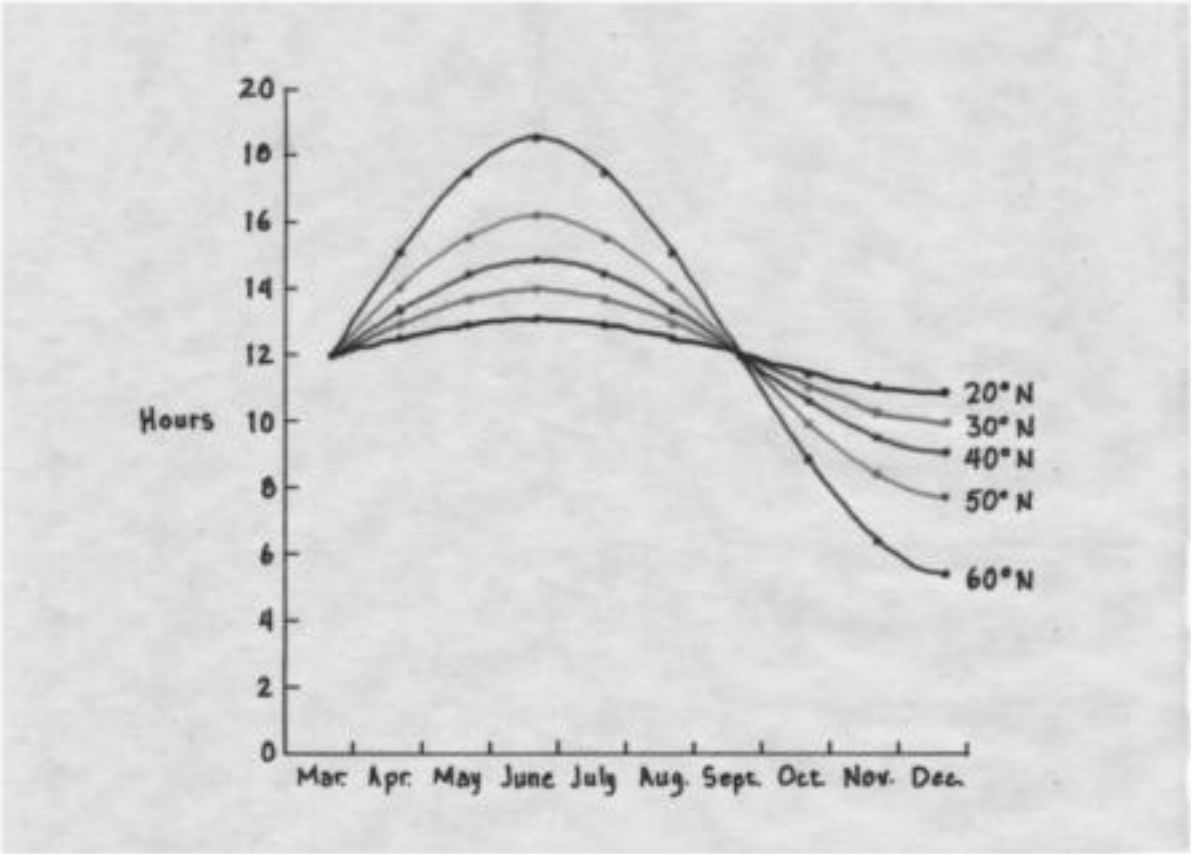
A graphical representation of a function—here the number of hours of daylight as a function of the time of year at various latitudes— is often the most natural and convenient way to represent the function.



Functions and Models

The fundamental objects that we deal with in calculus are functions. This chapter prepares the way for calculus by discussing the basic ideas concerning functions, their graphs, and ways of transforming and combining them. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that occur in calculus and describe the process of using these functions as mathematical models of real-world phenomena. We also discuss the use of graphing calculators and graphing software for computers.

1.1 Four Ways to Represent a Function

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

B. The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For instance,

$$P(1950) \approx 2{,}560{,}000{,}000$$

But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

C. The cost C of mailing a first-class letter depends on the weight w of the letter. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.

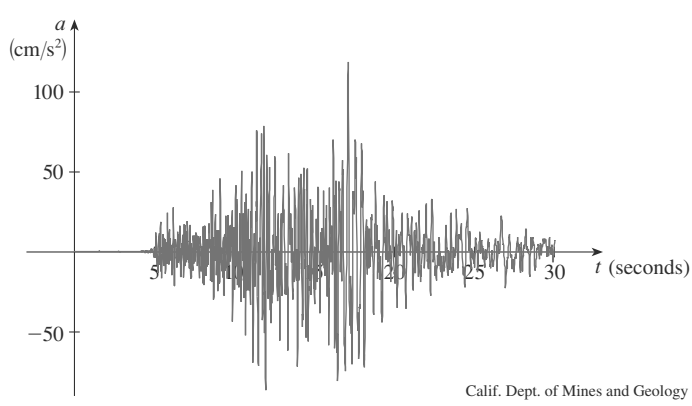


FIGURE 1
Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r, t, w, or t), another number (A, P, C, or a) is assigned. In each case we say that the second number is a function of the first number.

> A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

We usually consider functions for which the sets A and B are sets of real numbers. The set A is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read "f of x." The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

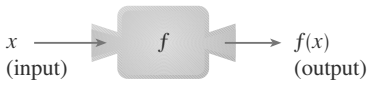


FIGURE 2
Machine diagram for a function f

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus, we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the square root key on your calculator computes such a function. You press the key labeled $\sqrt{\ }$ $\left(\text{or } \sqrt{x}\right)$ and enter the input x. If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input, and the calculator will indicate an error. If $x \geqslant 0$, then an *approximation* to $\sqrt{x}$ will appear in the display. Thus, the $\sqrt{x}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$.

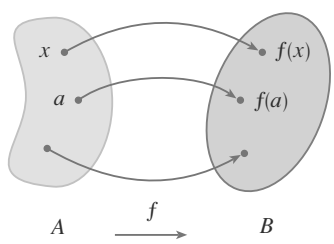


FIGURE 3
Arrow diagram for f

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of A to an element of B. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

The most common method for visualizing a function is its graph. If f is a function with domain A, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4). The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis as in Figure 5.

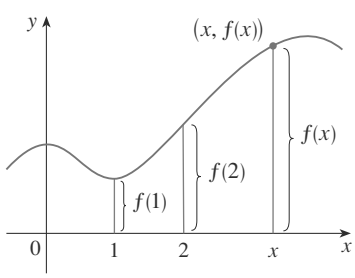


FIGURE 4

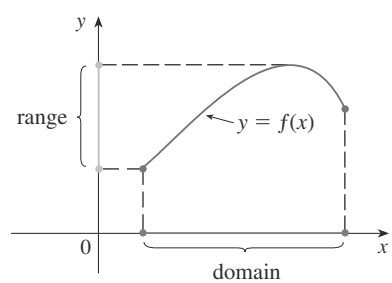


FIGURE 5

EXAMPLE 1 The graph of a function f is shown in Figure 6.
(a) Find the values of $f(1)$ and $f(5)$.
(b) What are the domain and range of f?

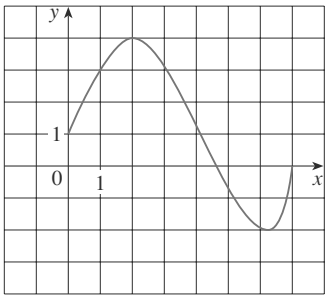


FIGURE 6

SOLUTION
(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 unit below the x-axis, so we estimate that $f(5) \approx -0.7$.

|||| The notation for intervals is given in Appendix A.

(b) We see that $f(x)$ is defined when $0 \leqslant x \leqslant 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leqslant y \leqslant 4\} = [-2, 4]$$

EXAMPLE 2 Sketch the graph and find the domain and range of each function.
(a) $f(x) = 2x - 1$ (b) $g(x) = x^2$

SOLUTION
(a) The equation of the graph is $y = 2x - 1$, and we recognize this as being the equation of a line with slope 2 and y-intercept -1. (Recall the slope-intercept form of the equation of a line: $y = mx + b$. See Appendix B.) This enables us to sketch the graph of f in Figure 7. The expression $2x - 1$ is defined for all real numbers, so the domain of f is the set of all real numbers, which we denote by $\mathbb{R}$. The graph shows that the range is also $\mathbb{R}$.

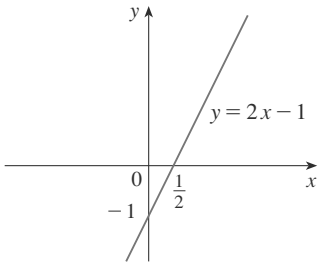


FIGURE 7

(b) Since $g(2) = 2^2 = 4$ and $g(-1) = (-1)^2 = 1$, we could plot the points $(2, 4)$ and $(-1, 1)$, together with a few other points on the graph, and join them to produce the graph (Figure 8). The equation of the graph is $y = x^2$, which represents a parabola (see Appendix C). The domain of g is $\mathbb{R}$. The range of g consists of all values of $g(x)$, that is, all numbers of the form x^2. But $x^2 \geqslant 0$ for all numbers x and any positive number y is a square. So the range of g is $\{y \mid y \geqslant 0\} = [0, \infty)$. This can also be seen from Figure 8.

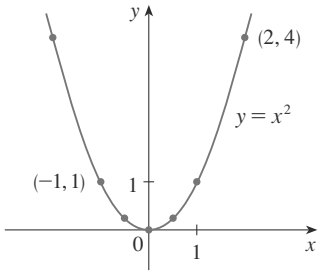


FIGURE 8

Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it is often useful to go from one representation to another to gain additional insight into the function. (In Example 2, for instance, we started with algebraic formulas and then obtained the graphs.) But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A(r) = \pi r^2$, though it is possible to compile a table of values or to sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.

B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t. The table of values of world population on page 11 provides a convenient representation of this function. If we plot these values, we get the graph (called a *scatter plot*) in Figure 9. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t. But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.5, we obtain the approximation

$$P(t) \approx f(t) = (0.008079266) \cdot (1.013731)^t$$

and Figure 10 shows that it is a reasonably good "fit." The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

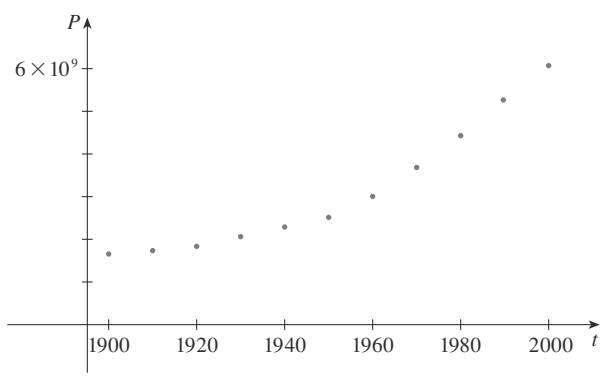


FIGURE 9

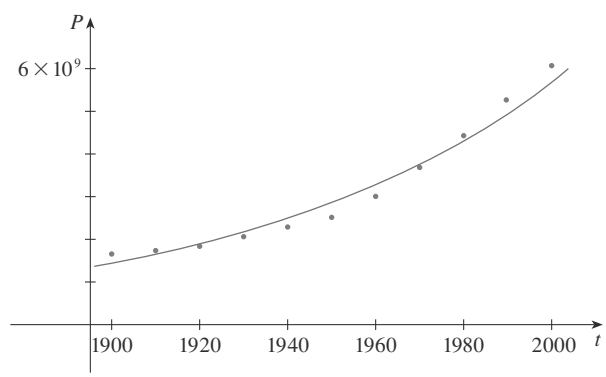


FIGURE 10

The function is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.

C. Again the function is described in words: is the cost of mailing a first-class letter with weight . The rule that the U.S. Postal Service used as of 2002 is as follows: The cost is 37 cents for up to one ounce, plus 23 cents for each successive ounce up to 11 ounces. The table of values shown in the margin is the most convenient representation for this function, though it is possible to sketch a graph (see Example 10).

D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function . It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.) Figures 11 and 12 show the graphs of the north-south and east-west accelerations for the Northridge earthquake; when used in conjunction with Figure 1, they provide a great deal of information about the earthquake.

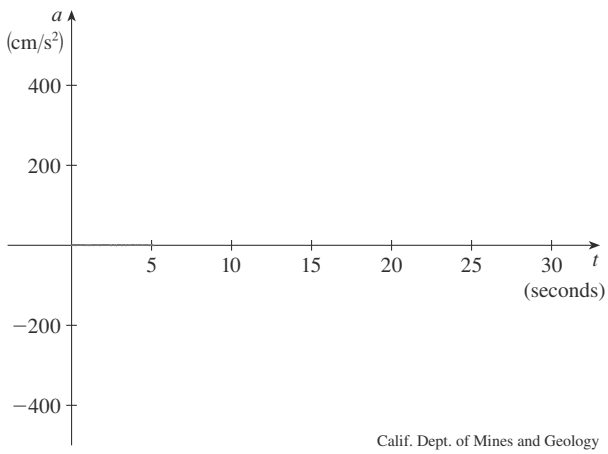


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FIGURE 11 North-south acceleration for the Northridge earthquake

In the next example we sketch the graph of a function that is defined verbally.

EXAMPLE 3 When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

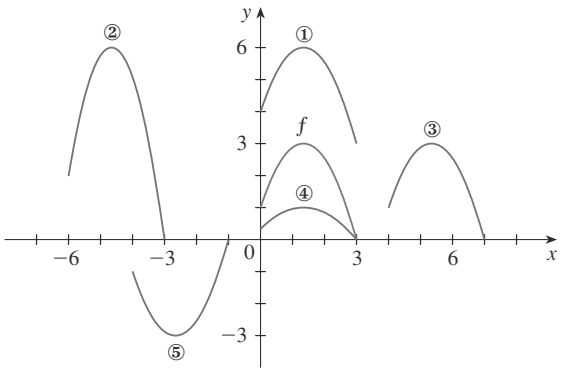
SOLUTION The initial temperature of the running water is close to room temperature because of the water that has been sitting in the pipes. When the water from the hot-water tank starts coming out, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 13.

2. Explain how the following graphs are obtained from the graph of $y = f(x)$.

(a) $y = 5f(x)$ (b) $y = f(x - 5)$
(c) $y = -f(x)$ (d) $y = -5f(x)$
(e) $y = f(5x)$ (f) $y = 5f(x) - 3$

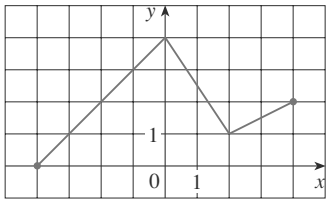
3. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.

(a) $y = f(x - 4)$ (b) $y = f(x) + 3$
(c) $y = \frac{1}{3}f(x)$ (d) $y = -f(x + 4)$
(e) $y = 2f(x + 6)$



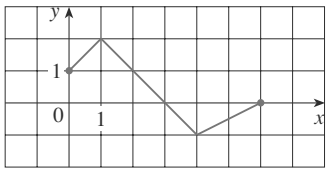
4. The graph of f is given. Draw the graphs of the following functions.

(a) $y = f(x + 4)$ (b) $y = f(x) + 4$
(c) $y = 2f(x)$ (d) $y = -\frac{1}{2}f(x) + 3$

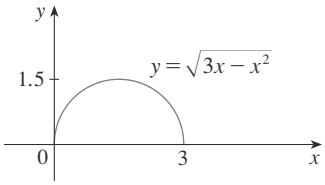


5. The graph of f is given. Use it to graph the following functions.

(a) $y = f(2x)$ (b) $y = f\left(\frac{1}{2}x\right)$
(c) $y = f(-x)$ (d) $y = -f(-x)$



6–7 The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



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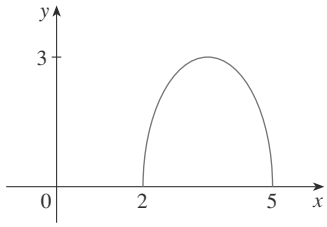

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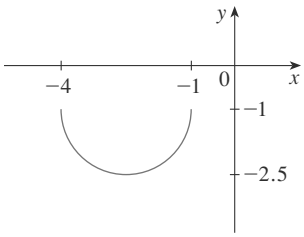


8. (a) How is the graph of $y = 2 \sin x$ related to the graph of $y = \sin x$? Use your answer and Figure 6 to sketch the graph of $y = 2 \sin x$.

(b) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 4(a) to sketch the graph of $y = 1 + \sqrt{x}$.

9–24 Graph the function, not by plotting points, but by starting with the graph of one of the standard functions given in Section 1.2, and then applying the appropriate transformations.

9. $y = -x^3$ **10.** $y = 1 - x^2$

11. $y = (x + 1)^2$ **12.** $y = x^2 - 4x + 3$

13. $y = 1 + 2 \cos x$ **14.** $y = 4 \sin 3x$

15. $y = \sin(x/2)$ **16.** $y = \dfrac{1}{x - 4}$

17. $y = \sqrt{x + 3}$ **18.** $y = (x + 2)^4 + 3$

19. $y = \frac{1}{2}(x^2 + 8x)$ **20.** $y = 1 + \sqrt[3]{x - 1}$

21. $y = \dfrac{2}{x + 1}$ **22.** $y = \dfrac{1}{4} \tan\left(x - \dfrac{\pi}{4}\right)$

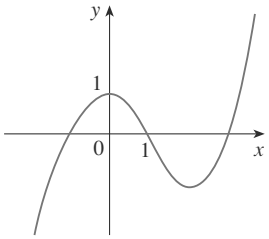
23. $y = |\sin x|$ **24.** $y = |x^2 - 2x|$

25. The city of New Orleans is located at latitude 30°N. Use Figure 9 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. Use the fact that on March 31 the Sun rises at 5:51 A.M. and sets at 6:18 P.M. in New Orleans to check the accuracy of your model.

26. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

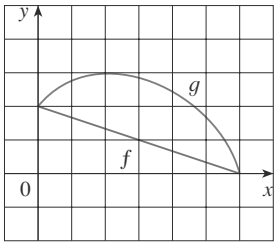
27. (a) How is the graph of $y = f(|x|)$ related to the graph of f?
(b) Sketch the graph of $y = \sin|x|$.
(c) Sketch the graph of $y = \sqrt{|x|}$.

28. Use the given graph of f to sketch the graph of $y = 1/f(x)$. Which features of f are the most important in sketching $y = 1/f(x)$? Explain how they are used.



29–30 Use graphical addition to sketch the graph of $f + g$.

29.



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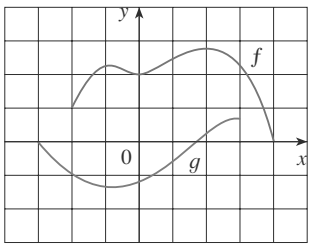


31–32 Find $f + g$, $f - g$, fg, and f/g and state their domains.

31. $f(x) = x^3 + 2x^2, \quad g(x) = 3x^2 - 1$

32. $f(x) = \sqrt{1 + x}, \quad g(x) = \sqrt{1 - x}$

33–34 Use the graphs of f and g and the method of graphical addition to sketch the graph of $f + g$.

33. $f(x) = x, \quad g(x) = 1/x$

34. $f(x) = x^3, \quad g(x) = -x^2$

35–40 Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

35. $f(x) = 2x^2 - x, \quad g(x) = 3x + 2$

36. $f(x) = 1 - x^3, \quad g(x) = 1/x$

37. $f(x) = \sin x, \quad g(x) = 1 - \sqrt{x}$

38. $f(x) = 1 - 3x, \quad g(x) = 5x^2 + 3x + 2$

39. $f(x) = x + \dfrac{1}{x}, \quad g(x) = \dfrac{x + 1}{x + 2}$

40. $f(x) = \sqrt{2x + 3}, \quad g(x) = x^2 + 1$

41–44 Find $f \circ g \circ h$.

41. $f(x) = x + 1, \quad g(x) = 2x, \quad h(x) = x - 1$

42. $f(x) = 2x - 1, \quad g(x) = x^2, \quad h(x) = 1 - x$

43. $f(x) = \sqrt{x - 1}, \quad g(x) = x^2 + 2, \quad h(x) = x + 3$

44. $f(x) = \dfrac{2}{x + 1}, \quad g(x) = \cos x, \quad h(x) = \sqrt{x + 3}$

45–50 Express the function in the form $f \circ g$.

45. $F(x) = (x^2 + 1)^{10}$

46. $F(x) = \sin(\sqrt{x})$

47. $G(x) = \dfrac{x^2}{x^2 + 4}$

48. $G(x) = \dfrac{1}{x + 3}$

49. $u(t) = \sqrt{\cos t}$

50. $u(t) = \dfrac{\tan t}{1 + \tan t}$

51–53 Express the function in the form $f \circ g \circ h$.

51. $H(x) = 1 - 3^{x^2}$

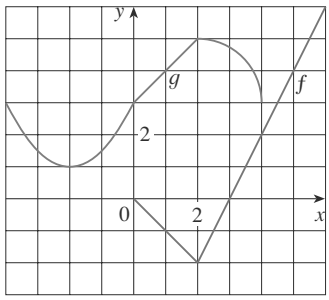
52. $H(x) = \sqrt[3]{\sqrt{x} - 1}$

53. $H(x) = \sec^4(\sqrt{x})$

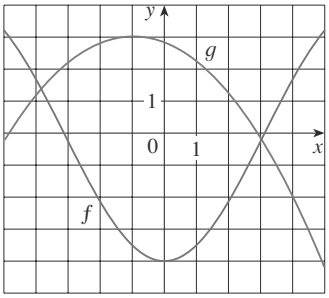
54. Use the table to evaluate each expression.
(a) $f(g(1))$ (b) $g(f(1))$ (c) $f(f(1))$
(d) $g(g(1))$ (e) $(g \circ f)(3)$ (f) $(f \circ g)(6)$

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

55. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.
(a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
(d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



56. Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \ldots, 5$. Use these estimates to sketch a rough graph of $f \circ g$.



57. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
(a) Express the radius r of this circle as a function of the time t (in seconds).
(b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.

58. An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.
(a) Express the horizontal distance d (in miles) that the plane has flown as a function of t.
(b) Express the distance s between the plane and the radar station as a function of d.
(c) Use composition to express s as a function of t.

59. The **Heaviside function** H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.
(a) Sketch the graph of the Heaviside function.
(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.
(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

60. The Heaviside function defined in Exercise 59 can also be used to define the **ramp function** $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.
(a) Sketch the graph of the ramp function $y = tH(t)$.
(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 60$.
(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 32$.

61. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)
(b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

62. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.

63. Suppose g is an even function and let $h = f \circ g$. Is h always an even function?

64. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

1.4 Graphing Calculators and Computers

In this section we assume that you have access to a graphing calculator or a computer with graphing software. We will see that the use of such a device enables us to graph more complicated functions and to solve more complex problems than would otherwise be possible. We also point out some of the pitfalls that can occur with these machines.

Graphing calculators and computers can give very accurate graphs of functions. But we will see in Chapter 4 that only through the use of calculus can we be sure that we have uncovered all the interesting aspects of a graph.

A graphing calculator or computer displays a rectangular portion of the graph of a function in a **display window** or **viewing screen**, which we refer to as a **viewing rectangle**. The default screen often gives an incomplete or misleading picture, so it is important to choose the viewing rectangle with care. If we choose the x-values to range from a minimum value of $Xmin = a$ to a maximum value of $Xmax = b$ and the y-values to range from

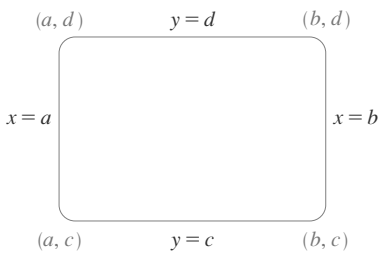


FIGURE 1
The viewing rectangle $[a, b]$ by $[c, d]$

a minimum of $Ymin = c$ to a maximum of $Ymax = d$, then the visible portion of the graph lies in the rectangle

$$[a, b] \times [c, d] = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$$

shown in Figure 1. We refer to this rectangle as the $[a, b]$ *by* $[c, d]$ *viewing rectangle.*

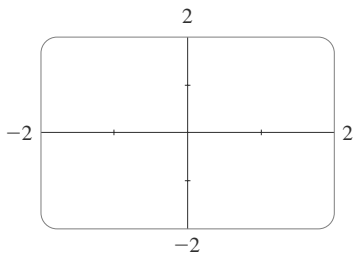
The machine draws the graph of a function f much as you would. It plots points of the form $(x, f(x))$ for a certain number of equally spaced values of x between a and b. If an x-value is not in the domain of f, or if $f(x)$ lies outside the viewing rectangle, it moves on to the next x-value. The machine connects each point to the preceding plotted point to form a representation of the graph of f.

EXAMPLE 1 Draw the graph of the function $f(x) = x^2 + 3$ in each of the following viewing rectangles.

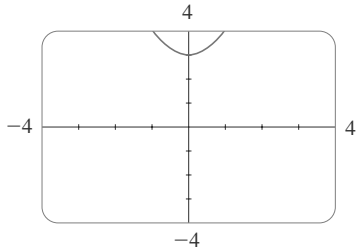
(a) $[-2, 2]$ by $[-2, 2]$ (b) $[-4, 4]$ by $[-4, 4]$
(c) $[-10, 10]$ by $[-5, 30]$ (d) $[-50, 50]$ by $[-100, 1000]$

SOLUTION For part (a) we select the range by setting $Xmin = -2$, $Xmax = 2$, $Ymin = -2$, and $Ymax = 2$. The resulting graph is shown in Figure 2(a). The display window is blank! A moment's thought provides the explanation: Notice that $x^2 \geqslant 0$ for all x, so $x^2 + 3 \geqslant 3$ for all x. Thus, the range of the function $f(x) = x^2 + 3$ is $[3, \infty)$. This means that the graph of f lies entirely outside the viewing rectangle $[-2, 2]$ by $[-2, 2]$.

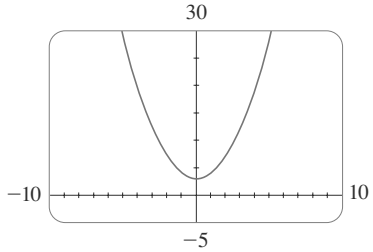
The graphs for the viewing rectangles in parts (b), (c), and (d) are also shown in Figure 2. Observe that we get a more complete picture in parts (c) and (d), but in part (d) it is not clear that the y-intercept is 3.



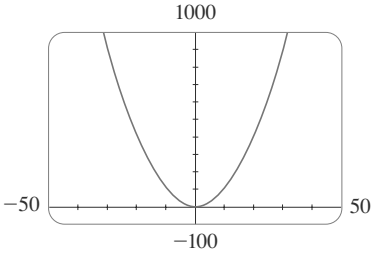
(a) $[-2, 2]$ by $[-2, 2]$



(b) $[-4, 4]$ by $[-4, 4]$



(c) $[-10, 10]$ by $[-5, 30]$



(d) $[-50, 50]$ by $[-100, 1000]$

FIGURE 2 Graphs of $f(x) = x^2 + 3$

We see from Example 1 that the choice of a viewing rectangle can make a big difference in the appearance of a graph. Sometimes it's necessary to change to a larger viewing rectangle to obtain a more complete picture, a more global view, of the graph. In the next example we see that knowledge of the domain and range of a function sometimes provides us with enough information to select a good viewing rectangle.

EXAMPLE 2 Determine an appropriate viewing rectangle for the function $f(x) = \sqrt{8 - 2x^2}$ and use it to graph f.

SOLUTION The expression for $f(x)$ is defined when

$$\begin{aligned} 8 - 2x^2 \geqslant 0 \iff 2x^2 \leqslant 8 &\iff x^2 \leqslant 4 \\ \iff |x| \leqslant 2 &\iff -2 \leqslant x \leqslant 2 \end{aligned}$$

Therefore, the domain of f is the interval $[-2, 2]$. Also,

$$0 \leqslant \sqrt{8 - 2x^2} \leqslant \sqrt{8} = 2\sqrt{2} \approx 2.83$$

so the range of f is the interval $\left[0, 2\sqrt{2}\right]$.

We choose the viewing rectangle so that the x-interval is somewhat larger than the domain and the y-interval is larger than the range. Taking the viewing rectangle to be $[-3, 3]$ by $[-1, 4]$, we get the graph shown in Figure 3.

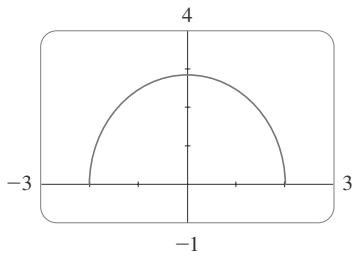


FIGURE 3

EXAMPLE 3 Graph the function $y = x^3 - 150x$.

SOLUTION Here the domain is $\mathbb{R}$, the set of all real numbers. That doesn't help us choose a viewing rectangle. Let's experiment. If we start with the viewing rectangle $[-5, 5]$ by $[-5, 5]$, we get the graph in Figure 4. It appears blank, but actually the graph is so nearly vertical that it blends in with the y-axis.

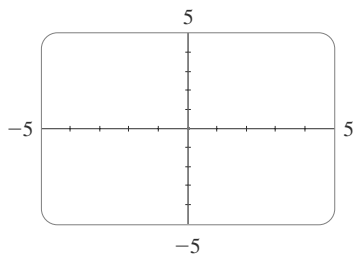


FIGURE 4

If we change the viewing rectangle to $[-20, 20]$ by $[-20, 20]$, we get the picture shown in Figure 5(a). The graph appears to consist of vertical lines, but we know that can't be correct. If we look carefully while the graph is being drawn, we see that the graph leaves the screen and reappears during the graphing process. This indicates that we need to see more in the vertical direction, so we change the viewing rectangle to $[-20, 20]$ by $[-500, 500]$. The resulting graph is shown in Figure 5(b). It still doesn't quite reveal all the main features of the function, so we try $[-20, 20]$ by $[-1000, 1000]$ in Figure 5(c). Now we are more confident that we have arrived at an appropriate viewing rectangle. In Chapter 4 we will be able to see that the graph shown in Figure 5(c) does indeed reveal all the main features of the function.

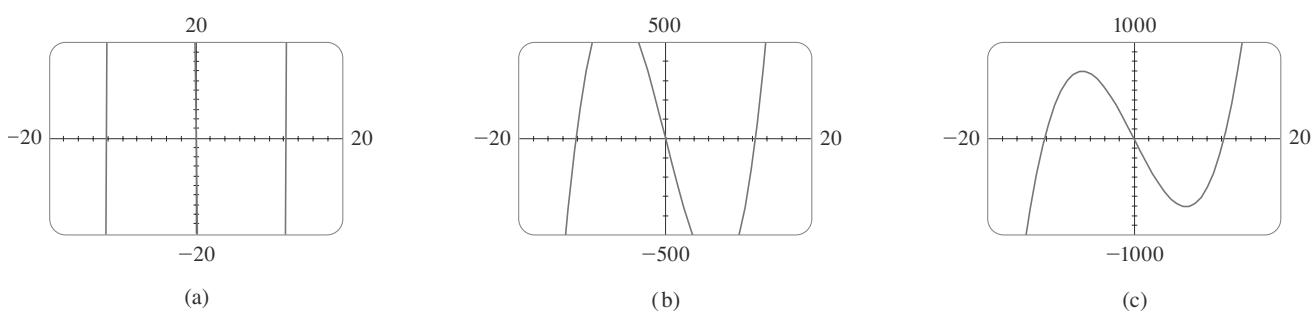


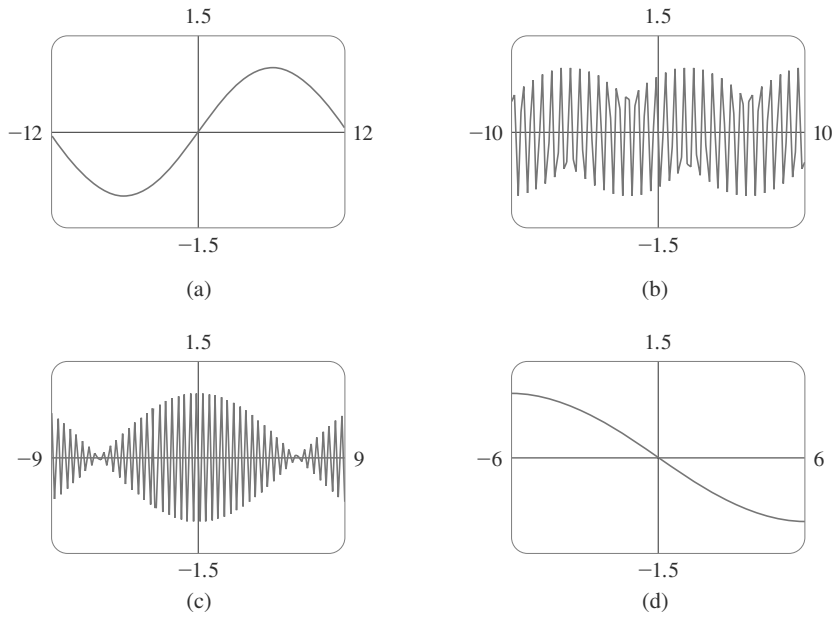
FIGURE 5 $f(x) = x^3 - 150x$

EXAMPLE 4 Graph the function $f(x) = \sin 50x$ in an appropriate viewing rectangle.

SOLUTION Figure 6(a) shows the graph of f produced by a graphing calculator using the viewing rectangle $[-12, 12]$ by $[-1.5, 1.5]$. At first glance the graph appears to be reasonable. But if we change the viewing rectangle to the ones shown in the following parts of Figure 6, the graphs look very different. Something strange is happening.

In order to explain the big differences in appearance of these graphs and to find an appropriate viewing rectangle, we need to find the period of the function $y = \sin 50x$. We know that the function $y = \sin x$ has period 2π and the graph of $y = \sin 50x$ is compressed horizontally by a factor of 50, so the period of $y = \sin 50x$ is

$$\frac{2\pi}{50} = \frac{\pi}{25} \approx 0.126$$



IIII The appearance of the graphs in Figure 6 depends on the machine used. The graphs you get with your own graphing device might not look like these figures, but they will also be quite inaccurate.

FIGURE 6
Graphs of $f(x) = \sin 50x$ in four viewing rectangles

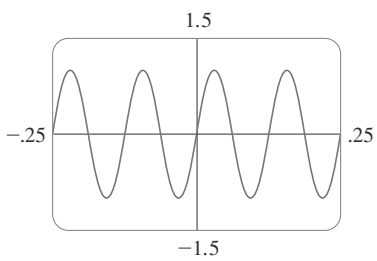


FIGURE 7
$f(x) = \sin 50x$

This suggests that we should deal only with small values of x in order to show just a few oscillations of the graph. If we choose the viewing rectangle $[-0.25, 0.25]$ by $[-1.5, 1.5]$, we get the graph shown in Figure 7.

Now we see what went wrong in Figure 6. The oscillations of $y = \sin 50x$ are so rapid that when the calculator plots points and joins them, it misses most of the maximum and minimum points and therefore gives a very misleading impression of the graph.

We have seen that the use of an inappropriate viewing rectangle can give a misleading impression of the graph of a function. In Examples 1 and 3 we solved the problem by changing to a larger viewing rectangle. In Example 4 we had to make the viewing rectangle smaller. In the next example we look at a function for which there is no single viewing rectangle that reveals the true shape of the graph.

EXAMPLE 5 Graph the function $f(x) = \sin x + \frac{1}{100} \cos 100x$.

SOLUTION Figure 8 shows the graph of f produced by a graphing calculator with viewing rectangle $[-6.5, 6.5]$ by $[-1.5, 1.5]$. It looks much like the graph of $y = \sin x$, but perhaps with some bumps attached. If we zoom in to the viewing rectangle $[-0.1, 0.1]$ by $[-0.1, 0.1]$, we can see much more clearly the shape of these bumps in Figure 9. The reason for this behavior is that the second term, $\frac{1}{100} \cos 100x$, is very small in comparison with the first term, $\sin x$. Thus, we really need two graphs to see the true nature of this function.

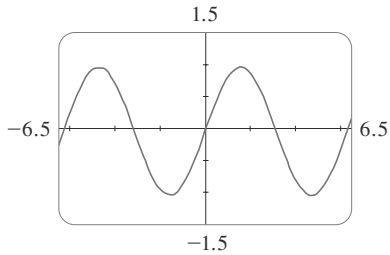


FIGURE 8

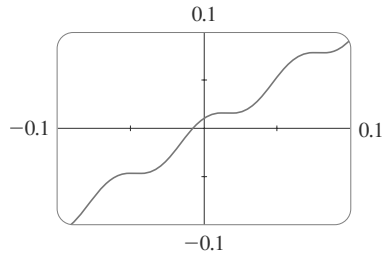


FIGURE 9

EXAMPLE 6 Draw the graph of the function $y = \dfrac{1}{1 - x}$.

SOLUTION Figure 10(a) shows the graph produced by a graphing calculator with viewing rectangle $[-9, 9]$ by $[-9, 9]$. In connecting successive points on the graph, the calculator produced a steep line segment from the top to the bottom of the screen. That line segment is not truly part of the graph. Notice that the domain of the function $y = 1/(1 - x)$ is $\{x \mid x \neq 1\}$. We can eliminate the extraneous near-vertical line by experimenting with a change of scale. When we change to the smaller viewing rectangle $[-4.7, 4.7]$ by $[-4.7, 4.7]$ on this particular calculator, we obtain the much better graph in Figure 10(b).

|||| Another way to avoid the extraneous line is to change the graphing mode on the calculator so that the dots are not connected. Alternatively, we could zoom in using the Zoom Decimal mode.

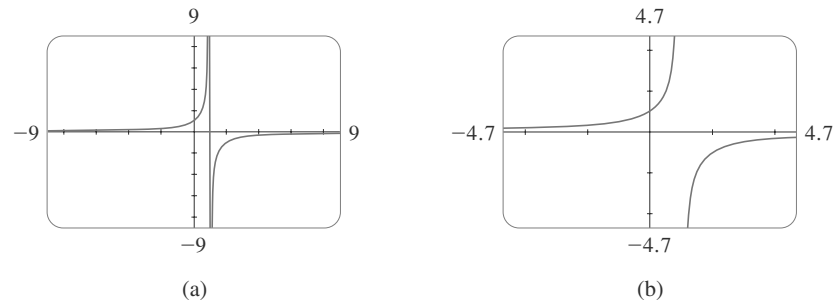


FIGURE 10
$y = \dfrac{1}{1 - x}$

EXAMPLE 7 Graph the function $y = \sqrt[3]{x}$.

SOLUTION Some graphing devices display the graph shown in Figure 11, whereas others produce a graph like that in Figure 12. We know from Section 1.2 (Figure 13) that the graph in Figure 12 is correct, so what happened in Figure 11? The explanation is that some machines compute the cube root of x using a logarithm, which is not defined if x is negative, so only the right half of the graph is produced.

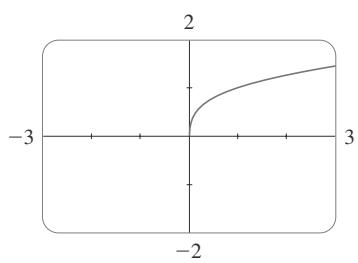


FIGURE 11

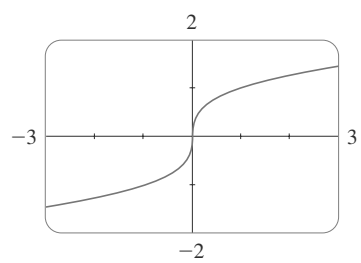


FIGURE 12

You should experiment with your own machine to see which of these two graphs is produced. If you get the graph in Figure 11, you can obtain the correct picture by graphing the function

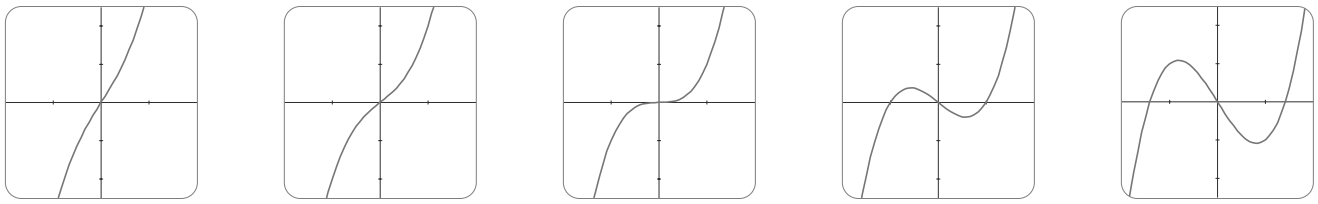
$$f(x) = \frac{x}{|x|} \cdot |x|^{1/3}$$

Notice that this function is equal to $\sqrt[3]{x}$ (except when $x = 0$).

To understand how the expression for a function relates to its graph, it's helpful to graph a **family of functions**, that is, a collection of functions whose equations are related. In the next example we graph members of a family of cubic polynomials.

EXAMPLE 8 Graph the function $y = x^3 + cx$ for various values of the number c. How does the graph change when c is changed?

SOLUTION Figure 13 shows the graphs of $y = x^3 + cx$ for $c = 2, 1, 0, -1$, and -2. We see that, for positive values of c, the graph increases from left to right with no maximum or minimum points (peaks or valleys). When $c = 0$, the curve is flat at the origin. When c is negative, the curve has a maximum point and a minimum point. As c decreases, the maximum point becomes higher and the minimum point lower.

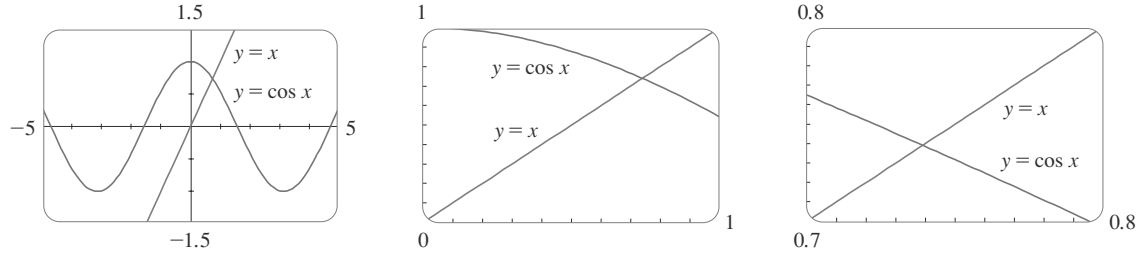


(a) $y = x^3 + 2x$ (b) $y = x^3 + x$ (c) $y = x^3$ (d) $y = x^3 - x$ (e) $y = x^3 - 2x$

FIGURE 13
Several members of the family of functions $y = x^3 + cx$, all graphed in the viewing rectangle $[-2, 2]$ by $[-2.5, 2.5]$

EXAMPLE 9 Find the solution of the equation $\cos x = x$ correct to two decimal places.

SOLUTION The solutions of the equation $\cos x = x$ are the x-coordinates of the points of intersection of the curves $y = \cos x$ and $y = x$. From Figure 14(a) we see that there is only one solution and it lies between 0 and 1. Zooming in to the viewing rectangle $[0, 1]$ by $[0, 1]$, we see from Figure 14(b) that the root lies between 0.7 and 0.8. So we zoom in further to the viewing rectangle $[0.7, 0.8]$ by $[0.7, 0.8]$ in Figure 14(c). By moving the cursor to the intersection point of the two curves, or by inspection and the fact that the x-scale is 0.01, we see that the root of the equation is about 0.74. (Many calculators have a built-in intersection feature.)



(a) $[-5, 5]$ by $[-1.5, 1.5]$, x-scale = 1 (b) $[0, 1]$ by $[0, 1]$, x-scale = 0.1 (c) $[0.7, 0.8]$ by $[0.7, 0.8]$, x-scale = 0.01

FIGURE 14
Locating the roots of $\cos x = x$

1.4 Exercises

1. Use a graphing calculator or computer to determine which of the given viewing rectangles produces the most appropriate graph of the function $f(x) = x^4 + 2$.
(a) $[-2, 2]$ by $[-2, 2]$ (b) $[0, 4]$ by $[0, 4]$
(c) $[-4, 4]$ by $[-4, 4]$
(d) $[-8, 8]$ by $[-4, 40]$
(e) $[-40, 40]$ by $[-80, 800]$

2. Use a graphing calculator or computer to determine which of the given viewing rectangles produces the most appropriate graph of the function $f(x) = x^2 + 7x + 6$.
(a) $[-5, 5]$ by $[-5, 5]$
(b) $[0, 10]$ by $[-20, 100]$
(c) $[-15, 8]$ by $[-20, 100]$
(d) $[-10, 3]$ by $[-100, 20]$

3. Use a graphing calculator or computer to determine which of the given viewing rectangles produces the most appropriate graph of the function $f(x) = 10 + 25x - x^3$.
(a) $[-4, 4]$ by $[-4, 4]$
(b) $[-10, 10]$ by $[-10, 10]$
(c) $[-20, 20]$ by $[-100, 100]$
(d) $[-100, 100]$ by $[-200, 200]$

4. Use a graphing calculator or computer to determine which of the given viewing rectangles produces the most appropriate graph of the function $f(x) = \sqrt{8x - x^2}$.
(a) $[-4, 4]$ by $[-4, 4]$
(b) $[-5, 5]$ by $[0, 100]$
(c) $[-10, 10]$ by $[-10, 40]$
(d) $[-2, 10]$ by $[-2, 6]$

5–18 Determine an appropriate viewing rectangle for the given function and use it to draw the graph.

5. $f(x) = 5 + 20x - x^2$

6. $f(x) = x^3 + 30x^2 + 200x$

7. $f(x) = 0.01x^3 - x^2 + 5$

8. $f(x) = x(x + 6)(x - 9)$

9. $f(x) = \sqrt[4]{81 - x^4}$

10. $f(x) = \sqrt{0.1x + 20}$

11. $f(x) = x^2 + \dfrac{100}{x}$

12. $f(x) = \dfrac{x}{x^2 + 100}$

13. $f(x) = \cos 100x$

14. $f(x) = 3 \sin 120x$

15. $f(x) = \sin(x/40)$

16. $y = \tan 25x$

17. $y = 3^{\cos(x^2)}$

18. $y = x^2 + 0.02 \sin 50x$

19. Graph the ellipse $4x^2 + 2y^2 = 1$ by graphing the functions whose graphs are the upper and lower halves of the ellipse.

20. Graph the hyperbola $y^2 - 9x^2 = 1$ by graphing the functions whose graphs are the upper and lower branches of the hyperbola.

21–23 Find all solutions of the equation correct to two decimal places.

21. $x^3 - 9x^2 - 4 = 0$

22. $x^3 = 4x - 1$

23. $x^2 = \sin x$

24. We saw in Example 9 that the equation $\cos x = x$ has exactly one solution.
(a) Use a graph to show that the equation $\cos x = 0.3x$ has three solutions and find their values correct to two decimal places.
(b) Find an approximate value of m such that the equation $\cos x = mx$ has exactly two solutions.

25. Use graphs to determine which of the functions $f(x) = 10x^2$ and $g(x) = x^3/10$ is eventually larger (that is, larger when x is very large).

26. Use graphs to determine which of the functions $f(x) = x^4 - 100x^3$ and $g(x) = x^3$ is eventually larger.

27. For what values of x is it true that $|\sin x - x| < 0.1$?

28. Graph the polynomials $P(x) = 3x^5 - 5x^3 + 2x$ and $Q(x) = 3x^5$ on the same screen, first using the viewing rectangle $[-2, 2]$ by $[-2, 2]$ and then changing to $[-10, 10]$ by $[-10{,}000, 10{,}000]$. What do you observe from these graphs?

29. In this exercise we consider the family of root functions $f(x) = \sqrt[n]{x}$, where n is a positive integer.
(a) Graph the functions $y = \sqrt{x}$, $y = \sqrt[4]{x}$, and $y = \sqrt[6]{x}$ on the same screen using the viewing rectangle $[-1, 4]$ by $[-1, 3]$.
(b) Graph the functions $y = x$, $y = \sqrt[3]{x}$, and $y = \sqrt[5]{x}$ on the same screen using the viewing rectangle $[-3, 3]$ by $[-2, 2]$. (See Example 7.)
(c) Graph the functions $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$, and $y = \sqrt[5]{x}$ on the same screen using the viewing rectangle $[-1, 3]$ by $[-1, 2]$.
(d) What conclusions can you make from these graphs?

30. In this exercise we consider the family of functions $f(x) = 1/x^n$, where n is a positive integer.
(a) Graph the functions $y = 1/x$ and $y = 1/x^3$ on the same screen using the viewing rectangle $[-3, 3]$ by $[-3, 3]$.
(b) Graph the functions $y = 1/x^2$ and $y = 1/x^4$ on the same screen using the same viewing rectangle as in part (a).
(c) Graph all of the functions in parts (a) and (b) on the same screen using the viewing rectangle $[-1, 3]$ by $[-1, 3]$.
(d) What conclusions can you make from these graphs?

31. Graph the function $f(x) = x^4 + cx^2 + x$ for several values of c. How does the graph change when c changes?

32. Graph the function $f(x) = \sqrt{1 + cx^2}$ for various values of c. Describe how changing the value of c affects the graph.

33. Graph the function $y = x^n 2^{-x}$, $x \geqslant 0$, for $n = 1, 2, 3, 4, 5$, and 6. How does the graph change as n increases?

34. The curves with equations

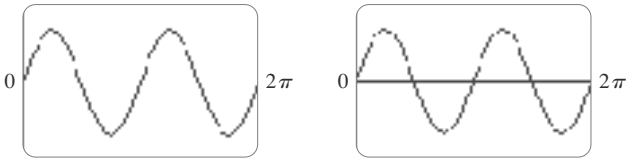
$$y = \frac{|x|}{\sqrt{c - x^2}}$$

are called **bullet-nose curves**. Graph some of these curves to see why. What happens as c increases?

35. What happens to the graph of the equation $y^2 = cx^3 + x^2$ as c varies?

36. This exercise explores the effect of the inner function g on a composite function $y = f(g(x))$.
(a) Graph the function $y = \sin(\sqrt{x})$ using the viewing rectangle $[0, 400]$ by $[-1.5, 1.5]$. How does this graph differ from the graph of the sine function?
(b) Graph the function $y = \sin(x^2)$ using the viewing rectangle $[-5, 5]$ by $[-1.5, 1.5]$. How does this graph differ from the graph of the sine function?

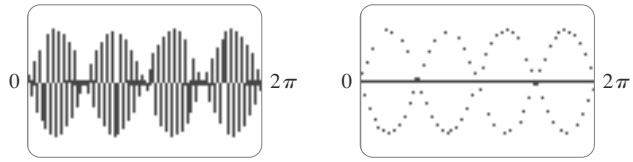
37. The figure shows the graphs of $y = \sin 96x$ and $y = \sin 2x$ as displayed by a TI-83 graphing calculator.



$y = \sin 96x$ $\qquad$ $y = \sin 2x$

The first graph is inaccurate. Explain why the two graphs appear identical. [*Hint:* The TI-83's graphing window is 95 pixels wide. What specific points does the calculator plot?]

38. The first graph in the figure is that of $y = \sin 45x$ as displayed by a TI-83 graphing calculator. It is inaccurate and so, to help explain its appearance, we replot the curve in dot mode in the second graph.



What two sine curves does the calculator appear to be plotting? Show that each point on the graph of $y = \sin 45x$ that the TI-83 chooses to plot is in fact on one of these two curves. (The TI-83's graphing window is 95 pixels wide.)

1.5 Exponential Functions

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

In general, an **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant. Let's recall what this means.

If $x = n$, a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

If $x = 0$, then $a^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$a^{-n} = \frac{1}{a^n}$$

If x is a rational number, $x = p/q$, where p and q are integers and $q > 0$, then

$$a^x = a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

But what is the meaning of a^x if x is an irrational number? For instance, what is meant by $2^{\sqrt{3}}$ or 5^{π}?

To help us answer this question we first look at the graph of the function $y = 2^x$, where x is rational. A representation of this graph is shown in Figure 1. We want to enlarge the domain of $y = 2^x$ to include both rational and irrational numbers.

There are holes in the graph in Figure 1 corresponding to irrational values of x. We want to fill in the holes by defining $f(x) = 2^x$, where $x \in \mathbb{R}$, so that f is an increasing function. In particular, since the irrational number $\sqrt{3}$ satisfies

$$1.7 < \sqrt{3} < 1.8$$

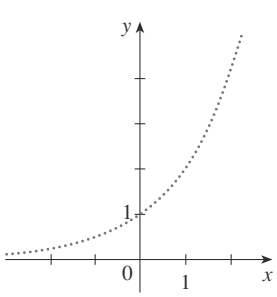


FIGURE 1
Representation of $y = 2^x$, x rational

we must have

$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what $2^{1.7}$ and $2^{1.8}$ mean because 1.7 and 1.8 are rational numbers. Similarly, if we use better approximations for $\sqrt{3}$, we obtain better approximations for $2^{\sqrt{3}}$:

$$\begin{array}{rcl}
1.73 < \sqrt{3} < 1.74 & \Rightarrow & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\
1.732 < \sqrt{3} < 1.733 & \Rightarrow & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\
1.7320 < \sqrt{3} < 1.7321 & \Rightarrow & 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\
1.73205 < \sqrt{3} < 1.73206 & \Rightarrow & 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\
\vdots \qquad \vdots & & \vdots \qquad \vdots
\end{array}$$

It can be shown that there is exactly one number that is greater than all of the numbers

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad \ldots$$

and less than all of the numbers

$$2^{1.8}, \quad 2^{1.74}, \quad 2^{1.733}, \quad 2^{1.7321}, \quad 2^{1.73206}, \quad \ldots$$

We define $2^{\sqrt{3}}$ to be this number. Using the preceding approximation process we can compute it correct to six decimal places:

$$2^{\sqrt{3}} \approx 3.321997$$

Similarly, we can define 2^x (or a^x, if $a > 0$) where x is any irrational number. Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function $f(x) = 2^x$, $x \in \mathbb{R}$.

The graphs of members of the family of functions $y = a^x$ are shown in Figure 3 for various values of the base a. Notice that all of these graphs pass through the same point (0, 1) because $a^0 = 1$ for $a \neq 0$. Notice also that as the base a gets larger, the exponential function grows more rapidly (for $x > 0$).

|||| A proof of this fact is given in J. Marsden and A. Weinstein, *Calculus Unlimited* (Menlo Park, CA: Benjamin/Cummings, 1980).

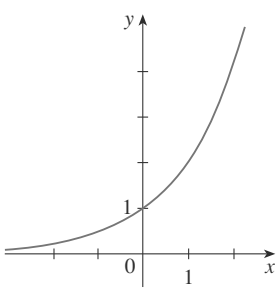


FIGURE 2
$y = 2^x$, x real

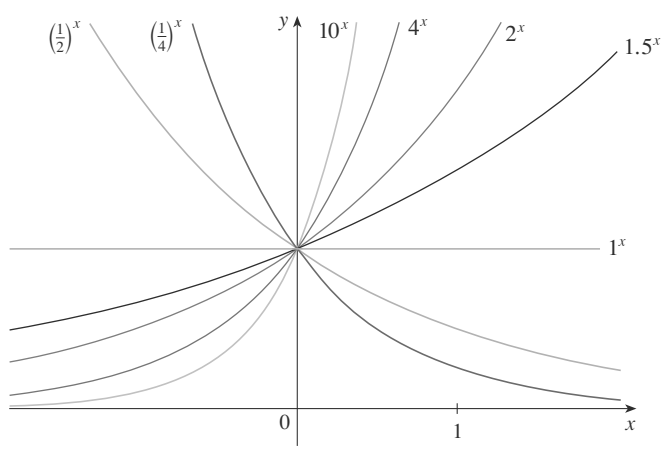
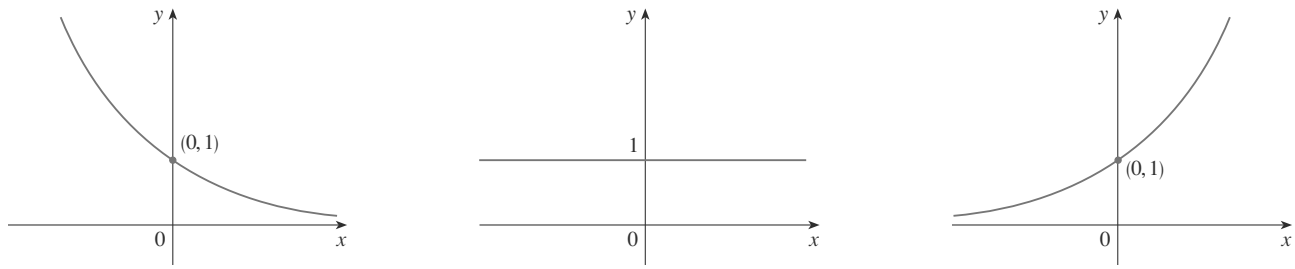


FIGURE 3

|||| If $0 < a < 1$, then a^x approaches 0 as x becomes large. If $a > 1$, then a^x approaches 0 as x decreases through negative values. In both cases the x-axis is a horizontal asymptote. These matters are discussed in Section 2.6.

You can see from Figure 3 that there are basically three kinds of exponential functions $y = a^x$. If $0 < a < 1$, the exponential function decreases; if $a = 1$, it is a constant; and if $a > 1$, it increases. These three cases are illustrated in Figure 4. Observe that if $a \neq 1$,

then the exponential function $y = a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. Notice also that, since $(1/a)^x = 1/a^x = a^{-x}$, the graph of $y = (1/a)^x$ is just the reflection of the graph of $y = a^x$ about the y-axis.



(a) $y = a^x,\ 0 < a < 1$ (b) $y = 1^x$ (c) $y = a^x,\ a > 1$

FIGURE 4

One reason for the importance of the exponential function lies in the following properties. If x and y are rational numbers, then these laws are well known from elementary algebra. It can be proved that they remain true for arbitrary real numbers x and y.

IIII In Section 5.6 we will present a definition of the exponential function that will enable us to give an easy proof of the Laws of Exponents.

Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

1. $a^{x+y} = a^x a^y$ **2.** $a^{x-y} = \dfrac{a^x}{a^y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

EXAMPLE 1 Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

IIII For a review of reflecting and shifting graphs, see Section 1.3.

SOLUTION First we reflect the graph of $y = 2^x$ (shown in Figure 2) about the x-axis to get the graph of $y = -2^x$ in Figure 5(b). Then we shift the graph of $y = -2^x$ upward 3 units to obtain the graph of $y = 3 - 2^x$ in Figure 5(c). The domain is $\mathbb{R}$ and the range is $(-\infty, 3)$.

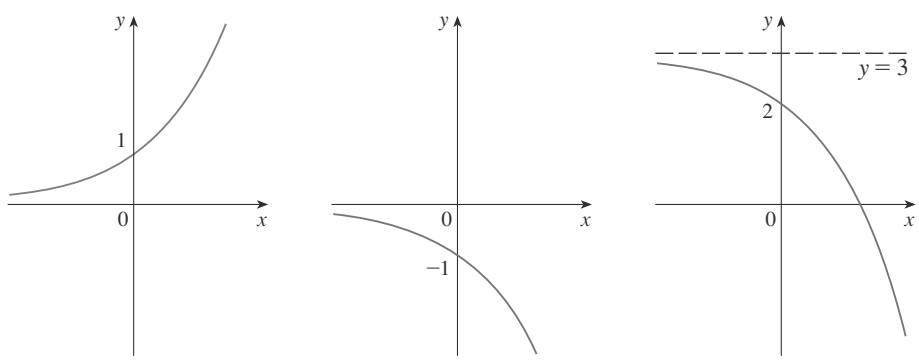


FIGURE 5 (a) $y = 2^x$ (b) $y = -2^x$ (c) $y = 3 - 2^x$

EXAMPLE 2 Use a graphing device to compare the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$. Which function grows more quickly when x is large?

SOLUTION Figure 6 shows both functions graphed in the viewing rectangle $[-2, 6]$ by $[0, 40]$. We see that the graphs intersect three times, but for $x > 4$ the graph of

stays above the graph of . Figure 7 gives a more global view and shows that for large values of x, the exponential function grows far more rapidly than the power function .

Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth and radioactive decay. In later chapters we will pursue these and other applications in greater detail.

First we consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the number of bacteria at time t is , where t is measured in hours, and the initial population is , then we have

$$p(3) = 2p(2) = 2^3$$

It seems from this pattern that, in general,

$$p(t) = 2^t \times 1000 = (1000)2^t$$

This population function is a constant multiple of the exponential function $y = 2^t$, so it exhibits the rapid growth that we observed in Figures 2 and 7. Under ideal conditions (unlimited space and nutrition and freedom from disease) this exponential growth is typical of what actually occurs in nature.

What about the human population? Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

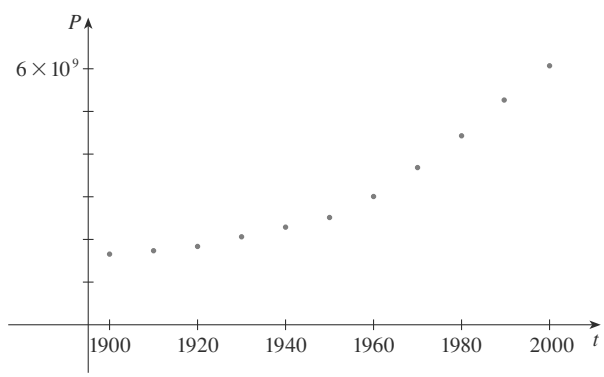


FIGURE 8 Scatter plot for world population growth

EXAMPLE 14 Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad \text{(since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2\text{)}$$

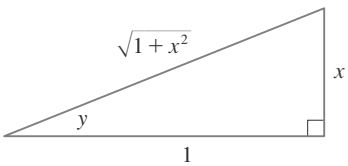


FIGURE 24

Thus $$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 24 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$

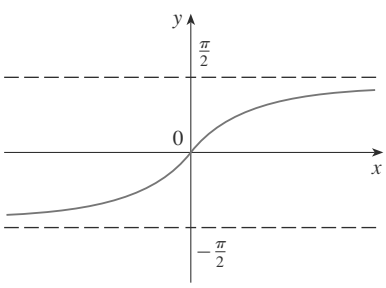


FIGURE 25
$y = \tan^{-1}x = \arctan x$

The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 25.

We know that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan. Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$.

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

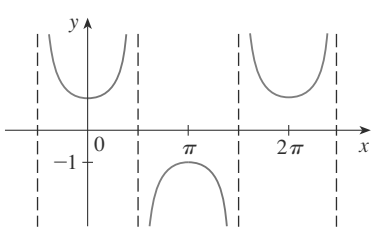


FIGURE 26
$y = \sec x$

11
$$y = \csc^{-1}x \ (|x| \geqslant 1) \iff \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$
$$y = \sec^{-1}x \ (|x| \geqslant 1) \iff \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$
$$y = \cot^{-1}x \ (x \in \mathbb{R}) \iff \cot y = x \quad \text{and} \quad y \in (0, \pi)$$

The choice of intervals for y in the definitions of $\csc^{-1}$ and $\sec^{-1}$ is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of $\sec^{-1}$. [You can see from the graph of the secant function in Figure 26 that both this choice and the one in (11) will work.]

1.6 Exercises

1. (a) What is a one-to-one function?
(b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
(b) If you are given a formula for f, how do you find a formula for f^{-1}?
(c) If you are given the graph of f, how do you find the graph of f^{-1}?

3–14 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

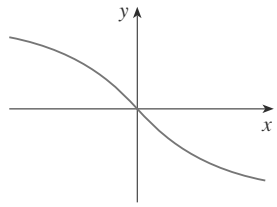
3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

4.

x	1	2	3	4	5	6
$f(x)$	1	2	4	8	16	32

5.



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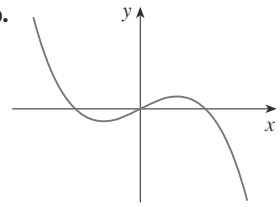

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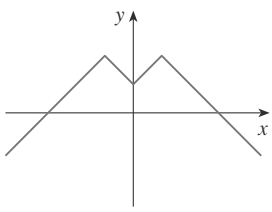

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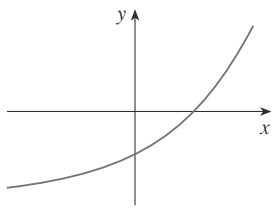


9. $f(x) = \frac{1}{2}(x + 5)$

10. $f(x) = 1 + 4x - x^2$

11. $g(x) = |x|$

12. $g(x) = \sqrt{x}$

13. $f(t)$ is the height of a football t seconds after kickoff.

14. $f(t)$ is your height at age t.

15–16 Use a graph to decide whether f is one-to-one.

15. $f(x) = x^3 - x$

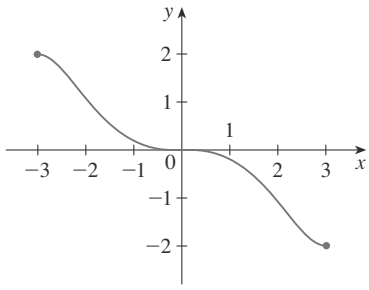
16. $f(x) = x^3 + x$

17. If f is a one-to-one function such that $f(2) = 9$, what is $f^{-1}(9)$?

18. Let $f(x) = 3 + x^2 + \tan(\pi x/2)$, where $-1 < x < 1$.
(a) Find $f^{-1}(3)$.
(b) Find $f(f^{-1}(5))$.

19. If $g(x) = 3 + x + e^x$, find $g^{-1}(4)$.

20. The graph of f is given.
(a) Why is f one-to-one?
(b) State the domain and range of f^{-1}.
(c) Estimate the value of $f^{-1}(1)$.



21. The formula $C = \frac{5}{9}(F - 32)$, where $F \geqslant -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

22. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

23–28 Find a formula for the inverse of the function.

23. $f(x) = \sqrt{10 - 3x}$

24. $f(x) = \dfrac{4x - 1}{2x + 3}$

25. $f(x) = e^{x^3}$

26. $y = 2x^3 + 3$

27. $y = \ln(x + 3)$

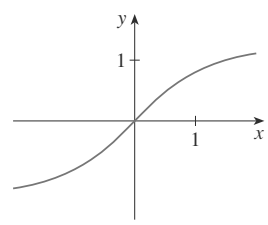
28. $y = \dfrac{1 + e^x}{1 - e^x}$

29–30 Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about the line.

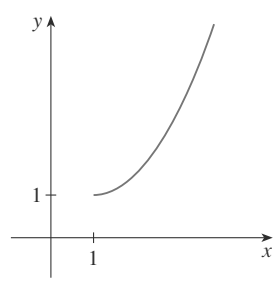
29. $f(x) = 1 - 2/x^2, \quad x > 0$

30. $f(x) = \sqrt{x^2 + 2x}, \quad x > 0$

31. Use the given graph of f to sketch the graph of f^{-1}.



32. Use the given graph of f to sketch the graphs of f^{-1} and $1/f$.



33. (a) How is the logarithmic function $y = \log_a x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_a x$ if $a > 1$.

34. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

35–38 Find the exact value of each expression.

35. (a) $\log_2 64$ (b) $\log_6 \frac{1}{36}$

36. (a) $\log_8 2$ (b) $\ln e^{\sqrt{2}}$

37. (a) $\log_{10} 1.25 + \log_{10} 80$
(b) $\log_5 10 + \log_5 20 - 3 \log_5 2$

38. (a) $2^{(\log_2 3 + \log_2 5)}$ (b) $e^{3 \ln 2}$

39–41 Express the given quantity as a single logarithm.

39. $2 \ln 4 - \ln 2$

40. $\ln x + a \ln y - b \ln z$

41. $\ln(1 + x^2) + \frac{1}{2} \ln x - \ln \sin x$

42. Use Formula 10 to evaluate each logarithm correct to six decimal places.
(a) $\log_{12} 10$ (b) $\log_2 8.4$

43–44 Use Formula 10 to graph the given functions on a common screen. How are these graphs related?

43. $y = \log_{1.5} x, \quad y = \ln x, \quad y = \log_{10} x, \quad y = \log_{50} x$

44. $y = \ln x, \quad y = \log_{10} x, \quad y = e^x, \quad y = 10^x$

45. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

46. Compare the functions $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?

47–48 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 12 and 13 and, if necessary, the transformations of Section 1.3.

47. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

48. (a) $y = \ln(-x)$ (b) $y = \ln |x|$

49–52 Solve each equation for x.

49. (a) $2 \ln x = 1$ (b) $e^{-x} = 5$

50. (a) $e^{2x+3} - 7 = 0$ (b) $\ln(5 - 2x) = -3$

51. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

52. (a) $\ln(\ln x) = 1$ (b) $e^{ax} = Ce^{bx}$, where $a \neq b$

53–54 Solve each inequality for x.

53. (a) $e^x < 10$ (b) $\ln x > -1$

54. (a) $2 < \ln x < 9$ (b) $e^{2-3x} > 4$

55–56 Find (a) the domain of f and (b) f^{-1} and its domain.

55. $f(x) = \sqrt{3 - e^{2x}}$

56. $f(x) = \ln(2 + \ln x)$

CAS **57.** Graph the function $f(x) = \sqrt{x^3 + x^2 + x + 1}$ and explain why it is one-to-one. Then use a computer algebra system to find an explicit expression for $f^{-1}(x)$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

CAS **58.** (a) If $g(x) = x^6 + x^4$, $x \geqslant 0$, use a computer algebra system to find an expression for $g^{-1}(x)$.
(b) Use the expression in part (a) to graph $y = g(x)$, $y = x$, and $y = g^{-1}(x)$ on the same screen.

59. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$. (See Exercise 25 in Section 1.5.)
(a) Find the inverse of this function and explain its meaning.
(b) When will the population reach 50,000?

60. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)
(a) Find the inverse of this function and explain its meaning.
(b) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

61. Starting with the graph of $y = \ln x$, find the equation of the graph that results from
(a) shifting 3 units upward
(b) shifting 3 units to the left
(c) reflecting about the x-axis
(d) reflecting about the y-axis
(e) reflecting about the line $y = x$
(f) reflecting about the x-axis and then about the line $y = x$
(g) reflecting about the y-axis and then about the line $y = x$
(h) shifting 3 units to the left and then reflecting about the line $y = x$

62. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.
(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

63–68 Find the exact value of each expression.

63. (a) $\sin^{-1}(\sqrt{3}/2)$ (b) $\cos^{-1}(-1)$

64. (a) $\arctan(-1)$ (b) $\csc^{-1} 2$

65. (a) $\tan^{-1}\sqrt{3}$ (b) $\arcsin(-1/\sqrt{2})$

66. (a) $\sec^{-1}\sqrt{2}$ (b) $\arcsin 1$

67. (a) $\sin(\sin^{-1} 0.7)$ (b) $\tan^{-1}\left(\tan \dfrac{4\pi}{3}\right)$

68. (a) $\sec(\arctan 2)$ (b) $\cos\left(2 \sin^{-1}\left(\frac{5}{13}\right)\right)$

69. Prove that $\cos(\sin^{-1}x) = \sqrt{1 - x^2}$.

70–72 Simplify the expression.

70. $\tan(\sin^{-1}x)$

71. $\sin(\tan^{-1}x)$

72. $\sin(2 \cos^{-1}x)$

73–74 Graph the given functions on the same screen. How are these graphs related?

73. $y = \sin x, \ -\pi/2 \leqslant x \leqslant \pi/2; \quad y = \sin^{-1}x; \quad y = x$

74. $y = \tan x, \ -\pi/2 < x < \pi/2; \quad y = \tan^{-1}x; \quad y = x$

75. Find the domain and range of the function

$$g(x) = \sin^{-1}(3x + 1)$$

76. (a) Graph the function $f(x) = \sin(\sin^{-1}x)$ and explain the appearance of the graph.
(b) Graph the function $g(x) = \sin^{-1}(\sin x)$. How do you explain the appearance of this graph?

1 Review

CONCEPT CHECK

1. (a) What is a function? What are its domain and range?
(b) What is the graph of a function?
(c) How can you tell whether a given curve is the graph of a function?

2. Discuss four ways of representing a function. Illustrate your discussion with examples.

3. (a) What is an even function? How can you tell if a function is even by looking at its graph?
(b) What is an odd function? How can you tell if a function is odd by looking at its graph?

4. What is an increasing function?

5. What is a mathematical model?

6. Give an example of each type of function.
(a) Linear function (b) Power function
(c) Exponential function (d) Quadratic function
(e) Polynomial of degree 5 (f) Rational function

7. Sketch by hand, on the same axes, the graphs of the following functions.
(a) $f(x) = x$ (b) $g(x) = x^2$
(c) $h(x) = x^3$ (d) $j(x) = x^4$

8. Draw, by hand, a rough sketch of the graph of each function.
(a) $y = \sin x$ (b) $y = \tan x$
(c) $y = e^x$ (d) $y = \ln x$
(e) $y = 1/x$ (f) $y = |x|$
(g) $y = \sqrt{x}$ (h) $y = \tan^{-1}x$

9. Suppose that f has domain A and g has domain B.
(a) What is the domain of $f + g$?
(b) What is the domain of fg?
(c) What is the domain of f/g?

10. How is the composite function $f \circ g$ defined? What is its domain?

11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.
(a) Shift 2 units upward.
(b) Shift 2 units downward.
(c) Shift 2 units to the right.
(d) Shift 2 units to the left.
(e) Reflect about the x-axis.
(f) Reflect about the y-axis.
(g) Stretch vertically by a factor of 2.
(h) Shrink vertically by a factor of 2.
(i) Stretch horizontally by a factor of 2.
(j) Shrink horizontally by a factor of 2.

12. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
(b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f?

13. (a) How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?
(b) How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?
(c) How is the inverse tangent function $f(x) = \tan^{-1}x$ defined? What are its domain and range?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f is a function, then $f(s + t) = f(s) + f(t)$.

2. If $f(s) = f(t)$, then $s = t$.

3. If f is a function, then $f(3x) = 3f(x)$.

4. If $x_1 < x_2$ and f is a decreasing function, then $f(x_1) > f(x_2)$.

5. A vertical line intersects the graph of a function at most once.

6. If f and g are functions, then $f \circ g = g \circ f$.

7. If f is one-to-one, then $f^{-1}(x) = \dfrac{1}{f(x)}$.

8. You can always divide by e^x.

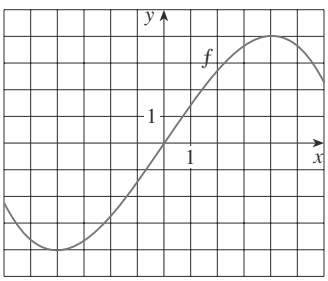
9. If $0 < a < b$, then $\ln a < \ln b$.

10. If $x > 0$, then $(\ln x)^6 = 6 \ln x$.

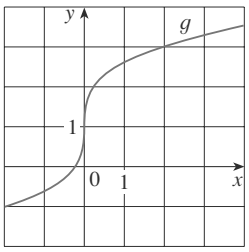
11. If $x > 0$ and $a > 1$, then $\dfrac{\ln x}{\ln a} = \ln \dfrac{x}{a}$.

EXERCISES

1. Let f be the function whose graph is given.
(a) Estimate the value of $f(2)$.
(b) Estimate the values of x such that $f(x) = 3$.
(c) State the domain of f.
(d) State the range of f.
(e) On what interval is f increasing?
(f) Is f one-to-one? Explain.
(g) Is f even, odd, or neither even nor odd? Explain.



2. The graph of g is given.
(a) State the value of $g(2)$.
(b) Why is g one-to-one?
(c) Estimate the value of $g^{-1}(2)$.
(d) Estimate the domain of g^{-1}.
(e) Sketch the graph of g^{-1}.



3. The distance traveled by a car is given by the values in the table.

t (seconds)	0	1	2	3	4	5
d (feet)	0	10	32	70	119	178

(a) Use the data to sketch the graph of d as a function of t.
(b) Use the graph to estimate the distance traveled after 4.5 seconds.

4. Sketch a rough graph of the yield of a crop as a function of the amount of fertilizer used.

5–8 Find the domain and range of the function.

5. $f(x) = \sqrt{4 - 3x^2}$

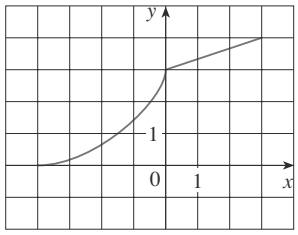
6. $g(x) = 1/(x + 1)$

7. $y = 1 + \sin x$

8. $y = \ln \ln x$

9. Suppose that the graph of f is given. Describe how the graphs of the following functions can be obtained from the graph of f.
(a) $y = f(x) + 8$
(b) $y = f(x + 8)$
(c) $y = 1 + 2f(x)$
(d) $y = f(x - 2) - 2$
(e) $y = -f(x)$
(f) $y = f^{-1}(x)$

10. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x - 8)$
(b) $y = -f(x)$
(c) $y = 2 - f(x)$
(d) $y = \frac{1}{2}f(x) - 1$
(e) $y = f^{-1}(x)$
(f) $y = f^{-1}(x + 3)$



11–16 Use transformations to sketch the graph of the function.

11. $y = -\sin 2x$

12. $y = 3\ln(x - 2)$

13. $y = (1 + e^x)/2$

14. $y = 2 - \sqrt{x}$

15. $f(x) = \dfrac{1}{x + 2}$

16. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geqslant 0 \end{cases}$

17. Determine whether f is even, odd, or neither even nor odd.
(a) $f(x) = 2x^5 - 3x^2 + 2$
(b) $f(x) = x^3 - x^7$
(c) $f(x) = e^{-x^2}$
(d) $f(x) = 1 + \sin x$

18. Find an expression for the function whose graph consists of the line segment from the point $(-2, 2)$ to the point $(-1, 0)$ together with the top half of the circle with center the origin and radius 1.

19. If $f(x) = \ln x$ and $g(x) = x^2 - 9$, find the functions $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$, and their domains.

20. Express the function $F(x) = 1/\sqrt{x + \sqrt{x}}$ as a composition of three functions.

21. Life expectancy improved dramatically in the 20th century. The table gives the life expectancy at birth (in years) of males born in the United States.

Birth year	Life expectancy
1900	48.3
1910	51.1
1920	55.2
1930	57.4
1940	62.5
1950	65.6
1960	66.6
1970	67.1
1980	70.0
1990	71.8
2000	73.0

Use a scatter plot to choose an appropriate type of model. Use your model to predict the life span of a male born in the year 2010.

22. A small-appliance manufacturer finds that it costs \$9000 to produce 1000 toaster ovens a week and \$12,000 to produce 1500 toaster ovens a week.
(a) Express the cost as a function of the number of toaster ovens produced, assuming that it is linear. Then sketch the graph.
(b) What is the slope of the graph and what does it represent?
(c) What is the y-intercept of the graph and what does it represent?

