ON THE CONTINUOUS ABSORPTION COEFFICIENT OF THE NEGATIVE HYDROGEN ION. II

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ABSTRACT

In this paper the continuous absorption coefficient of the negative hydrogen ion, determined in an earlier paper (*Ap. J.*, **102**, 223, 1945) in terms of the matrix elements of the momentum operator, is further improved to take into account the effect of the static field of the hydrogen atom on the motion of the ejected electron. It is shown that for wave functions for the ground state of H^- of the forms generally considered, the formula for the absorption cross-section can be reduced to the form

$$\kappa = \frac{3.7062}{k(k^2+0.05512)} \left| \int_0^\infty W_2(r)\chi_1(r)dr \right|^2 \times 10^{-18}\ \text{cm}^2 ,$$

where k denotes the momentum of the ejected electron in atomic units, $W_2(r)$ a certain weight function which can be tabulated, and $\chi_1(r)$ the radial part of a p-spherical wave in the Hartree field of a hydrogen atom which tends to unit amplitude at infinity.

The absorption cross-sections of H^- have been evaluated according to the foregoing formula for various wave lengths. It is found that the new values are larger than those obtained with a plane-wave representation of the outgoing electron by about 5 per cent in the visual and the near infrared regions. The new absorption-curve places the maximum at about λ 8500 A; at this wave length the atomic absorption coefficient has the value 4.52×10^{-17} cm^2.

1. *Introduction.*—In an earlier paper[1] it was shown that the continuous absorption coefficient of the negative hydrogen ion is most reliably determined in terms of the matrix elements of the momentum operator. However, in the actual evaluation of the absorption cross-sections by this method, the plane-wave approximation for the ejected electron was used (see I, eq. [15]). In this paper we propose to consider certain refinements in this direction.

2. *Formulae for the evaluation of the continuous absorption coefficient of* H^- *using the wave functions of an electron in the Hartree field of a hydrogen atom.*—It would seem that, if we abandon the plane-wave approximation[2] for the ejected electron, the next simplest thing to do will be to use the wave functions of an electron moving in the static field of a hydrogen atom.[3] In other words, it would appear that in the "next approximation" we use for the wave functions Ψ_c describing the continuous states of H^- that of a hydrogen atom in its ground state, together with an electron moving in the Hartree field,

$$-\left(\frac{1}{r}+1\right) e^{-2r} . \qquad (1)^{4}$$

On this approximation Ψ_c will have the form

$$\Psi_c = \frac{1}{\sqrt{2\pi}} \{ e^{-r_2}\phi(r_1) + e^{-r_1}\phi(r_2) \} , \qquad (2)$$

where $\phi(r)$ satisfies the wave equation

$$\nabla^2\phi + \left[k^2 + 2\left(1+\frac{1}{r}\right) e^{-2r} \right] \phi = 0 \qquad (3)$$

[1] S. Chandrasekhar, *Ap. J.*, **102**, 223, 1945. This paper will be referred to as "I."

[2] First suggested in this connection by H. S. W. Massey and D. R. Bates, *Ap. J.*, **91**, 202, 1940.

[3] A. Wheeler and R. Wildt, *Ap. J.*, **95**, 281, 1942; also S. Chandrasekhar, *Ap. J.*, **100**, 176, 1944.

[4] In writing this equation, we have adopted the atomic system of units. These units will be used in all our formal developments.

and tends asymptotically at infinity to a plane wave of unit amplitude along some chosen direction. If this direction, in which the ejected electron moves at infinity, be chosen as the polar axis of a spherical system of co-ordinates, the requirement is that

$$\phi(r) \to e^{ikr\cos\vartheta} \quad \text{as} \quad r \to \infty . \tag{4}$$

