
\begin{aligned}
&-\frac{g_W g_B}{4} W_\mu^3 B^\mu \varphi_0{}^2 \\
&= -\frac{g_W g_B}{4} (A_\mu \sin\theta_W + Z_\mu \cos\theta_W)(A^\mu \cos\theta_W - Z^\mu \sin\theta_W)\varphi_0{}^2 \\
&= -\frac{g_W g_B}{4} (A_\mu A^\mu \sin\theta_W \cos\theta_W - A_\mu Z^\mu \sin^2\theta_W \\
&\quad + Z_\mu A^\mu \cos^2\theta_W - Z_\mu Z^\mu \sin\theta_W \cos\theta_W)\varphi_0{}^2
\end{aligned} \tag{10.42}
$$

The second cross term becomes

$$
\begin{aligned}
&-\frac{g_B g_W}{4} B_\mu W^{3,\mu} \varphi_0{}^2 \\
&= -\frac{g_B g_W}{4} (A_\mu \cos\theta_W - Z_\mu \sin\theta_W)(A^\mu \sin\theta_W + Z^\mu \cos\theta_W)\,\varphi_0{}^2 \\
&= -\frac{g_B g_W}{4} (A_\mu A^\mu \sin\theta_W \cos\theta_W + A_\mu Z^\mu \cos^2\theta_W \\
&\quad - Z_\mu A^\mu \sin^2\theta_W - Z_\mu Z^\mu \sin\theta_W \cos\theta_W)\varphi_0{}^2
\end{aligned} \tag{10.43}
$$

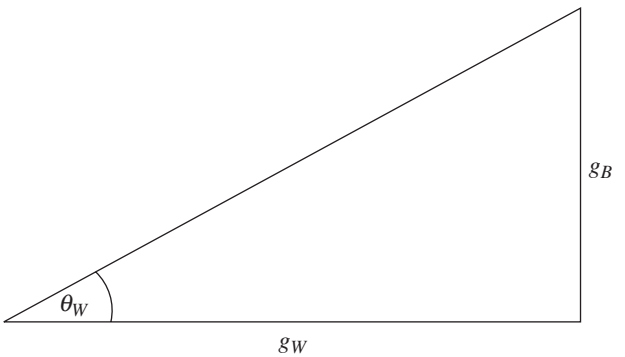
And the final term is

$$
\begin{aligned}
&\frac{g_B{}^2}{4} B_\mu B^\mu \varphi_0{}^2 \\
&= \frac{g_B{}^2}{4} (A_\mu \cos\theta_W - Z_\mu \sin\theta_W)(A^\mu \cos\theta_W - Z^\mu \sin\theta_W)\varphi_0{}^2 \\
&= \frac{g_B{}^2}{4} (A_\mu A^\mu \cos^2\theta_W - A_\mu Z^\mu \sin\theta_W \cos\theta_W \\
&\quad - Z_\mu A^\mu \sin\theta_W \cos\theta_W + Z_\mu Z^\mu \sin^2\theta_W)\varphi_0{}^2
\end{aligned} \tag{10.44}
$$

Now we add up Eqs. (10.41) through (10.44). We are looking for mass terms in the field, so we will ignore mixed terms that describe interactions like $A_\mu Z^\mu$. Let's look at each combination of the fields. We have

$$
\begin{aligned}
&A_\mu A^\mu \left[-\frac{g_B{}^2}{4}\cos^2\theta_W - \frac{g_B g_W}{4}\sin\theta_W \cos\theta_W - \frac{g_W g_B}{4}\sin\theta_W \cos\theta_W + \frac{g_W{}^2}{4}\sin^2\theta_W \right] \\
&= A_\mu A^\mu \left[\frac{g_B{}^2}{4}\cos^2\theta_W + \frac{g_W{}^2}{4}\sin^2\theta_W - \frac{g_B g_W}{2}\sin\theta_W \cos\theta_W \right]
\end{aligned}
$$

The mixing angle describes how the two forces mix. As you can see from the figure below, if $\theta_w = 0$, then we have pure coupling to the W bosons, and no coupling to the Z boson. A nonzero θ_w less than 90° indicates coupling to both fields (thus the term weak mixing angle) as shown in the following illustration.



From the diagram we see that

$$\cos\theta_W = \frac{g_W}{\sqrt{g_B^2 + g_W^2}} \qquad \sin\theta_W = \frac{g_B}{\sqrt{g_B^2 + g_W^2}} \tag{10.45}$$

and this becomes

$$A_\mu A^\mu \left[\frac{g_B{}^2}{4}\cos^2\theta_W + \frac{g_W{}^2}{4}\sin^2\theta_W - \frac{g_B g_W}{2}\sin\theta_W\cos\theta_W \right]$$

$$= A_\mu A^\mu \left[\frac{g_B{}^2}{4}\frac{g_W^2}{\left(g_B^2 + g_W^2\right)} + \frac{g_W{}^2}{4}\frac{g_B^2}{\left(g_B^2 + g_W^2\right)} - \frac{g_B g_W}{2}\frac{g_B}{\sqrt{g_B^2 + g_W^2}}\frac{g_W}{\sqrt{g_B^2 + g_W^2}} \right]$$

$$= 0$$

This tells us that A_μ is a massless field. In fact it couples to the electric charge, so we know this is our photon field. Now, for the Z field we get

$$Z_\mu Z^\mu \left[\frac{g_B{}^2}{4}\cos^2\theta_W + \frac{g_W{}^2}{4}\sin^2\theta_W + \frac{g_B g_W}{2}\sin\theta_W\cos\theta_W \right] {\varphi_0}^2$$

$$= Z_\mu Z^\mu \left[\frac{g_B{}^2}{4}\frac{g_W^2}{\left(g_B^2 + g_W^2\right)} + \frac{g_W{}^2}{4}\frac{g_B^2}{\left(g_B^2 + g_W^2\right)} + \frac{g_B g_W}{2}\frac{g_B}{\sqrt{g_B^2 + g_W^2}}\frac{g_W}{\sqrt{g_B^2 + g_W^2}} \right] {\varphi_0}^2$$

$$= Z_\mu Z^\mu \left(\frac{g_B^2 + g_W^2}{4} \right) {\varphi_0}^2$$

Therefore, the mass of the Z particle is

$$M_Z = \left(\frac{\sqrt{g_B^2 + g_W^2}}{2} \right) \varphi_0 \tag{10.46}$$

A similar exercise shows that the mass of the $W_\mu^\pm$ is

$$M_W = \frac{\varphi_0 g_W}{2} \tag{10.47}$$

We can also write

$$\sin^2 \theta_W = 1 - \left(\frac{M_W}{M_Z} \right)^2 \tag{10.48}$$

Theoretical bounds on the masses are

$$\begin{aligned} M_W &= \frac{\varphi_0 g_W}{2} = \frac{38}{\sin \theta_W} \text{ GeV } \geq 38 \text{ GeV} \\ M_Z &= \frac{\varphi_0 g_W}{2 \cos \theta_W} = \frac{76}{\sin \theta_W} \text{ GeV } \geq 76 \text{ GeV} \end{aligned} \tag{10.49}$$

Hence $M_W < M_Z$, which is born out in experiment. Measured masses indicate that

$$\sin^2 \theta_W = 0.222 \tag{10.50}$$

which tells us that θ_w is approximately 28.1 (which is the angle used to draw the figure).

The ability of the Weinberg-Salam model to predict the masses in this fashion is what earned its discoverers the Nobel Prize. Moreover, it gives us confidence in the Higgs mechanism as a valid means for adding mass to the standard model despite the fact that experimental observation as yet eludes us. However, at the time of writing solid experimental observation of the Higgs still eludes us. Researchers are

confident it will be seen when the Large Hadron Collider (LHC) begins operation in Switzerland sometime in 2008—but we will have to wait and see.

Summary

The Weinberg-Salam model combines leptons and gauge bosons into a single Lagrangian that has a $SU(2) \otimes U(1)$ symmetry. The original formulation of the Lagrangian is for massless particles, including a massless electron neutrino, massless electron, and four massless gauge fields. Spontaneous symmetry breaking using the Higgs mechanism breaks the $SU(2)$ symmetry and gives rise to masses for the electron and the gauge bosons $W^{\pm}$ and Z^0 that mediate the weak force. The symmetry breaking also introduces the photon field in the model, unifying the electromagnetic and weak interactions into a single Lagrangian.

Quiz

1. In electroweak theory, the neutrino has
   (a) No charge
   (b) $I_L^3 = +\frac{1}{2}, Y_L = -1, Q_L = 0$
   (c) $I_R^3 = 0, Y_L = -2, Q_R = -1$
   (d) $I_L^3 = -\frac{1}{2}, Y_L = -1, Q_L = -1$
2. In electroweak theory, the charges on the electron for a left-handed spinor are
   (a) $I_L^3 = -\frac{1}{2}, Y_L = -1, Q_L = -1$
   (b) $I_L^3 = \frac{1}{2}, Y_L = -1, Q_L = -1$
   (c) $I_R^3 = 0, Y_L = -2, Q_R = -1$
   (d) $I_L^3 = +\frac{1}{2}, Y_L = -1, Q_L = +1$
3. The introduction of the gauge field B_μ forces us to change the derivative operator as
   (a) $\partial_\mu \rightarrow \partial_\mu - ig_B \frac{Y}{2} B_\mu$
   (b) $\partial_\mu \rightarrow \partial_\mu + ig_B \frac{\vec{\tau}}{2} \cdot \vec{B}_\mu$
   (c) The derivative is unchanged
   (d) $\partial_\mu \rightarrow \partial_\mu + ig_B \frac{Y}{2} B_\mu$

4. In the weak interaction, invariance under a $U(1)$ transformation is associated with
   (a) The supercharge S_Y
   (b) The hypercharge Y
   (c) The weak isospin charge I
   (d) Coupling of the weak isospin charge to the weak hypercharge Y
5. Under a weak interaction $SU(2)$ transformation, the left-handed field transforms as
   (a) $\psi_L \rightarrow \psi'_L = e^{-i(\tau\cdot\alpha)/2}\psi_L = \psi_L$
   (b) $\psi_L \rightarrow \psi'_L = e^{-i(\tau\cdot\alpha)/2}\psi_L$
   (c) $\psi_L \rightarrow \psi_R$
   (d) There is no $SU(2)$ transformation in weak theory
6. The Weinberg angle
   (a) Describes scattering angles between leptons undergoing weak mediated collisions
   (b) Describes the ratio of the weak hypercharge to the weak isospin charge
   (c) Is related to the mixing of the coupling constants associated with the gauge fields of the weak theory
   (d) Describes the ratio of the weak isospin charge to the weak hypercharge
7. The mass of the W and Z particles are related as
   (a) $M_Z = \frac{M_W}{\cos\theta_W}$
   (b) $M_Z = \frac{M_W}{\sin\theta_W}$
   (c) Cannot be related directly
   (d) $M_Z = \frac{M_W}{\tan\theta_W}$
8. If we define the standard electron charge as measured by experiment as q, the charge on the electron predicted by electroweak theory
   (a) Is greater than or equal to the electron charge q
   (b) Is the same as the electron charge q
   (c) The charge cannot be predicted from the theory

CHAPTER 11

Path Integrals

One approach to quantum field theory, which is helpful in advanced contexts like string theory, is to use what is known as the *path integral* approach. A path integral is really a method for calculating amplitude for a quantum transition from one state to another. Our treatment will be very brief and introductory; for a detailed description see the text *Quantum Field Theory* by Lowell Brown.

Gaussian Integrals

It turns out that most calculations involving path integrals boil down to a simple looking integral known as a *Gaussian integral.* So before jumping into path integration directly, we will quickly summarize what a Gaussian integral is and how it is calculated.

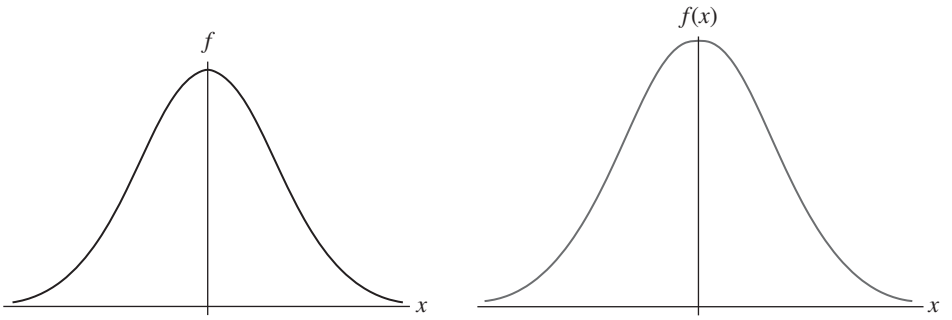


Figure 11.1 A plot of $f(x) = e^{-x^2}$.