23. If $f(x) = 2x + \ln x$, find $f^{-1}(2)$.

24. Find the inverse function of $f(x) = \dfrac{x + 1}{2x + 1}$.

25. Find the exact value of each expression.
(a) $e^{2\ln 3}$
(b) $\log_{10} 25 + \log_{10} 4$
(c) $\tan\left(\arcsin \frac{1}{2}\right)$
(d) $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$

26. Solve each equation for x.
(a) $e^x = 5$
(b) $\ln x = 2$
(c) $e^{e^x} = 2$
(d) $\tan^{-1} x = 1$

27. The half-life of palladium-100, ^{100}Pd, is four days. (So half of any given quantity of ^{100}Pd will disintegrate in four days.) The initial mass of a sample is one gram.
(a) Find the mass that remains after 16 days.
(b) Find the mass $m(t)$ that remains after t days.
(c) Find the inverse of this function and explain its meaning.
(d) When will the mass be reduced to 0.01 g?

28. The population of a certain species in a limited environment with initial population 100 and carrying capacity 1000 is

$$P(t) = \frac{100{,}000}{100 + 900e^{-t}}$$

where t is measured in years.
(a) Graph this function and estimate how long it takes for the population to reach 900.
(b) Find the inverse of this function and explain its meaning.
(c) Use the inverse function to find the time required for the population to reach 900. Compare with the result of part (a).

29. Graph members of the family of functions $f(x) = \ln(x^2 - c)$ for several values of c. How does the graph change when c changes?

30. Graph the three functions $y = x^a$, $y = a^x$, and $y = \log_a x$ on the same screen for two or three values of $a > 1$. For large values of x, which of these functions has the largest values and which has the smallest values?

PRINCIPLES OF PROBLEM SOLVING

There are no hard and fast rules that will ensure success in solving problems. However, it is possible to outline some general steps in the problem-solving process and to give some principles that may be useful in the solution of certain problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's book *How To Solve It*.

1 UNDERSTAND THE PROBLEM

The first step is to read the problem and make sure that you understand it clearly. Ask yourself the following questions:

What is the unknown?
What are the given quantities?
What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram.

Usually it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities we often use letters such as a, b, c, m, n, x, and y, but in some cases it helps to use initials as suggestive symbols; for instance, V for volume or t for time.

2 THINK OF A PLAN

Find a connection between the given information and the unknown that will enable you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

Try to Recognize Something Familiar Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

Try to Recognize Patterns Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, you might be able to guess what the continuing pattern is and then prove it.

Use Analogy Try to think of an analogous problem, that is, a similar problem, a related problem, but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem involves three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

Introduce Something Extra It may sometimes be necessary to introduce something new, an auxiliary aid, to help make the connection between the given and the unknown. For instance, in a problem where a diagram is useful the auxiliary aid could be a new line drawn in a diagram. In a more algebraic problem it could be a new unknown that is related to the original unknown.

Take Cases We may sometimes have to split a problem into several cases and give a different argument for each of the cases. For instance, we often have to use this strategy in dealing with absolute value.

Work Backward Sometimes it is useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x - 5 = 7$, we suppose that x is a number that satisfies $3x - 5 = 7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x = 4$. Since each of these steps can be reversed, we have solved the problem.

Establish Subgoals In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If we can first reach these subgoals, then we may be able to build on them to reach our final goal.

Indirect Reasoning Sometimes it is appropriate to attack a problem indirectly. In using proof by contradiction to prove that P implies Q, we assume that P is true and Q is false and try to see why this can't happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

Mathematical Induction In proving statements that involve a positive integer n, it is frequently helpful to use the following principle.

> **Principle of Mathematical Induction** Let S_n be a statement about the positive integer n. Suppose that
>
> **1.** S_1 is true.
>
> **2.** S_{k+1} is true whenever S_k is true.
>
> Then S_n is true for all positive integers n.

This is reasonable because, since S_1 is true, it follows from condition 2 (with $k = 1$) that S_2 is true. Then, using condition 2 with $k = 2$, we see that S_3 is true. Again using condition 2, this time with $k = 3$, we have that S_4 is true. This procedure can be followed indefinitely.

3 CARRY OUT THE PLAN

In Step 2 a plan was devised. In carrying out that plan we have to check each stage of the plan and write the details that prove that each stage is correct.

4 LOOK BACK

Having completed our solution, it is wise to look back over it, partly to see if we have made errors in the solution and partly to see if we can think of an easier way to solve the problem. Another reason for looking back is that it will familiarize us with the method of solution and this may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems."

These principles of problem solving are illustrated in the following examples. Before you look at the solutions, try to solve these problems yourself, referring to these Principles of Problem Solving if you get stuck. You may find it useful to refer to this section from time to time as you solve the exercises in the remaining chapters of this book.

EXAMPLE 1 Express the hypotenuse h of a right triangle with area 25 m^2 as a function of its perimeter P.

|||| Understand the problem

SOLUTION Let's first sort out the information by identifying the unknown quantity and the data:

Unknown: hypotenuse h

Given quantities: perimeter P, area 25 m^2

|||| Draw a diagram

It helps to draw a diagram and we do so in Figure 1.

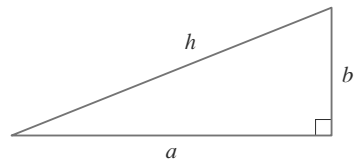


FIGURE 1

|||| Connect the given with the unknown

|||| Introduce something extra

In order to connect the given quantities to the unknown, we introduce two extra variables a and b, which are the lengths of the other two sides of the triangle. This enables us to express the given condition, which is that the triangle is right-angled, by the Pythagorean Theorem:

$$h^2 = a^2 + b^2$$

The other connections among the variables come by writing expressions for the area and perimeter:

$$25 = \tfrac{1}{2}ab \qquad P = a + b + h$$

Since P is given, notice that we now have three equations in the three unknowns a, b, and h:

$$\boxed{1} \qquad h^2 = a^2 + b^2$$

$$\boxed{2} \qquad 25 = \tfrac{1}{2}ab$$

$$\boxed{3} \qquad P = a + b + h$$

|||| Relate to the familiar

Although we have the correct number of equations, they are not easy to solve in a straightforward fashion. But if we use the problem-solving strategy of trying to recognize something familiar, then we can solve these equations by an easier method. Look at the right sides of Equations 1, 2, and 3. Do these expressions remind you of anything familiar? Notice that they contain the ingredients of a familiar formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Using this idea, we express $(a + b)^2$ in two ways. From Equations 1 and 2 we have

$$(a + b)^2 = (a^2 + b^2) + 2ab = h^2 + 4(25)$$

From Equation 3 we have

$$(a + b)^2 = (P - h)^2 = P^2 - 2Ph + h^2$$

Thus

$$h^2 + 100 = P^2 - 2Ph + h^2$$

$$2Ph = P^2 - 100$$

$$h = \frac{P^2 - 100}{2P}$$

This is the required expression for h as a function of P.

As the next example illustrates, it is often necessary to use the problem-solving principle of *taking cases* when dealing with absolute values.

EXAMPLE 2 Solve the inequality $|x - 3| + |x + 2| < 11$.

SOLUTION Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows that

$$|x - 3| = \begin{cases} x - 3 & \text{if } x - 3 \geqslant 0 \\ -(x - 3) & \text{if } x - 3 < 0 \end{cases}$$
$$= \begin{cases} x - 3 & \text{if } x \geqslant 3 \\ -x + 3 & \text{if } x < 3 \end{cases}$$

Similarly

$$|x + 2| = \begin{cases} x + 2 & \text{if } x + 2 \geqslant 0 \\ -(x + 2) & \text{if } x + 2 < 0 \end{cases}$$
$$= \begin{cases} x + 2 & \text{if } x \geqslant -2 \\ -x - 2 & \text{if } x < -2 \end{cases}$$

|||| Take cases

These expressions show that we must consider three cases:

$$x < -2 \qquad -2 \leqslant x < 3 \qquad x \geqslant 3$$

CASE I ▫ If $x < -2$, we have

$$\begin{aligned} |x - 3| + |x + 2| &< 11 \\ -x + 3 - x - 2 &< 11 \\ -2x &< 10 \\ x &> -5 \end{aligned}$$

CASE II ▫ If $-2 \leqslant x < 3$, the given inequality becomes

$$\begin{aligned} -x + 3 + x + 2 &< 11 \\ 5 &< 11 \qquad \text{(always true)} \end{aligned}$$

CASE III ▫ If $x \geqslant 3$, the inequality becomes

$$\begin{aligned} x - 3 + x + 2 &< 11 \\ 2x &< 12 \\ x &< 6 \end{aligned}$$

Combining cases I, II, and III, we see that the inequality is satisfied when $-5 < x < 6$. So the solution is the interval $(-5, 6)$.

In the following example we first guess the answer by looking at special cases and recognizing a pattern. Then we prove it by mathematical induction.

In using the Principle of Mathematical Induction, we follow three steps:

STEP 1 Prove that S_n is true when $n = 1$.

STEP 2 Assume that S_n is true when $n = k$ and deduce that S_n is true when $n = k + 1$.

STEP 3 Conclude that S_n is true for all n by the Principle of Mathematical Induction.

EXAMPLE 3 If $f_0(x) = x/(x+1)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

|||| Analogy: Try a similar, simpler problem

SOLUTION We start by finding formulas for $f_n(x)$ for the special cases $n = 1$, 2, and 3.

$$f_1(x) = (f_0 \circ f_0)(x) = f_0(f_0(x)) = f_0\left(\frac{x}{x+1}\right)$$

$$= \frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1} + 1} = \frac{\dfrac{x}{x+1}}{\dfrac{2x+1}{x+1}} = \frac{x}{2x+1}$$

$$f_2(x) = (f_0 \circ f_1)(x) = f_0(f_1(x)) = f_0\left(\frac{x}{2x+1}\right)$$

$$= \frac{\dfrac{x}{2x+1}}{\dfrac{x}{2x+1} + 1} = \frac{\dfrac{x}{2x+1}}{\dfrac{3x+1}{2x+1}} = \frac{x}{3x+1}$$

|||| Look for a pattern

$$f_3(x) = (f_0 \circ f_2)(x) = f_0(f_2(x)) = f_0\left(\frac{x}{3x+1}\right)$$

$$= \frac{\dfrac{x}{3x+1}}{\dfrac{x}{3x+1} + 1} = \frac{\dfrac{x}{3x+1}}{\dfrac{4x+1}{3x+1}} = \frac{x}{4x+1}$$

We notice a pattern: The coefficient of x in the denominator of $f_n(x)$ is $n + 1$ in the three cases we have computed. So we make the guess that, in general,

$$f_n(x) = \frac{x}{(n+1)x + 1} \tag{4}$$

To prove this, we use the Principle of Mathematical Induction. We have already verified that (4) is true for $n = 1$. Assume that it is true for $n = k$, that is,

$$f_k(x) = \frac{x}{(k+1)x + 1}$$

Then $$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{x}{(k+1)x+1}\right)$$

$$= \frac{\dfrac{x}{(k+1)x+1}}{\dfrac{x}{(k+1)x+1} + 1} = \frac{\dfrac{x}{(k+1)x+1}}{\dfrac{(k+2)x+1}{(k+1)x+1}} = \frac{x}{(k+2)x+1}$$

This expression shows that (4) is true for $n = k + 1$. Therefore, by mathematical induction, it is true for all positive integers n.

PROBLEMS

1. One of the legs of a right triangle has length 4 cm. Express the length of the altitude perpendicular to the hypotenuse as a function of the length of the hypotenuse.

2. The altitude perpendicular to the hypotenuse of a right triangle is 12 cm. Express the length of the hypotenuse as a function of the perimeter.

3. Solve the equation $|2x - 1| - |x + 5| = 3$.

4. Solve the inequality $|x - 1| - |x - 3| \geqslant 5$.

5. Sketch the graph of the function $f(x) = \left|x^2 - 4|x| + 3\right|$.

6. Sketch the graph of the function $g(x) = |x^2 - 1| - |x^2 - 4|$.

7. Draw the graph of the equation $x + |x| = y + |y|$.

8. Draw the graph of the equation $x^4 - 4x^2 - x^2y^2 + 4y^2 = 0$.

9. Sketch the region in the plane consisting of all points (x, y) such that $|x| + |y| \leqslant 1$.

10. Sketch the region in the plane consisting of all points (x, y) such that

$$|x - y| + |x| - |y| \leqslant 2$$

11. Evaluate $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{31} 32)$.

12. (a) Show that the function $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ is an odd function.
(b) Find the inverse function of f.

13. Solve the inequality $\ln(x^2 - 2x - 2) \leqslant 0$.

14. Use indirect reasoning to prove that $\log_2 5$ is an irrational number.

15. A driver sets out on a journey. For the first half of the distance she drives at the leisurely pace of 30 mi/h; she drives the second half at 60 mi/h. What is her average speed on this trip?

16. Is it true that $f \circ (g + h) = f \circ g + f \circ h$?

17. Prove that if n is a positive integer, then $7^n - 1$ is divisible by 6.

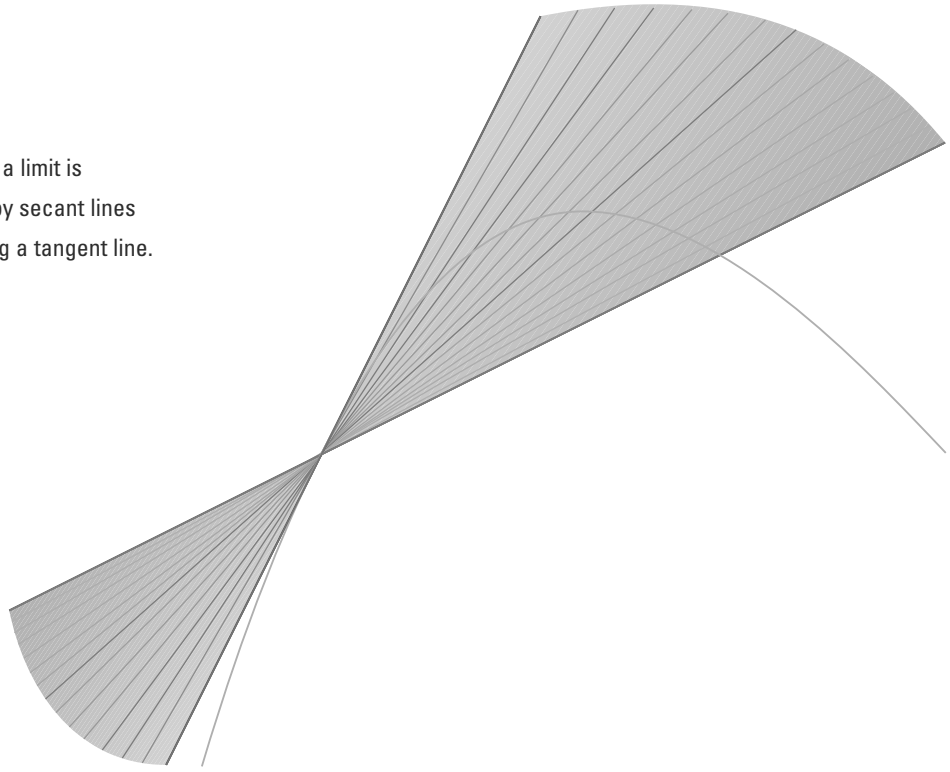
18. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

19. If $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

20. (a) If $f_0(x) = \dfrac{1}{2 - x}$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find an expression for $f_n(x)$ and use mathematical induction to prove it.

(b) Graph f_0, f_1, f_2, f_3 on the same screen and describe the effects of repeated composition.

The idea of a limit is illustrated by secant lines approaching a tangent line.



Limits and Derivatives

In *A Preview of Calculus* (page 2) we saw how the idea of a limit underlies the various branches of calculus. It is therefore appropriate to begin our study of calculus by investigating limits and their properties. The special type of limit that is used to find tangents and velocities gives rise to the central idea in differential calculus, the derivative.

2.1 The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means "touching." Thus, a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

Locate tangents interactively and explore them numerically.

Resources / Module 1 / Tangents / What Is a Tangent?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once as in Figure 1(a). For more complicated curves this definition is inadequate. Figure l(b) shows two lines l and t passing through a point P on a curve C. The line l intersects C only once, but it certainly does not look like what we think of as a tangent. The line t, on the other hand, looks like a tangent but it intersects C twice.

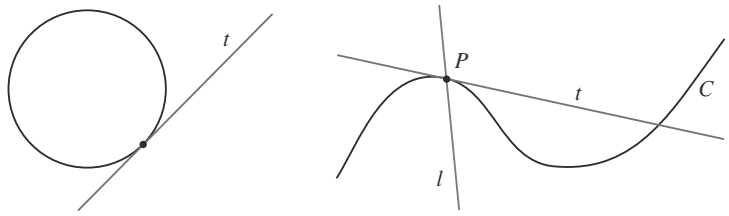


FIGURE 1 (a) (b)

To be specific, let's look at the problem of trying to find a tangent line t to the parabola $y = x^2$ in the following example.

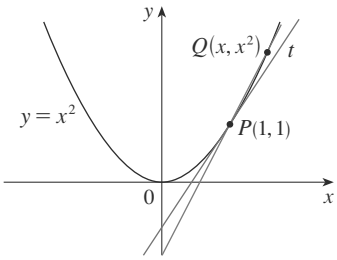


FIGURE 2

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION We will be able to find an equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ.

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line t should be $m = 2$.

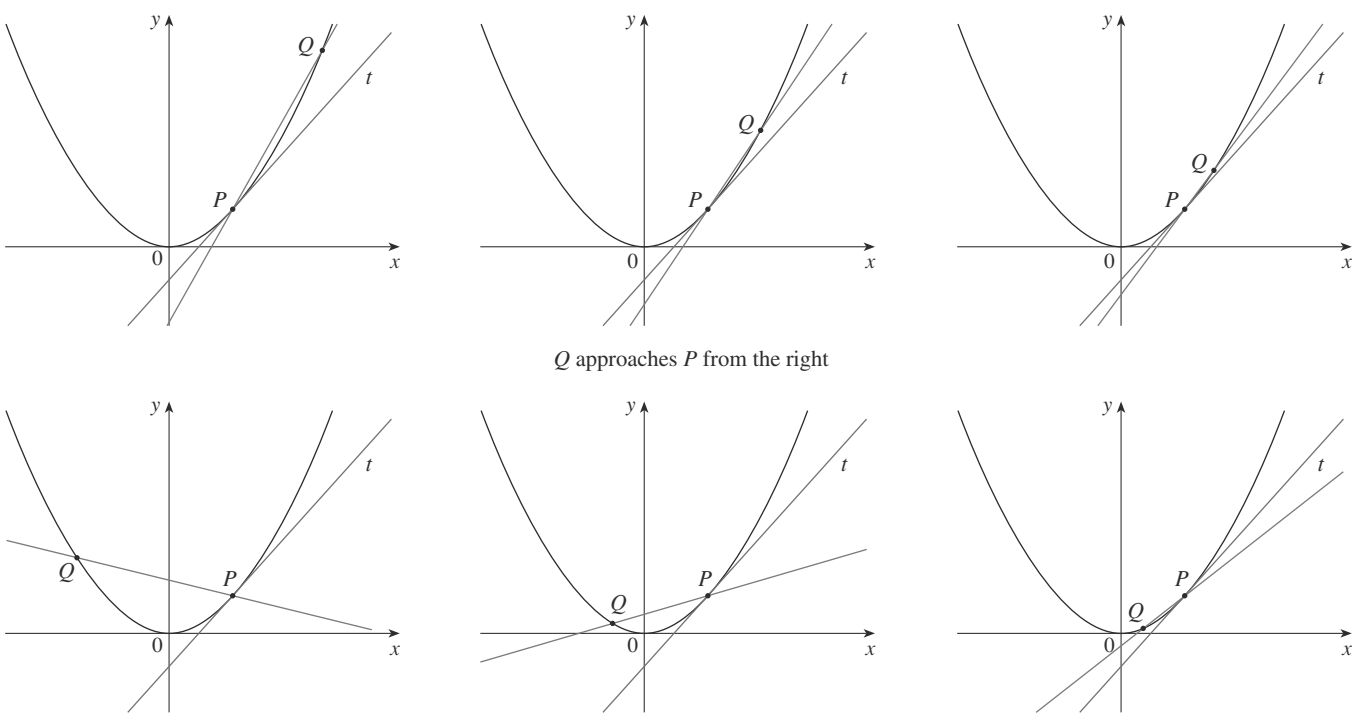
We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line (see Appendix B) to write the equation of the tangent line through (1, 1) as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Figure 3 illustrates the limiting process that occurs in this example. As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line t.



Q approaches P from the right

Q approaches P from the left

FIGURE 3

In Module 2.1 you can see how the process in Figure 3 works for five additional functions.

Many functions that occur in science are not described by explicit equations; they are defined by experimental data. The next example shows how to estimate the slope of the tangent line to the graph of such a function.

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

EXAMPLE 2 The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data at the left describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds after the flash goes off). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. [*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

SOLUTION In Figure 4 we plot the given data and use them to sketch a curve that approximates the graph of the function.

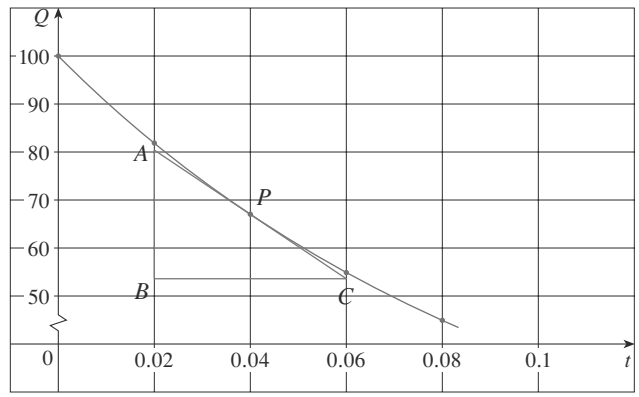


FIGURE 4

Given the points $P(0.04, 67.03)$ and $R(0.00, 100.00)$ on the graph, we find that the slope of the secant line PR is

$$m_{PR} = \frac{100.00 - 67.03}{0.00 - 0.04} = -824.25$$

R	m_{PR}
(0.00, 100.00)	−824.25
(0.02, 81.87)	−742.00
(0.06, 54.88)	−607.50
(0.08, 44.93)	−552.50
(0.10, 36.76)	−504.50

The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between -742 and -607.5. In fact, the average of the slopes of the two closest secant lines is

$$\tfrac{1}{2}(-742 - 607.5) = -674.75$$

So, by this method, we estimate the slope of the tangent line to be -675.

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC, as in Figure 4. This gives an estimate of the slope of the tangent line as

$$-\frac{|AB|}{|BC|} \approx -\frac{80.4 - 53.6}{0.06 - 0.02} = -670$$

|||| The physical meaning of the answer in Example 2 is that the electric current flowing from the capacitor to the flash bulb after 0.04 second is about −670 microamperes.

|||| The Velocity Problem

If you watch the speedometer of a car as you travel in city traffic, you see that the needle doesn't stay still for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.

The CN Tower in Toronto is currently the tallest freestanding building in the world.

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

SOLUTION Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\begin{aligned} \text{average velocity} &= \frac{\text{distance traveled}}{\text{time elapsed}} \\ &= \frac{s(5.1) - s(5)}{0.1} \\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s} \end{aligned}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus, the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 5) and we consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(a + h)^2 - 4.9a^2}{(a + h) - a}$$

which is the same as the average velocity over the time interval $[a, a + h]$. Therefore, the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).

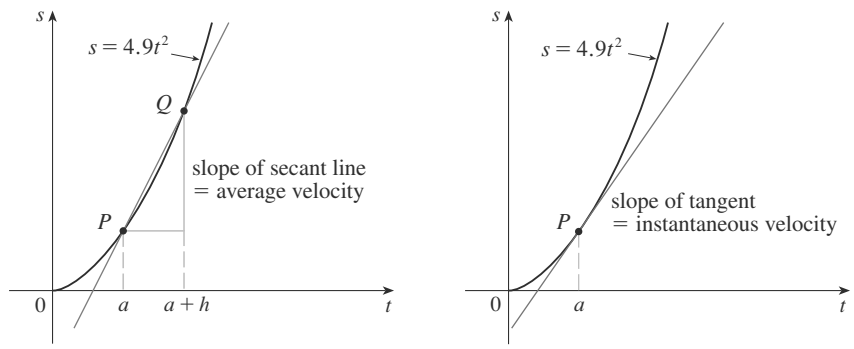


FIGURE 5

Examples 1 and 3 show that in order to solve tangent and velocity problems we must be able to find limits. After studying methods for computing limits in the next five sections, we will return to the problems of finding tangents and velocities in Section 2.7.

2.1 Exercises

1. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes.

t (min)	5	10	15	20	25	30
V (gal)	694	444	250	111	28	0

(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5$, 10, 20, 25, and 30.
(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.
(c) Use a graph of the function to estimate the slope of the tangent line at P. (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of t.
(a) $t = 36$ and $t = 42$ (b) $t = 38$ and $t = 42$
(c) $t = 40$ and $t = 42$ (d) $t = 42$ and $t = 44$
What are your conclusions?

3. The point $P\left(1, \frac{1}{2}\right)$ lies on the curve $y = x/(1 + x)$.
(a) If Q is the point $(x, x/(1 + x))$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 0.5 (ii) 0.9
(iii) 0.99 (iv) 0.999
(v) 1.5 (vi) 1.1
(vii) 1.01 (viii) 1.001
(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P\left(1, \frac{1}{2}\right)$.
(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P\left(1, \frac{1}{2}\right)$.

4. The point $P(2, \ln 2)$ lies on the curve $y = \ln x$.
(a) If Q is the point $(x, \ln x)$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 1.5 (ii) 1.9
(iii) 1.99 (iv) 1.999
(v) 2.5 (vi) 2.1
(vii) 2.01 (viii) 2.001
(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, \ln 2)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, \ln 2)$.
(d) Sketch the curve, two of the secant lines, and the tangent line.

5. If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet after t seconds is given by $y = 40t - 16t^2$.
(a) Find the average velocity for the time period beginning when $t = 2$ and lasting
(i) 0.5 second (ii) 0.1 second
(iii) 0.05 second (iv) 0.01 second
(b) Find the instantaneous velocity when $t = 2$.

6. If an arrow is shot upward on the moon with a velocity of 58 m/s, its height in meters after t seconds is given by $h = 58t - 0.83t^2$.
(a) Find the average velocity over the given time intervals:
(i) [1, 2] (ii) [1, 1.5] (iii) [1, 1.1]
(iv) [1, 1.01] (v) [1, 1.001]
(b) Find the instantaneous velocity after one second.

7. The displacement (in feet) of a certain particle moving in a straight line is given by $s = t^3/6$, where t is measured in seconds.
(a) Find the average velocity over the following time periods:
(i) [1, 3] (ii) [1, 2]
(iii) [1, 1.5] (iv) [1, 1.1]
(b) Find the instantaneous velocity when $t = 1$.
(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities found in part (a).
(d) Draw the tangent line whose slope is the instantaneous velocity from part (b).

8. The position of a car is given by the values in the table.

t (seconds)	0	1	2	3	4	5
s (feet)	0	10	32	70	119	178

(a) Find the average velocity for the time period beginning when $t = 2$ and lasting
(i) 3 seconds (ii) 2 seconds (iii) 1 second
(b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 2$.

9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.
(a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2$, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8, and 0.9. Do the slopes appear to be approaching a limit?
(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.
(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P.

2.2 The Limit of a Function

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2, but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

From the table and the graph of f (a parabola) shown in Figure 1 we see that when x is close to 2 (on either side of 2), $f(x)$ is close to 4. In fact, it appears that we can make the

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 Show that $\lim_{x \to 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

|||| The result of Example 7 looks plausible from Figure 3.

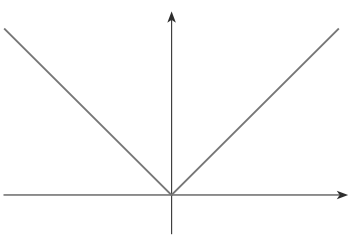


FIGURE 3

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0$$

Therefore, by Theorem 1,

$$\lim_{x \to 0} |x| = 0$$

EXAMPLE 8 Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

SOLUTION

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1$$

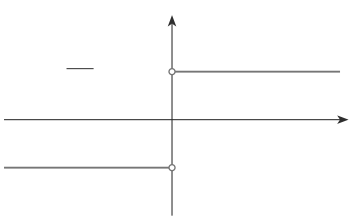


FIGURE 4

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. The graph of the function $f(x) = \frac{|x|}{x}$ is shown in Figure 4 and supports the one-sided limits that we found.

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

|||| It is shown in Example 3 in Section 2.4 that $\lim_{x \to 0^+} \sqrt{x} = 0$.

SOLUTION Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

Since $f\ x\ = 8\quad 2x$ for $x\quad 4$, we have

$$\lim_{x\ \ 4} f\ x\ = \lim_{x\ \ 4}\ 8\quad 2x\ = 8\quad 2\ \ 4 = 0$$

The right- and left-hand limits are equal. Thus, the limit exists and

$$\lim_{x\ \ 4} f\ x\ = 0$$

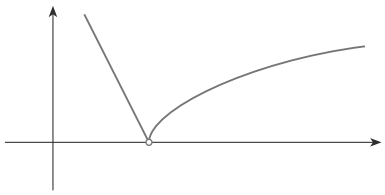


FIGURE 5

The graph of f is shown in Figure 5.

EXAMPLE 10 The **greatest integer function** is defined by $x\ =$ the largest integer that is less than or equal to x. (For instance, $4\ = 4$, $4.8\ = 4$, $\quad = 3$, $\overline{2}\ = 1$, $\frac{1}{2}\ =\quad 1$.) Show that $\lim_{x\ \ 3}\ x$ does not exist.

|||| Other notations for are and .

SOLUTION The graph of the greatest integer function is shown in Figure 6. Since $x\ = 3$ for $3\quad x\quad 4$, we have

$$\lim_{x\ \ 3}\ x\ = \lim_{x\ \ 3}\ 3 = 3$$

Since $x\ = 2$ for $2\quad x\quad 3$, we have

$$\lim_{x\ \ 3}\ x\ = \lim_{x\ \ 3}\ 2 = 2$$

Because these one-sided limits are not equal, $\lim_{x\ \ 3}\ x$ does not exist by Theorem 1.

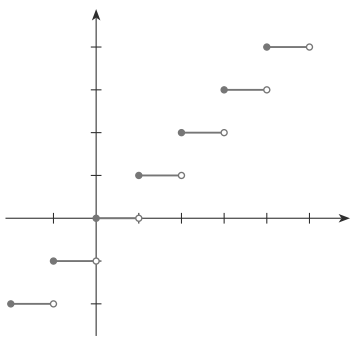


FIGURE 6
Greatest integer function

The next two theorems give two additional properties of limits. Their proofs can be found in Appendix F.

2 Theorem If $f\ x\qquad x$ when x is near a (except possibly at a) and the limits of f and both exist as x approaches a, then

$$\lim_{x\ \ a} f\ x\qquad \lim_{x\ \ a}\ x$$

3 The Squeeze Theorem If $f\ x\qquad x\qquad h\ x$ when x is near a (except possibly at a) and

$$\lim_{x\ \ a} f\ x\ = \lim_{x\ \ a} h\ x\ = L$$

then

$$\lim_{x\ \ a}\ x\ = L$$

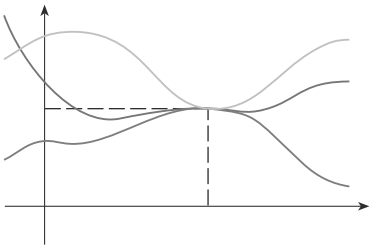


FIGURE 7

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7. It says that if x is squeezed between $f\ x$ and $h\ x$ near a, and if f and h have the same limit L at a, then is forced to have the same limit L at a.

EXAMPLE 11 Show that $\lim_{x \quad 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we *cannot* use

$$\lim_{x \quad 0} x^2 \sin \frac{1}{x} = \lim_{x \quad 0} x^2 \quad \lim_{x \quad 0} \sin \frac{1}{x}$$

because $\lim_{x \quad 0} \sin 1 \ x$ does not exist (see Example 4 in Section 2.2). However, since

$$1 \quad \sin \frac{1}{x} \quad 1$$

we have, as illustrated by Figure 8,

$$x^2 \quad x^2 \sin \frac{1}{x} \quad x^2$$

Watch an animation of a similar limit.
Resources / Module 2
/ Basics of Limits
/ Sound of a Limit that Exists

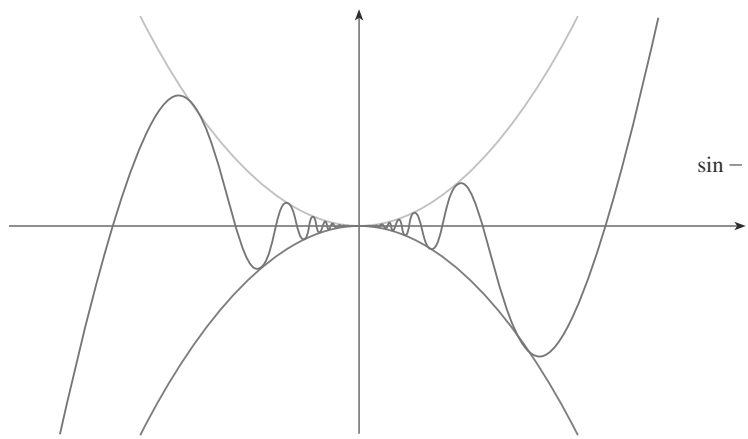


FIGURE 8

We know that

$$\lim_{x \quad 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \quad 0} \quad x^2 \ = 0$$

Taking $f \ x \ = \quad x^2, \quad x \ = x^2 \sin 1 \ x$, and $h \ x \ = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \quad 0} x^2 \sin \frac{1}{x} = 0$$

2.3 Exercises

1. Given that

$$\lim_{x \quad a} f \ x \ = \quad 3 \qquad \lim_{x \quad a} \quad x \ = 0 \qquad \lim_{x \quad a} h \ x \ = 8$$

Þnd the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x \quad a} \ f \ x \quad h \ x$

(b) $\lim_{x \quad a} \ f \ x \ ^2$

(c) $\lim_{x \quad a} \sqrt[3]{h \ x}$

(d) $\lim_{x \quad a} \frac{1}{f \ x}$

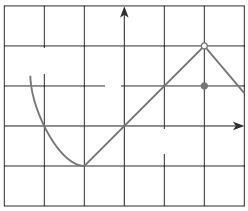
(e) $\lim_{x \quad a} \frac{f \ x}{h \ x}$

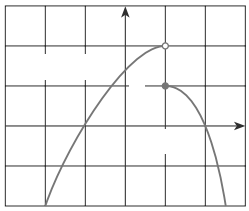
(f) $\lim_{x \quad a} \frac{x}{f \ x}$

(g) $\lim_{x \quad a} \frac{f \ x}{x}$

(h) $\lim_{x \quad a} \frac{2f \ x}{h \ x \quad f \ x}$

2. The graphs of f and are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.





(a) $\lim_{x \to 2} f(x) \quad x$

(b) $\lim_{x \to 1} f(x) \quad x$

(c) $\lim_{x \to 0} f(x) \quad x$

(d) $\lim_{x \to 1} \dfrac{f(x)}{x}$

(e) $\lim_{x \to 2} x^3 f(x)$

(f) $\lim_{x \to 1} \sqrt{3 \quad f(x)}$

3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \to 2} (3x^4 \quad 2x^2 \quad x \quad 1)$

4. $\lim_{x \to 2} \dfrac{2x^2 \quad 1}{x^2 \quad 6x \quad 4}$

5. $\lim_{x \to 3} (x^2 \quad 4)(x^3 \quad 5x \quad 1)$

6. $\lim_{t \to 1} (t^2 \quad 1)^3 (t \quad 3)^5$

7. $\lim_{x \to 1} \left(\dfrac{1 \quad 3x}{1 \quad 4x^2 \quad 3x^4}\right)^3$

8. $\lim_{u \to 2} \sqrt{u^4 \quad 3u \quad 6}$

9. $\lim_{x \to 4} \sqrt{16 \quad x^2}$

10. (a) What is wrong with the following equation?

$$\frac{x^2 \quad x \quad 6}{x \quad 2} = x \quad 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 \quad x \quad 6}{x \quad 2} = \lim_{x \to 2} (x \quad 3)$$

is correct.

11–30 Evaluate the limit, if it exists.

11. $\lim_{x \to 2} \dfrac{x^2 \quad x \quad 6}{x \quad 2}$

12. $\lim_{x \to 4} \dfrac{x^2 \quad 5x \quad 4}{x^2 \quad 3x \quad 4}$

13. $\lim_{x \to 2} \dfrac{x^2 \quad x \quad 6}{x \quad 2}$

14. $\lim_{x \to 4} \dfrac{x^2 \quad 4x}{x^2 \quad 3x \quad 4}$

15. $\lim_{t \to 3} \dfrac{t^2 \quad 9}{2t^2 \quad 7t \quad 3}$

16. $\lim_{x \to 1} \dfrac{x^2 \quad 4x}{x^2 \quad 3x \quad 4}$

17. $\lim_{h \to 0} \dfrac{(4 \quad h)^2 \quad 16}{h}$

18. $\lim_{x \to 1} \dfrac{x^3 \quad 1}{x^2 \quad 1}$

19. $\lim_{h \to 0} \dfrac{(1 \quad h)^4 \quad 1}{h}$

20. $\lim_{h \to 0} \dfrac{(2 \quad h)^3 \quad 8}{h}$

21. $\lim_{t \to 9} \dfrac{9 \quad t}{3 \quad \sqrt{t}}$

22. $\lim_{h \to 0} \dfrac{\sqrt{1 \quad h} \quad 1}{h}$

23. $\lim_{x \to 7} \dfrac{\sqrt{x \quad 2} \quad 3}{x \quad 7}$

24. $\lim_{x \to 2} \dfrac{x^4 \quad 16}{x \quad 2}$

25. $\lim_{x \to 4} \dfrac{\frac{1}{4} \quad \frac{1}{x}}{4 \quad x}$

26. $\lim_{t \to 0} \left(\dfrac{1}{t} \quad \dfrac{1}{t^2 \quad t}\right)$

27. $\lim_{x \to 9} \dfrac{x^2 \quad 81}{\sqrt{x} \quad 3}$

28. $\lim_{h \to 0} \dfrac{(3 \quad h)^{1} \quad 3^{1}}{h}$

29. $\lim_{t \to 0} \left(\dfrac{1}{t\sqrt{1 \quad t}} \quad \dfrac{1}{t}\right)$

30. $\lim_{x \to 1} \left(\dfrac{\sqrt{x} \quad x^2}{1 \quad \sqrt{x}}\right)$

31. (a) Estimate the value of

$$\lim_{x \to 0} \frac{x}{\sqrt{1 \quad 3x} \quad 1}$$

by graphing the function $f(x) = x\left(\sqrt{1 \quad 3x} \quad 1\right)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

32. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 \quad x} \quad \sqrt{3}}{x}$$

to estimate the value of $\lim_{x \to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

33. Use the Squeeze Theorem to show that $\lim_{x \to 0} x^2 \cos 20\pi x = 0$. Illustrate by graphing the functions $f(x) = \quad x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

34. Use the Squeeze Theorem to show that

$$\lim_{x \to 0} \sqrt{x^3 \quad x^2} \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

35. If $1 \quad f(x) \quad x^2 \quad 2x \quad 2$ for all x, find $\lim_{x \to 1} f(x)$.

36. If $3x \quad f(x) \quad x^3 \quad 2$ for $0 \quad x \quad 2$, evaluate $\lim_{x \to 1} f(x)$.

37. Prove that $\lim_{x \to 0} x^4 \cos \dfrac{2}{x} = 0$.

38. Prove that $\lim_{x \to 0} \sqrt{x}\, e^{\sin \pi x} = 0$.

39–44 Find the limit, if it exists. If the limit does not exist, explain why.

39. $\lim_{x \to 4} (x \quad 4)$

40. $\lim_{x \to 4} \dfrac{x \quad 4}{x \quad 4}$

41. $\lim_{x \quad 2} \dfrac{x \quad 2}{x \quad 2}$

42. $\lim_{x \quad 1.5} \dfrac{2x^2 \quad 3x}{2x \quad 3}$

43. $\lim_{x \quad 0} \dfrac{1}{x} \quad \dfrac{1}{x}$

44. $\lim_{x \quad 0} \dfrac{1}{x} \quad \dfrac{1}{x}$

45. The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{array}{ll} 1 & \text{if } x \quad 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x \quad 0 \end{array}$$

(a) Sketch the graph of this function.
(b) Find each of the following limits or explain why it does not exist.
(i) $\lim_{x \quad 0} \operatorname{sgn} x$ (ii) $\lim_{x \quad 0} \operatorname{sgn} x$
(iii) $\lim_{x \quad 0} \operatorname{sgn} x$ (iv) $\lim_{x \quad 0} \operatorname{sgn} x$

46. Let

$$f\ x = \begin{array}{ll} 4 \quad x^2 & \text{if } x \quad 2 \\ x \quad 1 & \text{if } x \quad 2 \end{array}$$

(a) Find $\lim_{x \quad 2} f\ x$ and $\lim_{x \quad 2} f\ x$.
(b) Does $\lim_{x \quad 2} f\ x$ exist?
(c) Sketch the graph of f.

47. Let $F\ x = \dfrac{x^2 \quad 1}{x \quad 1}$.

(a) Find
(i) $\lim_{x \quad 1} F\ x$ (ii) $\lim_{x \quad 1} F\ x$
(b) Does $\lim_{x \quad 1} F\ x$ exist?
(c) Sketch the graph of F.

48. Let

$$h\ x = \begin{array}{ll} x & \text{if } x \quad 0 \\ x^2 & \text{if } 0 \quad x \quad 2 \\ 8 \quad x & \text{if } x \quad 2 \end{array}$$

(a) Evaluate each of the following limits, if it exists.
(i) $\lim_{x \quad 0} h\ x$ (ii) $\lim_{x \quad 0} h\ x$ (iii) $\lim_{x \quad 1} h\ x$
(iv) $\lim_{x \quad 2} h\ x$ (v) $\lim_{x \quad 2} h\ x$ (vi) $\lim_{x \quad 2} h\ x$
(b) Sketch the graph of h.

49. (a) If the symbol denotes the greatest integer function defined in Example 10, evaluate
(i) $\lim_{x \quad 2} \ x$ (ii) $\lim_{x \quad 2} \ x$ (iii) $\lim_{x \quad 2.4} \ x$
(b) If n is an integer, evaluate
(i) $\lim_{x \quad n} \ x$ (ii) $\lim_{x \quad n} \ x$
(c) For what values of a does $\lim_{x \quad a} \ x$ exist?

50. Let $f\ x = x \quad x$.
(a) Sketch the graph of f.
(b) If n is an integer, evaluate
(i) $\lim_{x \quad n} f\ x$ (ii) $\lim_{x \quad n} f\ x$
(c) For what values of a does $\lim_{x \quad a} f\ x$ exist?

51. If $f\ x = x \quad x$, show that $\lim_{x \quad 2} f\ x$ exists but is not equal to $f\ 2$.

52. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \quad \overline{1 \quad v^2\ c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \quad c} L$ and interpret the result. Why is a left-hand limit necessary?

53. If p is a polynomial, show that $\lim_{x \quad a} p\ x = p\ a$.

54. If r is a rational function, use Exercise 53 to show that $\lim_{x \quad a} r\ x = r\ a$ for every number a in the domain of r.

55. If

$$f\ x = \begin{array}{ll} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{array}$$

prove that $\lim_{x \quad 0} f\ x = 0$.

56. Show by means of an example that $\lim_{x \quad a} f\ x \quad x$ may exist even though neither $\lim_{x \quad a} f\ x$ nor $\lim_{x \quad a} \ x$ exists.

57. Show by means of an example that $\lim_{x \quad a} f\ x \quad x$ may exist even though neither $\lim_{x \quad a} f\ x$ nor $\lim_{x \quad a} \ x$ exists.

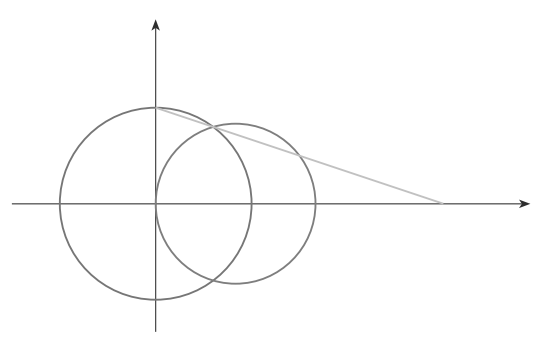
58. Evaluate $\lim_{x \quad 2} \dfrac{\overline{6 \quad x} \quad 2}{\overline{3 \quad x} \quad 1}$.

59. Is there a number a such that

$$\lim_{x \quad 2} \frac{3x^2 \quad ax \quad a \quad 3}{x^2 \quad x \quad 2}$$

exists? If so, find the value of a and the value of the limit.

60. The figure shows a fixed circle C_1 with equation $x \quad 1^2 \quad y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $0, r$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \quad 0$?



2.4 The Precise Definition of a Limit

The intuitive definition of a limit given in Section 2.2 is inadequate for some purposes because such phrases as "x is close to 2" and "$f(x)$ gets closer and closer to L" are vague. In order to be able to prove conclusively that

$$\lim_{x \to 0} x^3 \frac{\cos 5x}{10{,}000} = 0.0001 \qquad \text{or} \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x \to 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:

> How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad |x - 3| < \delta \quad \text{but } x \neq 3$$

|||| It is traditional to use the Greek letter δ (delta) in this situation.

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < \delta$$

Notice that if $0 < |x - 3| < \delta = 0.1/2 = 0.05$, then

$$|f(x) - 5| = |2x + 1 - 5| = |2x - 6| = 2|x - 3| < 0.1$$

that is,

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < 0.05$$

Thus, an answer to the problem is given by $\delta = 0.05$; that is, if x is within a distance of 0.05 from 3, then $f(x)$ will be within a distance of 0.1 from 5.

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that $f(x)$ will differ from 5 by less than 0.01 provided that x differs from 3 by less than $(0.01)/2 = 0.005$:

$$|f(x) - 5| < 0.01 \quad \text{if} \quad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \quad \text{if} \quad 0 < |x - 3| < 0.0005$$

The numbers 0.1, 0.01, and 0.001 that we have considered are *error tolerances* that we might allow. For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we

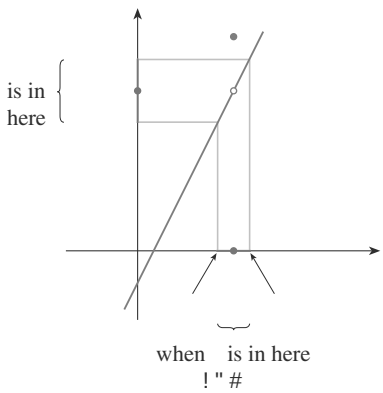


FIGURE 1

must be able to bring it below *any* positive number. And, by the same reasoning, we can! If we write (the Greek letter epsilon) for an arbitrary positive number, then we find as before that

$$\boxed{1} \qquad f\ x \quad 5 \qquad \text{if} \quad 0 \quad x \quad 3 \qquad = \frac{}{2}$$

This is a precise way of saying that $f\ x$ is close to 5 when x is close to 3 because (1) says that we can make the values of $f\ x$ within an arbitrary distance from 5 by taking the values of x within a distance 2 from 3 (but $x \neq 3$).

Note that (1) can be rewritten as

$$5 \qquad f\ x \quad 5 \qquad \text{whenever} \quad 3 \qquad x \quad 3 \qquad x \neq 3$$

and this is illustrated in Figure 1. By taking the values of x ($\neq 3$) to lie in the interval $3 \quad , 3$ we can make the values of $f\ x$ lie in the interval $5 \quad , 5$.

Using (1) as a model, we give a precise definition of a limit.

> **2 Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f\ x$ as x approaches a is L**, and we write
>
> $$\lim_{x \quad a} f\ x \ = L$$
>
> if for every number 0 there is a number 0 such that
>
> $$f\ x \quad L \qquad \text{whenever} \quad 0 \quad x \quad a$$

Another way of writing the last line of this definition is

$$\text{if} \quad 0 \quad x \quad a \qquad \text{then} \qquad f\ x \quad L$$

Since $x \quad a$ is the distance from x to a and $f\ x \quad L$ is the distance from $f\ x$ to L, and since can be arbitrarily small, the definition of a limit can be expressed in words as follows:

$\lim_{x \quad a} f\ x \ = L$ means that the distance between $f\ x$ and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0).

Alternatively,

$\lim_{x \quad a} f\ x \ = L$ means that the values of $f\ x$ can be made as close as we please to L by taking x close enough to a (but not equal to a).

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $x \quad a$ is equivalent to $x \quad a$, which in turn can be written as $a \quad x \quad a$. Also $0 \quad x \quad a$ is true if and only if $x \quad a \neq 0$, that is, $x \neq a$. Similarly, the inequality $f\ x \quad L$ is equivalent to the pair of inequalities $L \quad f\ x \quad L$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

$\lim_{x \quad a} f\ x \ = L$ means that for every 0 (no matter how small is) we can find 0 such that if x lies in the open interval $a \quad , a$ and $x \neq a$, then $f\ x$ lies in the open interval $L \quad , L$.

We interpret this statement geometrically by representing a function by an arrow diagram as in Figure 2, where f maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$.

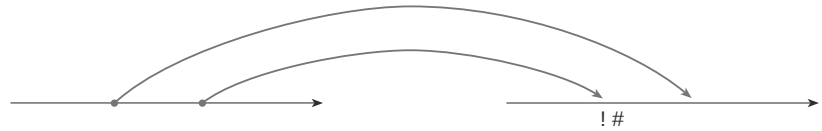


FIGURE 2

The definition of limit says that if any small interval $L \quad , L \quad$ is given around L, then we can find an interval $a \quad , a \quad$ around a such that f maps all the points in $a \quad , a \quad$ (except possibly a) into the interval $L \quad , L \quad$. (See Figure 3.)

FIGURE 3

Another geometric interpretation of limits can be given in terms of the graph of a function. If $\quad 0$ is given, then we draw the horizontal lines $y = L \quad$ and $y = L \quad$ and the graph of f (see Figure 4). If $\lim_{x \quad a} f \; x \; = L$, then we can find a number $\quad 0$ such that if we restrict x to lie in the interval $a \quad , a \quad$ and take $x \neq a$, then the curve $y = f \; x$ lies between the lines $y = L \quad$ and $y = L \quad$. (See Figure 5.) You can see that if such a $\quad$ has been found, then any smaller $\quad$ will also work.

It is important to realize that the process illustrated in Figures 4 and 5 must work for *every* positive number $\quad$ no matter how small it is chosen. Figure 6 shows that if a smaller $\quad$ is chosen, then a smaller $\quad$ may be required.

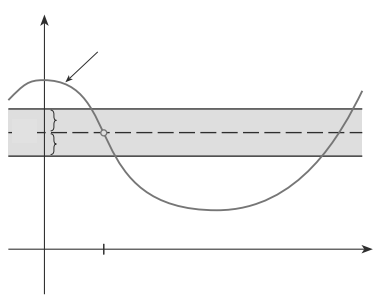


FIGURE 4

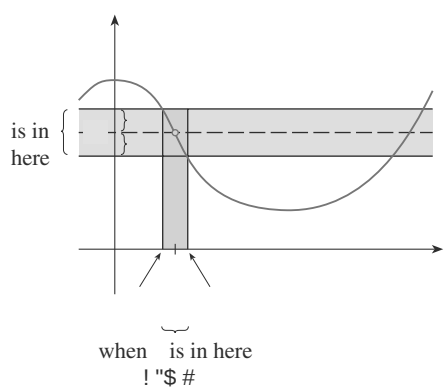


FIGURE 5

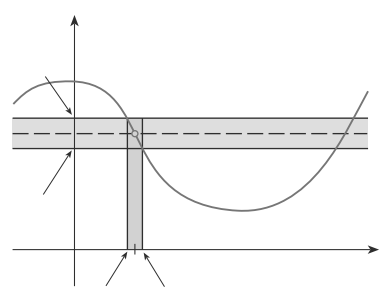


FIGURE 6

EXAMPLE 1 Use a graph to find a number $\quad$ such that

$$x^3 \quad 5x \quad 6 \quad 2 \quad 0.2 \qquad \text{whenever} \qquad x \quad 1$$

In other words, find a number $\quad$ that corresponds to $\quad = 0.2$ in the definition of a limit for the function $f \; x \; = x^3 \quad 5x \quad 6$ with $a = 1$ and $L = 2$.

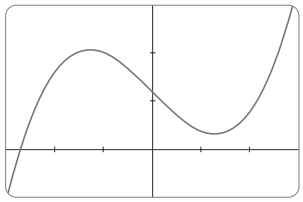


FIGURE 7

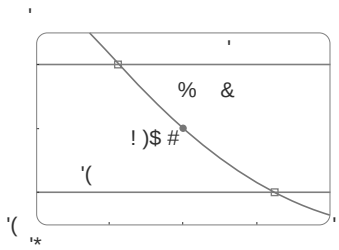


FIGURE 8

SOLUTION A graph of f is shown in Figure 7; we are interested in the region near the point $1, 2$. Notice that we can rewrite the inequality

$$x^3 \quad 5x \quad 6 \quad 2 \quad 0.2$$

as

$$1.8 \quad x^3 \quad 5x \quad 6 \quad 2.2$$

So we need to determine the values of x for which the curve $y = x^3 \quad 5x \quad 6$ lies between the horizontal lines $y = 1.8$ and $y = 2.2$. Therefore, we graph the curves $y = x^3 \quad 5x \quad 6$, $y = 1.8$, and $y = 2.2$ near the point $1, 2$ in Figure 8. Then we use the cursor to estimate that the x-coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 \quad 5x \quad 6$ is about 0.911. Similarly, $y = x^3 \quad 5x \quad 6$ intersects the line $y = 1.8$ when $x \quad 1.124$. So, rounding to be safe, we can say that

$$1.8 \quad x^3 \quad 5x \quad 6 \quad 2.2 \qquad \text{whenever} \qquad 0.92 \quad x \quad 1.12$$

This interval $0.92, 1.12$ is not symmetric about $x = 1$. The distance from $x = 1$ to the left endpoint is $1 \quad 0.92 = 0.08$ and the distance to the right endpoint is 0.12. We can choose to be the smaller of these numbers, that is, $= 0.08$. Then we can rewrite our inequalities in terms of distances as follows:

$$x^3 \quad 5x \quad 6 \quad 2 \quad 0.2 \qquad \text{whenever} \qquad x \quad 1 \quad 0.08$$

This just says that by keeping x within 0.08 of 1, we are able to keep $f \; x$ within 0.2 of 2.

Although we chose $= 0.08$, any smaller positive value of would also have worked.

The graphical procedure in Example 1 gives an illustration of the definition for $= 0.2$, but it does not *prove* that the limit is equal to 2. A proof has to provide a for *every* .

In proving limit statements it may be helpful to think of the definition of limit as a challenge. First it challenges you with a number . Then you must be able to produce a suitable . You have to be able to do this for *every* 0, not just a particular .

Imagine a contest between two people, A and B, and imagine yourself to be B. Person A stipulates that the fixed number L should be approximated by the values of $f \; x$ to within a degree of accuracy (say, 0.01). Person B then responds by finding a number such that $f \; x \quad L$ whenever $0 \quad x \quad a$. Then A may become more exacting and challenge B with a smaller value of (say, 0.0001). Again B has to respond by finding a corresponding . Usually the smaller the value of , the smaller the corresponding value of must be. If B always wins, no matter how small A makes , then $\lim_{x \quad a} f \; x = L$.

EXAMPLE 2 Prove that $\lim_{x \quad 3} 4x \quad 5 = 7$.

SOLUTION

1. *Preliminary analysis of the problem* (*guessing a value for*). Let be a given positive number. We want to find a number such that

$$4x \quad 5 \quad 7 \qquad \text{whenever} \qquad 0 \quad x \quad 3$$

But $4x \quad 5 \quad 7 = 4x \quad 12 = 4 \; x \quad 3 = 4 \; x \quad 3$. Therefore, we want

$$4 \; x \quad 3 \qquad \text{whenever} \qquad 0 \quad x \quad 3$$

that is, $\quad x \quad 3 \quad \frac{}{4} \quad$ whenever $\quad 0 \quad x \quad 3$

This suggests that we should choose $\quad = \quad 4$.

2. *Proof* (*showing that this works*). Given $\quad 0$, choose $\quad = \quad 4$. If $0 \quad x \quad 3 \quad$, then

$$4x \quad 5 \quad 7 \; = \; 4x \quad 12 \; = 4 \; x \quad 3 \quad 4 \; = 4 \; \frac{}{4} \; =$$

Thus

$$4x \quad 5 \quad 7 \qquad \text{whenever} \quad 0 \quad x \quad 3$$

Therefore, by the deÞnition of a limit,

$$\lim_{x \quad 3} 4x \quad 5 \; = 7$$

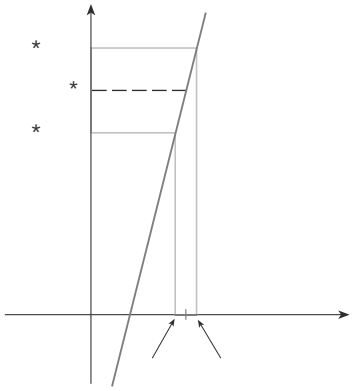


FIGURE 9

This example is illustrated by Figure 9.

Note that in the solution of Example 2 there were two stagesÑ guessing and proving. We made a preliminary analysis that enabled us to guess a value for . But then in the second stage we had to go back and prove in a careful, logical fashion that we had made a correct guess. This procedure is typical of much of mathematics. Sometimes it is necessary to Þrst make an intelligent guess about the answer to a problem and then prove that the guess is correct.

The intuitive deÞnitions of one-sided limits that were given in Section 2.2 can be precisely reformulated as follows.

3 Definition of Left-Hand Limit

$$\lim_{x \quad a} f \; x \; = L$$

if for every number $\quad 0$ there is a number $\quad 0$ such that

$$f \; x \quad L \qquad \text{whenever} \quad a \qquad x \quad a$$

4 Definition of Right-Hand Limit

$$\lim_{x \quad a} f \; x \; = L$$

if for every number $\quad 0$ there is a number $\quad 0$ such that

$$f \; x \quad L \qquad \text{whenever} \quad a \quad x \quad a$$

Notice that DeÞnition 3 is the same as DeÞnition 2 except that x is restricted to lie in the *left* half $a \quad , a$ of the interval $a \quad , a \quad$. In DeÞnition 4, x is restricted to lie in the *right* half $a, a \quad$ of the interval $a \quad , a \quad$.

EXAMPLE 3 Use DeÞnition 4 to prove that $\lim_{x \quad 0} \quad \overline{x} = 0$.

After the invention of calculus in the 17th century, there followed a period of free development of the subject in the 18th century. Mathematicians like the Bernoulli brothers and Euler were eager to exploit the power of calculus and boldly explored the consequences of this new and wonderful mathematical theory without worrying too much about whether their proofs were completely correct.

The 19th century, by contrast, was the Age of Rigor in mathematics. There was a movement to go back to the foundations of the subjectÑto provide careful deÞnitions and rigorous proofs. At the forefront of this movement was the French mathematician Augustin-Louis Cauchy (1789Ð1857), who started out as a military engineer before becoming a mathematics professor in Paris. Cauchy took NewtonÕs idea of a limit, which was kept alive in the 18th century by the French mathematician Jean dÕAlembert, and made it more precise. His deÞnition of a limit reads as follows: ÒWhen the successive values attributed to a variable approach indeÞnitely a Þxed value so as to end by differing from it by as little as one wishes, this last is called the *limit* of all the others.Ó But when Cauchy used this deÞnition in examples and proofs, he often employed delta-epsilon inequalities similar to the ones in this section. A typical Cauchy proof starts with: ÒDesignate by and two very small numbers; . . .Ó He used because of the correspondence between epsilon and the French word *erreur*. Later, the German mathematician Karl Weierstrass (1815Ð1897) stated the deÞnition of a limit exactly as in our DeÞnition 2.

SOLUTION

1. *Guessing a value for* . Let be a given positive number. Here $a = 0$ and $L = 0$, so we want to Þnd a number such that

$$\bar{x} \quad 0 \qquad \text{whenever} \qquad 0 \quad x$$

that is,

$$\bar{x} \qquad \text{whenever} \qquad 0 \quad x$$

or, squaring both sides of the inequality $\bar{x}$, we get

$$x \quad {}^{2} \qquad \text{whenever} \qquad 0 \quad x$$

This suggests that we should choose $= {}^{2}$.

2. *Showing that this works.* Given 0, let $= {}^{2}$. If $0 \quad x$, then

$$\bar{x} \quad {}^{-} = \overline{\ {}^{2}} =$$

$$\bar{x} \quad 0$$

so

According to DeÞnition 4, this shows that $\lim_{x \quad 0} \bar{x} = 0$.

EXAMPLE 4 Prove that $\lim_{x \quad 3} x^2 = 9$.

SOLUTION

1. *Guessing a value for* . Let 0 be given. We have to Þnd a number 0 such that

$$x^2 \quad 9 \qquad \text{whenever} \qquad 0 \quad x \quad 3$$

To connect $x^2 \quad 9$ with $x \quad 3$ we write $x^2 \quad 9 = x \quad 3 \ x \quad 3$. Then we want

$$x \quad 3 \ x \quad 3 \qquad \text{whenever} \qquad 0 \quad x \quad 3$$

Notice that if we can Þnd a positive constant C such that $x \quad 3 \quad C$, then

$$x \quad 3 \ x \quad 3 \quad C \ x \quad 3$$

and we can make $C \ x \quad 3$ by taking $x \quad 3 \quad C =$.

We can Þnd such a number C if we restrict x to lie in some interval centered at 3. In fact, since we are interested only in values of x that are close to 3, it is reasonable to assume that x is within a distance l from 3, that is, $x \quad 3 \quad 1$. Then $2 \quad x \quad 4$, so $5 \quad x \quad 3 \quad 7$. Thus, we have $x \quad 3 \quad 7$, and so $C = 7$ is a suitable choice for the constant.

But now there are two restrictions on $x \quad 3$, namely

$$x \quad 3 \quad 1 \qquad \text{and} \qquad x \quad 3 \quad \frac{}{C} = \frac{}{7}$$

To make sure that both of these inequalities are satisÞed, we take to be the smaller of the two numbers 1 and 7. The notation for this is $= \min \ 1, \quad 7$.

2. *Showing that this works.* Given 0, let $= \min \ 1, \quad 7$. If $0 \quad x \quad 3$, then $x \quad 3 \quad 1 \quad 2 \quad x \quad 4 \quad x \quad 3 \quad 7$ (as in part l). We also have $x \quad 3 \quad 7$, so

$$x^2 \quad 9 = x \quad 3 \ x \quad 3 \quad 7 \ \frac{}{7} =$$

This shows that $\lim_{x \quad 3} x^2 = 9$.

As Example 4 shows, it is not always easy to prove that limit statements are true using the , definition. In fact, if we had been given a more complicated function such as $f\ x\ =\ 6x^2\quad 8x\quad 9\quad 2x^2\quad 1$, a proof would require a great deal of ingenuity. Fortunately this is unnecessary because the Limit Laws stated in Section 2.3 can be proved using Definition 2, and then the limits of complicated functions can be found rigorously from the Limit Laws without resorting to the definition directly.

For instance, we prove the Sum Law: If $\lim_{x\ \ a} f\ x\ = L$ and $\lim_{x\ \ a}\ x\ = M$ both exist, then

$$\lim_{x\ \ a}\ f\ x\qquad x\ = L\quad M$$

The remaining laws are proved in the exercises and in Appendix F.

Proof of the Sum Law Let 0 be given. We must find 0 such that

$$f\ x\qquad x\qquad L\quad M\qquad\qquad \text{whenever}\qquad 0\qquad x\quad a$$

|||| Triangle Inequality:

$$a\quad b\qquad a\qquad b$$

(See Appendix A.)

Using the Triangle Inequality we can write

$$\boxed{5}\qquad f\ x\qquad x\qquad L\quad M\ =\ f\ x\quad L\qquad x\quad M$$
$$f\ x\quad L\qquad x\quad M$$

We make $f\ x\qquad x\qquad L\quad M$ less than by making each of the terms $f\ x\qquad L$ and $x\quad M$ less than 2.

Since 2 0 and $\lim_{x\ \ a} f\ x\ = L$, there exists a number $_1$ 0 such that

$$f\ x\quad L\quad \frac{}{2}\qquad \text{whenever}\qquad 0\qquad x\quad a\qquad {}_1$$

Similarly, since $\lim_{x\ \ a}\ x\ = M$, there exists a number $_2$ 0 such that

$$x\quad M\quad \frac{}{2}\qquad \text{whenever}\qquad 0\qquad x\quad a\qquad {}_2$$

Let $= \min\ {}_1,\ {}_2$. Notice that

$$\text{if}\quad 0\qquad x\quad a\qquad \text{then}\quad 0\qquad x\quad a\qquad {}_1\quad \text{and}\quad 0\qquad x\quad a\qquad {}_2$$

and so $\qquad f\ x\quad L\quad \frac{}{2}\qquad$ and $\qquad x\quad M\quad \frac{}{2}$

Therefore, by (5),

$$f\ x\qquad x\qquad L\quad M\qquad f\ x\quad L\qquad x\quad M$$
$$\frac{}{2}\quad \frac{}{2} =$$

To summarize,

$$f\ x\qquad x\qquad L\quad M\qquad\qquad \text{whenever}\qquad 0\qquad x\quad a$$

Thus, by the definition of a limit,

$$\lim_{x\ \ a}\ f\ x\qquad x\ = L\quad M$$

Infinite Limits

Infinite limits can also be defined in a precise wa[illegible] The follow[illegible]s a precise version of Definition 4 in Section 2.2.

Definition Let be a function defined [illegible]ome open in[illegible]l that contains the number , except possibly at itself. The[illegible]

means that for every positive number [illegible]re is a posi[illegible]umber such that

This says that the values of can b[illegible]itrar[illegible]e (larger than any given number) by taking x close enough to a (w[illegible] distanc[illegible] where depends on M, but with $x \neq a$). A geometric illustration is shown in Figure 10.

Given any horizontal line $y = M$, we can find a number 0 such that if we restrict x to lie in the interval a , a but $x \neq a$, then the curve $y = f\ x$ lies above the line $y = M$. You can see that if a larger M is chosen, then a smaller may be required.

EXAMPLE 5 Use Definition 6 to prove that $\lim_{x \quad 0} \frac{1}{x^2} =$.

SOLUTION

1. *Guessing a value for* . Given M 0, we want to find 0 such that

$$\frac{1}{x^2} \quad M \qquad \text{whenever} \qquad 0 \quad x \quad 0$$

that is,

$$x^2 \quad \frac{1}{M} \qquad \text{whenever} \qquad 0 \quad x$$

or

$$x \quad \frac{1}{\overline{M}} \qquad \text{whenever} \qquad 0 \quad x$$

This suggests that we should take $= 1 \quad \overline{M}$.

2. *Showing that this works.* If M 0 is given, let $= 1 \quad \overline{M}$. If 0 x 0 , then

$$x \qquad x^2 \quad {}^2$$

$$\frac{1}{x^2} \quad \frac{1}{{}^2} = M$$

Thus

$$\frac{1}{x^2} \quad M \qquad \text{whenever} \qquad 0 \quad x \quad 0$$

Therefore, by Definition 6,

$$\lim_{x \quad 0} \frac{1}{x^2} =$$

44. Suppose that $\lim_{x \to a} f(x) = \quad$ and $\lim_{x \to a} \quad x = c$, where c is a real number. Prove each statement.

(a) $\lim_{x \to a} \quad f(x) \quad x =$

(b) $\lim_{x \to a} \quad f(x) \quad x = \quad$ if $c \quad 0$

(c) $\lim_{x \to a} \quad f(x) \quad x = \quad$ if $c \quad 0$

2.5 Continuity

Explore continuous functions interactively.
Resources / Module 2
/ Continuity
/ Start of Continuity

We noticed in Section 2.3 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

1 Definition A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.

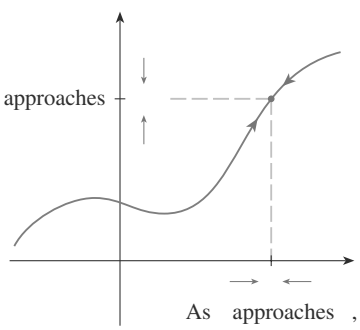


FIGURE 1

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. Thus, a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is discontinuous at a, or f has a discontinuity at a, if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents. [See Example 6 in Section 2.2, where the Heaviside function is discontinuous at 0 because $\lim_{t \to 0} H(t)$ does not exist.]

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

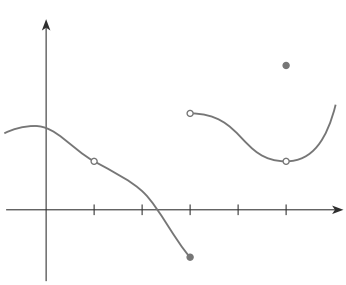


FIGURE 2

EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?

SOLUTION It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that f is discontinuous at 1 is that $f(1)$ is not defined.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x \to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x \to 5} f(x)$ exists (because the left and right limits are the same). But

$$\lim_{x \to 5} f(x) \neq f(5)$$

So f is discontinuous at 5.

Now let's see how to detect discontinuities when a function is defined by a formula.

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EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = \lfloor x \rfloor$

SOLUTION

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. (See Example 8 in Section 2.2.) So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3$$

exists. But

$$\lim_{x \to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

(d) The greatest integer function $f(x) = \lfloor x \rfloor$ has discontinuities at all of the integers because $\lim_{x \to n} \lfloor x \rfloor$ does not exist if n is an integer. (See Example 10 and Exercise 49 in Section 2.3.)

Figure 3 shows the graphs of the functions in Example 2. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.

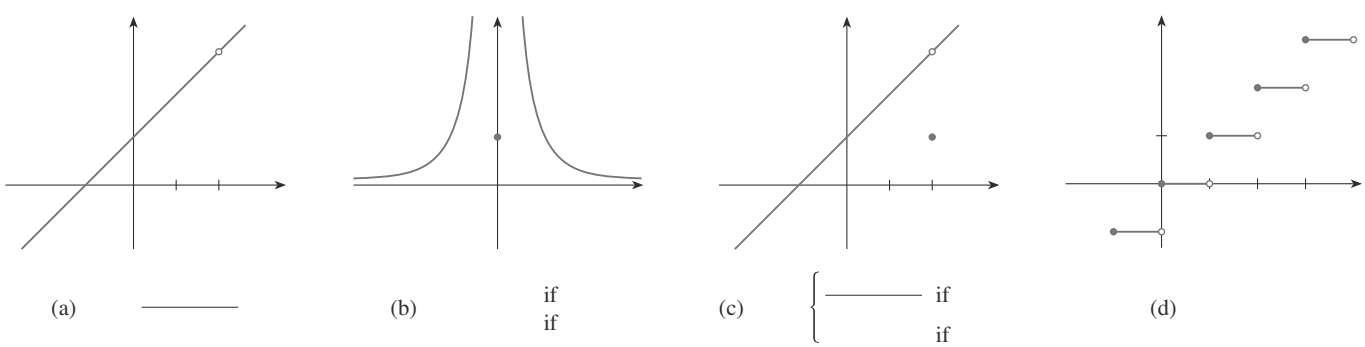


FIGURE 3 Graphs of the functions in Example 2

2 Definition A function f is continuous from the right at a number a if

$$\lim_{x \quad a} f\ x\ = f\ a$$

and f is continuous from the left at a if

$$\lim_{x \quad a} f\ x\ = f\ a$$

EXAMPLE 3 At each integer n, the function $f\ x\ =\ x$ [see Figure 3(d)] is continuous from the right but discontinuous from the left because

$$\lim_{x \quad n} f\ x\ = \lim_{x \quad n}\ x\ = n = f\ n$$

but

$$\lim_{x \quad n} f\ x\ = \lim_{x \quad n}\ x\ = n \qquad 1 \neq f\ n$$

3 Definition A function f is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*)

EXAMPLE 4 Show that the function $f\ x\ = 1 \qquad \overline{1 \quad x^2}$ is continuous on the interval $\quad 1, 1\ .$

SOLUTION If $\quad 1 \quad a \quad 1$, then using the Limit Laws, we have

$$\begin{aligned}
\lim_{x \quad a} f\ x\ &= \lim_{x \quad a} \left(1 \qquad \overline{1 \quad x^2}\right) \\
&= 1 \quad \lim_{x \quad a} \overline{1 \quad x^2} && \text{(by Laws 2 and 7)} \\
&= 1 \quad \overline{\lim_{x \quad a} 1 \quad x^2} && \text{(by 11)} \\
&= 1 \quad \overline{1 \quad a^2} && \text{(by 2, 7, and 9)} \\
&= f\ a
\end{aligned}$$

Thus, by Definition l, f is continuous at a if $\quad 1 \quad a \quad 1$. Similar calculations show that

$$\lim_{x \quad 1} f\ x\ = 1 = f \quad 1 \qquad \text{and} \qquad \lim_{x \quad 1} f\ x\ = 1 = f\ 1$$

so f is continuous from the right at $\quad 1$ and continuous from the left at 1. Therefore, according to Definition 3, f is continuous on $\quad 1, 1\ .$

The graph of f is sketched in Figure 4. It is the lower half of the circle

$$x^2 \qquad y \quad 1\ ^2 = 1$$

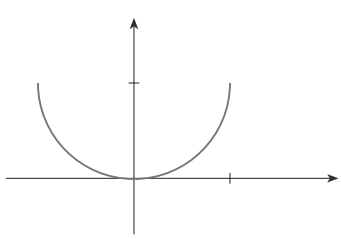


FIGURE 4

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function as we did in Example 4, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and are continuous at a and c is a constant, then the following functions are also continuous at a:

1. f **2.** f **3.** cf

4. f **5.** $\dfrac{f}{}$ if $a \neq 0$

Proof Each of the five parts of this theorem follows from the corresponding Limit Law in Section 2.3. For instance, we give the proof of part 1. Since f and are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to a} (x) = (a)$$

Therefore

$$\begin{aligned}
\lim_{x \to a} (f +)(x) &= \lim_{x \to a} (f(x) + (x)) \\
&= \lim_{x \to a} f(x) + \lim_{x \to a} (x) \qquad \text{(by Law 1)} \\
&= f(a) + (a) \\
&= (f +)(a)
\end{aligned}$$

This shows that $f +$ is continuous at a.

It follows from Theorem 4 and Definition 3 that if f and are continuous on an interval, then so are the functions $f -$, $f +$, cf, f, and (if is never 0) $f/$. The following theorem was stated in Section 2.3 as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $(-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Proof

(a) A polynomial is a function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \ldots, c_n$ are constants. We know that

$$\lim_{x \to a} c_0 = c_0 \qquad \text{(by Law 7)}$$

and

$$\lim_{x \to a} x^m = a^m \qquad m = 1, 2, \ldots, n \qquad \text{(by 9)}$$

This equation is precisely the statement that the function $f(x) = x^m$ is a continuous function. Thus, by part 3 of Theorem 4, the function $(x) = cx^m$ is continuous. Since P is a sum of functions of this form and a constant function, it follows from part 1 of Theorem 4 that P is continuous.

(b) A rational function is a function of the form

$$f\ x\ = \frac{P\ x}{Q\ x}$$

where P and Q are polynomials. The domain of f is $D = \ x \in \quad Q\ x\ \neq 0\ $. We know from part (a) that P and Q are continuous everywhere. Thus, by part 5 of Theorem 4, f is continuous at every number in D.

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V\ r\ = \frac{4}{3}\ \ r^3$ shows that V is a polynomial function of r. Likewise, if a ball is thrown vertically into the air with a velocity of 50 ft s, then the height of the ball in feet after t seconds is given by the formula $h = 50t \quad 16t^2$. Again this is a polynomial function, so the height is a continuous function of the elapsed time.

Knowledge of which functions are continuous enables us to evaluate some limits very quickly, as the following example shows. Compare it with Example 2(b) in Section 2.3.

EXAMPLE 5 Find $\lim_{x \quad 2} \dfrac{x^3 \quad 2x^2 \quad 1}{5 \quad 3x}$.

SOLUTION The function

$$f\ x\ = \frac{x^3 \quad 2x^2 \quad 1}{5 \quad 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\left\{x\ \ x \neq \frac{5}{3}\right\}$. Therefore

$$\lim_{x \quad 2} \frac{x^3 \quad 2x^2 \quad 1}{5 \quad 3x} = \lim_{x \quad 2} f\ x\ = f \quad 2$$

$$= \frac{2\ ^3 \quad 2 \quad 2\ ^2 \quad 1}{5 \quad 3 \quad 2} = \quad \frac{1}{11}$$

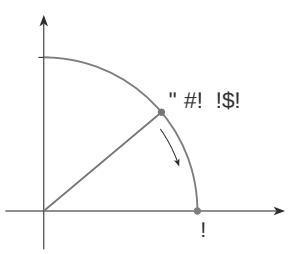


FIGURE 5

It turns out that most of the familiar functions are continuous at every number in their domains. For instance, Limit Law 10 (page 106) implies that root functions are continuous. [Example 3 in Section 2.4 shows that $f\ x\ = \ \sqrt{x}$ is continuous from the right at 0.]

From the appearance of the graphs of the sine and cosine functions (Figure 18 in Section 1.2), we would certainly guess that they are continuous. We know from the definitions of sin and cos that the coordinates of the point P in Figure 5 are cos , sin . As 0, we see that P approaches the point 1, 0 and so cos 1 and sin 0. Thus

$$\boxed{6} \qquad \lim_{0} \cos \ = 1 \qquad \lim_{0} \sin \ = 0$$

Since $\cos 0 = 1$ and $\sin 0 = 0$, the equations in (6) assert that the cosine and sine functions are continuous at 0. The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere (see Exercises 56 and 57).

It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

|||| Another way to establish the limits in (6) is to use the Squeeze Theorem with the inequality sin (for 0), which is proved in Section 3.4.

is continuous except where $\cos x = 0$. This happens when x is an odd integer multiple of 2, so $y = \tan x$ has infinite discontinuities when $x =$ 2, 3 2, 5 2, and so on (see Figure 6).

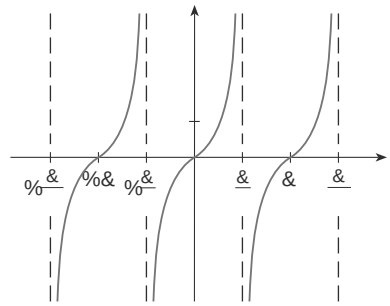


FIGURE 6
tan

The inverse trigonometric functions are reviewed in Section 1.6.

The inverse function of any continuous function is also continuous. (The graph of f^{1} is obtained by reflecting the graph of f about the line $y = x$. So if the graph of f has no break in it, neither does the graph of f^{1}.) Thus, the inverse trigonometric functions are continuous.

In Section 1.5 we defined the exponential function $y = a^x$ so as to fill in the holes in the graph of $y = a^x$ where x is rational. In other words, the very definition of $y = a^x$ makes it a continuous function on . Therefore, its inverse function $y = \log_a x$ is continuous on 0, .