On the other hand, since

$$\left.\begin{aligned} e^{ikr\cos\vartheta} &= \left(\frac{\pi}{2kr}\right)^{1/2} \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\vartheta) J_{l+1/2}(kr) \\ &\to \sum_{l=0}^{\infty} \frac{i^l(2l+1)}{kr} P_l(\cos\vartheta)\cos\left(kr - \tfrac{1}{2}l\pi - \tfrac{1}{2}\pi\right), \end{aligned}\right\} \tag{5}$$

it is evident that the solution for ϕ appropriate to our problem is

$$\phi = \sum_{l=0}^{\infty} \frac{i^l(2l+1)}{kr} P_l(\cos\vartheta)\,\chi_l(r) , \tag{6}$$

where the radial function $\chi_l(r)$ is a solution of the equation

$$\frac{d^2\chi_l}{dr^2} + \left\{k^2 - \frac{l(l+1)}{r^2} + 2\left(1+\frac{1}{r}\right)e^{-2r}\right\}\chi_l = 0 , \tag{7}$$

which tends to a pure sinusoidal wave of unit amplitude at infinity.

Thus, on our present approximation, the wave function can be expressed in the form (cf. I, eq. [33])

$$\left.\begin{aligned} \Psi_c = \frac{1}{\sqrt{2\pi}} \Big\{ e^{-r_2} \sum_{l=0}^{\infty} \frac{i^l(2l+1)}{kr_1} P_l(\mu_1)\,\chi_l(r_1;\, k) \\ + e^{-r_1} \sum_{l=0}^{\infty} \frac{i^l(2l+1)}{kr_2} P_l(\mu_2)\,\chi_l(r_2;\, k) \Big\}, \end{aligned}\right\} \tag{8}$$

where we have used μ_1 and μ_2 to denote $\cos\vartheta_1$ and $\cos\vartheta_2$, respectively.

For the evaluation of the absorption cross-sections according to formula II of paper I, we need the quantity

$$\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right)\Psi_c . \tag{9}$$

For Ψ_c given by equation (8) it can be readily shown that

$$\left.\begin{aligned} \left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right)\Psi_c = \frac{1}{\sqrt{2\pi}} \Big[&-\mu_1 e^{-r_1} \sum_{l=0}^{\infty} \frac{i^l(2l+1)}{kr_2} P_l(\mu_2)\,\chi_l(r_2) \\ &-\mu_2 e^{-r_2} \sum_{l=0}^{\infty} \frac{i^l(2l+1)}{kr_1} P_l(\mu_1)\,\chi_l(r_1) \\ &+ e^{-r_1} \sum_{l=0}^{\infty} \frac{i^l}{k} \{ lP_{l-1}(\mu_2) S_l(r_2) + (l+1) P_{l+1}(\mu_2) T_l(r_2) \} \\ &+ e^{-r_2} \sum_{l=0}^{\infty} \frac{i^l}{k} \{ lP_{l-1}(\mu_1) S_l(r_1) + (l+1) P_{l+1}(\mu_1) T_l(r_1) \} \Big], \end{aligned}\right\} \tag{10}$$

where

$$S_l(r)=\frac{\partial}{\partial r}\left(\frac{\chi_l}{r}\right)+(l+1)\frac{\chi_l}{r^2} \qquad (11)$$

and

$$T_l(r)=\frac{\partial}{\partial r}\left(\frac{\chi_l}{r}\right)-l\frac{\chi_l}{r^2}. \qquad (12)$$