The simplest Gaussian integral is an integral over all space in one dimension of the Gaussian function e^{-x^2}. A plot of this function, shown in Fig. 11.1, shows that it is localized in a small region about the origin.

As a result, we expect that the integral converges to a small finite value. Unfortunately, there is no elementary way to calculate

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

We must instead use a trick. The motivation is this: we can't evaluate this integral on the real line, but looking at the x^2 term, we want to see if we will have better luck in the plane using polar coordinates. First we square the integral. Now, since x is just a dummy variable when it appears in the integrand, we can call it something else—and we will opt for y for good reason. So we have

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$$

Next, we will change to polar coordinates. Recall that

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and so

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

In transforming to polar coordinates, the element of area changes as

$$dxdy \to rdrd\theta$$

This leads to

$$I^2 = \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r dr$$

and we have tamed the exponential term. The integral over r can be handled using a basic substitution $u = r^2$:

$$\int_0^{\infty} e^{-r^2} r dr = \frac{1}{2} \int_0^{\infty} e^{-u} du = \frac{1}{2}$$

so,

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r dr = \frac{1}{2} \int_0^{2\pi} d\theta = \frac{1}{2} 2\pi = \pi$$

Taking the square root we obtain the result

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{11.1}$$

The integral in Eq. (11.1) can be extended into more complicated situations. Start by considering multiplication by constant a

$$I' = \int_{-\infty}^{\infty} e^{-ax^2} dx$$

Knowing the result of Eq. (11.1), this can be done using a substitution technique. We set $y = \sqrt{a}x$ and then

$$I' = \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{a}} \tag{11.2}$$

The next extension of the basic Gaussian integral in Eq. (11.1) is to add powers of x. For example,

$$I = \int_{-\infty}^{\infty} x^n e^{-ax^2} dx$$

If n is odd, it is easy to see that this is just 0. Let $n = 1$. A plot of $f(x) = xe^{-x^2}$ in Fig. 11.2 shows this is an odd function, so it must integrate to zero since we are integrating over the entire real line. The integration over $(-\infty, 0)$ exactly cancels the integration over $(0, \infty)$.

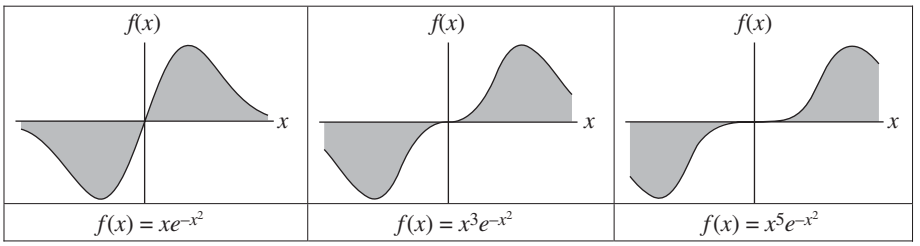


Figure 11.2 A plot of the first few odd powers on the exponential, all of which obviously integrate to zero.

Hence,

$$I = \int_{-\infty}^{\infty} x^n e^{-ax^2} dx = 0 \text{ for } n \text{ odd} \tag{11.3}$$

To obtain the result for even powers of n, we resort to more trickery. Start with Eq. (11.2), and take the derivative of both sides with respect to a. On the left we get

$$\frac{d}{da}\int_{-\infty}^{\infty} e^{-ax^2} dx = -\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx$$

On the right we get

$$\frac{d}{da}\sqrt{\frac{\pi}{a}} = -\frac{\sqrt{\pi}}{2a^{3/2}}$$

Equating the two, we've found that

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{2a^{3/2}}$$

A few plots of even powers of x against the exponential are illustrated in Figure 11.3.

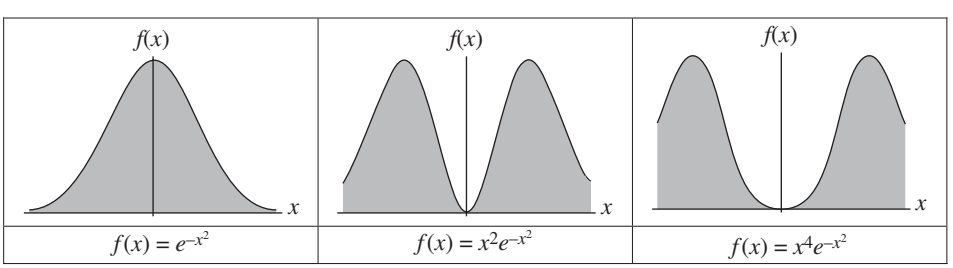
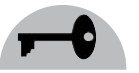


Figure 11.3 A plot of the first few even powers on the exponential. These integrals require some tinkering to compute.



This iteration procedure can be carried out ad infinitum to obtain more and more results. It turns out that for some general even n

$$\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = \frac{1\cdot 3\cdot 5\cdots(n+1)\sqrt{\pi}}{2^{n/2} a^{(n+1)/2}} \tag{11.4}$$

We can continue playing this game of introducing new terms into the Gaussian integral. One important Gaussian integral is

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx$$

This integral can be transformed into one of the form in Eq. (11.1) by completing the square. We write

$$-a\frac{x^2}{2} + bx = -\frac{a}{2}\left(x^2 - \frac{2b}{a}x\right) = -\frac{a}{2}\left(x - \frac{b}{a}\right)^2 + \frac{b^2}{2a}$$

Hence

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-ax^2+bx} dx &= \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x-\frac{b}{a}\right)^2} e^{b^2/2a} dx \\ &= e^{b^2/2a} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x-\frac{b}{a}\right)^2} dx \\ &= e^{b^2/2a} \sqrt{\frac{2\pi}{a}}\end{aligned} \tag{11.5}$$

In n dimensions, a Gaussian integral takes the form

$$\int e^{-x^T A x} d^n x$$

where A is an $n \times n$ matrix of coefficients, x is an n dimensional column vector, and $d^n x = dx_1 dx_2 \ldots dx_n$. Let's consider a two-dimensional case where

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad x = \begin{pmatrix} x \\ y \end{pmatrix}$$

The argument of the exponential reduces to a scalar

$$x^T A x = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x(ax+by) + y(bx+cy)$$

Then we have

$$\begin{aligned}\int e^{-x^T Ax} d^n x &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(ax^2+2bxy+cy^2)} dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x+b/ay)^2-\left(\frac{ac-b^2}{a}\right)y^2} dxdy \\ &= \sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty} e^{-\left(\frac{ac-b^2}{a}\right)y^2} dy = \frac{\pi}{\sqrt{ac-b^2}}\end{aligned}$$

But $ac - b^2$ is just the determinant of the matrix A. In general, we can write

$$\int e^{-x^T Ax} d^n x = \frac{\pi^{n/2}}{\sqrt{\det A}} \tag{11.6}$$

This can be extended to the results

$$\int e^{-\frac{1}{2}x^T Ax + J\cdot x} d^n x = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} e^{\frac{J}{2}\cdot A^{-1}\cdot J} \tag{11.7}$$

$$\int e^{-\frac{i}{2}x^T Ax + iJ\cdot x} d^n x = \frac{(2\pi i)^{n/2}}{\sqrt{\det A}} e^{-i\frac{J}{2}\cdot A^{-1}\cdot J} \tag{11.8}$$

Basic Path Integrals

Now that we know how to do some Gaussian integrals, we are ready to take a look at path integration. Our development follows closely the clear expositions by Zee and Hatfield, but it can be found in many quantum field theory books. The basic trick is to reduce the problem to a Riemann sum.

A path integral is a way to calculate the amplitude for a system that starts off in some state $|i\rangle$ to end up in a state $|f\rangle$ by adding up the amplitudes for the system to pass through all possible paths from $|i\rangle$ to $|f\rangle$. As a specific example, the path integral can be constructed by considering the amplitude for a particle to pass from a point x_0 to a point x_N. If the dynamics of the system are described by a Hamiltonian H, then the amplitude in question is

$$\langle x_N | e^{-i\hat{H}t} | x_0 \rangle \tag{11.9}$$

We can consider the simplest possible case, where

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

for a free particle of mass m. If we want to rewrite the amplitude in Eq. (11.9) so that we consider every possible path from x_0 to x_N, we start by splitting up the path into smaller pieces and then use a limiting procedure. We begin by dividing up the time interval into N equally spaced intervals Δt.

$$\Delta t = \frac{t}{N} \tag{11.10}$$

Then

$$\begin{aligned} e^{-i\hat{H}t} &= e^{-i\hat{H}(N\Delta t)} \\ &= e^{-i\hat{H}(\Delta t+\Delta t+\cdots+\Delta t)} = e^{-i\hat{H}\Delta t}e^{-i\hat{H}\Delta t}e^{-i\hat{H}\Delta t}\cdots e^{-i\hat{H}\Delta t} \end{aligned}$$

So the amplitude can be rewritten as

$$\langle x_N | e^{-i\hat{H}t} | x_0 \rangle = \langle x_N | e^{-i\hat{H}\Delta t}e^{-i\hat{H}\Delta t}e^{-i\hat{H}\Delta t}\cdots e^{-i\hat{H}\Delta t} | x_0 \rangle$$

Now we use the fact that the position eigenstates form a complete set of states. That is,

$$\int dx | x \rangle\langle x | = I \tag{11.11}$$

We can split up the interval into N parts and each little part will be a complete set of states, that is,

$$\int dx_j | x_j \rangle\langle x_j | = I$$

We stick these in between the exponential factors to give

$$\begin{aligned} \langle x_N | e^{-i\hat{H}t} | x_0 \rangle &= \langle x_N | e^{-i\hat{H}\Delta t}e^{-i\hat{H}\Delta t}e^{-i\hat{H}\Delta t}\cdots e^{-i\hat{H}\Delta t} | x_0 \rangle \\ &= \langle x_N | e^{-i\hat{H}\Delta t} I e^{-i\hat{H}\Delta t} I e^{-i\hat{H}\Delta t} I \cdots I e^{-i\hat{H}\Delta t} | x_0 \rangle \\ &= \langle x_N | e^{-i\hat{H}\Delta t} \int dx_{N-1} | x_{N-1} \rangle\langle x_{N-1} | e^{-i\hat{H}\Delta t} \int dx_{N-2} | x_{N-2} \rangle \\ &\quad \times \langle x_{N-2} | e^{-i\hat{H}\Delta t} \cdots \int dx_1 | x_1 \rangle\langle x_1 | e^{-i\hat{H}\Delta t} | x_0 \rangle \end{aligned}$$

This expression can be written more compactly as a product, like

$$\langle x_N | e^{-i\hat{H}t} | x_0 \rangle = \prod_{j=1}^{N-1} \int dx_j \langle x_N | e^{-i\hat{H}\Delta t} | x_{N-1} \rangle \langle x_{N-1} | e^{-i\hat{H}\Delta t} | x_{N-2} \rangle \langle x_{N-2} | e^{-i\hat{H}\Delta t} \cdots | x_1 \rangle \langle x_1 | e^{-i\hat{H}\Delta t} | x_0 \rangle$$

Right now the situation probably looks pretty hopeless. But we can calculate each individual term by bringing up the Gaussian integrals in the last section. Getting there will take some additional complications. We begin by recalling that the momentum eigenstates also form a complete set.

$$\frac{1}{2\pi} \int dp | p \rangle \langle p | \tag{11.12}$$

Noting that

$$\langle x | p \rangle = e^{ipx} \qquad \langle p | x \rangle = e^{-ipx} \tag{11.13}$$

and that for a free particle

$$\hat{H} | p \rangle = \frac{\hat{p}^2}{2m} | p \rangle = \frac{p^2}{2m} | p \rangle$$
$$\Rightarrow e^{-i\hat{H}\Delta t} | p \rangle = e^{-ip^2/2m\Delta t} | p \rangle$$

we can get expressions like $\langle x_j | e^{-i\hat{H}\Delta t} | x_{j-1} \rangle$ into a form that can be readily integrated. We have

$$\begin{aligned}
\langle x_j | e^{-i\hat{H}\Delta t} | x_{j-1} \rangle &= \langle x_j | e^{-i\hat{H}\Delta t} \left(\frac{1}{2\pi} \int dp | p \rangle \langle p | \right) | x_{j-1} \rangle \\
&= \frac{1}{2\pi} \langle x_j | \int dp \left(e^{-i\hat{H}\Delta t} | p \rangle \right) \langle p | x_{j-1} \rangle \\
&= \frac{1}{2\pi} \langle x_j | \int dp \left(e^{-i\hat{H}\Delta t} | p \rangle \right) e^{-ipx_{j-1}} \\
&= \frac{1}{2\pi} \langle x_j | \int dp \left(e^{-ip^2/2m\Delta t} | p \rangle \right) e^{-ipx_{j-1}} \\
&= \frac{1}{2\pi} \int dp \left(e^{-ip^2/2m\Delta t} \langle x_j | p \rangle \right) e^{-ipx_{j-1}} \\
&= \frac{1}{2\pi} \int dp \, e^{-ip^2/2m\Delta t} e^{ip(x_j - x_{j-1})}
\end{aligned}$$