7 Theorem The following types of functions are continuous at every number in their domains:

polynomials rational functions root functions

trigonometric functions inverse trigonometric functions

exponential functions logarithmic functions

EXAMPLE 6 Where is the function $f\ x = \dfrac{\ln x \quad \tan^{1} x}{x^2 \quad 1}$ continuous?

SOLUTION We know from Theorem 7 that the function $y = \ln x$ is continuous for $x \quad 0$ and $y = \tan^{1} x$ is continuous on . Thus, by part 1 of Theorem 4, $y = \ln x \quad \tan^{1} x$ is continuous on 0, . The denominator, $y = x^2 \quad 1$, is a polynomial, so it is continuous everywhere. Therefore, by part 5 of Theorem 4, f is continuous at all positive numbers x except where $x^2 \quad 1 = 0$. So f is continuous on the intervals 0, 1 and 1, .

Another way of combining continuous functions f and to get a new continuous function is to form the composite function f . This fact is a consequence of the following theorem.

This theorem says that a limit symbol can be moved through a function symbol if the function is continuous and the limit exists. In other words, the order of these two symbols can be reversed.

8 Theorem If f is continuous at b and $\lim_{x \ a} x = b$, then $\lim_{x \ a} f \quad x = f\ b$.
In other words,

$$\lim_{x \ a} f \quad x = f\left(\lim_{x \ a} x\right)$$

Intuitively, Theorem 8 is reasonable because if x is close to a, then x is close to b, and since f is continuous at b, if x is close to b, then $f\ x$ is close to $f\ b$. A proof of Theorem 8 is given in Appendix F.

EXAMPLE 7 Evaluate $\lim_{x\ 1} \arcsin \frac{1\ \ \bar{x}}{1\ \ x}$.

SOLUTION Because arcsin is a continuous function, we can apply Theorem 8:

$$\begin{aligned}
\lim_{x\ 1} \arcsin \frac{1\ \ \bar{x}}{1\ \ x} &= \arcsin \lim_{x\ 1} \frac{1\ \ \bar{x}}{1\ \ x} \\
&= \arcsin \lim_{x\ 1} \frac{1\ \ \bar{x}}{(1\ \ \bar{x})(1\ \ \bar{x})} \\
&= \arcsin \lim_{x\ 1} \frac{1}{1\ \ \bar{x}} \\
&= \arcsin \frac{1}{2} = \frac{}{6}
\end{aligned}$$

Let's now apply Theorem 8 in the special case where $f\ x\ =\ ^{n}\bar{x}$, with n being a positive integer. Then

$$f\ \ x\ = \ ^{n}\overline{\ \ x}$$

and

$$f\left(\lim_{x\ a}\ x\right) = \ ^{n}\overline{\lim_{x\ a}\ x}$$

If we put these expressions into Theorem 8, we get

$$\lim_{x\ a}\ ^{n}\overline{\ \ x} = \ ^{n}\overline{\lim_{x\ a}\ x}$$

and so Limit Law 11 has now been proved. (We assume that the roots exist.)

> **9 Theorem** If is continuous at a and f is continuous at a, then the composite function f given by $f\ \ x\ = f\ \ x$ is continuous at a.

This theorem is often expressed informally by saying "a continuous function of a continuous function is a continuous function."

Proof Since is continuous at a, we have

$$\lim_{x\ a}\ x\ =\ a$$

Since f is continuous at $b =\ a$, we can apply Theorem 8 to obtain

$$\lim_{x\ a} f\ \ x\ = f\ \ a$$

which is precisely the statement that the function $h\ x\ = f\ \ x$ is continuous at a; that is, f is continuous at a.

EXAMPLE 8 Where are the following functions continuous?

(a) $h\ x\ = \sin\ x^2$ (b) $F\ x\ = \ln\ 1\quad \cos x$

SOLUTION

(a) We have $h\ x\ = f\quad x\ $, where

$$x\ = x^2 \qquad \text{and} \qquad f\ x\ = \sin x$$

Now is continuous on since it is a polynomial, and f is also continuous everywhere. Thus, $h = f$ is continuous on by Theorem 9.

(b) We know from Theorem 7 that $f\ x\ = \ln x$ is continuous and $x\ = 1\quad \cos x$ is continuous (because both $y = 1$ and $y = \cos x$ are continuous). Therefore, by Theorem 9, $F\ x\ = f\quad x$ is continuous wherever it is defined. Now $\ln\ 1\quad \cos x$ is defined when $1\quad \cos x\quad 0$. So it is undefined when $\cos x =\quad 1$, and this happens when $x =\quad ,\quad 3\ , \ldots$. Thus, F has discontinuities when x is an odd multiple of and is continuous on the intervals between these values (see Figure 7).

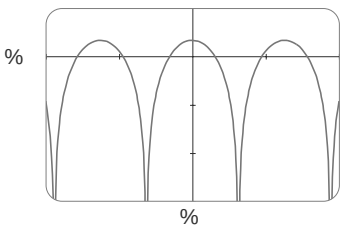


FIGURE 7
$\ln(\quad \cos\ )$

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

> **10 The Intermediate Value Theorem** Suppose that f is continuous on the closed interval a, b and let N be any number between $f\ a$ and $f\ b$, where $f\ a\ \neq f\ b$. Then there exists a number c in a, b such that $f\ c\ = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f\ a$ and $f\ b$. It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

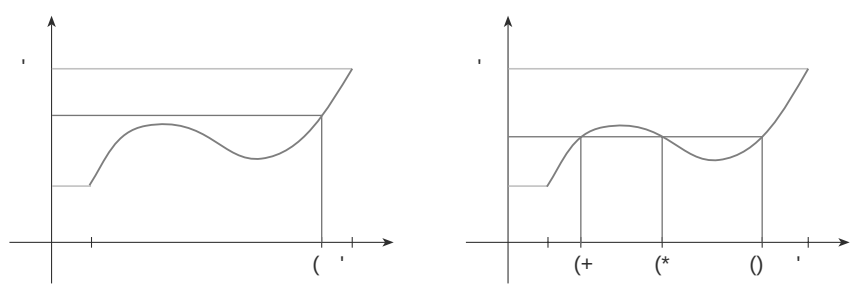


FIGURE 8 (a) (b)

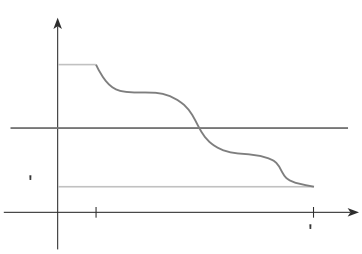


FIGURE 9

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true. In geometric terms it says that if any horizontal line $y = N$ is given between $y = f\ a$ and $y = f\ b$ as in Figure 9, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

It is important that the function f in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions (see Exercise 44).

One use of the Intermediate Value Theorem is in locating roots of equations as in the following example.

EXAMPLE 9 Show that there is a root of the equation

$$4x^3 \quad 6x^2 \quad 3x \quad 2 = 0$$

between 1 and 2.

SOLUTION Let $f\ x\ = 4x^3 \quad 6x^2 \quad 3x \quad 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f\ c\ = 0$. Therefore, we take $a = 1$, $b = 2$, and $N = 0$ in Theorem 10. We have

$$f\ 1\ = 4 \quad 6 \quad 3 \quad 2 = \quad 1 \quad 0$$

and

$$f\ 2\ = 32 \quad 24 \quad 6 \quad 2 = 12 \quad 0$$

Thus, $f\ 1 \quad 0 \quad f\ 2$; that is, $N = 0$ is a number between $f\ 1$ and $f\ 2$. Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f\ c\ = 0$. In other words, the equation $4x^3 \quad 6x^2 \quad 3x \quad 2 = 0$ has at least one root c in the interval $1, 2$.

In fact, we can locate a root more precisely by using the Intermediate Value Theorem again. Since

$$f\ 1.2\ = \quad 0.128 \quad 0 \qquad \text{and} \qquad f\ 1.3\ = 0.548 \quad 0$$

a root must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f\ 1.22\ = \quad 0.007008 \quad 0 \qquad \text{and} \qquad f\ 1.23\ = 0.056068 \quad 0$$

so a root lies in the interval $1.22, 1.23$.

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem in Example 9. Figure 10 shows the graph of f in the viewing rectangle $1, 3$ by $3, 3$ and you can see that the graph crosses the x-axis between 1 and 2. Figure 11 shows the result of zooming in to the viewing rectangle $1.2, 1.3$ by $0.2, 0.2$.

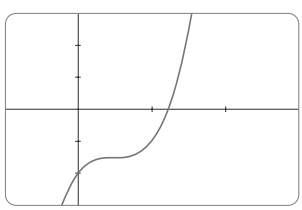


FIGURE 10

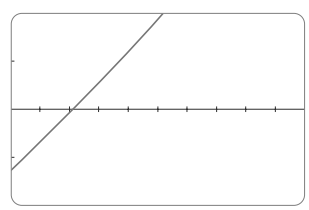


FIGURE 11

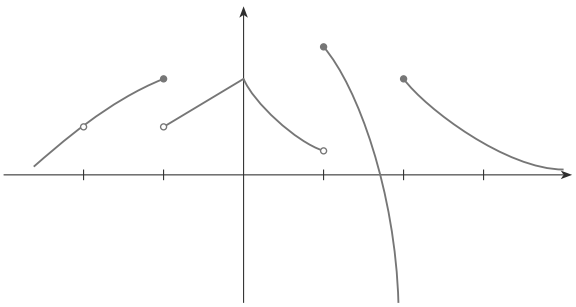
In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore connects the pixels by turning on the intermediate pixels.

2.5 Exercises

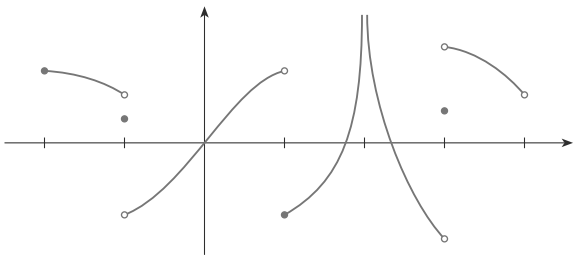
1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on , , what can you say about its graph?

3. (a) From the graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



4. From the graph of , state the intervals on which is continuous.



5. Sketch the graph of a function that is continuous everywhere except at $x = 3$ and is continuous from the left at 3.

6. Sketch the graph of a function that has a jump discontinuity at $x = 2$ and a removable discontinuity at $x = 4$, but is continuous elsewhere.

7. A parking lot charges \$3 for the first hour (or part of an hour) and \$2 for each succeeding hour (or part), up to a daily maximum of \$10.
(a) Sketch a graph of the cost of parking at this lot as a function of the time parked there.
(b) Discuss the discontinuities of this function and their significance to someone who parks in the lot.

8. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City
(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

9. If f and are continuous functions with $f\ 3 = 5$ and $\lim_{x\to 3} 2f\ x \quad x = 4$, find 3 .

10–12 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number.

10. $f\ x = x^2 \quad \overline{7 \quad x}, \quad a = 4$

11. $f\ x = x \quad 2x^3\ ^4, \quad a = \quad 1$

12. $x = \dfrac{x \quad 1}{2x^2 \quad 1}, \quad a = 4$

13–14 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

13. $f\ x = \dfrac{2x \quad 3}{x \quad 2}, \quad 2,$

14. $x = 2\ \overline{3 \quad x}, \quad , 3$

15–20 Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

15. $f\ x = \ln\ x \quad 2 \qquad a = 2$

16. $f\ x = \begin{cases} \dfrac{1}{x \quad 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases} \qquad a = 1$

17. $f\ x = \begin{cases} e^x & \text{if } x \quad 0 \\ x^2 & \text{if } x \quad 0 \end{cases} \qquad a = 0$

18. $f\ x = \begin{cases} \dfrac{x^2 \quad x}{x^2 \quad 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \qquad a = 1$

19. $f\ x = \begin{cases} \dfrac{x^2 \quad x \quad 12}{x \quad 3} & \text{if } x \neq \quad 3 \\ 5 & \text{if } x = \quad 3 \end{cases} \qquad a = \quad 3$

20. $f\ x = \begin{cases} 1 \quad x^2 & \text{if } x \quad 1 \\ 4 \quad x & \text{if } x \quad 1 \end{cases} \qquad a = 1$

21–28 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

21. $F\ x = \dfrac{x}{x^2 \quad 5x \quad 6}$

22. $G\ x = \sqrt[3]{x\ 1} \quad x^3$

23. $R(x) = x^2 \sqrt{2x - 1}$

24. $h(x) = \dfrac{\sin x}{x - 1}$

25. $f(x) = e^x \sin 5x$

26. $F(x) = \sin^{-1} x^2 - 1$

27. $G(t) = \ln(t^4 - 1)$

28. $H(x) = \cos(e^{\sqrt{x}})$

29–30 Locate the discontinuities of the function and illustrate by graphing.

29. $y = \dfrac{1}{1 - e^{1/x}}$

30. $y = \ln \tan^2 x$

31–34 Use continuity to evaluate the limit.

31. $\lim\limits_{x \to 4} \dfrac{5 - \sqrt{x}}{5 - \sqrt{x}}$

32. $\lim\limits_{x \to \infty} \sin(x - \sin x)$

33. $\lim\limits_{x \to 1} e^{x^2 - x}$

34. $\lim\limits_{x \to 2} \arctan \dfrac{x^2 - 4}{3x^2 - 6x}$

35–36 Show that f is continuous on $(-\infty, \infty)$.

35. $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ \sqrt{x} & \text{if } x > 1 \end{cases}$

36. $f(x) = \begin{cases} \sin x & \text{if } x < \dfrac{\pi}{4} \\ \cos x & \text{if } x \ge \dfrac{\pi}{4} \end{cases}$

37–39 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

37. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \le 0 \\ 2 - x & \text{if } 0 < x \le 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$

38. $f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ 1 + x & \text{if } 1 \le x \le 3 \\ \sqrt{x - 3} & \text{if } x > 3 \end{cases}$

39. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

40. The gravitational force exerted by Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \ge R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

41. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx + 1 & \text{if } x < 3 \\ cx^2 - 1 & \text{if } x \ge 3 \end{cases}$$

42. Find the constant c that makes f continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \ge 4 \end{cases}$$

43. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function that agrees with f for $x \neq a$ and is continuous on $\mathbb{R}$.

(a) $f(x) = \dfrac{x^2 - 2x - 8}{x - 2}, \quad a = 2$

(b) $f(x) = \dfrac{x - 7}{x - 7}, \quad a = 7$

(c) $f(x) = \dfrac{x^3 - 64}{x - 4}, \quad a = 4$

(d) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x}, \quad a = 9$

44. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

45. If $f(x) = x^3 - x^2 - x$, show that there is a number c such that $f(c) = 10$.

46. Use the Intermediate Value Theorem to prove that there is a positive number c such that $c^2 = 2$. (This proves the existence of the number $\sqrt{2}$.)

47–50 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

47. $x^4 - x - 3 = 0, \quad (1, 2)$

48. $\sqrt[3]{x} = 1 - x, \quad (0, 1)$

49. $\cos x = x, \quad (0, 1)$

50. $\ln x = e^{-x}, \quad (1, 2)$

51–52 (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

51. $e^x = 2 - x$

52. $x^5 - x^2 - 2x - 3 = 0$

53–54 (a) Prove that the equation has at least one real root.
(b) Use your graphing device to find the root correct to three decimal places.

53. $x^5 - x^2 - 4 = 0$

54. $\sqrt{x - 5} = \dfrac{1}{x - 3}$

55. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f\ a \quad h = f\ a$$

56. To prove that sine is continuous, we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. By Exercise 55 an equivalent statement is that

$$\lim_{h \to 0} \sin\ a \quad h = \sin a$$

Use (6) to show that this is true.

57. Prove that cosine is a continuous function.

58. (a) Prove Theorem 4, part 3.
(b) Prove Theorem 4, part 5.

59. For what values of x is f continuous?

$$f\ x = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

60. For what values of x is continuous?

$$x = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

61. Is there a number that is exactly 1 more than its cube?

62. (a) Show that the absolute value function $F\ x = \ x\ $ is continuous everywhere.
(b) Prove that if f is a continuous function on an interval, then so is f .
(c) Is the converse of the statement in part (b) also true? In other words, if f is continuous, does it follow that f is continuous? If so, prove it. If not, Þnd a counterexample.

63. A Tibetan monk leaves the monastery at 7:00 A.M. and takes his usual path to the top of the mountain, arriving at 7:00 P.M. The following morning, he starts at 7:00 A.M. at the top and takes the same path back, arriving at the monastery at 7:00 P.M. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

In Sections 2.2 and 2.4 we investigated inÞnite limits and vertical asymptotes. There we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). In this section we let x become arbitrarily large (positive or negative) and see what happens to y.

Let's begin by investigating the behavior of the function f deÞned by

$$f\ x = \frac{x^2 \quad 1}{x^2 \quad 1}$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

x	$f\ x$
0	1
1	0
2	0.600000
3	0.800000
4	0.882353
5	0.923077
10	0.980198
50	0.999200
100	0.999800
1000	0.999998

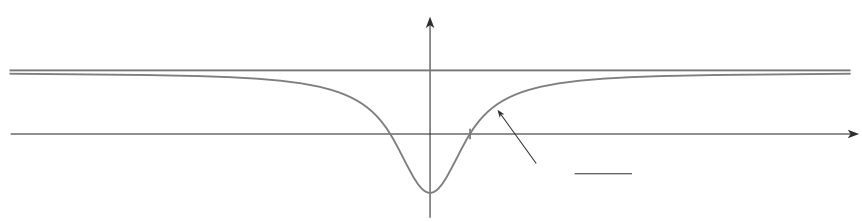


FIGURE 1

As x grows larger and larger you can see that the values of $f\ x$ get closer and closer to 1. In fact, it seems that we can make the values of $f\ x$ as close as we like to 1 by taking x sufÞciently large. This situation is expressed symbolically by writing

$$\lim_{x \to} \frac{x^2 \quad 1}{x^2 \quad 1} = 1$$

In general, we use the notation

$$\lim_{x \to} f\ x = L$$

to indicate that the values of $f\ x$ become closer and closer to L as x becomes larger and larger.

1 Definition Let f be a function defined on some interval $a, \quad$. Then

$$\lim_{x \to} f\ x = L$$

means that the values of $f\ x$ can be made arbitrarily close to L by taking x sufficiently large.

Another notation for $\lim_{x \to} f\ x = L$ is

$$f\ x \to L \quad \text{as} \quad x \to$$

The symbol $\quad$ does not represent a number. Nonetheless, the expression $\lim_{x \to} f\ x = L$ is often read as

"the limit of $f\ x$, as x approaches infinity, is L"

or "the limit of $f\ x$, as x becomes infinite, is L"

or "the limit of $f\ x$, as x increases without bound, is L"

The meaning of such phrases is given by Definition 1. A more precise definition, similar to the $\quad,\quad$ definition of Section 2.4, is given at the end of this section.

Geometric illustrations of Definition 1 are shown in Figure 2. Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a **horizontal asymptote**) as we look to the far right of each graph.

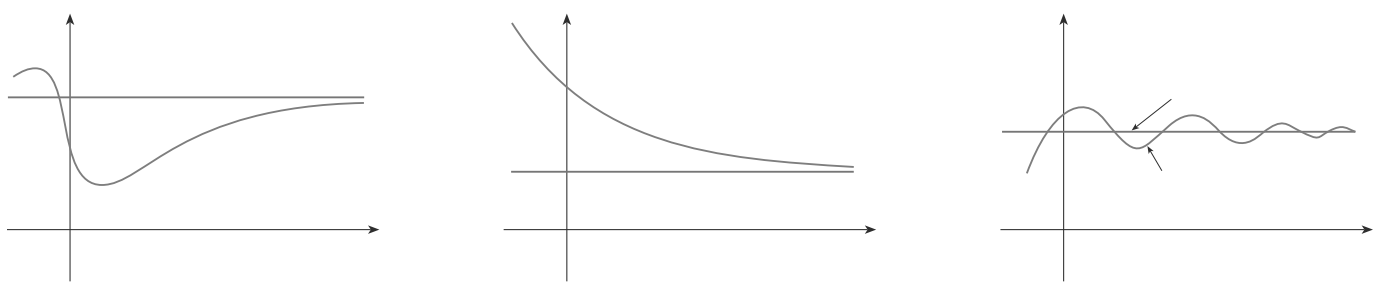


FIGURE 2
Examples illustrating $\lim_{\to}$

Referring back to Figure 1, we see that for numerically large negative values of x, the values of $f\ x$ are close to 1. By letting x decrease through negative values without bound, we can make $f\ x$ as close as we like to 1. This is expressed by writing

$$\lim_{x \to} \frac{x^2 \quad 1}{x^2 \quad 1} = 1$$

The general definition is as follows.

2 Definition Let f be a function defined on some interval $\quad, a$. Then

$$\lim_{x \to} f\ x = L$$

means that the values of $f\ x$ can be made arbitrarily close to L by taking x sufficiently large negative.

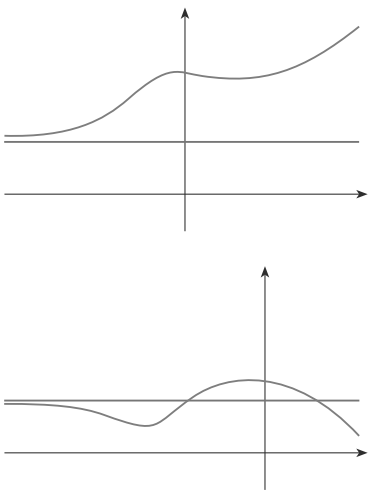


FIGURE 3
Examples illustrating $\lim_{\to}$

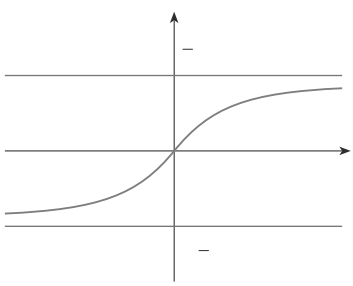


FIGURE 4
tan

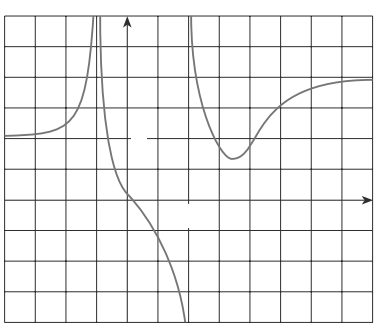


FIGURE 5

Again, the symbol does not represent a number, but the expression $\lim_{x\to} f\ x = L$ is often read as

"The limit of $f\ x$, as x approaches negative infinity, is L"

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.

> **3 Definition** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f\ x$ if either
>
> $$\lim_{x\to} f\ x = L \qquad \text{or} \qquad \lim_{x\to} f\ x = L$$

For instance, the curve illustrated in Figure 1 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x\to} \frac{x^2 \quad 1}{x^2 \quad 1} = 1$$

An example of a curve with two horizontal asymptotes is $y = \tan\ ^{1}x$. (See Figure 4.) In fact,

> **4** $$\lim_{x\to} \tan\ ^{1}x = \frac{}{2} \qquad \lim_{x\to} \tan\ ^{1}x = \frac{}{2}$$

so both of the lines $y =$ 2 and $y =$ 2 are horizontal asymptotes. (This follows from the fact that the lines $x =$ 2 are vertical asymptotes of the graph of tan.)

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

SOLUTION We see that the values of $f\ x$ become large as $x \to$ 1 from both sides, so

$$\lim_{x\to 1} f\ x =$$

Notice that $f\ x$ becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x\to 2} f\ x = \qquad \text{and} \qquad \lim_{x\to 2} f\ x =$$

Thus, both of the lines $x =$ 1 and $x = 2$ are vertical asymptotes.

As x becomes large, it appears that $f\ x$ approaches 4. But as x decreases through negative values, $f\ x$ approaches 2. So

$$\lim_{x\to} f\ x = 4 \qquad \text{and} \qquad \lim_{x\to} f\ x = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes.

EXAMPLE 2 Find $\lim_{x\to} \frac{1}{x}$ and $\lim_{x\to} \frac{1}{x}$.

SOLUTION Observe that when x is large, $1 \quad x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1 \quad x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x\to} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1 \quad x$ is small negative, so we also have

$$\lim_{x\to} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1 \quad x$. (This is an equilateral hyperbola; see Figure 6.)

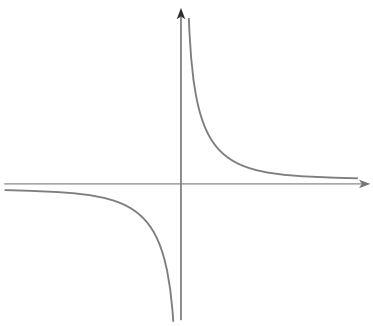


FIGURE 6

$\lim_{\to} - \qquad \lim_{\to} -$

Most of the Limit Laws that were given in Section 2.3 also hold for limits at infinity. It can be proved that the Limit Laws listed in Section 2.3 (with the exception of Laws 9 and 10) are also valid if $x \to a$ is replaced by $x \to$ or $x \to$. In particular, if we combine Laws 6 and 11 with the results of Example 2, we obtain the following important rule for calculating limits.

> **5 Theorem** If $r \quad 0$ is a rational number, then
>
> $$\lim_{x\to} \frac{1}{x^r} = 0$$
>
> If $r \quad 0$ is a rational number such that x^r is defined for all x, then
>
> $$\lim_{x\to} \frac{1}{x^r} = 0$$

EXAMPLE 3 Evaluate

$$\lim_{x\to} \frac{3x^2 \quad x \quad 2}{5x^2 \quad 4x \quad 1}$$

and indicate which properties of limits are used at each stage.

SOLUTION As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x.) In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x\to} \frac{3x^2 \quad x \quad 2}{5x^2 \quad 4x \quad 1} = \lim_{x\to} \frac{\dfrac{3x^2 \quad x \quad 2}{x^2}}{\dfrac{5x^2 \quad 4x \quad 1}{x^2}} = \lim_{x\to} \frac{3 \quad \dfrac{1}{x} \quad \dfrac{2}{x^2}}{5 \quad \dfrac{4}{x} \quad \dfrac{1}{x^2}}$$

$$= \frac{\lim_{x\to} \quad 3 \quad \dfrac{1}{x} \quad \dfrac{2}{x^2}}{\lim_{x\to} \quad 5 \quad \dfrac{4}{x} \quad \dfrac{1}{x^2}} \qquad \text{(by Limit Law 5)}$$

$$= \frac{\lim_{x\to} 3 \quad \lim_{x\to} \dfrac{1}{x} \quad 2 \lim_{x\to} \dfrac{1}{x^2}}{\lim_{x\to} 5 \quad 4 \lim_{x\to} \dfrac{1}{x} \quad \lim_{x\to} \dfrac{1}{x^2}} \qquad \text{(by 1, 2, and 3)}$$

$$= \frac{3 \quad 0 \quad 0}{5 \quad 0 \quad 0} \qquad \text{(by 7 and Theorem 5)}$$

$$= \frac{3}{5}$$

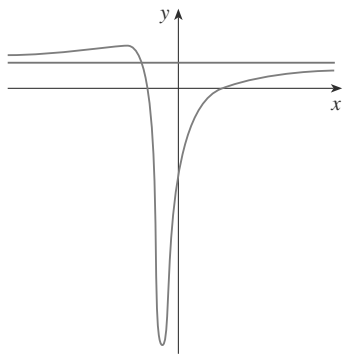


FIGURE 7

A similar calculation shows that the limit as $x \to \quad$ is also $\frac{3}{5}$. Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$.

EXAMPLE 4 Find the horizontal and vertical asymptotes of the graph of the function

$$f \; x \; = \frac{\overline{2x^2 \quad 1}}{3x \quad 5}$$

SOLUTION Dividing both numerator and denominator by x and using the properties of limits, we have

$$\lim_{x\to} \frac{\overline{2x^2 \quad 1}}{3x \quad 5} = \lim_{x\to} \frac{\overline{2 \quad \dfrac{1}{x^2}}}{3 \quad \dfrac{5}{x}} \qquad \text{(since } \overline{x^2} = x \text{ for } x \quad 0\text{)}$$

$$= \frac{\lim_{x\to} \quad \overline{2 \quad \dfrac{1}{x^2}}}{\lim_{x\to} \quad 3 \quad \dfrac{5}{x}} = \frac{\overline{\lim_{x\to} 2 \quad \lim_{x\to} \dfrac{1}{x^2}}}{\lim_{x\to} 3 \quad 5 \lim_{x\to} \dfrac{1}{x}}$$

$$= \frac{\overline{2 \quad 0}}{3 \quad 5 \quad 0} = \frac{\overline{2}}{3}$$

Therefore, the line $y = \quad \overline{2} \;\; 3$ is a horizontal asymptote of the graph of f.

In computing the limit as $x \to \quad$, we must remember that for $x \quad 0$, we have $\overline{x^2} = \;\; x \;\; = \quad x$. So when we divide the numerator by x, for $x \quad 0$ we get

$$\frac{1}{x} \quad \overline{2x^2 \quad 1} = \quad \frac{1}{\overline{x^2}} \quad \overline{2x^2 \quad 1} = \quad \overline{2 \quad \frac{1}{x^2}}$$

Therefore

Thus, the line is also a horizontal asymptote.

A vertical asymptote is likely to occur when the denominator, , is 0, that is, when . If is close to and , then the denominator is close to 0 and is positive. The numerator is always positive, so is positive. Therefore

If is close to but , then and so is large negative. Thus

The vertical asymptote is . All three asymptotes are shown in Figure 8.

EXAMPLE 5 Compute .

SOLUTION Because both and x are large when x is large, it's difficult to see what happens to their difference, so we use algebra to rewrite the function. We first multiply numerator and denominator by the conjugate radical:

The Squeeze Theorem could be used to show that this limit is 0. But an easier method is to divide numerator and denominator by . Doing this and using the Limit Laws, we obtain

Figure 9 illustrates this result.

The graph of the natural exponential function $y = e^x$ has the line $y = 0$ (the x-axis) as a horizontal asymptote. (The same is true of any exponential function with base a 1.) In

2.8 Derivatives

In Section 2.7 we defined the slope of the tangent to a curve with equation $y = f(x)$ at the point where $x = a$ to be

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

We also saw that the velocity of an object with position function $s = f(t)$ at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

2 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

$f'(a)$ is read "f prime of a"

If we write $x = a + h$, then $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore, an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3}$$

EXAMPLE 1 Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

SOLUTION From Definition 2 we have

Try problems like this one.
Resources / Module 3 / Derivative at a Point / Problem Wizard

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{(a+h)^2 - 8(a+h) + 9 - (a^2 - 8a + 9)}{h} \\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\
&= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) \\
&= 2a - 8
\end{aligned}$$

Interpretation of the Derivative as the Slope of a Tangent

In Section 2.7 we defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given by Equation 1. Since, by Definition 2, this is the same as the derivative $f'(a)$, we can now say the following.

> The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Thus, the geometric interpretation of a derivative [as defined by either (2) or (3)] is as shown in Figure 1.

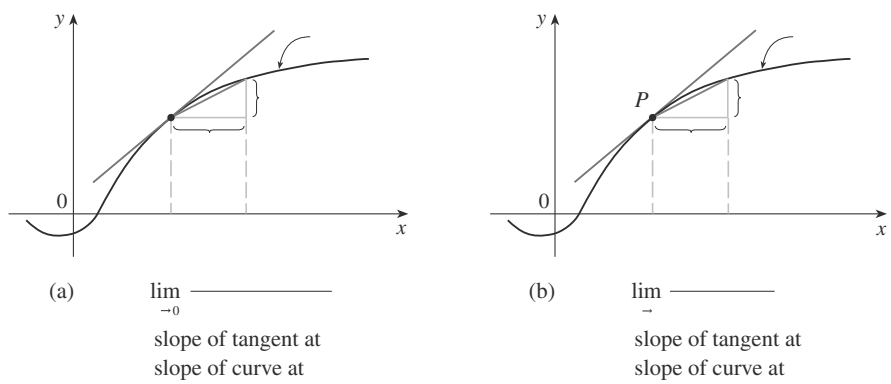


FIGURE 1
Geometric interpretation of the derivative

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

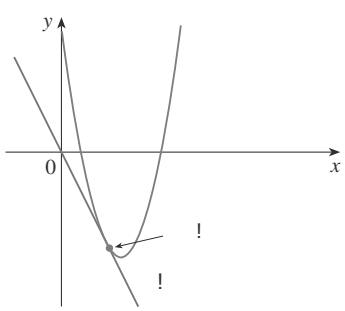


FIGURE 2

EXAMPLE 2 Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

SOLUTION From Example 1 we know that the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$. Therefore, the slope of the tangent line at $(3, -6)$ is $f'(3) = 2(3) - 8 = -2$. Thus, an equation of the tangent line, shown in Figure 2, is

$$y - (-6) = -2(x - 3) \qquad \text{or} \qquad y = -2x$$

EXAMPLE 3 Let $f(x) = 2^x$. Estimate the value of $f'(0)$ in two ways:
(a) By using Definition 2 and taking successively smaller values of h.
(b) By interpreting $f'(0)$ as the slope of a tangent and using a graphing calculator to zoom in on the graph of $y = 2^x$.

SOLUTION
(a) From Definition 2 we have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{2^h - 1}{h}$$

h	$\dfrac{2^h \quad 1}{h}$
0.1	0.718
0.01	0.696
0.001	0.693
0.0001	0.693
0.1	0.670
0.01	0.691
0.001	0.693
0.0001	0.693

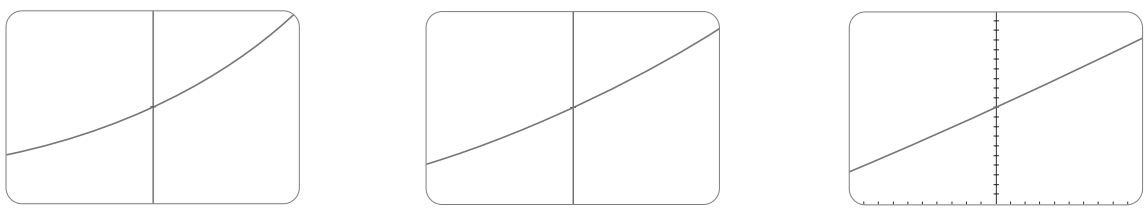
Since we are not yet able to evaluate this limit exactly, we use a calculator to approximate the values of $2^h \quad 1 \quad h$. From the numerical evidence in the table at the left we see that as h approaches 0, these values appear to approach a number near 0.69. So our estimate is

$$f \quad 0 \qquad 0.69$$

(b) In Figure 3 we graph the curve $y = 2^x$ and zoom in toward the point $0, 1$. We see that the closer we get to $0, 1$, the more the curve looks like a straight line. In fact, in Figure 3(c) the curve is practically indistinguishable from its tangent line at $0, 1$. Since the x-scale and the y-scale are both 0.01, we estimate that the slope of this line is

$$\frac{0.14}{0.20} = 0.7$$

So our estimate of the derivative is $f \quad 0 \qquad 0.7$. In Section 3.5 we will show that, correct to six decimal places, $f \quad 0 \qquad 0.693147$.



(a) ! by (b) ! " " by " " (c) ! " " by 0.9, 1.1

FIGURE 3 Zooming in on the graph of # near

Interpretation of the Derivative as a Rate of Change

In Section 2.7 we defined the instantaneous rate of change of $y = f \quad x$ with respect to x at $x = x_1$ as the limit of the average rates of change over smaller and smaller intervals. If the interval is x_1, x_2 , then the change in x is $\quad x = x_2 \qquad x_1$, the corresponding change in y is

$$y = f \quad x_2 \qquad f \quad x_1$$

and

$$\boxed{4} \qquad \text{instantaneous rate of change} = \lim_{x \to 0} \frac{y}{x} = \lim_{x_2 \to x_1} \frac{f \quad x_2 \qquad f \quad x_1}{x_2 \qquad x_1}$$

From Equation 3 we recognize this limit as being the derivative of f at x_1, that is, $f \quad x_1$. This gives a second interpretation of the derivative:

> The derivative $f \quad a$ is the instantaneous rate of change of $y = f \quad x$ with respect to x when $x = a$.

The connection with the first interpretation is that if we sketch the curve $y = f \quad x$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where

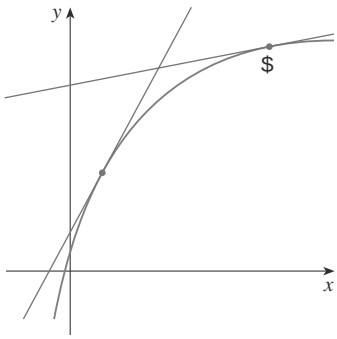


FIGURE 4
The y-values are changing rapidly at P and slowly at Q.

$x = a$. This means that when the derivative is large (and therefore the curve is steep, as at the point P in Figure 4), the y-values change rapidly. When the derivative is small, the curve is relatively ßat and the y-values change slowly.

In particular, if $s = f\ t$ is the position function of a particle that moves along a straight line, then $f\ \ a$ is the rate of change of the displacement s with respect to the time t. In other words, $f\ \ a$ *is the velocity of the particle at time* $t = a$. (See Section 2.7.) The *speed* of the particle is the absolute value of the velocity, that is, $\ \ f\ \ a\ \ \ $.

EXAMPLE 4 The position of a particle is given by the equation of motion $s = f\ t\ = 1\ \ 1\ \ \ \ t$, where t is measured in seconds and s in meters. Find the velocity and the speed after 2 seconds.

SOLUTION The derivative of f when $t = 2$ is

$$f\ \ 2\ = \lim_{h\to 0} \frac{f\ 2\ \ \ h\ \ \ \ f\ 2}{h} = \lim_{h\to 0} \frac{\dfrac{1}{1\ \ \ 2\ \ \ h} \ \ \ \dfrac{1}{1\ \ \ 2}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{3\ \ \ h} \ \ \ \dfrac{1}{3}}{h} = \lim_{h\to 0} \frac{\dfrac{3\ \ \ 3\ \ \ h}{3\ 3\ \ \ h}}{h}$$

$$= \lim_{h\to 0} \frac{h}{3\ 3\ \ \ \ h\ h} = \lim_{h\to 0} \frac{1}{3\ 3\ \ \ \ h} = \ \ \frac{1}{9}$$

Thus, the velocity after 2 seconds is $f\ \ 2\ \ =\ \ \ \frac{1}{9}$ m s, and the speed is $\ f\ \ 2\ \ =\ \ \ \frac{1}{9}\ = \frac{1}{9}$ m s.

In Module 2.8 you are asked to compare and order the slopes of tangent and secant lines at several points on a curve.

EXAMPLE 5 A manufacturer produces bolts of a fabric with a Þxed width. The cost of producing x yards of this fabric is $C = f\ x$ dollars.
(a) What is the meaning of the derivative $f\ \ x$? What are its units?
(b) In practical terms, what does it mean to say that $f\ \ 1000\ \ = 9$?
(c) Which do you think is greater, $f\ \ 50$ or $f\ \ 500$? What about $f\ \ 5000$?

SOLUTION
(a) The derivative $f\ \ x$ is the instantaneous rate of change of C with respect to x; that is, $f\ \ x$ means the rate of change of the production cost with respect to the number of yards produced. (Economists call this rate of change the *marginal cost.* This idea is discussed in more detail in Sections 3.3 and 4.8.)

Because

$$f\ \ x\ \ = \lim_{x\to 0} \frac{C}{x}$$

the units for $f\ \ x$ are the same as the units for the difference quotient $\ \ C\ \ \ \ x$. Since $\ \ C$ is measured in dollars and $\ \ x$ in yards, it follows that the units for $f\ \ x$ are dollars per yard.

(b) The statement that $f\ \ 1000\ \ = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9 yard. (When $x = 1000$, C is increasing 9 times as fast as x.)

Since $x = 1$ is small compared with $x = 1000$, we could use the approximation

$$f\ 1000 \qquad \frac{C}{x} = \frac{C}{1} = \quad C$$

Here we are assuming that the cost function is well behaved; in other words, C doesnÕt oscillate rapidly near $x = 1000$.

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efÞcient use of the Þxed costs of production.) So

$$f\ 50 \qquad f\ 500$$

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus, it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

$$f\ 5000 \qquad f\ 500$$

The following example shows how to estimate the derivative of a tabular function, that is, a function deÞned not by a formula but by a table of values.

t	$D\ t$
1980	930.2
1985	1945.9
1990	3233.3
1995	4974.0
2000	5674.2

EXAMPLE 6 Let $D\ t$ be the U.S. national debt at time t. The table in the margin gives approximate values of this function by providing end of year estimates, in billions of dollars, from 1980 to 2000. Interpret and estimate the value of $D\ 1990$.

SOLUTION The derivative $D\ 1990$ means the rate of change of D with respect to t when $t = 1990$, that is, the rate of increase of the national debt in 1990.

According to Equation 3,

$$D\ 1990 = \lim_{t \to 1990} \frac{D\ t \quad D\ 1990}{t \quad 1990}$$

So we compute and tabulate values of the difference quotient (the average rates of change) as follows.

t	$\dfrac{D\ t \quad D\ 1990}{t \quad 1990}$
1980	230.31
1985	257.48
1995	348.14
2000	244.09

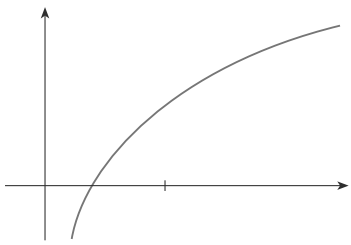
Another method is to plot the debt function and estimate the slope of the tangent line when $t = 1990$. (See Example 5 in Section 2.7.)

From this table we see that $D\ 1990$ lies somewhere between 257.48 and 348.14 billion dollars per year. [Here we are making the reasonable assumption that the debt didnÕt ßuctuate wildly between 1980 and 2000.] We estimate that the rate of increase of the national debt of the United States in 1990 was the average of these two numbers, namely

$$D\ 1990 \qquad 303 \text{ billion dollars per year}$$

2.8 Exercises

1. On the given graph of f, mark lengths that represent $f\ 2$, $f\ 2\quad h$, $f\ 2\quad h\quad f\ 2$, and h. (Choose $h\quad 0$.) What line has slope $\dfrac{f\ 2\quad h\quad f\ 2}{h}$?

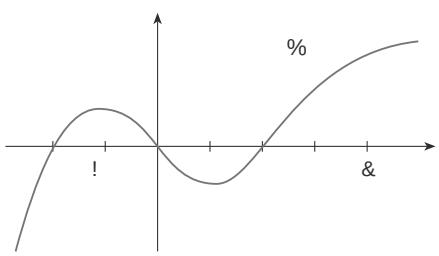


2. For the function f whose graph is shown in Exercise 1, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad f\ 2 \qquad f\ 3 \qquad f\ 2 \qquad \tfrac{1}{2}\ f\ 4 \qquad f\ 2$$

3. For the function whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad 2 \qquad 0 \qquad 2 \qquad 4$$



4. If the tangent line to $y = f\ x$ at (4, 3) passes through the point (0, 2), find $f\ 4$ and $f\ 4$.

5. Sketch the graph of a function f for which $f\ 0\ = 0$, $f\ 0\ = 3$, $f\ 1\ = 0$, and $f\ 2\ =\ \ 1$.

6. Sketch the graph of a function for which $0\ = 0$, $0\ = 3$, $1\ = 0$, and $2\ = 1$.

7. If $f\ x\ = 3x^2\quad 5x$, find $f\ 2$ and use it to find an equation of the tangent line to the parabola $y = 3x^2\quad 5x$ at the point $2, 2$.

8. If $x\ = 1\quad x^3$, find 0 and use it to find an equation of the tangent line to the curve $y = 1\quad x^3$ at the point $0, 1$.

9. (a) If $F\ x\ = x^3\quad 5x\quad 1$, find $F\ 1$ and use it to find an equation of the tangent line to the curve $y = x^3\quad 5x\quad 1$ at the point $1,\ \ 3$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

10. (a) If $G\ x\ = x\ \ 1\quad 2x$, find $G\ a$ and use it to find an equation of the tangent line to the curve $y = x\ \ 1\quad 2x$ at the point $\left(\ \tfrac{1}{4},\ \ \tfrac{1}{2}\right)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

11. Let $f\ x\ = 3^x$. Estimate the value of $f\ \ 1$ in two ways:
(a) By using Definition 2 and taking successively smaller values of h.
(b) By zooming in on the graph of $y = 3^x$ and estimating the slope.

12. Let $x\ = \tan x$. Estimate the value of 4 in two ways:
(a) By using Definition 2 and taking successively smaller values of h.
(b) By zooming in on the graph of $y = \tan x$ and estimating the slope.

13–18 Find $f\ \ a$.

13. $f\ x\ = 3\quad 2x\quad 4x^2$

14. $f\ t\ = t^4\quad 5t$

15. $f\ t\ = \dfrac{2t\quad 1}{t\quad 3}$

16. $f\ x\ = \dfrac{x^2\quad 1}{x\quad 2}$

17. $f\ x\ = \dfrac{1}{x\quad 2}$

18. $f\ x\ = \sqrt{3x\quad 1}$

19–24 Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

19. $\displaystyle\lim_{h\to 0} \frac{1\quad h\ ^{10}\quad 1}{h}$

20. $\displaystyle\lim_{h\to 0} \frac{\sqrt[4]{16\quad h}\quad 2}{h}$

21. $\displaystyle\lim_{x\to 5} \frac{2^x\quad 32}{x\quad 5}$

22. $\displaystyle\lim_{x\to\ \ 4} \frac{\tan x\quad 1}{x\quad 4}$

23. $\displaystyle\lim_{h\to 0} \frac{\cos\quad h\quad 1}{h}$

24. $\displaystyle\lim_{t\to 1} \frac{t^4\quad t\quad 2}{t\quad 1}$

25–26 A particle moves along a straight line with equation of motion $s = f\ t$, where s is measured in meters and t in seconds. Find the velocity when $t = 2$.

25. $f\ t\ = t^2\quad 6t\quad 5$

26. $f\ t\ = 2t^3\quad t\quad 1$

27. The cost of producing x ounces of gold from a new gold mine is $C = f\ x$ dollars.
(a) What is the meaning of the derivative $f\ \ x$? What are its units?
(b) What does the statement $f\ \ 800\ = 17$ mean?
(c) Do you think the values of $f\ \ x$ will increase or decrease in the short term? What about the long term? Explain.

28. The number of bacteria after t hours in a controlled laboratory experiment is $n = f\ t$.

(a) What is the meaning of the derivative $f\ \ 5$? What are its units?

(b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f\ \ 5$ or $f\ \ 10$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

29. The fuel consumption (measured in gallons per hour) of a car traveling at a speed of v miles per hour is $c = f\ v$.

(a) What is the meaning of the derivative $f\ \ v$? What are its units?

(b) Write a sentence (in laymanÕs terms) that explains the meaning of the equation $f\ \ 20\ \ =\ \ 0.05$.

30. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f\ \ p$.

(a) What is the meaning of the derivative $f\ \ 8$? What are its units?

(b) Is $f\ \ 8$ positive or negative? Explain.

31. Let $T\ t$ be the temperature (in F) in Dallas t hours after midnight on June 2, 2001. The table shows values of this function recorded every two hours. What is the meaning of $T\ \ 10$? Estimate its value.

t	0	2	4	6	8	10	12	14
T	73	73	70	69	72	81	88	91

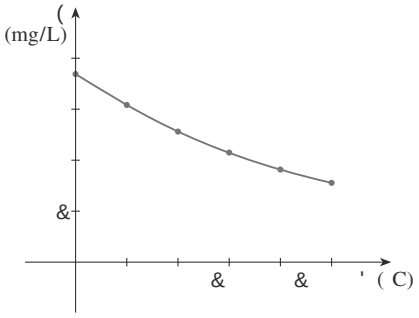
32. Life expectancy improved dramatically in the 20th century. The table gives values of $E\ t$, the life expectancy at birth (in years) of a male born in the year t in the United States. Interpret and estimate the values of $E\ \ 1910$ and $E\ \ 1950$.

t	$E\ t$	t	$E\ t$
1900	48.3	1960	66.6
1910	51.1	1970	67.1
1920	55.2	1980	70.0
1930	57.4	1990	71.8
1940	62.5	2000	74.1
1950	65.6		

33. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution inßuences the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T.

(a) What is the meaning of the derivative $S\ \ T$? What are its units?

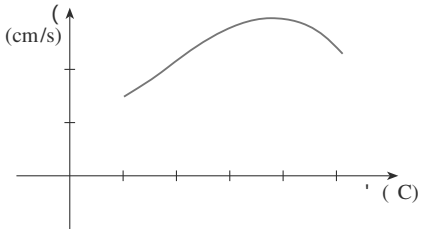
(b) Estimate the value of $S\ \ 16$ and interpret it.



34. The graph shows the inßuence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.

(a) What is the meaning of the derivative $S\ \ T$? What are its units?

(b) Estimate the values of $S\ \ 15$ and $S\ \ 25$ and interpret them.



35–36 Determine whether $f\ \ 0$ exists.

35. $f\ x\ \ =\ \ \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

36. $f\ x\ \ =\ \ \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

WRITING PROJECT

Early Methods for Finding Tangents

The Þrst person to formulate explicitly the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that ÒIf I have seen further than other men, it is because I have stood on the shoulders of giants.ÓTwo of those giants were Pierre Fermat (1601Ð1665) and NewtonÕs teacher at Cambridge, Isaac Barrow (1630Ð1677). Newton was familiar with the methods that these men used to Þnd tangent lines, and their methods played a role in NewtonÕs eventual formulation of calculus.

The following references contain explanations of these methods. Read one or more of the references and write a report comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.8 to Þnd an equation of the tangent line to the curve $y = x^3 \quad 2x$ at the point (1, 3) and show how either Fermat or Barrow would have solved the same problem. Although you used derivatives and they did not, point out similarities between the methods.

1. Carl Boyer and Uta Merzbach, *A History of Mathematics* (New York: Wiley, 1989), pp. 389, 432.
2. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.
3. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990), pp. 391, 395.
4. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 344, 346.

2.9 The Derivative as a Function

In the preceding section we considered the derivative of a function f at a Þxed number a:

$$f\ a\ = \lim_{h \to 0} \frac{f\ a\quad h\quad f\ a}{h} \tag{1}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x, we obtain

$$\boxed{f\ x\ = \lim_{h \to 0} \frac{f\ x\quad h\quad f\ x}{h}} \tag{2}$$

Given any number x for which this limit exists, we assign to x the number $f\ \ x$. So we can regard f as a new function, called the **derivative of *f*** and deÞned by Equation 2. We know that the value of f at x, $f\ \ x$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $x, f\ x$.

The function f is called the derivative of f because it has been ÒderivedÓ from f by the limiting operation in Equation 2. The domain of f is the set $x\ \ f\ \ x$ exists and may be smaller than the domain of f.

EXAMPLE 1 The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f .

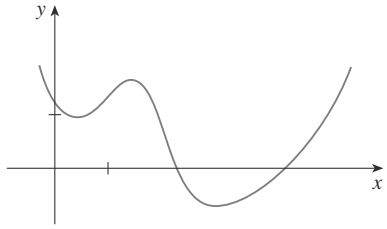


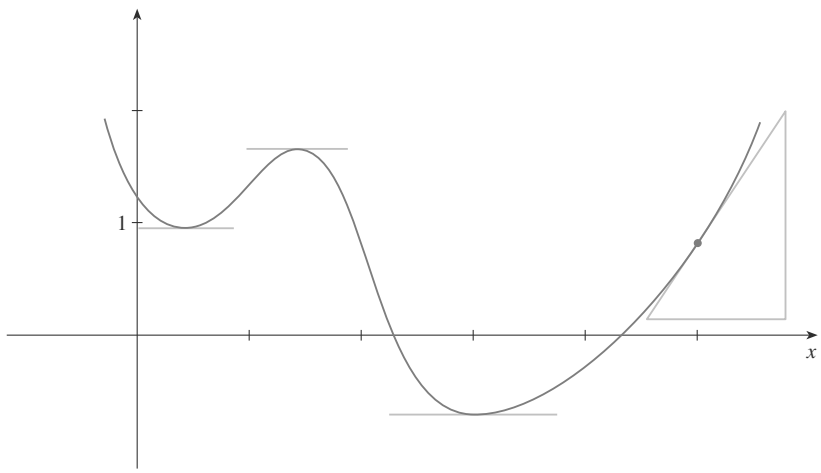
FIGURE 1

Watch an animation of the relation between a function and its derivative.

Resources / Module 3
/ Derivatives as Functions
/ Mars Rover

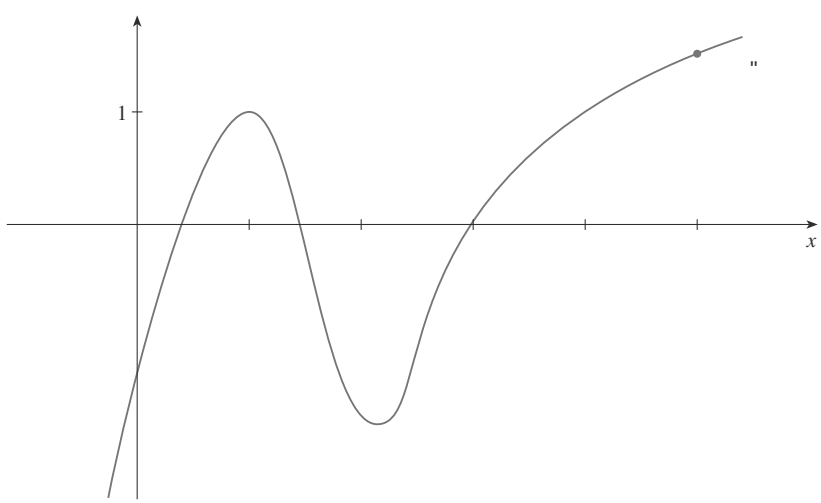
Resources / Module 3
/ Slope-a-Scope
/ Derivative of a Cubic

SOLUTION We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 5$ we draw the tangent at P in Figure 2(a) and estimate its slope to be about $\frac{3}{2}$, so $f'(5) \approx 1.5$. This allows us to plot the point $(5, 1.5)$ on the graph of f' directly beneath P. Repeating this procedure at several points, we get the graph shown in Figure 2(b). Notice that the tangents at A, B, and C are horizontal, so the derivative is 0 there and the graph of f' crosses the x-axis at the points A', B', and C', directly beneath A, B, and C. Between A and B the tangents have positive slope, so $f'(x)$ is positive there. But between B and C the tangents have negative slope, so $f'(x)$ is negative there.



(a)

Notice that where the derivative is positive (to the right of A and between B and C), the function f is increasing. Where f' is negative (to the left of A and between B and C), f is decreasing. In Section 4.3 we will prove that this is true for all functions.



(b)

FIGURE 2

If a function is defined by a table of values, then we can construct a table of approximate values of its derivative, as in the next example.

t	$B\ t$
1980	9,847
1982	9,856
1984	9,855
1986	9,862
1988	9,884
1990	9,962
1992	10,036
1994	10,109
1996	10,152
1998	10,175
2000	10,186

EXAMPLE 2 Let $B\ t$ be the population of Belgium at time t. The table at the left gives midyear values of $B\ t$, in thousands, from 1980 to 2000. Construct a table of values for the derivative of this function.

SOLUTION We assume that there were no wild ßuctuations in the population between the stated values. LetÕs start by approximating $B\ \ 1988$, the rate of increase of the population of Belgium in mid-1988. Since

$$B\ \ 1988 = \lim_{h \to 0} \frac{B\ 1988 \quad h \quad B\ 1988}{h}$$

we have

$$B\ \ 1988 \quad \frac{B\ 1988 \quad h \quad B\ 1988}{h}$$

for small values of h.

For $h = 2$, we get

$$B\ \ 1988 \quad \frac{B\ 1990 \quad B\ 1988}{2} = \frac{9962 \quad 9884}{2} = 39$$

(This is the average rate of increase between 1988 and 1990.) For $h = \quad 2$, we have

$$B\ \ 1988 \quad \frac{B\ 1986 \quad B\ 1988}{2} = \frac{9862 \quad 9884}{2} = 11$$

which is the average rate of increase between 1986 and 1988. We get a more accurate approximation if we take the average of these rates of change:

$$B\ \ 1988 \quad \tfrac{1}{2}\ 39 \quad 11\ = 25$$

This means that in 1988 the population was increasing at a rate of about 25,000 people per year.

Making similar calculations for the other values (except at the endpoints), we get the table at the left, which shows the approximate values for the derivative.

t	$B\ \ t$
1980	4.5
1982	2.0
1984	1.5
1986	7.3
1988	25.0
1990	38.0
1992	36.8
1994	29.0
1996	16.5
1998	8.5
2000	5.5

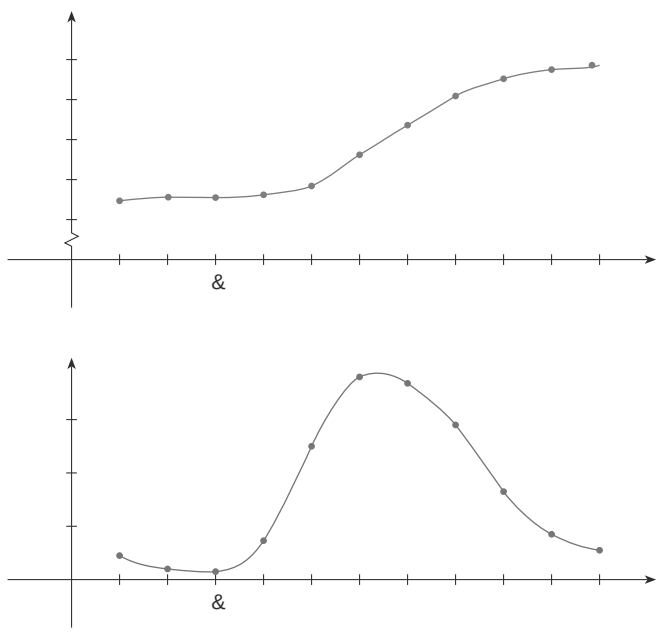


FIGURE 3

|||| Figure 3 illustrates Example 2 by showing graphs of the population function $B\ t$ and its derivative $B\ \ t$. Notice how the rate of population growth increases to a maximum in 1990 and decreases thereafter.

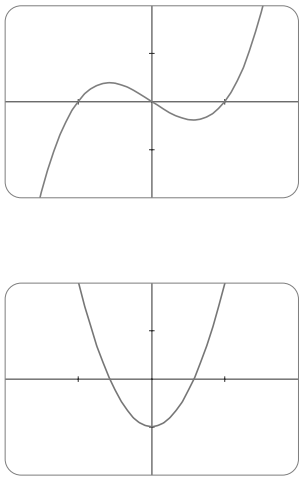


FIGURE 4

EXAMPLE 3

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

(b) Illustrate by comparing the graphs of f and f'.

SOLUTION

(a) When using Equation 2 to compute a derivative, we must remember that the variable is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x - h - (x^3 - x)}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\
&= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1
\end{aligned}
$$

(b) We use a graphing device to graph f and f' in Figure 4. Notice that $f'(x) = 0$ when f has horizontal tangents and $f'(x)$ is positive when the tangents have positive slope. So these graphs serve as a check on our work in part (a).

EXAMPLE 4 If $f(x) = \sqrt{x - 1}$, find the derivative of f. State the domain of f'.

See more problems like these.
Resources / Module 3
/ How to Calculate
/ The Essential Examples

SOLUTION

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x + h - 1} - \sqrt{x - 1}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x + h - 1} - \sqrt{x - 1}}{h} \cdot \frac{\sqrt{x + h - 1} + \sqrt{x - 1}}{\sqrt{x + h - 1} + \sqrt{x - 1}} \\
&= \lim_{h \to 0} \frac{(x + h - 1) - (x - 1)}{h\left(\sqrt{x + h - 1} + \sqrt{x - 1}\right)} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x + h - 1} + \sqrt{x - 1}} \\
&= \frac{1}{\sqrt{x - 1} + \sqrt{x - 1}} = \frac{1}{2\sqrt{x - 1}}
\end{aligned}
$$

Here we rationalize the numerator.

We see that $f'(x)$ exists if $x > 1$, so the domain of f' is $(1, \infty)$. This is smaller than the domain of f, which is $[1, \infty)$.

Let's check to see that the result of Example 4 is reasonable by looking at the graphs of f and f' in Figure 5. When x is close to 1, $\sqrt{x - 1}$ is close to 0, so $f'(x) = 1/(2\sqrt{x - 1})$ is very large; this corresponds to the steep tangent lines near $(1, 0)$ in Figure 5(a) and the large values of $f'(x)$ just to the right of 1 in Figure 5(b). When x is large, $f'(x)$ is very small; this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f'.

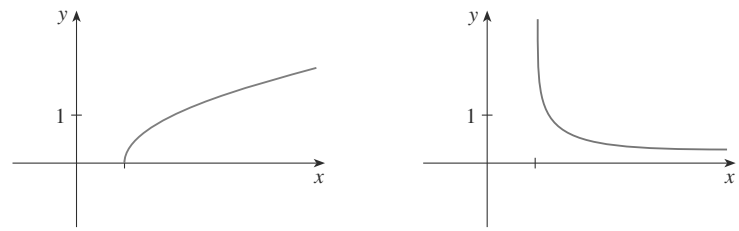


FIGURE 5 (a) (b) $\frac{1}{2}$

EXAMPLE 5 Find f if $f\ x\ = \frac{1}{2}\ \frac{x}{x}$.

SOLUTION

$$f\ x\ = \lim_{h\to 0} \frac{f\ x\ h\ f\ x}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}\ \frac{x\ h}{x\ h}\ \frac{1}{2}\ \frac{x}{x}}{h}$$

$$\frac{\frac{a}{b}\ \frac{c}{d}}{e} = \frac{ad\ bc}{bd}\ \frac{1}{e}$$

$$= \lim_{h\to 0} \frac{1\ x\ h\ 2\ x\ 1\ x\ 2\ x\ h}{h\ 2\ x\ h\ 2\ x}$$

$$= \lim_{h\to 0} \frac{2\ x\ 2h\ x^2\ xh\ 2\ x\ h\ x^2\ xh}{h\ 2\ x\ h\ 2\ x}$$

$$= \lim_{h\to 0} \frac{3h}{h\ 2\ x\ h\ 2\ x}$$

$$= \lim_{h\to 0} \frac{3}{2\ x\ h\ 2\ x} = \frac{3}{2\ x^2}$$

Other Notations

If we use the traditional notation $y = f\ x$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f\ x\ = y\ = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f\ x\ = Df\ x\ = D_x f\ x$$

The symbols D and $d\ dx$ are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

The symbol $dy\ dx$, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f\ x$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation. Referring to Equation 2.8.4, we can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{x\to 0} \frac{y}{x}$$

|||| Gottfried Wilhelm Leibniz was born in Leipzig in 1646 and studied law, theology, philosophy, and mathematics at the university there, graduating with a bachelorÕs degree at age 17. After earning his doctorate in law at age 20, Leibniz entered the diplomatic service and spent most of his life traveling to the capitals of Europe on political missions. In particular, he worked to avert a French military threat against Germany and attempted to reconcile the Catholic and Protestant churches.

His serious study of mathematics did not begin until 1672 while he was on a diplomatic mission in Paris. There he built a calculating machine and met scientists, like Huygens, who directed his attention to the latest developments in mathematics and science. Leibniz sought to develop a symbolic logic and system of notation that would simplify logical reasoning. In particular, the version of calculus that he published in 1684 established the notation and the rules for Þnding derivatives that we use today.

Unfortunately, a dreadful priority dispute arose in the 1690s between the followers of Newton and those of Leibniz as to who had invented calculus Þrst. Leibniz was even accused of plagiarism by members of the Royal Society in England. The truth is that each man invented calculus independently. Newton arrived at his version of calculus Þrst but, because of his fear of controversy, did not publish it immediately. So LeibnizÕs 1684 account of calculus was the Þrst to be published.

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

which is a synonym for $f'(a)$.

3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞), or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLE 6 Where is the function $f(x) = |x|$ differentiable?

SOLUTION If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$ we have

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$. Therefore, for $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{-(x + h) - (-x)}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and so f is differentiable for any $x < 0$.

For $x = 0$ we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0 + h| - 0}{h} \qquad \text{if it exists}$$

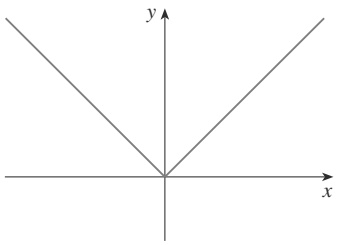
LetÕs compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|0 + h| - 0}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

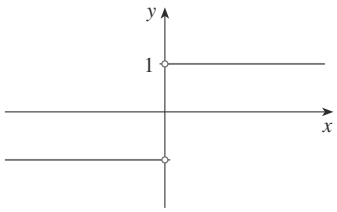
and

$$\lim_{h \to 0^-} \frac{|0 + h| - 0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus, f is differentiable at all x except 0.



(a)



(b)

FIGURE 6

A formula for f is given by

$$f\ x\ = \begin{matrix} 1 & \text{if } x & 0 \\ & 1 & \text{if } x & 0 \end{matrix}$$

and its graph is shown in Figure 6(b). The fact that $f\ 0$ does not exist is reßected geometrically in the fact that the curve $y = \ x$ does not have a tangent line at $0, 0$. [See Figure 6(a).]

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

4 Theorem If f is differentiable at a, then f is continuous at a.

Proof To prove that f is continuous at a, we have to show that $\lim_{x\to a} f\ x\ = f\ a$. We do this by showing that the difference $f\ x \quad f\ a$ approaches 0.

The given information is that f is differentiable at a, that is,

$$f\ \ a\ = \lim_{x\to a} \frac{f\ x \quad f\ a}{x \quad a}$$

exists (see Equation 2.8.3). To connect the given and the unknown, we divide and multiply $f\ x \quad f\ a$ by $x \quad a$ (which we can do when $x \neq a$):

$$f\ x \quad f\ a\ = \frac{f\ x \quad f\ a}{x \quad a}\ x \quad a$$

Thus, using the Product Law and (2.8.3), we can write

$$\begin{aligned} \lim_{x\to a}\ f\ x \quad f\ a\ &= \lim_{x\to a} \frac{f\ x \quad f\ a}{x \quad a}\ x \quad a \\ &= \lim_{x\to a} \frac{f\ x \quad f\ a}{x \quad a} \lim_{x\to a}\ x \quad a \\ &= f\ \ a \quad 0 = 0 \end{aligned}$$

To use what we have just proved, we start with $f\ x$ and add and subtract $f\ a$:

$$\begin{aligned} \lim_{x\to a} f\ x\ &= \lim_{x\to a}\ f\ a \qquad f\ x \quad f\ a \\ &= \lim_{x\to a} f\ a \qquad \lim_{x\to a}\ f\ x \quad f\ a \\ &= f\ a \qquad 0 = f\ a \end{aligned}$$

Therefore, f is continuous at a.

NOTE ▫ The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f\ x\ = \ x$ is continuous at 0 because

$$\lim_{x\to 0} f\ x\ = \lim_{x\to 0}\ x\ = 0 = f\ 0$$

(See Example 7 in Section 2.3.) But in Example 6 we showed that f is not differentiable at 0.

How Can a Function Fail to Be Differentiable?

We saw that the function $y = |x|$ in Example 6 is not differentiable at 0 and Figure 6(a) shows that its graph changes direction abruptly when $x = 0$. In general, if the graph of a function f has a Òcorner Ó or ÒkinkÓ in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we Þnd that the left and right limits are different.]

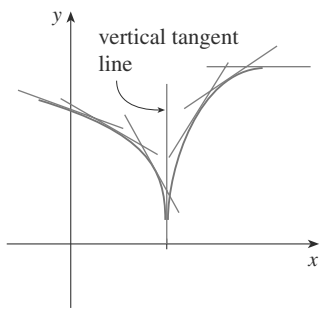


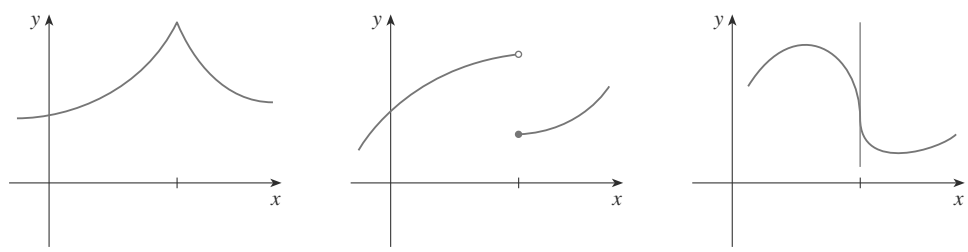
FIGURE 7

Theorem 4 gives another way for a function not to have a derivative. It says that if f is not continuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and

$$\lim_{x \to a} f'(x) = \infty$$

This means that the tangent lines become steeper and steeper as $x \to a$. Figure 7 shows one way that this can happen; Figure 8(c) shows another. Figure 8 illustrates the three possibilities that we have discussed.



(a) A corner (b) A discontinuity (c) A vertical tangent

FIGURE 8
Three ways for Ä not to be differentiable at a

A graphing calculator or computer provides another way of looking at differentiability. If f is differentiable at a, then when we zoom in toward the point $(a, f(a))$ the graph straightens out and appears more and more like a line. (See Figure 9. We saw a speciÞc example of this in Figure 3 in Section 2.8.) But no matter how much we zoom in toward a point like the ones in Figures 7 and 8(a), we canÕt eliminate the sharp point or corner (see Figure 10).

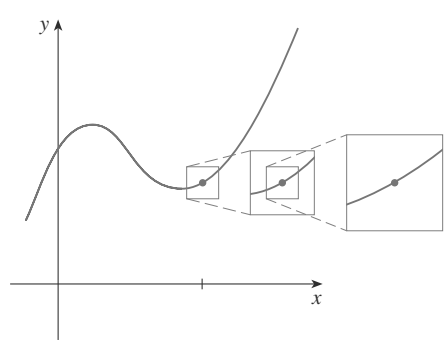


FIGURE 9
Ä is differentiable at a.

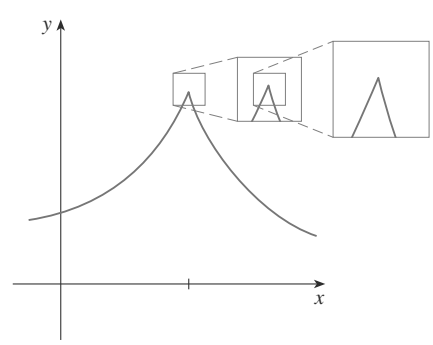
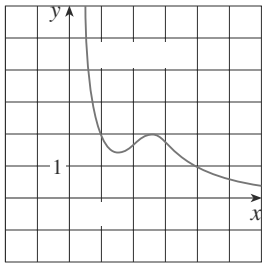


FIGURE 10
Ä is not differentiable at a.

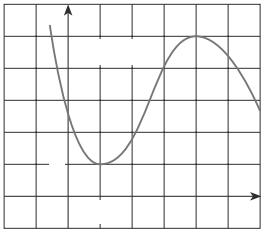
2.9 Exercises

1–3 Use the given graph to estimate the value of each derivative. Then sketch the graph of f .

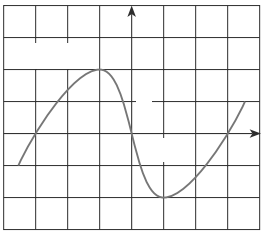
1. (a) f 1
(b) f 2
(c) f 3
(d) f 4



2. (a) f 0
(b) f 1
(c) f 2
(d) f 3
(e) f 4
(f) f 5

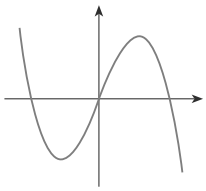


3. (a) f 3
(b) f 2
(c) f 1
(d) f 0
(e) f 1
(f) f 2
(g) f 3

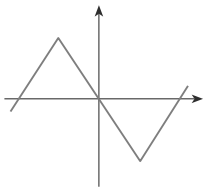


4. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.

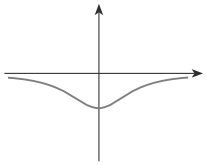
(a)



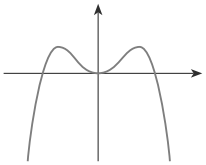
(b)



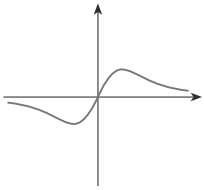
(c)



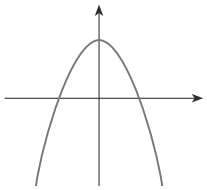
(d)



I



II



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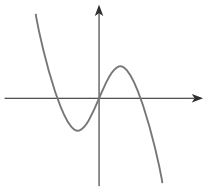
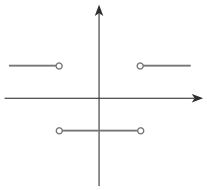
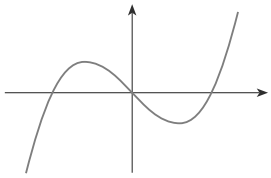


IV



5–13 Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f below it.

5.



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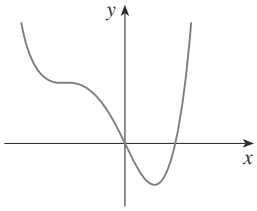

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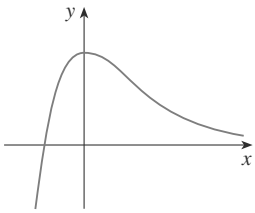

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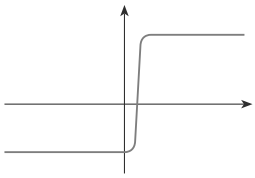

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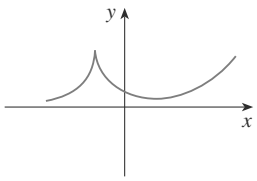

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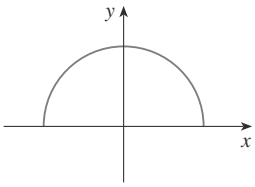

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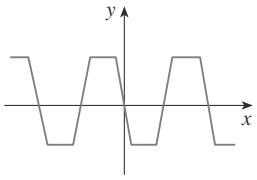

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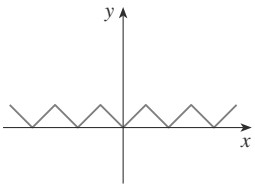

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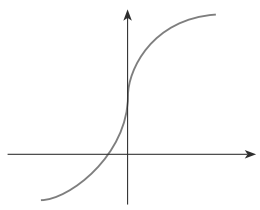
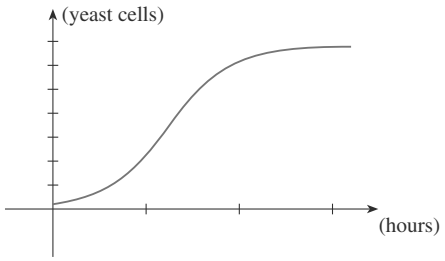
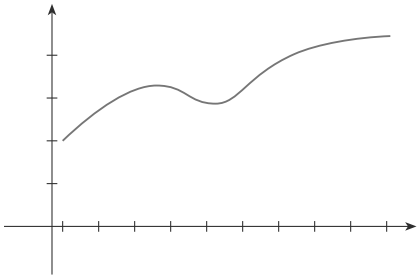


14. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



15. The graph shows how the average age of first marriage of Japanese men varied in the last half of the 20th century. Sketch the graph of the derivative function $M'(t)$. During which years was the derivative negative?



16–18 Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 5–13. Can you guess a formula for $f'(x)$ from its graph?

16. $f(x) = \sin x$

17. $f(x) = e^x$

18. $f(x) = \ln x$

19. Let $f(x) = x^2$.

(a) Estimate the values of $f'(0)$, $f'\left(\frac{1}{2}\right)$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of $f'\left(-\frac{1}{2}\right)$, $f'(-1)$, and $f'(-2)$.

(c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.

(d) Use the definition of a derivative to prove that your guess in part (c) is correct.

20. Let .

(a) Estimate the values of , , , , and by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of , , , and .

(c) Use the values from parts (a) and (b) to graph .

(d) Guess a formula for .

(e) Use the definition of a derivative to prove that your guess in part (d) is correct.

21–31 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

21.

22.

23.

24.

25.

26.

28.

30.

31.

32. (a) Sketch the graph of by starting with the graph of and using the transformations of Section 1.3.

(b) Use the graph from part (a) to sketch the graph of .

(c) Use the definition of a derivative to find . What are the domains of f and ?

(d) Use a graphing device to graph and compare with your sketch in part (b).

33. (a) If , find .

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of and .

34. (a) If $f(t) = 6(1 \quad)$, find $f'(t)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

35. The unemployment rate $U(t)$ varies with time. The table (from the Bureau of Labor Statistics) gives the percentage of unemployed in the U.S. labor force from 1991 to 2000.

(a) What is the meaning of $U'(t)$? What are its units?

(b) Construct a table of values for $U'(t)$.

Geometrically, this means that of all the possible exponential functions $y = a^x$, the function $f(x) = e^x$ is the one whose tangent line at $(0, 1)$ has a slope that is exactly 1. (See Figures 6 and 7.)

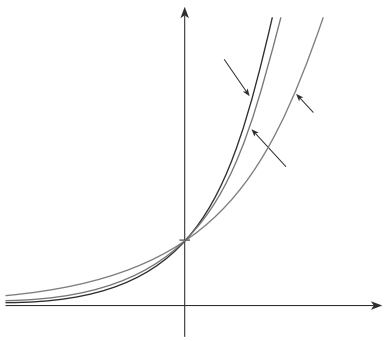


FIGURE 6

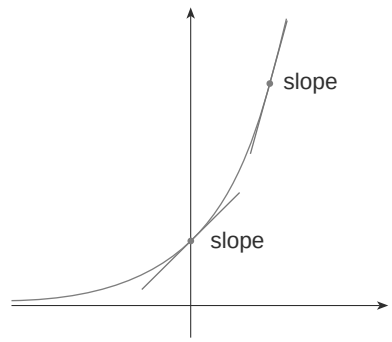


FIGURE 7

If we put $a = e$ and, therefore, $f(0) = 1$ in Equation 4, it becomes the following important differentiation formula.

Derivative of the Natural Exponential Function

$$\frac{d}{dx} e^x = e^x$$

Thus, the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ is equal to the y-coordinate of the point (see Figure 7).

EXAMPLE 1 If $f(x) = e^x - x$, find f'. Compare the graphs of f and f'.