For a wave function of the ground state of the form I, equation (3), we find after some lengthy but straightforward calculations that

$$\left.\begin{aligned}&\int\Psi_d\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)\Psi_c d\tau\\&=-\frac{(2048\pi^3)^{1/2}\mathfrak{N}i}{k(1+a)^3}\left\{\int_0^\infty dr S_1(r)\left[e^{-ar}\sum_{j=0}^{7}s_j r^{j+1}+e^{-(1+2a)r}\sum_{j=0}^{+1}\sigma_j r^{j+1}\right]\right.\\&\qquad\left.-\int_0^\infty dr\chi_1(r)\left[e^{-ar}\sum_{j=-1}^{6}l_j r^j+e^{-(1+2a)r}\sum_{j=-1}^{+1}\lambda_j r^j\right]\right\},\end{aligned}\right\} \qquad (13)$$

where s_j, σ_j, l_j, and λ_j have the same meanings as in I, equations (29)-(32). After some further reductions, equation (13) can be simplified to the form

$$\int\Psi_d\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)\Psi_c d\tau=-\frac{(2048\pi^3)^{1/2}\mathfrak{N}i}{k(1+a)^3}\int_0^\infty W_2(r)\chi_1(r)dr, \qquad (14)$$

where

$$\left.\begin{aligned}W_2(r)&=e^{-ar}\sum_{j=-1}^{7}[as_j-js_{j+1}-l_j]r^j\\&\qquad+e^{-(1+2a)r}\sum_{j=-1}^{+1}[(1+2a)\sigma_j-j\sigma_{j+1}-\lambda_j]r^j,\end{aligned}\right\} \qquad (15)$$

with the understanding that

$$s_{-1}=s_8=\sigma_{-1}=\sigma_2=l_7=0. \qquad (16)$$

Formula II (paper I) for the absorption cross-section now becomes

$$\kappa=2.725\times10^{-19}\frac{2048\pi^3\mathfrak{N}^2}{(1+a)^6k(k^2+2I)}\left|\int_0^\infty W_2(r)\chi_1(r)dr\right|^2. \qquad (17)$$

Inserting the numerical values for the various constants in equations (15) and (17), we find:

$$\left.\begin{aligned}W_2(r)&=e^{-0.707735r}(0.1573261r^{-1}+0.2686713+0.9780967r\\&-0.2397504r^2+0.06594301r^3-0.006107001r^4+0.0008050248r^5\\&-0.00007231322r^6+0.000004366 32r^7)-e^{-2.415470r}(0.1573261r^{-1}\\&\qquad+0.5373426+0.2294096r)\end{aligned}\right\} \qquad (18)$$

and

$$\kappa=\frac{3.7062}{k(k^2+0.055118)}\left|\int_0^\infty W_2(r)\chi_1(r)dr\right|^2\times10^{-18}\text{ cm}^2. \qquad (\text{II}_1)$$

For the purposes of evaluating the absorption cross-sections according to the foregoing formulae, it is convenient to have a fairly extensive table of the "weight function," $W_2(r)$. This is provided in Table 1.