This integral is nothing other than Eq. (11.5). Taking

$$a = \frac{i}{m}\Delta t \qquad J = x_j - x_{j-1}$$

we get

$$\left\langle x_j \left| e^{-i\hat{H}\Delta t} \right| x_{j-1} \right\rangle = \frac{1}{2\pi}\int dp\, e^{-ip^2/2m\Delta t} e^{ip(x_j - x_{j-1})}$$
$$= \frac{1}{2\pi}\left(-i\frac{2\pi m}{\Delta t}\right)^{1/2} e^{i(x_j - x_{j-1})^2 m/2\Delta t}$$

Using this expression for each term in the original product allows us to write the compact expression as follows:

$$\left\langle x_N \left| e^{-i\hat{H}t} \right| x_0 \right\rangle = \left(-i\frac{m}{2\pi\,\Delta t}\right)^{N/2} \prod_{j=1}^{N-1}\int dx_j e^{\sum_{j=0}^{N-1} im/2\Delta t\left[(x_j - x_{j-1})/\Delta t\right]^2} \tag{11.14}$$

Now we let $\Delta t \to 0$. In the limit, the term $(x_j - x_{j-1})/\Delta t$ just becomes a derivative

$$\lim_{\Delta t\to 0}\frac{x_j - x_{j-1}}{\Delta t} = \frac{dx}{dt} = \dot{x}$$

and the summation $\sum_{j=0}^{N-1} \to \int_0^t dt'$, so Eq. (11.14) becomes

$$\left\langle x_N \left| e^{-i\hat{H}t} \right| x_0 \right\rangle = \lim_{N\to\infty}\left(-i\frac{m}{2\pi\,\Delta t}\right)^{N/2} \prod_{j=1}^{N-1}\int dx_j e^{i\int_0^t dt' \frac{m}{2}\dot{x}^2}$$

The path integral measure is then

$$Dx = \lim_{N\to\infty}\prod_{j=1}^{N-1}\int dx_j \left(-i\frac{m}{2\pi\,\Delta t}\right)^{N/2} \tag{11.15}$$

where $\Delta t = \frac{t}{N}$. We can then write the path integral as the compact expression

$$\left\langle x_N \left| e^{-i\hat{H}t} \right| x_0 \right\rangle = \int Dx\, e^{i\int_0^t dt' \frac{m}{2}\dot{x}^2} \tag{11.16}$$

This tells us that the amplitude for a particle to take the path from x_0 to x_N is proportional to the exponential of the action *S*, which you recall as

$$S = \int L\, dt$$

In this example, we have considered a free particle, so $L = \frac{1}{2} m\dot{x}^2$. This is a nice result which ties quantum theory to classical mechanics. We can write the general amplitude for a system with Lagrangian *L* as

$$\langle F | e^{-i\hat{H}t} | I \rangle = \int Dx\, e^{i\int_0^t dt' L(q,\dot{q})} \tag{11.17}$$

where $|I\rangle$ and $|F\rangle$ are the initial and final states of the system. In a quantum field theory, we can use a path integral to compute the amplitude for a transition from a state $\varphi_1(t_1)$ to a state $\varphi_2(t_2)$.

$$\begin{aligned} \langle \varphi_2(t_2) | \varphi_1(t_1) \rangle &= \langle \varphi_2 | e^{-i\hat{H}(t_2 - t_1)} | \varphi_1 \rangle \\ &= \int_{\varphi_1}^{\varphi_2} D\varphi \exp\left(i \int L\, d^4x \right) \end{aligned} \tag{11.18}$$

For example, if we let $\varphi_1 = \varphi_2 = \varphi_0$ be the ground state, we can calculate the energy of the vacuum using path integral methods. When an external source $J(x)$ is present, the path integral becomes

$$\langle \varphi_2(t_2) | \varphi_1(t_1) \rangle = \int_{\varphi_1}^{\varphi_2} D\varphi \exp\left(i \int \left[L + \varphi(x) J(x) \right] d^4x \right) \tag{11.19}$$

We define the vacuum to vacuum amplitude by

$$Z[J] = e^{iW[J]} = \langle 0 | 0 \rangle_J = \int D\varphi \exp\left[\int d^4x \left(L + \varphi(x) J(x) \right) \right] \tag{11.20}$$

Summary

In this chapter we have succinctly introduced the notion of a path integral. This is a method that can be used to calculate the amplitude for a system to transition from one state to another by considering all possible paths between the two states. Path integrals can be used to calculate any quantity in field theory such as the vacuum expectation value for a field. The use of Gaussian integrals is important in the calculation of path integrals.

Quiz

1. The integral $\int_{-\infty}^{\infty} x^3 e^{-x^2} dx$ is
   (a) $\frac{4\sqrt{\pi}}{2^{3/2} a^2}$
   (b) 0
   (c) Indeterminate
2. The integral $\int_0^{\infty} x^3 e^{-x^2} dx$ is
   (a) ½
   (b) 0
   (c) $\int_0^{\infty} x^3 e^{-x^2} dx \to \infty$ unbounded
3. Compute $\int e^{-x^T A x} d^n x$ when $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 1 & 0 & 2 \end{pmatrix}$.
   (a) $i \frac{\pi^{3/2}}{\sqrt{2}}$
   (b) $\frac{\pi^{3/2}}{\sqrt{2}}$
   (c) 0
4. Path integrals often involve Gaussians because
   (a) Path integrals are the square of an energy integral.
   (b) Path integrals do not involve Gaussians.
   (c) A path integral contains the Lagrangian, which has a quadratic dependence on momentum.
5. The relationship between the quantum mechanical path integral and classical mechanics can be stated as
   (a) There is no relation.
   (b) A path integral involves an exponential of the action *S*.
   (c) The action *S* can be recovered from a path integral by explicit calculation.
   (d) The square of the argument to the exponential gives the action.
6. The vacuum to vacuum amplitude with a source is
   (a) $Z[J] = e^{iW[J]} = \langle 0|0\rangle_J = \int D\varphi \exp\left[\int d^4x (L + \varphi^2(x) J^2(x))\right]$
   (b) $Z[J] = e^{iW[J]} = \langle 0|0\rangle_J = \int D\varphi \exp\left[\int d^4x (L - \varphi(x) J(x))\right]$
   (c) $Z[J] = e^{iW[J]} = \langle 0|0\rangle_J = \int D\varphi \exp\left[\int d^4x (L + J(x))\right]$
   (d) $Z[J] = e^{iW[J]} = \langle 0|0\rangle_J = \int D\varphi \exp\left[\int d^4x (L + \varphi(x) J(x))\right]$

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CHAPTER 12

Supersymmetry

Latter progress in theoretical particle physics prior to string theory came about in the 1970s under the name of *supersymmetry* (SUSY). This is a symmetry that relates or mixes (unites) fermions and bosons. Fermions are particles with half-integral spin, while bosons are particles with integer spin. The idea of supersymmetry is that for every fermion, there is a corresponding boson. We know that the force-carrying or mediator bosons have spin-0 or spin-1, so what we would hope to find is that corresponding to spin-1/2 particles like quarks and electrons, there would be spin-0 or spin-1 particles (denoted the *selectron* and *squark*). Also, corresponding to each spin-0 or spin-1 particle there would be a half-integral spin particle. The proposed particles go by the fanciful names *photino, wino,* and *gluino,* which would correspond to the photon, *W*, and gluon, for example.

At the time of writing, there is no experimental evidence for supersymmetry despite diligent experiment. However, it is hoped that when the Large Hadron Collider (LHC) begins operation sometime in 2008, experimental evidence for supersymmetry will be found or alternatively that supersymmetry can be effectively ruled out, making it an exercise in mathematics and history.

In this chapter we will provide a very cursory introduction to the subject. Interested readers are urged to consult advanced physics books and published articles to learn more about the theory.

Basic Overview of Supersymmetry

As mentioned in the introduction, supersymmetry proposes that to each fermion there exists a boson and vice versa. So we can think of *supersymmetry as proposing that a symmetry exists between bosons and fermions, and that in nature, there are equal numbers of fermion and boson states*. Ideally, these would be the same mass. So as the positron is the antiparticle of the electron, essentially it's an electron with the same mass but opposite charge. There would be a bosonic selectron which would have the same mass as the electron but would have whole integer spin rather than being spin-1/2. Obviously, no such particle has been seen. This means that if supersymmetry is real and selectrons do exist, their mass must be much heavier than that of real electrons or they are cloaked by mechanisms unknown. This suggests why we have not detected them yet—we need larger particle accelerators to attain the higher energies necessary to create selectrons. Since the masses actually differ in nature, supersymmetry must be broken.

If supersymmetry is not observed, why does it garner so much attention? Because supersymmetry solves many outstanding "mysteries" in particle physics. One important problem that stands out is called the *hierarchy problem.* It is believed that the mass of the Higgs boson, which we will denote by m_h, is much smaller than a fundamental quantity that physicists call the *Planck mass.* The Planck mass is computed from the fundamental constants.

$$m_p = \sqrt{\frac{\hbar c}{G}} \approx 10^{19}\ \text{GeV/c}^2 \tag{12.1}$$

This is an astonishingly large 22 μg[1]. At the present time it is believed that

$$m_h \ll m_p \tag{12.2}$$

In quantum field theory, there are quadratic corrections to the Higgs mass that cause its mass to diverge. Therefore the natural value for the mass of the Higgs boson would be extremely large, that is, we could expect it to be on the order of the Planck mass. This would make it much larger than what is expected to be the experimentally observed value of the Higgs mass, which we are denoting by m_h. This is the hierarchy problem. Although standard theory requires the mass of the

[1]A tiny grain of SiO_2 sand of dimension 0.1 mm weighs 2.6 μg.

Higgs boson to be as large as possible, this is not what is seen (or at least what we think will be seen). Why is the Higgs mass so much smaller than its natural value? It turns out that supersymmetry does away with the corrections to the Higgs mass, and allows a Higgs mass more like what is expected to be seen experimentally.

There are also other reasons to hope that supersymmetry turns out to be valid. One is the so-called *gauge unification problem.* It is believed that the coupling constants of the standard model would unify (become the same strength) at a certain energy. Using standard model quantum field theory, this does not quite work out. But supersymmetry solves this problem, giving an energy at which the coupling constants of the different interactions would converge to a single value.

Supercharge

We can start to understand supersymmetry by taking a look at the *supercharge operator Q.* The supercharge operator acts to transform fermions into bosons and bosons into fermions. For the moment let's denote fermionic states by $|F\rangle$ and bosonic states by $|B\rangle$. Then the action of the supercharge is as follows:

$$Q|F\rangle = |B\rangle \tag{12.3}$$

$$Q|B\rangle = |F\rangle \tag{12.4}$$

The number of supercharges in the theory characterizes the theory. If there is a single supercharge Q, then we say that we have an $N = 1$ supersymmetry. If there are two supercharges, then there is an $N = 2$ supersymmetry. If there are three supercharges, then we have an $N = 3$ supersymmetry and so on.