SOLUTION Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx} e^x - \frac{d}{dx} x = e^x - 1$$

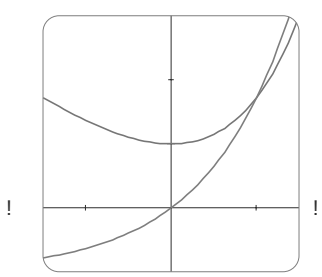


FIGURE 8

The function f and its derivative f' are graphed in Figure 8. Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing. ∎

EXAMPLE 2 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

SOLUTION Since $y = e^x$, we have $y' = e^x$. Let the x-coordinate of the point in question be a. Then the slope of the tangent line at that point is e^a. This tangent line will be parallel to the line $y = 2x$ if it has the same slope, that is, 2. Equating slopes, we get

$$e^a = 2 \qquad a = \ln 2$$

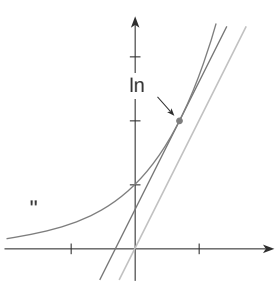


FIGURE 9

Therefore, the required point is $(a, e^a) = (\ln 2, 2)$. (See Figure 9.) ∎

3.1 Exercises

1. (a) How is the number e defined?
   (b) Use a calculator to estimate the values of the limits

$$\lim_{h \to 0} \frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{2.8^h - 1}{h}$$

   correct to two decimal places. What can you conclude about the value of e?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y-axis. What fact allows you to do this?
   (b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g.
   (c) Which of the two functions in part (b) grows more rapidly when x is large?

3–32 Differentiate the function.

3. $f(x) = 186.5$
4. $f(x) = \sqrt{30}$
5. $f(x) = 5x - 1$
6. $F(x) = 4x^{10}$
7. $f(x) = x^2 - 3x + 4$
8. $x = 5x^8 - 2x^5 + 6$
9. $f(t) = \frac{1}{4}t^4 - 8$
10. $f(t) = \frac{1}{2}t^6 - 3t^4 + t$
11. $y = (x^2)^5$
12. $y = 5e^x - 3$
13. $V(r) = \frac{4}{3}\pi r^3$
14. $R(t) = 5t^{3/5}$
15. $Y(t) = 6t^{-9}$
16. $R(x) = \dfrac{\sqrt{10}}{x^7}$
17. $G(x) = \sqrt{x} + 2e^x$
18. $y = \sqrt[3]{x}$
19. $F(x) = \left(\frac{1}{2}x\right)^5$
20. $f(t) = \sqrt{t} - \dfrac{1}{\sqrt{t}}$
21. $x = x^2 - \dfrac{1}{x^2}$
22. $y = \sqrt{x}(x - 1)$
23. $y = \dfrac{x^2 + 4x - 3}{\sqrt{x}}$
24. $y = \dfrac{x^2 + 2\sqrt{x}}{x}$
25. $y = (4 - x)^2$
26. $u = \sqrt{2}u + \sqrt{3u}$
27. $y = ax^2 + bx + c$
28. $y = ae^v + \dfrac{b}{v} + \dfrac{c}{v^2}$
29. $v = t^2 - \dfrac{1}{\sqrt[4]{t^3}}$
30. $u = \sqrt[3]{t^2}(2 + \sqrt{t^3})$
31. $\dfrac{A}{y^{10}} + Be^y$
32. $y = (e^x - 1)^2 + 1$

33–36 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

33. $f(x) = e^x + 5x$
34. $f(x) = 3x^5 - 20x^3 + 50x$
35. $f(x) = 3x^{15} - 5x^3 + 3$
36. $f(x) = x + \dfrac{1}{x}$

37–38 Estimate the value of $f'(a)$ by zooming in on the graph of f. Then differentiate f to find the exact value of $f'(a)$ and compare with your estimate.

37. $f(x) = 3x^2 - x^3, \quad a = 1$
38. $f(x) = 1 + \sqrt{x}, \quad a = 4$

39–40 Find an equation of the tangent line to the curve at the given point.

39. $y = x^4 + 2e^x, \quad (0, 2)$
40. $y = (1 + 2x)^2, \quad (1, 9)$

41–42 Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

41. $y = 3x^2 - x^3, \quad (1, 2)$
42. $y = x + \sqrt{x}, \quad (4, 8)$

43. (a) Use a graphing calculator or computer to graph the function $f(x) = x^4 + 3x^3 - 6x^2 - 7x + 30$ in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.
    (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 1 in Section 2.9.)
    (c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (b).

44. (a) Use a graphing calculator or computer to graph the function $f(x) = e^x + 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.
    (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 1 in Section 2.9.)
    (c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (b).

45. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 1$ where the tangent is horizontal.

46. For what values of x does the graph of $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent?

47. Show that the curve $y = 6x^3 + 5x + 3$ has no tangent line with slope 4.

48. At what point on the curve $y = 1 + 2e^x + 3x$ is the tangent line parallel to the line $3x + y = 5$? Illustrate by graphing the curve and both lines.

49. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, 4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

50. Find equations of both lines through the point $(2, 3)$ that are tangent to the parabola $y = x^2 - x$.

Velocity, density, and current are not the only rates of change that are important in physics. Others include power (the rate at which work is done), the rate of heat ßow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.

Chemistry

EXAMPLE A chemical reaction results in the formation of one or more substances (called *products*) from one or more starting materials (called *reactants*). For instance, the ÒequationÓ

$$2H_2 \quad O_2 \quad 2H_2O$$

indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water. LetÕs consider the reaction

$$A \quad B \quad C$$

where A and B are the reactants and C is the product. The *concentration* of a reactant A is the number of moles (1 mole $6.022 \quad 10^{23}$ molecules) per liter and is denoted by A. The concentration varies during a reaction, so B, C, and are all functions of time t. The average rate of reaction of the product C over a time interval $t_1 \quad t_2$ is

$$\frac{C}{t} \quad \frac{C\ t_2 \quad C\ t_1}{t_2 \quad t_1}$$

But chemists are more interested in the *instantaneous rate of reaction*, which is obtained by taking the limit of the average rate of reaction as the time interval t approaches 0:

$$\text{rate of reaction} \quad \lim_{t \quad 0} \frac{C}{t} \quad \frac{d\ C}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $d\ C\ \ dt$ will be positive. (You can see intuitively that the slope of the tangent to the graph of an increasing function is positive.) Thus, the rate of reaction of C is positive. The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives $d\ A\ \ dt$ and $d\ B\ \ dt$. Since A and B each decrease at the same rate that C increases, we have

$$\text{rate of reaction} \quad \frac{d\ C}{dt} \quad \frac{d\ A}{dt} \quad \frac{d\ B}{dt}$$

More generally, it turns out that for a reaction of the form

$$aA \quad bB \quad cC \quad dD$$

we have

$$\frac{1}{a}\frac{d\ A}{dt} \quad \frac{1}{b}\frac{d\ B}{dt} \quad \frac{1}{c}\frac{d\ C}{dt} \quad \frac{1}{d}\frac{d\ D}{dt}$$

The rate of reaction can be determined by graphical methods (see Exercise 22). In some cases we can use the rate of reaction to find explicit formulas for the concentrations as functions of time (see Exercises 9.3).

EXAMPLE 5 One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P. We can consider the rate of change of volume with respect to pressure, namely, the derivative dV/dP. As P increases, V decreases, so $dV/dP < 0$. The compressibility is defined by introducing a minus sign and dividing this derivative by the volume V:

$$\text{isothermal compressibility} = \frac{1}{V}\frac{dV}{dP}$$

Thus, measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume V (in cubic meters) of a sample of air at 25 °C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

The rate of change of V with respect to P when $P = 50$ kPa is

$$\left.\frac{dV}{dP}\right|_{P=50} = \left.\frac{5.3}{P^2}\right|_{P=50} = \frac{5.3}{2500} = 0.00212 \text{ m}^3/\text{kPa}$$

The compressibility at that pressure is

$$\left.\frac{1}{V}\frac{dV}{dP}\right|_{P=50} = \frac{0.00212}{\frac{5.3}{50}} = 0.02 \text{ m}^3/(\text{kPa}\cdot\text{m}^3)$$

Biology

EXAMPLE 6 Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \le t \le t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The instantaneous rate of growth is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and, therefore, not differentiable. However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in Figure 5.

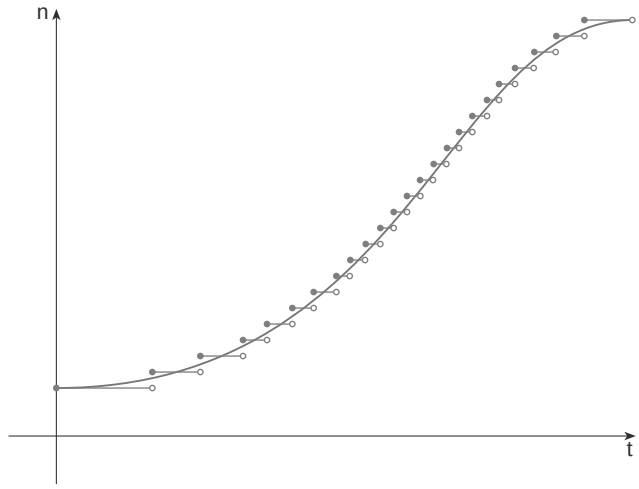


FIGURE 5
A smooth curve approximating a growth function

To be more specific, consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the initial population n_0 is and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$

$$f(2) = 2f(1) = 2^2 n_0$$

$$f(3) = 2f(2) = 2^3 n_0$$

and, in general,

$$f(t) = 2^t n_0$$

The population function is $n = n_0 2^t$.

In Section 3.1 we discussed derivatives of exponential functions and found that

$$\frac{d}{dx} 2^x = 0.69\, 2^x$$

So the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt} n_0 2^t = n_0\, 0.69\, 2^t$$

For example, suppose that we start with an initial population n_0 of 100 bacteria. Then the rate of growth after 4 hours is

$$\left.\frac{dn}{dt}\right|_{t=4} = 100\, 0.69\, 2^4 = 1104$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1100 bacteria per hour.

EXAMPLE 1 When we consider the flow of blood through a blood vessel, such as a vein or artery, we can take the shape of the blood vessel to be a cylindrical tube with radius R and length l as illustrated in Figure 6.

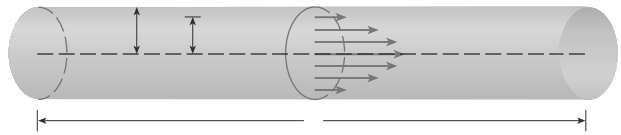


FIGURE 6
Blood flow in an artery

Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relationship between v and r is given by the law of laminar flow discovered by the French physician Jean-Louis-Marie Poiseuille in 1840. This states that

$$v = \frac{P}{4\eta l}(R^2 - r^2) \tag{1}$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$. [For more detailed information, see W. Nichols and M. O'Rourke (eds.), McDonald's Blood Flow in Arteries: Theoretic, Experimental, and Clinical Principles, 4th ed. (New York: Oxford University Press, 1998).]

The average rate of change of the velocity as we move from r_1 outward to r_2 is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let $\Delta r \to 0$, we obtain the velocity gradient, that is, the instantaneous rate of change of velocity with respect to r:

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Using Equation 1, we obtain

$$\frac{dv}{dr} = \frac{P}{4\eta l}(0 - 2r) = -\frac{Pr}{2\eta l}$$

For one of the smaller human arteries we can take $\eta = 0.027$ R, $l = 0.008$ cm, and $P = 4000$ dynes/cm^2, which gives

$$v = \frac{4000}{4(0.027)(2)}(0.000064 - r^2)$$

$$= (1.85 \times 10^4)(6.4 \times 10^{-5} - r^2)$$

At $r = 0.002$ cm the blood is flowing at a speed of

$$v(0.002) = (1.85 \times 10^4)(64 \times 10^{-6} - 4 \times 10^{-6})$$

$$= 1.11 \text{ cm/s}$$

and the velocity gradient at that point is

$$\left.\frac{dv}{dr}\right|_{r=0.002} = \frac{4000(0.002)}{2(0.0272)} = 74 \text{ cm/s/cm}$$

To get a feeling for what this statement means, let us change our units from centimeters to micrometers (1 cm = 10,000 μm). Then the radius of the artery is $r_0 = 80$ μm. The velocity at the central axis is 11,850 μm/s, which decreases to 11,110 μm/s at a distance of $r = 20$ μm. The fact that $dv/dr = -74$ (μm/s)/μm means that, when $r = 20$ μm, the velocity is decreasing at a rate of about 74 μm/s for each micrometer that we proceed away from the center.

Economics

EXAMPLE 8 Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a cost function. If the number of items produced is increased from x_1 to x_2, the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the marginal cost by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since x often takes on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6.]

Taking $\Delta x = 1$ and n large (so that $\Delta x = 1$ is small compared to n), we have

$$C'(n) \approx C(n+1) - C(n)$$

Thus, the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n+1)$st unit].

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to x, but labor costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10{,}000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C\ 500 \quad 5 \quad 0.02\,500 \quad \$15\ \text{item}$$

This gives the rate at which costs are increasing with respect to the production level when $x \quad 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$C\ 501 \quad C\ 500 \quad 10{,}000 \quad 5\ 501 \quad 0.01\,501^2$$

$$10{,}000 \quad 5\ 500 \quad 0.01\,500^2$$

$$\$15.01$$

Notice that $C\ 500 \quad C\ 501 \quad C\ 500$.

Economists also study marginal demand, marginal revenue, and marginal profit, which are the derivatives of the demand, revenue, and profit functions. These will be considered in Chapter 4 after we have developed techniques for finding the maximum and minimum values of functions.

Other Sciences

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An engineer wants to know the rate at which water flows into or out of a reservoir. An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height (see Exercise 17 in Section 9.4).

In psychology, those interested in learning theory study the so-called learning curve, which graphs the performance $P\ t$ of someone learning a skill as a function of the training time t . Of particular interest is the rate at which performance improves as time passes, that is, $dP\ dt$.

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions). If $p\ t$ denotes the proportion of a population that knows a rumor by time t , then the derivative $dp\ dt$ represents the rate of spread of the rumor (see Exercise 70 in Section 3.5).

Summary

Velocity, density, current, power, and temperature gradient in physics, rate of reaction and compressibility in chemistry, rate of growth and blood velocity gradient in biology, marginal cost and marginal profit in economics, rate of heat flow in geology, rate of improvement of performance in psychology, rate of spread of a rumor in sociology: these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier (1768–1830) put it succinctly: “Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.”

3.3 Exercises

1–6 A particle moves according to a law of motion $s = f(t)$, $t \geq 0$, where t is measured in seconds and s in feet.

(a) Find the velocity at time t.
(b) What is the velocity after 3 s?
(c) When is the particle at rest?
(d) When is the particle moving in the positive direction?
(e) Find the total distance traveled during the first 8 s.
(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.

1. $f(t) = t^2 - 10t + 12$
2. $f(t) = t^3 - 9t^2 + 15t - 10$
3. $f(t) = t^3 - 12t^2 + 36t$
4. $f(t) = t^4 - 4t + 1$
5. $s = \dfrac{t}{t^2 + 1}$
6. $s = \bar{t}\ 3t^2 - 35t + 90$

7. The position function of a particle is given by

$$s = t^3 - 4.5t^2 - 7t \qquad t \geq 0$$

When does the particle reach a velocity of 5 m/s?

8. If a ball is given a push so that it has an initial velocity of 5 m/s down a certain inclined plane, then the distance it has rolled after t seconds is $s = 5t + 3t^2$.
   (a) Find the velocity after 2 s.
   (b) How long does it take for the velocity to reach 35 m/s?

9. If a stone is thrown vertically upward from the surface of the moon with a velocity of 10 m/s, its height (in meters) after t seconds is $h = 10t - 0.83t^2$.
   (a) What is the velocity of the stone after 3 s?
   (b) What is the velocity of the stone after it has risen 25 m?

10. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$.
   (a) What is the maximum height reached by the ball?
   (b) What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

11. (a) A company makes computer chips from square wafers of silicon. It wants to keep the side length of a wafer very close to 15 mm and it wants to know how the area $A(x)$ of a wafer changes when the side length x changes. Find $A(15)$ and explain its meaning in this situation.
   (b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain geometrically why this is true by drawing a square whose side length x is increased by an amount Δx. How can you approximate the resulting change in area ΔA if Δx is small?

12. (a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly. If $V(x)$ is the volume of such a cube with side length x, calculate dV/dx when $x = 3$ mm and explain its meaning.
   (b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 11(b).

13. (a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from
   (i) 2 to 3 (ii) 2 to 2.5 (iii) 2 to 2.1
   (b) Find the instantaneous rate of change when $r = 2$.
   (c) Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small?

14. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after (a) 1 s, (b) 3 s, and (c) 5 s. What can you conclude?

15. A spherical balloon is being inflated. Find the rate of increase of the surface area $S = 4\pi r^2$ with respect to the radius r when r is (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make?

16. (a) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1\ \mu m = 10^{-6}$ m). Find the average rate of change of V with respect to r when r changes from
   (i) 5 to 8 μm (ii) 5 to 6 μm (iii) 5 to 5.1 μm
   (b) Find the instantaneous rate of change of V with respect to r when $r = 5$ μm.
   (c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 13(c).

17. The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density (see Example 2) when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?

18. If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000\left(1 - \frac{t}{40}\right)^2 \qquad 0 \leq t \leq 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

19. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is

given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when (a) $t = 0.5$ s and (b) $t = 1$ s. [See Example 3. The unit of current is an ampere (1 A = 1 C/s).] At what time is the current lowest?

20. Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.
(a) Find dF/dr and explain its meaning. What does the minus sign indicate?
(b) Suppose it is known that Earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20{,}000$ km. How fast does this force change when $r = 10{,}000$ km?

21. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant: $PV = C$.
(a) Find the rate of change of volume with respect to pressure.
(b) A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain.
(c) Prove that the isothermal compressibility (see Example 5) is given by $1/P$.

22. The data in the table concern the lactonization of hydroxyvaleric acid at 25°C. They give the concentration $C(t)$ of this acid in moles per liter after t minutes.

t	0	2	4	6	8
$C(t)$	0.0800	0.0570	0.0408	0.0295	0.0210

(a) Find the average rate of reaction for the following time intervals:
(i) $2 \le t \le 6$ (ii) $2 \le t \le 4$ (iii) $0 \le t \le 2$
(b) Plot the points from the table and draw a smooth curve through them as an approximation to the graph of the concentration function. Then draw the tangent at $t = 2$ and use it to estimate the instantaneous rate of reaction when $t = 2$.

23. The table gives the population of the world in the 20th century.

Year	Population (in millions)	Year	Population (in millions)
1900	1650	1960	3040
1910	1750	1970	3710
1920	1860	1980	4450
1930	2070	1990	5280
1940	2300	2000	6080
1950	2560		

(a) Estimate the rate of population growth in 1920 and in 1980 by averaging the slopes of two secant lines.
(b) Use a graphing calculator or computer to find a cubic function (a third-degree polynomial) that models the data. (See Section 1.2.)
(c) Use your model in part (b) to find a model for the rate of population growth in the 20th century.
(d) Use part (c) to estimate the rates of growth in 1920 and 1980. Compare with your estimates in part (a).
(e) Estimate the rate of growth in 1985.

24. The table shows how the average age $A(t)$ of marriage of Japanese women varied in the last half of the 20th century.

t	$A(t)$	t	$A(t)$
1950	23.0	1975	24.7
1955	23.8	1980	25.2
1960	24.4	1985	25.5
1965	24.5	1990	25.9
1970	24.2	1995	26.3

(a) Use a graphing calculator or computer to model these data with a fourth-degree polynomial.
(b) Use part (a) to find a model for $A'(t)$.
(c) Estimate the rate of change of marriage age for women in 1990.
(d) Graph the data points and the models A and A'.

25. If, in Example 4, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value $[A]_0 = [B]_0 = a$ moles/L, then

$$C = a^2kt/(akt + 1)$$

where k is a constant.
(a) Find the rate of reaction at time t.
(b) Show that if $x = [C]$, then

$$\frac{dx}{dt} = k(a - x)^2$$

(c) What happens to the concentration x as $t \to \infty$?
(d) What happens to the rate of reaction as $t \to \infty$?
(e) What do the results of parts (c) and (d) mean in practical terms?

26. Suppose that a bacteria population starts with 500 bacteria and triples every hour.
(a) What is the population after 3 hours? After 4 hours? After t hours?
(b) Use (5) in Section 3.1 to estimate the rate of increase of the bacteria population after 6 hours.

27. Refer to the law of laminar flow given in Example 7. Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 3000 dynes/cm^2, and viscosity $\eta = 0.027$.
(a) Find the velocity of the blood along the centerline ($r = 0$), at radius $r = 0.005$ cm, and at the wall ($r = R = 0.01$ cm).

(b) Find the velocity gradient at $t = 0$, $r = 0.005$, and $r = 0.01$.
(c) Where is the velocity the greatest? Where is the velocity changing most?

28. The frequency of vibrations of a vibrating violin string is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

where L is the length of the string, T is its tension, and ρ is its linear density. [See Chapter 11 in Donald E. Hall, *Musical Acoustics*, 3d ed. (Pacific Grove, CA: Brooks/Cole, 2002).]
(a) Find the rate of change of the frequency with respect to
  (i) the length (when T and ρ are constant),
  (ii) the tension (when L and ρ are constant), and
  (iii) the linear density (when L and T are constant).
(b) The pitch of a note (how high or low the note sounds) is determined by the frequency f. (The higher the frequency, the higher the pitch.) Use the signs of the derivatives in part (a) to determine what happens to the pitch of a note
  (i) when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates,
  (ii) when the tension is increased by turning a tuning peg,
  (iii) when the linear density is increased by switching to another string.

29. Suppose that the cost (in dollars) for a company to produce x pairs of a new line of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$$

(a) Find the marginal cost function.
(b) Find $C'(100)$ and explain its meaning. What does it predict?
(c) Compare $C'(100)$ with the cost of manufacturing the 101st pair of jeans.

30. The cost function for a certain commodity is

$$C(x) = 84 + 0.16x + 0.0006x^2 + 0.000003x^3$$

(a) Find and interpret $C'(100)$.
(b) Compare $C'(100)$ with the cost of producing the 101st item.

31. If $p(x)$ is the total value of the production when there are x workers in a plant, then the *average productivity* of the workforce at the plant is

$$A(x) = \frac{p(x)}{x}$$

(a) Find $A'(x)$. Why does the company want to hire more workers if $A'(x) > 0$?
(b) Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity.

32. If R denotes the reaction of the body to some stimulus of strength x, the *sensitivity* S is defined to be the rate of change of the reaction with respect to x. A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence R of x on R when R is measured in square millimeters and x is measured in appropriate units of brightness.
(a) Find the sensitivity.
(b) Illustrate part (a) by graphing both R and S as functions of x. Comment on the values of R and S at low levels of brightness. Is this what you would expect?

33. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V (in liters) is $PV = nRT$, where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ mol.

34. In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0\left(1 - \frac{P(t)}{P_c}\right)P(t) - \alpha P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the *carrying capacity*), and α is the percentage of the population that is harvested.
(a) What value of dP/dt corresponds to a stable population?
(b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.
(c) What happens if α is raised to 5%?

35. In the study of ecosystems, *predator-prey models* are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bCW \qquad \frac{dW}{dt} = cW - dCW$$

(a) What values of dC/dt and dW/dt correspond to stable populations?
(b) How would the statement "The caribou go extinct" be represented mathematically?
(c) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct?

3.4 Derivatives of Trigonometric Functions

|||| A review of the trigonometric functions is given in Appendix D.

Before starting this section, you might need to review the trigonometric functions. In particular, it is important to remember that when we talk about the function defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that $\sin x$ means the sine of the angle whose radian measure is x. A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot. Recall from Section 2.5 that all of the trigonometric functions are continuous at every number in their domains.

If we sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f' (see Exercise 16 in Section 2.9), then it looks as if the graph of f' may be the same as the cosine curve (see Figure 1 and also page 182).

See an animation of Figure 1.
Resources / Module 4
/ Trigonometric Models
/ Slope-A-Scope for Sine

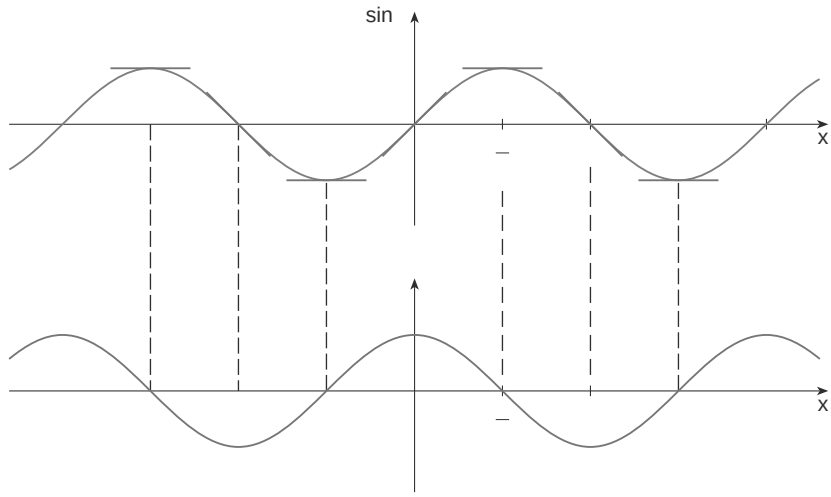


FIGURE 1

Let's try to confirm our guess that if $f(x) = \sin x$, then $f'(x) = \cos x$. From the definition of a derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

|||| We have used the addition formula for sine. See Appendix D.

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}$$

$$= \lim_{h \to 0} \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}$$

$$= \lim_{h \to 0} \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \lim_{h \to 0} \frac{\sin h}{h} \qquad (1)$$

Velocity, density, and current are not the only rates of change that are important in physics. Others include power (the rate at which work is done), the rate of heat ßow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.

Chemistry

EXAMPLE A chemical reaction results in the formation of one or more substances (called *products*) from one or more starting materials (called *reactants*). For instance, the ÒequationÓ

$$2H_2 \quad O_2 \quad 2H_2O$$

indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water. LetÕs consider the reaction

$$A \quad B \quad C$$

where A and B are the reactants and C is the product. The *concentration* of a reactant A is the number of moles (1 mole $6.022 \quad 10^{23}$ molecules) per liter and is denoted by A. The concentration varies during a reaction, so B, C, and are all functions of time t. The average rate of reaction of the product C over a time interval $t_1 \quad t_2$ is

$$\frac{C}{t} \quad \frac{C\ t_2 \quad C\ t_1}{t_2 \quad t_1}$$

But chemists are more interested in the *instantaneous rate of reaction*, which is obtained by taking the limit of the average rate of reaction as the time interval t approaches 0:

$$\text{rate of reaction} \quad \lim_{t \quad 0} \frac{C}{t} \quad \frac{d\ C}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $d\ C\ \ dt$ will be positive. (You can see intuitively that the slope of the tangent to the graph of an increasing function is positive.) Thus, the rate of reaction of C is positive. The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives $d\ A\ \ dt$ and $d\ B\ \ dt$. Since A and B each decrease at the same rate that C increases, we have

$$\text{rate of reaction} \quad \frac{d\ C}{dt} \quad \frac{d\ A}{dt} \quad \frac{d\ B}{dt}$$

More generally, it turns out that for a reaction of the form

$$aA \quad bB \quad cC \quad dD$$

we have

$$\frac{1}{a}\frac{d\ A}{dt} \quad \frac{1}{b}\frac{d\ B}{dt} \quad \frac{1}{c}\frac{d\ C}{dt} \quad \frac{1}{d}\frac{d\ D}{dt}$$

The rate of reaction can be determined by graphical methods (see Exercise 22). In some cases we can use the rate of reaction to find explicit formulas for the concentrations as functions of time (see Exercises 9.3).

EXAMPLE 5 One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P. We can consider the rate of change of volume with respect to pressure, namely, the derivative dV/dP. As P increases, V decreases, so $dV/dP < 0$. The compressibility is defined by introducing a minus sign and dividing this derivative by the volume V:

$$\text{isothermal compressibility} = \frac{1}{V}\frac{dV}{dP}$$

Thus, measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume V (in cubic meters) of a sample of air at 25 °C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

The rate of change of V with respect to P when $P = 50$ kPa is

$$\left.\frac{dV}{dP}\right|_{P=50} = \left.\frac{5.3}{P^2}\right|_{P=50} = \frac{5.3}{2500} = 0.00212 \text{ m}^3/\text{kPa}$$

The compressibility at that pressure is

$$\left.\frac{1}{V}\frac{dV}{dP}\right|_{P=50} = \frac{0.00212}{\frac{5.3}{50}} = 0.02 \text{ m}^3/(\text{kPa}\cdot\text{m}^3)$$

Biology

EXAMPLE 6 Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \le t \le t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The instantaneous rate of growth is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and, therefore, not differentiable. However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in Figure 5.

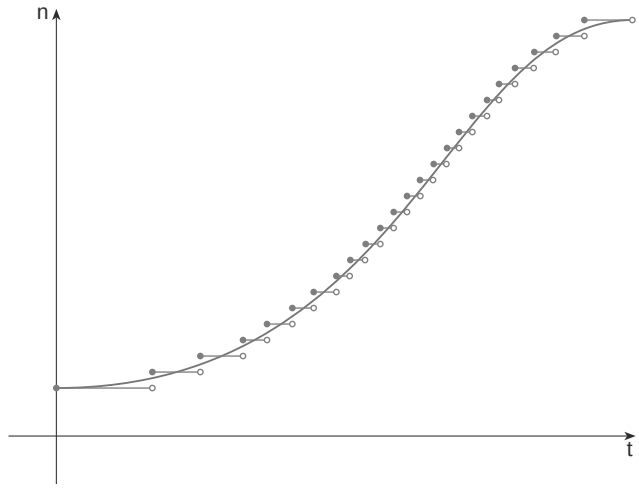


FIGURE 5
A smooth curve approximating a growth function

To be more specific, consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the initial population n_0 is and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$

$$f(2) = 2f(1) = 2^2 n_0$$

$$f(3) = 2f(2) = 2^3 n_0$$

and, in general,

$$f(t) = 2^t n_0$$

The population function is $n = n_0 2^t$.

In Section 3.1 we discussed derivatives of exponential functions and found that

$$\frac{d}{dx} 2^x = 0.69\, 2^x$$

So the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt} n_0 2^t = n_0\, 0.69\, 2^t$$

For example, suppose that we start with an initial population n_0 of 100 bacteria. Then the rate of growth after 4 hours is

$$\left.\frac{dn}{dt}\right|_{t=4} = 100\, 0.69\, 2^4 = 1104$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1100 bacteria per hour.

EXAMPLE 1 When we consider the flow of blood through a blood vessel, such as a vein or artery, we can take the shape of the blood vessel to be a cylindrical tube with radius R and length l as illustrated in Figure 6.

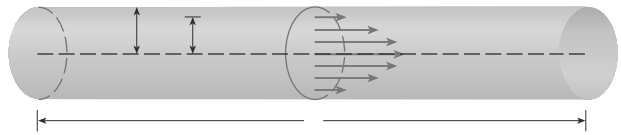


FIGURE 6
Blood flow in an artery

Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relationship between v and r is given by the law of laminar flow discovered by the French physician Jean-Louis-Marie Poiseuille in 1840. This states that

$$v = \frac{P}{4\eta l}(R^2 - r^2) \tag{1}$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$. [For more detailed information, see W. Nichols and M. O'Rourke (eds.), McDonald's Blood Flow in Arteries: Theoretic, Experimental, and Clinical Principles, 4th ed. (New York: Oxford University Press, 1998).]

The average rate of change of the velocity as we move from r_1 outward to r_2 is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let $\Delta r \to 0$, we obtain the velocity gradient, that is, the instantaneous rate of change of velocity with respect to r:

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Using Equation 1, we obtain

$$\frac{dv}{dr} = \frac{P}{4\eta l}(0 - 2r) = -\frac{Pr}{2\eta l}$$

For one of the smaller human arteries we can take $\eta = 0.027$ R, $l = 0.008$ cm, and $P = 4000$ dynes/cm^2, which gives

$$v = \frac{4000}{4(0.027)(2)}(0.000064 - r^2)$$

$$= (1.85 \times 10^4)(6.4 \times 10^{-5} - r^2)$$

At $r = 0.002$ cm the blood is flowing at a speed of

$$v(0.002) = (1.85 \times 10^4)(64 \times 10^{-6} - 4 \times 10^{-6})$$

$$= 1.11 \text{ cm/s}$$

and the velocity gradient at that point is

$$\left.\frac{dv}{dr}\right|_{r=0.002} = \frac{4000(0.002)}{2(0.0272)} = 74 \text{ cm/s/cm}$$

To get a feeling for what this statement means, let us change our units from centimeters to micrometers (1 cm = 10,000 μm). Then the radius of the artery is $r_0 = 80$ μm. The velocity at the central axis is 11,850 μm/s, which decreases to 11,110 μm/s at a distance of $r = 20$ μm. The fact that $dv/dr = -74$ (μm/s)/μm means that, when $r = 20$ μm, the velocity is decreasing at a rate of about 74 μm/s for each micrometer that we proceed away from the center.

Economics

EXAMPLE 8 Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a cost function. If the number of items produced is increased from x_1 to x_2, the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the marginal cost by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since x often takes on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6.]

Taking $\Delta x = 1$ and n large (so that $\Delta x = 1$ is small compared to n), we have

$$C'(n) \approx C(n+1) - C(n)$$

Thus, the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n+1)$st unit].

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to x, but labor costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10{,}000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C\ 500 \quad 5 \quad 0.02\,500 \quad \$15\ \text{item}$$

This gives the rate at which costs are increasing with respect to the production level when $x \quad 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$C\ 501 \quad C\ 500 \quad 10{,}000 \quad 5\ 501 \quad 0.01\,501^2$$

$$10{,}000 \quad 5\ 500 \quad 0.01\,500^2$$

$$\$15.01$$

Notice that $C\ 500 \quad C\ 501 \quad C\ 500$.

Economists also study marginal demand, marginal revenue, and marginal profit, which are the derivatives of the demand, revenue, and profit functions. These will be considered in Chapter 4 after we have developed techniques for finding the maximum and minimum values of functions.

Other Sciences

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An engineer wants to know the rate at which water flows into or out of a reservoir. An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height (see Exercise 17 in Section 9.4).

In psychology, those interested in learning theory study the so-called learning curve, which graphs the performance $P\ t$ of someone learning a skill as a function of the training time t . Of particular interest is the rate at which performance improves as time passes, that is, $dP\ dt$.

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions). If $p\ t$ denotes the proportion of a population that knows a rumor by time t , then the derivative $dp\ dt$ represents the rate of spread of the rumor (see Exercise 70 in Section 3.5).

Summary

Velocity, density, current, power, and temperature gradient in physics, rate of reaction and compressibility in chemistry, rate of growth and blood velocity gradient in biology, marginal cost and marginal profit in economics, rate of heat flow in geology, rate of improvement of performance in psychology, rate of spread of a rumor in sociology: these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier (1768–1830) put it succinctly: “Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.”

3.3 Exercises

1–6 A particle moves according to a law of motion $s = f(t)$, $t \geq 0$, where t is measured in seconds and s in feet.

(a) Find the velocity at time t.
(b) What is the velocity after 3 s?
(c) When is the particle at rest?
(d) When is the particle moving in the positive direction?
(e) Find the total distance traveled during the first 8 s.
(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.

1. $f(t) = t^2 - 10t + 12$
2. $f(t) = t^3 - 9t^2 + 15t - 10$
3. $f(t) = t^3 - 12t^2 + 36t$
4. $f(t) = t^4 - 4t + 1$
5. $s = \dfrac{t}{t^2 + 1}$
6. $s = \bar{t}\ 3t^2 - 35t + 90$

7. The position function of a particle is given by

$$s = t^3 - 4.5t^2 - 7t \qquad t \geq 0$$

When does the particle reach a velocity of 5 m/s?

8. If a ball is given a push so that it has an initial velocity of 5 m/s down a certain inclined plane, then the distance it has rolled after t seconds is $s = 5t + 3t^2$.
   (a) Find the velocity after 2 s.
   (b) How long does it take for the velocity to reach 35 m/s?

9. If a stone is thrown vertically upward from the surface of the moon with a velocity of 10 m/s, its height (in meters) after t seconds is $h = 10t - 0.83t^2$.
   (a) What is the velocity of the stone after 3 s?
   (b) What is the velocity of the stone after it has risen 25 m?

10. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$.
   (a) What is the maximum height reached by the ball?
   (b) What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

11. (a) A company makes computer chips from square wafers of silicon. It wants to keep the side length of a wafer very close to 15 mm and it wants to know how the area $A(x)$ of a wafer changes when the side length x changes. Find $A(15)$ and explain its meaning in this situation.
   (b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain geometrically why this is true by drawing a square whose side length x is increased by an amount Δx. How can you approximate the resulting change in area ΔA if Δx is small?

12. (a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly. If $V(x)$ is the volume of such a cube with side length x, calculate dV/dx when $x = 3$ mm and explain its meaning.
   (b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 11(b).

13. (a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from
   (i) 2 to 3 (ii) 2 to 2.5 (iii) 2 to 2.1
   (b) Find the instantaneous rate of change when $r = 2$.
   (c) Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small?

14. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after (a) 1 s, (b) 3 s, and (c) 5 s. What can you conclude?

15. A spherical balloon is being inflated. Find the rate of increase of the surface area $S = 4\pi r^2$ with respect to the radius r when r is (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make?

16. (a) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1\ \mu m = 10^{-6}$ m). Find the average rate of change of V with respect to r when r changes from
   (i) 5 to 8 μm (ii) 5 to 6 μm (iii) 5 to 5.1 μm
   (b) Find the instantaneous rate of change of V with respect to r when $r = 5$ μm.
   (c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 13(c).

17. The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density (see Example 2) when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?

18. If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000\left(1 - \frac{t}{40}\right)^2 \qquad 0 \leq t \leq 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

19. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is

given by . Find the current when
(a) s and (b) s. [See Example 3. The unit of cur-
rent is an ampere (A CRATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES

Proof of the Chain Rule Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y, then we can use Equation 7 to write

$$\boxed{8} \qquad \Delta u = g'(a)\,\Delta x + \varepsilon_1 \Delta x = [g'(a) + \varepsilon_1]\,\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly

$$\boxed{9} \qquad \Delta y = f'(b)\,\Delta u + \varepsilon_2 \Delta u = [f'(b) + \varepsilon_2]\,\Delta u$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. If we now substitute the expression for Δu from Equation 8 into Equation 9, we get

$$\Delta y = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\,\Delta x$$

so

$$\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

As $\Delta x \to 0$, Equation 8 shows that $\Delta u \to 0$. So both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$. Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

$$= f'(b)\,g'(a) = f'(g(a))\,g'(a)$$

This proves the Chain Rule. ∎

3.5 Exercises

1–6 Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx.

1. $y = \sin 4x$
2. $y = \sqrt{4 - 3x}$
3. $y = (1 + x^2)^{10}$
4. $y = \tan(\sin x)$
5. $y = e^{\sqrt{x}}$
6. $y = \sin(e^x)$

7–42 Find the derivative of the function.

7. $F(x) = (x^3 - 4x)^7$
8. $F(x) = (x^2 + x + 1)^3$
9. $F(x) = \sqrt[4]{1 + 2x + x^3}$
10. $f(x) = (1 + x^4)^{2/3}$
11. $t(t) = \dfrac{1}{(t^4 + 1)^3}$
12. $f(t) = \sqrt[3]{1 + \tan t}$
13. $y = \cos(a^3 + x^3)$
14. $y = a^3 + \cos^3 x$
15. $y = e^{mx}$
16. $y = 4\sec(5x)$
17. $x(t) = (1 + 4x)^5(3 - x + x^2)^8$
18. $h(t) = (t^4 + 1)^3(t^3 + 1)^4$
19. $y = (2x - 5)^4(8x^2 + 5)^{-3}$
20. $y = (x^2 + 1)^3\sqrt{x^2 + 2}$
21. $y = xe^{-x^2}$
22. $y = e^{-5x}\cos 3x$
23. $y = e^{x\cos x}$
24. $y = 10^{(1 - x^2)}$
25. $F(t) = \sqrt{\dfrac{t - 1}{t + 1}}$
26. $G(y) = \dfrac{(y - 1)^4}{(y^2 + 2y)^5}$
27. $y = \dfrac{r}{\sqrt{r^2 + 1}}$
28. $y = \dfrac{e^{2u}}{(e^u + e^{-u})^2}$
29. $y = \tan(\cos x)$
30. $y = \dfrac{\sin^2 x}{\cos x}$
31. $y = 2^{\sin x}$
32. $y = \tan^2(3\theta)$
33. $y = (1 + \cos^2 x)^6$
34. $y = x\sin\dfrac{1}{x}$
35. $y = \sec^2 x - \tan^2 x$
36. $y = e^{k\tan\sqrt{x}}$
37. $y = \cot^2(\sin\theta)$
38. $y = \sin(\sin(\sin x))$
39. $y = \sqrt{x + \sqrt{x}}$
40. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
41. $y = \sin(\tan\sqrt{\sin x})$
42. $y = 2^{3^{x^2}}$

43–46 Find an equation of the tangent line to the curve at the given point.

43. $y = 1 + 2x^{10}$, $(0, 1)$

44. $y = \sin x + \sin^2 x$, $(0, 0)$

45. $y = \sin \sin x$, $(\pi, 0)$

46. $y = x^2 e^{-x}$, $(1, 1/e)$

47. (a) Find an equation of the tangent line to the curve $y = 2/(1 + e^{-x})$ at the point $(0, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

48. (a) The curve $y = x\sqrt{2 - x^2}$ is called a bullet-nose curve. Find an equation of the tangent line to this curve at the point $(1, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

49. (a) If $f(x) = \sqrt{1 - x^2}$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

50. The function $f(x) = \sin x + \sin 2x$, $0 \le x \le \pi$, arises in applications to frequency modulation (FM) synthesis.
(a) Use a graph of f produced by a graphing device to make a rough sketch of the graph of f'.
(b) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (a).

51. Find all points on the graph of the function

$$f(x) = 2\sin x + \sin^2 x$$

at which the tangent line is horizontal.

52. Find the x-coordinates of all points on the curve $y = \sin 2x - 2\sin x$ at which the tangent line is horizontal.

53. Suppose that $F(x) = f(x)g(x)$ and $g(3) = 6$, $g'(3) = 4$, $f(3) = 2$, and $f'(6) = 7$. Find $F'(3)$.

54. Suppose that $w = u/v$ and $u(0) = 1$, $v(0) = 2$, $u'(0) = 3$, $u'(2) = 4$, $v'(0) = 5$, and $v'(2) = 6$. Find $w'(0)$.

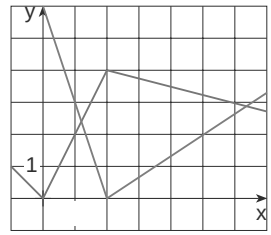
55. A table of values for f, f', g, and g' is given.

x	$f(x)$	x	$f'(x)$	x
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

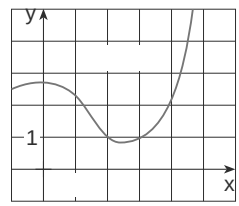
(a) If $h(x) = f(x)g(x)$, find $h'(1)$.
(b) If $H(x) = f(x)/g(x)$, find $H'(1)$.

56. Let f and g be the functions in Exercise 55.
(a) If $F(x) = f(f(x))$, find $F'(2)$.
(b) If $G(x) = x/g(x)$, find $G'(3)$.

57. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$, $v(x) = f(x)/g(x)$, and $w(x) = g(x)/f(x)$. Find each derivative, if it exists. If it does not exist, explain why.
(a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$



58. If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.
(a) $h'(2)$ (b) $g'(2)$



59. Use the table to estimate the value of $h'(0.5)$, where $h(x) = f(f(x))$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	12.6	14.8	18.4	23.0	25.9	27.5	29.1
$f'(x)$	0.58	0.40	0.37	0.26	0.17	0.10	0.05

60. If $g(x) = f(f(x))$, use the table to estimate the value of $g'(1)$.

x	0.0	0.5	1.0	1.5	2.0	2.5
$f(x)$	1.7	1.8	2.0	2.4	3.1	4.4

61. Suppose f is differentiable on $\mathbb{R}$. Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

62. Suppose f is differentiable on $\mathbb{R}$ and a is a real number. Let $F(x) = f(x) - ax$ and $G(x) = f(x) - x$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

63. Suppose L is a function such that $L(x) = 1 + x$ for $x \ge 0$. Find an expression for the derivative of each function.
(a) $f(x) = L(x^4)$ (b) $g(x) = L(4x)$
(c) $F(x) = L(x^4)$ (d) $G(x) = L(1 + x)$

64. Let $r(x) = f(h(x))$, where $h(1) = 2$, $h'(1) = 4$, $h'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

65. The displacement of a particle on a vibrating string is given by the equation

$$s(t) = 10 + \tfrac{1}{4}\sin(10\pi t)$$

where s is measured in centimeters and t in seconds. Find the velocity of the particle after t seconds.

66. If the equation of motion of a particle is given by $s = A\cos(\omega t + \delta)$, the particle is said to undergo simple harmonic motion.
(a) Find the velocity of the particle at time t.
(b) When is the velocity 0?

67. A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by 0.35. In view of these data, the brightness of Delta Cephei at time t, where t is measured in days, has been modeled by the function

$$B(t) = 4.0 + 0.35\sin(2\pi t/5.4)$$

(a) Find the rate of change of the brightness after t days.
(b) Find, correct to two decimal places, the rate of increase after one day.

68. In Example 4 in Section 1.3 we arrived at a model for the length of daylight (in hours) in Philadelphia on the tth day of the year:

$$L(t) = 12 + 2.8\sin\left(\frac{2\pi}{365}(t - 80)\right)$$

Use this model to compare how the number of hours of daylight is increasing in Philadelphia on March 21 and May 21.

69. The motion of a spring that is subject to a frictional force or a damping force (such as a shock absorber in a car) is often modeled by the product of an exponential function and a sine or cosine function. Suppose the equation of motion of a point on such a spring is

$$s(t) = 2e^{-1.5t}\sin(2\pi t)$$

where s is measured in centimeters and t in seconds. Find the velocity after t seconds and graph both the position and velocity functions for $0 \le t \le 2$.

70. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In Section 9.5 we will see that this is a reasonable equation for $p(t)$.]
(a) Find $\lim_{t\to\infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

71. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

(a) Use a graphing calculator or computer to find an exponential model for the charge. (See Section 1.5.)
(b) The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when $t = 0.04$ s. Compare with the result of Example 2 in Section 2.1.

72. The table gives the U.S. population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?
(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.
(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).
(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

CAS 73. Computer algebra systems have commands that differentiate functions, but the form of the answer may not be convenient and so further commands may be necessary to simplify the answer.
(a) Use a CAS to find the derivative in Example 5 and compare with the answer in that example. Then use the simplify command and compare again.
(b) Use a CAS to find the derivative in Example 6. What happens if you use the simplify command? What happens if you use the factor command? Which form of the answer would be best for locating horizontal tangents?

CAS 74. (a) Use a CAS to differentiate the function

$$f(x) = \frac{\sqrt{x^4 + x + 1}}{x^4 + x + 1}$$

and to simplify the result.

(b) Where does the graph of f have horizontal tangents?

(c) Graph f and f' on the same screen. Are the graphs consistent with your answer to part (b)?

75. Use the Chain Rule to prove the following.

(a) The derivative of an even function is an odd function.

(b) The derivative of an odd function is an even function.

76. Use the Chain Rule and the Product Rule to give an alternative proof of the Quotient Rule.
[Hint: Write $f(x) = x \cdot f(x) \cdot x^{-1}$.]

77. (a) If n is a positive integer, prove that

$$\frac{d}{dx}\sin^n x \cos nx = n \sin^{n-1} x \cos(n+1)x$$

(b) Find a formula for the derivative of

$$y = \cos^n x \cos nx$$

that is similar to the one in part (a).

78. Suppose $y = f(x)$ is a curve that always lies above x-the -axis and never has a horizontal tangent, where f is differentiable everywhere. For what value of y is the rate of change of y^5 with respect to x eighty times the rate of change of y with respect to x?

79. Use the Chain Rule to show that if x is measured in degrees, then

$$\frac{d}{dx}\sin x = \frac{\pi}{180}\cos x$$

(This gives one reason for the convention that radian measure is always used when dealing with trigonometric functions in calculus: The differentiation formulas would not be as simple if we used degree measure.)

80. (a) Write $|x| = \sqrt{x^2}$ and use the Chain Rule to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}$$

(b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of f and f'. Where is f not differentiable?

(c) If $g(x) = \sin|x|$, find $g'(x)$ and sketch the graphs of g and g'. Where is g not differentiable?

81. Suppose P and Q are polynomials and n is a positive integer. Use mathematical induction to prove that the nth derivative of the rational function $f(x) = \dfrac{P(x)}{Q(x)}$ can be written as a rational function with denominator $[Q(x)]^{n+1}$. In other words, there is a polynomial A_n such that $f^{(n)}(x) = \dfrac{A_n(x)}{[Q(x)]^{n+1}}$.

3.6 Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y, such as

$$x^2 + y^2 = 25 \tag{1}$$

or

$$x^3 + y^3 = 6xy \tag{2}$$

In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x. For instance, if we solve Equation 1 for y, we get $y = \sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$. The graphs of f and g are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)

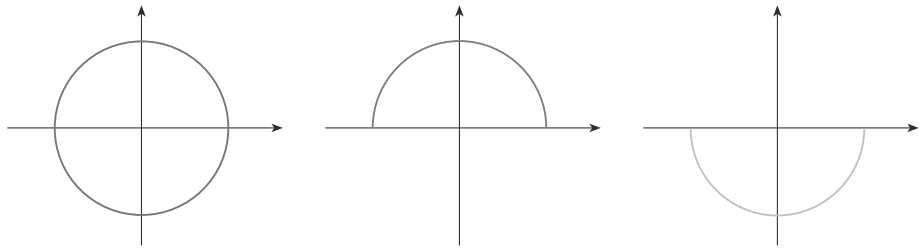


FIGURE 1 (a) (b) (c)

It's not easy to solve Equation 2 for y explicitly as a function of x by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.) Nonetheless, (2) is the equation of a curve called the folium of Descartes shown in Figure 2 and it implicitly defines y as several functions of x. The graphs of three such functions are shown in Figure 3. When we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6x\,f(x)$$

is true for all values of x in the domain of f.

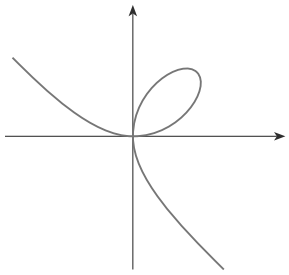


FIGURE 2 The folium of Descartes

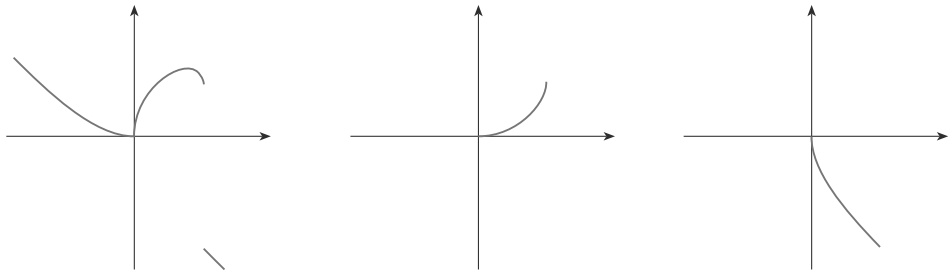


FIGURE 3 Graphs of three functions defined by the folium of Descartes

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of *implicit differentiation*. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'. In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

EXAMPLE 1

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

SOLUTION 1

(a) Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}\,y^2 = \frac{d}{dy}\,y^2\,\frac{dy}{dx} = 2y\,\frac{dy}{dx}$$

Thus $$2x + 2y\,\frac{dy}{dx} = 0$$

Now we solve this equation for dy/dx:

$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) At the point $(3, 4)$ we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at $(3, 4)$ is therefore

$$y - 4 = -\tfrac{3}{4}(x - 3) \quad \text{or} \quad 3x + 4y = 25$$

SOLUTION 2

(b) Solving the equation $x^2 + y^2 = 25$, we get $y = \sqrt{25 - x^2}$. The point $(3, 4)$ lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$. Differentiating f using the Chain Rule, we have

$$f'(x) = \tfrac{1}{2}(25 - x^2)^{-1/2}\,\frac{d}{dx}(25 - x^2)$$

$$= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}$$

So $$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and, as in Solution 1, an equation of the tangent is $3x + 4y = 25$. ∎

NOTE 1 Example 1 illustrates that even when it is possible to solve an equation explicitly for y in terms of x, it may be easier to use implicit differentiation.

NOTE 2 The expression $dy/dx = -x/y$ gives the derivative in terms of both x and y. It is correct no matter which function y is determined by the given equation. For instance, for $y = f(x) = \sqrt{25 - x^2}$ we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{25 - x^2}}$$

whereas for $y = -\sqrt{25 - x^2}$ we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{-\sqrt{25 - x^2}} = \frac{x}{\sqrt{25 - x^2}}$$

EXAMPLE 2

(a) Find y if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

(c) At what points on the curve is the tangent line horizontal?

SOLUTION

(a) Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the y^3 term and the Product Rule on the $6xy$ term, we get

$$3x^2 + 3y^2 y' = 6y + 6xy'$$

or

$$x^2 + y^2 y' = 2y + 2xy'$$

We now solve for y':

$$y^2 y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)\, y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

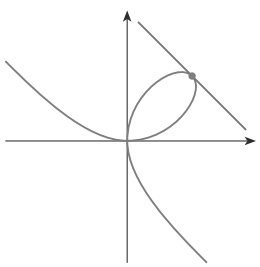


FIGURE 4

(b) When $x = y = 3$,

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

and a glance at Figure 4 confirms that this is a reasonable value for the slope at $(3, 3)$. So an equation of the tangent to the folium at $(3, 3)$ is

$$y - 3 = -1(x - 3) \quad \text{or} \quad x + y = 6$$

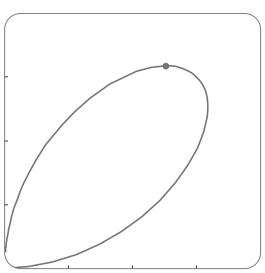


FIGURE 5

(c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $2y - x^2 = 0$. Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get

$$x^3 + \left(\tfrac{1}{2}x^2\right)^3 = 6x\left(\tfrac{1}{2}x^2\right)$$

which simplifies to $x^6 = 16x^3$. So either $x = 0$ or $x^3 = 16$. If $x = 16^{1/3} = 2^{4/3}$, then $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$. Thus, the tangent is horizontal at $(0, 0)$ and at $(2^{4/3}, 2^{5/3})$, which is approximately (2.5198, 3.1748). Looking at Figure 5, we see that our answer is reasonable. ∎

NOTE 3 There is a formula for the three roots of a cubic equation that is like the quadratic formula but much more complicated. If we use this formula (or a computer algebra system) to solve the equation $x^3 + y^3 = 6xy$ for y in terms of x, we get three functions determined by the equation:

$$y = f(x) = \sqrt[3]{\tfrac{1}{2}x^3 + \sqrt{\tfrac{1}{4}x^6 - 8x^3}} + \sqrt[3]{\tfrac{1}{2}x^3 - \sqrt{\tfrac{1}{4}x^6 - 8x^3}}$$

and

$$y = \tfrac{1}{2}\left[f(x) \pm \sqrt{3}\left(\sqrt[3]{\tfrac{1}{2}x^3 + \sqrt{\tfrac{1}{4}x^6 - 8x^3}} - \sqrt[3]{\tfrac{1}{2}x^3 - \sqrt{\tfrac{1}{4}x^6 - 8x^3}}\right)\right]$$

(These are the three functions whose graphs are shown in Figure 3.) You can see that the method of implicit differentiation saves an enormous amount of work in cases such as this.

|||| The Norwegian mathematician Niels Abel proved in 1824 that no general formula can be given for the roots of a fifth-degree equation in terms of radicals. Later the French mathematician Evariste Galois proved that it is impossible to find a general formula for the roots of an nth-degree equation (in terms of algebraic operations on the coefficients) if n is any integer larger than 4.

Moreover, implicit differentiation works just as easily for equations such as

$$y^5 \quad 3x^2y^2 \quad 5x^4 \quad 12$$

for which it is impossible to find a similar expression for y in terms of x.

EXAMPLE 5 Find y if $\sin x \quad y \quad y^2 \cos x$.

SOLUTION Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos x \quad y \quad 1 \quad y \quad 2yy \cos x \quad y^2 \quad \sin x$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y, we get

$$\cos x \quad y \quad y^2 \sin x \quad 2y \cos x\, y \quad \cos x \quad y \quad y$$

So

$$y \quad \frac{y^2 \sin x \quad \cos x \quad y}{2y \cos x \quad \cos x \quad y}$$

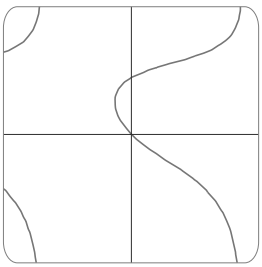


FIGURE 6

Figure 6, drawn with the implicit-plotting command of a computer algebra system, shows part of the curve $\sin x \quad y \quad y^2 \cos x$. As a check on our calculation, notice that $y \quad 1$ when $x \quad y \quad 0$ and it appears from the graph that the slope is approximately 1 at the origin.

Orthogonal Trajectories

Two curves are called *orthogonal* if at each point of intersection their tangent lines are perpendicular. In the next example we use implicit differentiation to show that two families of curves are *orthogonal trajectories* of each other; that is, every curve in one family is orthogonal to every curve in the other family. Orthogonal families arise in several areas of physics. For example, the lines of force in an electrostatic field are orthogonal to the lines of constant potential. In thermodynamics, the isotherms (curves of equal temperature) are orthogonal to the flow lines of heat. In aerodynamics, the streamlines (curves of direction of airflow) are orthogonal trajectories of the velocity-equipotential curves.

EXAMPLE 7 The equation

$$xy \quad c \qquad c \quad 0 \tag{3}$$

represents a family of hyperbolas. (Different values of the constant c give different hyperbolas. See Figure 7.) The equation

$$x^2 \quad y^2 \quad k \qquad k \quad 0 \tag{4}$$

represents another family of hyperbolas with asymptotes $x \quad y$. Show that every curve in the family (3) is orthogonal to every curve in the family (4); that is, the families are orthogonal trajectories of each other.

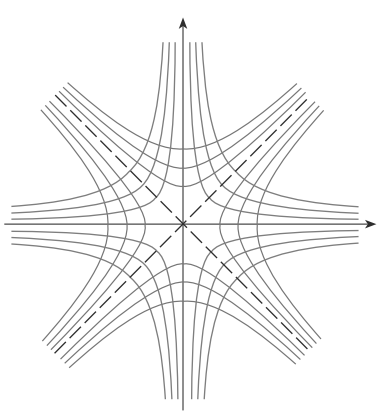


FIGURE 7

SOLUTION Implicit differentiation of Equation 3 gives

$$x \frac{dy}{dx} \quad y \quad 0 \qquad \text{so} \qquad \frac{dy}{dx} \quad \frac{y}{x} \tag{5}$$

Implicit differentiation of Equation 4 gives

$$2x \quad 2y\frac{dy}{dx} \quad 0 \qquad \text{so} \qquad \frac{dy}{dx} \quad \frac{x}{y} \tag{6}$$

From (5) and (6) we see that at any point of intersection of curves from each family, the slopes of the tangents are negative reciprocals of each other. Therefore, the curves intersect at right angles; that is, they are orthogonal.

Derivatives of Inverse Trigonometric Functions

The inverse trigonometric functions were reviewed in Section 1.6. We discussed their continuity in Section 2.5 and their asymptotes in Section 2.6. Here we use implicit differentiation to find the derivatives of the inverse trigonometric functions, assuming that these functions are differentiable. [In fact, if f is any one-to-one differentiable function, it can be proved that its inverse function f^{1} is also differentiable, except where its tangents are vertical. This is plausible because the graph of a differentiable function has no corner or kink and so if we reflect it about $y \quad x$, the graph of its inverse function also has no corner or kink.]

Recall the definition of the arcsine function:

$$y \quad \sin^{1}x \qquad \text{means} \qquad \sin y \quad x \quad \text{and} \quad \frac{}{2} \quad y \quad \frac{}{2}$$

Differentiating $\sin y \quad x$ implicitly with respect to x, we obtain

$$\cos y\frac{dy}{dx} \quad 1 \qquad \text{or} \qquad \frac{dy}{dx} \quad \frac{1}{\cos y}$$

Now $\cos y \quad 0$, since $\quad 2 \quad y \quad 2$, so

$$\cos y \quad \sqrt{1 \quad \sin^2 y} \quad \sqrt{1 \quad x^2}$$

Therefore

$$\frac{dy}{dx} \quad \frac{1}{\cos y} \quad \frac{1}{\sqrt{1 \quad x^2}}$$

$$\boxed{\frac{d}{dx}\sin^{1}x \quad \frac{1}{\sqrt{1 \quad x^2}}}$$

The same method can be used to find a formula for the derivative of any inverse function. See Exercise 67.

Figure 8 shows the graph of $f(x) \quad \tan^{1}x$ and its derivative $f'(x) \quad 1 \quad 1 \quad x^2$. Notice that f is increasing and $f'(x)$ is always positive. The fact that $\tan^{1}x \quad 2$ as x is reflected in the fact that $f'(x) \quad 0$ as x .

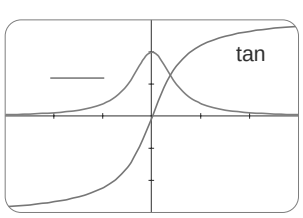


FIGURE 8

The formula for the derivative of the arctangent function is derived in a similar way. If $y \quad \tan^{1}x$, then $\tan y \quad x$. Differentiating this latter equation implicitly with respect to x, we have

$$\sec^2 y\frac{dy}{dx} \quad 1$$

$$\frac{dy}{dx} \quad \frac{1}{\sec^2 y} \quad \frac{1}{1 \quad \tan^2 y} \quad \frac{1}{1 \quad x^2}$$

$$\boxed{\frac{d}{dx}\tan^{1}x \quad \frac{1}{1 \quad x^2}}$$

EXAMPLE 5 Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x \arctan \sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}x\right)^{-1} = -\left(\sin^{-1}x\right)^{-2}\frac{d}{dx}\sin^{-1}x$$
$$= \frac{-1}{\left(\sin^{-1}x\right)^{2}\sqrt{1-x^2}}$$

|||| Recall that $\arctan x$ is an alternative notation for $\tan^{-1}x$.

(b)
$$f'(x) = x\frac{1}{1+\left(\sqrt{x}\right)^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$
$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$$

The inverse trigonometric functions that occur most frequently are the ones that we have just discussed. The derivatives of the remaining four are given in the following table. The proofs of the formulas are left as exercises.

Derivatives of Inverse Trigonometric Functions

$\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}\csc^{-1}x = \dfrac{1}{x\sqrt{x^2-1}}$
$\dfrac{d}{dx}\cos^{-1}x = \dfrac{-1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{x\sqrt{x^2-1}}$
$\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$	$\dfrac{d}{dx}\cot^{-1}x = \dfrac{-1}{1+x^2}$

|||| The formulas for the derivatives of $\csc^{-1}x$ and $\sec^{-1}x$ depend on the definitions that are used for these functions. See Exercise 54.

|||| 3.6 Exercises

1–4 ||||

(a) Find dy/dx by implicit differentiation.

(b) Solve the equation explicitly for y and differentiate to get dy/dx in terms of x.

(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

1. $xy + 2x = 3x^2 - 4$
2. $4x^2 + 9y^2 = 36$
3. $\dfrac{1}{x} + \dfrac{1}{y} = 1$
4. $\sqrt{x} + \sqrt{y} = 4$

5–20 |||| Find dy/dx by implicit differentiation.

5. $x^2 + y^2 = 1$
6. $x^2 - y^2 = 1$
7. $x^3 + x^2y - 4y^2 = 6$
8. $x^2 + 2xy = y^3 + c$
9. $x^2y + xy^2 = 3x$
10. $y^5 + x^2y^3 = 1 + ye^{x^2}$
11. $x^2y^2 = x\sin y + 4$
12. $1 + x = \sin xy^2$
13. $4\cos x \sin y = 1$
14. $y\sin x^2 = x\sin y^2$
15. $e^{x^2y} = x + y$
16. $\sqrt{x+y} = 1 + x^2y^2$
17. $\sqrt{xy} = 1 + x^2y$
18. $\tan x + y = \dfrac{y}{1+x^2}$
19. $xy = \cot xy$
20. $\sin x + \cos y = \sin x \cos y$
21. If $1 + f(x) = x^2 + f(x)^3$, $f(0) = 2$, find $f'(1)$.
22. If $x + x\sin x = x^2$ and $f(1) = 0$, find $f'(1)$.

23–24 |||| Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy.

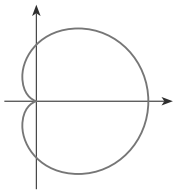
23. $y^4 + x^2y^2 = yx^4 + y + 1$
24. $(x^2 + y^2)^2 = ax^2y$

25–30 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

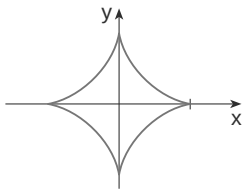
25. $x^2 \quad xy \quad y^2 \quad 3, \quad 1, 1$ (ellipse)

26. $x^2 \quad 2xy \quad y^2 \quad x \quad 2, \quad 1, 2$ (hyperbola)

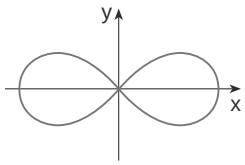
27. $x^2 \quad y^2 \quad 2x^2 \quad 2y^2 \quad x^2$
$\left(0, \frac{1}{2}\right)$
(cardioid)



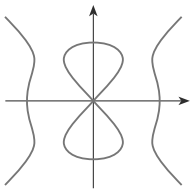
28. $x^{2/3} \quad y^{2/3} \quad 4$
$\left(\ \ 3 \quad \sqrt{3}, 1\right)$
(astroid)



29. $2\,x^2 \quad y^2{}^2 \quad 25\,x^2 \quad y^2$
(3, 1)
(lemniscate)



30. $y^2\,y^2 \quad 4 \quad x^2\,x^2 \quad 5$
(0, 2)
(devil's curve)



31. (a) The curve with equation $y^2 \quad 5x^4 \quad x^2$ is called a kampyle of Eudoxus. Find an equation of the tangent line to this curve at the point $1, 2$.
(b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (If your graphing device will graph implicitly defined curves, then use that capability. If not, you can still graph this curve by graphing its upper and lower halves separately.)

32. (a) The curve with equation $y^2 \quad x^3 \quad 3x^2$ is called the Tschirnhausen cubic. Find an equation of the tangent line to this curve at the point $1, \quad 2$.
(b) At what points does this curve have a horizontal tangent?
(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

CAS 33. Fanciful shapes can be created by using the implicit plotting capabilities of computer algebra systems.
(a) Graph the curve with equation

$$y\,y^2 \quad 1\,y \quad 2 \quad x\,x \quad 1\,x \quad 2$$

At how many points does this curve have horizontal tangents? Estimate the x-coordinates of these points.
(b) Find equations of the tangent lines at the points (0, 1) and (0, 2).
(c) Find the exact x-coordinates of the points in part (a).
(d) Create even more fanciful curves by modifying the equation in part (a).

CAS 34. (a) The curve with equation

$$2y^3 \quad y^2 \quad y^5 \quad x^4 \quad 2x^3 \quad x^2$$

has been likened to a bouncing wagon. Use a computer algebra system to graph this curve and discover why.
(b) At how many points does this curve have horizontal tangent lines? Find the x-coordinates of these points.

35. Find the points on the lemniscate in Exercise 29 where the tangent is horizontal.

36. Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} \quad \frac{y^2}{b^2} \quad 1$$

at the point x_0, y_0 is

$$\frac{x_0 x}{a^2} \quad \frac{y_0 y}{b^2} \quad 1$$

37. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} \quad \frac{y^2}{b^2} \quad 1$$

at the point x_0, y_0.

38. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} \quad \sqrt{y} \quad \sqrt{c}$ is equal to c.

39. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

40. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n \quad p/q$, and $y \quad f(x) \quad x^n$ is assumed beforehand to be a differentiable function. If $y \quad x^{p/q}$, then $y^q \quad x^p$. Use implicit differentiation to show that

$$y' \quad \frac{p}{q}\,x^{(p/q) \quad 1}$$

41–50 Find the derivative of the function. Simplify where possible.

41. $y \quad \tan^{-1}\sqrt{x}$

42. $y \quad \sqrt{\tan^{-1}x}$

43. $y \quad \sin^{-1}(2x \quad 1)$

44. $h(x) \quad \sqrt{1 \quad x^2}\,\arcsin x$

45. $H(x) \quad (1 \quad x^2)\arctan x$

46. $y \quad \tan^{-1}\left(x \quad \sqrt{1 \quad x^2}\right)$

47. $h(t) \quad \cot^{-1}(t) \quad \cot^{-1}(1/t)$

48. $y \quad x\cos^{-1}x \quad \sqrt{1 \quad x^2}$

49. $y \quad \cos^{-1}(e^{2x})$

50. $y \quad \arctan(\cos\ )$

51–52 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

51. $f(x) \quad e^x \quad x^2\arctan x$

52. $f(x) \quad x\arcsin(1 \quad x^2)$

53. Prove the formula for $d/dx\ \cos^{-1}x$ by the same method as for $d/dx\ \sin^{-1}x$.

54. (a) One way of defining $\sec^{-1}x$ is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \le y \le \pi/2$ or $\pi \le y < 3\pi/2$. Show that, with this definition,

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$

(b) Another way of defining $\sec^{-1}x$ that is sometimes used is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \le y \le \pi$, $y \ne \pi/2$. Show that, with this definition,

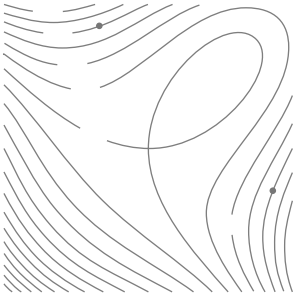
$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$

55–56 Show that the given curves are orthogonal.

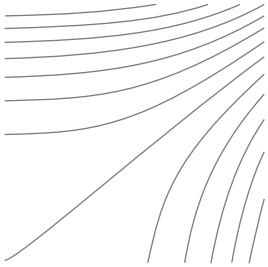
55. $2x^2 - y^2 = 3$, $x = y^2$

56. $x^2 - y^2 = 5$, $4x^2 + 9y^2 = 72$

57. Contour lines on a map of a hilly region are curves that join points with the same elevation. A ball rolling down a hill follows a curve of steepest descent, which is orthogonal to the contour lines. Given the contour map of a hill in the figure, sketch the paths of balls that start at positions A and B.



58. TV meteorologists often present maps showing pressure fronts. Such maps display isobars—curves along which the air pressure is constant. Consider the family of isobars shown in the figure. Sketch several members of the family of orthogonal trajectories of the isobars. Given the fact that wind blows from regions of high air pressure to regions of low air pressure, what does the orthogonal family represent?



59–62 Show that the given families of curves are orthogonal trajectories of each other. Sketch both families of curves on the same axes.

59. $x^2 + y^2 = r^2$, $ax + by = 0$

60. $x^2 + y^2 = ax$, $x^2 + y^2 = by$

61. $y = cx^2$, $x^2 + 2y^2 = k$

62. $y = ax^3$, $x^2 + 3y^2 = b$

63. The equation $x^2 - xy + y^2 = 3$ represents a rotated ellipse, that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.

64. (a) Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(1, 1)$ intersect the ellipse a second time? (See page 192 for the definition of a normal line.)
(b) Illustrate part (a) by graphing the ellipse and the normal line.

65. Find all points on the curve $x^2y^2 - xy = 2$ where the slope of the tangent line is -1.

66. Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

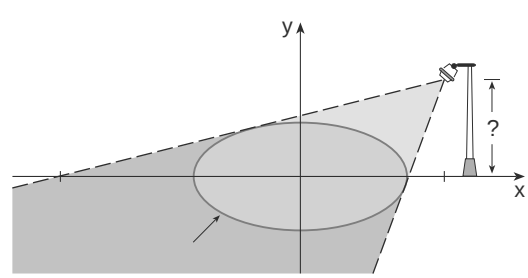
67. (a) Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.
(b) If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $f^{-1}(5)$.

68. (a) Show that $f(x) = 2x + \cos x$ is one-to-one.
(b) What is the value of $f^{-1}(1)$?
(c) Use the formula from Exercise 67(a) to find $f^{-1}(1)$.

69. The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 = 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



3.7 Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')'$. This new function $(f')'$ is called the second derivative of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Another notation is $f''(x) = D^2f(x)$.

In Module 3.7A you can see how changing the coefficients of a polynomial affects the appearance of the graphs of f, f', and f''.

EXAMPLE 1 If $f(x) = x\cos x$, find and interpret $f''(x)$.

SOLUTION Using the Product Rule, we have

$$f'(x) = x\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(x)$$

$$= -x\sin x + \cos x$$

To find $f''(x)$ we differentiate $f'(x)$:

$$f''(x) = \frac{d}{dx}(-x\sin x + \cos x)$$

$$= -x\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(x) + \frac{d}{dx}(\cos x)$$

$$= -x\cos x - \sin x - \sin x$$

$$= -x\cos x - 2\sin x$$

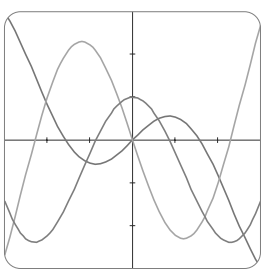


FIGURE 1
The graphs of $f(x) = x\cos x$ and its first and second derivatives

The graphs of f, f', and f'' are shown in Figure 1.

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 1 that $f''(x) = 0$ whenever $y = f'(x)$ has a horizontal tangent. Also, $f''(x)$ is positive when $y = f'(x)$ has positive slope and negative when $f'(x)$ has negative slope. So the graphs serve as a check on our calculations. ■

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is acceleration, which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

EXAMPLE 7 The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 - 9t$$

where t is measured in seconds and s in meters.

(a) Find the acceleration at time t. What is the acceleration after 4 s?

(b) Graph the position, velocity, and acceleration functions for $0 \le t \le 5$.

(c) When is the particle speeding up? When is it slowing down?

SOLUTION

(a) The velocity function is the derivative of the position function:

$$s = f(t) = t^3 - 6t^2 - 9t$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t - 9$$

The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12$$

$$a(4) = 6 \cdot 4 - 12 = 12 \text{ m/s}^2$$

|||| The units for acceleration are meters per second per second, written as m/s^2.

(b) Figure 2 shows the graphs of s, v, and a.

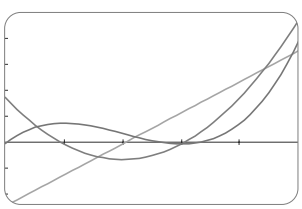


FIGURE 2

(c) The particle speeds up when the velocity is positive and increasing (a is also both positive) and also when the velocity is negative and decreasing (a is also both negative). In other words, the particle speeds up when the velocity and acceleration have the same sign. (The particle is pushed in the same direction it is moving.) From Figure 2 we see that this happens when $1 < t < 2$ and when $t > 3$. The particle slows down when v and a have opposite signs, that is, when $0 < t < 1$ and when $2 < t < 3$. Figure 3 summarizes the motion of the particle.

In Module 3.7B you can see an animation of Figure 3 with an expression for s that you can choose yourself.

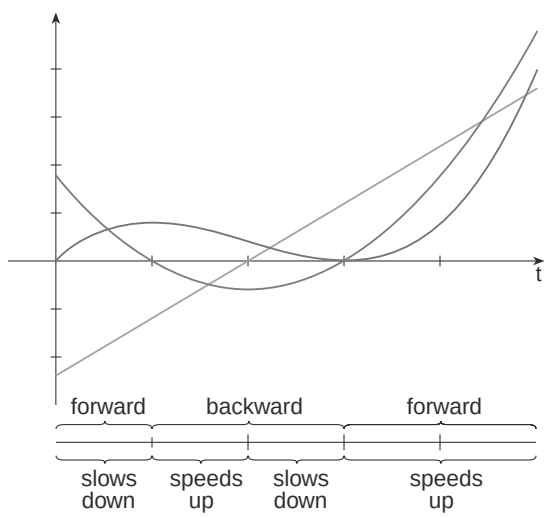


FIGURE 3

The third derivative f is the derivative of the second derivative f: f . So $f\ x$ can be interpreted as the slope of the curve $y\ f\ x$ or as the rate of change of $f\ x$. If $y\ f\ x$, then alternative notations for the third derivative are

$$y\ f\ x\ \frac{d}{dx}\ \frac{d^2y}{dx^2}\ \frac{d^3y}{dx^3}\ D^3f\ x$$

The process can be continued. The fourth derivative is usually denoted by f^4. In general, the nth derivative of f is denoted by f^n and is obtained from by differentiating n times. If $y\ f\ x$, we write

$$y^n\ f^n\ x\ \frac{d^ny}{dx^n}\ D^nf\ x$$

We can interpret the third derivative physically in the case where the function is the position function $s\ s\ t$ of an object that moves along a straight line. Because $s\ s\ a$, the third derivative of the position function is the derivative of the acceleration function and is called the *jerk*:

$$j\ \frac{da}{dt}\ \frac{d^3s}{dt^3}$$

Thus, the jerk j is the rate of change of acceleration. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

EXAMPLE 3 If $y\ x^3\ 6x^2\ 5x\ 3$

then $y\ 3x^2\ 12x\ 5$

$y\ 6x\ 12$

$y\ 6$

$y^4\ 0$

and in fact $y^n\ 0$ for all $n\ 4$.

EXAMPLE 4 If $f\ x\ \frac{1}{x}$, find $f^n\ x$.

SOLUTION $f\ x\ \frac{1}{x}\ x^{1}$

$f\ x\ x^{2}\ \frac{1}{x^2}$

$f\ x\ 2\ 1\ x^{3}\ \frac{2}{x^3}$

$f\ x\ 3\ 2\ 1\ x^{4}$

$f^4\ x\ 4\ 3\ 2\ 1\ x^{5}$

$f^5\ x\ 5\ 4\ 3\ 2\ 1\ x^{6}\ 5!\ x^{6}$

$\vdots$

$f^n\ x\ 1^n n\ n\ 1\ n\ 2\ 2\ 1\ x^{n\ 1}$

|||| The factor 1^n occurs in the formula for $f^n\ x$ because we introduce another negative sign every time we differentiate. Since the successive values of 1^n are $1\ 1\ ,1\ ,1\ ,1\ ,\ ,$ $1, \ldots$, the presence of 1^n indicates that the sign changes with each successive derivative.

or

Here we have used the factorial symbol for the product of the first positive integers.

The following example shows how to find the second derivative of a function that is defined implicitly.

EXAMPLE 5 Find if .

SOLUTION Differentiating the equation implicitly with respect to , we get

Solving for gives

To find we differentiate this expression for using the Quotient Rule and remembering that is a function of :

If we now substitute Equation 1 into this expression, we get

But the values of and must satisfy the original equation x^4 . So the answer simplifies to

$$y \qquad \frac{3x^2 \quad 16}{y^7} \qquad 48\frac{x^2}{y^7}$$

EXAMPLE 6 Find $D^{27}\cos x$.