TABLE 1

THE WEIGHT FUNCTION $W_2(r)$

r	$W_2(r)$	r	$W_2(r)$	r	$W_2(r)$	r	$W_2(r)$
0	0	4.4	0.18818	8.8	0.05010	16.4	0.00676
0.1	0.12932	4.5	.18220	8.9	.04872	16.6	.00638
0.2	.23292	4.6	.17644	9.0	.04738	16.8	.00601
0.3	.31486	4.7	.17089	9.1	.04609	17.0	.00567
0.4	.37865	4.8	.16554	9.2	.04484	17.2	.00530
0.5	.42731	4.9	.16039	9.3	.04363	17.4	.00502
0.6	.46340	5.0	.15541	9.4	.04245	17.6	.00473
0.7	.48911	5.1	.15061	9.5	.04132	17.8	.00443
0.8	.50628	5.2	.14598	9.6	.04021	18.0	.00417
0.9	.51649	5.3	.14150	9.7	.03915	18.2	.00391
1.0	.52105	5.4	.13718	9.8	.03811	18.4	.00366
1.1	.52106	5.5	.13300	9.9	.03711	18.6	.00342
1.2	.51744	5.6	.12896	10.0	.03614	18.8	.00321
1.3	.51097	5.7	.12506	10.2	.03428	19.0	.00302
1.4	.50229	5.8	.12128	10.4	.03253	19.2	.00281
1.5	.49192	5.9	.11763	10.6	.03089	19.4	.00264
1.6	.48030	6.0	.11409	10.8	.02934	19.6	.00246
1.7	.46778	6.1	.11067	11.0	.02787	19.8	.00230
1.8	.45467	6.2	.10736	11.2	.02649	20.0	.00214
1.9	.44118	6.3	.10415	11.4	.02518	20.2	.00200
2.0	.42753	6.4	.10105	11.6	.02394	20.4	.00187
2.1	.41384	6.5	.09804	11.8	.02277	20.6	.00174
2.2	.40026	6.6	.09514	12.0	.02165	20.8	.00162
2.3	.38686	6.7	.09232	12.2	.02059	21.0	.00151
2.4	.37373	6.8	.08959	12.4	.01959	21.2	.00140
2.5	.36090	6.9	.08695	12.6	.01863	21.4	.00130
2.6	.34844	7.0	.08440	12.8	.01771	21.6	.00121
2.7	.33635	7.1	.08192	13.0	.01684	21.8	.00112
2.8	.32465	7.2	.07952	13.2	.01601	22.0	.00104
2.9	.31338	7.3	.07720	13.4	.01522	22.2	.00097
3.0	.30250	7.4	.07496	13.6	.01446	22.4	.00090
3.1	.29204	7.5	.07278	13.8	.01373	22.6	.00083
3.2	.28197	7.6	.07068	14.0	.01304	22.8	.00077
3.3	.27230	7.7	.06864	14.2	.01239	23.0	.00071
3.4	.26301	7.8	.06666	14.4	.01175	23.2	.00066
3.5	.25410	7.9	.06475	14.6	.01115	23.4	.00061
3.6	.24553	8.0	.06291	14.8	.01057	23.6	.00056
3.7	.23732	8.1	.06112	15.0	.01001	23.8	.00052
3.8	.22943	8.2	.05938	15.2	.00949	24.0	.00048
3.9	.22185	8.3	.05771	15.4	.00899	24.2	.00044
4.0	.21457	8.4	.05608	15.6	.00849	24.4	.00041
4.1	.20758	8.5	.05451	15.8	.00803	24.6	.00038
4.2	.20086	8.6	.05299	16.0	.00758	24.8	.00035
4.3	0.19440	8.7	0.05152	16.2	0.00715	25.0	0.00032

In an analogous manner it is found that, under the same assumptions, formulae I and III of paper I become

$$\kappa = 9.266 \frac{k^2 + 0.055118}{k} \left| \int_0^\infty W_1(r)\, \chi_1(r)\, dr \right|^2 \times 10^{-19}\ \text{cm}^2 \qquad (\text{I}_1)$$

and

$$\kappa = 1.4825 \frac{1}{k(k^2+0.055118)^3} \left| \int_0^\infty W_3(r)\chi_1(r)\,dr \right|^2 \times 10^{-17}\ \mathrm{cm}^2 , \quad (\mathrm{III}_1)$$

where

$$\left.\begin{aligned} W_1(r) &= e^{-0.707735r}(0.7810175r + 1.0260897r^2 - 0.107743r^3 \\ &+ 0.0748147r^4 - 0.0043258\,7r^5 + 0.000771209r^6 - 0.0000547250r^7 \\ &+ 0.00000616943r^8) - e^{-2.415407r}(0.7810175r + 1.333771r^2 \\ &\qquad + 1.138864r^3 + 0.3241463r^4) \end{aligned}\right\} \quad (19)$$

and

$$\left.\begin{aligned} W_3(r) &= e^{-0.707735r}(0.3796213r^{-2} + 1.0311631r^{-1} - 0.1091749 \\ &- 0.00361463r + 0.00176560r^2 + 0.000847872r^3 - 0.0000857360r^4 \\ &+ 0.00001105546r^5 - 0.000001669208r^6 - 0.07846795r^4 \\ &\times Ei[1.707735r]) - e^{-2.415470r}(0.3796213r^{-2} + 0.1944878r^{-1} \\ &\qquad + 0.0553556 - 0.03151090r + 0.02690613r^2 - 0.04594855r^3) . \end{aligned}\right\} \quad (20)$$