The operators Q and $Q^\dagger$ transform as spin-1/2 operators under Lorentz transformations. If we let P^μ be a conserved 4-momentum, then Q is a spinor that satisfies anticommutation relations of the form

$$\begin{aligned} \{Q, Q^\dagger\} &= P^\mu \\ \{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0 \end{aligned} \tag{12.5}$$

In addition, the operators Q and $Q^\dagger$ commute with the 4-momentum P^μ.

$$[Q, P^\mu] = [Q^\dagger, P^\mu] = 0 \tag{12.6}$$

Let the fermion and boson states of a given type be defined by $|\psi_F\rangle$ and $|\psi_B\rangle$, respectively. For example, if the particle we are talking about is an electron, then it

represents the electron state while $|\psi_B\rangle$ is the superpartner selectron state. They transform into one another as

$$Q|\psi_F\rangle = |\psi_B\rangle$$
$$Q|\psi_B\rangle = |\psi_F\rangle$$

The square of the 4-momentum operator gives the mass of the state. That is,

$$P^2|\psi_F\rangle = -m_F^2|\psi_F\rangle \tag{12.7}$$

Since Q and $Q^\dagger$ commute with the 4-momentum P^μ, it follows using $[A, BC] = [A,B]C + B[A,C]$ that they also commute with the square of the 4-momentum operator. Hence

$$[P^2, Q] = P^2Q - QP^2 = 0$$
$$\Rightarrow P^2Q = QP^2$$

This result indicates that the superpartners will have the same mass. First we note that

$$P^2Q|\psi_F\rangle = P^2|\psi_B\rangle$$

However, since $P^2Q = QP^2$, we can write

$$P^2Q|\psi_F\rangle = QP^2|\psi_F\rangle = -Qm_F^2|\psi_F\rangle$$

but m_F^2 is just a scalar. Hence

$$P^2Q|\psi_F\rangle = -Qm_F^2|\psi_F\rangle = -m_F^2Q|\psi_F\rangle = -m_F^2|\psi_B\rangle$$

Putting this together with $P^2Q|\psi_F\rangle = -Qm_F^2|\psi_F\rangle$, we see that

$$P^2|\psi_B\rangle = -m_F^2|\psi_B\rangle$$

This proves that if the anticommutation relations are satisfied, the partner and superpartner states have the same mass. We've already noted that this is not what is seen in nature—otherwise low mass partners like the selectron would have been

detected experimentally long ago. Supersymmetry, if it exists, is broken in nature and the superpartners have much larger mass.

The basic program of supersymmetry is to add one or more supercharges to the fields of the standard model, and determine what happens when we vary the action with respect to the supercharge. The result will be some leftover terms. We then add more terms to cancel the unwanted ones. In the end, the action remains invariant under the supersymmetry transformation.

The added terms are the new fields and associated particles we mentioned earlier. So, bosonic partners to each fermion need to be added to the standard model such as selectrons and squarks to keep it invariant under a supersymmetry transformation. There are also fermionic fields like the *Higgsino* that correspond to the bosonic particles (in this case the Higgs) added to the action to keep it invariant. The $N = 1$ supersymmetry case with a single supercharge is called the *minimally supersymmetric standard model* or MSSM.

We will explain supersymmetry in the standard model in more detail below. However, first let's take a look at a simpler way to introduce supersymmetry using what is called *supersymmetric quantum mechanics.*

Supersymmetric Quantum Mechanics

Supersymmetric quantum mechanics was developed to apply some of the ideas of supersymmetry to the simpler nonrelativistic quantum mechanics to gain insight into some of the problems of supersymmetry, such as some understanding into why supersymmetry is broken—meaning that superpartners of the same mass to known particles do not exist in nature. In supersymmetric quantum mechanics there exist N supercharges Q_i that commute with the Hamiltonian.

$$[Q_i, H] = 0 \tag{12.8}$$

These supercharges satisfy a set of anticommutation relations relating them to H

$$\{Q_i, Q_j\} = \delta_{ij} H \tag{12.9}$$

and there exists a *superpotential* $W(x)$ whose role will become clear in a moment. We consider a two-state system in one spatial dimension x with a wave function given by

$$\psi(x) = \begin{pmatrix} \alpha(x) \\ \beta(x) \end{pmatrix} \tag{12.10}$$

There exist two supercharges Q_1 and Q_2. They can be defined using the Pauli matrices together with the superpotential $W(x)$ as

$$\begin{aligned} Q_1 &= \frac{1}{2}\left[\sigma_1 p + \sigma_2 W(x)\right] \\ Q_2 &= \frac{1}{2}\left[\sigma_2 p - \sigma_1 W(x)\right] \end{aligned} \tag{12.11}$$

EXAMPLE 12.1
Compute the Hamiltonian corresponding to the supercharges defined in Eq. (12.11).

SOLUTION
We find the Hamiltonian using $\{Q_i, Q_j\} = \delta_{ij} H$. Let us use Q_1. Now

$$\begin{aligned} \{Q_1, Q_1\} &= Q_1 Q_1 + Q_1 Q_1 \\ &= 2Q_1 Q_1 \\ &= 2Q_1^2 \end{aligned}$$

Hence we see that the Hamiltonian can be written directly in terms of the supercharge as $H = 2Q_1^2$. Explicitly

$$\begin{aligned} Q_1 Q_1 &= \frac{1}{2}\left[\sigma_1 p + \sigma_2 W(x)\right]\frac{1}{2}\left[\sigma_1 p + \sigma_2 W(x)\right] \\ &= \frac{1}{4}\left[\sigma_1 p \sigma_1 p + \sigma_1 p \sigma_2 W(x) + \sigma_2 W(x)\sigma_1 p + \sigma_2{}^2 W^2(x)\right] \end{aligned}$$

However, we know that the Pauli matrices square to the identity. Therefore

$$\sigma_2^2 = I$$

Moreover, we can move the Pauli matrices and momentum operators past each other with impunity. Hence the first term is

$$\sigma_1 p \sigma_1 p = \sigma_1^2 p^2 = p^2$$

Now, recall the Lie algebra of these matrices, which is as follows:

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3$$

$$\sigma_2\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3$$

And, recall that the commutator of any function of position with the momentum operator is

$$\left[F(x), p\right] = F(x)p - pF(x) = i\frac{dF}{dx}$$

Therefore,

$$\begin{aligned} \sigma_1\sigma_2 PW + \sigma_2\sigma_1 WP &= \sigma_1\sigma_2 PW + \sigma_2\sigma_1\left(PW + i\frac{dW}{dx}\right) \\ &= (\sigma_1\sigma_2 + \sigma_2\sigma_1)PW + i\sigma_2\sigma_1\frac{dW}{dx} \\ &= (i\sigma_3 - i\sigma_3)PW + \sigma_3\frac{dW}{dx} \\ &= \sigma_3\frac{dW}{dx} \end{aligned}$$

Putting all of these results together, we find

$$Q_1^2 = \frac{1}{4}\left(p^2 + \sigma_3\frac{dW}{dx} + W^2(x)\right)$$

Hence, the Hamiltonian is

$$H = 2Q_1^2 = \frac{1}{2}\left(p^2 + \sigma_3\frac{dW}{dx} + W^2(x)\right)$$

Take a look back at the action of a supercharge in a quantum field theory, as described schematically in Eqs. (12.3) and (12.4). When we look at the action of the

supercharges in this example of nonrelativistic quantum mechanics, we will see how we are constructing a simple model of a supersymmetric theory. We define spin-up and spin-down states as

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now we have

$$\begin{aligned} Q_1|+\rangle &= \frac{1}{2}(\sigma_1 p + \sigma_2 W(x))|+\rangle \\ &= \frac{p}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{W(x)}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{p}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{W(x)}{2}\begin{pmatrix} 0 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ p + iW \end{pmatrix} \end{aligned}$$

That is, the supercharge Q_1 turns a spin-up state into a spin-down state, analogous to a supercharge converting a fermion to a boson, say. Next consider the action of Q_1 on a spin-down state as follows:

$$\begin{aligned} Q_1|-\rangle &= \frac{1}{2}(\sigma_1 p + \sigma_2 W(x))|-\rangle \\ &= \frac{p}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{W(x)}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{p}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{W(x)}{2}\begin{pmatrix} -i \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p - iW \\ 0 \end{pmatrix} \end{aligned}$$

If supersymmetric quantum mechanics is unbroken, then the states $\psi(x)$ are left invariant by a unitary transformation of the form

$$U = e^{-i\alpha_i Q_i}$$

Moreover there exists a state $\psi_0(x)$ that is annihilated by the supercharges, that is,

$$Q_1\psi_0(x) = 0 \tag{12.12}$$

This implies that the ground state of the system has zero energy since

$$H\psi_0(x) = 2Q_1^{\,2}\psi_0(x) = 0$$

EXAMPLE 12.2
Derive the form of the ground state $\psi_0(x)$.

SOLUTION
We use the fact that the supercharge annihilates the state Eq. (12.12) together with Eq. (12.11).

$$\begin{aligned} Q_1\psi_0(x) &= 0 \\ \Rightarrow 0 &= \frac{1}{2}(\sigma_1 p + \sigma_2 W(x))\psi_0(x) \\ &= \sigma_1 p\psi_0(x) + \sigma_2 W(x)\psi_0(x) \\ &= -i\sigma_1 \frac{d\psi_0}{dx} + \sigma_2 W(x)\psi_0(x) \end{aligned}$$

Multiplying by σ_1 and using $\sigma_1^2 = I$ together with $\sigma_1\sigma_2 = i\sigma_3$ gives

$$\begin{aligned} 0 &= -i\frac{d\psi_0}{dx} + i\sigma_3 W(x)\psi_0(x) \\ \Rightarrow \frac{d\psi_0}{dx} &= \sigma_3 W(x)\psi_0(x) \end{aligned}$$

Integrating and taking the initial condition to be $\psi_0(0)$ gives

$$\psi_0(x) = \exp\int_0^x dx'\sigma_3 W(x')\,\psi_0(0)$$

The Simplified Wess-Zumino Model

Now that we have gotten a taste of supersymmetry by taking a brief look at supersymmetric quantum mechanics, we are ready to dive into a simple supersymmetric quantum field theory. We will discuss the Wess-Zumino model, a supersymmetric theory proposed in 1974 that starts with a Lagrangian consisting of

bosonic and fermionic fields and shows that it is possible to develop a transformation that mixes bosonic and fermionic fields.

THE CHIRAL REPRESENTATION

In supersymmetry, it is convenient to work in the representation where the Dirac matrices are given by

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \tag{12.13}$$

where we have introduced the barred Pauli matrices which are obtained with a sign change as

$$\begin{aligned} \bar{\sigma}^0 &= \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \bar{\sigma}^1 &= -\sigma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \bar{\sigma}^2 = -\sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ \bar{\sigma}^3 &= -\sigma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{12.14}$$

Note that lowering the index with the metric introduces a sign change when $i = 1, 2, 3$ so that

$$\bar{\sigma}_3 = -\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for example. The following anticommutation relations are useful.

$$\begin{aligned} \{\sigma^\mu, \bar{\sigma}^\nu\} &= \sigma^\mu \bar{\sigma}^\nu + \bar{\sigma}^\nu \sigma^\mu = -2g^{\mu\nu} \\ \{\bar{\sigma}^\mu, \sigma^\nu\} &= \bar{\sigma}^\mu \sigma^\nu + \sigma^\nu \bar{\sigma}^\mu = -2g^{\mu\nu} \end{aligned} \tag{12.15}$$

A Simple SUSY Lagrangian

The key to supersymmetry is writing down a Lagrangian and then relying on our old friend: computing a variation of the bosonic and fermionic states together and requiring that the variation of the action be zero. This will introduce a *supercurrent* that is conserved, meaning that if S is the supercurrent then the total divergence is 0.

$$\partial_\mu S^\mu = 0$$

The supercurrent will allow us to calculate the supercharges Q. A simple model that illustrates how supersymmetry works is the *Wess-Zumino model*. In the following, we will consider the simplest possible example, a model that consists of a single massless spin-0 boson field A and a single massless spin-1/2 fermion field ψ. If we let the scalar field for the spin-0 boson be a complex field, then the Lagrangian is

$$L_B = -\partial_\mu A^* \partial^\mu A$$

If we take the spinor field to be right handed, the Lagrangian for a massless spin-1/2 field can be written as

$$L_F = \frac{i}{2}\partial_\mu \psi^\dagger \bar{\sigma}^\mu \psi - \frac{i}{2}\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi$$

In order to close the SUSY algebra, it will be necessary to add an *auxiliary field F*. The auxiliary Lagrangian is the simple term

$$L_{\text{aux}} = F^* F$$

The supersymmetric Lagrangian is the sum of these individual Lagrangians, that is,

$$\begin{aligned} L &= L_B + L_F + L_{\text{aux}} \\ &= -\partial_\mu A^* \partial^\mu A + \frac{i}{2}\partial_\mu \psi^\dagger \bar{\sigma}^\mu \psi - \frac{i}{2}\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + F^* F \end{aligned} \tag{12.16}$$

For the Lagrangian to be invariant under supersymmetry, we require that

$$\delta S = \delta \int L\, d^4x = 0$$

where the supersymmetric variation converts bosons into fermions and vice versa. We need variations that will work with each field in the Lagrangian, so obtain

$$\begin{aligned} &\delta A = \sqrt{2}\varepsilon\psi \qquad \delta A^* = \sqrt{2}\varepsilon^\dagger \psi^\dagger \\ &\delta\psi = i\sqrt{2}\sigma^\mu \varepsilon^\dagger \partial_\mu A + \sqrt{2}\varepsilon F \qquad \delta\psi^\dagger = -i\sqrt{2}\varepsilon\sigma^\mu \partial_\mu A^* + \sqrt{2}\varepsilon^\dagger F^* \\ &\delta F = i\sqrt{2}\varepsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi \qquad \delta F^* = -i\sqrt{2}\partial_\mu \bar{\psi}\bar{\sigma}^\mu \varepsilon \end{aligned} \tag{12.17}$$

We have introduced a new quantity here, a two-component Weyl spinor ε. When a supersymmetry is global, ε does not depend on spacetime and so

$$\partial_\mu \varepsilon = 0 \tag{12.18}$$

This will not be the case when a supersymmetric transformation is local.

