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Citation: American Journal of Physics **64**, 812 (1996); doi: 10.1119/1.18181
View online: http://dx.doi.org/10.1119/1.18181
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/64/6?ver=pdfcov
Published by the American Association of Physics Teachers

Direct calculation of time dilation

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(Received 27 January 1995; accepted 28 August 1995)

The possibility of explaining time dilation as a dynamic cause-and-effect phenomenon is explored by calculating the rates of three elementary electromagnetic clocks in a stationary and in a moving reference frame. The operation of the clocks is based on the interaction between a field-experiencing electric point charge and different field-producing electric charge configurations. The calculations show that, when the clocks move, they run as predicted by the special relativity theory. The slowing down of the three moving clocks is due to a different electromagnetic field produced by the moving field-producing charges (and hence due to a different force acting on the field-experiencing charges) as well as due to a change of the effective mass of the moving field-experiencing charges. Thus, for the clocks under consideration, time dilation can be considered a dynamic cause-and-effect phenomenon and not merely a kinematic effect, as time dilation is usually explained in conventional presentations of the special relativity theory.

I. INTRODUCTION

A fundamental concept in Einstein's special relativity theory is the concept of time dilation, according to which time slows down in a reference frame moving relative to the reference frame assumed to be at rest (laboratory).[1] In conventional presentations of the special relativity theory, time dilation is treated as a kinematic effect. The question of whether or not time dilation can also be explained as a dynamic cause-and-effect phenomenon is usually not discussed. Naturally, insofar as time dilation is supposed to hold for any clock mechanism whatsoever, an all-inclusive dynamic (causal) interpretation of time dilation is hardly possible. But it should be possible to provide a causal interpretation of time dilation for some specific clock mechanisms. The aim of this paper is to provide such an interpretation of time dilation for some very simple clocks. The basic idea of the paper is as follows.

As a physical entity, time is defined in terms of specific measurement procedures, which for the purpose of the present discussion may be described simply as "observing the rate of clocks." Therefore the operational manifestation of time dilation is the slowing down of the rate of clocks that are located in moving reference frames. But a clock is a physical apparatus or device and is subject to the laws of physics in accordance with which the clock is constructed. Hence, if the clock runs slower when it is located in a moving reference frame, its slower rate should be explainable on the basis of the specific laws responsible for the operation of the clock. The laws of electromagnetism are especially well known for both stationary and moving systems. For the purpose of this paper it is natural therefore to consider clocks operating on the basis of electromagnetic laws.

II. THE THEORY

Clock #1. Consider a ring of radius a carrying a uniformly distributed charge q_1. Let the axis of the ring be the x axis. The electric field on the axis of the ring is[2]

$$\mathbf{E}=\frac{q_1 x}{4\pi\epsilon_0(a^2+x^2)^{3/2}}\,\mathbf{i}. \tag{1}$$

A charge q_2, whose polarity is opposite to that of q_1, is placed on the x axis near the center of the ring at a distance x from the center and is constrained to move only along the axis.[3] If q_2 is sufficiently close to the center, so that $x \ll a$, which we assume to be the case, the force on q_2 is essentially

$$\mathbf{F}=-\frac{q_1 q_2 x}{4\pi\epsilon_0 a^3}\,\mathbf{i}. \tag{2}$$

Let the ring be fixed in the laboratory and let the mass of q_2 be m_0. Since the force given by Eq. (2) is a linear restoring force, the ring and the charge constitute a simple harmonic oscillator, and the period of oscillations of q_2 is[4]

$$T=2\pi\left(\frac{m_0}{F/x}\right)^{1/2}=4\pi^{3/2}a^{3/2}\left(\frac{m_0\epsilon_0}{q_1 q_2}\right)^{1/2}. \tag{3}$$

Clearly, the ring and the charge may be considered to constitute a clock and can be used for measuring time in terms of the period of oscillations T.

Let us now assume that the same ring and the charge q_2 are located in a reference frame moving along the x axis with velocity $\mathbf{v}$ relative to the laboratory. By symmetry, the electric field on the axis of the ring is the same as the x component of the electric field of a moving point charge q_1 whose perpendicular distance from the axis is a. The electric field of a moving point charge is given by[5]

$$\mathbf{E}_m=\frac{q(1-v^2/c^2)}{4\pi\epsilon_0 r^3[1-(v^2/c^2)\sin^2\theta]^{3/2}}\,\mathbf{r}, \tag{4}$$

where v is the velocity of the charge, c is the velocity of light, $\mathbf{r}$ is the vector from the position of the charge to the point of observation, and θ is the angle between $\mathbf{r}$ and $\mathbf{v}$; the subscript m is used to indicate that the field under consideration is that of the moving charge. Thus the electric field on the axis of the ring is now

$$\mathbf{E}_m=\frac{q_1(1-v^2/c^2)x}{4\pi\epsilon_0(a^2+x^2)^{3/2}[1-v^2a^2/c^2(a^2+x^2)]^{3/2}}\,\mathbf{i}. \tag{5}$$