A brief table of the function $W_1(r)$ has been given in an earlier paper.[5] We now provide a similar tabulation of the function $W_3(r)$ (see Table 2). In Figure 1 we have further illustrated the dependence of the functions W_1, W_2, and W_3 on r. It is seen that the three weight functions are of entirely different orders of magnitude at large distances; it is in terms of these differences that the remarks made in paper I, §§ 2 and 5, have to be understood.

TABLE 2

THE WEIGHT FUNCTION $W_3(r)$

r	$W_3(r)$	r	$W_3(r)$	r	$W_3(r)$	r	$W_3(r)$
0.	∞	0.6	1.408	1.4	0.293	3.5	0.021
0.1	13.49	0.7	1.100	1.6	.215	4.0	.013
.2	6.13	0.8	0.879	1.8	.160	4.5	.009
.3	3.721	0.9	0.713	2.0	.121	5.0	.006
.4	2.526	1.0	0.586	2.5	.064	5.5	0.004
0.5	1.853	1.2	0.408	3.0	0.036		

3. *The continuous absorption coefficient of* H^- *evaluated according to formula* (II_1).—The evaluation of the absorption cross-sections according to formula (II_1) requires a knowledge of the p-spherical waves $\chi_1(r)$ of an electron in the Hartree field of a hydrogen atom. The writer has at his disposal a large number of tables of these functions for various values of k^2 in the range $1.75 \geq k^2 > 0$.[6] Using these tables and the table of the weight function $W_2(r)$, we have evaluated the absorption cross-sections according to formula (II_1) by straightforward numerical quadratures. The results of these integrations are given in Table 3.

[5] S. Chandrasekhar, *Ap. J.*, **100**, 176, 1944 (Table 1).

[6] These tables were computed by Mrs. Frances H. Breen and the writer. It is hoped to publish these (and similar tables of the s-waves, which have also been completed) in the near future.

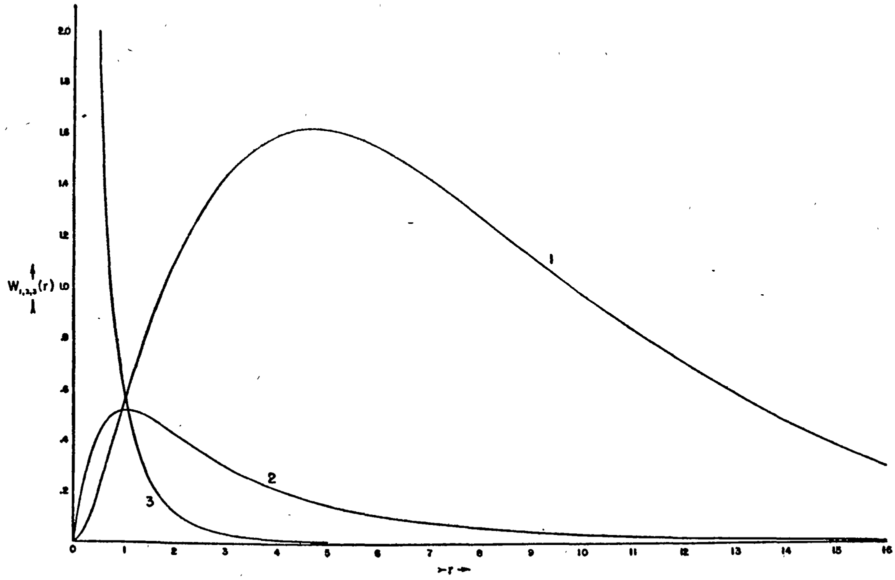

FIG. 1.—A comparison of the weight functions $W_1(r)$ (curve 1), $W_2(r)$ (curve 2), and $W_3(r)$ (curve 3) which occur in the formulae (I_1, II_1, and III_1) for the absorption cross-sections of H^- in terms of th[e] matrix elements of the dipole moment, momentum, and acceleration, respectively.