We proceed by considering the variation of the Lagrangian term by term. Starting with the first term in Eq. (12.16),

$$\begin{aligned}\delta L_B &= \delta(\partial_\mu A^* \partial^\mu A) \\ &= \delta(\partial_\mu A^*)\partial^\mu A + \partial_\mu A^* \delta(\partial^\mu A) \\ &= \partial_\mu(\sqrt{2}\varepsilon^\dagger \psi^\dagger)\partial^\mu A + \partial_\mu A^* \partial^\mu(\sqrt{2}\varepsilon\psi) \\ &= \sqrt{2}\varepsilon^\dagger \partial_\mu \psi^\dagger \partial^\mu A + \sqrt{2}\varepsilon\,\partial_\mu A^* \partial^\mu \psi \end{aligned} \tag{12.19}$$

Now you remember that $\delta(\partial_\mu \varphi) = \partial_\mu(\delta\varphi)$, a trick we used in moving from the second line to the third line. Next, we compute the variation of the second term in Eq. (12.16), which is

$$\begin{aligned}\delta\frac{i}{2}\partial_\mu \psi^\dagger \bar{\sigma}^\mu \psi &= \frac{i}{2}\partial_\mu(\delta\psi^\dagger)\bar{\sigma}^\mu\psi + \frac{i}{2}\partial_\mu\psi^\dagger\bar{\sigma}^\mu(\delta\psi) \\ &= \frac{i}{2}\partial_\mu(-i\sqrt{2}\varepsilon\sigma^\nu\partial_\nu A^* + \sqrt{2}\varepsilon^\dagger F^*)\bar{\sigma}^\mu\psi \\ &\quad + \frac{i}{2}\partial_\mu\psi^\dagger\bar{\sigma}^\mu(i\sqrt{2}\sigma^\nu\varepsilon^\dagger\partial_\nu A + \sqrt{2}\varepsilon F) \\ &= \frac{1}{\sqrt{2}}\varepsilon\sigma^\nu\bar{\sigma}^\mu(\partial_\mu\partial_\nu A^*)\psi + \frac{i}{\sqrt{2}}\varepsilon^\dagger\partial_\mu F^*\bar{\sigma}^\mu\psi \\ &\quad - \frac{1}{\sqrt{2}}\partial_\mu\psi^\dagger\bar{\sigma}^\mu\sigma^\nu\varepsilon^\dagger\partial_\nu A + \frac{i}{\sqrt{2}}\partial_\mu\psi^\dagger\bar{\sigma}^\mu\varepsilon F \end{aligned} \tag{12.20}$$

Continuing, the next term is similar.

$$\begin{aligned}\delta\left(-\frac{i}{2}\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi\right) &= -\frac{i}{2}(\delta\psi^\dagger)\bar{\sigma}^\mu\partial_\mu\psi - \frac{i}{2}\psi^\dagger\bar{\sigma}^\mu\partial_\mu(\delta\psi) \\ &= -\frac{i}{2}(-i\sqrt{2}\varepsilon\sigma^\nu\partial_\nu A^* + \sqrt{2}\varepsilon^\dagger F^*)\bar{\sigma}^\mu\partial_\mu\psi \\ &\quad - \frac{i}{2}\psi^\dagger\bar{\sigma}^\mu\partial_\mu(i\sqrt{2}\sigma^\nu\varepsilon^\dagger\partial_\nu A + \sqrt{2}\varepsilon F) \\ &= -\frac{1}{\sqrt{2}}\varepsilon\sigma^\nu\bar{\sigma}^\mu\partial_\nu A^*\partial_\mu\psi - \frac{i}{\sqrt{2}}\varepsilon^\dagger F^*\bar{\sigma}^\mu\partial_\mu\psi \\ &\quad + \frac{1}{\sqrt{2}}\psi^\dagger\bar{\sigma}^\mu\sigma^\nu\varepsilon^\dagger\partial_\mu\partial_\nu A - \frac{i}{\sqrt{2}}\psi^\dagger\bar{\sigma}^\mu\varepsilon\partial_\mu F \end{aligned} \tag{12.21}$$

Remember, the Pauli matrices only operate on the spinors, so we can move them past the bosonic fields however we like. Finally, we compute δFF^*.

$$\begin{aligned}\delta FF^* &= (\delta F)F^* + F(\delta F^*) \\ &= (i\sqrt{2}\varepsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi)F^* + F(-i\sqrt{2}\partial_\mu\psi^\dagger\bar{\sigma}^\mu\varepsilon)\end{aligned} \tag{12.22}$$

Now it is useful to consider terms dependent on ε only. Grouping them together using Eqs. (12.19) through (12.22), we have

$$\begin{aligned}\delta L_\varepsilon = &-\sqrt{2}\varepsilon\partial_\mu A^*\partial^\mu\psi + \frac{1}{\sqrt{2}}\varepsilon\sigma^\nu\bar{\sigma}^\mu(\partial_\mu\partial_\nu A^*)\psi - \frac{1}{\sqrt{2}}\varepsilon\sigma^\nu\bar{\sigma}^\mu\partial_\nu A^*\partial_\mu\psi \\ &-\frac{i}{\sqrt{2}}\partial_\mu\psi^\dagger\bar{\sigma}^\mu\varepsilon F - \frac{i}{\sqrt{2}}\psi^\dagger\bar{\sigma}^\mu\varepsilon\partial_\mu F\end{aligned} \tag{12.23}$$

We wish to simplify this expression so that it can be written as a total derivative, which would allow us to satisfy $\delta S = \delta\int L\,d^4x = 0$. First, notice that Eq. (12.23) can be written as

$$\begin{aligned}\delta L_\varepsilon = &-\sqrt{2}\varepsilon\partial_\mu A^*\partial^\mu\psi + \frac{1}{\sqrt{2}}\varepsilon\sigma^\nu\bar{\sigma}^\mu(\partial_\mu\partial_\nu A^*)\psi - \frac{1}{\sqrt{2}}\varepsilon\sigma^\nu\bar{\sigma}^\mu\partial_\nu A^*\partial_\mu\psi \\ &-\frac{i}{\sqrt{2}}\partial_\mu\psi^\dagger\bar{\sigma}^\mu\varepsilon F - \frac{i}{\sqrt{2}}\psi^\dagger\bar{\sigma}^\mu\varepsilon\partial_\mu F \\ = &\,\partial_\mu\left(-\frac{i}{\sqrt{2}}\psi^\dagger\bar{\sigma}^\mu\varepsilon F\right) - \sqrt{2}\varepsilon\partial_\mu A^*\partial^\mu\psi + \frac{1}{\sqrt{2}}\varepsilon\sigma^\nu\bar{\sigma}^\mu(\partial_\mu\partial_\nu A^*)\psi \\ &-\frac{1}{\sqrt{2}}\varepsilon\sigma^\nu\bar{\sigma}^\mu\partial_\nu A^*\partial_\mu\psi\end{aligned}$$

Now we can apply Eq. (12.15) to simplify this even further. Recall that

$$\begin{aligned}&\{\bar{\sigma}^\mu,\sigma^\nu\} = \bar{\sigma}^\mu\sigma^\nu + \sigma^\nu\bar{\sigma}^\mu = -2g^{\mu\nu} \\ &\Rightarrow \sigma^\nu\bar{\sigma}^\mu = -2g^{\mu\nu} - \bar{\sigma}^\mu\sigma^\nu\end{aligned}$$

Also note that repeated indices are *dummy indices,* so we can change them. The next to last term can then be rewritten as

$$\frac{1}{\sqrt{2}}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}(\partial_{\mu}\partial_{\nu}A^{*})\psi = \frac{1}{\sqrt{2}}\varepsilon\sigma^{\mu}\bar{\sigma}^{\nu}(\partial_{\nu}\partial_{\mu}A^{*})\psi$$

$$= \frac{1}{\sqrt{2}}\varepsilon\sigma^{\mu}\bar{\sigma}^{\nu}(\partial_{\mu}\partial_{\nu}A^{*})\psi$$

Now, applying the anticommutation relation and raising an index with the metric,

$$\frac{1}{\sqrt{2}}\varepsilon\sigma^{\mu}\bar{\sigma}^{\nu}(\partial_{\mu}\partial_{\nu}A^{*})\psi = \frac{1}{\sqrt{2}}\varepsilon(-2g^{\mu\nu} - \sigma^{\nu}\bar{\sigma}^{\mu})(\partial_{\mu}\partial_{\nu}A^{*})\psi$$

$$= -\sqrt{2}\,\varepsilon\, g^{\mu\nu}(\partial_{\mu}\partial_{\nu}A^{*})\psi - \frac{1}{\sqrt{2}}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}(\partial_{\mu}\partial_{\nu}A^{*})\psi$$

$$= -\sqrt{2}\,\varepsilon(\partial_{\mu}\partial^{\mu}A^{*})\psi - \frac{1}{\sqrt{2}}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}(\partial_{\mu}\partial_{\nu}A^{*})\psi$$

Therefore,

$$-\sqrt{2}\varepsilon\,\partial_{\mu}A^{*}\partial^{\mu}\psi + \frac{1}{\sqrt{2}}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}(\partial_{\mu}\partial_{\nu}A^{*})\psi - \frac{1}{\sqrt{2}}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}\partial_{\nu}A^{*}\partial_{\mu}\psi$$

$$= -\sqrt{2}\varepsilon\,\partial_{\mu}A^{*}\partial^{\mu}\psi - \sqrt{2}\,\varepsilon(\partial_{\mu}\partial^{\mu}A^{*})\psi - \frac{1}{\sqrt{2}}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}(\partial_{\mu}\partial_{\nu}A^{*})\psi$$

$$-\frac{1}{\sqrt{2}}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}\partial_{\nu}A^{*}\partial_{\mu}\psi$$

$$= -\partial_{\mu}(\sqrt{2}\varepsilon\,\partial_{\mu}A^{*}\psi) - \partial_{\mu}\left(\frac{1}{\sqrt{2}}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}\partial_{\nu}A^{*}\psi\right)$$

So here is the variation of the Lagrangian, considering terms dependent upon ε, written as a total derivative.

$$\delta L_{\varepsilon} = \partial_{\mu}\left(-\frac{i}{\sqrt{2}}\psi^{\dagger}\bar{\sigma}^{\mu}\varepsilon F\right) - \partial_{\mu}(\sqrt{2}\varepsilon\,\partial_{\mu}A^{*}\psi) - \partial_{\mu}\left(\frac{1}{\sqrt{2}}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}\partial_{\nu}A^{*}\psi\right)$$

We use this to identify a current.

$$K^{\mu}{}_{\varepsilon} = -\frac{i}{\sqrt{2}}\psi^{\dagger}\bar{\sigma}^{\mu}\varepsilon F - \sqrt{2}\varepsilon\,\partial_{\mu}A^{*}\psi - \frac{1}{\sqrt{2}}\varepsilon\,\sigma^{\nu}\bar{\sigma}^{\mu}\partial_{\nu}A^{*}\psi \tag{12.24}$$

A similar procedure can be used to define a current dependent upon $\varepsilon^{\dagger}$, giving us a total current.

$$K^{\mu} = K^{\mu}{}_{\varepsilon} + K^{\mu}{}_{\varepsilon^{\dagger}} \tag{12.25}$$

By calculating the Noether current J^{μ}, we can arrive at a supercurrent that is conserved.

$$\begin{aligned} S^{\mu} &= J^{\mu} - K^{\mu} \\ \partial_{\mu}S^{\mu} &= 0 \end{aligned} \tag{12.26}$$

The Noether current is calculated as follows. We take the kinetic part of the Lagrangian, which is given by

$$J^{\mu} = \frac{i}{2}(\delta\psi^{\dagger})\bar{\sigma}^{\mu}\psi - \frac{i}{2}\psi^{\dagger}\bar{\sigma}^{\mu}(\delta\psi) - (\delta A^{*})\partial^{\mu}A - \partial^{\mu}A^{*}(\delta A)$$