SOLUTION The first few derivatives of $\cos x$ are as follows:

$$D\cos x \qquad \sin x$$

$$D^2\cos x \qquad \cos x$$

$$D^3\cos x \qquad \sin x$$

$$D^4\cos x \qquad \cos x$$

$$D^5\cos x \qquad \sin x$$

3.8 Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_a x$ and, in particular, the natural logarithmic function $y = \ln x$. We assume that logarithmic functions are differentiable; this is certainly plausible from their graphs (see Figure 12 in Section 1.6).

1
$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{x \ln a}}$$

Proof Let $y = \log_a x$. Then

$$a^y = x$$

Formula 3.5.5 says that

$$\frac{d}{dx}a^x = a^x \ln a$$

Differentiating this equation implicitly with respect to x, using Formula 3.5.5, we get

$$a^y \ln a \frac{dy}{dx} = 1$$

and so
$$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

If we put $a = e$ in Formula 1, then the factor $\ln a$ on the right side becomes $\ln e = 1$ and we get the formula for the derivative of the natural logarithmic function $y = \log_e x = \ln x$:

2
$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

By comparing Formulas 1 and 2, we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: The differentiation formula is simplest when $a = e$ because $\ln e = 1$.

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}\,3x^2 = \frac{3x^2}{x^3 + 1}$$

In general, if we combine Formula 2 with the Chain Rule as in Example 1, we get

3
$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}} \quad \text{or} \quad \boxed{\frac{d}{dx}\ln|x| = \frac{x}{x}}$$

EXAMPLE 2 Find $\frac{d}{dx} \ln \sin x$.

SOLUTION Using (3), we have

$$\frac{d}{dx} \ln \sin x = \frac{1}{\sin x} \frac{d}{dx} \sin x = \frac{1}{\sin x} \cos x = \cot x$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f(x) = \tfrac{1}{2} \ln x^{1/2} \frac{d}{dx} \ln x = \frac{1}{2\sqrt{\ln x}} \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

EXAMPLE 4 Differentiate $f(x) = \log_{10}(2 + \sin x)$.

SOLUTION Using Formula 1 with $a = 10$, we have

$$f'(x) = \frac{d}{dx} \log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x) \ln 10} \frac{d}{dx}(2 + \sin x)$$

$$= \frac{\cos x}{(2 + \sin x) \ln 10}$$

EXAMPLE 5 Find $\frac{d}{dx} \ln \frac{x + 1}{\sqrt{x + 2}}$.

SOLUTION 1

$$\frac{d}{dx} \ln \frac{x + 1}{\sqrt{x + 2}} = \frac{1}{\frac{x + 1}{\sqrt{x + 2}}} \frac{d}{dx} \frac{x + 1}{\sqrt{x + 2}}$$

$$= \frac{\sqrt{x + 2}}{x + 1} \frac{\sqrt{x + 2}\,(1) - (x + 1)\left(\frac{1}{2}\right)(x + 2)^{-1/2}}{x + 2}$$

$$= \frac{x + 2 - \frac{1}{2}(x + 1)}{(x + 1)(x + 2)} = \frac{x + 5}{2(x + 1)(x + 2)}$$

|||| Figure 1 shows the graph of the function of Example 5 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when x is rapidly decreasing.

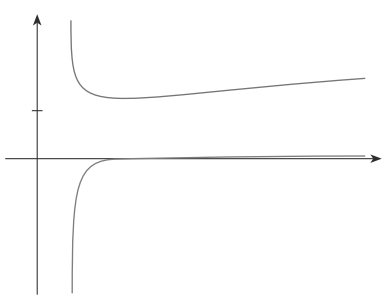


FIGURE 1

SOLUTION 2 If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx} \ln \frac{x + 1}{\sqrt{x + 2}} = \frac{d}{dx}\left[\ln(x + 1) - \tfrac{1}{2} \ln(x + 2)\right]$$

$$= \frac{1}{x + 1} - \frac{1}{2} \frac{1}{x + 2}$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.)

|||| Figure 2 shows the graph of the function $f(x) = \ln|x|$ in Example 6 and its derivative $f'(x) = 1/x$. Notice that when x is small, the graph of $y = \ln|x|$ is steep and so $f'(x)$ is large (positive or negative).

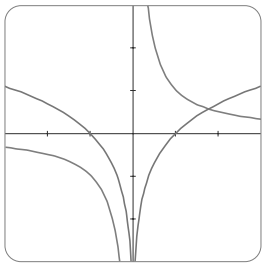


FIGURE 2

EXAMPLE 6 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{(-x)}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus, $f'(x) = 1/x$ for all $x \neq 0$.

The result of Example 6 is worth remembering:

4 $$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

|||| Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called *logarithmic differentiation*.

EXAMPLE 7 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} - 5\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

|||| If we hadn't used logarithmic differentiation in Example 7, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

If $f(x) = 0$ for some values of x, then $\ln f(x)$ is not defined, but we can write $y = |f(x)|$ and use Equation 4. We illustrate this procedure by proving the general version of the Power Rule, as promised in Section 3.1.

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

Proof Let $y = x^n$ and use logarithmic differentiation:

|||| If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

$$\ln|y| = \ln|x^n| = n\ln|x| \qquad x \neq 0$$

Therefore
$$\frac{y'}{y} = \frac{n}{x}$$

Hence
$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

You should distinguish carefully between the Power Rule $(x^n)' = nx^{n-1}$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $(a^x)' = a^x \ln a$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

1. $\dfrac{d}{dx} a^b = 0$ (a and b are constants)
2. $\dfrac{d}{dx} [f(x)]^b = b[f(x)]^{b-1} f'(x)$
3. $\dfrac{d}{dx} a^{f(x)} = a^{f(x)} \ln a \, f'(x)$
4. To find $\dfrac{d}{dx} [f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

|||| Figure 3 illustrates Example 8 by showing the graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

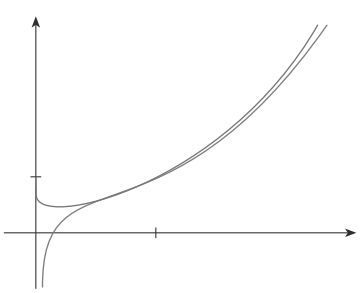


FIGURE 3

EXAMPLE 8 Differentiate $y = x^{\sqrt{x}}$.

SOLUTION Using logarithmic differentiation, we have

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$

$$\frac{y'}{y} = \sqrt{x}\,\frac{1}{x} + \ln x\,\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

SOLUTION 2 Another method is to write $x^{\bar{x}} = e^{\ln x^{\bar{x}}}$:

$$\frac{d}{dx}\left(x^{\bar{x}}\right) = \frac{d}{dx}\left(e^{\bar{x}\ln x}\right) = e^{\bar{x}\ln x}\frac{d}{dx}\left(\bar{x}\ln x\right)$$

$$= x^{\bar{x}}\,\frac{2 - \ln x}{2\bar{x}} \qquad \text{(as in Solution 1)}$$

The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus, $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h \to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x \to 0}\frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x \to 0}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0}\frac{1}{x}\ln(1+x)$$

$$= \lim_{x \to 0}\ln(1+x)^{1/x}$$

Because $f'(1) = 1$, we have

$$\lim_{x \to 0}\ln(1+x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^{1} = e^{\lim_{x \to 0}\ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0}(1+x)^{1/x}$$

5 $$\boxed{e = \lim_{x \to 0}(1+x)^{1/x}}$$

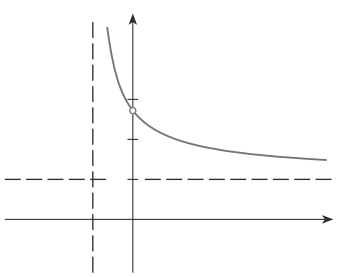


FIGURE 4

Formula 5 is illustrated by the graph of the function $y = (1+x)^{1/x}$ in Figure 4 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

$$e = 2.7182818$$

x	$(1+x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

If we put $n = 1/x$ in Formula 5, then $n \to \infty$ as $x \to 0$ and so an alternative expression for e is

6 $$\boxed{e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^{n}}$$

3.8 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

2–20 Differentiate the function.

2. $f(x) = \ln(x^2 + 10)$

3. $f(\ ) = \ln \cos(\ )$

4. $f(x) = \cos \ln x$

5. $f(x) = \log_2(1 + 3x)$

6. $f(x) = \log_{10}\left(\dfrac{x}{x+1}\right)$

7. $f(x) = \sqrt[5]{\ln x}$

8. $f(x) = \ln \sqrt[5]{x}$

9. $f(x) = \sqrt{x}\ln x$

10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

11. $F(t) = \ln\left(\dfrac{2t+1}{3t-1}\right)^{3/4}$

12. $h(x) = \ln\left(x + \sqrt{x^2+1}\right)$

13. $(\ ) = \ln\left(\dfrac{a+x}{a-x}\right)$

14. $F(y) = y\ln(1 + e^y)$

15. $f(u) = \dfrac{\ln u}{1 + \ln 2u}$

16. $y = \ln(x^4 \sin^2 x)$

17. $y = \ln(2 + x + 5x^2)$

18. $G(u) = \ln\sqrt{\dfrac{3u+2}{3u-2}}$

19. $y = \ln(e^{-x} + xe^{-x})$

20. $y = \ln(1 + e^{x^2})$

21–24 Find y' and y''.

21. $y = x\ln x$

22. $y = \dfrac{\ln x}{x^2}$

23. $y = \log_{10} x$

24. $y = \ln(\sec x + \tan x)$

25–28 Differentiate f and find the domain of f.

25. $f(x) = \dfrac{x}{1 + \ln x + 1}$

26. $f(x) = \dfrac{1}{1 + \ln x}$

27. $f(x) = x^2 \ln(1 + x^2)$

28. $f(x) = \ln \ln \ln x$

29. If $f(x) = \dfrac{x}{\ln x}$, find $f'(e)$.

30. If $f(x) = x^2 \ln x$, find $f'(1)$.

31–32 Find an equation of the tangent line to the curve at the given point.

31. $y = \ln \ln x$, $(e, 0)$

32. $y = \ln(x^3 + 7)$, $(2, 0)$

33. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

34. Find equations of the tangent lines to the curve $y = \ln x - x$ at the points $(1, 0)$ and $(e, 1 - e)$. Illustrate by graphing the curve and its tangent lines.

35–46 Use logarithmic differentiation to find the derivative of the function.

35. $y = (2x + 1)^5 (x^4 - 3)^6$

36. $y = \sqrt{x}\, e^{x^2} (x^2 + 1)^{10}$

37. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

38. $y = \sqrt[4]{\dfrac{x^2 - 1}{x^2 + 1}}$

39. $y = x^x$

40. $y = x^{1/x}$

41. $y = x^{\sin x}$

42. $y = (\sin x)^x$

43. $y = (\ln x)^x$

44. $y = x^{\ln x}$

45. $y = x^{e^x}$

46. $y = (\ln x)^{\cos x}$

47. Find y' if $y = \ln(x^2 + y^2)$.

48. Find y' if $x^y = y^x$.

49. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x + 1)$.

50. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

51. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

52. Show that $\lim_{n \to \infty}\left(1 + \dfrac{x}{n}\right)^n = e^x$ for any $x > 0$.

3.9 Hyperbolic Functions

Certain combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called *hyperbolic functions* and individually called *hyperbolic sine*, *hyperbolic cosine*, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

The graphs of hyperbolic sine and cosine can be sketched using graphical addition as in Figures 1 and 2.

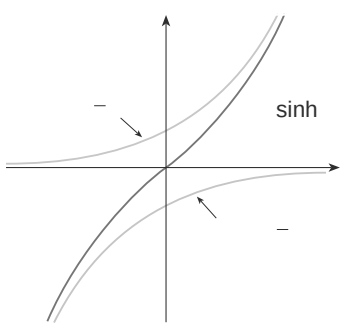


FIGURE 1
sinh

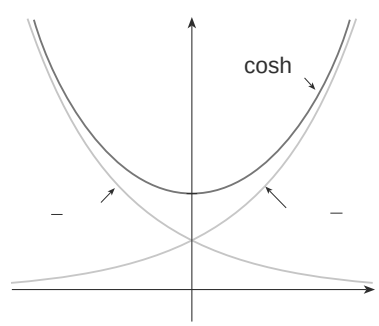


FIGURE 2
cosh

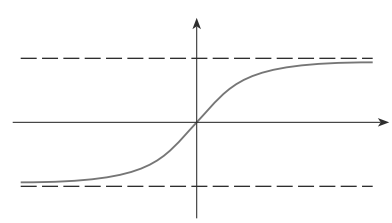


FIGURE 3
tanh

Note that sinh has domain and range , while cosh has domain and range 1, . The graph of tanh is shown in Figure 3. It has the horizontal asymptotes 1 . (See Exercise 23.)

Some of the mathematical uses of hyperbolic functions will be seen in Chapter 7. Applications to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished, for the decay can be represented by hyperbolic functions. The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire. It can be proved that if a flexible cable (such as a telephone or power line) is suspended between two points at the same height, then it takes the shape of a curve with equation $y = c + a\cosh x/a$ called a catenary (see Figure 4). (The Latin word *catena* means "chain.")

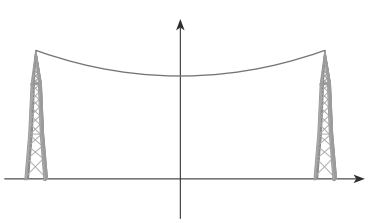


FIGURE 4
A catenary cosh

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities. We list some of them here and leave most of the proofs to the exercises.

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

EXAMPLE 1 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

SOLUTION

(a)
$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$
$$= \frac{4}{4} = 1$$

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

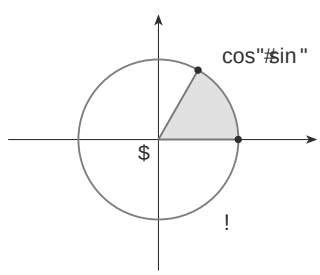


FIGURE 5

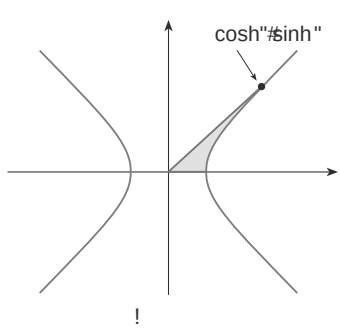


FIGURE 6

The identity proved in Example 1(a) gives a clue to the reason for the term "hyperbolic" functions:

If t is any real number, then the point $P(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ because $\cos^2 t + \sin^2 t = 1$. In fact, t can be interpreted as the radian measure of $\angle POQ$ in Figure 5. For this reason the trigonometric functions are sometimes called circular functions.

Likewise, if t is any real number, then the point $P(\cosh t, \sinh t)$ lies on the right branch of the hyperbola $x^2 - y^2 = 1$ because $\cosh^2 t - \sinh^2 t = 1$ and $\cosh t \geq 1$. This time, t does not represent the measure of an angle. However, it turns out that t represents twice the area of the shaded hyperbolic sector in Figure 6, just as in the trigonometric case t represents twice the area of the shaded circular sector in Figure 5.

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

We list the differentiation formulas for the hyperbolic functions as Table 1. The remaining proofs are left as exercises. Note the analogy with the differentiation formulas for trigonometric functions, but beware that the signs are different in some cases.

1 Derivatives of Hyperbolic Functions

$\frac{d}{dx}\sinh x = \cosh x$		$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$	
$\frac{d}{dx}\cosh x = \sinh x$		$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$	
$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$		$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$	

EXAMPLE 2 Any of these differentiation rules can be combined with the Chain Rule. For instance,

$$\frac{d}{dx}(\cosh \sqrt{x}) = \sinh \sqrt{x}\,\frac{d}{dx}\sqrt{x} = \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

Inverse Hyperbolic Functions

You can see from Figures 1 and 3 that $\sinh$ and $\tanh$ are one-to-one functions and so they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$. Figure 2 shows that $\cosh$ is not one-to-one, but when restricted to the domain $[0, \infty)$ it becomes one-to-one. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

2

$$y = \sinh^{-1} x \iff \sinh y = x$$

$$y = \cosh^{-1} x \iff \cosh y = x \text{ and } y \geq 0$$

$$y = \tanh^{-1} x \iff \tanh y = x$$

The remaining inverse hyperbolic functions are defined similarly (see Exercise 28).

We can sketch the graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ in Figures 7, 8, and 9 by using Figures 1, 2, and 3.

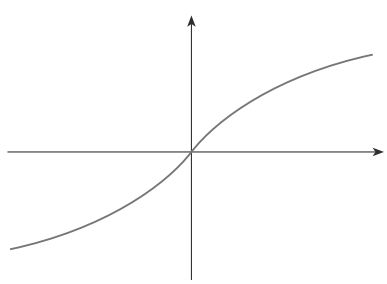


FIGURE 7 $\sinh^{-1}$
domain $\mathbb{R}$ range $\mathbb{R}$

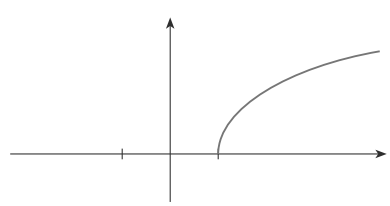


FIGURE 8 $\cosh^{-1}$
domain $[1, \infty)$ range $[0, \infty)$

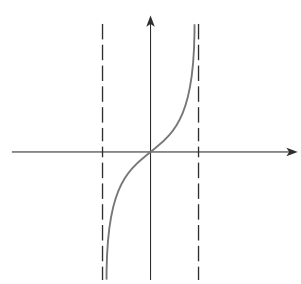


FIGURE 9 $\tanh^{-1}$
domain $(-1, 1)$ range $\mathbb{R}$

Since the hyperbolic functions are defined in terms of exponential functions, it is not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms. In particular, we have:

|||| Formula 3 is proved in Example 3. The proofs of Formulas 4 and 5 are requested in Exercises 26 and 27.

3 $\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$

4 $\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geq 1$

5 $\tanh^{-1}x = \frac{1}{2}\ln\frac{1 + x}{1 - x} \qquad -1 < x < 1$

EXAMPLE 3 Show that $\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

SOLUTION Let $y = \sinh^{-1}x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

so

$$e^y - 2x - e^{-y} = 0$$

or, multiplying by e^y,

$$e^{2y} - 2xe^y - 1 = 0$$

This is really a quadratic equation in e^y:

$$\left(e^y\right)^2 - 2x\left(e^y\right) - 1 = 0$$

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2 + 1} < 0$ (because $x < \sqrt{x^2 + 1}$). Thus, the minus sign is inadmissible and we have

$$e^y = x + \sqrt{x^2 + 1}$$

Therefore $\qquad y = \ln e^y = \ln\left(x + \sqrt{x^2 + 1}\right)$

(See Exercise 25 for another method.)

6 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1 + x^2}} \qquad \frac{d}{dx}\operatorname{csch}^{-1}x = \frac{-1}{|x|\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d}{dx}\operatorname{sech}^{-1}x = \frac{-1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\tanh^{-1}x = \frac{1}{1 - x^2} \qquad \frac{d}{dx}\coth^{-1}x = \frac{1}{1 - x^2}$$

|||| Notice that the formulas for the derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1}x$ is defined for $-1 < x < 1$ whereas $\coth^{-1}x$ is defined for $|x| > 1$.

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable. The formulas in Table 6 can be proved either by the method for inverse functions or by differentiating Formulas 3, 4, and 5.

EXAMPLE 4 Prove that $\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$.

SOLUTION 1 Let $y = \sinh^{-1}x$. Then $\sinh y = x$. If we differentiate this equation implicitly with respect to x, we get

$$\cosh y\,\frac{dy}{dx} = 1$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y > 0$, we have $\cosh y = \sqrt{1+\sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

SOLUTION 2 From Equation 3 (proved in Example 3), we have

$$\begin{aligned}
\frac{d}{dx}\sinh^{-1}x &= \frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right) \\
&= \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right) \\
&= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) \\
&= \frac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}} \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

EXAMPLE 5 Find $\frac{d}{dx}\tanh^{-1}\sin x$.

SOLUTION Using Table 6 and the Chain Rule, we have

$$\begin{aligned}
\frac{d}{dx}\tanh^{-1}\sin x &= \frac{1}{1-(\sin x)^2}\frac{d}{dx}\sin x \\
&= \frac{1}{1-\sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x
\end{aligned}$$

3.9 Exercises

1–6 Find the numerical value of each expression.

1. (a) $\sinh 0$ (b) $\cosh 0$
2. (a) $\tanh 0$ (b) $\tanh 1$
3. (a) $\sinh \ln 2$ (b) $\sinh 2$
4. (a) $\cosh 3$ (b) $\cosh \ln 3$
5. (a) $\operatorname{sech} 0$ (b) $\cosh^{-1} 1$
6. (a) $\sinh 1$ (b) $\sinh^{-1} 1$

7–19 Prove the identity.

7. $\sinh(-x) = -\sinh x$
(This shows that $\sinh x$ is an odd function.)

8. $\cosh(-x) = \cosh x$
(This shows that $\cosh x$ is an even function.)

9. $\cosh x + \sinh x = e^x$

10. $\cosh x - \sinh x = e^{-x}$

11. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

12. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

13. $\coth^2 x - 1 = \operatorname{csch}^2 x$

14. $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

15. $\sinh 2x = 2 \sinh x \cosh x$

16. $\cosh 2x = \cosh^2 x + \sinh^2 x$

17. $\tanh(\ln x) = \dfrac{x^2 - 1}{x^2 + 1}$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

19. $\cosh nx + \sinh nx = (\cosh x + \sinh x)^n$
(n any real number)

20. If $\sinh x = \frac{3}{4}$, find the values of the other hyperbolic functions at x.

21. If $\tanh x = \frac{4}{5}$, find the values of the other hyperbolic functions at x.

22. (a) Use the graphs of $\sinh$, $\cosh$, and $\tanh$ in Figures 8–11 to draw the graphs of csch, sech, and $\coth$.
(b) Check the graphs that you sketched in part (a) by using a graphing device to produce them.

23. Use the definitions of the hyperbolic functions to find each of the following limits.
(a) $\lim_{x \to \infty} \tanh x$ (b) $\lim_{x \to -\infty} \tanh x$
(c) $\lim_{x \to \infty} \sinh x$ (d) $\lim_{x \to -\infty} \sinh x$
(e) $\lim_{x \to \infty} \operatorname{sech} x$ (f) $\lim_{x \to \infty} \coth x$
(g) $\lim_{x \to 0^+} \coth x$ (h) $\lim_{x \to 0^-} \coth x$
(i) $\lim_{x \to \infty} \operatorname{csch} x$

24. Prove the formulas given in Table 1 for the derivatives of the functions (a) $\cosh$, (b) $\tanh$, (c) csch, (d) sech, and (e) $\coth$.

25. Give an alternative solution to Example 3 by letting $y = \sinh^{-1} x$ and then using Exercise 9 and Example 1(a) with x replaced by y.

26. Prove Equation 4.

27. Prove Equation 5 using (a) the method of Example 3 and (b) Exercise 18 with x replaced by y.

28. For each of the following functions (i) give a definition like those in (2), (ii) sketch the graph, and (iii) find a formula similar to Equation 3.
(a) csch^{-1} (b) sech^{-1} (c) $\coth^{-1}$

29. Prove the formulas given in Table 6 for the derivatives of the following functions.
(a) $\cosh^{-1}$ (b) $\tanh^{-1}$ (c) csch^{-1}
(d) sech^{-1} (e) $\coth^{-1}$

30–47 Find the derivative.

30. $f(x) = \tanh 4x$

31. $f(x) = x \cosh x$

32. $f(x) = \sinh^2 x$

33. $h(x) = \sinh(x^2)$

34. $F(x) = \sinh x \tanh x$

35. $G(x) = \dfrac{1 + \cosh x}{1 - \cosh x}$

36. $f(t) = e^t \operatorname{sech} t$

37. $h(t) = \coth\sqrt{1 + t^2}$

38. $f(t) = \ln \sinh t$

39. $H(t) = \tanh(e^t)$

40. $y = \sinh(\cosh x)$

41. $y = e^{\cosh 3x}$

42. $y = x^2 \sinh^{-1} 2x$

43. $y = \tanh^{-1}(\sqrt{x})$

44. $y = x \tanh^{-1} x + \ln\sqrt{1 - x^2}$

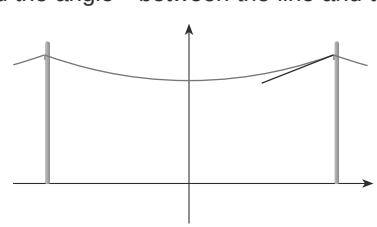
45. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2}$

46. $y = \operatorname{sech}^{-1}(\sqrt{1 - x^2})$, $x > 0$

47. $y = \coth^{-1}(\sqrt{x^2 + 1})$

48. A flexible cable always hangs in the shape of a catenary $y = c + a \cosh(x/a)$, where c and a are constants and $a > 0$ (see Figure 4 and Exercise 50). Graph several members of the family of functions $y = a \cosh(x/a)$. How does the graph change as a varies?

49. A telephone line hangs between two poles 14 m apart in the shape of the catenary $y = 20 \cosh(x/20) - 15$, where x and y are measured in meters.
(a) Find the slope of this curve where it meets the right pole.
(b) Find the angle θ between the line and the pole.



50. Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{\rho}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where is the linear density of the cable, is the acceleration due to gravity, and T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately. Verify that the function

$$y \quad f\ x \quad \frac{T}{\ } \cosh \frac{x}{T}$$

is a solution of this differential equation.

51. (a) Show that any function of the form

$$y \quad A \sinh mx \quad B \cosh mx$$

satisfies the differential equation $y'' \quad m^2 y$.

(b) Find $y \quad y\ x$ such that $y'' \quad 9y \quad y\ 0 \quad 4$, and $y'\ 0 \quad 6$.

52. Evaluate $\lim_{x} \frac{\sinh x}{e^x}$.

53. At what point of the curve $y \quad \cosh x$ does the tangent have slope 1?

54. If $x \quad \ln \sec \quad \tan$, show that $\sec \quad \cosh x$.

55. Show that if $a \quad 0$ and $b \quad 0$, then there exist numbers and such that $ae^x \quad be^{\ x}$ equals either $\sinh x$ or $\cosh x$. In other words, almost every function of the form $f\ x \quad ae^x \quad be^{\ x}$ is a shifted and stretched hyperbolic sine or cosine function.

3.10 Related Rates

Explore an expanding balloon interactively.
Resources / Module 5
/ Related Rates
/ Start of Related Rates

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

EXAMPLE Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

According to the Principles of Problem Solving discussed on page 80, the first step is to understand the problem. This includes reading the problem carefully, identifying the given and the unknown, and introducing suitable notation.

SOLUTION We start by identifying two things:

the given information:

the rate of increase of the volume of air is 100 cm^3/s

and the unknown:

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathematically, we introduce some suggestive notation:

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t. The rate of increase of the volume with respect to time is the derivative dV/dt, and the rate of increase of the radius is dr/dt. We can therefore restate the given and the unknown as follows:

$$\text{Given:} \quad \frac{dV}{dt} \quad 100 \text{ cm}^3/\text{s}$$

$$\text{Unknown:} \quad \frac{dr}{dt} \quad \text{when } r \quad 25 \text{ cm}$$

|||| The second stage of problem solving is to think of a plan for connecting the given and the unknown.

In order to connect dV/dt and dr/dt, we first relate V and r by the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

In order to use the given information, we differentiate each side of this equation with respect to t. To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

|||| Notice that, although dV/dt is constant, dr/dt is not constant.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100\pi$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100\pi = \frac{1}{25}$$

The radius of the balloon is increasing at the rate of $\frac{1}{25}$ cm/s.

How high will a fireman get while climbing a sliding ladder?

Resources / Module 5 / Related Rates / Start of the Sliding Fireman

EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

SOLUTION We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time).

We are given that $dx/dt = 1$ ft/s and we are asked to find dy/dt when $x = 6$ ft (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem:

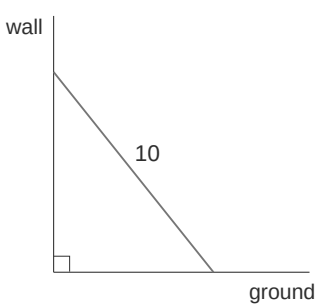


FIGURE 1

$$x^2 + y^2 = 100$$

Differentiating each side with respect to t using the Chain Rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

and solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

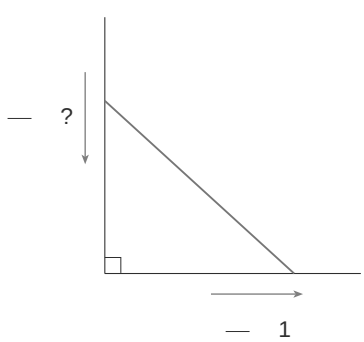


FIGURE 2

When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values and $dx/dt = 1$, we have

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \text{ ft/s}$$

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is decreasing at a rate of $\frac{3}{4}$ ft/s. In other words, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ ft/s.

EXAMPLE 4 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2\ \text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3 m deep.

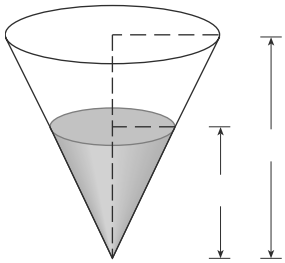


FIGURE 3

SOLUTION We first sketch the cone and label it as in Figure 3. Let V, r, and h be the volume of the water, the radius of the surface, and the height at time t, where t is measured in minutes.

We are given that $dV/dt = 2\ \text{m}^3/\text{min}$ and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$

but it is very useful to express V as a function of h alone. In order to eliminate r, we use the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \qquad r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now we can differentiate each side with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3$ m and $dV/dt = 2\ \text{m}^3/\text{min}$, we have

$$\frac{dh}{dt} = \frac{4}{\pi 3^2}\,2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $8/9\pi \approx 0.28$ m/min. ■

|||| Look back: What have we learned from Examples 1–3 that will help us solve future problems?

Strategy It is useful to recall some of the problem-solving principles from page 80 and adapt them to related rates in light of our experience in Examples 1–3.

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (as in Example 3).
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

WARNING A common error is to substitute the given numerical information (for quantities that vary with time) too early. This should be done only *after* the differentiation. (Step 7 follows Step 6.) For instance, in Example 3 we dealt with general values of h until we finally substituted $h = 3$ at the last stage. (If we had put $h = 3$ earlier, we would have gotten $dV/dt = 0$, which is clearly wrong.)

The following examples are further illustrations of the strategy.

EXAMPLE 2 Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

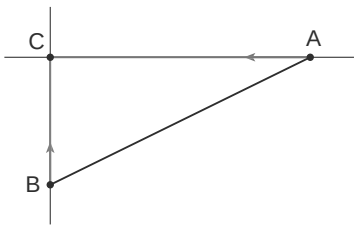


FIGURE 4

SOLUTION We draw Figure 4, where C is the intersection of the roads. At a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars, where x, y, and z are measured in miles.

We are given that $dx/dt = -50$ mi/h and $dy/dt = -60$ mi/h. (The derivatives are negative because x and y are decreasing.) We are asked for dz/dt. The equation that relates x, y, and z is given by the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiating each side with respect to t, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

When $x = 0.3$ mi and $y = 0.4$ mi, the Pythagorean Theorem gives $z = 0.5$ mi, so

$$\frac{dz}{dt} = \frac{1}{0.5}(0.3)(-50) + (0.4)(-60)$$

$$= -78 \text{ mi/h}$$

The cars are approaching each other at a rate of 78 mi/h. ■

EXAMPLE 3 A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

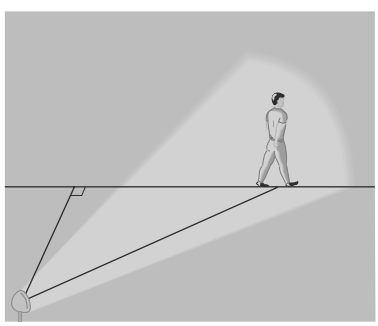


FIGURE 5

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that $dx/dt = 4$ ft/s and are asked for $d\theta/dt$ when $x = 15$. The equation that relates x and θ can be written from Figure 5:

$$\frac{x}{20} = \tan\theta \qquad x = 20\tan\theta$$

Differentiating each side with respect to t, we get

$$\frac{dx}{dt} = 20\sec^2\theta\frac{d\theta}{dt}$$

so

$$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta\frac{dx}{dt} = \frac{1}{20}\cos^2\theta(4) = \frac{1}{5}\cos^2\theta$$

When $x = 15$, the length of the beam is 25, so $\cos\theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of 0.128 rad/s. ■

3.10 Exercises

1. If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt.

2. (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.
   (b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

3. If $y = x^3 - 2x$ and $dx/dt = 5$, find dy/dt when $x = 2$.

4. If $x^2 + y^2 = 25$ and $dy/dt = 6$, find dx/dt when $y = 4$.

5. If $z^2 = x^2 + y^2$, $dx/dt = 2$, and $dy/dt = 3$, find dz/dt when $x = 5$ and $y = 12$.

6. A particle moves along the curve $y = \sqrt{1 + x^3}$. As it reaches the point $(2, 3)$, the y-coordinate is increasing at a rate of 4 cm/s. How fast is the x-coordinate of the point changing at that instant?

7–10 ||||
(a) What quantities are given in the problem?
(b) What is the unknown?
(c) Draw a picture of the situation for any time t.
(d) Write an equation that relates the quantities.
(e) Finish solving the problem.

7. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

8. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

9. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

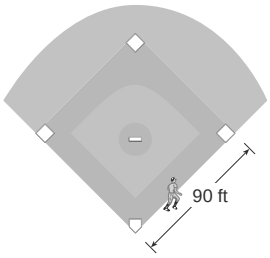
10. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00?

11. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

12. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

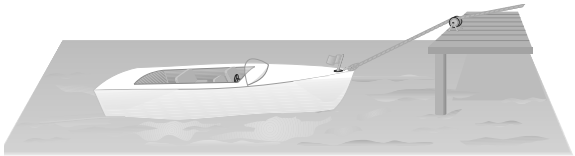
13. A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 min after the woman starts walking?

14. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.
   (a) At what rate is his distance from second base decreasing when he is halfway to first base?
   (b) At what rate is his distance from third base increasing at the same moment?



15. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

16. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



17. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00?

18. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?

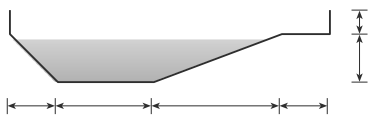
19. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

20. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min,

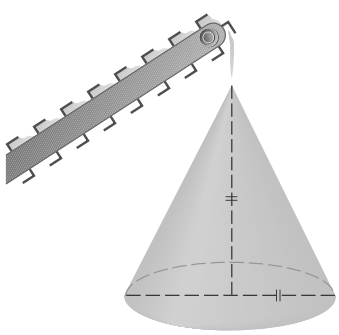
how fast is the water level rising when the water is 6 inches deep?

21. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of $0.2\ m^3/min$, how fast is the water level rising when the water is 30 cm deep?

22. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of $0.8\ ft^3/min$, how fast is the water level rising when the depth at the deepest point is 5 ft?



23. Gravel is being dumped from a conveyor belt at a rate of $30\ ft^3/min$, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?



24. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?

25. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\frac{\pi}{3}$.

26. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of 2°/min. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60°?

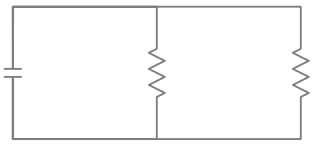
27. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is $600\ cm^3$, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

28. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is $400\ cm^3$ and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?

29. If two resistors with resistances R_1 and R_2 are connected in parallel, as in the figure, then the total resistance R, measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

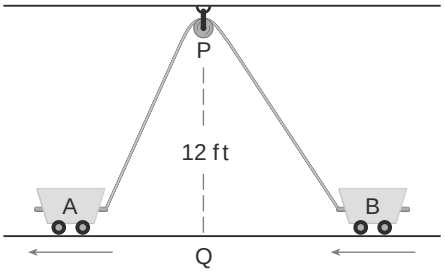
If R_1 and R_2 are increasing at rates of $0.3\ \Omega/s$ and $0.2\ \Omega/s$, respectively, how fast is R changing when $R_1 = 80\ \Omega$ and $R_2 = 100\ \Omega$?



30. Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when the average length was 18 cm?

31. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 1 ft/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$ rad?

32. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P (see the figure). The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?



33. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising

rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.

(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

34. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

35. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later?

36. Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

37. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

38. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

3.11 Linear Approximations and Differentials

Resources / Module 3
/ Linear Approximation
/ Start of Linear Approximation

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. (See Figure 2 in Section 2.7 and Figure 3 in Section 2.8.) This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

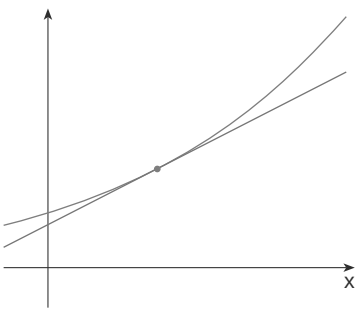


FIGURE 1

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a) \tag{1}$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a) \tag{2}$$

is called the **linearization** of f at a.

The following example is typical of situations in which we use a linear approximation to predict the future behavior of a function given by empirical data.

EXAMPLE 1 Suppose that after you stuff a turkey its temperature is 50°F and you then put it in a 325°F oven. After an hour the meat thermometer indicates that the temperature of the turkey is 93°F and after two hours it indicates 129°F. Predict the temperature of the turkey after three hours.

SOLUTION If $T(t)$ represents the temperature of the turkey at time t hours, we are given that $T(0) = 50$, $T(1) = 93$, and $T(2) = 129$. In order to make a linear approximation with $a = 2$, we need an estimate for the derivative $T'(2)$. Because

$$T'(2) = \lim_{t \to 2} \frac{T(t) - T(2)}{t - 2}$$

we could estimate $T\ 2$ by the difference quotient with $t\ 1$:

$$T\ 2 \quad \frac{T\ 1 \quad T\ 2}{1 \quad 2} \quad \frac{93 \quad 129}{1} \quad 36$$

This amounts to approximating the instantaneous rate of temperature change by the average rate of change between $t\ 1$ and $t\ 2$, which is $36\ F\ h$. With this estimate, the linear approximation (1) for the temperature after 3 h is

$$T\ 3 \quad T\ 2 \quad T\ 2\ 3 \quad 2$$
$$129 \quad 36\ 1 \quad 165$$

So the predicted temperature after three hours is $165\ F$.

We obtain a more accurate estimate for $T\ 2$ by plotting the given data, as in Figure 2, and estimating the slope of the tangent line at $t\ 2$ to be

$$T\ 2 \quad 33$$

Then our linear approximation becomes

$$T\ 3 \quad T\ 2 \quad T\ 2\ 1 \quad 129 \quad 33 \quad 162$$

and our improved estimate for the temperature is $162\ F$.

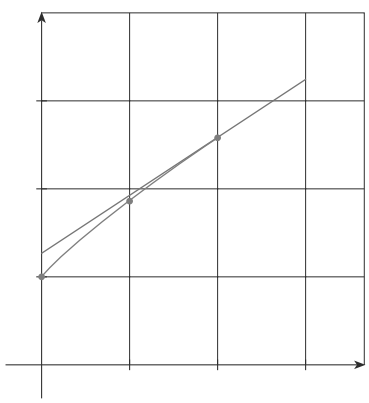


FIGURE 2

Because the temperature curve lies below the tangent line, it appears that the actual temperature after three hours will be somewhat less than $162\ F$, perhaps closer to $160\ F$.

EXAMPLE 2 Find the linearization of the function $f\ x \quad \sqrt{x \quad 3}$ at 1 and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

SOLUTION The derivative of $f\ x \quad x \quad 3^{1\ 2}$ is

$$f\ x \quad \tfrac{1}{2}\ x \quad 3^{\ 1\ 2} \quad \frac{1}{2\ \sqrt{x \quad 3}}$$

and so we have $f\ 1 \quad 2$ and $f\ 1 \quad \frac{1}{4}$. Putting these values into Equation 2, we see that the linearization is

$$L\ x \quad f\ 1 \quad f\ 1\ x \quad 1 \quad 2 \quad \tfrac{1}{4}\ x \quad 1 \quad \frac{7}{4} \quad \frac{x}{4}$$

The corresponding linear approximation (1) is

$$\sqrt{x \quad 3} \quad \frac{7}{4} \quad \frac{x}{4} \qquad (\text{when } x \text{ is near } 1)$$

In particular, we have

$$\sqrt{3.98} \quad \tfrac{7}{4} \quad \tfrac{0.98}{4} \quad 1.995 \qquad \text{and} \qquad \sqrt{4.05} \quad \tfrac{7}{4} \quad \tfrac{1.05}{4} \quad 2.0125$$

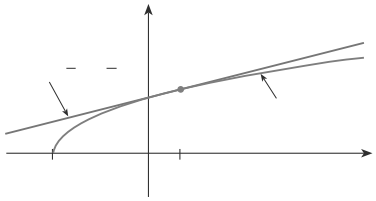


FIGURE 3

The linear approximation is illustrated in Figure 3. We see that, indeed, the tangent line approximation is a good approximation to the given function when x is near 1. We also see that our approximations are overestimates because the tangent line lies above the curve.

Of course, a calculator could give us approximations to $\sqrt{3.98}$ and $\sqrt{4.05}$, but the linear approximation gives an approximation over an entire interval.

In the following table we compare the estimates from the linear approximation in Example 2 with the true values. Notice from this table, and also from Figure 3, that the tangent line approximation gives good estimates when x is close to 1 but the accuracy of the approximation deteriorates when x is farther away from 1.

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.9748417 . . .
$\sqrt{3.98}$	0.98	1.995	1.9949937 . . .
$\sqrt{4}$	1	2	2.0000000 . . .
$\sqrt{4.05}$	1.05	2.0125	2.0124611 . . .
$\sqrt{4.1}$	1.1	2.025	2.0248456 . . .
$\sqrt{5}$	2	2.25	2.2360679 . . .
$\sqrt{6}$	3	2.5	2.4494897 . . .

How good is the approximation that we obtained in Example 2? The next example shows that by using a graphing calculator or computer we can determine an interval throughout which a linear approximation provides a specified accuracy.

EXAMPLE 5 For what values of x is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

SOLUTION Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left|\sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4}\right)\right| < 0.5$$

Equivalently, we could write

$$-0.5 < \sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4}\right) < 0.5$$

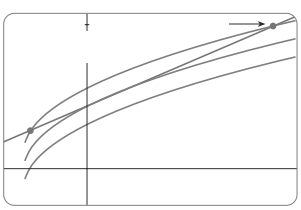


FIGURE 4

This says that the linear approximation should lie between the curves obtained by shifting the curve $y = \sqrt{x+3}$ upward and downward by an amount 0.5. Figure 4 shows the tangent line $y = (7 + x)/4$ intersecting the upper curve $y = \sqrt{x+3} + 0.5$ at P and Q. Zooming in and using the cursor, we estimate that the x-coordinate of P is about -2.66 and the x-coordinate of Q is about 8.66. Thus, we see from the graph that the approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

is accurate to within 0.5 when $-2.6 < x < 8.6$. (We have rounded to be safe.)

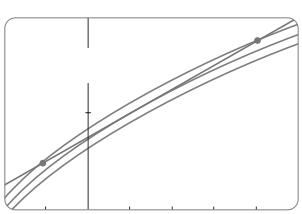


FIGURE 5

Similarly, from Figure 5 we see that the approximation is accurate to within 0.1 when $-1.1 < x < 3.9$.

Applications to Physics

Linear approximations are often used in physics. In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation. For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain the expression $a = \quad \sin \quad$ for tangential acceleration and then replace $\sin \quad$ by $\quad$ with the remark that $\sin \quad$ is very close to $\quad$ if $\quad$ is not too large. [See, for example, *Physics: Calculus*, 2d ed., by Eugene Hecht (Pacific Grove, CA: Brooks/Cole, 2000), p. 431.] You can verify that the linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$ and so the linear approximation at 0 is

$$\sin x \approx x$$

(see Exercise 48). So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called *paraxial rays*. In paraxial (or Gaussian) optics, both $\sin \quad$ and $\cos \quad$ are replaced by their linearizations. In other words, the linear approximations

$$\sin \approx \quad \text{and} \quad \cos \approx 1$$

are used because $\quad$ is close to 0. The results of calculations made with these approximations became the basic theoretical tool used to design lenses. [See *Optics*, 4th ed., by Eugene Hecht (Reading, MA: Addison-Wesley, 2002), p. 154.]

In Section 11.12 we will present several other applications of the idea of linear approximations to physics.

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials*. If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation

$$dy = f'(x)\,dx \tag{3}$$

So dy is a dependent variable; it depends on the values of x and dx. If x is given a specific value and dx is taken to be some specific number in the domain of f', then the numerical value of dy is determined.

The geometric meaning of differentials is shown in Figure 6. Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is the derivative $f'(x)$. Thus, the directed distance from S to R is $f'(x)\,dx = dy$. Therefore, dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount $dx = \Delta x$.

If $dx \neq 0$, we can divide both sides of Equation 3 by dx to obtain

$$\frac{dy}{dx} = f'(x)$$

We have seen similar equations before, but now the left side can genuinely be interpreted as a ratio of differentials.

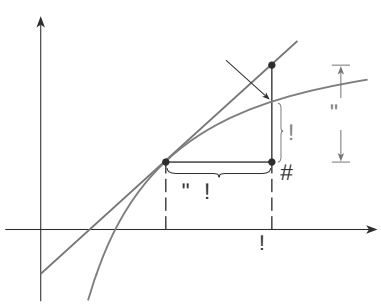


FIGURE 6

EXAMPLE 4 Compare the values of y and dy if $f(x) = x^3 - x^2 + 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

SOLUTION

(a) We have

$$f(2) = 2^3 - 2^2 + 2(2) + 1 = 9$$

$$f(2.05) = (2.05)^3 - (2.05)^2 + 2(2.05) + 1 = 9.717625$$

$$\Delta y = f(2.05) - f(2) = 0.717625$$

In general,

$$dy = f'(x)\,dx = (3x^2 - 2x + 2)\,dx$$

When $x = 2$ and $dx = \Delta x = 0.05$, this becomes

$$dy = (3(2)^2 - 2(2) + 2)(0.05) = 0.7$$

(b)

$$f(2.01) = (2.01)^3 - (2.01)^2 + 2(2.01) + 1 = 9.140701$$

$$\Delta y = f(2.01) - f(2) = 0.140701$$

When $dx = \Delta x = 0.01$,

$$dy = (3(2)^2 - 2(2) + 2)(0.01) = 0.14$$

|||| Figure 7 shows the function in Example 4 and a comparison of Δy and dy when $x = 2$. The viewing rectangle is $[1.8, 2.5]$ by $[6, 18]$.

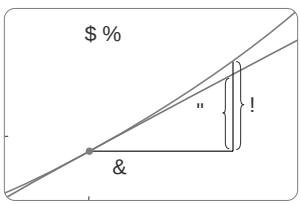


FIGURE 7

Notice that the approximation $\Delta y \approx dy$ becomes better as dx becomes smaller in Example 4. Notice also that dy was easier to compute than Δy. For more complicated functions it may be impossible to compute Δy exactly. In such cases the approximation by differentials is especially useful.

In the notation of differentials, the linear approximation (1) can be written as

$$f(a + dx) \approx f(a) + dy$$

For instance, for the function $f(x) = \sqrt{x + 3}$ in Example 2, we have

$$dy = f'(x)\,dx = \frac{dx}{2\sqrt{x + 3}}$$

If $a = 1$ and $dx = \Delta x = 0.05$, then

$$dy = \frac{0.05}{2\sqrt{1 + 3}} = 0.0125$$

and

$$\sqrt{4.05} = f(1.05) \approx f(1) + dy = 2.0125$$

just as we found in Example 2.

Our final example illustrates the use of differentials in estimating the errors that occur because of approximate measurements.

EXAMPLE 5 The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

SOLUTION If r is the radius of the sphere, then its volume is $V = \frac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by Δr, then the corresponding error in the calcu-

lated value of V is V, which can be approximated by the differential

$$dV \quad 4 \quad r^2 dr$$

When $r \quad 21$ and $dr \quad 0.05$, this becomes

$$dV \quad 4 \quad 21^2 0.05 \quad 277$$

The maximum error in the calculated volume is about 277 cm^3.

NOTE Although the possible error in Example 5 may appear to be rather large, a better picture of the error is given by the *relative error*, which is computed by dividing the error by the total volume:

$$\frac{V}{V} \quad \frac{dV}{V} \quad \frac{4 \quad r^2 dr}{\frac{4}{3} \quad r^3} \quad 3\frac{dr}{r}$$

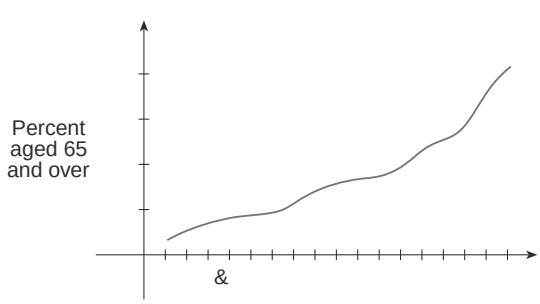
Thus, the relative error in the volume is about three times the relative error in the radius. In Example 5 the relative error in the radius is approximately $\frac{dr}{r} \quad \frac{0.05}{21} \quad 0.0024$ and it produces a relative error of about 0.007 in the volume. The errors could also be expressed as *percentage errors* of 0.24% in the radius and 0.7% in the volume.

3.11 Exercises

1. The turkey in Example 1 is removed from the oven when its temperature reaches 185 F and is placed on a table in a room where the temperature is 75 F. After 10 minutes the temperature of the turkey is 172 F and after 20 minutes it is 160 F. Use a linear approximation to predict the temperature of the turkey after half an hour. Do you think your prediction is an overestimate or an underestimate? Why?

2. Atmospheric pressure P decreases as altitude h increases. At a temperature of 15 C, the pressure is 101.3 kilopascals (kPa) at sea level, 87.1 kPa at 1 km, and 74.9 kPa at 2 km. Use a linear approximation to estimate the atmospheric pressure at an altitude of 3 km.

3. The graph indicates how Australia's population is aging by showing the past and projected percentage of the population aged 65 and over. Use a linear approximation to predict the percentage of the population that will be 65 and over in the years 2040 and 2050. Do you think your predictions are too high or too low? Why?



4. The table shows the population of Nepal (in millions) as of June 30 of the given year. Use a linear approximation to estimate the population at midyear in 1984. Use another linear approximation to predict the population in 2006.

t	1980	1985	1990	1995	2000
N t	15.0	17.0	19.3	22.0	24.9

5–8 Find the linearization $L \quad x$ of the function a at .

5. $f \quad x \quad x^3, \quad a \quad 1$
6. $f \quad x \quad \ln x, \quad a \quad 1$
7. $f \quad x \quad \cos x, \quad a \quad 2$
8. $f \quad x \quad \sqrt[3]{x}, \quad a \quad 8$

9. Find the linear approximation of the function $f \quad x \quad \sqrt{1 \quad x}$ at $a \quad 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. Illustrate by graphing f and the tangent line.

10. Find the linear approximation of the function $f \quad x \quad \sqrt[3]{1 \quad x}$ at $a \quad 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Illustrate by graphing and the tangent line.

11–14 Verify the given linear approximation at $a \quad 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

11. $\sqrt[3]{1 \quad x} \quad 1 \quad \frac{1}{3}x$
12. $\tan x \quad x$
13. $1 \quad 1 \quad 2x^{-4} \quad 1 \quad 8x$
14. $e^x \quad 1 \quad x$

15–20 Find the differential of the function.

15. $y = x^4 - 5x$

16. $y = \cos x$

17. $y = x \ln x$

18. $y = \sqrt{1 - t^2}$

19. $y = \dfrac{u - 1}{u + 1}$

20. $y = (1 + 2r)^{-4}$

21–26 (a) Find the differential dy and (b) evaluate dy for the given values of x and dx.

21. $y = x^2 - 2x,\quad x = 3,\quad dx = \frac{1}{2}$

22. $y = e^{x^4},\quad x = 0,\quad dx = 0.1$

23. $y = \sqrt{4 - 5x},\quad x = 0,\quad dx = 0.04$

24. $y = \dfrac{1 - x}{1 + x},\quad x = 1,\quad dx = 0.01$

25. $y = \tan x,\quad x = \pi/4,\quad dx = 0.1$

26. $y = \cos x,\quad x = \pi/3,\quad dx = 0.05$

27–30 Compute Δy and dy for the given values of x and $dx = \Delta x$. Then sketch a diagram like Figure 6 showing the line segments with lengths Δx, dx, Δy, and dy.

27. $y = x^2,\quad x = 1,\quad \Delta x = 0.5$

28. $y = \sqrt{x},\quad x = 1,\quad \Delta x = 1$

29. $y = 6 - x^2,\quad x = 2,\quad \Delta x = 0.4$

30. $y = 16/x,\quad x = 4,\quad \Delta x = 1$

31–36 Use differentials (or, equivalently, a linear approximation) to estimate the given number.

31. $(2.001)^5$

32. $\sqrt{99.8}$

33. $(8.06)^{2/3}$

34. $1/1002$

35. $\tan 44°$

36. $\ln 1.07$

37–39 Explain, in terms of linear approximations or differentials, why the approximation is reasonable.

37. $\sec 0.08 \approx 1$

38. $(1.01)^6 \approx 1.06$

39. $\ln 1.05 \approx 0.05$

40. Let $f(x) = (x + 1)^2$ and $g(x) = e^{2x}$ and $h(x) = 1 + \ln(1 + 2x)$.

(a) Find the linearizations of f, g, and h at $a = 0$. What do you notice? How do you explain what happened?

(b) Graph f, g, and h and their linear approximation. For which function is the linear approximation best? For which is it worst? Explain.

41. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

42. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.
(a) Use differentials to estimate the maximum error in the calculated area of the disk.
(b) What is the relative error? What is the percentage error?

43. The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.
(a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error?
(b) Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

44. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

45. (a) Use differentials to find a formula for the approximate volume of a thin cylindrical shell with height h, inner radius r, and thickness Δr.
(b) What is the error involved in using the formula from part (a)?

46. When blood flows along a blood vessel, the flow F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

(This is known as Poiseuille's Law; we will show why it is true in Section 8.4.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

Show that the relative change in F is about four times the relative change in R. How will a 5% increase in the radius affect the flow of blood?

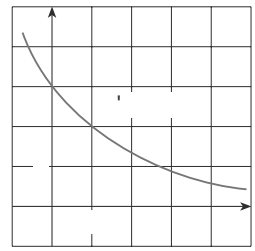
47. Establish the following rules for working with differentials (where c denotes a constant and u and v are functions of x).
(a) $dc = 0$
(b) $d(cu) = c\,du$
(c) $d(u + v) = du + dv$
(d) $d(uv) = u\,dv + v\,du$
(e) $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$
(f) $d(x^n) = nx^{n-1}dx$

48. On page 431 of *Physics: Calculus*, 2d ed., by Eugene Hecht (Pacific Grove, CA: Brooks/Cole, 2000), in the course of deriving the formula $T \quad 2 \quad \overline{L}$ for the period of a pendulum of length L, the author obtains the equation $\quad \sin$ for the tangential acceleration of the bob of the pendulum. He then says, "For small angles, the value of in radians is very nearly the value of $\sin$; they differ by less than 2% out to about 20°."
(a) Verify the linear approximation at 0 for the sine function:

$$\sin x \quad x$$

(b) Use a graphing device to determine the values of x for which $\sin x$ and x differ by less than 2%. Then verify Hecht's statement by converting from radians to degrees.

49. Suppose that the only information we have about a function f is that $f \; 1 \quad 5$ and the graph of its derivative is as shown.
(a) Use a linear approximation to estimate $f \; 0.9$ and $f \; 1.1$.
(b) Are your estimates in part (a) too large or too small? Explain.



50. Suppose that we don't have a formula for x but we know that $2 \quad 4$ and $x \quad \overline{x^2 \quad 5}$ for all x.
(a) Use a linear approximation to estimate 1.95 and 2.05.
(b) Are your estimates in part (a) too large or too small? Explain.

LABORATORY PROJECT

Taylor Polynomials

The tangent line approximation $L x$ is the best first-degree (linear) approximation to $f x$ near $x \quad a$ because $f x$ and $L x$ have the same rate of change (derivative) at a. For a better approximation than a linear one, let's try a second-degree (quadratic) approximation $P x$. In other words, we approximate a curve by a parabola instead of by a straight line. To make sure that the approximation is a good one, we stipulate the following:

(i) $P a \quad f a$ (P and f should have the same value at a.)

(ii) $P \; a \quad f \; a$ (P and f should have the same rate of change at a.)

(iii) $P \; a \quad f \; a$ (The slopes of P and f should change at the same rate.)

1. Find the quadratic approximation $P x \quad A \quad Bx \quad Cx^2$ to the function $f x \quad \cos x$ that satisfies conditions (i), (ii), and (iii) with $a \quad 0$. Graph P, f, and the linear approximation $L x \quad 1$ on a common screen. Comment on how well the functions L and P approximate f.

2. Determine the values of x for which the quadratic approximation $P x$ in Problem 1 is accurate to within 0.1. [Hint: Graph $y \quad P x$, $y \quad \cos x \quad 0.1$, and $y \quad \cos x \quad 0.1$ on a common screen.]

3. To approximate a function f by a quadratic function P near a number a, it is best to write P in the form

$$P x \quad A \quad B x \quad a \quad C x \quad a^2$$

Show that the quadratic function that satisfies conditions (i), (ii), and (iii) is

$$P x \quad f a \quad f \; a \; x \quad a \quad \tfrac{1}{2} f \; a \; x \quad a^2$$

4. Find the quadratic approximation to $x \quad \overline{x \quad 3}$ near 1. Graph f, the quadratic approximation, and the linear approximation from Example 3 in Section 3.11 on a common screen. What do you conclude?

5. Instead of being satisfied with a linear or quadratic approximation to $f x$ near a, let's try to find better approximations with higher-degree polynomials. We look for an nth-degree polynomial

$$T_n x \quad c_0 \quad c_1 x \quad a \quad c_2 x \quad a^2 \quad c_3 x \quad a^3 \quad \quad c_n x \quad a^n$$

such that T_n and its first n derivatives have the same values as f at a and its n derivatives. By differentiating repeatedly and setting $x = a$, show that these conditions are satisfied if $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{2}f''(a)$, and in general

$$c_k = \frac{f^{(k)}(a)}{k!}$$

where $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots k$. The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

is called the nth-degree Taylor polynomial of f centered at a.

6. Find the 8th-degree Taylor polynomial centered at 0 for the function $f(x) = \cos x$. Graph f together with the Taylor polynomials T_2, T_4, T_6, T_8 in the viewing rectangle $[-5, 5]$ by $[-1.4, 1.4]$ and comment on how well they approximate f.

3 Review

CONCEPT CHECK

1. State each of the following differentiation rules both in symbols and in words.
   (a) The Power Rule
   (b) The Constant Multiple Rule
   (c) The Sum Rule
   (d) The Difference Rule
   (e) The Product Rule
   (f) The Quotient Rule
   (g) The Chain Rule

2. State the derivative of each function.

(a) $y = x^n$	(b) $y = e^x$	(c) $y = a^x$
(d) $y = \ln x$	(e) $y = \log_a x$	(f) $y = \sin x$
(g) $y = \cos x$	(h) $y = \tan x$	(i) $y = \csc x$
(j) $y = \sec x$	(k) $y = \cot x$	(l) $y = \sin^{-1} x$
(m) $y = \cos^{-1} x$	(n) $y = \tan^{-1} x$	(o) $y = \sinh x$
(p) $y = \cosh x$	(q) $y = \tanh x$	(r) $y = \sinh^{-1} x$
(s) $y = \cosh^{-1} x$	(t) $y = \tanh^{-1} x$	

3. (a) How is the number e defined?
   (b) Express e as a limit.
   (c) Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = a^x$?
   (d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_a x$?

4. (a) Explain how implicit differentiation works.
   (b) Explain how logarithmic differentiation works.

5. What are the second and third derivatives of a function f? If f is the position function of an object, how can you interpret f'' and f'''?

6. (a) Write an expression for the linearization of f at a.
   (b) If $y = f(x)$, write an expression for the differential dy.
   (c) If $dx = \Delta x$, draw a picture showing the geometric meanings of Δy and dy.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

2. If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x) + f(x)g(x)$$

3. If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

4. If f is differentiable, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$.

5. If f is differentiable, then $\frac{d}{dx}f(\sqrt{x}) = \frac{f'(x)}{2\sqrt{x}}$.

6. If $y = e^2$, then $y' = 2e$.

7. $\frac{d}{dx}\ 10^x \quad x10^{x\ \ 1}$

8. $\frac{d}{dx}\ \ln 10 \quad \frac{1}{10}$

9. $\frac{d}{dx}\ \tan^2 x \quad \frac{d}{dx}\ \sec^2 x$

10. $\frac{d}{dx}\ x^2 \quad x \quad 2x \quad 1$

11. If $x \quad x^5$, then $\lim_{x\ \ 2} \frac{x \quad 2}{x \quad 2} \quad 80$.

12. $\frac{d^2y}{dx^2} \quad \frac{dy}{dx}^{\ 2}$

13. An equation of the tangent line to the parabola $y \quad x^2$ at $2, 4$ is $y \quad 4 \quad 2x\ x \quad 2$.

EXERCISES

1Ð48 |||| Calculate y .

1. $y \quad x^4 \quad 3x^2 \quad 5^{\ 3}$

2. $y \quad \cos\tan x$

3. $y \quad \bar{x} \quad \frac{1}{\sqrt[3]{x^4}}$

4. $y \quad \frac{3x \quad 2}{\sqrt{2x \quad 1}}$

5. $y \quad 2x\sqrt{x^2 \quad 1}$

6. $y \quad \frac{e^x}{1 \quad x^2}$

7. $y \quad e^{\sin 2}$

8. $y \quad e^{\ t}\ t^2 \quad 2t \quad 2$

9. $y \quad \frac{t}{1 \quad t^2}$

10. $y \quad \sin^{\ 1} e^x$

11. $y \quad xe^{\ 1\ x}$

12. $y \quad x^r e^{sx}$

13. $y \quad \tan\sqrt{1 \quad x}$

14. $y \quad \frac{1}{\sin x \quad \sin x}$

15. $xy^4 \quad x^2y \quad x \quad 3y$

16. $y \quad \ln\csc 5x$

17. $y \quad \frac{\sec 2}{1 \quad \tan 2}$

18. $x^2\cos y \quad \sin 2y \quad xy$

19. $y \quad e^{cx}\ c\sin x \quad \cos x$

20. $y \quad \ln x^2e^x$

21. $y \quad e^{e^x}$

22. $y \quad \sec 1 \quad x^2$

23. $y \quad 1 \quad x^{\ 1}\ \ ^{1}$

24. $y \quad 1\ \sqrt[3]{x \quad \bar{x}}$

25. $\sin xy \quad x^2 \quad y$

26. $y \quad \sqrt{\sin \quad \bar{x}}$

27. $y \quad \log_5 1 \quad 2x$

28. $y \quad \cos x^{\ x}$

29. $y \quad \ln\sin x \quad \frac{1}{2}\sin^2 x$

30. $y \quad \frac{x^2 \quad 1^{\ 4}}{2x \quad 1^{\ 3}\ 3x \quad 1^{\ 5}}$

31. $y \quad x\tan^{\ 1} 4x$

32. $y \quad e^{\cos x} \quad \cos e^x$

33. $y \quad \ln\ \sec 5x \quad \tan 5x$

34. $y \quad 10^{\tan}$

35. $y \quad \cot 3x^2 \quad 5$

36. $y \quad \sqrt{t\ln t^4}$

37. $y \quad \sin\left(\tan\sqrt{1 \quad x^3}\right)$

38. $y \quad \arctan\left(\arcsin \quad \bar{x}\right)$

39. $y \quad \tan^2 \sin$

40. $xe^y \quad y \quad 1$

41. $y \quad \frac{\sqrt{x \quad 1}\ 2 \quad x^{\ 5}}{x \quad 3^{\ 7}}$

42. $y \quad \frac{x \quad ^{4}}{x^4 \quad ^{4}}$

43. $y \quad x\sinh x^2$

44. $y \quad \frac{\sin mx}{x}$

45. $y \quad \ln\cosh 3x$

46. $y \quad \ln\frac{x^2 \quad 4}{2x \quad 5}$

47. $y \quad \cosh^{\ 1}\sinh x$

48. $y \quad x\tanh^{\ 1} \bar{x}$

49. If $f\ t \quad \sqrt{4t \quad 1}$, Þnd $f \quad 2$.

50. If $\sin$, Þnd 6 .

51. Find y if $x^6 \quad y^6 \quad 1$.

52. Find $f^{\ n}\ x$ if $f\ x \quad 1\ \ 2 \quad x$.

53. Use mathematical induction to show that if $f\ x \quad xe^x$, then $f^{\ n}\ x \quad x \quad n\ e^x$.

54. Evaluate $\lim_{t\ \ 0} \frac{t^3}{\tan^3 2t}$.

55Ð59 |||| Find an equation of the tangent to the curve at the given point.

55. $y \quad 4\sin^2 x, \quad 6, 1$

56. $y \quad \frac{x^2 \quad 1}{x^2 \quad 1}, \quad 0, \quad 1$

57. $y \quad \sqrt{1 \quad 4\sin x}, \quad 0, 1$

58. $x^2 \quad 4xy \quad y^2 \quad 13, \quad 2, 1$

59. $y \quad 2 \quad x\ e^{\ x}, \quad 0, 2$

60. If $f\ x \quad xe^{\sin x}$, Þnd $f \quad x$. Graph f and f on the same screen and comment.

61. (a) If $f\ x \quad x\ \sqrt{5 \quad x}$, Þnd $f \quad x$.
(b) Find equations of the tangent lines to the curve $y \quad x\ \sqrt{5 \quad x}$ at the points $1, 2$ and $4, 4$.
(c) Illustrate part (b) by graphing the curve and tangent lines on the same screen.
(d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f .

62. (a) If $f\ x \quad 4x \quad \tan x, \quad 2 \quad x \quad 2$, Þnd f and f .
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, , and f .

63. At what points on the curve $y = \sin x + \cos x$ $0 \le x \le 2\pi$, is the tangent line horizontal?

64. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

65. If $f(x) = (x - a)(x - b)(x - c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$$

66. (a) By differentiating the double-angle formula

$$\cos 2x = \cos^2 x - \sin^2 x$$

obtain the double-angle formula for the sine function.

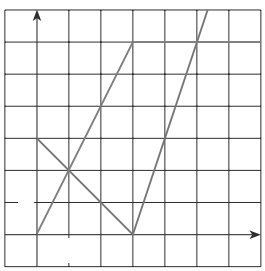
(b) By differentiating the addition formula

$$\sin(x + a) = \sin x \cos a + \cos x \sin a$$

obtain the addition formula for the cosine function.

67. Suppose that $h(x) = f(x)g(x)$ and $F(x) = f(x)/g(x)$, where $f(2) = 3$, $g(2) = 5$, $f'(2) = 4$, $g'(2) = 2$, and $f(5) = 11$. Find (a) $h'(2)$ and (b) $F'(2)$.

68. If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find (a) $P'(2)$, (b) $Q'(2)$, and (c) $C'(2)$.



69–76 Find f' in terms of g'.

69. $f(x) = x^2 g(x)$

70. $f(x) = [g(x)]^2$

71. $f(x) = [g(x)]^2$

72. $f(x) = g(x^2)$

73. $f(x) = e^{g(x)}$

74. $f(x) = e^{-g(x)}$

75. $f(x) = \ln[g(x)]$

76. $f(x) = \ln g(x)$

77–79 Find h' in terms of f' and g'.

77. $h(x) = \dfrac{f(x)g(x)}{f(x) - g(x)}$

78. $h(x) = \dfrac{\sqrt{f(x)}}{g(x)}$

79. $h(x) = f(\sin 4x)$

80. (a) Graph the function $f(x) = x + 2\sin x$ in the viewing rectangle $[0, 8]$ by $[-2, 8]$.

(b) On which interval is the average rate of change larger: $[1, 2]$ or $[2, 3]$?

(c) At which value of x is the instantaneous rate of change larger: $x = 2$ or $x = 5$?

(d) Check your visual estimates in part (c) by computing $f'(x)$ and comparing the numerical values for $x = 2$ and $x = 5$.

81. At what point on the curve $y = \ln(x^2 + 4)$ is the tangent horizontal?

82. (a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.

(b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

83. Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = 1$ and $x = 5$ have slopes 6 and -2, respectively.

84. The function $C(t) = K(e^{-at} - e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.

(a) Show that $\lim_{t \to \infty} C(t) = 0$.

(b) Find $C'(t)$, the rate at which the drug is cleared from circulation.

(c) When is this rate equal to 0?

85. An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

86. A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2 t^2}$, $t \ge 0$, where b and c are positive constants.

(a) Find the velocity and acceleration functions.

(b) Show that the particle always moves in the positive direction.

87. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t - 3$, $t \ge 0$.

(a) Find the velocity and acceleration functions.

(b) When is the particle moving upward and when is it moving downward?

(c) Find the distance that the particle travels in the time interval $0 \le t \le 3$.

88. The volume of a right circular cone is $V = \frac{\pi}{3} r^2 h$, where r is the radius of the base and h is the height.

(a) Find the rate of change of the volume with respect to the height if the radius is constant.

(b) Find the rate of change of the volume with respect to the radius if the height is constant.

89. The mass of part of a wire is $s(x) = \sqrt{1 + x}$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 4$ m.

90. The cost, in dollars, of producing x units of a certain commodity is

$$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$$

(a) Find the marginal cost function.

(b) Find $C'(100)$ and explain its meaning.

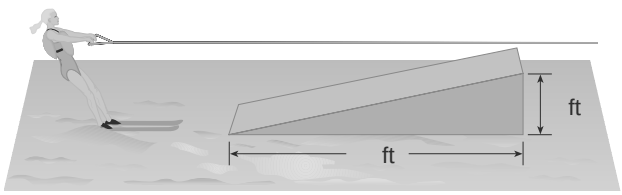
(c) Compare $C'(100)$ with the cost of producing the 101st item.

91. The volume of a cube is increasing at a rate of $10\ \text{cm}^3/\text{min}$. How fast is the surface area increasing when the length of an edge is 30 cm?

92. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

93. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon, it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?

94. A waterskier skis over the ramp shown in the Þgure at a speed of 30 ft/s. How fast is she rising as she leaves the ramp?



95. The angle of elevation of the Sun is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the Sun is $\pi/6$?

96. (a) Find the linear approximation for $f(x) = \sqrt{25 - x^2}$ near 3.
(b) Illustrate part (a) by graphing f and the linear approximation.
(c) For what values of x is the linear approximation accurate to within 0.1?

97. (a) Find the linearization of $f(x) = \sqrt[3]{1 + 3x}$ at $a = 0$. State the corresponding linear approximation and use it to give an approximate value for $\sqrt[3]{1.03}$.
(b) Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1.

98. Evaluate dy if $y = x^3 - 2x^2 + 1$, $x = 2$, and $dx = 0.2$.

99. A window has the shape of a square surmounted by a semicircle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in computing the area of the window.

100–102 Express the limit as a derivative and evaluate.

100. $\lim_{x \to 1} \dfrac{x^{17} - 1}{x - 1}$

101. $\lim_{h \to 0} \dfrac{\sqrt[4]{16 + h} - 2}{h}$

102. $\lim_{\theta \to \pi/3} \dfrac{\cos\theta - 0.5}{\theta - \pi/3}$

103. Evaluate $\lim_{x \to 0} \dfrac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$.

104. Suppose f is a differentiable function such that $f(x) = x$ and $f'(x) = 1 + f(x)^2$. Show that $f(x) = \tan(x)$.

105. Find $f(x)$ if it is known that

$$\frac{d}{dx} f(2x) = x^2$$

106. Show that the length of the portion of any tangent line to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant.

PROBLEMS PLUS

Before you look at the example, cover up the solution and try it yourself first.

EXAMPLE 1 How many lines are tangent to both of the parabolas $y = 1 + x^2$ and $y = 1 - x^2$? Find the coordinates of the points at which these tangents touch the parabolas.

SOLUTION In gain insight into this problem, it is essential to draw a diagram. So we sketch the parabolas $y = 1 + x^2$ (which is the standard parabola $y = x^2$ shifted 1 unit upward) and $y = 1 - x^2$ (which is obtained by reflecting the first parabola about the x-axis). If we try to draw a line tangent to both parabolas, we soon discover that there are only two possibilities, as illustrated in Figure 1.

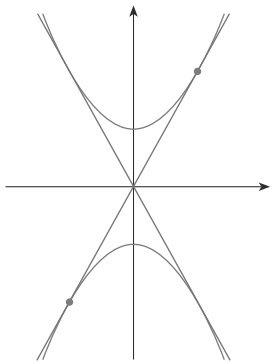


FIGURE 1

Let P be a point at which one of these tangents touches the upper parabola, a and let be its x-coordinate. (The choice of notation for the unknown is important. Of course we could have used b or c or x_0 or x_1 instead of a. However, it's not advisable to use place of a because that could be confused with the variable x in the equation of the parabola.) Then, since P lies on the parabola $y = 1 + x^2$, its y-coordinate must be $1 + a^2$. Because of the symmetry shown in Figure 1, the coordinates of the Q point where the tangent touches the lower parabola must be $a, -1 - a^2$.

To use the given information that the line is a tangent, we equate the slope of the line PQ to the slope of the tangent line at P. We have

$$m_{PQ} = \frac{1 + a^2 - (-1 - a^2)}{a - a} = \frac{1 + a^2}{a}$$

If $f(x) = 1 + x^2$, then the slope of the tangent line at P is $f'(a) = 2a$. Thus, the condition that we need to use is that

$$\frac{1 + a^2}{a} = 2a$$

Solving this equation, we get $1 + a^2 = 2a^2$, so $a^2 = 1$ and $a = 1$. Therefore, the points are (1, 2) and (1, -2). By symmetry, the two remaining points are $(-1, 2)$ and $(-1, -2)$.

EXAMPLE 2 For what values of c does the equation $\ln x = cx^2$ have exactly one solution?

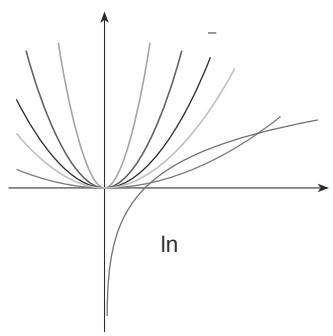


FIGURE 2

SOLUTION One of the most important principles of problem solving is to draw a diagram, even if the problem as stated doesn't explicitly mention a geometric situation. Our present problem can be reformulated geometrically as follows: For what values of c does the curve $y = \ln x$ intersect the curve $y = cx^2$ in exactly one point?

Let's start by graphing $y = \ln x$ and $y = cx^2$ for various values of c. We know that, for $c = 0$, $y = cx^2$ is a parabola that opens upward if $c > 0$ and downward if $c < 0$. Figure 2 shows the parabola $y = cx^2$ for several positive values of c. Most of them don't intersect $y = \ln x$ at all and one intersects twice. We have the feeling that there must be a value of c (somewhere between $c = 0.1$ and $c = 0.3$) for which the curves intersect exactly once, as in Figure 3.

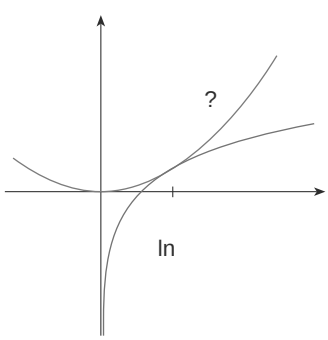


FIGURE 3

To find that particular value of c, we let a be the x-coordinate of the single point of intersection. In other words, $\ln a = ca^2$, so a is the unique solution of the given equation. We see from Figure 3 that the curves just touch, so they have a common tangent

line when $x = a$. That means the curves $y = \ln x$ and $y = cx^2$ have the same slope when $x = a$. Therefore

$$\frac{1}{a} = 2ca$$

Solving the equations $\ln a = ca^2$ and $\frac{1}{a} = 2ca$, we get

$$\ln a = ca^2 = c\frac{1}{2c} = \frac{1}{2}$$

Thus, $a = e^{1/2}$ and

$$c = \frac{\ln a}{a^2} = \frac{\ln e^{1/2}}{e} = \frac{1}{2e}$$

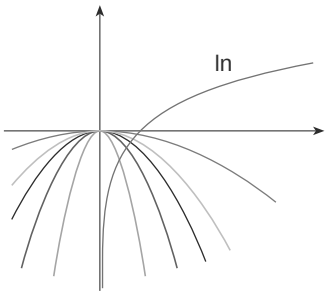


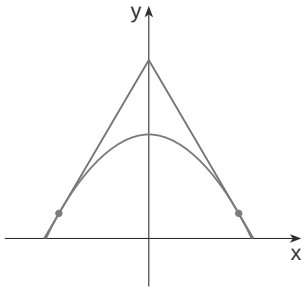
FIGURE 4

For negative values of c we have the situation illustrated in Figure 4: All parabolas $y = cx^2$ with negative values of c intersect $y = \ln x$ exactly once. And don't forget about $c = 0$: The curve $y = 0x^2 = 0$ is just the x-axis, which intersects $y = \ln x$ exactly once.

To summarize, the required values of c are $\frac{1}{2e}$ and 0. ∎

PROBLEMS

1. Find points P and Q on the parabola $y = 1 - x^2$ so that the triangle ABC formed by the x-axis and the tangent lines at P and Q is an equilateral triangle.



2. Find the point where the curves $y = x^3 - 3x - 4$ and $y = 3x^2 - x$ are tangent to each other, that is, have a common tangent line. Illustrate by sketching both curves and the common tangent.

3. Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.

4. A car is traveling at night along a highway shaped like a parabola with its vertex at the origin (see the Figure). The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a statue located 100 m east and 50 m north of the origin. At what point on the highway will the car headlights illuminate the statue?