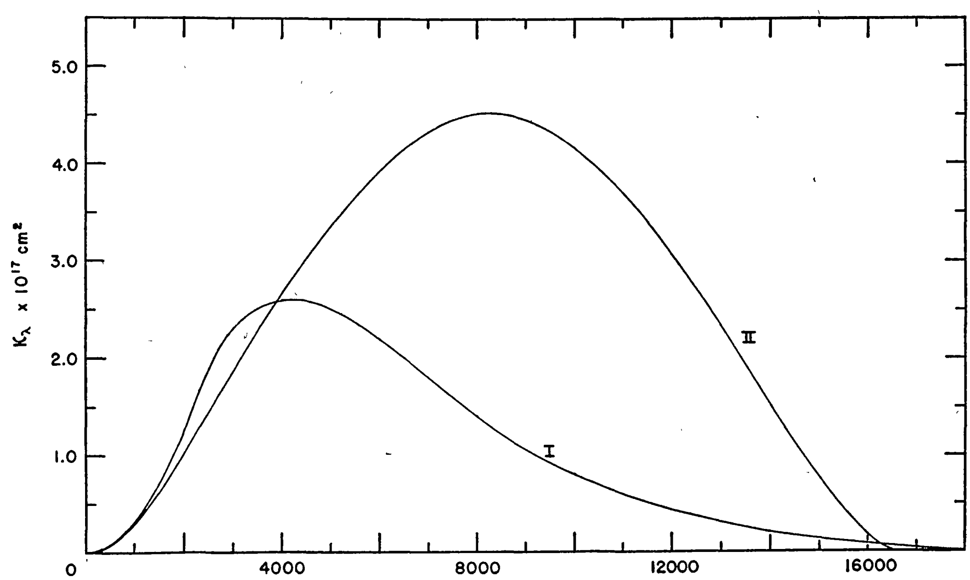

FIG. 2.—A comparison of the continuous absorption coefficient of H^- as determined by Massey and Bates (curve I) with the determination of the present paper (curve II).

It is seen that our present values differ from those obtained earlier on the plane-wave representation of the ejected electron by about 5 per cent in the entire spectral range of astrophysical interest. The maximum is, however, not appreciably shifted and is at the same place (λ 8500 A); here the atomic absorption coefficient has the value 4.52×10^{-17} cm.[2] In view of the many determinations of the continuous absorption coefficient of H^- that have been made, it is perhaps of interest to compare the first of these determinations by Massey and Bates with that of the present paper. This is done in Figure 2.

TABLE 3

THE CONTINUOUS ABSORPTION COEFFICIENT OF H^- COMPUTED ACCORDING TO FORMULA (II_1)

k^2	λ(A)	$\kappa_\lambda \times 10^{17}$ Cm²	k^2	λ(A)	$\kappa_\lambda \times 10^{17}$ Cm²	k^2	λ(A)	$\kappa_\lambda \times 10^{17}$ Cm²
1.75	505	0.0657	0.175	3960	2.62	0.055	8275	4.52
0.80	1066	0.333	.150	4443	2.97	.050	8669	4.50
0.50	1642	0.740	.125	5059	3.39	.045	9102	4.44
0.35	2249	1.231	.100	5875	3.87	.035	10,111	4.13
0.25	2987	1.84	.090	6280	4.06	.020	12,131	2.96
0.20	3572	2.32	0.070	7283	4.41	0.010	13,994	1.50

It is again a pleasure to record my indebtedness to Mrs. Frances H. Breen for valuable assistance with the numerical work.