Assuming, as before, that $x \ll a$, we then have for the force on q_2

$$\mathbf{F}_m=-\frac{q_1 q_2 x}{4\pi\epsilon_0 a^3(1-v^2/c^2)^{1/2}}\,\mathbf{i}. \tag{6}$$

Let us also assume that the velocity $\mathbf{v}$ of the moving reference frame is much larger than the maximum velocity of q_2 relative to the ring. In this case the velocity of q_2 relative to the laboratory is essentially v, and the *longitudinal mass*[6] of q_2 is

$$m_l=\frac{m_0}{(1-v^2/c^2)^{3/2}}. \quad (7)$$

The period of oscillations of q_2 is therefore

$$T_m=2\pi\left(\frac{m_l}{F_m/x}\right)^{1/2}$$
$$=2\pi\left(\frac{m_0 4\pi\epsilon_0 a^3(1-v^2/c^2)^{1/2}}{(1-v^2/c^2)^{3/2}q_1q_2}\right)^{1/2}$$
$$=4\pi^{3/2}a^{3/2}\left(\frac{m_0\epsilon_0}{(1-v^2/c^2)q_1q_2}\right)^{1/2}, \quad (8)$$

so that

$$T_m=\frac{1}{(1-v^2/c^2)^{1/2}}T. \quad (9)$$

Thus the period of oscillations of q_2 in the moving reference frame is by the factor $(1-v^2/c^2)^{-1/2}$ *longer* than the period of oscillations of q_2 in the laboratory. Hence our clock consisting of the charged ring and the point charge runs *slower* when the clock is moving, and the rate of the moving clock is $(1-v^2/c^2)^{-1/2}$ times the rate of the same stationary clock.[7]

Clock #2. Consider two point charges of the same magnitude and polarity located on the z axis at distances $\pm a$ from the origin. Let the magnitude of each charge be q_1 and let the charges be fixed in the laboratory. A point charge q_2, whose polarity is opposite to that of the first two charges and whose mass is m_0, is placed on the y axis at a distance y close to the origin ($y \ll a$) and is constrained to move along the axis. The electric field at the location of q_2 is now

$$\mathbf{E}=\frac{q_1 y}{2\pi\epsilon_0(a^2+y^2)^{3/2}}\mathbf{j}, \quad (10)$$

which, after neglecting y^2 in the denominator, becomes

$$\mathbf{E}=\frac{q_1 y}{2\pi\epsilon_0 a^3}\mathbf{j}. \quad (11)$$

The force on q_2 is therefore

$$\mathbf{F}=-\frac{q_1q_2y}{2\pi\epsilon_0 a^3}\mathbf{j}. \quad (12)$$

Except for the direction and the factor 2 instead of 4 in the denominator, this is the same force as that given by Eq. (2). Therefore q_2 executes a simple harmonic motion with the period

$$T=2\pi\left(\frac{m_0}{F/y}\right)^{1/2}=(2\pi a)^{3/2}\left(\frac{m_0\epsilon_0}{q_1q_2}\right)^{1/2}. \quad (13)$$

Let us now assume that the three charges are placed in a reference frame moving along the x axis with velocity $\mathbf{v}$ relative to the laboratory. In determining the force on q_2, we must now take into account that q_2 is subjected not only to the electric field but also to the magnetic field. As seen from the laboratory, the force on q_2 is therefore the Lorentz force[8]

$$\mathbf{F}_L=q_2(\mathbf{E}_m+\mathbf{v}\times\mathbf{B}_m), \quad (14)$$

where $\mathbf{E}_m$ is the electric field, and $\mathbf{B}_m$ is the magnetic flux density field produced at the location of q_2 by the moving charges q_1.