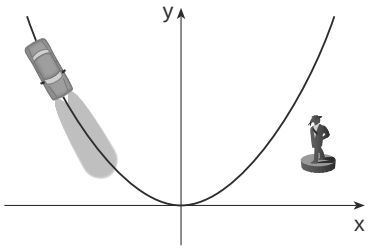


FIGURE FOR PROBLEM 4

5. Prove that $\frac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}(\cos 4x + n - 2)$.

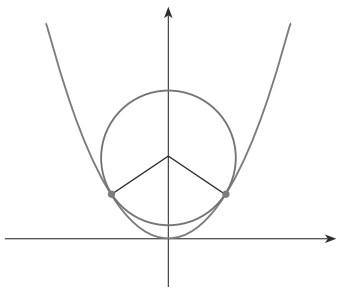


FIGURE FOR PROBLEM 7

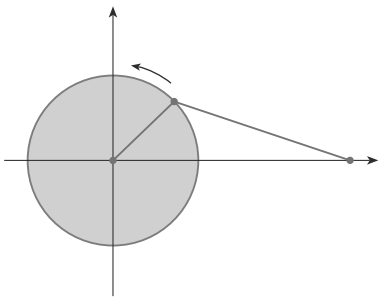


FIGURE FOR PROBLEM 9

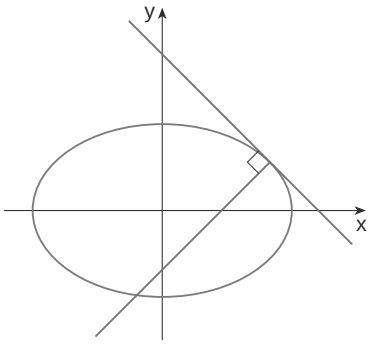


FIGURE FOR PROBLEM 13

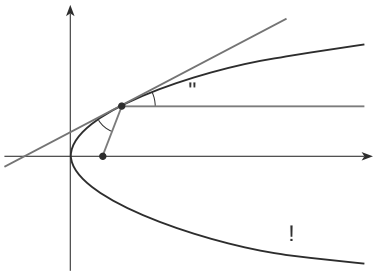


FIGURE FOR PROBLEM 16

6. Find the nth derivative of the function $f(x) = x^n/(1-x)$.

7. The figure shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.

8. If f is differentiable at a, where $a \neq 0$, evaluate the following limit in terms of f and a:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x^2 - a^2}$$

9. The figure shows a rotating wheel with radius 40 cm and a connecting rod AP with length 1.2 m. The pin P slides back and forth along the x-axis as the wheel rotates counterclockwise at a rate of 360 revolutions per minute.
   (a) Find the angular velocity of the connecting rod, $d\theta/dt$, in radians per second, when $\theta = \pi/3$.
   (b) Express the distance $x = OP$ in terms of θ.
   (c) Find an expression for the velocity of the pin P in terms of θ.

10. Tangent lines T_1 and T_2 are drawn at two points P_1 and P_2 on the parabola $y = x^2$ and they intersect at a point P. Another tangent line T is drawn at a point between P_1 and P_2; it intersects T_1 at Q_1 and T_2 at Q_2. Show that

$$\frac{PQ_1}{PP_1} + \frac{PQ_2}{PP_2} = 1$$

11. Show that

$$\frac{d^n}{dx^n}\left(e^{ax}\sin bx\right) = r^n e^{ax}\sin(bx + n\theta)$$

where a and b are positive numbers, $r = \sqrt{a^2 + b^2}$, and $\theta = \tan^{-1}(b/a)$.

12. Evaluate $\lim_{x \to 0} \dfrac{e^{\sin x} - 1}{x}$.

13. Let T and N be the tangent and normal lines to the ellipse $x^2/9 + y^2/4 = 1$ at any point P on the ellipse in the first quadrant. Let x_T and y_T be the x- and y-intercepts of T and x_N and y_N be the intercepts of N. As P moves along the ellipse in the first quadrant (but not on the axes), what values can x_T, y_T, x_N, and y_N take on? First try to guess the answers just by looking at the figure. Then use calculus to solve the problem and see how good your intuition is.

14. Evaluate $\lim_{x \to 0} \dfrac{\sin(3 + x^2) - \sin 9}{x}$.

15. (a) Use the identity for $\tan(x - y)$ (see Equation 14b in Appendix D) to show that if two lines L_1 and L_2 intersect at an angle θ, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

   where m_1 and m_2 are the slopes of L_1 and L_2, respectively.
   (b) The angle between the curves C_1 and C_2 at a point of intersection P is defined to be the angle between the tangent lines to C_1 and C_2 at P (if these tangent lines exist). Use part (a) to find, correct to the nearest degree, the angle between each pair of curves at each point of intersection.
   (i) $y = x^2$ and $y = (x - 2)^2$
   (ii) $x^2 + y^2 = 3$ and $x^2 + 4x + y^2 = 3 + 0$

16. Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4px$ with focus $F(p, 0)$. Let α be the angle between the parabola and the line segment FP, and let β be the angle between the horizontal line $y = y_1$ and the parabola as in the figure. Prove that $\alpha = \beta$. (Thus, by a principle of geometrical optics, light from a source placed at F will be reflected along a line parallel to the x-axis. This explains why paraboloids, the surfaces obtained by rotating parabolas about their axes, are used as the shape of some automobile headlights and mirrors for telescopes.)

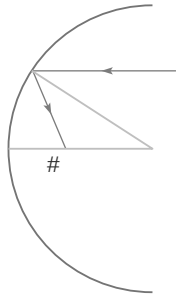


FIGURE FOR PROBLEM 17

17. Suppose that we replace the parabolic mirror of Problem 16 by a spherical mirror. Although the mirror has no focus, we can show the existence of an approximate focus. In the figure, C is a semicircle with center O. A ray of light coming in toward the mirror parallel to the axis along the line PQ will be reflected to the point R on the axis so that PQO OQR (the angle of incidence is equal to the angle of reflection). What happens to the point R as P is taken closer and closer to the axis?

18. If f and are differentiable functions with f 0 0 0 and 0 0, show that

$$\lim_{x \to 0} \frac{f(x)}{x} = \frac{f(0)}{0}$$

19. Evaluate $\lim_{x \to 0} \dfrac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}$.

CAS 20. (a) The cubic function $f(x) = x(x - 2)(x - 6)$ has three distinct zeros: 0, 2, and 6. Graph and its tangent lines at the average of each pair of zeros. What do you notice?
(b) Suppose the cubic function $f(x) = (x - a)(x - b)(x - c)$ has three distinct zeros: , , and c. Prove, with the help of a computer algebra system, that a tangent line drawn at the average of the zeros a and b intersects the graph of f at the third zero.

21. For what value of k does the equation $e^{2x} = kx$ have exactly one solution?

22. For which positive numbers a is it true that $a^x \geq 1 + x$ for x all ?

23. If

$$y = \frac{x}{a^2 + 1} - \frac{2}{a^2 + 1}\arctan\frac{a\sin x}{a^2 + 1 - \cos x}$$

show that $y = \dfrac{1}{a - \cos x}$.

24. Given an ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a > b > 0$, find the equation of the set of all points from which there are two tangents to the curve whose slopes are (a) reciprocals and (b) negative reciprocals.

25. Find the two points on the curve $y = x^4 + 2x^2 - x$ that have a common tangent line.

26. Suppose that three points on the parabola $y = ax^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x-coordinates is 0.

27. A lattice point in the plane is a point with integer coordinates. Suppose that circles with radius r are drawn using all lattice points as centers. Find the smallest value of r such that any line with slope $\frac{2}{5}$ intersects some of these circles.

28. A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

29. A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes through the sides at a rate proportional to the area of the container that is in contact with the liquid. (The surface area of a cone is , where r is the radius and l is the slant height.) If we pour the liquid into the container at a rate of 2 cm^3/min, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container?

Scientists have tried to explain how rainbows are formed since the time of Aristotle. In the project on page 288, you will be able to use the principles of differential calculus to explain the formation, location, and colors of the rainbow.

Applications of Differen

We have already investigated some of the applications of derivatives, but now that we know the differentiation rules we are in a better position to pursue the applications of differentiation in greater depth. Here we learn how derivatives affect the shape of a graph of a function and, in particular, how they help us locate maximum and minimum values of functions. Many practical problems require us to minimize a cost or maximize an area or somehow Þnd the best possible outcome of a situation. In particular, we will be able to investigate the optimal shape of a can and to explain the location of rainbows in the sky.

4.1 Maximum and Minimum Values

Some of the most important applications of differential calculus are optimization problems, in which we are required to Þnd the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter:

What is the shape of a can that minimizes manufacturing costs?

What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)

What is the radius of a contracted windpipe that expels air most rapidly during a cough?

At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to Þnding the maximum or minimum values of a function. LetÕs Þrst explain exactly what we mean by maximum and minimum values.

1 Definition A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and minimum values of f are called **extreme values** of f.

Figure 1 shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point.

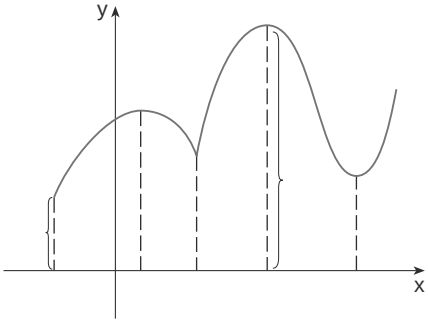


FIGURE 1
Minimum value $f(a)$, maximum value $f(d)$

In Figure 1, if we consider only values x of near [for instance, if we restrict our attention to the interval a, c], then $f\ b$ is the largest of those values $f\ x$ of and is called a local maximum value of f. Likewise, $f\ c$ is called a local minimum value of f because $f\ c \quad f\ x$ for x near c [in the interval b, d , for instance]. The function also has a local minimum at e. In general, we have the following definition.

> **2 Definition** A function f has a local maximum (or relative maximum) at c if $f\ c \quad f\ x$ when x is near c. [This means that $f\ c \quad f\ x$ for all x in some open interval containing c.] Similarly, f has a local minimum at c if $f\ c \quad f\ x$ when x is near c.

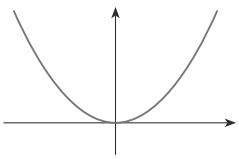


FIGURE 2
Minimum value 0, no maximum

EXAMPLE 1 The function $f\ x \quad \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2n \quad 1$ for any integer and $1 \quad \cos x \quad 1$ for all x. Likewise, $\cos 2n \quad 1 \quad 1$ is its minimum value, where is any integer.

EXAMPLE 2 If $f\ x \quad x^2$, then $f\ x \quad f\ 0$ because $x^2 \quad 0$ for all . Therefore $f\ 0 \quad 0$ is the absolute (and local) minimum value of . This corresponds to the fact that the origin is the lowest point on the parabola $y \quad x^2$. (See Figure 2.) However, there is no highest point on the parabola and so this function has no maximum value.

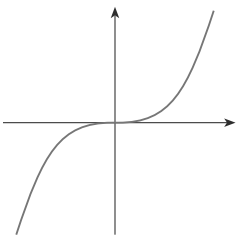


FIGURE 3
No minimum, no maximum

EXAMPLE 3 From the graph of the function $f\ x \quad x^3$, shown in Figure 3, we see that this function has neither an absolute maximum value nor an absolute minimum value. In fact, it has no local extreme values either.

EXAMPLE 4 The graph of the function

$$f\ x \quad 3x^4 \quad 16x^3 \quad 18x^2 \quad 1 \quad x \quad 4$$

is shown in Figure 4. You can see that $f\ 1 \quad 5$ is a local maximum, whereas the absolute maximum is $f\ 1 \quad 37$. (This absolute maximum is not a local maximum because it occurs at an endpoint.) Also $f\ 0 \quad 0$ is a local minimum and $f\ 3 \quad 27$ is both a local and an absolute minimum. Note that has neither a local nor an absolute maximum at $x \quad 4$.

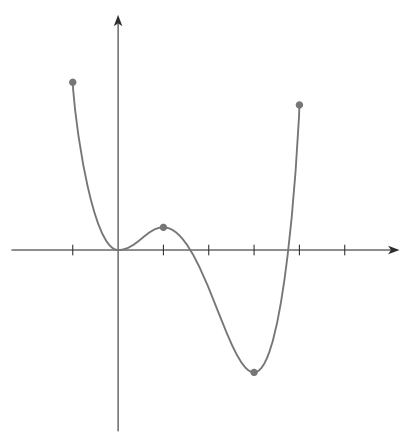


FIGURE 4

We have seen that some functions have extreme values, whereas others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Extreme Value Theorem is illustrated in Figure 5. Note that an extreme value can be taken on more than once. Although the Extreme Value Theorem is intuitively very plausible, it is difficult to prove and so we omit the proof.

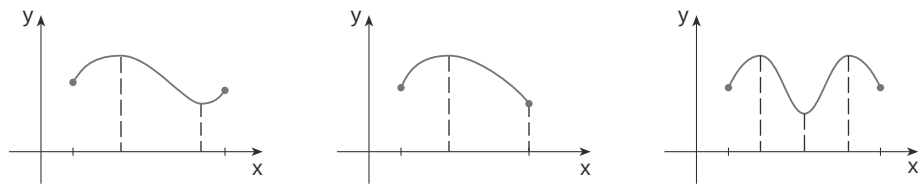


FIGURE 5

Figures 6 and 7 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

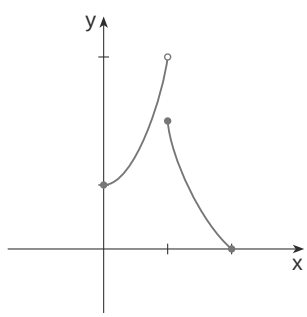


FIGURE 6
This function has minimum value , but no maximum value.

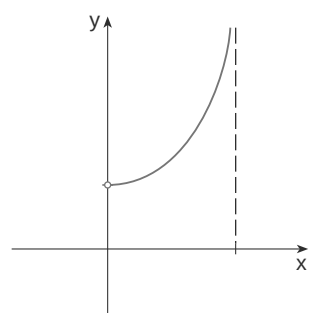


FIGURE 7
This continuous function has no maximum or minimum.

The function f whose graph is shown in Figure 6 is defined on the closed interval $[0, 2]$ but has no maximum value. (Notice that the range of f is $[0, 3)$. The function takes on values arbitrarily close to 3, but never actually attains the value 3.) This does not contradict the Extreme Value Theorem because f is not continuous. [Nonetheless, a discontinuous function could have maximum and minimum values. See Exercise 13(b).]

The function t shown in Figure 7 is continuous on the open interval $(0, 2)$ but has neither a maximum nor a minimum value. [The range of t is $[1, \infty)$. The function takes on arbitrarily large values.] This does not contradict the Extreme Value Theorem because the interval $(0, 2)$ is not closed.

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values. We start by looking for local extreme values.

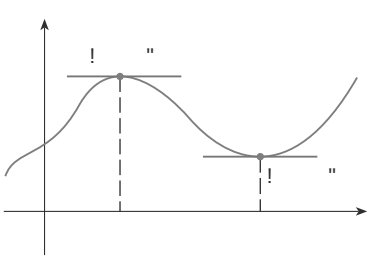


FIGURE 8

Figure 8 shows the graph of a function f with a local maximum at c and a local minimum at d. It appears that at the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0. We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$. The following theorem says that this is always true for differentiable functions.

|||| FermatÕs Theorem is named after Pierre Fermat (1601Ð1665), a French lawyer who took up mathematics as a hobby. Despite his amateur status, Fermat was one of the two inventors of analytic geometry (Descartes was the other). His methods for Þnding tangents to curves and maximum and minimum values (before the invention of limits and derivatives) made him a forerunner of Newton in the creation of differential calculus.

4 FermatÕs Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof Suppose, for the sake of definiteness, that f has a local maximum at c. Then, according to Definition 2, $f(c) \geqslant f(x)$ if x is sufÞciently close to c. This implies that if h is sufÞciently close to 0, with h being positive or negative, then

$$f(c) \geqslant f(c+h)$$

and therefore

$$f(c+h) - f(c) \leqslant 0 \tag{5}$$

We can divide both sides of an inequality by a positive number. Thus, if $0 < h$ and h is sufÞciently small, we have

$$\frac{f(c+h) - f(c)}{h} \leqslant 0$$

Taking the right-hand limit of both sides of this inequality (using Theorem 2.3.2), we get

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leqslant \lim_{h \to 0^+} 0 = 0$$

But since $f'(c)$ exists, we have

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

and so we have shown that $f'(c) \leqslant 0$.

If $h < 0$, then the direction of the inequality (5) is reversed when we divide by h:

$$\frac{f(c+h) - f(c)}{h} \geqslant 0 \qquad h < 0$$

So, taking the left-hand limit, we have

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geqslant 0$$

We have shown that $f'(c) \leqslant 0$ and also that $f'(c) \geqslant 0$. Since both of these inequalities must be true, the only possibility is that $f'(c) = 0$.

We have proved FermatÕs Theorem for the case of a local maximum. The case of a local minimum can be proved in a similar manner, or we could use Exercise 76 to deduce it from the case we have just proved (see Exercise 77). ∎

The following examples caution us against reading too much into FermatÕs Theorem. We canÕt expect to locate extreme values simply by setting $f'(x) = 0$ and solving for x.

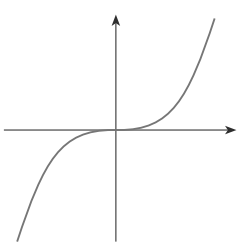


FIGURE 9
If # , then $ but Ä has no maximum or minimum.

EXAMPLE 5 If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$. But f has no maximum or minimum at 0, as you can see from its graph in Figure 9. (Or observe that $f(0) < f(x)$ for $x \neq 0$ but $x^3 > 0$ for $x > 0$.) The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a

horizontal tangent at $(0, 0)$. Instead of having a maximum or minimum at $(0, 0)$, the curve crosses its horizontal tangent there.

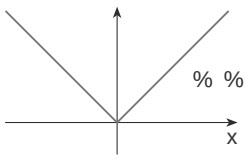


FIGURE 10
If # % % , then is a minimum value, but $ does not exist.

EXAMPLE 5 The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value cannot be found by setting $f'(x) = 0$ because, as was shown in Example 6 in Section 2.9, $f'(0)$ does not exist. (See Figure 10.)

WARNING Examples 5 and 6 show that we must be careful when using Fermat's Theorem. Example 5 demonstrates that even when $f'(c) = 0$ there need not be a maximum or minimum at c. (In other words, the converse of Fermat's Theorem is false in general.) Furthermore, there may be an extreme value even when $f'(c)$ does not exist (as in Example 6).

Fermat's Theorem does suggest that we should at least be looking for extreme values of $f(x)$ at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.

> **6 Definition** A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

|||| Figure 11 shows a graph of the function f in Example 7. It supports our answer because there is a horizontal tangent when $x = 1.5$ and a vertical tangent when $x = 0$.

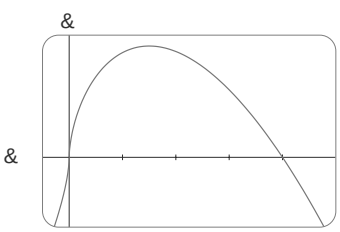


FIGURE 11

EXAMPLE 7 Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

SOLUTION The Product Rule gives

$$f'(x) = \tfrac{3}{5}x^{-2/5}(4 - x) - x^{3/5}(1) = \frac{3(4 - x)}{5x^{2/5}} - x^{3/5} = \frac{3(4 - x) - 5x}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

[The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.] Therefore, $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$. Thus, the critical numbers are $\frac{3}{2}$ and 0.

In terms of critical numbers, Fermat's Theorem can be rephrased as follows (compare Definition 6 with Theorem 4):

> **7** If f has a local maximum or minimum at c, then c is a critical number of f.

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local [in which case it occurs at a critical number by (7)] or it occurs at an endpoint of the interval. Thus, the following three-step procedure always works.

> **The Closed Interval Method** To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:
>
> 1. Find the values of f at the critical numbers of f in (a, b).
> 2. Find the values of f at the endpoints of the interval.
> 3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad \tfrac{1}{2} \le x \le 4$$

SOLUTION Since f is continuous on $\left[\tfrac{1}{2}, 4\right]$, we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

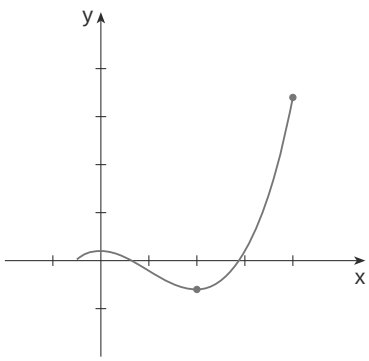


FIGURE 12

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $\left(\tfrac{1}{2}, 4\right)$. The values of f at these critical numbers are

$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the endpoints of the interval are

$$f\left(\tfrac{1}{2}\right) = \tfrac{1}{8} \qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Note that in this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number. The graph of f is sketched in Figure 12.

If you have a graphing calculator or a computer with graphing software, it is possible to estimate maximum and minimum values very easily. But, as the next example shows, calculus is needed to find the exact values.

EXAMPLE 9

(a) Use a graphing device to estimate the absolute minimum and maximum values of the function $f(x) = x - 2\sin x$, $0 \le x \le 2\pi$.

(b) Use calculus to find the exact minimum and maximum values.

SOLUTION

(a) Figure 13 shows a graph of f in the viewing rectangle $0 \le x \le 2\pi$, $-1 \le y \le 8$. By moving the cursor close to the maximum point, we see that the y-coordinate does not change very much in the vicinity of the maximum. The absolute maximum value is about 6.97 and it occurs when $x = 5.2$. Similarly, by moving the cursor close to the minimum point, we see that the absolute minimum value is about -0.68 and it occurs when $x = 1.0$. It is possible to get more accurate estimates by zooming in toward the maximum and minimum points, but instead let's use calculus.

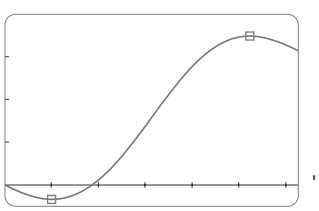


FIGURE 13

(b) The function $f(x) = x - 2\sin x$ is continuous on $[0, 2\pi]$. Since $f'(x) = 1 - 2\cos x$, we have $f'(x) = 0$ when $\cos x = \tfrac{1}{2}$ and this occurs when $x = \tfrac{\pi}{3}$ or $\tfrac{5\pi}{3}$. The values of f at these critical points are

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3} = -0.684853$$

and

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} = 6.968039$$

The values of f at the endpoints are

$$f(0) = 0 \quad \text{and} \quad f(2\pi) = 2\pi \approx 6.28$$

Comparing these four numbers and using the Closed Interval Method, we see that the absolute minimum value is $f(\pi/3) = \pi/3$ and the absolute maximum value is $f(5\pi/3) = 5\pi/3 = \sqrt{3}$. The values from part (a) serve as a check on our work.

EXAMPLE 10 The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s, is given by

$$v(t) = -0.001302t^3 + 0.09029t^2 - 23.61t + 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the acceleration of the shuttle between liftoff and the jettisoning of the boosters.

SOLUTION We are asked for the extreme values not of the given velocity function, but rather of the acceleration function. So we first need to differentiate to find the acceleration:

$$a(t) = v'(t) = \frac{d}{dt}\left(-0.001302t^3 + 0.09029t^2 - 23.61t + 3.083\right)$$

$$= -0.003906t^2 + 0.18058t - 23.61$$

We now apply the Closed Interval Method to the continuous function $a(t)$ on the interval $0 \le t \le 126$. Its derivative is

$$a'(t) = -0.007812t + 0.18058$$

The only critical number occurs when $a'(t) = 0$:

$$t_1 = \frac{0.18058}{0.007812} \approx 23.12$$

Evaluating $a(t)$ at the critical number and at the endpoints, we have

$$a(0) = -23.61 \qquad a(t_1) \approx -21.52 \qquad a(126) \approx 62.87$$

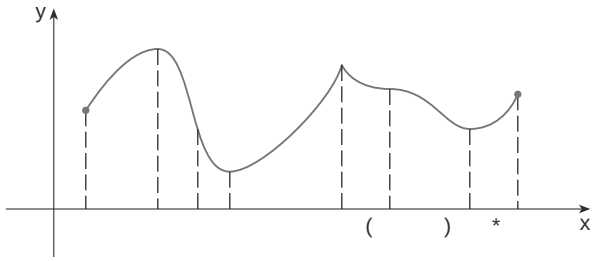
So the maximum acceleration is about 62.87 ft/s^2 and the minimum acceleration is about -21.52 ft/s^2.

4.1 Exercises

1. Explain the difference between an absolute minimum and a local minimum.
2. Suppose f is a continuous function defined on a closed interval $[a, b]$.
   (a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for f?
   (b) What steps would you take to find those maximum and minimum values?

3–4 ||| For each of the numbers a, b, c, d, e, r, s, and t, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

3.



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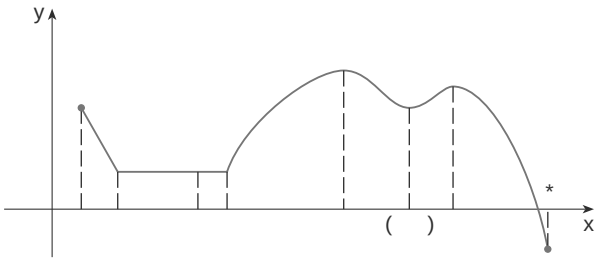
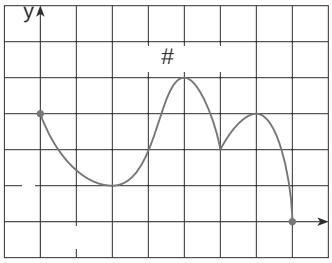
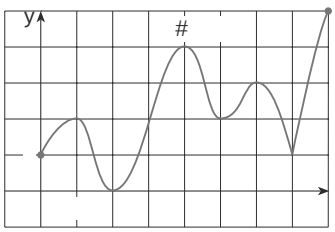


5–6 ||| Use the graph to state the absolute and local maximum and minimum values of the function.

5.



6.



7–10 ||| Sketch the graph of a function f that is continuous on [1, 5] and has the given properties.

7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4

8. Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4

9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4

10. f has no local maximum or minimum, but 2 and 4 are critical numbers

11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
(b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
(c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.

12. (a) Sketch the graph of a function on $[1, 2]$ that has an absolute maximum but no local maximum.
(b) Sketch the graph of a function on $[1, 2]$ that has a local maximum but no absolute maximum.

13. (a) Sketch the graph of a function on $[1, 2]$ that has an absolute maximum but no absolute minimum.
(b) Sketch the graph of a function on $[1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.

14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
(b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

15–30 ||| Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f. (Use the graphs and transformations of Sections 1.2 and 1.3.)

15. $f(x) = 8 - 3x, \quad x \geq 1$

16. $f(x) = 3 - 2x, \quad x \geq 5$

17. $f(x) = x^2, \quad 0 \leq x \leq 2$

18. $f(x) = x^2, \quad 0 \leq x \leq 2$

19. $f(x) = x^2, \quad 0 \leq x \leq 2$

20. $f(x) = x^2, \quad 0 \leq x \leq 2$

21. $f(x) = x^2, \quad -3 \leq x \leq 2$

22. $f(x) = |x - 1|^2, \quad -2 \leq x \leq 5$

23. $f(t) = 1 - |t|, \quad 0 \leq t \leq 1$

24. $f(t) = 1 - |t|, \quad 0 \leq t \leq 1$

25. $f(\ ) = \sin(\ ), \quad -2 \leq \ \leq 2$

26. $f(\ ) = \tan(\ ), \quad -4 \leq \ \leq 2$

27. $f(x) = 1 - \sqrt{x}$

28. $f(x) = e^x$

29. $f(x) = \begin{cases} 1 - x & \text{if } 0 \leq x \leq 2 \\ 2x - 4 & \text{if } 2 \leq x \leq 3 \end{cases}$

30. $f(x) = \begin{cases} x^2 & \text{if } -1 \leq x \leq 0 \\ 2 - x^2 & \text{if } 0 \leq x \leq 1 \end{cases}$

31–46 Find the critical numbers of the function.

31. $f(x) = 5x^2 + 4x$

32. $f(x) = x^3 - x^2 + x$

33. $f(x) = x^3 - 3x^2 - 24x$

34. $f(x) = x^3 - x^2 - x$

35. $s(t) = 3t^4 - 4t^3 - 6t^2$

36. $f(\ ) = \dfrac{1}{\ ^2 + 1}$

37. $(\ ) = \ - 2x + 3$

38. $(x) = x^{1/3}(x - 2)^{2/3}$

39. $(t) = 5t^{2/3} - t^{5/3}$

40. $(t) = \sqrt{t}(1 - t)$

41. $F(x) = x^{4/5}(x - 4)^2$

42. $G(x) = \sqrt[3]{x^2 - x}$

43. $f(\ ) = 2\cos\ + \sin^2\ $

44. $(\ ) = 4\ + \tan\ $

45. $f(x) = x\ln x$

46. $f(x) = xe^{2x}$

47–62 Find the absolute maximum and absolute minimum values of f on the given interval.

47. $f(x) = 3x^2 - 12x + 5,\quad [0, 3]$

48. $f(x) = x^3 - 3x + 1,\quad [0, 3]$

49. $f(x) = 2x^3 - 3x^2 - 12x + 1,\quad [-2, 3]$

50. $f(x) = x^3 - 6x^2 + 9x + 2,\quad [-1, 4]$

51. $f(x) = x^4 - 2x^2 + 3,\quad [-2, 3]$

52. $f(x) = (x^2 - 1)^3,\quad [-1, 2]$

53. $f(x) = \dfrac{x}{x^2 + 1},\quad [0, 2]$

54. $f(x) = \dfrac{x^2 - 4}{x^2 + 4},\quad [-4, 4]$

55. $f(t) = t\sqrt{4 - t^2},\quad [-1, 2]$

56. $f(t) = \sqrt[3]{t} - \dfrac{8}{\ }t,\quad [0, 8]$

57. $f(x) = \sin x + \cos x,\quad [0, \ ]$

58. $f(x) = x + 2\cos x,\quad [\ , \ ]$

59. $f(x) = xe^{-x},\quad [0, 2]$

60. $f(x) = \ln x - x,\quad [1, 3]$

61. $f(x) = x + 3\ln x,\quad [1, 4]$

62. $f(x) = e^{-x} - e^{-2x},\quad [0, 1]$

63. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1 - x)^b,\ 0 \le x \le 1$.

64. Use a graph to estimate the critical numbers of $f(x) = x^3 - 3x^2 + 2$ correct to one decimal place.

65–68

(a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.

(b) Use calculus to find the exact maximum and minimum values.

65. $f(x) = x^3 - 8x + 1,\quad -3 \le x \le 3$

66. $f(x) = e^{x^3 - x},\quad -1 \le x \le 0$

67. $f(x) = x\sqrt{x - x^2}$

68. $f(x) = \cos x + 2\sin x,\quad 0 \le x \le 2\ $

69. Between $0°C$ and $30°C$, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given approximately by the formula

$$V = 999.87 + 0.06426T - 0.0085043T^2 + 0.0000679T^3$$

Find the temperature at which water has its maximum density.

70. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle $\ $ with the plane, then the magnitude of the force is

$$F = \frac{W}{\sin\ + \ \cos\ }$$

where $\ $ is a positive constant called the coefficient of friction and where $0 \le \ \le \frac{\pi}{2}$. Show that F is minimized when $\tan\ = \ $.

71. A model for the food-price index (the price of a representative "basket" of foods) between 1984 and 1994 is given by the function

$$I(t) = 0.000904t^5 - 0.00143t^4 + 0.0656t^3 - 0.459t^2 + 0.627t + 99.3$$

where t is measured in years since midyear 1984, so $0 \le t \le 10$, and $I(t)$ is measured in 1987 dollars and scaled such that $I(3) = 100$. Estimate the times when food was cheapest and most expensive during the period 1984–1994.

72. On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

Event	Time (s)	Velocity (fts)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

(a) Use a graphing calculator or computer to find the cubic polynomial that best models the velocity of the shuttle for the time interval $t \in [0, 125]$. Then graph this polynomial.

(b) Find a model for the acceleration of the shuttle and use it to estimate the maximum and minimum values of the acceleration during the first 125 seconds.

73. When a foreign object lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward causing an increase in pressure in the lungs. This is accompanied by a contraction of the trachea, making a narrower channel for the expelled air to flow through. For a given amount of air to escape in a fixed time, it must move faster through the narrower channel than the wider one. The greater the velocity of the airstream, the greater the force on the foreign object. X rays show that the radius of the circular tracheal tube contracts to about two-thirds of its normal radius during a cough. According to a mathematical model of coughing, the velocity v of the airstream is related to the radius r of the trachea by the equation

$$v(r) = k(r_0 - r)r^2 \qquad \tfrac{1}{2}r_0 \le r \le r_0$$

where k is a constant and r_0 is the normal radius of the trachea. The restriction on r is due to the fact that the tracheal wall stiffens under pressure and a contraction greater than $\frac{1}{2}r_0$ is prevented (otherwise the person would suffocate).

(a) Determine the value of r in the interval $[\frac{1}{2}r_0, r_0]$ at which v has an absolute maximum. How does this compare with experimental evidence?

(b) What is the absolute maximum value of v on the interval?

(c) Sketch the graph of v on the interval $[0, r_0]$.

74. Show that 5 is a critical number of the function

$$f(x) = (x - 2)(x - 5)^3$$

but f does not have a local extreme value at 5.

75. Prove that the function

$$f(x) = x^{101} - x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

76. If f has a minimum value at c, show that the function $g(x) = -f(x)$ has a maximum value at c.

77. Prove Fermat's Theorem for the case in which f has a local minimum at c.

78. A cubic function is a polynomial of degree 3; that is, it has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

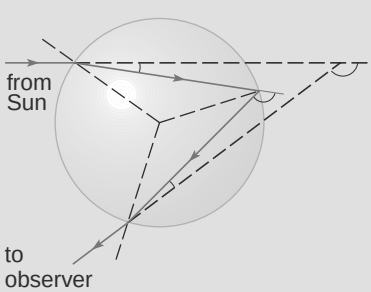
(a) Show that a cubic function can have two, one, or no critical number(s). Give examples and sketches to illustrate the three possibilities.

(b) How many local extreme values can a cubic function have?

APPLIED PROJECT

The Calculus of Rainbows

Rainbows are created when raindrops scatter sunlight. They have fascinated mankind since ancient times and have inspired attempts at scientific explanation since the time of Aristotle. In this project we use the ideas of Descartes and Newton to explain the shape, location, and colors of rainbows.



Formation of the primary rainbow

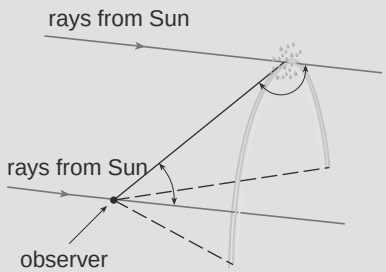
1. The figure shows a ray of sunlight entering a spherical raindrop at A. Some of the light is reflected, but the line AB shows the path of the part that enters the drop. Notice that the light is refracted toward the normal line AO and in fact Snell's law says that $\sin\alpha = k\sin\beta$, where α is the angle of incidence, β is the angle of refraction, and $k = \frac{4}{3}$ is the index of refraction for water. At B some of the light passes through the drop and is refracted into the air, but the line BC shows the part that is reflected. (The angle of incidence equals the angle of reflection.) When the ray reaches C, part of it is reflected, but for the time being we are more interested in the part that leaves the raindrop at C. (Notice that it is refracted away from the normal line.) The angle of deviation D is the amount of clockwise rotation that the ray has undergone during this three-stage process. Thus

$$D = \alpha - \beta + \alpha - \beta + \alpha - \beta$$

Show that the minimum value of the deviation D is 138° and occurs when $\alpha = 59.4^\circ$.

The significance of the minimum deviation is that when $\alpha = 59.4^\circ$ we have $\frac{dD}{d\alpha} = 0$, so $D' = 0$. This means that many rays with $\alpha = 59.4^\circ$ become deviated by approximately the same amount. It is the concentration of rays coming from near the direction of minimum

deviation that creates the brightness of the primary rainbow. The following figure shows that the angle of elevation from the observer up to the highest point on the rainbow is $180 \quad 138 \quad 42$. (This angle is called the rainbow angle.)



2. Problem 1 explains the location of the primary rainbow, but how do we explain the colors? Sunlight comprises a range of wavelengths, from the red range through orange, yellow, green, blue, indigo, and violet. As Newton discovered in his prism experiments of 1666, the index of refraction is different for each color. (The effect is called dispersion.) For red light the refractive index is $k \quad 1.3318$ whereas for violet light it is $k \quad 1.3435$. By repeating the calculation of Problem 1 for these values of k , show that the rainbow angle is about 42.3 for the red bow and 40.6 for the violet bow. So the rainbow really consists of seven individual bows corresponding to the seven colors.

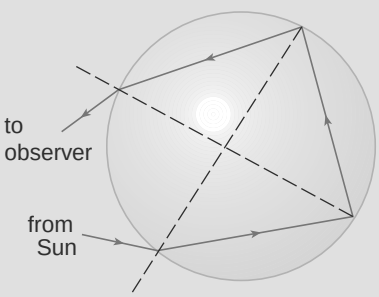
3. Perhaps you have seen a fainter secondary rainbow above the primary bow. That results from the part of a ray that enters a raindrop and is refracted at A , reflected twice (at B and C), and refracted as it leaves the drop at D (see Figure). This time the deviation angle D is the total amount of counterclockwise rotation that the ray undergoes in this four-stage process. Show that

$$D \quad 2 \quad 6 \quad 2$$

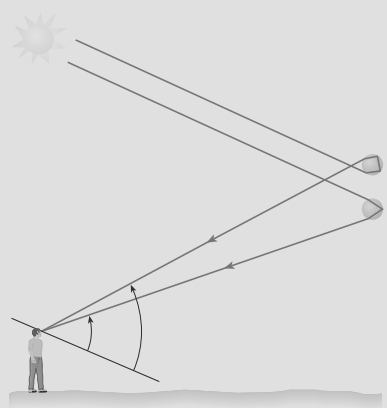
and D has a minimum value when

$$\cos \quad \frac{k^2 \quad 1}{8}$$

Taking $k \quad \frac{4}{3}$, show that the minimum deviation is about 129 and so the rainbow angle for the secondary rainbow is about 51 , as shown in Figure.



Formation of the secondary rainbow



4. Show that the colors in the secondary rainbow appear in the opposite order from those in the primary rainbow.

4.2 The Mean Value Theorem

We will see that many of the results of this chapter depend on one central fact, which is called the Mean Value Theorem. But to arrive at the Mean Value Theorem we first need the following result.

|||| RolleÕs Theorem was Þrst published in 1691 by the French mathematician Michel Rolle (1652Ð1719) in a book entitled

Méthode pour rŽsoudre les Žgalités. Later, however, he became a vocal critic of the methods of his day and attacked calculus as being a Òcollection of ingenious fallacies.Ó

RolleÕs Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Before giving the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses. Figure 1 shows the graphs of four such functions. In each case it appears that there is at least one point $(c, f(c))$ on the graph where the tangent is horizontal and therefore $f'(c) = 0$. Thus, RolleÕs Theorem is plausible.

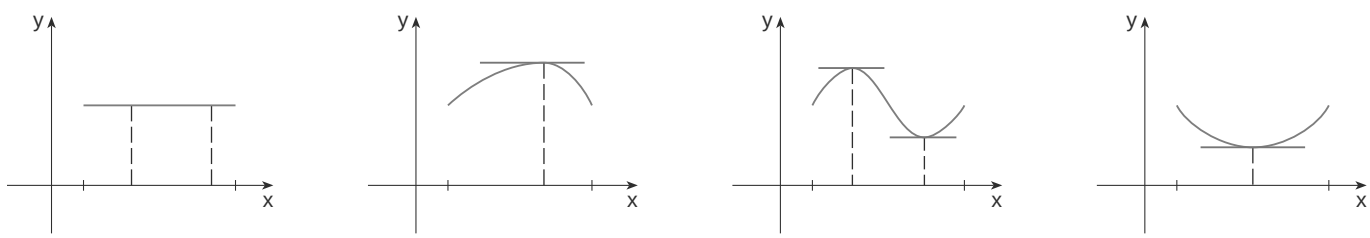


FIGURE 1

|||| Take cases

Proof There are three cases:

CASE I $f(x) = k$, a constant
Then $f'(x) = 0$, so the number c can be taken to be any number in (a, b).

CASE II $f(x) > f(a)$ for some x in (a, b) [as in Figure 1(b) or (c)]
By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a local maximum at c and, by hypothesis 2, f' is differentiable at c. Therefore $f'(c) = 0$ by Fermat's Theorem.

CASE III $f(x) < f(a)$ for some x in (a, b) [as in Figure 1(c) or (d)]
By the Extreme Value Theorem, f has a minimum value in $[a, b]$ and, since $f(a) = f(b)$, it attains this minimum value at a number c in (a, b). Again $f'(c) = 0$ by Fermat's Theorem. ■

EXAMPLE 1 Let's apply RolleÕs Theorem to the position function $s = f(t)$ of a moving object. If the object is in the same place at two different instants $t = a$ and $t = b$, then $f(a) = f(b)$. RolleÕs Theorem says that there is some instant of time c between a and b when $f'(c) = 0$; that is, the velocity is 0. (In particular, you can see that this is true when a ball is thrown directly upward.) ■

|||| Figure 2 shows a graph of the function $f(x) = x^3 + x + 1$ discussed in Example 2. Rolle's Theorem shows that, no matter how much we enlarge the viewing rectangle, we can never find a second x-intercept.

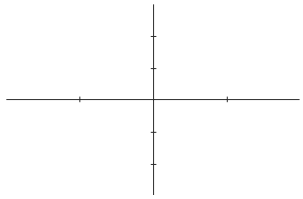


FIGURE 2

EXAMPLE 2 Prove that the equation $x^3 + x + 1 = 0$ has exactly one real root.

SOLUTION First we use the Intermediate Value Theorem (2.5.10) to show that a root exists. Let $f(x) = x^3 + x + 1$. Then $f(0) = 1 > 0$ and $f(1) = 1 < 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that $f(c) = 0$. Thus, the given equation has a root.

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction. Suppose that it had two roots a and b. Then $f(a) = 0 = f(b)$ and, since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$. But

$$f'(x) = 3x^2 + 1 \geq 1 \quad \text{for all } x$$

(since $x^2 \geq 0$) so $f'(x)$ can never be 0. This gives a contradiction. Therefore, the equation can't have two real roots. ∎

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a) \tag{2}$$

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions. The slope of the secant line AB is

$$m_{AB} = \frac{f(b) - f(a)}{b - a} \tag{3}$$

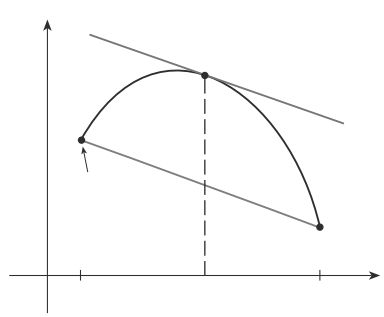


FIGURE 3

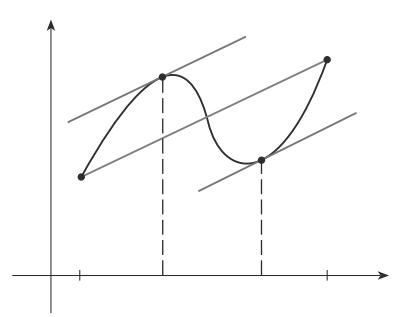


FIGURE 4

70. (a) Show that $e^x \geqslant 1 + x$ for $x \geqslant 0$.
(b) Deduce that $e^x \geqslant 1 + x + \frac{1}{2}x^2$ for $x \geqslant 0$.
(c) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer n,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

71. Show that a cubic function (a third-degree polynomial) always has exactly one point of inflection. If its graph has three x-intercepts x_1, x_2, and x_3, show that the x-coordinate of the inflection point is $(x_1 + x_2 + x_3)/3$.

72. For what values of c does the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? None? Illustrate by graphing P for several values of c. How does the graph change as c decreases?

73. Prove that if $(c, f(c))$ is a point of inflection of the graph of f and f'' exists in an open interval that contains c, then $f''(c) = 0$. [Hint: Apply the First Derivative Test and Fermat's Theorem to the function $= f'$.]

74. Show that if $f(x) = x^4$, then $f''(0) = 0$, but $(0, 0)$ is not an inflection point of the graph of f.

75. Show that the function $(x) = x|x|$ has an inflection point at $(0, 0)$ but $''(0)$ does not exist.

76. Suppose that f''' is continuous and $f'(c) = f''(c) = 0$, but $f'''(c) > 0$. Does f have a local maximum or minimum at c? Does f have a point of inflection at c?

4.4 Indeterminate Forms and L'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when $x = 1$, we need to know how F behaves near 1. In particular, we would like to know the value of the limit

$$\boxed{1} \qquad \lim_{x \to 1} \frac{\ln x}{x - 1}$$

In computing this limit we can't apply Law 5 of limits (the limit of a quotient is the quotient of the limits, see Section 2.3) because the limit of the denominator is 0. In fact, although the limit in (1) exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{(x)}$$

where both $f(x) \to 0$ and $(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type** $\frac{0}{0}$. We met some limits of this type in Chapter 2. For rational functions, we can cancel common factors:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}$$

We used a geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

But these methods do not work for limits such as (1), so in this section we introduce a systematic method, known as L'Hospital's Rule, for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of F and need to evaluate the limit

$$\lim_{x \to \infty} \frac{\ln x}{x - 1} \tag{2}$$

It isnÕt obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$. There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞; if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer may be some Þnite positive number.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an *indeterminate form of type* ∞/∞. We saw in Section 2.6 that this type of limit can be evaluated for certain functions, including rational functions, by dividing numerator and denominator by the highest power x^n of x that occurs in the denominator. For instance,

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{1 - \dfrac{1}{x^2}}{2 + \dfrac{1}{x^2}} = \frac{1 - 0}{2 + 0} = \frac{1}{2}$$

This method does not work for limits such as (2), but lÕHospitalÕs Rule also applies to this type of indeterminate form.

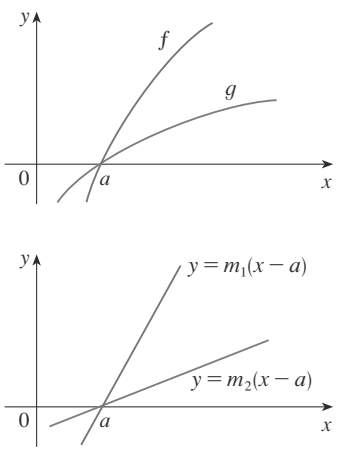


FIGURE 1

|||| Figure 1 suggests visually why lÕHospitalÕs Rule might be true. The Þrst graph shows two differentiable functions f and g, each of which approaches 0 as $x \to a$. If we were to zoom in toward the point $(a, 0)$, the graphs would start to look almost linear. But if the functions actually *were* linear, as in the second graph, then their ratio would be

$$\frac{m_1(x - a)}{m_2(x - a)} = \frac{m_1}{m_2}$$

which is the ratio of their derivatives. This suggests that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

LÕHospitalÕs Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $0/0$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

NOTE 1 LÕHospitalÕs Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisÞed. It is especially important to verify the conditions regarding the limits of f and g before using lÕHospitalÕs Rule.

NOTE 2 LÕHospitalÕs Rule is also valid for one-sided limits and for limits at inÞnity or negative inÞnity; that is, $x \to a$ Ó can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

NOTE 3 For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, it is easy to see why lÕHospitalÕs Rule is true. In fact, using the alternative form

|||| LÕHospitalÕs Rule is named after a French nobleman, the Marquis de lÕHospital (1661Ð1704), but was discovered by a Swiss mathematician, John Bernoulli (1667Ð1748). See Exercise 71 for the example that the Marquis used to illustrate his rule. See the project on page 315 for further historical details.

of the definition of a derivative, we have

$$\lim_{x \quad a} \frac{f'(x)}{\quad '(x)} \quad \frac{f'(a)}{\quad '(a)} \quad \frac{\lim_{x \quad a} \dfrac{f(x) - f(a)}{x - a}}{\lim_{x \quad a} \dfrac{(x) - \quad (a)}{x - a}}$$

$$\lim_{x \quad a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{(x) - \quad (a)}{x - a}}$$

$$\lim_{x \quad a} \frac{f(x) - f(a)}{(x) - \quad (a)}$$

$$\lim_{x \quad a} \frac{f(x)}{(x)}$$

It is more difficult to prove the general version of l'Hospital's Rule. See Appendix F.

EXAMPLE 1 Find $\lim_{x \quad 1} \dfrac{\ln x}{x - 1}$.

SOLUTION Since

$$\lim_{x \quad 1} \ln x \quad \ln 1 \quad 0 \qquad \text{and} \qquad \lim_{x \quad 1} (x - 1) \quad 0$$

we can apply l'Hospital's Rule:

Notice that when using lÕHospitalÕs Rule we differentiate the numerator and denominator *separately*. We do *not* use the Quotient Rule.

$$\lim_{x \quad 1} \frac{\ln x}{x - 1} \quad \lim_{x \quad 1} \frac{\dfrac{d}{dx}(\ln x)}{\dfrac{d}{dx}(x - 1)} \quad \lim_{x \quad 1} \frac{1/x}{1}$$

$$\lim_{x \quad 1} \frac{1}{x} \quad 1$$

EXAMPLE 2 Calculate $\lim_{x \quad \infty} \dfrac{e^x}{x^2}$.

SOLUTION We have $\lim_{x \quad \infty} e^x \quad \infty$ and $\lim_{x \quad \infty} x^2 \quad \infty$, so l'Hospital's Rule gives

$$\lim_{x \quad \infty} \frac{e^x}{x^2} \quad \lim_{x \quad \infty} \frac{\dfrac{d}{dx}(e^x)}{\dfrac{d}{dx}(x^2)} \quad \lim_{x \quad \infty} \frac{e^x}{2x}$$

Since $e^x \quad \infty$ and $2x \quad \infty$ as $x \quad \infty$, the limit on the right side is also indeterminate, but a second application of l'Hospital's Rule gives

$$\lim_{x \quad \infty} \frac{e^x}{x^2} \quad \lim_{x \quad \infty} \frac{e^x}{2x} \quad \lim_{x \quad \infty} \frac{e^x}{2} \quad \infty$$

|||| The graph of the function of Example 2 is shown in Figure 2. We have noticed previously that exponential functions grow far more rapidly than power functions, so the result of Example 2 is not unexpected. See also Exercise 67.

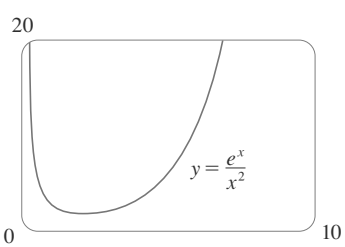


FIGURE 2

|||| The graph of the function of Example 3 is shown in Figure 3. We have discussed previously the slow growth of logarithms, so it isn't surprising that this ratio approaches 0 as $x \to \infty$. See also Exercise 68.

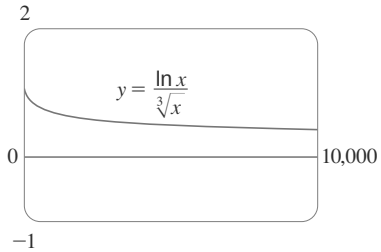


FIGURE 3

EXAMPLE 3 Calculate $\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$.

SOLUTION Since $\ln x \to \infty$ and $\sqrt[3]{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x \to \infty} \frac{3}{\sqrt[3]{x}} = 0$$

EXAMPLE 4 Find $\lim_{x \to 0} \frac{\tan x - x}{x^3}$. [See Exercise 36(d) in Section 2.2.]

SOLUTION Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use l'Hospital's Rule:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x}$$

|||| The graph in Figure 4 gives visual confirmation of the result of Example 4. If we were to zoom in too far, however, we would get an inaccurate graph because $\tan x$ is close to x when x is small. See Exercise 36(d) in Section 2.2.

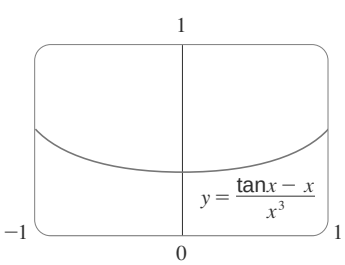


FIGURE 4

Because $\lim_{x \to 0} \sec^2 x = 1$, we simplify the calculation by writing

$$\lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x \to 0} \sec^2 x \lim_{x \to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \to 0} \frac{\tan x}{x}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits. Putting together all the steps, we get

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \to 0} \frac{\sec^2 x}{1} = \frac{1}{3}$$

EXAMPLE 5 Find $\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$.

SOLUTION If we blindly attempted to use l'Hospital's Rule, we would get

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$$

This is *wrong*! Although the numerator $\sin x \to 0$ as $x \to \pi^-$, notice that the denominator $(1 - \cos x)$ does not approach 0, so l'Hospital's Rule can't be applied here.

The required limit is, in fact, easy to find because the function is continuous and the denominator is nonzero at π:

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0$$

Example 5 shows what can go wrong if you use l'Hospital's Rule without thinking. Other limits *can* be found using l'Hospital's Rule but are more easily found by other methods. (See Examples 3 and 5 in Section 2.3, Example 3 in Section 2.6, and the discussion at the beginning of this section.) So when evaluating any limit, you should consider other methods before using l'Hospital's Rule.

Indeterminate Products

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x \to a} f(x)g(x)$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an *indeterminate form* of type $0 \cdot \infty$. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $0/0$ or ∞/∞ so that we can use l'Hospital's Rule.

EXAMPLE 6 Evaluate $\lim_{x \to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $\ln x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

NOTE In solving Example 6 another possible option would have been to write

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{x}{1/\ln x}$$

This gives an indeterminate form of the type $0/0$, but if we apply l'Hospital's Rule we get a more complicated expression than the one we started with. In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

Figure 5 shows the graph of the function in Example 6. Notice that the function is undefined at $x = 0$; the graph approaches the origin but never quite reaches it.

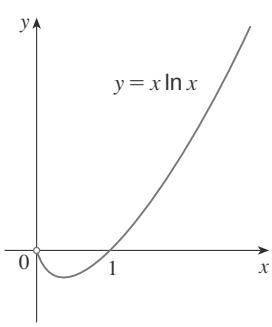


FIGURE 5

Indeterminate Differences

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

is called an *indeterminate form of type* $\infty - \infty$. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance,

by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type ∞/∞ or $\frac{0}{0}$.

EXAMPLE 2 Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$.

SOLUTION First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \to (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$ as $x \to (\pi/2)^-$. ∎

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let } y = [f(x)]^{g(x)}, \quad \text{then } \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

EXAMPLE 3 Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate. Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then

$$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\dfrac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y itself. To find this

we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$

EXAMPLE 9 Find $\lim_{x \to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed as in Example 8 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$$

The graph of the function $f(x) = x^x$, $x > 0$, is shown in Figure 6. Notice that although 0^0 is not defined, the values of the function approach 1 as $x \to 0^+$. This confirms the result of Example 9.

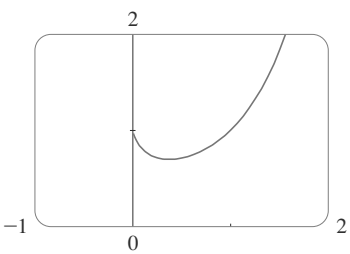


FIGURE 6

4.4 Exercises

1–4 Given that

$$\lim_{x \to a} f(x) = 0 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 1$$

$$\lim_{x \to a} p(x) = \infty \qquad \lim_{x \to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ (b) $\lim_{x \to a} \dfrac{f(x)}{p(x)}$ (c) $\lim_{x \to a} \dfrac{h(x)}{p(x)}$ (d) $\lim_{x \to a} \dfrac{p(x)}{f(x)}$ (e) $\lim_{x \to a} \dfrac{p(x)}{q(x)}$

2. (a) $\lim_{x \to a} [f(x)p(x)]$ (b) $\lim_{x \to a} [h(x)p(x)]$ (c) $\lim_{x \to a} [p(x)q(x)]$

3. (a) $\lim_{x \to a} [f(x) - p(x)]$ (b) $\lim_{x \to a} [p(x) - q(x)]$ (c) $\lim_{x \to a} [p(x) + q(x)]$

4. (a) $\lim_{x \to a} [f(x)]^{g(x)}$ (b) $\lim_{x \to a} [f(x)]^{p(x)}$ (c) $\lim_{x \to a} [h(x)]^{p(x)}$ (d) $\lim_{x \to a} [p(x)]^{f(x)}$ (e) $\lim_{x \to a} [p(x)]^{q(x)}$ (f) $\lim_{x \to a} [q(x)]^{p(x)}$

5–62 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

5. $\lim_{x \to -1} \dfrac{x^2 - 1}{x + 1}$

6. $\lim_{x \to -2} \dfrac{x + 2}{x^2 + 3x + 2}$

7. $\lim_{x \to 1} \dfrac{x^9 - 1}{x^5 - 1}$

8. $\lim_{x \to 1} \dfrac{x^a - 1}{x^b - 1}$

9. $\lim_{x \to (\pi/2)^+} \dfrac{\cos x}{1 - \sin x}$

10. $\lim_{x \to 0} \dfrac{x + \tan x}{\sin x}$

11. $\lim_{t \to 0} \dfrac{e^t - 1}{t^3}$

12. $\lim_{t \to 0} \dfrac{e^{3t} - 1}{t}$

13. $\lim_{x \to 0} \dfrac{\tan px}{\tan qx}$

14. $\lim_{\theta \to \pi/2} \dfrac{1 - \sin \theta}{\csc \theta}$

15. $\lim_{x \to \infty} \dfrac{\ln x}{x}$

16. $\lim_{x \to \infty} \dfrac{e^x}{x}$

17. $\lim_{x \to 0^+} \dfrac{\ln x}{x}$

18. $\lim_{x \to \infty} \dfrac{\ln \ln x}{x}$

19. $\lim_{t \to 0} \dfrac{5^t - 3^t}{t}$

20. $\lim_{x \to 1} \dfrac{\ln x}{\sin \pi x}$

21. $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$

22. $\lim_{x \to 0} \dfrac{e^x - 1 - x - (x^2/2)}{x^3}$

23. $\lim_{x \to \infty} \dfrac{e^x}{x^3}$

24. $\lim_{x \to 0} \dfrac{\sin x}{\sinh x}$

25. $\lim_{x \to 0} \dfrac{\sin^{-1} x}{x}$

26. $\lim_{x \to 0} \dfrac{\sin x - x}{x^3}$

27. $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$

28. $\lim_{x \to \infty} \dfrac{(\ln x)^2}{x}$

29. $\lim_{x \to 0} \dfrac{x + \sin x}{x + \cos x}$

30. $\lim_{x \to 0} \dfrac{\cos mx - \cos nx}{x^2}$

31. $\lim_{x \to \infty} \dfrac{x}{\ln(1 + 2e^x)}$

32. $\lim_{x \to 0} \dfrac{x}{\tan^{-1}(4x)}$

33. $\lim_{x \to 1} \dfrac{1 - x + \ln x}{1 + \cos \pi x}$

34. $\lim_{x \to \infty} \dfrac{\sqrt{x^2 + 2}}{2x^2 + 1}$

35. $\lim_{x \to 1} \dfrac{x^a - ax + a - 1}{(x - 1)^2}$

36. $\lim_{x \to 0} \dfrac{1 - e^{-2x}}{\sec x}$

37. $\lim_{x \to 0^+} \sqrt{x} \ln x$

38. $\lim_{x \to -\infty} x^2 e^x$

39. $\lim_{x \to 0} \cot 2x \sin 6x$

40. $\lim_{x \to 0^+} \sin x \ln x$

41. $\lim_{x \to \infty} x^3 e^{-x^2}$

42. $\lim_{x \to \pi/4} (1 - \tan x) \sec x$

43. $\lim_{x \to 1^+} \ln x \tan(\pi x/2)$

44. $\lim_{x \to \infty} x \tan(1/x)$

45. $\lim_{x \to 0} \left(\dfrac{1}{x} - \csc x \right)$

46. $\lim_{x \to 0} (\csc x - \cot x)$

47. $\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right)$

48. $\lim_{x \to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x - 1} \right)$

49. $\lim_{x \to \infty} (x - \ln x)$

50. $\lim_{x \to \infty} (xe^{1/x} - x)$

51. $\lim_{x \to 0^+} x^{x^2}$

52. $\lim_{x \to 0^+} (\tan 2x)^x$

53. $\lim_{x \to 0} (1 - 2x)^{1/x}$

54. $\lim_{x \to \infty} \left(1 + \dfrac{a}{x} \right)^{bx}$

55. $\lim_{x \to \infty} \left(1 + \dfrac{3}{x} + \dfrac{5}{x^2} \right)^x$

56. $\lim_{x \to \infty} x^{(\ln 2)/(1 + \ln x)}$

57. $\lim_{x \to \infty} x^{1/x}$

58. $\lim_{x \to \infty} (e^x + x)^{1/x}$

59. $\lim_{x \to \infty} \left(\dfrac{x}{x + 1} \right)^x$

60. $\lim_{x \to 0} (\cos 3x)^{5/x}$

61. $\lim_{x \to 0^+} (\cos x)^{1/x^2}$

62. $\lim_{x \to \infty} \left(\dfrac{2x - 3}{2x + 5} \right)^{2x+1}$

63–64 Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

63. $\lim_{x \to \infty} x [\ln(x + 5) - \ln x]$

64. $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$

65–66 Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$ to see that these ratios have the same limit as $x \to 0$. Also calculate the exact value of the limit.

65. $f(x) = e^x - 1, \quad g(x) = x^3 + 4x$

66. $f(x) = 2x \sin x, \quad g(x) = \sec x - 1$

67. Prove that

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$$

for any positive integer n. This shows that the exponential function approaches ∞ infinity faster than any power of x.

68. Prove that

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This shows that the logarithmic function approaches ∞ more slowly than any power of x.

69. If an initial amount A_0 of money is invested at an interest rate i compounded n times a year, the value of the investment after t years is

$$A = A_0 \left(1 + \frac{i}{n} \right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = A_0 e^{it}$$

70. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c} (1 - e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 9 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object.)

(a) Calculate $\lim_{t \to \infty} v$. What is the meaning of this limit?

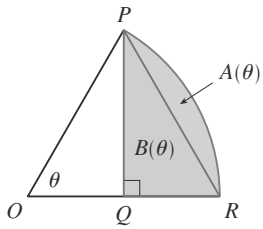
(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{m \to \infty} v$. What can you conclude about the speed of a very heavy falling object?

71. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus *textbook* ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

72. The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta \to 0^+} A(\theta)/B(\theta)$.



73. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2+3x) + f(2+5x)}{x}$$

74. For what values of a and b is the following equation true?

$$\lim_{x \to 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2} \right) = 0$$

75. If f' is continuous, use l'Hospital's Rule to show that

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x)$$

Explain the meaning of this equation with the aid of a diagram.

76. If f'' is continuous, show that

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x)$$

77. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.
(b) Show that f has derivatives of all orders that are defined on $\mathbb{R}$. [Hint: First show by induction that there is a polynomial $p_n(x)$ and a nonnegative integer k_n such that $f^{(n)}(x) = p_n(x) f(x)/x^{k_n}$ for $x \neq 0$.]

78. Let

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show that f is continuous at 0.
(b) Investigate graphically whether f is differentiable at 0 by zooming in several times toward the point $(0, 1)$ on the graph of f.
(c) Show that f is not differentiable at 0. How can you reconcile this fact with the appearance of the graphs in part (b)?

WRITING PROJECT

The Origins of L'Hospital's Rule

ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES
SECONDE EDITION
A PARIS,
M DCC XXVI

L'Hospital's Rule was first published in 1696 in the Marquis de l'Hospital's calculus textbook Analyse des Infiniment Petits, but the rule was discovered in 1694 by the Swiss mathematician John (Johann) Bernoulli. The explanation is that these two mathematicians had entered into a curious business arrangement whereby the Marquis de l'Hospital bought the rights to Bernoulli's mathematical discoveries. The details, including a translation of l'Hospital's letter to Bernoulli proposing the arrangement, can be found in the book by Eves [1].

Write a report on the historical and mathematical origins of l'Hospital's Rule. Start by providing brief biographical details of both men (the dictionary edited by Gillispie [2] is a good source) and outline the business deal between them. Then give l'Hospital's statement of his rule, which is found in Struik's sourcebook [4] and more briefly in the book of Katz [3]. Notice that l'Hospital and Bernoulli formulated the rule geometrically and gave the answer in terms of differentials. Compare their statement with the version of l'Hospital's Rule given in Section 4.4 and show that the two statements are essentially the same.

1. Howard Eves, In Mathematical Circles (Volume 2: Quadrants III and IV) (Boston: Prindle, Weber and Schmidt, 1969), pp. 32–22.
2. C. C. Gillispie, ed., Dictionary of Scientific Biography (New York: Scribner's, 1974). See the article on Johann Bernoulli by E. A. Fellmann and J. O. Fleckenstein in Volume II and the article on the Marquis de l'Hospital by Abraham Robinson in Volume VIII.
3. Victor Katz, A History of Mathematics: An Introduction (New York: HarperCollins, 1993), p. 484.
4. D. J. Struik, ed., A Sourcebook in Mathematics, 1200–1800 (Princeton, NJ: Princeton University Press, 1969), pp. 314–316.

|||| The Internet is another source of information for this project. See the web site

www.stewartcalculus.com

and click on *History of Mathematics*.

4.5 Summary of Curve Sketching

So far we have been concerned with some particular aspects of curve sketching: domain, range, and symmetry in Chapter 1; limits, continuity, and asymptotes in Chapter 2; derivatives and tangents in Chapters 2 and 3; and extreme values, intervals of increase and decrease, concavity, points of inflection, and l'Hospital's Rule in this chapter. It is now time to put all of this information together to sketch graphs that reveal the important features of functions.

You may ask: What is wrong with just using a calculator to plot points and then joining these points with a smooth curve? To see the pitfalls of this approach, suppose you have used a calculator to produce the table of values and corresponding points in Figure 1.

x	f(x)	x	f(x)
−5	22	1	7
−4	7	2	10
−3	−2	3	11
−2	−4	4	10
−1	−2	5	8
0	3	6	−8

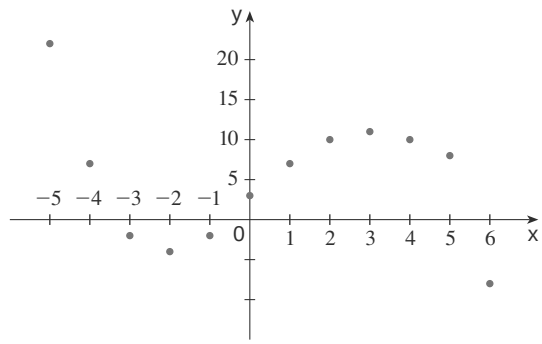


FIGURE 1

You might then join these points to produce the curve shown in Figure 2, but the correct graph might be the one shown in Figure 3. You can see the drawbacks of the method of plotting points. Certain essential features of the graph may be missed, such as the maximum and minimum values between −2 and −1 or between 2 and 5. If you just plot points, you don't know when to stop. (How far should you plot to the left or right?) But the use of calculus ensures that all the important aspects of the curve are illustrated.

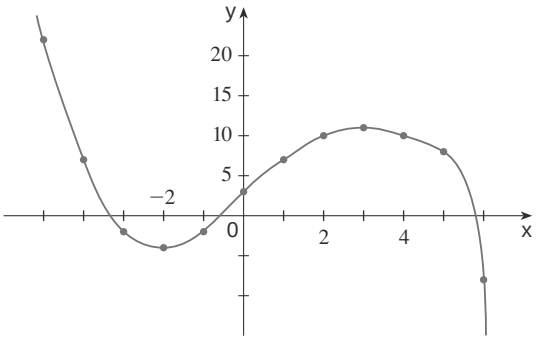


FIGURE 2

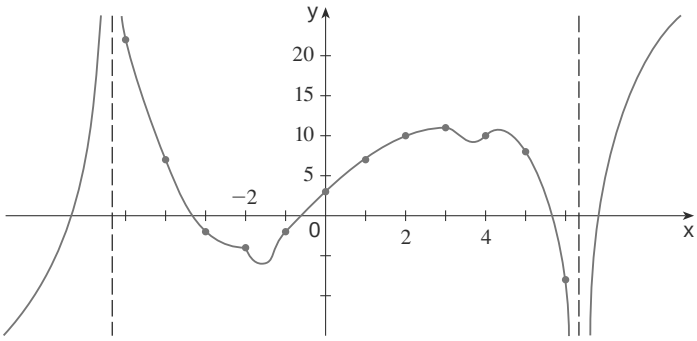


FIGURE 3

You might respond: Yes, but what about graphing calculators and computers? Don't they plot such a huge number of points that the sort of uncertainty demonstrated by Figures 2 and 3 is unlikely to happen?

It's true that modern technology is capable of producing very accurate graphs. But even the best graphing devices have to be used intelligently. We saw in Section 1.4 that it is extremely important to choose an appropriate viewing rectangle to avoid getting a misleading graph. (See especially Examples 1, 3, 4, and 5 in that section.) The use of calculus

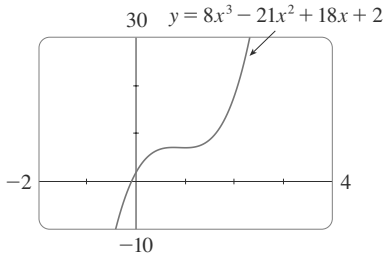


FIGURE 4

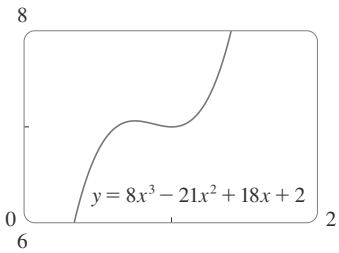


FIGURE 5

enables us to discover the most interesting aspects of graphs and in many cases to calculate maximum and minimum points and inflection points exactly instead of approximately.

For instance, Figure 4 shows the graph of $f(x) = 8x^3 - 21x^2 + 18x + 2$. At first glance it seems reasonable: It has the same shape as cubic curves like $y = x^3$, and it appears to have no maximum or minimum point. But if you compute the derivative, you will see that there is a maximum when $x = 0.75$ and a minimum when $x = 1$. Indeed, if we zoom in to this portion of the graph, we see that behavior exhibited in Figure 5. Without calculus, we could easily have overlooked it.

In the next section we will graph functions by using the interaction between calculus and graphing devices. In this section we draw graphs first by considering the following information. We don't assume that you have a graphing device, but if you do have one you should use it as a check on your work.

Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. **Domain** It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

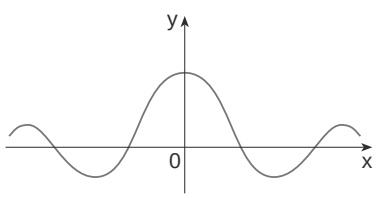
B. **Intercepts** The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. **Symmetry**

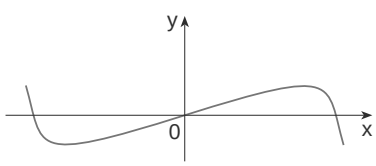
(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 6(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure 6(b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period**. For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 7).



(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

FIGURE 6

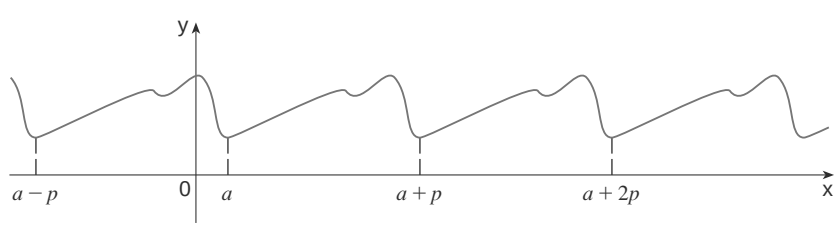


FIGURE 7
Periodic function: translational symmetry

D. **Asymptotes**

(i) **Horizontal Asymptotes.** Recall from Section 2.6 that if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve

$y = f(x)$. If it turns out that $\lim_{x \to \infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

(ii) Vertical Asymptotes. Recall from Section 2.2 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\boxed{1} \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (1) is true. If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$, whether or not this limit is infinite.

(iii) Slant Asymptotes. These are discussed at the end of this section.

E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if c is a critical number such that $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

H. **Sketch the Curve** Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

In Module 4.5 you can practice using information about f', , and asymptotes to determine the shape of the graph of .

EXAMPLE 1 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D.
$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

Therefore, the line $y = 2$ is a horizontal asymptote.

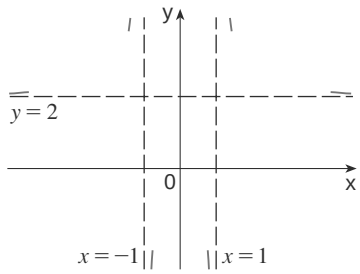


FIGURE 8
Preliminary sketch

|||| We have shown the curve approaching its horizontal asymptote from above in Figure 8. This is confirmed by the intervals of increase and decrease.

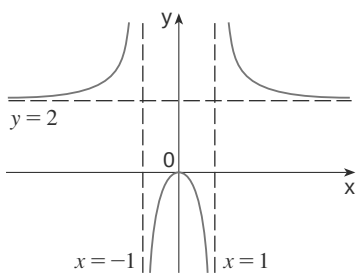


FIGURE 9
Finished sketch of $y = \dfrac{2x^2}{x^2 - 1}$

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Therefore, the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 8, showing the parts of the curve near the asymptotes.

E. $$f'(x) = \frac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

G. $$f''(x) = \frac{-4(x^2 - 1)^2 + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have

$$f''(x) > 0 \qquad x^2 - 1 > 0 \qquad |x| > 1$$

and $f''(x) < 0$ $|x| < 1$. Thus, the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

H. Using the information in E–G, we finish the sketch in Figure 9.

EXAMPLE 2 Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x + 1}}$.

A. Domain $= \{x \mid x + 1 > 0\} = \{x \mid x > -1\} = (-1, \infty)$

B. The x- and y-intercepts are both 0.

C. Symmetry: None

D. Since

$$\lim_{x \to \infty} \frac{x^2}{\sqrt{x + 1}} = \infty$$

there is no horizontal asymptote. Since $\sqrt{x + 1} \to 0$ as $x \to -1^+$ and $f(x)$ is always positive, we have

$$\lim_{x \to -1^+} \frac{x^2}{\sqrt{x + 1}} = \infty$$

and so the line $x = -1$ is a vertical asymptote.

E. $$f'(x) = \frac{2x\sqrt{x + 1} - x^2 \cdot 1/\left(2\sqrt{x + 1}\right)}{x + 1} = \frac{x(3x + 4)}{2(x + 1)^{3/2}}$$

We see that $f'(x) = 0$ when $x = 0$ (notice that $-\frac{4}{3}$ is not in the domain of f), so the only critical number is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

F. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

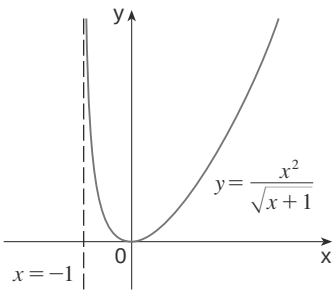


FIGURE 10

G. $$f''(x) = \frac{2(x+1)^{3/2}(6x+4) - (3x^2+4x)3(x+1)^{1/2}}{4(x+1)^3} = \frac{3x^2+8x+8}{4(x+1)^{5/2}}$$

Note that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant $b^2 - 4ac = -32$, which is negative, and the coefficient of x^2 is positive. Thus, $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.

H. The curve is sketched in Figure 10.

EXAMPLE 3 Sketch the graph of $f(x) = xe^x$.

A. The domain is $\mathbb{R}$.

B. The x- and y-intercepts are both 0.

C. Symmetry: None

D. Because both x and e^x become large as $x \to \infty$, we have $\lim_{x \to \infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminate product that requires the use of l'Hospital's Rule:

$$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \lim_{x \to -\infty} (-e^x) = 0$$

Thus, the x-axis is a horizontal asymptote.

E. $$f'(x) = xe^x + e^x = (x+1)e^x$$

Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(x) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

F. Because $f'(-1) = 0$ and f' changes from negative to positive at $x = -1$, $f(-1) = -e^{-1}$ is a local (and absolute) minimum.

G. $$f''(x) = (x+1)e^x + e^x = (x+2)e^x$$

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. The inflection point is $(-2, -2e^{-2})$.

H. We use this information to sketch the curve in Figure 11.

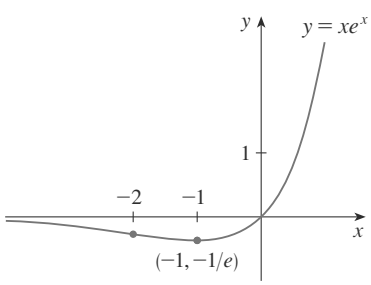


FIGURE 11

EXAMPLE 4 Sketch the graph of $f(x) = 2\cos x + \sin 2x$.

A. The domain is $\mathbb{R}$.

B. The y-intercept is $f(0) = 2$. The x-intercepts occur when

$$2\cos x + \sin 2x = 2\cos x + 2\sin x\cos x = 2\cos x(1 + \sin x) = 0$$

that is, when $\cos x = 0$ or $\sin x = -1$. Thus, in the interval $[0, 2\pi]$, the x-intercepts are $\pi/2$ and $3\pi/2$.

C. f is neither even nor odd, but $f(x + 2\pi) = f(x)$ for all x and so f is periodic and has period 2π. Thus, in what follows we need to consider only $0 \le x \le 2\pi$ and then extend the curve by translation in H.

D. Asymptotes: None

E. $$f'(x) = -2\sin x + 2\cos 2x = -2\sin x + 2(1 - 2\sin^2 x)$$
$$= -2(2\sin^2 x + \sin x - 1) = -2(2\sin x - 1)(\sin x + 1)$$

Thus, $f'(x) = 0$ when $\sin x = \frac{1}{2}$ or $\sin x = -1$, so if $[0, 2\pi]$ we have $x = \pi/6$, $5\pi/6$, and $3\pi/2$. In determining the sign of $f'(x)$ in the following chart, we use the

fact that $\sin x + 1 \geqslant 0$ for all x.

Interval	$f'(x)$	f
$0 < x < \pi/6$	$+$	increasing on $(0, \pi/6)$
$\pi/6 < x < 5\pi/6$	$-$	decreasing on $(\pi/6, 5\pi/6)$
$5\pi/6 < x < 3\pi/2$	$+$	increasing on $(5\pi/6, 3\pi/2)$
$3\pi/2 < x < 2\pi$	$+$	increasing on $(3\pi/2, 2\pi)$

F. From the chart in E the First Derivative Test says that $f(\pi/6) = 3\sqrt{3}/2$ is a local maximum and $f(5\pi/6) = -3\sqrt{3}/2$ is a local minimum, but f has no maximum or minimum at $3\pi/2$, only a horizontal tangent.

G. $$f''(x) = -2\cos x - 4\sin 2x = -2\cos x\,(1 + 4\sin x)$$

Thus, $f''(x) = 0$ when $\cos x = 0$ (so $x = \pi/2$ or $3\pi/2$) and when $\sin x = -\frac{1}{4}$. From Figure 12 we see that there are two values x of between 2π and for which $\sin x = -\frac{1}{4}$. Let's call them α_1 and α_2. Then $f''(x) > 0$ on $(\pi/2, \alpha_1)$ and $(3\pi/2, \alpha_2)$, so f is concave upward there. Also $f''(x) < 0$ on $(0, \pi/2)$, $(\alpha_1, 3\pi/2)$, and $(\alpha_2, 2\pi)$, so f is concave downward there. Inflection points occur when $x = \pi/2, \alpha_1, 3\pi/2$, and α_2.