The variation procedure is accomplished in the same manner as used to calculate $\delta\mathcal{L}$, using Eq. (12.17). Once again, we can extract ε dependent and $\varepsilon^{\dagger}$ dependent Noether currents. It can be shown that

$$J^{\mu}_{\varepsilon} = \frac{1}{\sqrt{2}}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}\psi\partial_{\nu}A^{*} - \frac{i}{\sqrt{2}}\psi^{\dagger}\bar{\sigma}^{\mu}\varepsilon F - \sqrt{2}\varepsilon\,\psi\,\partial^{\mu}A^{*}$$

Hence, the ε dependent supercurrent in Eq. (12.26) takes on a relatively simple form.

$$S^{\mu}_{\varepsilon} = \sqrt{2}\varepsilon\sigma^{\nu}\bar{\sigma}^{\mu}\psi\,\partial_{\nu}A^{*} \tag{12.27}$$

The supercharges are computed by integrating the time-component of the supercurrent, in a manner analogous to electrodynamics. For example,

$$Q_{\varepsilon} = \int d^{3}x\,S^{0}_{\varepsilon} = \int d^{3}x\sqrt{2}\varepsilon\sigma^{\nu}\bar{\sigma}^{0}\psi\,\partial_{\nu}A^{*} \tag{12.28}$$

Summary

Supersymmetry is a proposition to introduce a symmetry between fermions and bosons. If such a symmetry exists, then there are supercharge operators that convert fermion states into boson states and vice versa. In its most basic form, the theory predicts that the partners of known particles, obtained by applying the supercharge operators and known as superpartners, have the same mass. This has not been experimentally observed. If supersymmetry is real, the masses of the superpartners are much larger than the masses of known particles, and this explains why they have not yet been detected experimentally. The difference in masses breaks the supersymmetry. Hence we know supersymmetry is broken. Theorists have great hopes for the theory because it solves many outstanding problems in theoretical physics, such as the mass of the Higgs particle (the hierarchy problem). Supersymmetry may also explain the existence of mysterious dark matter particles and it is of fundamental importance to string theory.

Quiz

1. Consider the supersymmetric quantum mechanics described in Examples 12.1 and 12.2. Compute $\{Q_1, Q_2\}$ and $\{Q_2, Q_2\}$.

   For problems 2 to 4, consider the Lagrangian

   $$L = \frac{i}{2}\partial_n \chi \sigma^n \bar{\chi} - \frac{i}{2}\sigma^n \partial_n \bar{\chi} - \partial_n \bar{A}\,\partial^n A + \bar{F}F$$

   which describes a left handed spinor χ that does not interact with a complex spin-0 field A.
2. Find the field equations.
3. If the supersymmetry transformation is

   $$\begin{aligned}\delta A &= \sqrt{2}\chi\varepsilon \\ \delta\chi &= -i\sqrt{2}\bar{\varepsilon}\bar{\sigma}^m \partial_m A + \sqrt{2}\varepsilon F \\ \delta F &= -i\sqrt{2}\partial_m \chi \sigma^m \bar{\varepsilon}\end{aligned}$$

   find the SUSY current.
4. Find an expression for the supercharge.

5. Let an operator A be given by $A = (-1)^{2S}$ where S is the spin operator. Given that

$$\{Q, A\} = \{Q^{\dagger}, A\} = 0$$

calculate $\sum_i \langle i | (-1)^{2S} P^{\mu} | i \rangle$ where $|i\rangle$ is a set of fermion or boson states belonging to the same multiplet. (*Hint:* Assume that the states satisfy a completeness relation

$$\sum_j |j\rangle\langle j| = I$$

The result of this calculation implies that each supermultiplet contains the same number of bosonic and fermionic degrees of freedom.)

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Final Exam

1. Consider the following Einstein's equation relating energy, mass, and momentum.

   $$E^2 = \vec{p}^{\,2} + m^2$$

   Make the usual operator substitutions from quantum mechanics, that is,

   $$E \to i\frac{\partial}{\partial t} \qquad \vec{p} \to -i\vec{\nabla}$$

   Determine the resulting field equation.
2. Let $\mathcal{L} = \frac{1}{2}\{(\partial_\mu \varphi)^2 - m^2\varphi^2\}$ where φ is a real scalar field and determine the conserved quantity Q.
3. Consider an electromagnetic type field with mass, and let $S = \int(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu)$. Vary the action to determine and equation of motion.
4. Return to the action of the previous problem and the equation of motion you found. What condition on the vector potential does the equation of motion imply?

5. Suppose that $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + \frac{\lambda^3}{6}\varphi^3 + \frac{\rho^4}{24}\varphi^4$, but the field is invariant under a parity transformation $\varphi \to -\varphi$. What restrictions does this place on the Lagrangian?
6. Calculate $tr(\gamma^\mu\gamma^\nu)$.
7. Find $i\partial_\mu(\bar{\psi}\gamma^\mu\psi)$.
8. Calculate $\partial\!\!\!/\partial\!\!\!/$ where $a\!\!\!/ = \gamma^\mu a_\mu$.
9. The Dirac equation applies to
   (a) Spin-1 particles
   (b) Spin-3/2 particles
   (c) Spin-½ particles
   (d) Both (b) and (c)
10. If $\gamma^\mu\partial_\mu\psi = 0$ find $\gamma^\mu\partial_\mu(\gamma_5\psi)$.
11. Find $(I+\gamma_5)\psi$ if $\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$.
12. Supersymmetry is a symmetry that relates
   (a) Half-integral fields
   (b) Scalar fields to the higgs field
   (c) Quarks and antiquarks
   (d) Fermions and bosons
13. The Higgs field can be best described as responsible for
   (a) Dark energy
   (b) Dark matter
   (c) Giving mass to fundamental particles
   (d) Relating supersymmetry to $SU(5)$ transformations
14. If supersymmetry is unbroken then
   (a) The masses of a fermion and superpartner boson are the same.
   (b) The masses of a fermion and superpartner boson are inverse.
   (c) There is no relation between the mass of a fermion and its superpartner boson.
   (d) The mass of the Higgs particle is 100 GeV.
15. The commutation or anticommutation relation obeyed by a supercharge operator Q is
   (a) $[Q,Q^\dagger] = P^\mu$
   (b) $[Q,Q^\dagger] = 0$

(c) $\{Q,Q^{\dagger}\} = P^{\mu}$

(d) $\{Q,Q^{\dagger}\} = 0$

16. The commutation relation satisfied between the supercharge operator Q and 4-momentum is

(a) $[P^2,Q] = 0$

(b) $[P^2,Q] = \lambda$

(c) $[P^2,Q] = -Q$

(d) $[P^2,Q] = Q^{\dagger}$

17. In supersymmetric quantum mechanics, the supercharges satisfy

(a) $\{Q_i,Q_j\} = 0$

(b) $\{Q_i,Q_j\} = \delta_{ij}$

(c) $[Q_i,Q_j] = \delta_{ij}H$

(d) $\{Q_i,Q_j\} = \delta_{ij}H$

18. In the chiral representation, the Pauli matrices satisfy

(a) $\{\sigma^{\mu},\bar{\sigma}^{\nu}\} = \sigma^{\mu}\bar{\sigma}^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu} - 2g^{\mu\nu}$

(b) $\{\sigma^{\mu},\bar{\sigma}^{\nu}\} = 2g^{\mu\nu}$

(c) $\{\sigma^{\mu},\bar{\sigma}^{\nu}\} = \sigma^{\mu}\bar{\sigma}^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu} = -2g^{\mu\nu}$

19. The supercurrent satisfies

(a) $\partial_{\mu}S^{\mu} = J^{\mu}$, where J^{μ} is the Noether current

(b) $\partial_{\mu}S^{\mu} = 0$

(c) The supercurrent cannot be conserved

(d) An uncertainty relation with the supercharge

20. In order to close the supersymmetry algebra, it can be necessary to introduce

(a) An auxiliary field

(b) A supersymmetric Hamiltonian operator

(c) The uncertainty relations

(d) The Cauchy-Schwarz lemma

21. A group is abelian if group elements a and b satisfy

(a) $\{a,b\} = ab + ba = 0$

(b) $[ab - ba] = e$, where e is the identity element

(c) $[ab - ba] = 0$

(d) ab is closed

22. A Lie group
    (a) Depends on a finite set of continuous parameters θ_i
    (b) Depends on a finite set of discrete parameters that are periodic in 2π
    (c) Does not have derivatives with respect to group elements
    (d) Obeys an open algebra
23. In a Lie group, a generator X is related to a group element g through
    (a) $X = \frac{\partial g}{\partial \theta}\Big|_{\theta=\pi}$
    (b) $X = \frac{\partial^2 g}{\partial \theta^2}\Big|_{\theta=0}$
    (c) $X = \frac{\partial g}{\partial \theta}\Big|_{\theta=0} + \int_o^{2\pi} g(\phi) d\phi$
    (d) $X = \frac{\partial g}{\partial \theta}\Big|_{\theta=0}$
24. A representation D of a group can be related to the generators X using
    (a) $D(\varepsilon\theta) \approx 1 + i\varepsilon\theta X$
    (b) $D(\varepsilon\theta) = i\varepsilon\theta X$
    (c) $D(\varepsilon\theta) \approx \varepsilon \frac{\partial X}{\partial \theta}$
    (d) $D(\varepsilon\theta) \approx \lim_{\theta\to\infty} 1 - i\varepsilon\theta X$
25. If the generator of a group X is Hermitian then the representation D is
    (a) Anti-Hermitian
    (b) Unitary
    (c) Anti-unitary
26. The Lie algebra of a group is $[X_i, X_j] = if_{ijk}X_k$. We call the coefficients f_{ijk}
    (a) Representation constants
    (b) Group generators
    (c) The fine structure constants
    (d) The structure constants of the group
27. A group *SO(N)* is
    (a) Orthogonal $N \times N$ matrices with determinant + 1
    (b) Orthogonal $N \times N$ matrices with determinant − 1

(c) Unitary, orthogonal $N \times N$ matrices with determinant + 1

(d) Unitary $N \times N$ matrices with determinant + 1

28. The special unitary group $SU(2)$ has

(a) 1 generator

(b) 3 generators

(c) 2 generators

(d) 8 generators

29. The special unitary group $SU(3)$ has

(a) 1 generator

(b) 3 generators

(c) 4 generators

(d) 8 generators

30. The unitary group $U(1)$ can be represented by

(a) $U = e^{-i\theta}$

(b) $U = \int e^{-i\theta}$

(c) $U = \dfrac{dg}{d\theta}$

31. The Pauli matrices are a representation of

(a) $SU(3)$

(b) $U(1)$

(c) $SU(2)$

(d) $SU(1)$

32. The rank of the group is defined as

(a) The number of matrix representations of the generators that are diagonal

(b) The number of generators

(c) The number of generators minus one

(d) The number of matrix representations of the generators that are diagonal, minus one

33. A Casimir operator

(a) Forms a finite representation of the group

(b) Commutes with all the generators

(c) Does not commute with any generator

(d) Commutes with the rank representation

34. Consider a quantum number. If the quantum number is multiplicative

(a) For a composite system, the quantum number is the product of individual quantum numbers $\prod_i n_i$.

(b) For a composite system, the quantum number is the sum of individual quantum numbers.

(c) The quantum number is a product of fundamental constants.

(d) The quantum number is a sum of fundamental constants.

35. The eigenvalues of parity are

(a) $\alpha = 0, \pm 1$

(b) $\alpha = 0, 1$

(c) $\alpha = \pm 1$

36. If a wave function has even parity then

(a) $\psi(-x) = \psi(x)$

(b) $\psi(-x) = -\psi(x)$

(c) $\psi(-x) = 0$

(d) $\psi(-2x) = 2\psi(x)$

37. The parity operator P satisfies

(a) $P^2 = iI$

(b) $P^2 = 0$

(c) $P^2 = -I$

(d) $P^2 = I$

38. A parity operator acts on an angular momentum state as

(a) $P\left|L, m_z\right\rangle = (-1)^{m_z L}\left|L, m_z\right\rangle$

(b) $P\left|L, m_z\right\rangle = L\left|L, m_z\right\rangle$

(c) $P\left|L, m_z\right\rangle = -\left|L, m_z\right\rangle$

(d) $P\left|L, m_z\right\rangle = (-1)^L\left|L, m_z\right\rangle$

39. By convention

(a) An electron has positive parity.