The electric field at the location of q_2 is given by Eq. (4) with $q=q_1$, $\mathbf{r}=y\mathbf{j}$, $r=(a^2+y^2)^{1/2}$, $\sin\theta=1$, and with the factor 2 instead of 4 in the denominator, that is

$$\mathbf{E}_m=\frac{q_1(1-v^2/c^2)y}{2\pi\epsilon_0(a^2+y^2)^{3/2}[1-v^2/c^2]^{3/2}}\mathbf{j}, \quad (15)$$

which, after neglecting y^2, becomes

$$\mathbf{E}_m=\frac{q_1 y}{2\pi\epsilon_0 a^3(1-v^2/c^2)^{1/2}}\mathbf{j}. \quad (16)$$

Since the electric and magnetic fields of any uniformly moving charge distribution are connected by the formula[9]

$$\mathbf{B}_m=\frac{\mathbf{v}\times\mathbf{E}_m}{c^2}, \quad (17)$$

we have for the Lorentz force acting on q_2

$$\mathbf{F}_L=-\frac{q_1q_2}{2\pi\epsilon_0 a^3(1-v^2/c^2)^{1/2}}\left(\mathbf{j}+\frac{\mathbf{v}\times(\mathbf{v}\times\mathbf{j})}{c^2}\right), \quad (18)$$

or

$$\mathbf{F}_L=-\frac{q_1q_2}{2\pi\epsilon_0 a^3}\left(1-\frac{v^2}{c^2}\right)^{1/2}\mathbf{j}. \quad (19)$$

Using now the *transverse mass*[6] of q_2

$$m_t=\frac{m_0}{(1-v^2/c^2)^{1/2}}, \quad (20)$$

we obtain for the period of oscillations of q_2

$$T_m=2\pi\left(\frac{m_t}{F_L/x}\right)^{1/2}=(2\pi a)^{3/2}\left(\frac{m_0\epsilon_0}{(1-v^2/c^2)q_1q_2}\right)^{1/2}. \quad (21)$$

Once again therefore

$$T_m=\frac{1}{(1-v^2/c^2)^{1/2}}T. \quad (22)$$

Clock #3. Consider two point charges q_1 and q_2 of the same polarity located at a distance r one from the other. Let q_1 be fixed in the laboratory and let q_2 be free to move under the action of q_1. The force exerted by q_1 upon q_2 is

$$\mathbf{F}=\frac{q_1q_2}{4\pi\epsilon_0 r^3}\mathbf{r}. \quad (23)$$

If r is sufficiently large, and if q_2 moves only a short distance, which we assume to be the case, we can ignore the variation of the force with r, so that the force can be considered essentially constant.[10] Let the mass of q_2 be m_0. The distance traveled by q_2 during a time interval Δt (as measured by the "standard clock" in the laboratory) is then

$$d=\frac{F}{2m_0}(\Delta t)^2=\frac{q_1q_2}{8\pi\epsilon_0 m_0 r^2}(\Delta t)^2. \quad (24)$$

Hence we can use the two charges as a clock for measuring time intervals in terms of the distance d traveled by q_2. By Eq. (24), the formula for converting d into Δt is

$$\Delta t=\left(\frac{8\pi\epsilon_0 m_0 r^2}{q_1q_2}d\right)^{1/2}. \quad (25)$$

Note that the rate of our two-charge clock depends on how fast q_2 travels under the action of q_1: the larger is Δt corresponding to a given d, the slower is the rate of the clock.

Let us now assume that we have a second two-charge clock, identical with the one just described, but located in a reference frame that moves along the x axis with velocity $\mathbf{v}$ relative to the laboratory. Let us also assume that the line joining the two charges is perpendicular to $\mathbf{v}$, and let us assume that the velocity which q_2 acquires under the action of q_1 is much smaller than v. As seen from the laboratory, the force on q_2 is then the Lorentz force

$$\mathbf{F}_L = q_2(\mathbf{E}_m + \mathbf{v}\times\mathbf{B}_m), \tag{26}$$

where $\mathbf{E}_m$ is the electric field, and $\mathbf{B}_m$ is the magnetic flux density field produced at the location of q_2 by the moving q_1.

Since the line joining the two charges is perpendicular to $\mathbf{v}$, so that $\sin\theta=1$ in Eq. (4), the electric field $\mathbf{E}_m$ is

$$\mathbf{E}_m = \frac{q_1}{4\pi\epsilon_0 r^3(1-v^2/c^2)^{1/2}}\,\mathbf{r}, \tag{27}$$

and the magnetic flux density field is

$$\mathbf{B}_m = \frac{\mathbf{v}\times\mathbf{E}_m}{c^2} = \frac{q_1}{4\pi\epsilon_0 r^3 c^2(1-v^2/c^2)^{1/2}}\,\mathbf{v}\times\mathbf{r}. \tag{28}$$

Hence the Lorentz force on q_2 is

$$\mathbf{F}_L = \frac{q_1 q_2}{4\pi\epsilon_0 r^3(1-v^2/c^2)^{1/2}}\left(\mathbf{r} + \frac{\mathbf{v}\times(\mathbf{v}\times\mathbf{r})}{c^2}\right), \tag{29}$$

or

$$\mathbf{F}_L = \frac{q_1 q_2}{4\pi\epsilon_0 r^3}\left(1-\frac{v^2}{c^2}\right)^{1/2}\mathbf{r}. \tag{30}$$