$\alpha_1 = \pi + \arcsin\frac{1}{4}$

$\alpha_2 = 2\pi - \arcsin\frac{1}{4}$

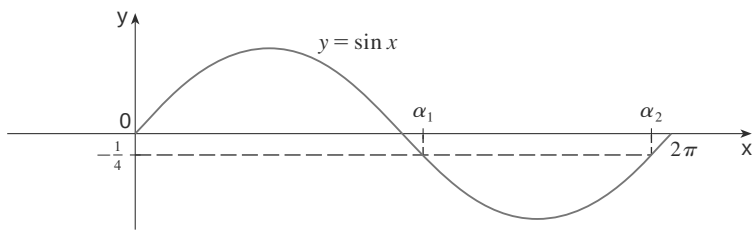


FIGURE 12

H. The graph of the function restricted to $0 \leqslant x \leqslant 2\pi$ is shown in Figure 13. Then it is extended, using periodicity, to the complete graph in Figure 14.

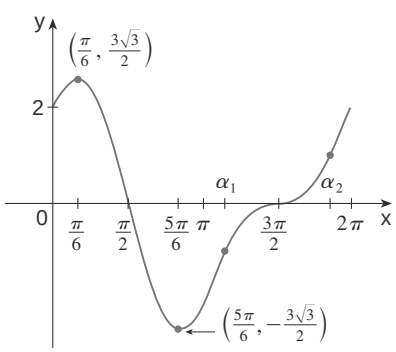


FIGURE 13

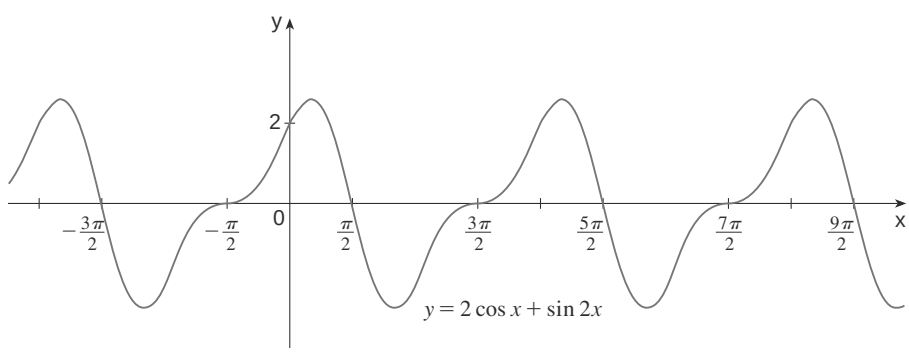


FIGURE 14

EXAMPLE 5 Sketch the graph of $y = \ln(4 - x^2)$.

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = \log_e 1 = 0$ (since $e^0 = 1$), so we have $4 - x^2 = 1$ $x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$$

Thus, the lines $x = 2$ and $x = -2$ are vertical asymptotes.

E.
$$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G.
$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.

H. Using this information, we sketch the curve in Figure 15.

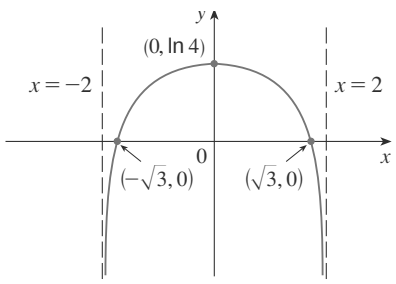


FIGURE 15
$y = \ln(4 - x^2)$

Slant Asymptotes

Some curves have asymptotes that are oblique, that is, neither horizontal nor vertical. If

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

then the line $y = mx + b$ is called a slant asymptote because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0, as in Figure 16. (A similar situation exists if we let $x \to -\infty$.) For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division as in the following example.

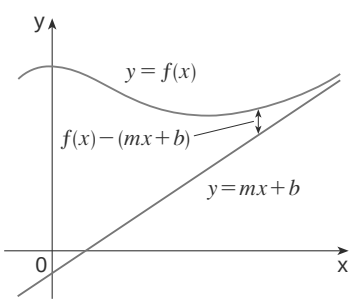


FIGURE 16

EXAMPLE 6 Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

A. The domain is $\mathbb{R} = (-\infty, \infty)$.

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = -f(x)$, f is odd and its graph is symmetric about the origin.

D. Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. But long division gives

$$f(x) = \frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1}$$

$$f(x) - x = -\frac{x}{x^2 + 1} = -\frac{\frac{1}{x}}{1 + \frac{1}{x^2}} \to 0 \quad \text{as} \quad x \to \pm\infty$$

So the line $y = x$ is a slant asymptote.

E. $$f'(x) = \frac{3x^2(x^2+1) - x^3 \cdot 2x}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2}$$

Since $f'(x) > 0$ for all x (except 0), f is increasing on $(-\infty, \infty)$.

F. Although $f'(0) = 0$, f' does not change sign at 0, so there is no local maximum or minimum.

G. $$f''(x) = \frac{(4x^3+6x)(x^2+1)^2 - (x^4+3x^2) \cdot 2(x^2+1)2x}{(x^2+1)^4} = \frac{2x(3-x^2)}{(x^2+1)^3}$$

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:

Interval	x	$3 - x^2$	$(x^2+1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD on $(\sqrt{3}, \infty)$

The points of inflection are $(-\sqrt{3}, -3\sqrt{3}/4)$, $(0, 0)$, and $(\sqrt{3}, 3\sqrt{3}/4)$.

H. The graph of f is sketched in Figure 17.

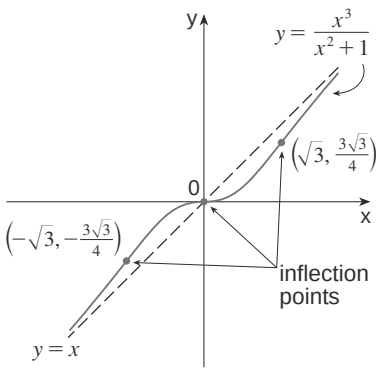


FIGURE 17

4.5 Exercises

1–52 Use the guidelines of this section to sketch the curve.

1. $y = x^3 + x$

2. $y = x^3 + 6x^2 + 9x$

3. $y = 2 - 15x + 9x^2 - x^3$

4. $y = 8x^2 - x^4$

5. $y = x^4 + 4x^3$

6. $y = x(x+2)^3$

7. $y = 2x^5 - 5x^2 + 1$

8. $y = 20x^3 - 3x^5$

9. $y = \dfrac{x}{x-1}$

10. $y = \dfrac{x}{(x-1)^2}$

11. $y = \dfrac{1}{x^2-9}$

12. $y = \dfrac{x}{x^2-9}$

13. $y = \dfrac{x}{x^2+9}$

14. $y = \dfrac{x^2}{x^2+9}$

15. $y = \dfrac{x-1}{x^2}$

16. $y = \dfrac{x^2-2}{x^4}$

17. $y = \dfrac{x^2}{x^2+3}$

18. $y = \dfrac{x^3-1}{x^3+1}$

19. $y = x\sqrt{5-x}$

20. $y = 2\sqrt{x} - x$

21. $y = \sqrt{x^2+1} - x$

22. $y = \sqrt{\dfrac{x}{x-5}}$

23. $y = \dfrac{x}{\sqrt{x^2+1}}$

24. $y = x\sqrt{2-x^2}$

25. $y = \dfrac{\sqrt{1-x^2}}{x}$

26. $y = \dfrac{x}{\sqrt{x^2-1}}$

27. $y = x - 3x^{1/3}$

28. $y = x^{5/3} - 5x^{2/3}$

29. $y = x + \sqrt{|x|}$

30. $y = \sqrt[3]{(x^2-1)^2}$

31. $y = 3\sin x - \sin^3 x$

32. $y = \sin x - \tan x$

33. $y = x\tan x, \quad -\pi/2 < x < \pi/2$

34. $y = 2x - \tan x, \quad -\pi/2 < x < \pi/2$

35. $y = \frac{1}{2}x - \sin x, \quad 0 < x < 3\pi$

36. $y = \cos^2 x - 2\sin x$

37. $y = \sin 2x - 2\sin x$

38. $y = \sin x - x$

39. $y = \dfrac{\sin x}{1+\cos x}$

40. $y = \dfrac{\cos x}{2+\sin x}$

41. $y = 1/(1+e^{-x})$

42. $y = e^{2x} - e^x$

43. $y = x\ln x$

44. $y = e^x/x$

45. $y = xe^{-x}$

46. $y = \ln(x^2 - 3x + 2)$

47. $y = \ln(\sin x)$

48. $y = x(\ln x)^2$

49. $y = xe^{-x^2}$

50. $y = e^x - 3e^{-x} - 4x$

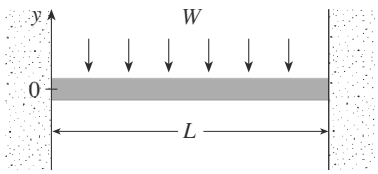
51. $y = e^{3x} + e^{-2x}$

52. $y = \tan^{-1}\left(\dfrac{x-1}{x+1}\right)$

53. The figure shows a beam of length L embedded in concrete walls. If a constant load W is distributed evenly along its length, the beam takes the shape of the deflection curve

$$y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2$$

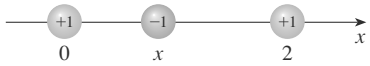
where E and I are positive constants. (E is Young's modulus of elasticity and I is the moment of inertia of a cross-section of the beam.) Sketch the graph of the deflection curve.



54. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The figure shows particles with charge 1 located at positions 0 and 2 on a coordinate line and a particle with charge -1 at a position x between them. It follows from Coulomb's Law that the net force acting on the middle particle is

$$F(x) = -\frac{k}{x^2} + \frac{k}{(x-2)^2} \qquad 0 < x < 2$$

where k is a positive constant. Sketch the graph of the net force function. What does the graph say about the force?



55–58 Find an equation of the slant asymptote. Do not sketch the curve.

55. $y = \dfrac{x^2 + 1}{x + 1}$

56. $y = \dfrac{2x^3 + x^2 + x + 3}{x^2 + 2x}$

57. $y = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3}$

58. $y = \dfrac{5x^4 + x^2 + x}{x^3 - x^2 + 2}$

59–64 Use the guidelines of this section to sketch the curve. In guideline D find an equation of the slant asymptote.

59. $y = \dfrac{-2x^2 + 5x - 1}{2x - 1}$

60. $y = \dfrac{x^2 + 12}{x - 2}$

61. $xy = x^2 + 4$

62. $y = e^x - x$

63. $y = \dfrac{2x^3 + x^2 + 1}{x^2 + 1}$

64. $y = \dfrac{(x+1)^3}{(x-1)^2}$

65. Show that the curve $y = x - \tan^{-1}x$ has two slant asymptotes: $y = x + \pi/2$ and $y = x - \pi/2$. Use this fact to help sketch the curve.

66. Show that the curve $y = \sqrt{x^2 + 4x}$ has two slant asymptotes: $y = x + 2$ and $y = -x - 2$. Use this fact to help sketch the curve.

67. Show that the lines $y = (b/a)x$ and $y = -(b/a)x$ are slant asymptotes of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$.

68. Let $f(x) = (x^3 + 1)/x$. Show that

$$\lim_{x \to \pm\infty} [f(x) - x^2] = 0$$

This shows that the graph of f approaches the graph of $y = x^2$, and we say that the curve $y = f(x)$ is asymptotic to the parabola $y = x^2$. Use this fact to help sketch the graph of f.

69. Discuss the asymptotic behavior of $f(x) = (x^4 + 1)/x$ in the same manner as in Exercise 68. Then use your results to help sketch the graph of f.

70. Use the asymptotic behavior of $f(x) = \cos x + 1/x^2$ to sketch its graph without going through the curve-sketching procedure of this section.

4.6 Graphing with Calculus and Calculators

If you have not already read Section 1.4, you should do so now. In particular, it explains how to avoid some of the pitfalls of graphing devices by choosing appropriate viewing rectangles.

The method we used to sketch curves in the preceding section was a culmination of much of our study of differential calculus. The graph was the final object that we produced. In this section our point of view is completely different. Here we start with a graph produced by a graphing calculator or computer and then we refine it. We use calculus to make sure that we reveal all the important aspects of the curve. And with the use of graphing devices we can tackle curves that would be far too complicated to consider without technology. The theme is the interaction between calculus and calculators.

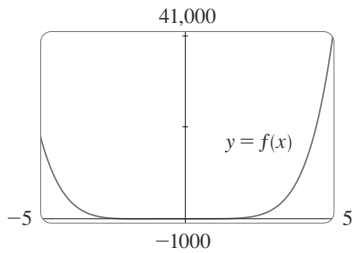


FIGURE 1

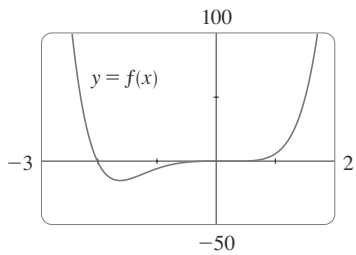


FIGURE 2

EXAMPLE 1 Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs of f' and f'' to estimate all maximum and minimum points and intervals of concavity.

SOLUTION If we specify a domain but not a range, many graphing devices will deduce a suitable range from the values computed. Figure 1 shows the plot from one such device if we specify that $-5 \leqslant x \leqslant 5$. Although this viewing rectangle is useful for showing that the asymptotic behavior (or end behavior) is the same as for $y = 2x^6$, it is obviously hiding some finer detail. So we change to the viewing rectangle $[-3, 2]$ by $[-50, 100]$ shown in Figure 2.

From this graph it appears that there is an absolute minimum value of about -15.3 when $x \approx -1.62$ (by using the cursor) and f is decreasing on $(-\infty, -1.62)$ and increasing on $(-1.62, \infty)$. Also there appears to be a horizontal tangent at the origin and inflection points when $x = 0$ and when x is somewhere between -2 and -1.

Now let's try to confirm these impressions using calculus. We differentiate and get

$$f'(x) = 12x^5 + 15x^4 + 9x^2 - 4x$$

$$f''(x) = 60x^4 + 60x^3 + 18x - 4$$

When we graph f' in Figure 3 we see that $f'(x)$ changes from negative to positive when $x \approx -1.62$; this confirms (by the First Derivative Test) the minimum value that we found earlier. But, perhaps to our surprise, we also notice that $f'(x)$ changes from positive to negative when $x = 0$ and from negative to positive when $x = 0.35$. This means that f has a local maximum at 0 and a local minimum when $x = 0.35$, but these were hidden in Figure 2. Indeed, if we now zoom in toward the origin in Figure 4, we see what we missed before: a local maximum value of 0 when $x = 0$ and a local minimum value of about -0.1 when $x \approx 0.35$.

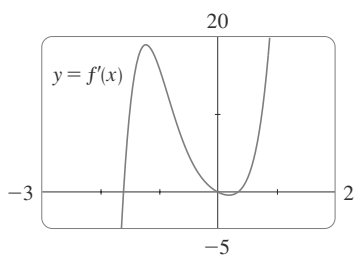


FIGURE 3

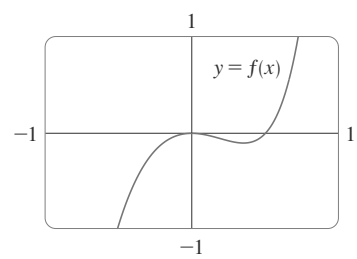


FIGURE 4

What about concavity and inflection points? From Figures 2 and 4 there appear to be inflection points when x is a little to the left of -1 and when x is a little to the right of 0. But it's difficult to determine inflection points from the graph of f, so we graph the second derivative f'' in Figure 5. We see that $f''(x)$ changes from positive to negative when $x \approx -1.23$ and from negative to positive when $x \approx 0.19$. So, correct to two decimal places, f is concave upward on $(-\infty, -1.23)$ and $(0.19, \infty)$ and concave downward on $(-1.23, 0.19)$. The inflection points are $(-1.23, -10.18)$ and $(0.19, -0.05)$.

We have discovered that no single graph reveals all the important features of this polynomial. But Figures 2 and 4, when taken together, do provide an accurate picture.

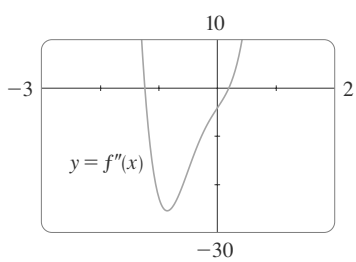


FIGURE 5

EXAMPLE 2 Draw the graph of the function

$$f(x) = \frac{x^2 + 7x + 3}{x^2}$$

in a viewing rectangle that contains all the important features of the function. Estimate

the maximum and minimum values and the intervals of concavity. Then use calculus to find these quantities exactly.

SOLUTION Figure 6, produced by a computer with automatic scaling, is a disaster. Some graphing calculators use $[-10, 10]$ by $[-10, 10]$ as the default viewing rectangle, so let's try it. We get the graph shown in Figure 7, a major improvement.

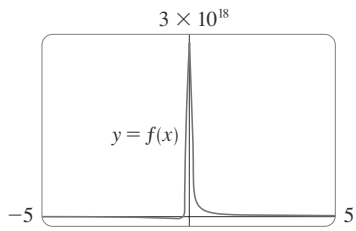


FIGURE 6

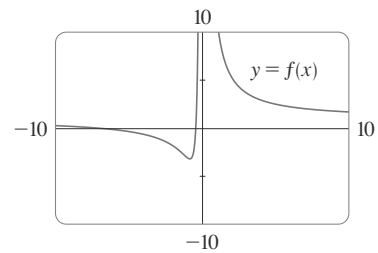


FIGURE 7

The y-axis appears to be a vertical asymptote and indeed it is because

$$\lim_{x \to 0} \frac{x^2 + 7x + 3}{x^2} = \infty$$

Figure 7 also allows us to estimate the x-intercepts: about -0.5 and -6.5. The exact values are obtained by using the quadratic formula to solve the equation $x^2 + 7x + 3 = 0$; we get $x = \left(-7 \pm \sqrt{37}\right)/2$.

To get a better look at horizontal asymptotes, we change to the viewing rectangle $[-20, 20]$ by $[-5, 10]$ in Figure 8. It appears that $y = 1$ is the horizontal asymptote and this is easily confirmed:

$$\lim_{x \to \pm\infty} \frac{x^2 + 7x + 3}{x^2} = \lim_{x \to \pm\infty} \left(1 + \frac{7}{x} + \frac{3}{x^2}\right) = 1$$

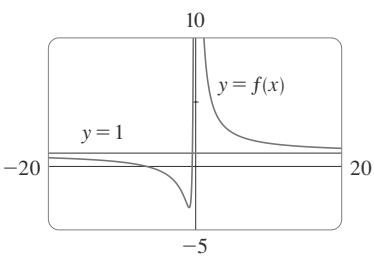


FIGURE 8

To estimate the minimum value we zoom in to the viewing rectangle $[-3, 0]$ by $[-4, 2]$ in Figure 9. The cursor indicates that the absolute minimum value is about -3.1 when $x \approx -0.9$, and we see that the function decreases on $(-\infty, -0.9)$ and $(0, \infty)$ and increases on $(-0.9, 0)$. The exact values are obtained by differentiating:

$$f'(x) = -\frac{7}{x^2} - \frac{6}{x^3} = -\frac{7x + 6}{x^3}$$

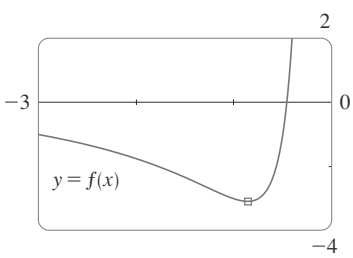


FIGURE 9

This shows that $f'(x) > 0$ when $-\frac{6}{7} < x < 0$ and $f'(x) < 0$ when $x < -\frac{6}{7}$ and when $x > 0$. The exact minimum value is $f\left(-\frac{6}{7}\right) = -\frac{37}{12} \approx -3.08$.

Figure 9 also shows that an inflection point occurs somewhere between $x = -1$ and $x = -2$. We could estimate it much more accurately using the graph of the second derivative, but in this case it's just as easy to find exact values. Since

$$f''(x) = \frac{14}{x^3} + \frac{18}{x^4} = 2\,\frac{7x + 9}{x^4}$$

we see that $f''(x) > 0$ when $x > -\frac{9}{7}$ $(x \neq 0)$. So f is concave upward on $\left(-\frac{9}{7}, 0\right)$ and $(0, \infty)$ and concave downward on $\left(-\infty, -\frac{9}{7}\right)$. The inflection point is $\left(-\frac{9}{7}, -\frac{71}{27}\right)$.

The analysis using the first two derivatives shows that Figures 7 and 8 display all the major aspects of the curve.

EXAMPLE 3 Graph the function $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4}$.

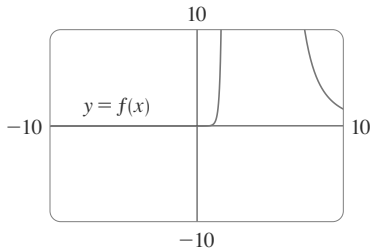


FIGURE 10

SOLUTION Drawing on our experience with a rational function in Example 2, we start by graphing f in the viewing rectangle $[-10, 10]$ by $[-10, 10]$. From Figure 10 we have the feeling that we are going to have to zoom in to see some detail and also zoom out to see the larger picture. But, as a guide to intelligent zooming, let's first take a close look at the expression for $f(x)$. Because of the factors $(x-2)^2$ and $(x-4)^4$ in the denominator, we expect $x = 2$ and $x = 4$ to be the vertical asymptotes. Indeed

$$\lim_{x \to 2} \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \infty \qquad \text{and} \qquad \lim_{x \to 4} \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \infty$$

To find the horizontal asymptotes we divide numerator and denominator by x^6 :

$$\frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \frac{\dfrac{1}{x}\left(1+\dfrac{1}{x}\right)^3}{\left(1-\dfrac{2}{x}\right)^2\left(1-\dfrac{4}{x}\right)^4} \to 0 \quad \text{as} \quad x \to \pm\infty$$

so the x -axis is the horizontal asymptote.

It is also very useful to consider the behavior of the graph near the x -intercepts using an analysis like that in Example 11 in Section 2.6. Since x^2 is positive, $f(x)$ does not change sign at 0 and so its graph does not cross the x -axis at 0. But, because of the factor $(x+1)^3$, the graph does cross the x -axis at -1 and has a horizontal tangent there. Putting all this information together, but without using derivatives, we see that the curve has to look something like the one in Figure 11.

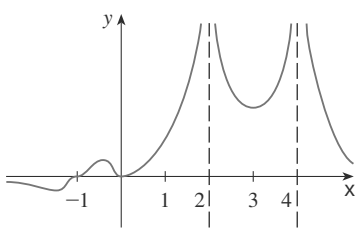


FIGURE 11

Now that we know what to look for, we zoom in (several times) to produce the graphs in Figures 12 and 13 and zoom out (several times) to get Figure 14.

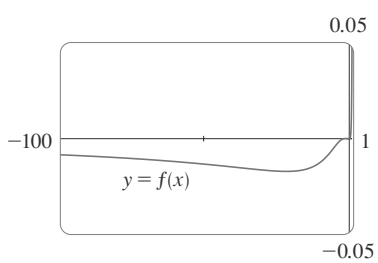


FIGURE 12

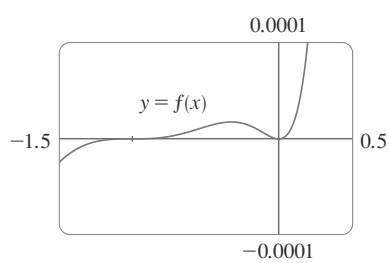


FIGURE 13

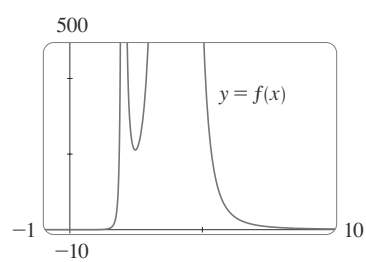


FIGURE 14

We can read from these graphs that the absolute minimum is about -0.02 and occurs when $x \approx -20$. There is also a local maximum ≈ 0.00002 when $x \approx -0.3$ and a local minimum ≈ 211 when $x \approx 2.5$. These graphs also show three inflection points near -35, -5, and -1 and two between -1 and 0 . To estimate the inflection points closely we would need to graph f'' , but to compute f'' by hand is an unreasonable chore. If you have a computer algebra system, then it is easy to do (see Exercise 17).

We have seen that, for this particular function, three graphs (Figures 12, 13, and 14) are necessary to convey all the useful information. The only way to display all these features of the function on a single graph is to draw it by hand. Despite the exaggerations and distortions, Figure 11 does manage to summarize the essential nature of the function.

|||| The family of functions

$$f(x) = \sin(x + \sin cx)$$

where c is a constant, occurs in applications to frequency modulation (FM) synthesis. A sine wave is modulated by a wave with a different frequency ($\sin cx$). The case where $c = 2$ is studied in Example 4. Exercise 25 explores another special case.

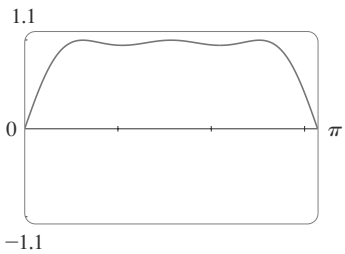


FIGURE 15

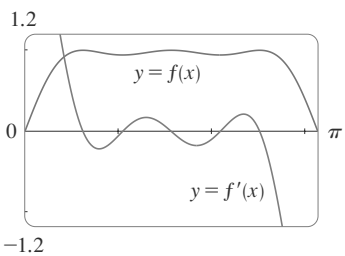


FIGURE 16

EXAMPLE 4 Graph the function $f(x) = \sin(x + \sin 2x)$. For $0 \leqslant x \leqslant \pi$, locate all maximum and minimum values, intervals of increase and decrease, and inflection points correct to one decimal place.

SOLUTION We first note that f is periodic with period 2π. Also, f is odd and $|f(x)| \leqslant 1$ for all x. So the choice of a viewing rectangle is not a problem for this function: We start with $[0, \pi]$ by $[-1.1, 1.1]$. (See Figure 15.) It appears that there are three local maximum values and two local minimum values in that window. To confirm this and locate them more accurately, we calculate that

$$f'(x) = \cos(x + \sin 2x) \cdot (1 + 2\cos 2x)$$

and graph both f and f' in Figure 16. Using zoom-in and the First Derivative Test, we find the following values to one decimal place.

Intervals of increase: $(0, 0.6)$, $(1.0, 1.6)$, $(2.1, 2.5)$

Intervals of decrease: $(0.6, 1.0)$, $(1.6, 2.1)$, $(2.5, \pi)$

Local maximum values: $f(0.6) \approx 1$, $f(1.6) \approx 1$, $f(2.5) \approx 1$

Local minimum values: $f(1.0) \approx 0.94$, $f(2.1) \approx 0.94$

The second derivative is

$$f''(x) = -(1 + 2\cos 2x)^2 \sin(x + \sin 2x) - 4\sin 2x \cos(x + \sin 2x)$$

Graphing both f and f'' in Figure 17, we obtain the following approximate values:

Concave upward on: $(0.8, 1.3)$, $(1.8, 2.3)$

Concave downward on: $(0, 0.8)$, $(1.3, 1.8)$, $(2.3, \pi)$

Inflection points: $(0, 0)$, $(0.8, 0.97)$, $(1.3, 0.97)$, $(1.8, 0.97)$, $(2.3, 0.97)$

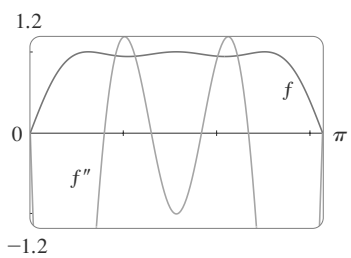


FIGURE 17

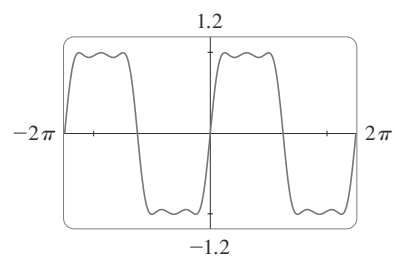


FIGURE 18

Having checked that Figure 15 does indeed represent f accurately for $0 \leqslant x \leqslant \pi$, we can state that the extended graph in Figure 18 represents f accurately for $-2\pi \leqslant x \leqslant 2\pi$.

Our final example is concerned with families of functions. As discussed in Section 1.4, this means that the functions in the family are related to each other by a formula that contains one or more arbitrary constants. Each value of the constant gives rise to a member of the family and the idea is to see how the graph of the function changes as the constant changes.

EXAMPLE 5 How does the graph of $f(x) = 1/(x^2 + 2x + c)$ vary as c varies?

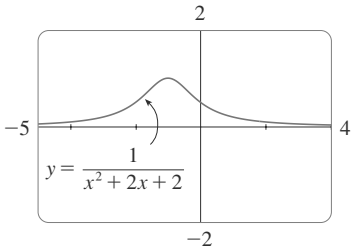


FIGURE 19 $c = 2$

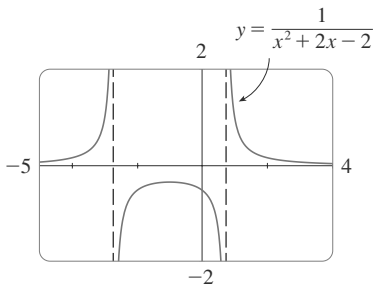


FIGURE 20 $c = -2$

SOLUTION The graphs in Figures 19 and 20 (the special cases $c = 2$ and $c = -2$) show two very different-looking curves. Before drawing any more graphs, let's see what members of this family have in common. Since

$$\lim_{x \to \pm\infty} \frac{1}{x^2 + 2x + c} = 0$$

for any value of c, they all have the x-axis as a horizontal asymptote. A vertical asymptote will occur when $x^2 + 2x + c = 0$. Solving this quadratic equation, we get $x = -1 \pm \sqrt{1 - c}$. When $c > 1$, there is no vertical asymptote (as in Figure 19). When $c = 1$, the graph has a single vertical asymptote $x = -1$ because

$$\lim_{x \to -1} \frac{1}{x^2 + 2x + 1} = \lim_{x \to -1} \frac{1}{(x + 1)^2} = \infty$$

When $c < 1$, there are two vertical asymptotes $x = -1 \pm \sqrt{1 - c}$ (as in Figure 20).

Now we compute the derivative:

$$f'(x) = -\frac{2x + 2}{(x^2 + 2x + c)^2}$$

This shows that $f'(x) = 0$ when $x = -1$ (if $c \neq 1$), $f'(x) > 0$ when $x < -1$, and $f'(x) < 0$ when $x > -1$. For $c \geqslant 1$, this means that f increases on $(-\infty, -1)$ and decreases on $(-1, \infty)$. For $c > 1$, there is an absolute maximum value $f(-1) = 1/(c - 1)$. For $c < 1$, $f(-1) = 1/(c - 1)$ is a local maximum value and the intervals of increase and decrease are interrupted at the vertical asymptotes.

Figure 21 is a "slide show" displaying five members of the family, all graphed in the viewing rectangle $[-5, 4]$ by $[-2, 2]$. As predicted, $c = 1$ is the value at which a transition takes place from two vertical asymptotes to one, and then to none. As c increases from 1, we see that the maximum point becomes lower; this is explained by the fact that $1/(c - 1) \to 0$ as $c \to \infty$. As c decreases from 1, the vertical asymptotes become more widely separated because the distance between them, $2\sqrt{1 - c}$, becomes large as $c \to -\infty$. Again, the maximum point approaches the x-axis because $1/(c - 1) \to 0$ as $c \to -\infty$.

See an animation of Figure 21.
Resources / Module 5
/ Max and Min
/ Families of Functions

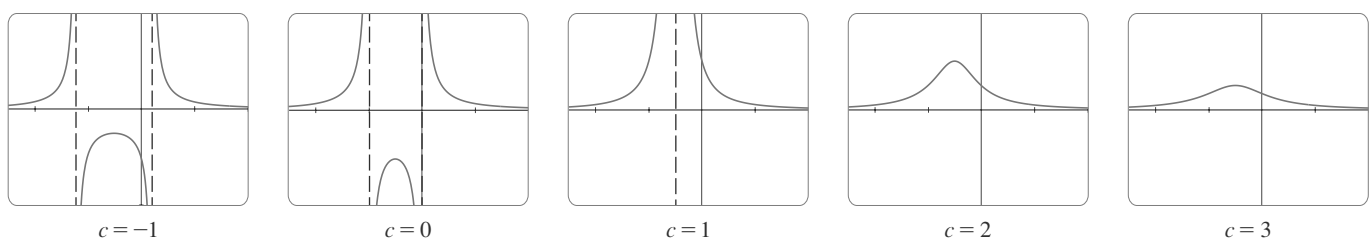


FIGURE 21 The family of functions $f(x) = 1/(x^2 + 2x + c)$

There is clearly no inflection point when $c \leqslant 1$. For $c > 1$ we calculate that

$$f''(x) = \frac{2(3x^2 + 6x + 4 - c)}{(x^2 + 2x + c)^3}$$

and deduce that inflection points occur when $x = -1 \pm \sqrt{(c - 1)/3}$. So the inflection points become more spread out as c increases and this seems plausible from the last two parts of Figure 21.

4.6 Exercises

1–8 Produce graphs of f that reveal all the important aspects of the curve. In particular, you should use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

1. $f(x) = 4x^4 - 32x^3 + 89x^2 - 95x + 29$

2. $f(x) = x^6 - 15x^5 + 75x^4 - 125x^3 - x$

3. $f(x) = \sqrt[3]{x^2 - 3x - 5}$

4. $f(x) = \dfrac{x^4 + x^3 - 2x^2 + 2}{x^2 + x - 2}$

5. $f(x) = \dfrac{x}{x^3 - x^2 - 4x + 1}$

6. $f(x) = \tan x + 5\cos x$

7. $f(x) = x^2 - 4x + 7\cos x, \quad -4 \leqslant x \leqslant 4$

8. $f(x) = \dfrac{e^x}{x^2 - 9}$

9–12 Produce graphs of f that reveal all the important aspects of the curve. Estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points, and use calculus to find these quantities exactly.

9. $f(x) = 8x^3 - 3x^2 - 10$

10. $f(x) = \dfrac{x^2 + 11x - 20}{x^2}$

11. $f(x) = x\sqrt{9 - x^2}$

12. $f(x) = x - 2\sin x, \quad -2\pi \leqslant x \leqslant 2\pi$

13–14
(a) Graph the function.
(b) Use l'Hospital's Rule to explain the behavior as $x \to 0$.
(c) Estimate the minimum value and intervals of concavity. Then use calculus to find the exact values.

13. $f(x) = x^2 \ln x$

14. $f(x) = xe^{1/x}$

15–16 Sketch the graph by hand using asymptotes and intercepts, but not derivatives. Then use your sketch as a guide to producing graphs (with a graphing device) that display the major features of the curve. Use these graphs to estimate the maximum and minimum values.

15. $f(x) = \dfrac{(x + 4)(x - 3)^2}{x^4(x - 1)}$

16. $f(x) = \dfrac{10x(x - 1)^4}{(x - 2)^3(x + 1)^2}$

CAS **17.** If f is the function considered in Example 3, use a computer algebra system to calculate f' and then graph it to confirm that all the maximum and minimum values are as given in the example. Calculate f'' and use it to estimate the intervals of concavity and inflection points.

CAS **18.** If f is the function of Exercise 16, find f' and f'' and use their graphs to estimate the intervals of increase and decrease and concavity of f.

CAS **19–22** Use a computer algebra system to graph f and find f' and f''. Use graphs of these derivatives to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points of f.

19. $f(x) = \dfrac{\sin^2 x}{x^2 + 1}, \quad 0 \leqslant x \leqslant 3\pi$

20. $f(x) = \dfrac{2x - 1}{\sqrt[4]{x^4 + x + 1}}$

21. $f(x) = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$

22. $f(x) = \dfrac{1}{1 + e^{\tan x}}$

CAS **23–24**
(a) Graph the function.
(b) Explain the shape of the graph by computing the limit as $x \to 0^+$ or as $x \to \infty$.
(c) Estimate the maximum and minimum values and then use calculus to find the exact values.
(d) Use a graph of f'' to estimate the x-coordinates of the inflection points.

23. $f(x) = x^{1/x}$

24. $f(x) = (\sin x)^{\sin x}$

25. In Example 4 we considered a member of the family of functions $f(x) = \sin(x + \sin cx)$ that occur in FM synthesis. Here we investigate the function with $c = 3$. Start by graphing f in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$. How many local maximum points do you see? The graph has more than are visible to the naked eye. To discover the hidden maximum and minimum points you will need to examine the graph of f' very carefully. In fact, it helps to look at the graphs of f and f' at the same time. Find all the maximum and minimum values and inflection points. Then graph f in the viewing rectangle $[-2\pi, 2\pi]$ by $[-1.2, 1.2]$ and comment on symmetry.

26–33 Describe how the graph of f varies as c varies. Graph several members of the family to illustrate the trends that you discover. In particular, you should investigate how maximum and

minimum points and inflection points move when c changes. You should also identify any transitional values of c at which the basic shape of the curve changes.

26. $f(x) = x^3 + cx$

27. $f(x) = x^4 + cx^2$

28. $f(x) = x^2\sqrt{c^2 - x^2}$

29. $f(x) = e^{-c/x^2}$

30. $f(x) = \ln(x^2 + c)$

31. $f(x) = \dfrac{cx}{1 + c^2x^2}$

32. $f(x) = \dfrac{1}{(1 - x^2)^2 + cx^2}$

33. $f(x) = cx + \sin x$

34. The family of functions $f(t) = C(e^{-at} - e^{-bt})$, where a, b, and C are positive numbers and $b > a$, has been used to model the concentration of a drug injected into the blood at time $t = 0$. Graph several members of this family. What do they have in common? For fixed values of C and a, discover graphically what happens as b increases. Then use calculus to prove what you have discovered.

35. Investigate the family of curves given by $f(x) = xe^{-cx}$, where c is a real number. Start by computing the limits as $x \to \pm\infty$. Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate by graphing several members of the family.

36. Investigate the family of curves given by the equation $f(x) = x^4 + cx^2 + x$. Start by determining the transitional value of c at which the number of inflection points changes. Then graph several members of the family to see what shapes are possible. There is another transitional value of c at which the number of critical numbers changes. Try to discover it graphically. Then prove what you have discovered.

37. (a) Investigate the family of polynomials given by the equation $f(x) = cx^4 - 2x^2 + 1$. For what values of c does the curve have minimum points?
(b) Show that the minimum and maximum points of every curve in the family lie on the parabola $y = 1 - x^2$. Illustrate by graphing this parabola and several members of the family.

38. (a) Investigate the family of polynomials given by the equation $f(x) = 2x^3 + cx^2 + 2x$. For what values of c does the curve have maximum and minimum points?
(b) Show that the minimum and maximum points of every curve in the family lie on the curve $y = x - x^3$. Illustrate by graphing this curve and several members of the family.

4.7 Optimization Problems

The methods we have learned in this chapter for finding extreme values have practical applications in many areas of life. A businessperson wants to minimize costs and maximize profits. A traveler wants to minimize transportation time. Fermat's principle in optics states that light follows the path that takes the least time. In this section and the next we solve such problems as maximizing areas, volumes, and profits and minimizing distances, times, and costs.

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized. Let's recall the problem-solving principles discussed on page 80 and adapt them to this situation:

Steps in Solving Optimization Problems

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.
4. Express Q in terms of some of the other symbols from Step 3.

5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of one variable x, say, $Q = f(x)$. Write the domain of this function.
6. Use the methods of Sections 4.1 and 4.3 to find the absolute maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

|||| Understand the problem
|||| Analogy: Try special cases
|||| Draw diagrams

SOLUTION In order to get a feeling for what is happening in this problem, let's experiment with some special cases. Figure 1 (not to scale) shows three possible ways of laying out the 2400 ft of fencing. We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

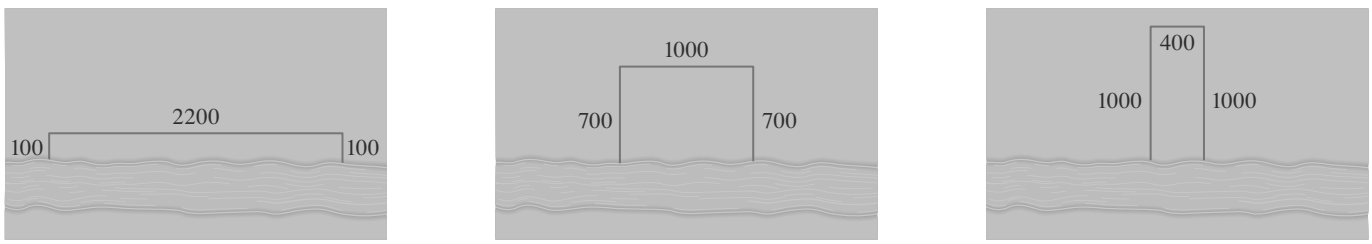


FIGURE 1

|||| Introduce notation

Figure 2 illustrates the general case. We wish to maximize the area A of the rectangle. Let x and y be the depth and width of the rectangle (in feet). Then we express A in terms of x and y:

$$A = xy$$

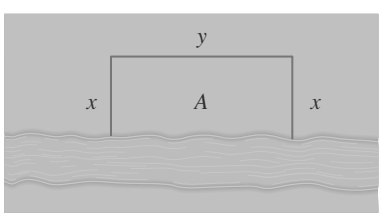


FIGURE 2

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 ft. Thus

$$2x + y = 2400$$

From this equation we have $y = 2400 - 2x$, which gives

$$A = x(2400 - 2x) = 2400x - 2x^2$$

Note that $x \geqslant 0$ and $x \leqslant 1200$ (otherwise $A < 0$). So the function that we wish to maximize is

$$A(x) = 2400x - 2x^2 \qquad 0 \leqslant x \leqslant 1200$$

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives $x = 600$. The maximum value of A must occur either at this critical number or at an endpoint of the interval. Since $A(0) = 0$, $A(600) = 720{,}000$, and $A(1200) = 0$, the Closed Interval Method gives the maximum value as $A(600) = 720{,}000$.

[Alternatively, we could have observed that $A''(x) = -4 < 0$ for all x, so A is always concave downward and the local maximum at $x = 600$ must be an absolute maximum.]

Thus, the rectangular field should be 600 ft deep and 1200 ft wide.

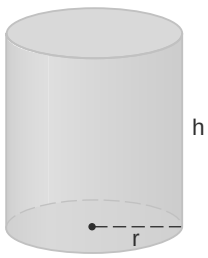


FIGURE 3

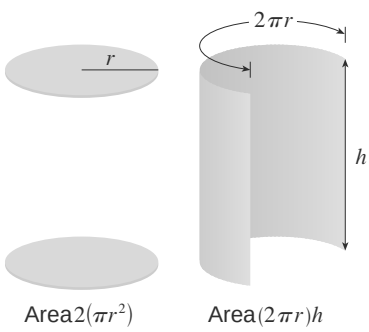


FIGURE 4

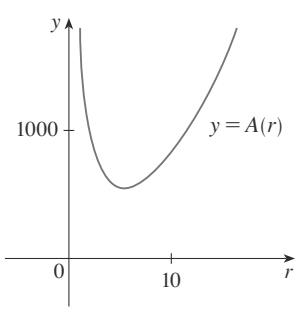


FIGURE 5

EXAMPLE 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

SOLUTION Draw the diagram as in Figure 3, where r is the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 4 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

To eliminate h, we use the fact that the volume is given as 1 L, which we take to be $1000\ \text{cm}^3$. Thus

$$\pi r^2 h = 1000$$

which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression for A gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

Therefore, the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

Since the domain of A is $(0, \infty)$, we cannot use the argument of Example 1 concerning endpoints. But we can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$, so A is decreasing for all r to the left of the critical number and increasing for all r to the right. Thus $r = \sqrt[3]{500/\pi}$ must give rise to an absolute minimum.

[Alternatively, we could argue that $A(r) \to \infty$ as $r \to 0^+$ and $A(r) \to \infty$ as $r \to \infty$, so there must be a minimum value of $A(r)$, which must occur at the critical number. See Figure 5.]

The value of h corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter.

|||| In the Applied Project on page 341 we investigate the most economical shape for a can by taking into account other manufacturing costs.

NOTE 1 The argument used in Example 2 to justify the absolute minimum is a variant of the First Derivative Test (which applies only to local maximum or minimum values) and is stated here for future reference.

Module 4.7 takes you through eight additional optimization problems, including animations of the physical situations.

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

NOTE 2 An alternative method for solving optimization problems is to use implicit differentiation. Let's look at Example 2 again to illustrate the method. We work with the same equations

$$A = 2\pi r^2 + 2\pi rh \qquad \pi r^2 h = 100$$

but instead of eliminating h, we differentiate both equations implicitly with respect to r:

$$A' = 4\pi r + 2\pi h + 2\pi rh' \qquad 2\pi rh + \pi r^2 h' = 0$$

The minimum occurs at a critical number, so we set $A' = 0$, simplify, and arrive at the equations

$$2r + h + rh' = 0 \qquad 2h + rh' = 0$$

and subtraction gives $2r - h = 0$, or $h = 2r$.

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

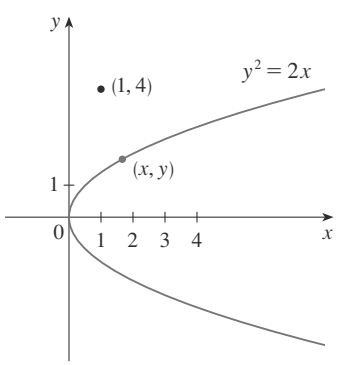


FIGURE 6

SOLUTION The distance between the point $(1, 4)$ and the point (x, y) is

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

(See Figure 6.) But if (x, y) lies on the parabola, then $x = y^2/2$, so the expression for d becomes

$$d = \sqrt{\left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2}$$

(Alternatively, we could have substituted $y = \sqrt{2x}$ to get d in terms of x alone.) Instead of minimizing d, we minimize its square:

$$d^2 = f(y) = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

(You should convince yourself that the minimum of d occurs at the same point as the minimum of d^2, but d^2 is easier to work with.) Differentiating, we obtain

$$f'(y) = 2\left(\tfrac{1}{2}y^2 - 1\right)y + 2(y-4) = y^3 - 8$$

so $f'(y) = 0$ when $y = 2$. Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$, so by the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$. (Or we could simply say that because of the geometric nature of the problem, it's obvious that there is a closest point but not a farthest point.) The corresponding value of x is $x = y^2/2 = 2$. Thus, the point on $y^2 = 2x$ closest to $(1, 4)$ is $(2, 2)$.

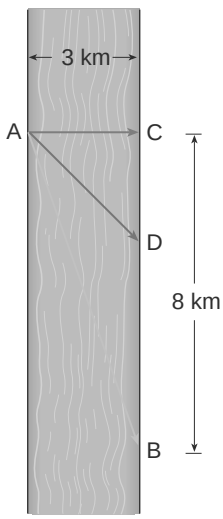


FIGURE 7

EXAMPLE 4 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)

SOLUTION If we let x be the distance from C to D, then the running distance is $|DB| = 8 - x$ and the Pythagorean Theorem gives the rowing distance as $|AD| = \sqrt{x^2 + 9}$. We use the equation

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

Then the rowing time is $\sqrt{x^2 + 9}/6$ and the running time is $(8 - x)/8$, so the total time T as a function of x is

$$T(x) = \frac{\sqrt{x^2 + 9}}{6} + \frac{8 - x}{8}$$

The domain of this function T is $[0, 8]$. Notice that if $x = 0$ he rows to C and if $x = 8$ he rows directly to B. The derivative of T is

$$T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}$$

Try another problem like this one.
Resources / Module 5
/ Max and Min
/ Start of Optimal Lifeguard

Thus, using the fact that $x \geqslant 0$, we have

$$T'(x) = 0 \qquad \frac{x}{6\sqrt{x^2 + 9}} = \frac{1}{8} \qquad 4x = 3\sqrt{x^2 + 9}$$

$$16x^2 = 9(x^2 + 9) \qquad 7x^2 = 81$$

$$x = \frac{9}{\sqrt{7}}$$

The only critical number is $x = 9/\sqrt{7}$. To see whether the minimum occurs at this critical number or at an endpoint of the domain $[0, 8]$, we evaluate T at all three points:

$$T(0) = 1.5 \qquad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \qquad T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Since the smallest of these values of T occurs when $x = 9/\sqrt{7}$, the absolute minimum value of T must occur there. Figure 8 illustrates this calculation by showing the graph of T.

Thus, the man should land the boat at a point $9/\sqrt{7} \approx 3.4$ km downstream from his starting point.

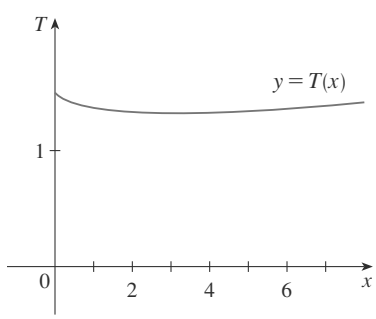


FIGURE 8

EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

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/ Max and Min
/ Start of Max and Min

SOLUTION Let's take the semicircle to be the upper half of the circle $x^2 + y^2 = r^2$ with center the origin. Then the word *inscribed* means that the rectangle has two vertices on the semicircle and two vertices on the x-axis as shown in Figure 9.

Let (x, y) be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths $2x$ and y, so its area is

$$A = 2xy$$

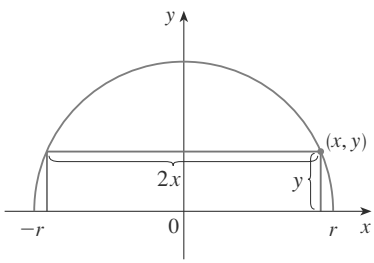


FIGURE 9

To eliminate y we use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$ and so $y = \sqrt{r^2 - x^2}$. Thus

$$A = 2x\sqrt{r^2 - x^2}$$

The domain of this function is $0 \leqslant x \leqslant r$. Its derivative is

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

which is 0 when $2x^2 = r^2$, that is, $x = r/\sqrt{2}$ (since $x \geqslant 0$). This value of x gives a maximum value of A since $A(0) = 0$ and $A(r) = 0$. Therefore, the area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$$

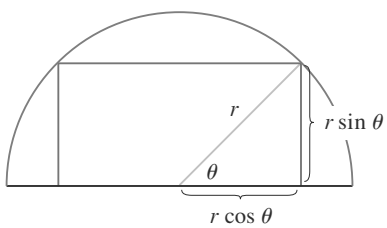


FIGURE 10

SOLUTION 2 A simpler solution is possible if we think of using an angle as a variable. Let θ be the angle shown in Figure 10. Then the area of the rectangle is

$$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\cos\theta) = r^2\sin 2\theta$$

We know that $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \pi/2$. Then $A(\theta)$ has a maximum value of r^2 and it occurs when $\theta = \pi/4$.

Notice that this trigonometric solution does not involve differentiation. In fact, we didn't need to use calculus at all.

4.7 Exercises

1. Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.
   (a) Make a table of values, like the following one, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.

First number	Second number	Product
1	22	22
2	21	42
3	20	60
⋮	⋮	⋮

   (b) Use calculus to solve the problem and compare with your answer to part (a).

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. Find a positive number such that the sum of the number and its reciprocal is as small as possible.

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

6. Find the dimensions of a rectangle with area 100 m^2 whose perimeter is as small as possible.

7. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

(a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the total area.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the total area as a function of one variable.
(f) Finish solving the problem and compare the answer with your estimate in part (a).

8. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the volume.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the volume as a function of one variable.
(f) Finish solving the problem and compare the answer with your estimate in part (a).

9. A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

10. A box with a square base and open top must have a volume of $32{,}000\ \text{cm}^3$. Find the dimensions of the box that minimize the amount of material used.

11. If $1200\ \text{cm}^2$ of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

12. A rectangular storage container with an open top is to have a volume of $10\ \text{m}^3$. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

13. Do Exercise 12 assuming the container has a lid that is made from the same material as the sides.

14. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.
(b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

15. Find the point on the line $y = 4x + 7$ that is closest to the origin.

16. Find the point on the line $6x + y = 9$ that is closest to the point $(-3, 1)$.

17. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

18. Find, correct to two decimal places, the coordinates of the point on the curve $y = \tan x$ that is closest to the point $(1, 1)$.

19. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

20. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

21. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

22. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

23. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

24. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.

25. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

26. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

27. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

28. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus, the diameter of the semicircle is equal to the width of the rectangle. See Exercise 52 on page 24.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

29. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at $384\ \text{cm}^2$, find the dimensions of the poster with the smallest area.

30. A poster is to have an area of $180\ \text{in}^2$ with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

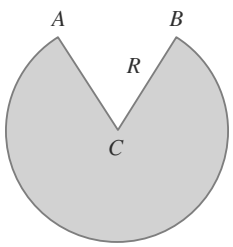
31. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

32. Answer Exercise 31 if one piece is bent into a square and the other into a circle.

33. A cylindrical can without a top is made to contain V cm^3 of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

34. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

35. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



36. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

37. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

38. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u $(u < v)$, then the time required to swim a distance L is $L/(v - u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u}$$

where a is the proportionality constant.
(a) Determine the value of v that minimizes E.
(b) Sketch the graph of E.

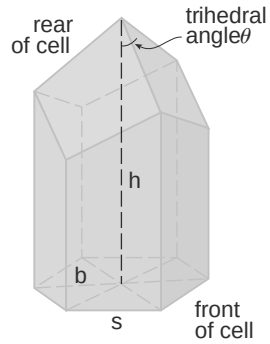
Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.

39. In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedral angle at the other end. It is believed that bees form their cells in such a way as to minimize the surface area for a given volume, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle θ is amazingly consistent. Based on the geometry of the cell, it can be shown that the surface area S is given by

$$S = 6sh - \tfrac{3}{2}s^2 \cot\theta + \left(3s^2 - 3\sqrt{3}/2\right) \csc\theta$$

where s, the length of the sides of the hexagon, and h, the height, are constants.
(a) Calculate $dS/d\theta$.
(b) What angle should the bees prefer?
(c) Determine the minimum surface area of the cell (in terms of s and h).

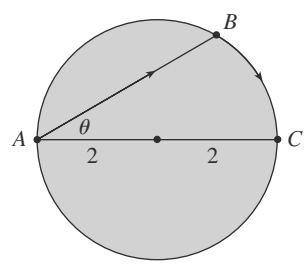
Note: Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than 2°.



40. A boat leaves a dock at 2:00 P.M. and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00. At what time were the two boats closest together?

41. Solve the problem in Example 4 if the river is 5 km wide and point B is only 5 km downstream from A.

42. A woman at a point A on the shore of a circular lake with radius 2 mi wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at the rate of 4 mi/h and row a boat at 2 mi/h. How should she proceed?



43. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 ft apart, where should an object be placed on the line between the sources so as to receive the least illumination?

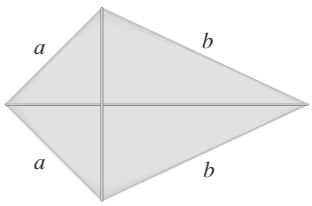
44. Find an equation of the line through the point $(3, 5)$ that cuts off the least area from the first quadrant.

45. Let a and b be positive numbers. Find the length of the shortest line segment that is cut off by the first quadrant and passes through the point (a, b).

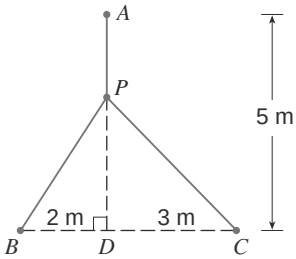
46. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

47. Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

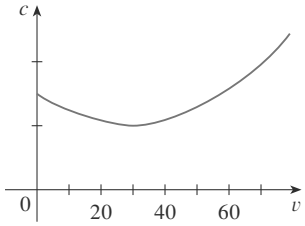
CAS **48.** The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?



49. A point P needs to be located somewhere on the line AD so that the total length L of cables linking P to the points A, B, and C is minimized (see the figure). Express L as a function of $x = |AP|$ and use the graphs of L and dL/dx to estimate the minimum value.



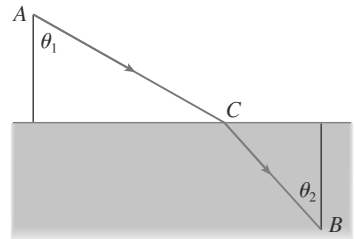
50. The graph shows the fuel consumption $c(v)$ of a car (measured in gallons per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 30$ mi/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in gallons per mile. Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.



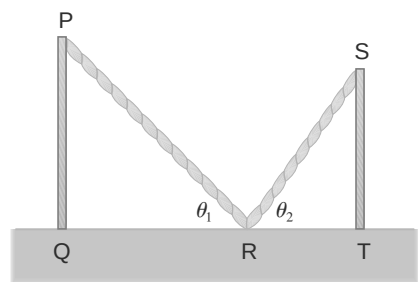
51. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

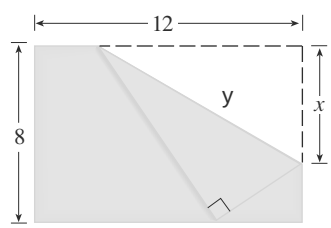
where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's law.



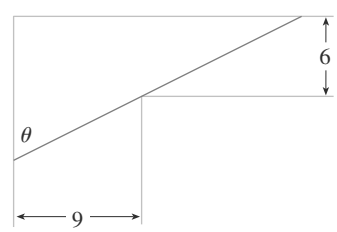
52. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure. Show that the shortest length of such a rope occurs when $\theta_1 = \theta_2$.



53. The upper right-hand corner of a piece of paper, 12 in. by 8 in., as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?



54. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?



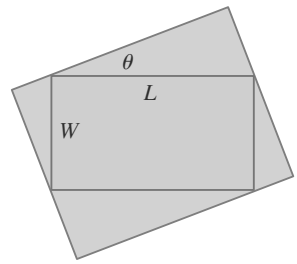
55. An observer stands at a point , one unit away from a track. Two runners start at the point in the figure and run along the track. One runner runs three times as fast as the other. Find the maximum value of the observer's angle of sight between the runners. [Hint: Maximize .]

56. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle . How should be chosen so that the gutter will carry the maximum amount of water?

Where should the point be chosen on the line segment so as to maximize the angle ?

58. A painting in an art gallery has height and is hung so that its lower edge is a distance above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle subtended at his eye by the painting?)

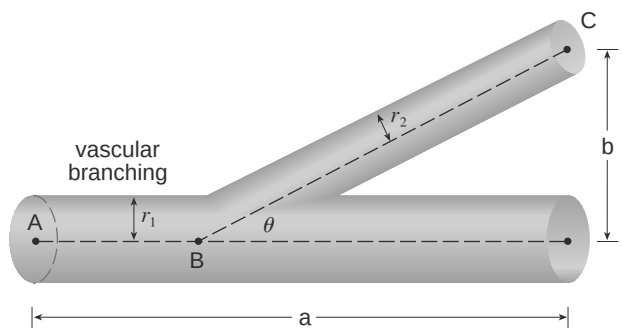
59. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length and width .



60. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's laws gives the resistance of the blood as

$$R = C\frac{L}{r^4}$$

where is the length of the blood vessel, is the radius, and is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally, but it also follows from Equation 8.4.2.) The figure shows a main blood vessel with radius branching at an angle into a smaller vessel with radius .



(a) Use Poiseuille's Law to show that the total resistance of the blood along the path ABC is

$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

where and are the distances shown in the figure.

(b) Prove that this resistance is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

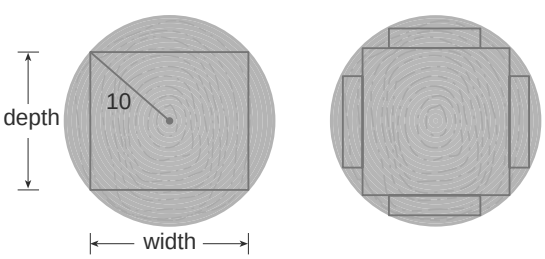
73. (a) If $f(x) = 0.1e^x + \sin x$, $-4 \leqslant x \leqslant 4$, use a graph of f to sketch a rough graph of the antiderivative F of f that satisfies $F(0) = 0$.
(b) Find an expression for $F(x)$.
(c) Graph F using the expression in part (b). Compare with your sketch in part (a).

74. Investigate the family of curves given by $f(x) = x^4 + x^3 + cx^2$. In particular you should determine the transitional value c of c at which the number of critical numbers changes and the transitional value at which the number of inflection points changes. Illustrate the various possible shapes with graphs.

75. A canister is dropped from a helicopter 500 m above the ground. Its parachute does not open, but the canister has been designed to withstand an impact velocity of 100 m/s. Will it burst?

76. In an automobile race along a straight road, car A passed car B twice. Prove that at some time during the race their accelerations were equal. State the assumptions that you make.

77. A rectangular beam will be cut from a cylindrical log of radius 10 inches.
(a) Show that the beam of maximal cross-sectional area is a square.
(b) Four rectangular planks will be cut from the four sections of the log that remain after cutting the square beam. Determine the dimensions of the planks that will have maximal cross-sectional area.
(c) Suppose that the strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from the cylindrical log.



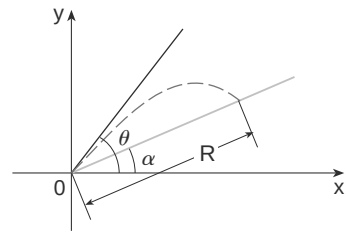
78. If a projectile is fired with an initial velocity v at an angle of inclination θ from the horizontal, then its trajectory, neglecting air resistance, is the parabola

$$y = (\tan\theta)x - \frac{}{2v^2\cos^2\theta}x^2 \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

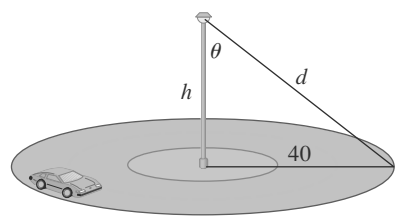
(a) Suppose the projectile is fired from the base of a plane that is inclined at an angle α, $\alpha > 0$, from the horizontal, as shown in the figure. Show that the range of the projectile, measured up the slope, is given by

$$R(\theta) = \frac{2v^2\cos\theta\,\sin(\theta - \alpha)}{\cos^2\alpha}$$

(b) Determine θ so that R is a maximum.
(c) Suppose the plane is at an angle α below the horizontal. Determine the range R in this case, and determine the angle at which the projectile should be fired to maximize R.



79. A light is to be placed atop a pole of height h feet to illuminate a busy traffic circle, which has a radius of 40 ft. The intensity of illumination I at any point P on the circle is directly proportional to the cosine of the angle θ (see figure) and inversely proportional to the square of the distance d from the source.
(a) How tall should the light pole be to maximize I?
(b) Suppose that the light pole is 50 feet tall and that a woman is walking away from the base of the pole at the rate of 4 ft/s. At what rate is the intensity of the light at the point on her back 4 ft above the ground decreasing when she reaches the outer edge of the traffic circle?



80. Water is flowing at a constant rate into a spherical tank. Let $V(t)$ be the volume of water in the tank and $H(t)$ be the height of the water in the tank at time t.
(a) What are the meanings of $V'(t)$ and $H'(t)$? Are these derivatives positive, negative, or zero?
(b) Is $V''(t)$ positive, negative, or zero? Explain.
(c) Let t_1, t_2, and t_3 be the times when the tank is one-quarter full, half full, and three-quarters full, respectively. Are the values $H''(t_1)$, $H''(t_2)$, and $H''(t_3)$ positive, negative, or zero? Why?

81. Show that, for $x > 0$,

$$\frac{x}{1 + x^2} < \tan^{-1}x < x$$

82. Sketch the graph of a function f such that $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x \to \pm\infty}[f(x) + x] = 0$.

PROBLEMS PLUS

One of the most important principles of problem solving is analogy (see page 80). If you are having trouble getting started on a problem, it is sometimes helpful to start by solving a similar, but simpler, problem. The following example illustrates the principle. Cover up the solution and try solving it yourself first.

EXAMPLE If x, y, and z are positive numbers, prove that

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geqslant 8$$

SOLUTION It may be difficult to get started on this problem. (Some students have tackled it by multiplying out the numerator, but that just creates a mess.) Try to think of a similar, simpler problem. When several variables are involved, it is often helpful to think of an analogous problem with fewer variables. In the present case we can reduce the number of variables from three to one and prove the analogous inequality

$$\frac{x^2+1}{x} \geqslant 2 \qquad \text{for } x > 0 \tag{1}$$

In fact, if we are able to prove (1), then the desired inequality follows because

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} = \left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right)\left(\frac{z^2+1}{z}\right) \geqslant 2 \cdot 2 \cdot 2 = 8$$

The key to proving (1) is to recognize that it is a disguised version of a minimum problem. If we let

$$f(x) = \frac{x^2+1}{x} = x + \frac{1}{x} \qquad x > 0$$

then $f'(x) = 1 - (1/x^2)$, so $f'(x) = 0$ when $x = 1$. Also, $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. Therefore, the absolute minimum value of f is $f(1) = 2$. This means that

$$\frac{x^2+1}{x} \geqslant 2 \qquad \text{for all positive values of } x$$

and, as previously mentioned, the given inequality follows by multiplication.

The inequality in (1) could also be proved without calculus. In fact, if $x > 0$, we have

$$\frac{x^2+1}{x} \geqslant 2 \qquad x^2+1 \geqslant 2x \qquad x^2 - 2x + 1 \geqslant 0$$

$$(x-1)^2 \geqslant 0$$

Because the last inequality is obviously true, the first one is true too. ∎

Look Back

What have we learned from the solution to this example?

- To solve a problem involving several variables, it might help to solve a similar problem with just one variable.
- When trying to prove an inequality, it might help to think of it as a maximum or minimum problem.

PROBLEMS

1. If a rectangle has its base on the x-axis and two vertices on the curve $y = e^{-x^2}$, show that the rectangle has the largest possible area when the two vertices are at the points of inflection of the curve.

2. Show that $|\sin x - \cos x| \leqslant \sqrt{2}$ for all x.

3. Show that, for all positive values of x and y,

$$\frac{e^{x+y}}{xy} \geqslant e^2$$

4. Show that $x^2y^2(4 - x^2)(4 - y^2) \leqslant 16$ for all numbers x and y such that $|x| \leqslant 2$ and $|y| \leqslant 2$.

5. Let a and b be positive numbers. Show that not both of the numbers $b(1-b)$ and $a(1-a)$ can be greater than $\frac{1}{4}$.

6. Find the point on the parabola $y = 1 - x^2$ at which the tangent line cuts from the first quadrant the triangle with the smallest area.

7. Find the highest and lowest points on the curve $x^2 - xy + y^2 = 12$.

8. Sketch the set of all points (x, y) such that $|x + y| \leqslant e^x$.

9. The line $y = mx + b$ intersects the parabola $y = x^2$ in points A and B (see the figure). Find the point P on the arc AOB of the parabola that maximizes the area of the triangle PAB.

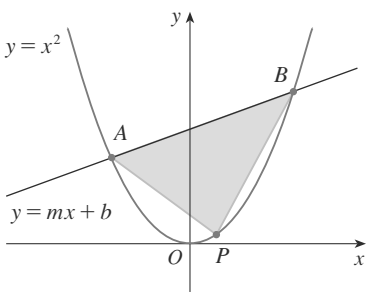


FIGURE FOR PROBLEM 9

10. Find a function f such that $f'(-1) = \frac{1}{2}$, $f'(0) = 0$, and $f'(x) > 0$ for all x, or prove that such a function cannot exist.

11. Determine the values of the number a for which the function f has no critical number:

$$f(x) = (a^2 + a - 6)\cos 2x + (a - 2)x + \cos 1$$

12. Sketch the region in the plane consisting of all points (x, y) such that

$$2xy \leqslant |x - y| \leqslant x^2 + y^2$$

13. Let ABC be a triangle with $\angle BAC = 120^\circ$ and $|AB| \cdot |AC| = 1$.
    (a) Express the length of the angle bisector AD in terms of $|AB|$.
    (b) Find the largest possible value of $|AD|$.

14. $ABCD$ is a square piece of paper with sides of length 1 m. A quarter-circle is drawn from B to D with center A. The piece of paper is folded along EF, with E on AB and F on AD, so A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF could have.

15. For which positive numbers a does the curve $y = a^x$ intersect the line $y = x$?

16. For what value of a is the following equation true?

$$\lim_{x \to \infty} \left(\frac{x + a}{x - a}\right)^x = e$$

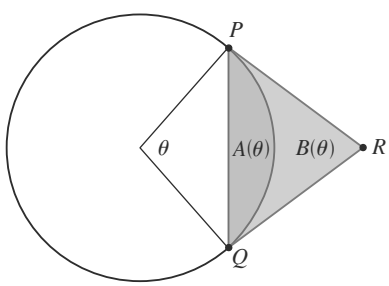


FIGURE FOR PROBLEM 18

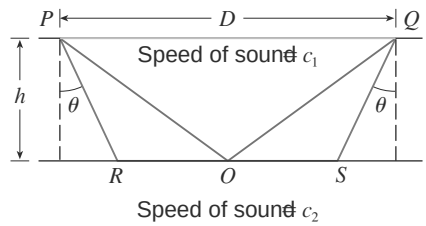
17. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where $a_1, a_2, \ldots, a_n$ are real numbers and n is a positive integer. If it is given that $|f(x)| \leqslant |\sin x|$ for all x, show that

$$|a_1 + 2a_2 + \cdots + na_n| \leqslant 1$$

18. An arc PQ of a circle subtends a central angle θ as in the figure. Let $A(\theta)$ be the area between the chord PQ and the arc PQ. Let $B(\theta)$ be the area between the tangent lines PR, QR, and the arc. Find

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$$

19. The speeds of sound c_1 in an upper layer and c_2 in a lower layer of rock and the thickness h of the upper layer can be determined by seismic exploration if the speed of sound in the lower layer is greater than the speed in the upper layer. A dynamite charge is detonated at P a point and the transmitted signals are recorded at a point Q, which is a distance D from P. The first signal to arrive at Q travels along the surface and takes T_1 seconds. The next signal travels from P to a point R, from R to S in the lower layer, and then to Q, taking T_2 seconds. The third signal is reflected off the lower layer at the midpoint O of RS and takes T_3 seconds to reach Q.
(a) Express T_1, T_2, and T_3 in terms of D, h, c_1, c_2, and θ.
(b) Show that T_2 is a minimum when $\sin\theta = c_1/c_2$.
(c) Suppose that $D = 1$ km, $T_1 = 0.26$ s, $T_2 = 0.32$ s, $T_3 = 0.34$ s. Find c_1, c_2, and h.



Note: Geophysicists use this technique when studying the structure of the Earth, whether searching for oil or examining fault lines.

20. For what values of c is there a straight line that intersects the curve

$$y = x^4 + cx^3 + 12x^2 - 5x + 2$$

in four distinct points?

21. One of the problems posed by the Marquis de l'Hôpital in his calculus textbook Analyse des Infiniment Petits concerns a pulley that is attached to the ceiling of a room at a point C by a rope of length r. At another point B on the ceiling, at a distance d from C (where $d > r$), a rope of length x is attached and passed through the pulley F at C and connected to a weight W. The weight is released and comes to rest at its equilibrium position D. L'Hôpital argued, this happens when the distance $|ED|$ is maximized. Show that when the system reaches equilibrium, the value of x is

$$\frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$$

Notice that this expression is independent of the weight W.

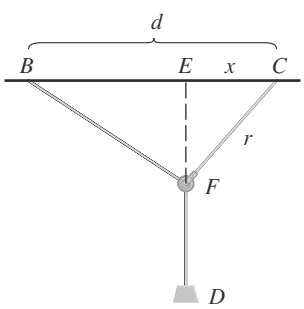


FIGURE FOR PROBLEM 21

22. Given a sphere with radius r, find the height of a pyramid of minimum volume whose base is a square and whose base and triangular faces are all tangent to the sphere. What if the base of the pyramid is a regular n-gon (a polygon with n equal sides and angles)? (Use the fact that the volume of a pyramid is $\frac{1}{3}Ah$, where A is the area of the base.)

23. Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hours for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely?

24. A hemispherical bubble is placed on a spherical bubble of radius 1. A smaller hemispherical bubble is then placed on the first one. This process is continued until n chambers, including the sphere, are formed. (The figure shows the case $n = 4$.) Use mathematical induction to prove that the maximum height of any bubble tower with n chambers is $\sqrt{n}$.

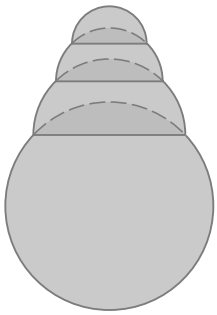
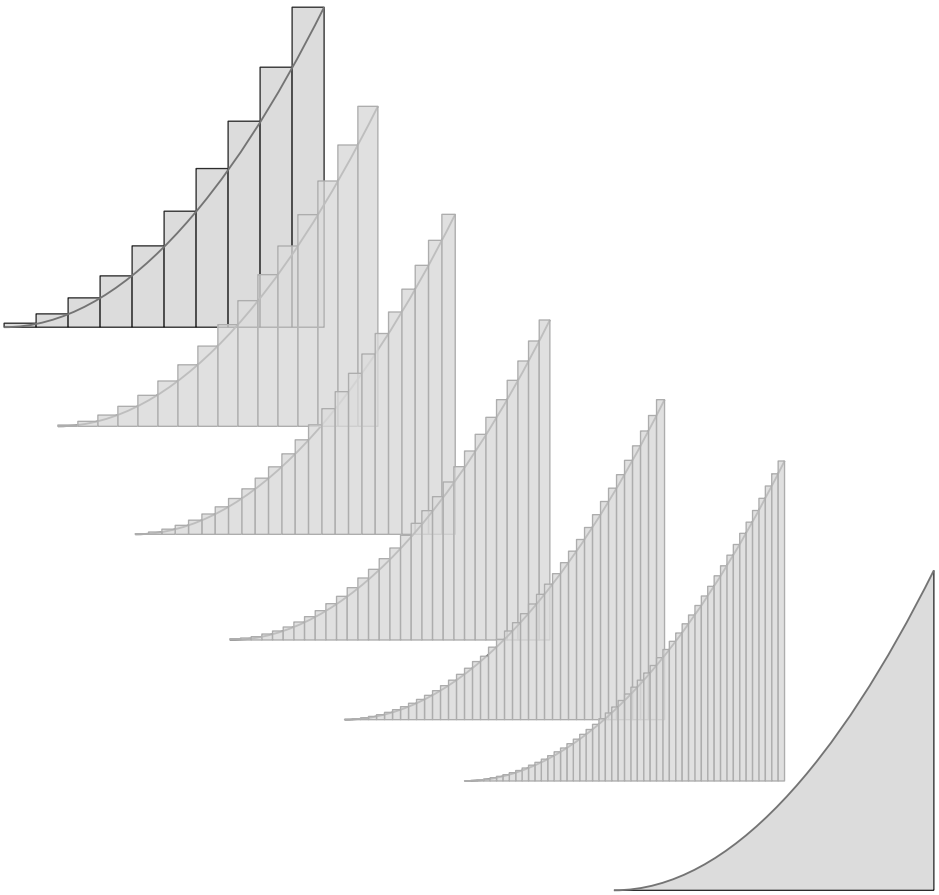


FIGURE FOR PROBLEM 24

To compute an area we approximate a region by rectangles and let the number of rectangles become large. The precise area is the limit of these sums of areas of rectangles.



Integrals

|||| Now is a good time to read (or reread) *A Preview of Calculus* (see page 2). It discusses the unifying ideas of calculus and helps put in perspective where we have been and where we are going.

In Chapter 2 we used the tangent and velocity problems to introduce the derivative, which is the central idea in differential calculus. In much the same way, this chapter starts with the area and distance problems and uses them to formulate the idea of a definite integral, which is the basic concept of integral calculus. We will see in Chapters 6 and 8 how to use the integral to solve problems concerning volumes, lengths of curves, population predictions, cardiac output, forces on a dam, work, consumer surplus, and baseball, among many others.

There is a connection between integral calculus and differential calculus. The Fundamental Theorem of Calculus relates the integral to the derivative, and we will see in this chapter that it greatly simplifies the solution of many problems.

5.1 Areas and Distances

In this section we discover that in trying to find the area under a curve or the distance traveled by a car, we end up with the same special type of limit.

The Area Problem

We begin by attempting to solve the *area problem:* Find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geqslant 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.

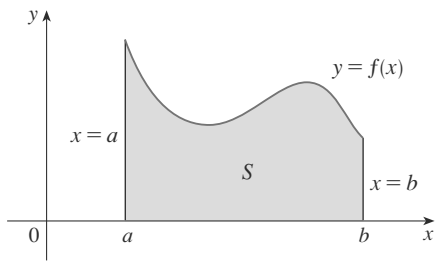


FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$

In trying to solve the area problem we have to ask ourselves: What is the meaning of the word *area*? This question is easy to answer for regions with straight sides. For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

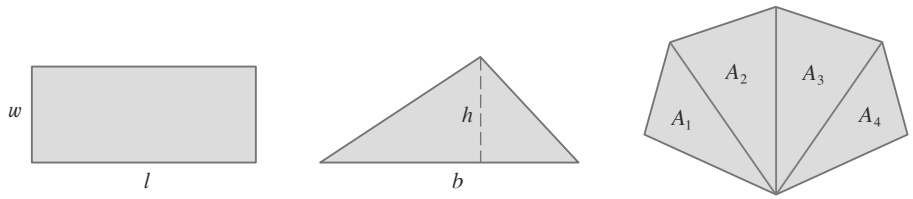


FIGURE 2
$A = lw$ $\quad$ $A = \frac{1}{2}bh$ $\quad$ $A = A_1 + A_2 + A_3 + A_4$

However, it isn't so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations. We pursue a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles. The following example illustrates the procedure.

Try placing rectangles to estimate the area.
Resources / Module 6
/ What Is Area?
/ Estimating Area under a Parabola

EXAMPLE 1 Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

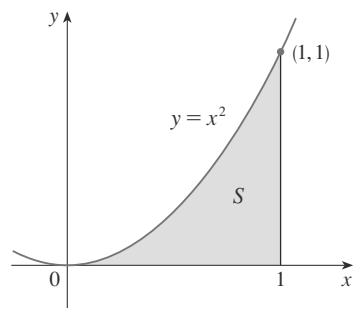


FIGURE 3

SOLUTION We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure 4(a).

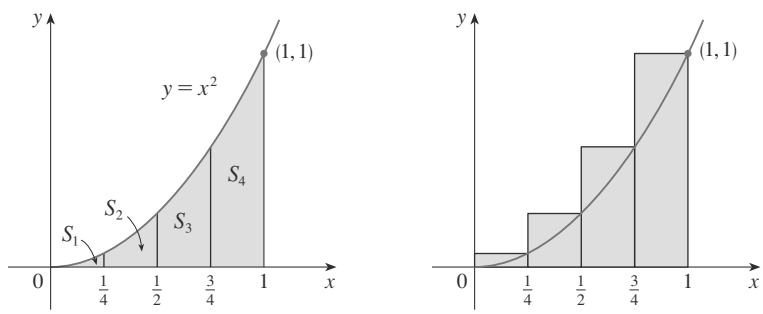


FIGURE 4

We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip [see Figure 4(b)]. In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right end points of the subintervals $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, and $[\frac{3}{4}, 1]$.