(b) An electron has negative parity.

(c) An antielectron has positive parity.

(d) The parity of an electron is indeterminate.

40. The parity of a composite system *ab* each with individual parity operators P_a, P_b is

(a) $-P_aP_b$

(b) P_aP_b

(c) $(-1)^{P_aP_b}P_aP_b$

(d) $(-1)^{P_a+P_b}P_aP_b$

41. Parity is

(a) Conserved in the electromagnetic and strong interactions

(b) Not conserved in the weak interaction

(c) Conserved in the weak interaction

(d) Both a and b

(e) Both a and c

42. A particle denoted by 0^-

(a) Has spin-0 and negative parity

(b) Has no parity and negative spin-½

(c) Is a scalar particle with negative parity

(d) Has zero charge and negative parity

43. Charge conjugation

(a) Only applies to vector particles

(b) Turns particles into antiparticles

(c) Turns particles into antiparticles with a sign change

(d) Flips charge for scalar bosons

44. The charge conjugation operator *C* acts on the electromagnetic field as

(a) $CA^\mu C^{-1} = -A^\mu$

(b) $CA^\mu C^{-1} = A^\mu$

(c) $CA^\mu C^{-1} = -J^\mu$

(d) $CA^\mu C^{-1} = J^\mu$

45. CP is
    (a) Never violated
    (b) Violated in the strong interaction
    (c) Violated in the weak interaction
    (d) Violated in the electroweak interaction
46. An operator A is antiunitary. It satisfies
    (a) $\langle A\phi | A\psi \rangle = \langle \phi | \psi \rangle$
    (b) $\langle A^{\dagger}\phi | A\psi \rangle = \langle \phi | \psi \rangle^{*}$
    (c) $\langle A\phi | A\psi \rangle = \langle \phi | \psi \rangle^{*}$
    (d) $\langle A\phi | A\psi \rangle = -\langle \phi | \psi \rangle^{*}$
47. An antilinear operator satisfies
    (a) $T\left(\alpha|\psi\rangle + \beta|\phi\rangle\right) = \alpha^{*}|\psi'\rangle + \beta^{*}|\phi'\rangle$
    (b) $T\left(\alpha|\psi\rangle + \beta|\phi\rangle\right) = -\alpha|\psi\rangle - \beta|\phi\rangle$
    (c) $T\left(\alpha|\psi\rangle + \beta|\phi\rangle\right) = \alpha^{*}\langle\psi| + \beta^{*}\langle\phi|$
48. The *CPT* theorem implies that
    (a) *CPT* is conserved in all interactions except the weak interaction.
    (b) *CPT* is conserved in all interactions.
    (c) *CPT* is conserved only in weak interactions.
    (d) *CPT* is conserved except in K meson decay.
49. The eigenvalues of charge conjugation are
    (a) $c = \pm 1$
    (b) $c = 0, \pm 1$
    (c) $c = \pm q$
    (d) $c = 0, \pm q$
50. If *CP* were conserved in weak interactions then
    (a) $|K_1\rangle \rightarrow$ would only decay into 2π mesons
    (b) $|K_1\rangle \rightarrow$ only decays into 3π mesons
    (c) Time invariance would not be satisfied
    (d) $|K_2\rangle \rightarrow$ only decays into 2π mesons

51. When interpreted as a single particle wave equation, the Klein-Gordon equation
    (a) Is plagued by infinities
    (b) Leads to negative probability densities
    (c) Always gives zero
    (d) Gives the same results as the Schrödinger equation
52. Under a Lorentz transformation $\Lambda^{\mu}{}_{\nu}$, a scalar field transforms as
    (a) $\varphi'(x) = -\varphi(\Lambda^{-1}x)$
    (b) $\varphi'(x) = \varphi(\Lambda x)$
    (c) $\varphi'(x) = -\varphi(\Lambda x)$
    (d) $\varphi'(x) = \varphi(\Lambda^{-1}x)$
53. The Klein-Gordon equation was deemed incorrect because it leads to solutions for the energy as
    (a) $E = \pm\sqrt{p^2 + m^2}$
    (b) $E = -\sqrt{p^2 + m^2}$
    (c) $E = \sqrt{p^2 + m^2}$
    (d) $E = p^2 + m^2$
54. The probability density of the Klein-Gordon equation is given by
    (a) $\rho = \varphi^* \varphi$
    (b) $\rho = -i\left(\varphi^* \dfrac{\partial \varphi}{\partial t} - \varphi \dfrac{\partial \varphi^*}{\partial t}\right)$
    (c) $\rho = i\left(\varphi^* \dfrac{\partial \varphi}{\partial t} - \varphi \dfrac{\partial \varphi^*}{\partial t}\right)$
    (d) $\rho = -\varphi^* \varphi$
55. Second quantization
    (a) Imposes *equal time* commutation relations on the fields and their conjugate momenta
    (b) Imposes *no time* commutation relations on the fields and their conjugate momenta
    (c) Promotes time to an operator
    (d) Imposes *equal time* commutation relations on the fields at the same spacetime point

56. Particles can be created and destroyed at relativistic energies. In particular
    (a) High energy processes tend to create antiparticles where $E = mc^2$.
    (b) To create an antiparticle, we need the rest mass energy $E = mc^2$.
    (c) To create a particle, we need the rest mass energy $E = mc^2$.
    (d) To create a particle, we need twice the rest mass energy $E = mc^2$ and this creates a particle–antiparticle pair.
57. The creation and annihilation operators of the harmonic oscillator satisfy
    (a) $[\hat{a}, \hat{a}^\dagger] = i$
    (b) $[\hat{a}, \hat{a}^\dagger] = 1$
    (c) $[\hat{a}, \hat{a}^\dagger] = -1$
    (d) $[\hat{a}, \hat{a}^\dagger] = -i$
58. The number operator is defined by
    (a) $\hat{N} = \hat{a}\hat{a}^\dagger$
    (b) $\hat{N} = \hat{a}^\dagger\hat{a}$
    (c) $\hat{N} = -\hat{a}^\dagger\hat{a}$
59. The number operator satisfies
    (a) $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$
    (b) $[\hat{N}, \hat{a}^\dagger] = -\hat{a}^\dagger$
    (c) $[\hat{N}, \hat{a}] = \hat{a}^\dagger$
    (d) $[\hat{N}, \hat{a}] = \hat{a}$
60. For a scalar field φ, the equal time commutation relation that is satisfied is
    (a) $\left[\hat{\varphi}(x), \hat{\pi}(y)\right] = 0$
    (b) $\left[\hat{\varphi}(x), \hat{\pi}(y)\right] = i\delta(\vec{x} - \vec{y})$
    (c) $\left[\hat{\varphi}(x), \hat{\pi}(y)\right] = i$
    (d) $[\hat{\varphi}(x), \hat{\pi}(y)] = -i\delta(\vec{x} - \vec{y})$
61. The Fourier expansion of a field is given by

$$\varphi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\left[a(\vec{p})e^{ipx} + a^\dagger(\vec{p})e^{-ipx}\right]$$

    Find the conjugate momentum.

62. In a field theory, the creation and annihilation operators satisfy
    (a) $\left[a(\vec{p}), a^{\dagger}(\vec{p}')\right] = \delta(\vec{p} - \vec{p}')$
    (b) $\left[a(\vec{p}), a^{\dagger}(\vec{p}')\right] = 0$
    (c) $\left[a(\vec{p}), a^{\dagger}(\vec{p}')\right] = i$
    (d) $\left[a(\vec{p}), a^{\dagger}(\vec{p}')\right] = \delta_{p,p'}$
63. To calculate the energy of the vacuum, we find
    (a) $\langle 0|0\rangle$
    (b) $\langle 0|\hat{a} + \hat{a}^{\dagger}|0\rangle$
    (c) It cannot be calculated
    (d) $\langle 0|\hat{H}|0\rangle$
64. The normal product is
    (a) $:\hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k}): = \hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})$
    (b) $:\hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k}): = \hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k})$
    (c) $:\hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k}): = -\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})$
    (d) $:\hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k}): = -\hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k})$
65. The time ordered product of two fields is
    (a) $T(\varphi(t_1)\psi(t_2)) = \psi(t_2)\varphi(t_1) - \varphi(t_1)\psi(t_2)$
    (b) $T(\varphi(t_1)\psi(t_2)) = \begin{cases} \varphi(t_1)\psi(t_2) \text{ if } t_1 > t_2 \\ \psi(t_2)\varphi(t_1) \text{ if } t_2 > t_1 \end{cases}$
    (c) $T(\varphi(t_1)\psi(t_2)) = \begin{cases} \psi(t_2)\varphi(t_1) \text{if } t_1 > t_2 \\ \varphi(t_1)\psi(t_2) \text{ if } t_2 > t_1 \end{cases}$
66. When including particles and antiparticles
    (a) We only include annihilation operators for antiparticles.
    (b) We only include creation operators for particles.
    (c) To get the field operator, we sum up negative frequency parts for particles together with positive frequency parts for antiparticles.
    (d) To get the field operator, we sum up positive frequency parts for particles together with negative frequency parts for antiparticles.

67. The creation and annihilation operators for antiparticles satisfy
    (a) $\left[\hat{b}(\vec{k}),\hat{b}^{\dagger}(\vec{k}')\right]=\delta(\vec{k}-\vec{k}')$
    (b) $\left[\hat{b}(\vec{k}),\hat{b}^{\dagger}(\vec{k}')\right]=-\delta(\vec{k}-\vec{k}')$
    (c) $\left[\hat{b}(\vec{k}),\hat{a}^{\dagger}(\vec{k}')\right]=\delta(\vec{k}-\vec{k}')$, where $\hat{a}^{\dagger}$ creates particles
    (d) $\left[\hat{b}(\vec{k}),\hat{b}^{\dagger}(\vec{k}')\right]=0$
68. For a charged field, the charge operator can be written as
    (a) $\hat{Q}=\int d^3k\,\vec{k}\left[\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})+\hat{b}^{\dagger}(\vec{k})\hat{b}(\vec{k})\right]=\hat{N}_{\hat{a}}-\hat{N}_{\hat{b}}$
    (b) $\hat{Q}=\int d^3k\left[\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})+\hat{b}^{\dagger}(\vec{k})\hat{b}(\vec{k})\right]=\hat{N}_{\hat{a}}-\hat{N}_{\hat{b}}$
    (c) $\hat{Q}=\int d^3k\,k\left[\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})+\hat{b}^{\dagger}(\vec{k})\hat{b}(\vec{k})\right]=\hat{N}_{\hat{a}}-\hat{N}_{\hat{b}}$
    (d) $\hat{Q}=\int d^3k\left[\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})-\hat{b}^{\dagger}(\vec{k})\hat{b}(\vec{k})\right]=\hat{N}_{\hat{a}}-\hat{N}_{\hat{b}}$
69. Find the energy for the vacuum using
$$\hat{H}_R=\hat{H}-\int d^3k=\int d^3k\,\omega_k\hat{N}(\vec{k})=\int d^3k\,\omega_k\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k})$$
70. State vectors in the interaction picture evolve in time according to
    (a) The interaction part of the Hamiltonian
    (b) The free part of the Hamiltonian
    (c) Are stationary in time
    (d) The full Hamiltonian
71. Interaction picture and Schrödinger picture operators are related by
    (a) $A_I=e^{iH_0t}A_Se^{-iH_It}$
    (b) $A_I=e^{iH_It}A_Se^{-iH_It}$
    (c) $A_I=e^{iH_0t}A_Se^{-iH_0t}$
    (d) $A_I=-e^{iH_0t}A_Se^{-iH_0t}$
72. In the interaction picture, the time evolution of operators is determined by
    (a) The full Hamiltonian
    (b) The interaction part of the Hamiltonian
    (c) The free part of the Hamiltonian
    (d) They are stationary in time

73. In quantum field theory, scattering
    (a) Results from the exchange of a force-carrying boson
    (b) Results from the exchange of a force-carrying fermion
74. In a Feynman diagram, if the arrow for a particle points against the direction of time flow
    (a) It is a force-carrying boson
    (b) It is an incoming or outgoing antiparticle
    (c) It is an incoming antiparticle or an outgoing particle
    (d) It is an incoming or outgoing particle
75. In a Feynman diagram, conservation of momentum at a vertex
    (a) Is enforced with a Dirac delta function $\delta(\Sigma p_i - q)$
    (b) Momentum is not conserved
    (c) Is enforced at the corresponding absorption vertex
    (d) 4-momentum is not conserved
76. At each vertex in a Feynman diagram
    (a) Add two factors of the coupling constant g
    (b) Take the product of two factors of the coupling constant g
    (c) Include one factor of the coupling constant g
    (d) Add one factor of the inverse coupling constant ig^{-1}
77. A propagator is associated with
    (a) An internal line in a Feynman diagram
    (b) Outgoing lines in a Feynman diagram
    (c) The coupling constant
78. The lifetime of a particle
    (a) Is proportional to the amplitude squared of a process
    (b) Is proportional to the magnitude of a process
    (c) Is proportional to the inverse of the amplitude squared of a process
    (d) Cannot be estimated by perturbation theory
79. The rate of decay of a process
    (a) Is proportional to the amplitude squared of the process
    (b) Cannot be calculated using perturbative expansions
    (c) Is proportional to the inverse of the amplitude squared of the process