Using now the *transverse mass* of q_2 [see Eq. (20)], we obtain for the distance traveled by q_2 under the action of q_1

$$d_m = \frac{F_L}{2m_t}(\Delta t_m)^2 = \frac{q_1 q_2(1-v^2/c^2)}{8\pi\epsilon_0 m_0 r^2}(\Delta t_m)^2, \tag{31}$$

where the subscripts m are used to indicate that we are now dealing with the moving two-charge clock. According to Eq. (31), the time interval needed for q_2 to travel through the distance d_m is

$$\Delta t_m = \left(\frac{8\pi\epsilon_0 m_0 r^2}{q_1 q_2(1-v^2/c^2)}\,d_m\right)^{1/2}. \tag{32}$$

Let us now compare Δt and Δt_m corresponding to equal distances traveled by q_2 under the action of q_1 in the stationary and in the moving two-charge clock, that is, corresponding to

$$d_m = d. \tag{33}$$

From Eqs. (25), (32), and (33) we have

$$\Delta t_m = \frac{1}{(1-v^2/c^2)^{1/2}}\,\Delta t. \tag{34}$$

Thus Δt_m is by the factor $(1-v^2/c^2)^{-1/2}$ *longer* than Δt. Hence our moving two-charge clock runs $(1-v^2/c^2)^{-1/2}$ times *slower* than the identical stationary clock.[11]

III. SUMMARY

Conventional presentations of the special relativity theory treat time dilation as a strictly kinematic effect and do not provide a dynamic cause-and-effect type explanation of the dilation. The calculations presented in this paper show that at least in some cases time dilation can be explained also dynamically.

[1]See, for example, A. R. French, *Special Relativity* (Norton, New York, 1968), pp. 97–104.

[2]See, for example, A. Hudson and R. Nelson, *University Physics* (Saunders, New York, 1990), 2nd ed., pp. 572–573.

[3]The charge must be constrained to stay on the axis because otherwise it is unstable with respect to a lateral displacement.

[4]See, for example, Ref. 2, pp. 338–339.

[5]This equation (in a different notation) was first derived by Oliver Heaviside. See Oliver Heaviside, "The Electromagnetic Effects of a Moving Charge," The Electrician **22**, 147–148 (1888); Oliver Heaviside, "On the Electromagnetic Effects due to the Motion of Electricity Through a Dielectric," Philos. Mag. **27**, 324–339 (1889). For a modern derivation see, for example, David J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, NJ, 1981) 2nd ed., pp. 421–425. For another derivation see Ref. 9(b), below.

[6]See, for example, Herbert Goldstein, *Classical Mechanics* (Addison-Wesley, Cambridge, MA, 1951), p. 205; Carl G. Adler, "Does mass really depend on velocity, dad?" Am. J. Phys. **55**(8), 739–743 (1987).

[7]Similar clocks can be constructed by replacing the charged ring by two or more equal point charges in the yz plane arranged symmetrically relative to the x axis. A different numerical factor will then appear in Eqs. (2) and (6), but Eq. (9) (the time dilation equation) will remain the same.

[8]See, for example, Ref. 2, p. 691.

[9]See, for example, (a) W. G. V. Rosser, *Classical Electromagnetism via Relativity* (Plenum, New York, 1968), p. 39; (b) for a more direct proof see Oleg D. Jefimenko, "Direct calculation of the electric and magnetic fields of an electric point charge moving with constant velocity," Am. J. Phys. **62**(1), 79–85 (1994).

[10]This approximation has no effect on the conclusions reached in the paper, because, as we shall presently see, the dependence of the force on r is exactly the same for the charges in the laboratory and for the charges in the moving reference frame. However, neglecting the dependence of the force on r simplifies the calculations and eliminates irrelevant algebraic manipulations.

[11]A clock similar to Clock #3 can be constructed by replacing the point charge q_1 by a long line charge of uniform line density λ lying along the z axis and having its midpoint at the origin. The point charge q_2 is then placed on the y axis at a distance r from the origin. The electric field produced by the stationary line charge at the location of q_2 is

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\mathbf{j},$$

and the electric field produced by the same line charge moving along the x axis is

$$\mathbf{E}_m = \frac{\lambda}{2\pi\epsilon_0 r(1-v^2/c^2)^{1/2}}\,\mathbf{j}.$$

(This formula was first obtained by Oliver Heaviside; see his articles cited in Ref. 5, above). The remaining equations are practically the same as for Clock #3, and the time dilation is once again given by Eq. (34).