80. Feynman diagrams can be best described as
    (a) A trick
    (b) Are a symbolic representation of a perturbative expansion
    (c) Are exact calculations
    (d) Can be used to exactly describe a process to second order
81. In quantum electrodynamics, the electromagnetic force results from
    (a) The exchange of photons
    (b) The exchange of photons and W particles
    (c) The exchange of photons, W particles, and Z particles
    (d) The field only
82. The 4-momentum and polarization vector of a photon state satisfy
    (a) $p_\mu \varepsilon^\mu = -1$
    (b) $p_\mu \varepsilon^\mu = 1$
    (c) $p_\mu \varepsilon^\nu = g_\mu{}^\nu$
    (d) $p_\mu \varepsilon^\mu = 0$
83. The gauge group of electromagnetism is
    (a) $SU(3)$
    (b) $SU(2) \otimes U(1)$
    (c) $SU(2)$
    (d) $U(1)$
84. The gauge group of electroweak theory is
    (a) $SU(3)$
    (b) $SU(2) \otimes U(1)$
    (c) $SU(2)$
    (d) $U(1)$
85. A Dirac particle is interacting with the electromagnetic field. The interaction Lagrangian is best written as
    (a) $L_{\text{int}} = -q\bar{\psi}\gamma^\mu \psi A_\mu$
    (b) $L_{\text{int}} = -\bar{\psi}\gamma^\mu \psi A_\mu$
    (c) $L_{\text{int}} = -q\bar{\psi}\psi A_\mu$
    (d) $L_{\text{int}} = m^2 \bar{\psi}\psi A$

86. A global $U(1)$ transformation can be written as
    (a) $\psi(x) \to e^{i\theta(x)}\psi(x)$
    (b) $\psi(x) \to e^{i\theta}\psi(x)$
    (c) $\psi(x) \to e^{-i\theta(x)}\psi(x)$
    (d) $\psi(x) \to e^{i\theta}\psi(x) + \partial_\mu\theta$
87. Consider a Feynman diagram for an electromagnetic process. An outgoing particle is represented by
    (a) $\bar{u}(p,s)$
    (b) $u(p,s)$
    (c) $-\bar{u}(p,s)$
    (d) $\bar{u}(-p,s)$
88. In a Feynman diagram for a QED process, at each vertex we add a factor of
    (a) $g_e = \sqrt{4\pi\alpha}$
    (b) $-ig_e\gamma^\mu$
    (c) $ig_e\gamma^\mu$
    (d) $ig_e\gamma^0$
89. For an internal line in a Feynman diagram in QED, an electron or positron is associated with a propagator of the form
    (a) $\dfrac{i\gamma^\mu q_\mu}{q^2 - m^2}$
    (b) $\dfrac{im}{q^2 - m^2}$
    (c) $\dfrac{i\gamma^\mu}{q^2 - m^2}$
    (d) $\dfrac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$
90. Spontaneous symmetry breaking can be best described as
    (a) Setting the minimum potential energy to the coupling constant
    (b) Reducing the minimum potential energy by the ground state energy
    (c) Shifting the minimum of the potential energy such that the energy of the ground state is nonzero

91. Consider scalar fields. A mass term in the Lagrangian can be recognized
    (a) Because it is quadratic in the fields
    (b) Because it is quartic in the fields
    (c) It is linear
    (d) It is real
92. A covariant derivative
    (a) Ensures that the Euler-Lagrange equations are satisfied
    (b) Ensures the Lagrangian is invariant under a Lorentz transformation
    (c) Is not used in quantum electrodynamics
    (d) Can only be used in weak theory
93. Let $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$ and compute $\frac{1}{2}(1-\gamma_5)\psi$.
94. The adjoint spinor is given by
    (a) $\psi^\dagger$
    (b) $\bar{\psi} = \psi^\dagger \gamma^0$
    (c) $\bar{\psi} = \gamma^0 \psi$
    (d) γ^0
95. For a spinor, the Lagrangian can be separated into left- and right-handed kinetic parts as
    (a) $L = i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R$
    (b) $L = i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L - i\bar{\psi}_R \gamma^\mu \partial_\mu \psi_R$
    (c) $L = i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + i(1+\gamma_5)\bar{\psi}_R$
    (d) $L = i\bar{\psi}_L (1-\gamma_5)\gamma^\mu \partial_\mu \psi_L + i\bar{\psi}_R (1+\gamma_5)\gamma^\mu \partial_\mu \psi_R$
96. The isospin of the neutrino is
    (a) 0
    (b) $I_\nu^3 = +\frac{3}{2}$
    (c) $I_\nu^3 = +\frac{1}{2}$
    (d) $I_\nu^3 = -\frac{1}{2}$
97. A right-handed electron
    (a) Has isospin +1/2
    (b) Has isospin −1/2

(c) Has isospin +3/2

(d) Has 0 isospin

98. In electroweak theory, conservation of hypercharge corresponds with

(a) A symmetry describing three gauge fields, $SU(2): W_\mu^1, W_\mu^2, W_\mu^3$

(b) A single gauge field $U(1): B_\mu$

(c) A symmetry describing three gauge fields, $SU(2): W_\mu^+, W_\mu^-, Z$

99. The $SU(2)$ transformation of electroweak theory can be written as

(a) $U(\alpha) = \exp(i\alpha_j \tau_j/2)$, where the generators are the Pauli matrices.

(b) $U(\alpha) = \exp(i\alpha_j \lambda_j/2)$, where λ_j are the Gell-Mann matrices.

(c) There is no $SU(2)$ transformation that leaves electroweak theory invariant.

100. The Weinberg angle

(a) Mixes $SU(2)$ and $SU(3)$ symmetries in quantum chromodynamics

(b) Gives a scattering cross section

(c) Mixes the gauge fields in electroweak theory giving rise to the massless electromagnetic field and the massive Z vector boson

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Solutions to Quizzes and Final Exam

Chapter 1

1. d
2. a
3. a
4. c
5. b
6. a
7. c
8. c
9. c
10. d

Chapter 2

1. $\frac{d^2x}{dt^2} + \omega^2 x = -\frac{\alpha}{m}$
2. (a) $\partial_\mu \partial^\mu \varphi - m^2 \varphi = \frac{\partial V}{\partial \varphi}$

   (b) $\pi = \dot{\varphi}$

   (c) $H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi) \right)$
3. $J^\mu = \partial^\mu \varphi$
4. Each field separately satisfies a Klein-Gordon equation, that is, $\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0$, $\partial_\mu \partial^\mu \varphi^\dagger + m^2 \varphi^\dagger = 0$. To get this result apply Eq. (2.14) to the Lagrangian Eq. (2.37).
5. $Q = i \int d^3x \left(\varphi^\dagger \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^\dagger}{\partial t} \right)$
6. The action is invariant under that transformation.

Chapter 3

1. $U = \cos\alpha + i\sigma_x \sin\alpha$
2. $2\delta_{ij}$
3. 1
4. Try $\vec{\sigma}^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$
5. $K_x = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
6. $K_y = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

7. No, because the algebra among the generators requires the introduction of the angular momentum operators. Therefore, Lorentz transformations together with rotations form a group.

Chapter 4

1. c
2. a
3. d
4. a
5. c

Chapter 5

1. $i\frac{\partial\bar{\psi}}{\partial x^{\mu}}\gamma^{\mu}+m\bar{\psi}$
2. 0
3. $v=\frac{\vec{k}\cdot\vec{\sigma}}{\omega_k+m}u$
4. $\psi(0)=\sqrt{2m}\begin{pmatrix}u\\v\end{pmatrix}$
5. $-\frac{i}{2}\begin{pmatrix}\sigma_1 & 0\\0 & -\sigma_1\end{pmatrix}$
6. $\gamma^{\mu}(i\partial_{\mu}-qA_{\mu})\psi-m\psi=0$

Chapter 6

1. 0
2. $\left[n(\vec{k})+1\right]\hat{a}^{\dagger}(\vec{k})\left|n(\vec{k})\right\rangle$
3. 0
4. $[H,Q]=0$

Chapter 7

1. $-i\dfrac{g^2}{(p_2-p_4)^2-m_B^2}$
2. $\dfrac{1}{g^2}$
3. b
4. c
5. a
6. $\alpha = 1/137$

Chapter 8

1. $iqF_{\mu\nu}$
2. a
3. $-g_e^2[v(k')\gamma^\mu \overline{v}(k)]\dfrac{g_{\mu\nu}}{(p-p')^2}\overline{u}(p')\gamma^\mu u(p)$
4. c
5. a

Chapter 9

1. It describes a massive particle. Use $\cosh ax = 1+\dfrac{a^2x^2}{2}+\dfrac{a^4x^4}{4!}+O(x^6)$.
2. b
3. $L = -\partial_\mu\psi\,\partial^\mu\psi - \psi^2\partial_\mu\theta\,\partial^\mu\theta - \dfrac{\lambda}{4}\left(\psi^4 - 2\dfrac{\mu^2}{\lambda}\psi + \dfrac{\mu^4}{\lambda^2}\right)$
4. $\dfrac{\mu^2}{2}\psi$
5. $-\dfrac{\lambda}{4}\psi^4$

Chapter 10

1. b
2. a
3. d
4. b
5. b
6. b
7. a
8. a

Chapter 11

1. b
2. a
3. a
4. c
5. b
6. d

Chapter 12

1. 0 $\quad \frac{1}{2}p^2+\frac{1}{2}W^2+\frac{1}{2}\sigma_3\frac{dW}{dx}$
2. $i\sigma^n\partial_n\bar{\chi}=0 \qquad \partial_n\partial^n A=0 \qquad F=0$
3. $S_\varepsilon^n=\sqrt{2}\chi\sigma^n\bar{\sigma}^m\varepsilon\partial_n\bar{A}$
4. $Q^a=\sqrt{2}\int d^3x(\chi\sigma^0\bar{\sigma}^m)^a\partial_m\bar{A}$
5. 0. The states are eigenstates of momentum, so that $P^\mu|i\rangle=p^\mu|i\rangle$. We have

$$\sum_i\langle i|(-1)^{2S}P^\mu|i\rangle=p^\mu Tr[(-1)^{2S}]=p^\mu(n_B-n_F)=0$$

$$\Rightarrow n_B=n_F$$

Final Exam

1. This is the Klein-Gordon equation $\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi + m^2\varphi = 0.$
2. There is no conserved quantity, $Q = 0.$
3. $\partial^\mu F_{\mu\nu} + m^2 A_\nu = 0$
4. $\partial^\mu A_\mu = 0$
5. The φ^3 term must be dropped, so $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + \frac{\rho^4}{24}\varphi^4.$
6. $4g^{\mu\nu}$
7. 0
8. $\partial_\mu\partial^\mu = \partial^2$
9. c
10. 0
11. $2\psi_R$
12. d
13. c
14. a
15. c
16. a
17. d
18. c
19. b
20. a
21. c
22. a
23. d
24. a
25. b
26. d
27. a
28. b
29. d
30. a
31. c
32. a
33. b
34. a
35. c
36. a
37. d
38. d
39. a
40. b
41. d
42. a
43. b
44. a
45. c
46. c
47. a
48. b
49. a
50. a
51. b
52. d
53. a
54. c
55. a
56. d
57. b
58. b
59. a
60. b

61. $i\int \frac{d^3p}{\sqrt{(2\pi)^3}}\sqrt{\frac{p^0}{2}}\left[a(\vec{p})e^{ipx} - a^{\dagger}(\vec{p})e^{-ipx}\right]$

62. a
63. d
64. a
65. b
66. d
67. a
68. b
69. 0
70. b
71. c
72. c
73. a
74. b
75. a
76. c
77. a
78. c
79. a
80. b
81. a
82. d
83. d
84. b
85. a
86. b
87. a
88. c
89. d
90. c
91. a
92. b
93. ψ_L
94. b
95. a
96. c
97. d
98. b
99. a
100. c

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