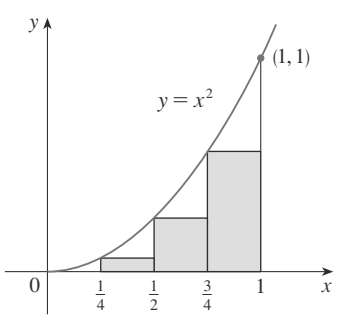


FIGURE 5

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4} \cdot 1^2 = \tfrac{15}{32} = 0.46875$$

From Figure 4(b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

Instead of using the rectangles in Figure 4(b) we could use the smaller rectangles in Figure 5 whose heights are the values of f at the left endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.) The sum of the areas of these

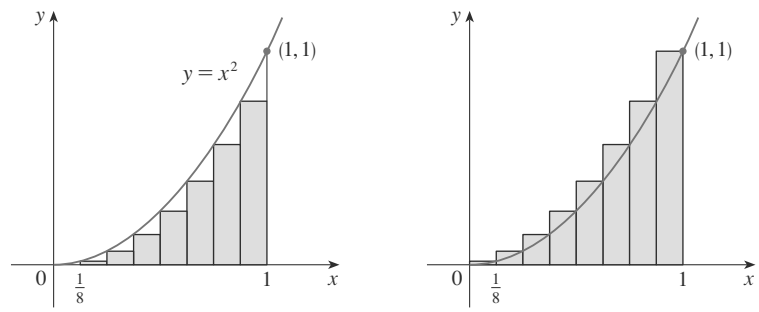
approximating rectangles is

$$L_4 = \tfrac{1}{4} \cdot 0^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 = \tfrac{7}{32} = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips. Figure 6 shows what happens when we divide the region S into eight strips of equal width.



(a) Using left endpoints (b) Using right endpoints

FIGURE 6
Approximating S with eight rectangles

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

We could obtain better estimates by increasing the number of strips. The table at the left shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n). In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335. A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$.

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

From the values in the table in Example 1, it looks as if R_n is approaching $\frac{1}{3}$ as n increases. We confirm this in the next example.

EXAMPLE 2 For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,

$$\lim_{n \to \infty} R_n = \tfrac{1}{3}$$

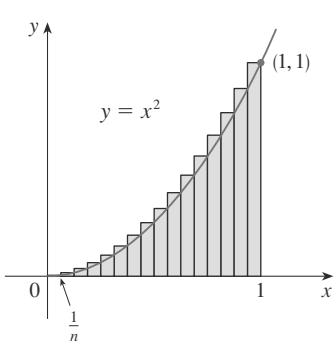


FIGURE 7

SOLUTION R_n is the sum of the areas of the n rectangles in Figure 7. Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \ldots, n/n$; that is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$.

Thus

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n} \cdot \frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

$$= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

The ideas in Examples 1 and 2 are explored in Module 5.1/5.2/7.7 for a variety of functions.

Here we need the formula for the sum of the squares of the first n positive integers:

$$\boxed{1} \qquad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Perhaps you have seen this formula before. It is proved in Example 5 in Appendix E.

Putting Formula 1 into our expression for R_n, we get

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Thus, we have

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

$$= \tfrac{1}{6} \cdot 1 \cdot 2 = \tfrac{1}{3}$$

|||| Here we are computing the limit of the sequence $\{R_n\}$. Sequences were discussed in *A Preview of Calculus* and will be studied in detail in Chapter 11. Their limits are calculated in the same way as limits at infinity (Section 2.6). In particular, we know that

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

It can be shown that the lower approximating sums also approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} L_n = \tfrac{1}{3}$$

From Figures 8 and 9 it appears that, as n increases, both L_n and R_n become better and better approximations to the area of S. Therefore, we *define* the area A to be the limit of the

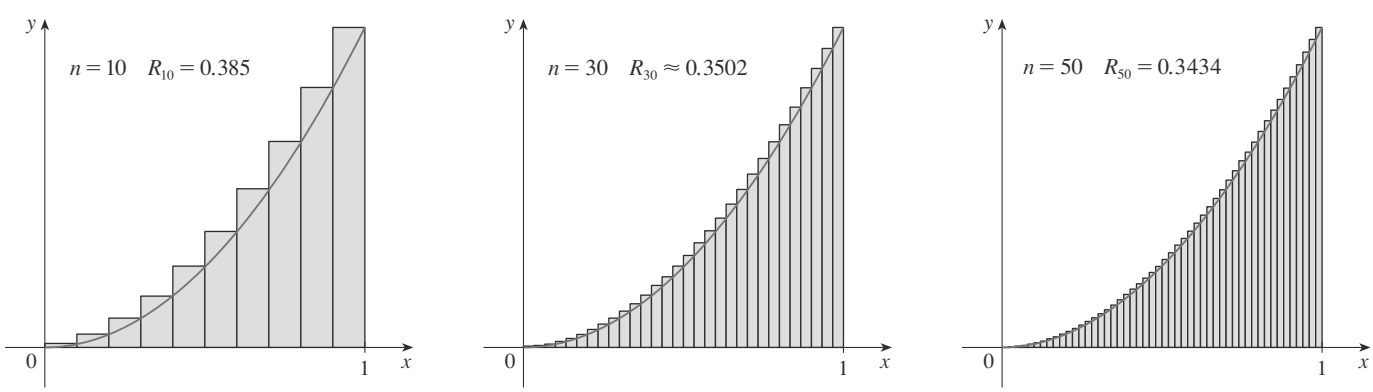


FIGURE 8

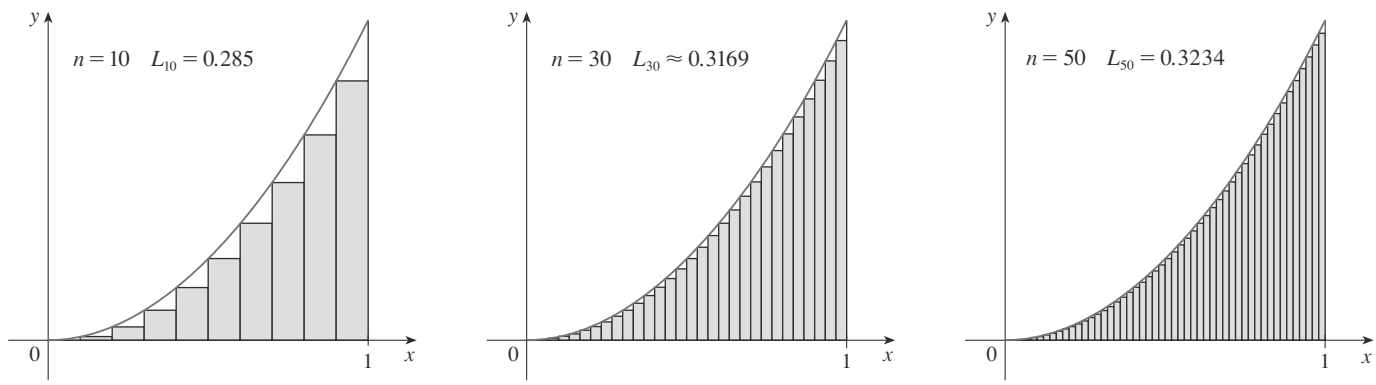


FIGURE 9
The area is the number that is smaller than all upper sums and larger than all lower sums

sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \tfrac{1}{3}$$

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10. The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x, \quad x_2 = a + 2\,\Delta x, \quad x_3 = a + 3\,\Delta x, \quad \ldots$$

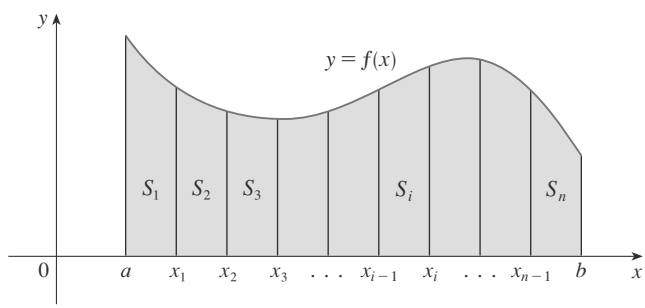


FIGURE 10

Let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure 11). Then the area of the ith rectangle

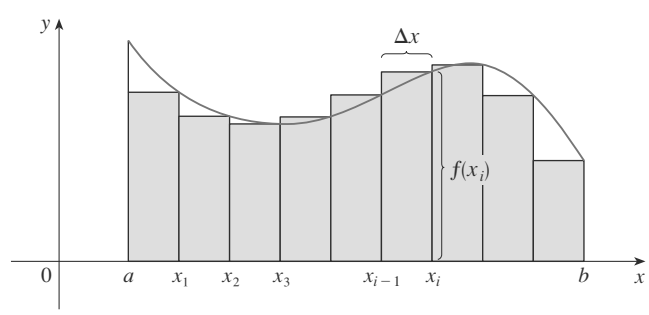


FIGURE 11

is $f(x_i)\,\Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

Figure 12 shows this approximation for $n = 2$, 4, 8, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore, we define the area A of the region S in the following way.

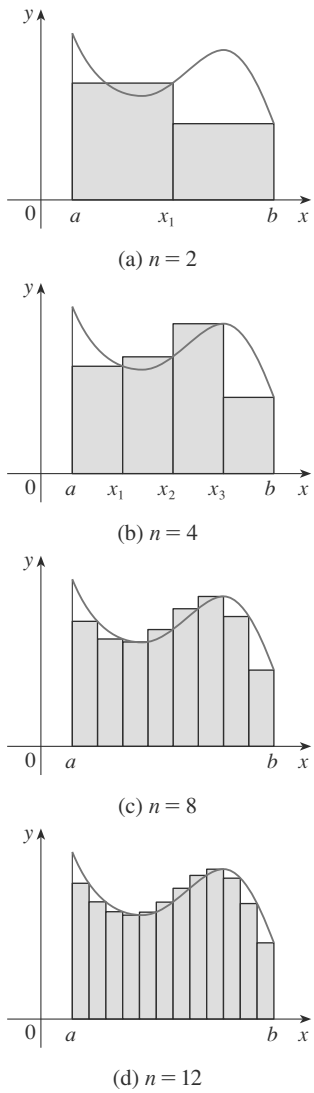


FIGURE 12

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming that f is continuous. It can also be shown that we get the same value if we use left endpoints:

3 $$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the ith rectangle to be the value of f at *any* number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ the **sample points**. Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints. So a more general expression for the area of S is

4 $$A = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

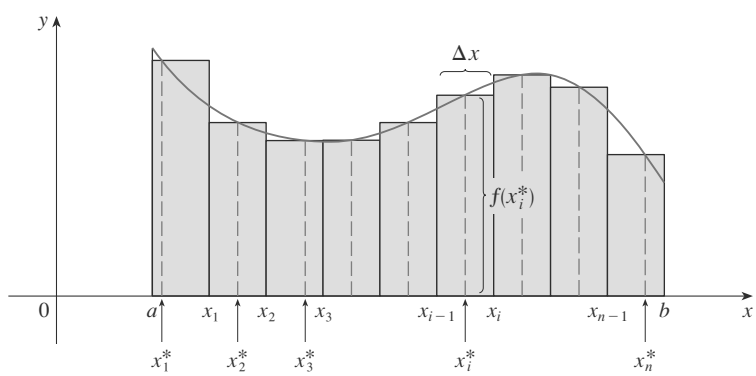


FIGURE 13

We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

This tells us to end with $i = n$.

This tells us to add.

$$\sum_{i=m}^{n} f(x_i)\,\Delta x$$

This tells us to start with $i = m$.

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

|||| If you need practice with sigma notation, look at the examples and try some of the exercises in Appendix E.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

We could also rewrite Formula 1 in the following way:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

EXAMPLE 3 Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.
(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.
(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

SOLUTION
(a) Since $a = 0$ and $b = 2$, the width of a subinterval is

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

So $x_1 = 2/n$, $x_2 = 4/n$, $x_3 = 6/n$, $x_i = 2i/n$, and $x_n = 2n/n$. The sum of the areas of the approximating rectangles is

$$\begin{aligned} R_n &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= e^{-x_1}\,\Delta x + e^{-x_2}\,\Delta x + \cdots + e^{-x_n}\,\Delta x \\ &= e^{-2/n}\left(\frac{2}{n}\right) + e^{-4/n}\left(\frac{2}{n}\right) + \cdots + e^{-2n/n}\left(\frac{2}{n}\right) \end{aligned}$$

According to Definition 2, the area is

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{2}{n}\left(e^{-2/n} + e^{-4/n} + e^{-6/n} + \cdots + e^{-2n/n}\right)$$

Using sigma notation we could write

$$A = \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

It is difficult to evaluate this limit directly by hand, but with the aid of a computer algebra system it isn't hard (see Exercise 24). In Section 5.3 we will be able to find A more easily using a different method.

(b) With $n = 4$ the subintervals of equal width $\Delta x = 0.5$ are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of these subintervals are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$,

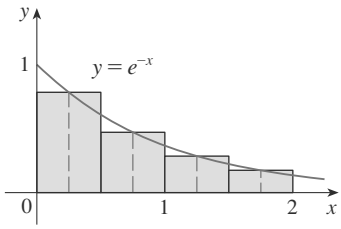


FIGURE 14

and $x_4^* = 1.75$, and the sum of the areas of the four approximating rectangles (see Figure 14) is

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(x_i^*)\,\Delta x \\ &= f(0.25)\,\Delta x + f(0.75)\,\Delta x + f(1.25)\,\Delta x + f(1.75)\,\Delta x \\ &= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5) \\ &= \tfrac{1}{2}(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}) \approx 0.8557 \end{aligned}$$

So an estimate for the area is

$$A \approx 0.8557$$

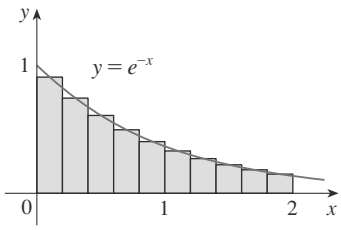


FIGURE 15

With $n = 10$ the subintervals are $[0, 0.2], [0.2, 0.4], \ldots, [1.8, 2]$ and the midpoints are $x_1^* = 0.1, x_2^* = 0.3, x_3^* = 0.5, \ldots, x_{10}^* = 1.9$. Thus

$$\begin{aligned} A \approx M_{10} &= f(0.1)\,\Delta x + f(0.3)\,\Delta x + f(0.5)\,\Delta x + \cdots + f(1.9)\,\Delta x \\ &= 0.2(e^{-0.1} + e^{-0.3} + e^{-0.5} + \cdots + e^{-1.9}) \approx 0.8632 \end{aligned}$$

From Figure 15 it appears that this estimate is better than the estimate with $n = 4$.

The Distance Problem

Now let's consider the *distance problem:* Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times. (In a sense this is the inverse problem of the velocity problem that we discussed in Section 2.1.) If the velocity remains constant, then the distance problem is easy to solve by means of the formula

$$\text{distance} = \text{velocity} \times \text{time}$$

But if the velocity varies, it's not so easy to find the distance traveled. We investigate the problem in the following example.

EXAMPLE 4 Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

In order to have the time and the velocity in consistent units, let's convert the velocity readings to feet per second (1 mi/h = 5280/3600 ft/s):

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

During the first five seconds the velocity doesn't change very much, so we can estimate the distance traveled during that time by assuming that the velocity is constant. If we take the velocity during that time interval to be the initial velocity (25 ft/s), then we

obtain the approximate distance traveled during the first five seconds:

$$25 \text{ ft/s} \times 5 \text{ s} = 125 \text{ ft}$$

Similarly, during the second time interval the velocity is approximately constant and we take it to be the velocity when $t = 5$ s. So our estimate for the distance traveled from $t = 5$ s to $t = 10$ s is

$$31 \text{ ft/s} \times 5 \text{ s} = 155 \text{ ft}$$

If we add similar estimates for the other time intervals, we obtain an estimate for the total distance traveled:

$$25 \times 5 + 31 \times 5 + 35 \times 5 + 43 \times 5 + 47 \times 5 + 46 \times 5 = 1135 \text{ ft}$$

We could just as well have used the velocity at the *end* of each time period instead of the velocity at the beginning as our assumed constant velocity. Then our estimate becomes

$$31 \times 5 + 35 \times 5 + 43 \times 5 + 47 \times 5 + 46 \times 5 + 41 \times 5 = 1215 \text{ ft}$$

If we had wanted a more accurate estimate, we could have taken velocity readings every two seconds, or even every second.

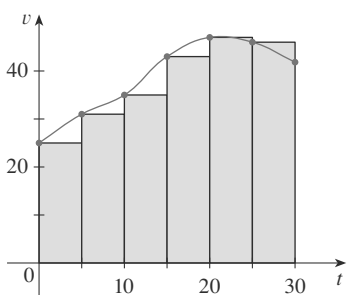


FIGURE 16

Perhaps the calculations in Example 4 remind you of the sums we used earlier to estimate areas. The similarity is explained when we sketch a graph of the velocity function of the car in Figure 16 and draw rectangles whose heights are the initial velocities for each time interval. The area of the first rectangle is $25 \times 5 = 125$, which is also our estimate for the distance traveled in the first five seconds. In fact, the area of each rectangle can be interpreted as a distance because the height represents velocity and the width represents time. The sum of the areas of the rectangles in Figure 16 is $L_6 = 1135$, which is our initial estimate for the total distance traveled.

In general, suppose an object moves with velocity $v = f(t)$, where $a \leqslant t \leqslant b$ and $f(t) \geqslant 0$ (so the object always moves in the positive direction). We take velocity readings at times $t_0\ (= a), t_1, t_2, \ldots, t_n\ (= b)$ so that the velocity is approximately constant on each subinterval. If these times are equally spaced, then the time between consecutive readings is $\Delta t = (b - a)/n$. During the first time interval the velocity is approximately $f(t_0)$ and so the distance traveled is approximately $f(t_0)\,\Delta t$. Similarly, the distance traveled during the second time interval is about $f(t_1)\,\Delta t$ and the total distance traveled during the time interval $[a, b]$ is approximately

$$f(t_0)\,\Delta t + f(t_1)\,\Delta t + \cdots + f(t_{n-1})\,\Delta t = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t$$

If we use the velocity at right endpoints instead of left endpoints, our estimate for the total distance becomes

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t = \sum_{i=1}^{n} f(t_i)\,\Delta t$$

The more frequently we measure the velocity, the more accurate our estimates become, so it seems plausible that the *exact* distance d traveled is the *limit* of such expressions:

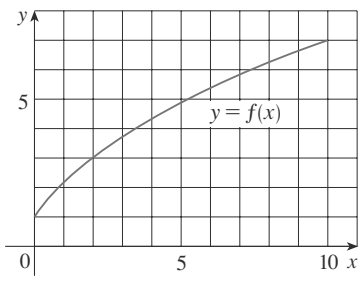
5 $$d = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

We will see in Section 5.4 that this is indeed true.

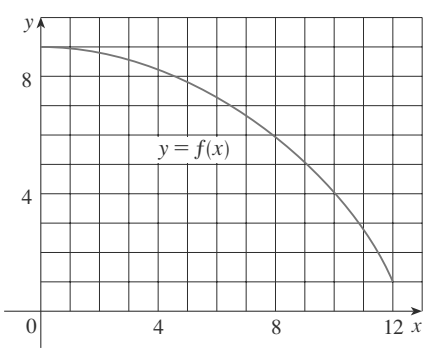
Because Equation 5 has the same form as our expressions for area in Equations 2 and 3, it follows that the distance traveled is equal to the area under the graph of the velocity function. In Chapters 6 and 8 we will see that other quantities of interest in the natural and social sciences~~such~~ as the work done by a variable force or the cardiac output of the heart~~can~~ also be interpreted as the area under a curve. So when we compute areas in this chapter, bear in mind that they can be interpreted in a variety of practical ways.

5.1 Exercises

1. (a) By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under the given graph of f from $x = 0$ to $x = 10$. In each case sketch the rectangles that you use.
(b) Find new estimates using 10 rectangles in each case.



2. (a) Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.
(i) L_6 (sample points are left endpoints)
(ii) R_6 (sample points are right endpoints)
(iii) M_6 (sample points are midpoints)
(b) Is L_6 an underestimate or overestimate of the true area?
(c) Is R_6 an underestimate or overestimate of the true area?
(d) Which of the numbers L_6, R_6, or M_6 gives the best estimate? Explain.



3. (a) Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 5$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?
(b) Repeat part (a) using left endpoints.

4. (a) Estimate the area under the graph of $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$ using five approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?
(b) Repeat part (a) using left endpoints.

5. (a) Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.
(b) Repeat part (a) using left endpoints.
(c) Repeat part (a) using midpoints.
(d) From your sketches in parts (a)~~(c)~~, which appears to be the best estimate?

6. (a) Graph the function $f(x) = e^{-x^2}$, $-2 \leqslant x \leqslant 2$.
(b) Estimate the area under the graph of f using four approximating rectangles and taking the sample points to be
(i) right endpoints (ii) midpoints
In each case sketch the curve and the rectangles.
(c) Improve your estimates in part (b) by using eight rectangles.

7–8 With a programmable calculator (or a computer), it is possible to evaluate the expressions for the sums of areas of approximating rectangles, even for large values of n, using looping. (On a TI use the Is> command or a For-EndFor loop, on a Casio use Isz, on an HP or in BASIC use a FOR-NEXT loop.) Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10$, 30, and 50. Then guess the value of the exact area.

7. The region under $y = \sin x$ from 0 to π

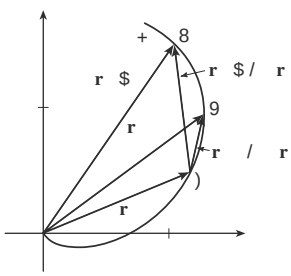
8. The region under $y = 1/x^2$ from 1 to 2

CAS **9.** Some computer algebra systems have commands that will draw approximating rectangles and evaluate the sums of their areas, at least if x_i^* is a left or right endpoint. (For instance, in Maple use `leftbox`, `rightbox`, `leftsum`, and `rightsum`.)
(a) If $f(x) = \sqrt{x}$, $1 \leqslant x \leqslant 4$, find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 4.6 and 4.7.

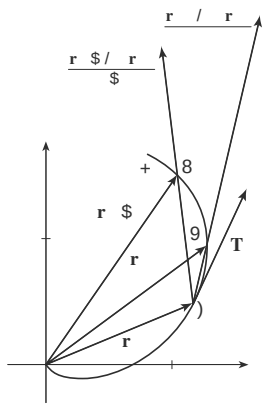
CAS **10.** (a) If $f(x) = \sin(\sin x)$, $0 \leqslant x \leqslant \pi/2$, use the commands discussed in Exercise 9 to find the left and right sums for $n = 10$, 30, and 50.

Exercises 13.2 page 861

1. (a)

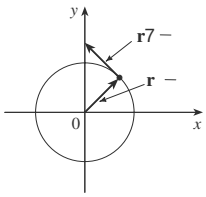


(b), (d)



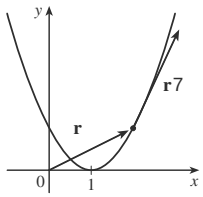
(c) **r** 4 $\lim_{h \quad 0} \dfrac{\mathbf{r}\ 4 \quad h \quad \mathbf{r}\ 4}{h}$; **T** 4 $\dfrac{\mathbf{r}\ 4}{\mathbf{r}\ 4}$

3. (a), (c)



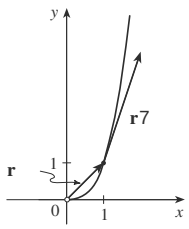
(b) **r** t $\sin t, \cos t$

5. (a), (c)



(b) **r** t **i** $2t$ **j**

7. (a), (c)



(b) **r** t e^t **i** $3e^{3t}$ **j**

9. **r** t $2t$, 1, 1 $(2 \quad \bar{t})$ **11.** **r** t $4e^{4t}$ **k**

13. **r** t $2te^{t^2}$ **i** [3 1 $3t$ **k** **15.** **r** t **b** $2t$**c**

17. 15 $\overline{262}$, 6 $\overline{262}$, 1 $\overline{262}$ **19.** $\frac{3}{5}$ **j** $\frac{4}{5}$ **k**

21. 1, $2t$, $3t^2$, 1 $\overline{14}$, 2 $\overline{14}$, 3 $\overline{14}$, 0, 2, $6t$, $6t^2$, $6t$, 2

23. x 1 $5t$, y 1 $4t$, 1 $3t$

25. x 1 t, y t, 1 t

27. x $\frac{1}{4}$ t, y 1 t, 1 t

29. (a) Not smooth (b) Smooth (c) Not smooth

31. 66¡ **33.** 4 **i** 3 **j** 5 **k** **35.** **i** **j** **k**

37. e^t **i** t^2 **j** $t \ln t$ t **k** **C**

39. $\frac{1}{3}t^3$**i** t^4 1 **j** $\frac{1}{3}t^3$**k**

45. 1 $4t \cos t$ $11t^2 \sin t$ $3t^3 \cos t$

Exercises 13.3 page 868

1. 20 $\overline{29}$ **3.** e e 1 **5.** $\frac{1}{27}$ $13^{3\ 2}$ 8 **7.** 9.5706

9. **r** t s $\dfrac{2}{\overline{29}}s$ **i** 1 $\dfrac{3}{\overline{29}}s$ **j** 5 $\dfrac{4}{\overline{29}}s$ **k**

11. **r** t s 3 sin s 5 **i** $4s$ 5 **j** 3 cos s 5 **k**

13. (a) $(2 \quad \overline{29}) \cos t$, 5 $\overline{29}$, $(\ 2 \quad \overline{29}) \sin t$,
sin t, 0, cos t

(b) $\frac{2}{29}$

15. (a) $\dfrac{1}{e^{2t} \quad 1}$ $\overline{2}e^t, e^{2t}$, 1 , $\dfrac{1}{e^{2t} \quad 1}$ 1 e^{2t}, $\overline{2}e^t$, $\overline{2}e^t$

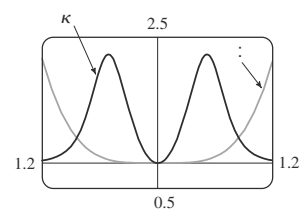
(b) $\overline{2}e^{2t}$ e^{2t} 1 ^{2}

17. 2 $4t^2$ 1 $^{3\ 2}$ **19.** $\frac{4}{25}$ **21.** $\frac{1}{7}$ $\frac{\overline{19}}{14}$

23. 6 x 1 $9x^4$ $^{3\ 2}$ **25.** 15 $\bar{x}$ 1 $100x^{3\ 3\ 2}$

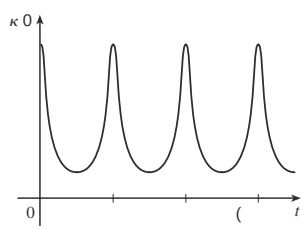
27. $(\ \frac{1}{2} \ln 2, 1 \quad \overline{2})$; approaches 0 **29.** (a) P (b) 1.3, 0.7

31.



33. a is y f x , b is y x

35. t $\dfrac{6 \quad \overline{4 \cos^2 t \quad 12 \cos t \quad 13}}{17 \quad 12 \cos t}$ $^{3\ 2}$

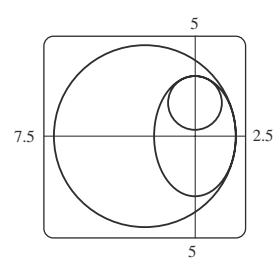


integer multiples of 2

37. 1 $(\ \overline{2}e^t)$ **39.** $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$, $\frac{1}{3}, \frac{2}{3}$, $\frac{2}{3}$, $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$

41. y $6x$, x $6y$ 6

43. $(x \quad \frac{5}{2})^2$ y^2 $\frac{81}{4}$, x^2 $(y \quad \frac{5}{3})^2$ $\frac{16}{9}$



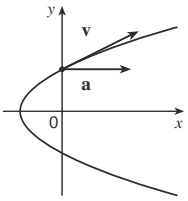
45. 1, 3, 1 **53.** 2 t^4 $4t^2$ 1

55. 2.07 10^{10} • 2 m

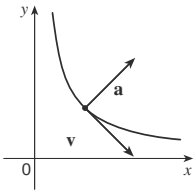
Exercises 13.4 page 878

1. (a) $1.8\mathbf{i}$ $3.8\mathbf{j}$ $0.7\mathbf{k}$, $2.0\mathbf{i}$ $2.4\mathbf{j}$ $0.6\mathbf{k}$,
$2.8\mathbf{i}$ $1.8\mathbf{j}$ $0.3\mathbf{k}$, $2.8\mathbf{i}$ $0.8\mathbf{j}$ $0.4\mathbf{k}$
(b) $2.4\mathbf{i}$ $0.8\mathbf{j}$ $0.5\mathbf{k}$, 2.58

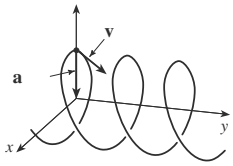
3. $\mathbf{v}\ t$ $2t, 1$
$\mathbf{a}\ t$ $2, 0$
$|\mathbf{v}\ t|$ $\sqrt{4t^2 \quad 1}$



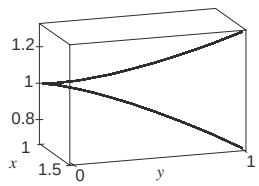
5. $\mathbf{v}\ t$ $e^t\mathbf{i}$ $e^{\ t}\mathbf{j}$
$\mathbf{a}\ t$ $e^t\mathbf{i}$ $e^{\ t}\mathbf{j}$
$|\mathbf{v}\ t|$ $\sqrt{e^{2t} \quad e^{\ 2t}}$



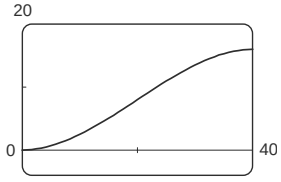
7. $\mathbf{v}\ t$ $\cos t\,\mathbf{i}$ $\mathbf{j}$ $\sin t\,\mathbf{k}$
$\mathbf{a}\ t$ $\sin t\,\mathbf{i}$ $\cos t\,\mathbf{k}$
$|\mathbf{v}\ t|$ $\sqrt{2}$

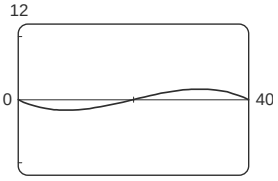


9. $2t, 3t^2, 2t$, $2, 6t, 2$, t $\sqrt{9t^2 \quad 8}$
11. $\sqrt{2}\,\mathbf{i}$ $e^t\mathbf{j}$ $e^{\ t}\mathbf{k}$, $e^t\mathbf{j}$ $e^{\ t}\mathbf{k}$, e^t $e^{\ t}$
13. e^t $\cos t$ $\sin t\,\mathbf{i}$ $\sin t$ $\cos t\,\mathbf{j}$ t $1\ \mathbf{k}$,
e^t $2\sin t\,\mathbf{i}$ $2\cos t\,\mathbf{j}$ t $2\ \mathbf{k}$, e^t $\sqrt{t^2 \quad 2t \quad 3}$
15. $\mathbf{v}\ t$ $\mathbf{i}$ $\mathbf{j}$ $t\,\mathbf{k}$, $\mathbf{r}\ t$ $t\,\mathbf{i}$ $t\,\mathbf{j}$ $\frac{1}{2}t^2\mathbf{k}$
17. (a) $\mathbf{r}\ t$ 1 $\frac{1}{2}t^2\ \mathbf{i}$ $t^2\,\mathbf{j}$ 1 $\frac{1}{3}t^3\ \mathbf{k}$
(b)



19. t 4
21. $\mathbf{r}\ t$ $t\,\mathbf{i}$ $t\,\mathbf{j}$ $\frac{5}{2}t^2\,\mathbf{k}$, $|\mathbf{v}\ t|$ $\sqrt{25t^2 \quad 2}$
23. (a) 22 km (b) 3.2 km (c) 500 m s
25. 30 m s
27. 10.2 , 79.8
29. (a) 16 m (b) 23.6 upstream





31. $6t, 6$ **33.** $0, 1$ **35.** e^t $e^{\ t}$, $\sqrt{2}$
37. 4.5 cm s^2, 9.0 cm s^2 **39.** t 1

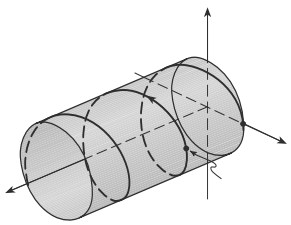
Chapter 13 Review page 882

True-False Quiz

1. True **3.** False **5.** False **7.** False **9.** True

Exercises

1. (a)



(b) $\mathbf{r}\ t$ $\mathbf{i}$ $\sin t\,\mathbf{j}$ $\cos t\,\mathbf{k}$,
$\mathbf{r}\ t$ $^2\cos t\,\mathbf{j}$ $^2\sin t\,\mathbf{k}$
3. $\mathbf{r}\ t$ $4\cos t\,\mathbf{i}$ $4\sin t\,\mathbf{j}$ 5 $4\cos t\ \mathbf{k}$, 0 t 2
5. $\frac{1}{3}\mathbf{i}$ 2 $^2\,\mathbf{j}$ $\sqrt{2 \quad \mathbf{k}}$ **7.** 15.9241 **9.** 2
11. (a) $t^2, t, 1$ $\sqrt{t^4 \quad t^2 \quad 1}$
(b) $2t, 1$ $t^4,$ $2t^3$ t $\sqrt{t^8 \quad 4t^6 \quad 2t^4 \quad 5t^2}$
(c) $\sqrt{t^8 \quad 4t^6 \quad 2t^4 \quad 5t^2}$ t^4 t^2 1 ^{2}
13. 12 $17^{3\ 2}$ **15.** x $2y$ 2 0
17. $\mathbf{v}\ t$ $\frac{1}{2}$ $\ln t\ \mathbf{i}$ $\mathbf{j}$ $e^{\ t}\mathbf{k}$,
$|\mathbf{v}\ t|$ $\sqrt{2 \quad 2\ln t \quad \ln t^{\ 2} \quad e^{\ 2t}}$, $\mathbf{a}\ t$ 1 $t\ \mathbf{i}$ $e^{\ t}\mathbf{k}$
19. (a) About 3.8 ft above the ground, 60.8 ft from the athlete
(b) 21.4 ft (c) 64.2 ft from the athlete
21. (c) $2e^{\ t}\mathbf{v}_d$ $e^{\ t}\mathbf{R}$

Problems Plus page 884

1. (a) $\mathbf{v}$ R $\sin t\,\mathbf{i}$ $\cos t\,\mathbf{j}$ (c) $\mathbf{a}$ $^2\mathbf{r}$
3. (a) 90 , v_0^2 2 **5.** (b) $\mathbf{R}\ t$ $m\ k$ 1 $e^{\ kt\ m}\ \mathbf{v}_0$
$m\ k$ $m\ k$ 1 $e^{\ kt\ m}$ $t\,\mathbf{j}$
7. (a) 0.94 ft to the right of the table's edge, 15 ft s
(b) 7.6 (c) 2.13 ft to the right of the table's edge

Chapter 14

Exercises 14.1 page 897

1. (a) 27; a temperature of 15 C with wind blowing at 40 km h feels equivalent to about 27 C without wind.
(b) When the temperature is 20 C, what wind speed gives a wind chill of 30 C? 20 km h
(c) With a wind speed of 20 km h, what temperature gives a wind chill of 49 C? 35 C
(d) A function of wind speed that gives wind-chill values when the temperature is 5 C
(e) A function of temperature that gives wind-chill values when the wind speed is 50 km h

3. Yes

5. (a) 25; a 40-knot wind blowing in the open sea for 15 h will create waves about 25 ft high.

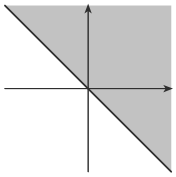
(b) $f\ 30, t$ is a function of t giving the wave heights produced by 30-knot winds blowing for t hours.

(c) $f\ v, 30$ is a function of v giving the wave heights produced by winds of speed v blowing for 30 hours.

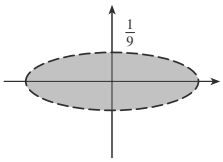
7. (a) 4 (b) ^{2} (c) 0,

9. (a) e (b) $x, y,$ x^2 y^2 (c) 1,

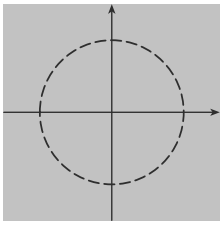
11. x, y y x



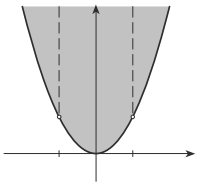
13. x, y $\frac{1}{9}x^2$ y^2 1



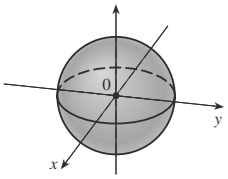
15. x, y x^2 y^2 4



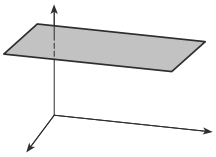
17. x, y y x^2, x 1



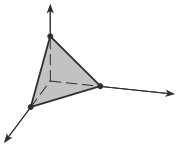
19. $x, y,$ x^2 y^2 ^{2} 1



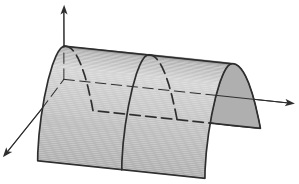
21. 3, horizontal plane



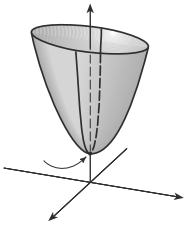
23. x y 1, plane



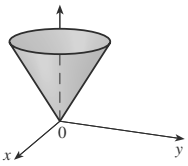
25. 1 x^2
parabolic cylinder



27. $4x^2$ y^2 1
elliptic paraboloid



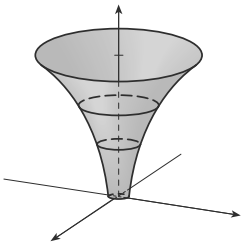
29. $\overline{x^2 \quad y^2}$,
top half of cone



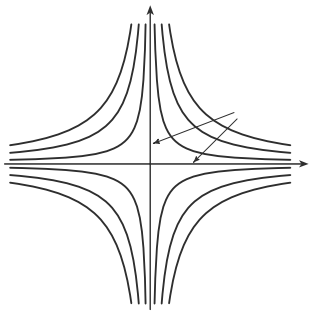
31. 56, 35

33. Steep; nearly ßat

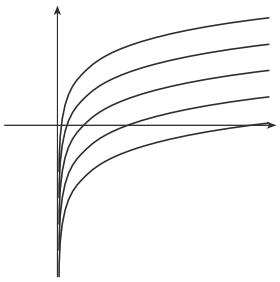
35.



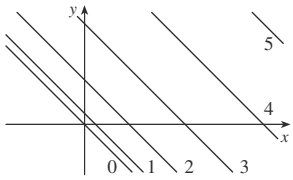
37. xy k



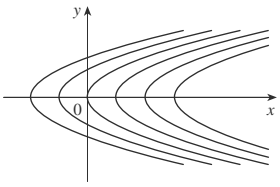
39.



41. $\overline{x \quad y}$ k



43. x y^2 k



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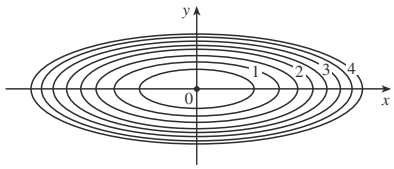

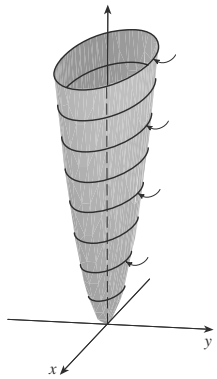

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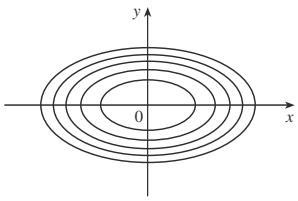

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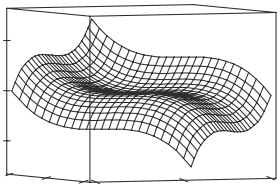
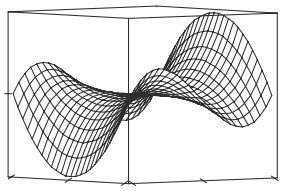


51.



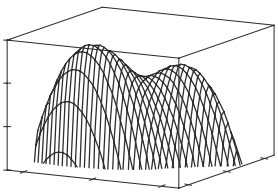
53. (a) B (b) III
55. (a) F (b) V
57. (a) D (b) IV

59. Family of parallel planes

61. Family of hyperboloids of one or two sheets with axis the y-axis

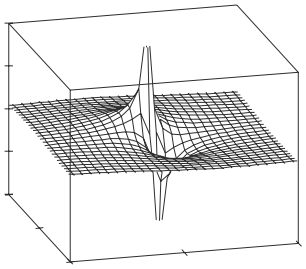
63. (a) Shift the graph of f upward 2 units
(b) Stretch the graph of f vertically by a factor of 2
(c) Reßect the graph of f about the xy-plane
(d) Reßect the graph of f about the xy-plane and then shift it upward 2 units

65.



f appears to have a maximum value of about 15. There are two local maximum points but no local minimum point.

67.



The function values approach 0 as x, y become large; as x, y approaches the origin, f approaches or 0, depending on the direction of approach.

69. If c 0, the graph is a cylindrical surface. For c 0, the level curves are ellipses. The graph curves upward as we leave the origin, and the steepness increases as c increases. For c 0, the level curves are hyperbolas. The graph curves upward in the y-direction and downward, approaching the xy-plane, in the x-direction giving a saddle-shaped appearance near (0, 0, 1).

71. (b) y $0.75x$ 0.01

Exercises 14.2 page 908

1. Nothing; if f is continuous, f 3, 1 6 **3.** $\frac{5}{2}$

5. 2025 **7.** Does not exist **9.** Does not exist

11. 0 **13.** Does not exist **15.** 2 **17.** 1

19. Does not exist

21. The graph shows that the function approaches different numbers along different lines.

23. h x, y $4x^2$ $9y^2$ $12xy$ $24x$ $36y$ 36
$\overline{2x \quad 3y \quad 6}$; x, y $2x$ $3y$ 6

25. Along the line y x

27. x, y y $e^{x\ 2}$ **29.** x, y y 0

31. x, y x^2 y^2 4 **33.** $x, y,$ y 0, y $\overline{x^2 \quad {}^2}$

35. x, y x, y 0, 0 **37.** 0 **39.** 0

Exercises 14.3 page 919

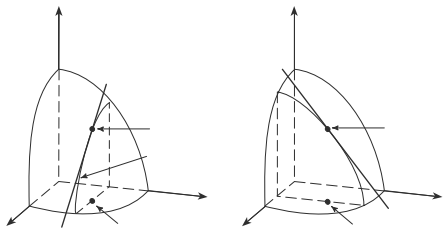
1. (a) The rate of change of temperature as longitude varies, with latitude and time Þxed; the rate of change as only latitude varies; the rate of change as only time varies.
(b) Positive, negative, positive

3. (a) f_T 15, 30 1.3; for a temperature of 15 C and wind speed of 30 km h, the wind-chill index rises by 1.3 C for each degree the temperature increases. f_v 15, 30 0.15; for a temperature of 15 C and wind speed of 30 km h, the wind-chill index decreases by 0.15 C for each km h the wind speed increases.
(b) Positive, negative (c) 0

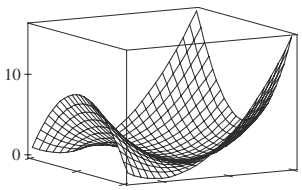
5. (a) Positive (b) Negative

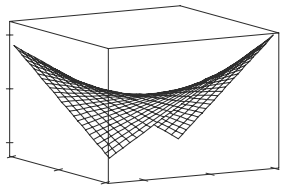
7. c f, b f_x, a f_y

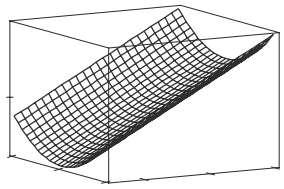
9. $f_x\ 1, 2\quad 8$ slope of C_1, $f_y\ 1, 2\quad 4$ slope of C_2



11. $f_x\quad 2x\quad 2xy$, $f_y\quad 2y\quad x^2$







13. $f_x\ x, y\quad 3$, $f_y\ x, y\quad 8y^3$

15. $\quad x\quad e^{3y}$, $\quad y\quad 3xe^{3y}$

17. $f_x\ x, y\quad 2y\quad x\quad y^2$, $f_y\ x, y\quad 2x\quad x\quad y^2$

19. $w\quad \cos\quad \cos\ ,\ w\quad \sin\quad \sin$

21. $f_r\ r, s\quad \dfrac{2r^2}{r^2\quad s^2}\quad \ln r^2\quad s^2$, $f_s\ r, s\quad \dfrac{2rs}{r^2\quad s^2}$

23. $u\quad t\quad e^{w\ t}(1\quad w\ t)$, $u\quad w\quad e^{w\ t}$

25. $f_x\quad y^2\ ^3$, $f_y\quad 2xy\ ^3\quad 3$, $f\quad 3xy^2\ ^2\quad 3y$

27. $w\quad x\quad 1\quad x\quad 2y\quad 3\ ,\ w\quad y\quad 2\quad x\quad 2y\quad 3\ ,$
$w\quad 3\quad x\quad 2y\quad 3$

29. $u\quad x\quad e^{\ t}\sin\ ,\ u\quad t\quad xe^{\ t}\sin\ ,\ u\quad xe^{\ t}\cos$

31. $f_x\quad y^{\ 2}\tan\ yt$, $f_y\quad xy^{\ 2}t\sec^2\ yt\quad x^{\ 2}\tan\ yt$,
$f\quad 2xy\ \tan\ yt$, $f_t\quad xy^2\ ^2\sec^2\ yt$

33. $u\quad x_i\quad x_i\quad \overline{x_1^2\quad x_2^2\quad\quad x_n^2}$ **35.** $\frac{3}{5}$ **37.** $\frac{1}{3}$

39. $f_x\ x, y\quad 2x\quad y$, $f_y\ x, y\quad 4y\quad x$

41. $\dfrac{\ }{x}\quad \dfrac{3y\quad 2x}{2\quad 3xy}$, $\dfrac{\ }{y}\quad \dfrac{3x\quad 2y}{2\quad 3xy}$

43. $\dfrac{\ }{x}\quad \dfrac{1\quad y^2\ ^2}{1\quad y\quad y^2\ ^2}$, $\dfrac{\ }{y}\quad \dfrac{\ }{1\quad y\quad y^2\ ^2}$

45. (a) $f\ x\ ,\quad y$ (b) $f\ x\quad y\ , f\ x\quad y$

47. $f_{xx}\quad 12x^2\quad 6y^3$, $f_{xy}\quad 18xy^2\quad f_{yx}$, $f_{yy}\quad 18x^2y$

49. $_{xx}\quad 2y\ x\quad y^3$, $_{xy}\quad x\quad y\ x\quad y^3\quad _{yx}$,
$_{yy}\quad 2x\ x\quad y^3$

51. $u_{ss}\quad e^{\ s}\sin t$, $u_{st}\quad e^{\ s}\cos t\quad u_{ts}$, $u_{tt}\quad e^{\ s}\sin t$

57. $12xy$, $72xy$

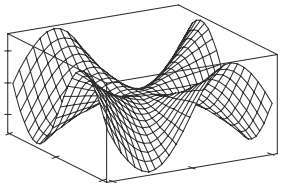
59. $24\sin 4x\quad 3y\quad 2\ ,\ 12\sin 4x\quad 3y\quad 2$

61. $e^r\ 2\sin\quad \cos\quad r\ \sin$

63. $4\quad y\quad 2\ ^3, 0$ **65.** 12.2, 16.8, 23.25 **77.** $R^2\ R_1^2$

83. No **85.** $x\quad 1\quad t, y\quad 2,\quad 2\quad 2t$ **89.** 2

91. (a)



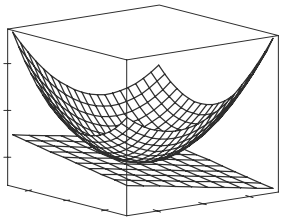
(b) $f_x\ x, y\quad \dfrac{x^4y\quad 4x^2y^3\quad y^5}{x^2\quad y^2\ ^2}$, $f_y\ x, y\quad \dfrac{x^5\quad 4x^3y^2\quad xy^4}{x^2\quad y^2\ ^2}$

(c) 0, 0 (e) No, since f_{xy} and f_{yx} are not continuous.

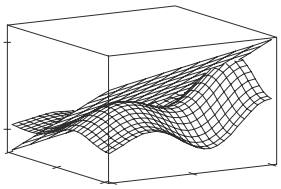
Exercises 14.4 page 930

1. $\quad 8x\quad 2y$ **3.** $x\quad 2y\quad 4$ **5.** $\quad y$

7.



9.



11. $2x\quad \frac{1}{4}y\quad 1$ **13.** $x\quad 1$ **15.** $\frac{1}{2}x\quad y\quad \frac{1}{4}\quad \frac{1}{2}$

17. $\frac{2}{3}x\quad \frac{7}{3}y\quad \frac{20}{3}$; $2.84\overline{6}$ **19.** $\frac{3}{7}x\quad \frac{2}{7}y\quad \frac{6}{7}$; 6.9914

21. $4T\quad H\quad 329$; 129 F

23. $d\quad 3x^2\ln y^2\ dx\quad 2x^3\ y\ dy$

25. $du\quad e^t\sin\quad dt\quad e^t\cos\quad d$

27. $dw\quad x^2\quad y^2\quad ^2\quad ^1\ x\,dx\quad y\,dy\quad d$

29. $\quad 0.9225$, $d\quad 0.9$ **31.** 5.4 cm^2 **33.** 16 cm^3

35. 150 **37.** $\frac{1}{17}\quad 0.059$ **39.** $_1\quad x$, $_2\quad y$

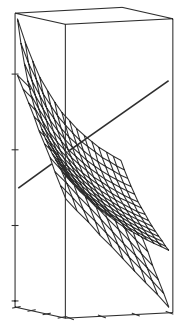
Exercises 14.5 page 938

1. 4 $2xy$ y^2 t^3 3 x^2 $2xy$ t^2
3. $\cos x \cos y$ $\sin x \sin y$ $(2 \quad \bar{t})$
5. e^y $2t$ x $2xy$ ^{2}
7. s $2x$ y xt $2yt$, t $2x$ y xs $2ys$
9. $\dfrac{}{s}$ $\dfrac{4st \quad \ln t}{1 \quad 2x \quad y^2}$, $\dfrac{}{t}$ $\dfrac{2s^2 \quad s\ t}{1 \quad 2x \quad y^2}$
11. $\dfrac{}{s}$ e^r $t \cos$ $\dfrac{s}{\sqrt{s^2 \quad t^2}} \sin$,
$\dfrac{}{t}$ e^r $s \cos$ $\dfrac{t}{\sqrt{s^2 \quad t^2}} \sin$
13. 62 **15.** 7, 2
17. $\dfrac{u}{r}$ $\dfrac{u}{x}\dfrac{x}{r}$ $\dfrac{u}{y}\dfrac{y}{r}$, $\dfrac{u}{s}$ $\dfrac{u}{x}\dfrac{x}{s}$ $\dfrac{u}{y}\dfrac{y}{s}$,
$\dfrac{u}{t}$ $\dfrac{u}{x}\dfrac{x}{t}$ $\dfrac{u}{y}\dfrac{y}{t}$
19. $\dfrac{v}{x}$ $\dfrac{v}{p}\dfrac{p}{x}$ $\dfrac{v}{q}\dfrac{q}{x}$ $\dfrac{v}{r}\dfrac{r}{x}$,
$\dfrac{v}{y}$ $\dfrac{v}{p}\dfrac{p}{y}$ $\dfrac{v}{q}\dfrac{q}{y}$ $\dfrac{v}{r}\dfrac{r}{y}$,
$\dfrac{v}{}$ $\dfrac{v}{p}\dfrac{p}{}$ $\dfrac{v}{q}\dfrac{q}{}$ $\dfrac{v}{r}\dfrac{r}{}$
21. 85, 178, 54 **23.** $\frac{9}{7}, \frac{9}{7}$ **25.** 36, 24, 30
27. $\dfrac{4\ xy^{\ 3\ 2} \quad y}{x \quad 2x^2 \quad \sqrt{xy}}$ **29.** $\dfrac{\sin x \quad y \quad e^y}{\sin x \quad y \quad xe^y}$
31. $\dfrac{3y \quad 2x}{2 \quad 3xy}$, $\dfrac{3x \quad 2y}{2 \quad 3xy}$
33. $\dfrac{1 \quad y^2\ ^2}{1 \quad y \quad y^2\ ^2}$, $\dfrac{}{1 \quad y \quad y^2\ ^2}$
35. 2 C s
37. 0.33 m s per minute
39. (a) 6 m^3 s (b) 10 m^2 s (c) 0 m s **41.** 0.27 L s
43. (a) r x cos y sin ,
$x\ r \sin$ $y\ r \cos$
49. $4rs$ ^{2} x^2 $4r^2$ $4s^2$ ^{2} x y $4rs$ ^{2} y^2 2 y

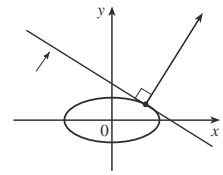
Exercises 14.6 page 950

1. 0.1 millibar mi **3.** 0.778 **5.** $\frac{5}{16}$ $\bar{3}$ $\frac{1}{4}$
7. (a) $f\ x, y$ $5y^2$ $12x^2y$, $10xy$ $4x^3$
(b) 4, 16 (c) 172 13
9. (a) e^{2y}, $2x\ e^{2y}$, $2xye^{2y}$ (b) 1, 12, 0 (c) $\frac{22}{3}$
11. 23 10 **13.** 4 $\bar{2}$ **15.** 4 9
17. 9 $(2 \quad \bar{5})$ **19.** 2 5 **21.** 4 $\bar{2}$, 1, 1 **23.** 1, 0, 1
25. $\bar{3}$, 1, 1, 1 **27.** (b) 12, 92
29. All points on the line y x 1 **31.** (a) 40 $(3 \quad \bar{3})$
33. (a) 32 $\bar{3}$ (b) 38, 6, 12 (c) 2 $\sqrt{406}$ **35.** $\frac{327}{13}$
39. (a) $4x$ $2y$ 3 21 (b) $\dfrac{x \quad 4}{8}$ $\dfrac{y \quad 1}{4}$ $\dfrac{}{6}$ $\dfrac{1}{}$
41. (a) $4x$ $5y$ 4 (b) $\dfrac{x \quad 2}{4}$ $\dfrac{y \quad 1}{5}$ $\dfrac{}{1}$ $\dfrac{1}{}$
43. (a) x y 1 (b) x 1 y

45.



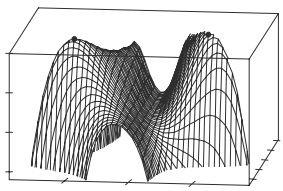
47. 4, 8 , x $2y$ 4



53. $(\quad \bar{6}\ 3, \quad 2\ \bar{6}\ 3, \quad \bar{6}\ 2)$
59. x 1 $10t$, y 1 $16t$, 2 $12t$
63. If **u** a, b and **v** c, d , then af_x bf_y and cf_x df_y are known, so we solve linear equations for f_x and f_y.

Exercises 14.7 page 961

1. (a) f has a local minimum at (1, 1).
(b) f has a saddle point at (1, 1).
3. Local minimum at (1, 1), saddle point at (0, 0)
5. Maximum $f(\quad 1, \frac{1}{2})$ 11
7. Minima f 1, 1 0, f 1, 1 0, saddle point at 0, 0
9. Saddle points at 1, 1 , 1, 1
11. f has a local maximum value of 1 at all points of the form x_0, x_0 .
13. None **15.** Saddle points 0, n , n an integer
17. Minimum f 0, 0 0, saddle points at 1, 0
19. Maximum f 0, 0 2, minimum f 0, 2 2,
saddle points 1, 1
21. Maximum f 3, 3 3 $\bar{3}$ 2,
minimum f 5 3, 5 3 3 $\bar{3}$ 2
23. Minima f 1.714, 0 9.200, f 1.402, 0 0.242,
saddle point (0.312, 0), lowest point 1.714, 0, 9.200
25. Maxima f 1.267, 0 1.310, f 1.629, 1.063 8.105,
saddle points 0.259, 0 , 1.526, 0 ,
highest points 1.629, 1.063, 8.105
27. Maximum f 2, 0 9, minimum f 0, 3 14
29. Maximum f 1, 1 7, minimum f 0, 0 4
31. Maximum f 3, 0 83, minimum f 1, 1 0
33. Maximum f 1, 0 2, minimum f 1, 0 2
35.



37. $\bar{3}$ **39.** 0, 0, 1 , 0, 0, 1 **41.** $\frac{100}{3}, \frac{100}{3}, \frac{100}{3}$
43. 16 $\bar{3}$ **45.** $\frac{4}{3}$ **47.** Cube, edge length c 12
49. Square base of side 40 cm, height 20 cm **51.** L^3 $(3 \quad \bar{3})$

Exercises 14.8 page 970

1. 59, 30 **3.** Maxima f 1, 0 1, minima f 0, 1 1
5. Maxima f 2, 1 4, minima f 2, 1 4
7. Maximum f 1, 3, 5 70, minimum f 1, 3, 5 70
9. Maximum 2 $\overline{3}$, minimum 2 $\overline{3}$
11. Maximum $\overline{3}$, minimum 1
13. Maximum $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ 2,
minimum $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ 2
15. Maximum $f(1, \overline{2}, \overline{2})$ 1 2 $\overline{2}$,
minimum $f(1, \overline{2}, \overline{2})$ 1 2 $\overline{2}$
17. Maximum $\frac{3}{2}$, minimum $\frac{1}{2}$
19. Maxima $f(1 \overline{2}, 1 (2 \overline{2}))$ $e^{1 4}$,
minima $f(1 \overline{2}, 1 (2 \overline{2}))$ $e^{1 4}$
25–37. See Exercises 37Ð51 in Section 14.7.
39. Nearest $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, farthest 1, 1, 2
41. Maximum 9.7938, minimum 5.3506
43. (a) c n (b) When x_1 x_2 x_n

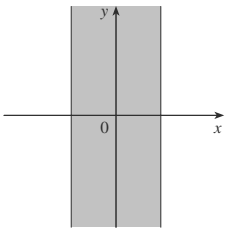
Chapter 14 Review page 975

True-False Quiz

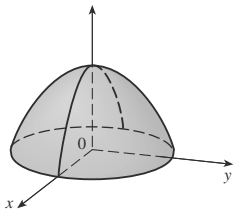
1. True **3.** False **5.** False **7.** True **9.** False
11. True

Exercises

1. x, y 1 x 1



3.



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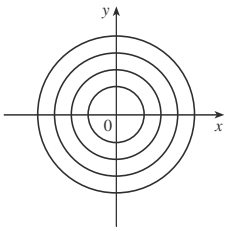

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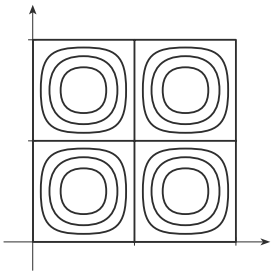


9. $\frac{2}{3}$
11. (a) 3.5 C m, 3.0 C m (b) 0.35 C m by Equation 14.6.9 (DeÞnition 14.6.2 gives 1.1 C m.) (c) 0.25
13. f_x 1 $\overline{2x \quad y^2}$, f_y y $\overline{2x \quad y^2}$
15. $_u$ $\tan^{1} v$, $_v$ u 1 v^2
17. T_p $\ln q$ e^r, T_q p q e^r, T_r pe^r q e^r
19. f_{xx} $24x$, f_{xy} $2y$ f_{yx}, f_{yy} $2x$
21. f_{xx} $k\ k$ 1 $x^{k\ 2} y^{l\ m}$, f_{xy} $klx^{k\ 1} y^{l\ 1\ m}$ f_{yx},
f_x $kmx^{k\ 1} y^{l\ m\ 1}$ f_x, f_{yy} $l\ l$ 1 $x^k y^{l\ 2\ m}$,
f_y $lmx^k y^{l\ 1\ m\ 1}$ f_y, f $m\ m$ 1 $x^k y^{l\ m\ 2}$
25. (a) $8x$ $4y$ 1 (b) $\frac{x \quad 1}{8}$ $\frac{y \quad 2}{4}$ 1
27. (a) $2x$ $2y$ 3 3 (b) $\frac{x \quad 2}{4}$ $\frac{y \quad 1}{4}$ $\frac{\quad 1}{6}$
29. (a) x $2y$ 5 0 (b) x 2 $\frac{y \quad 1}{2}$ $\frac{\quad}{5}$
31. $(\overline{2\ 7}, 1 \overline{14}, 3 \overline{14})$
33. $60x$ $\frac{24}{5} y$ $\frac{32}{5}$ 120; 38.656
35. e^t 2 y $3t^2$ 4 $2t$ $y^{2\ 2}$ **37.** 47, 108
43. $e^{x\ \bar{y}}$ $\bar{y}, x$ $(2 \bar{y})$, 2 **45.** $\frac{43}{5}$ **47.** $\overline{145}$ 2, 4, $\frac{9}{2}$
49. $\frac{5}{8}$ knot mi **51.** Minimum f 4, 1 11
53. Maximum f 1, 1 1; saddle points (0, 0), (0, 3), (3, 0)
55. Maximum f 1, 2 4, minimum f 2, 4 64
57. Maximum f 1, 0 2, minima f 1, 1 3,
saddle points 1, 1 , 1, 0
59. Maximum $f(\overline{2\ 3}, 1 \overline{3})$ 2 $(3 \overline{3})$,
minimum $f(\overline{2\ 3}, 1 \overline{3})$ 2 $(3 \overline{3})$
61. Maximum 1, minimum 1
63. $(3^{1 4}, 3^{1 4} \overline{2}, 3^{1 4})$, $(3^{1 4}, 3^{1 4} \overline{2}, 3^{1 4})$
65. $P(2 \overline{3})$, $P(3 \overline{3})$ 6, $P(2 \overline{3} 3)$ 3

Problems Plus page 978

1. L^2W^2, $\frac{1}{4}L^2W^2$ **3.** (a) x w 3, base w 3 (b) Yes
9. $\overline{6}$ 2, 3 $\overline{2}$ 2

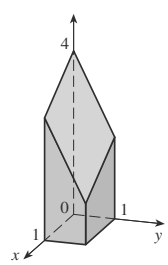
Chapter 15

Exercises 15.1 page 988

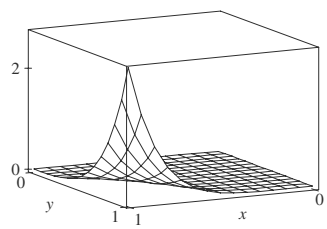
1. (a) 288 (b) 144 **3.** (a) ^{2} 2 4.935 (b) 0
5. (a) 6 (b) 3.5 **7.** U V L
9. (a) 248 (b) 15.5 **11.** 60
13. 3 **15.** 0.6065, 0.5694, 0.5606, 0.5585, 0.5579, 0.5578

Exercises 15.2 page 994

1. 9 $27y$, $8x$ $24x^2$ **3.** 10 **5.** 2 **7.** 261,632 45
9. $\frac{21}{2}$ ln 2 **11.** 6 **13.** $\frac{21}{2}$ **15.** 9 ln 2
17. $[(\overline{3} \quad 1) 2]$ 12 **19.** $\frac{1}{2} e^2$ 3
21.



23. 47.5 **25.** $\frac{166}{27}$ **27.** $\frac{4}{15}(2\ \ \overline{2}\ \ 1)$ **29.** 36
31. $21e\ \ 57$



33. $\frac{5}{6}$
35. FubiniÕS Theorem does not apply. The integrand has an inÞnite discontinuity at the origin.

Exercises 15.3 page 1002

1. $\frac{9}{20}$ **3.** $\frac{4}{9}e^{3\ \ 2}\ \ \ \frac{32}{45}$ **5.** $e\ \ \ 1$ **7.** $\frac{256}{21}$ **9.** $\frac{1}{2}\ln 2$
11. $\frac{1}{2}e^4\ \ \ 2e$ **13.** $1\ \ \ \cos 1\ \ 2$ **15.** $\frac{147}{20}$ **17.** 0
19. $\frac{7}{18}$ **21.** $\frac{31}{8}$ **23.** $\frac{1}{6}$ **25.** $\frac{128}{15}$
27. $\frac{1}{3}$ **29.** 0, 1.213, 0.713 **31.** $\frac{64}{3}$
33. 13,984,735,616 14,549,535 **35.** $\ \ \ \

29. $\int_{1}^{1}\int_{0}^{1}\int_{0}^{x^2}\int_{0}^{y} f(x, y, \quad)\, d\ dy\, dx$

$\int_{0}^{1}\int_{\sqrt{1-y}}^{\sqrt{1-y}}\int_{0}^{y} f(x, y, \quad)\, d\ dx\, dy$

$\int_{0}^{1}\int_{1}^{\sqrt{1-y}}\int_{\sqrt{1-y}}^{\sqrt{1-y}} f(x, y, \quad)\, dx\, dy\, d$

$\int_{0}^{1}\int_{0}^{y}\int_{\sqrt{1-y}}^{\sqrt{1-y}} f(x, y, \quad)\, dx\, d\ dy$

$\int_{1}^{1}\int_{0}^{1}\int_{x^2}^{1}\int_{x^2} f(x, y, \quad)\, dy\, d\ dx$

$\int_{0}^{1}\int_{\sqrt{1-}}^{1}\int_{x^2} f(x, y, \quad)\, dy\, dx\, d$

31. $\int_{0}^{1}\int_{\sqrt{x}}^{1}\int_{0}^{y} f(x, y, \quad)\, d\ dy\, dx$

$\int_{0}^{1}\int_{0}^{y^2}\int_{0}^{y} f(x, y, \quad)\, d\ dx\, dy$

$\int_{0}^{1}\int_{0}^{1}\int_{0}^{y^2} f(x, y, \quad)\, dx\, dy\, d$

$\int_{0}^{1}\int_{0}^{1}\int_{0}^{y}\int_{0}^{y^2} f(x, y, \quad)\, dx\, d\ dy$

$\int_{0}^{1}\int_{0}^{\sqrt{x}}\int_{\sqrt{x}}^{1} f(x, y, \quad)\, dy\, d\ dx$

$\int_{0}^{1}\int_{0}^{1}\int_{\sqrt{x}}^{1} f(x, y, \quad)\, dy\, dx\, d$

33. $\int_{0}^{1}\int_{0}^{x}\int_{0}^{y} f(x, y, \quad)\, d\ dy\, dx \qquad \int_{0}^{1}\int_{1}^{1}\int_{y}^{1} f(x, y, \quad)\, dx\, dy\, d$

$\int_{0}^{1}\int_{0}^{y}\int_{y}^{1} f(x, y, \quad)\, dx\, d\ dy \qquad \int_{0}^{1}\int_{0}^{x}\int_{x} f(x, y, \quad)\, dy\, d\ dx$

$\int_{0}^{1}\int_{1}^{x} f(x, y, \quad)\, dy\, dx\, d$

35. $\frac{79}{30}, \left(\frac{358}{553}, \frac{33}{79}, \frac{571}{553}\right)$ **37.** a^5, $7a\ 12, 7a\ 12, 7a\ 12$

39. (a) $m \int_{3}^{3}\int_{\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5}\int_{y} \sqrt{x^2 \quad y^2}\, d\ dy\, dx$

(b) $\bar{x}, \bar{y}, \bar{\ }$, where

$\bar{x} \quad 1\ m \int_{3}^{3}\int_{\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5}\int_{y} x\ \sqrt{x^2 \quad y^2}\, d\ dy\, dx$

$\bar{y} \quad 1\ m \int_{3}^{3}\int_{\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5}\int_{y} y\ \sqrt{x^2 \quad y^2}\, d\ dy\, dx$

$\bar{\ } \quad 1\ m \int_{3}^{3}\int_{\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5}\int_{y} \sqrt{x^2 \quad y^2}\, d\ dy\, dx$

(c) $\int_{3}^{3}\int_{\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5}\int_{y} x^2 \quad y^2\ {}^{3\ 2}\, d\ dy\, dx$

41. (a) $\frac{3}{32} \quad \frac{11}{24}$ (b) $\bar{x}, \bar{y}, \bar{\ }$, where $\bar{x} \quad 28\ 9 \quad 44$,
$\bar{y} \quad 2\ 15 \quad 64\ 5\ 9 \quad 44$, $\bar{\ } \quad 45 \quad 208\ 15\ 9 \quad 44$
(c) $68 \quad 15 \quad 240$

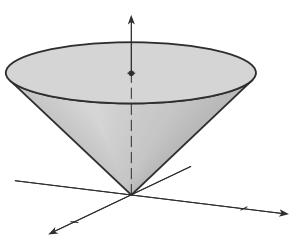
43. $I_x \quad I_y \quad I \quad \frac{2}{3}kL^5$ **45.** (a) $\frac{1}{8}$ (b) $\frac{1}{64}$ (c) $\frac{1}{5760}$

47. $L^3\ 8$

49. The region bounded by the ellipsoid $x^2 \quad 2y^2 \quad 3\ {}^{2} \quad 1$

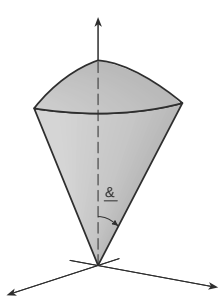
Exercises 15.8 page 1037

1.



$64 \quad 3$

3.



$9 \quad 4\ (2 \quad \sqrt{3})$

5. $\int_{0}^{2}\int_{0}^{3}\int_{0}^{2} f(r\cos \quad, r\sin \quad, \quad r\, d\ dr\, d$ **7.** 384

9. $e^6 \quad e \quad 5$ **11.** $2 \quad 5$

13. (a) 162 (b) $0, 0, 15$

15. $Ka^2\ 8,\ 0, 0, 2a\ 3$ **17.** $4 \quad 5$ **19.** $15 \quad 16$

21. $1562\ 15$ **23.** 10 **25.** $\left(0, \frac{525}{296}, 0\right)$

27. (a) $\left(0, 0, \frac{3}{8}a\right)$ (b) $4K\ a^5\ 15$

29. $2 \quad 3\ [1 \quad (1 \quad \sqrt{2})], (0, 0, 3\ [8(2 \quad \sqrt{2})])$ **31.** $5 \quad 6$

33. $8 \quad 35$ **35.** $243 \quad 5$ **37.** $136 \quad 99$

Exercises 15.9 page 1048

1. 14 **3.** 0 **5.** $2uvw$

7. The parallelogram with vertices (0, 0), (6, 3), (12, 1), (6, 2)

9. The region bounded by the line $y \quad 1$, the y-axis, and $y \quad \sqrt{x}$

11. 3 **13.** 6 **15.** $2 \ln 3$

17. (a) $\frac{4}{3}\ abc$ (b) $1.083 \quad 10^{12}\ \text{km}^3$

19. $\frac{8}{5} \ln 8$ **21.** $\frac{3}{2} \sin 1$ **23.** $e \quad e\ {}^{1}$

Chapter 15 Review page 1049

True-False Quiz

1. True **3.** True **5.** True **7.** False

Exercises

1. 64.0 **3.** $4e^2 \quad 4e \quad 3$ **5.** $\frac{1}{2} \sin 1$ **7.** $\frac{2}{3}$

9. $\int_{0}^{4}\int_{2} f(r\cos \quad, r\sin \quad r\, dr\, d$

11. The region inside the loop of the four-leaved rose $r \quad \sin 2$ in the Þrst quadrant

13. $\frac{1}{2} \sin 1$ **15.** $\frac{1}{2}e^6 \quad \frac{7}{2}$ **17.** $\frac{1}{4} \ln 2$

19. 8 **21.** $81 \quad 5$ **23.** 40.5 **25.** 96

27. $\frac{64}{15}$ **29.** 176 **31.** $\frac{2}{3}$ **33.** $2ma^3\ 9$

35. (a) $\frac{1}{4}$ (b) $\left(\frac{1}{3}, \frac{8}{15}\right)$

(c) $I_x \quad \frac{1}{12}, I_y \quad \frac{1}{24}; \bar{\bar{y}} \quad 1 \quad \sqrt{3}, \bar{\bar{x}} \quad 1 \quad \sqrt{6}$

37. (a) $0, 0, h\ 4$ (b) $a^4h\ 10$ **39.** $\ln(\sqrt{2} \quad \sqrt{3}) \quad \sqrt{2}\ 3$

41. 97.2 **43.** 0.0512 **45.** (a) $\frac{1}{15}$ (b) $\frac{1}{3}$ (c) $\frac{1}{45}$

47. $\int_{0}^{1}\int_{0}^{1}\int_{\bar{y}}^{\bar{y}} f(x, y, \quad dx\, dy\, d$ **49.** $\ln 2$ **51.** 0

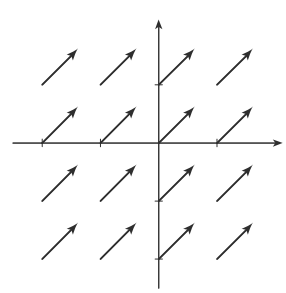
Problems Plus page 1052

1. 30 **3.** $\frac{1}{2} \sin 1$ **7.** (b) 0.90

Chapter 16

Exercises 16.1 page 1060

1.



33. (a) (b)

Exercises 16.2 page 1071

1. **3.** 1638.4 **5.** **7.**
9. 320 **11.** **13.** **15.**
17. (a) Positive (b) Negative **19.**
21.
23.

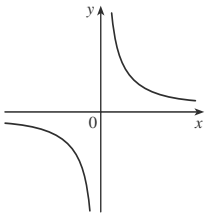
25. (a) (b)

27. **29.** 0.052 **31.**
33. (a) ,
,
, where
(b)
35. I_x , I_y $k\left(\quad 2 \quad \frac{2}{3} \right)$
37. 2 ^{2} **39.** 26 **41.** 1.67 10^4 ft-lb **43.** (b) Yes
45. 22 J

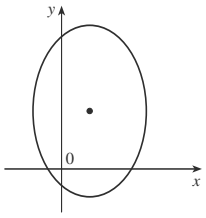
Exercises 16.3 page 1081

1. 40 **3.** $f\ x, y \quad 3x^2 \quad 5xy \quad 2y^2 \quad K$
5. Not conservative **7.** $f\ x, y \quad x^2 \cos y \quad y \sin x \quad K$
9. $f\ x, y \quad ye^x \quad x \sin y \quad K$ **11.** (b) 16
13. (a) $f\ x, y \quad \frac{1}{4}x^4y^4$ (b) 4
15. (a) $f\ x, y, \quad xy \quad ^{2}$ (b) 77
17. (a) $f\ x, y, \quad xy^2 \cos$ (b) 0
19. 2 **21.** 30 **23.** No **25.** No
29. (a) Yes (b) Yes (c) Yes
31. (a) Yes (b) Yes (c) No

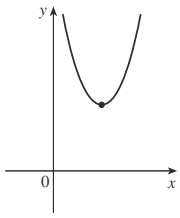
23. Hyperbola



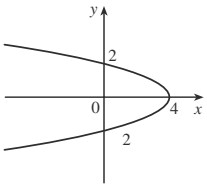
25. Ellipse



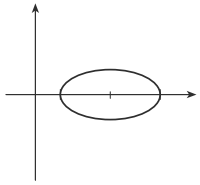
27. Parabola



29. Parabola



31. Ellipse



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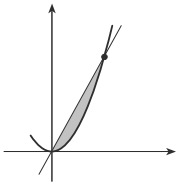
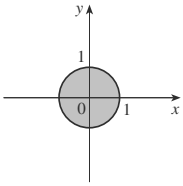
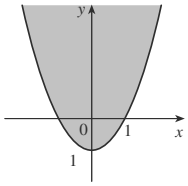


35. $y \quad x^2 \quad 2x$

37.



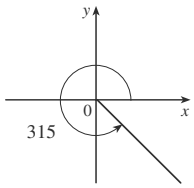
39.



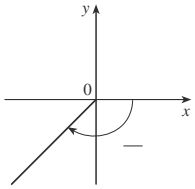
Exercises D page A32

1. 7 6 **3.** 20 **5.** 5 **7.** 720¡ **9.** 75¡
11. 67.5 **13.** 3 cm **15.** $\frac{2}{3}$ rad 120

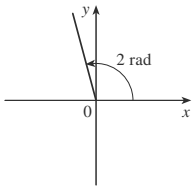
17.



19.



21.



23. sin 3 4 1 $\overline{2}$, cos 3 4 1 $\overline{2}$, tan 3 4 1, csc 3 4 $\overline{2}$, sec 3 4 $\overline{2}$, cot 3 4 1

25. sin 9 2 1, cos 9 2 0, csc 9 2 1, cot 9 2 0, tan 9 2 andsec 9 2 undeÞned

27. sin 5 6 $\frac{1}{2}$, cos 5 6 $\overline{3}$ 2, tan 5 6 1 $\overline{3}$, csc 5 6 2, sec 5 6 2 $\overline{3}$, cot 5 6 $\overline{3}$

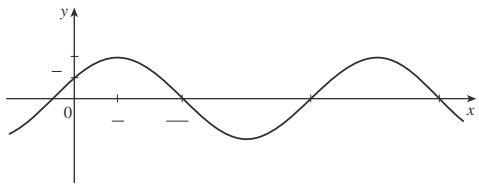
29. cos $\frac{4}{5}$, tan $\frac{3}{4}$, csc $\frac{5}{3}$, sec $\frac{5}{4}$, cot $\frac{4}{3}$

31. sin $\overline{5}$ 3, cos $\frac{2}{3}$, tan $\overline{5}$ 2, csc 3 $\overline{5}$, cot 2 $\overline{5}$

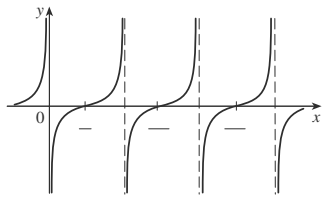
33. sin 1 $\overline{10}$, cos 3 $\overline{10}$, tan $\frac{1}{3}$, csc $\overline{10}$, sec $\overline{10}$ 3

35. 5.73576 cm **37.** 24.62147 cm **59.** (4 6 $\overline{2}$) 15
61. (3 8 $\overline{2}$) 15 **63.** $\frac{24}{25}$ **65.** 3, 5 3
67. 4, 3 4, 5 4, 7 4 **69.** 6, 2, 5 6, 3 2
71. 0, , 2 **73.** 0 x 6 and5 6 x 2
75. 0 x 4, 3 4 x 5 4, 7 4 x 2

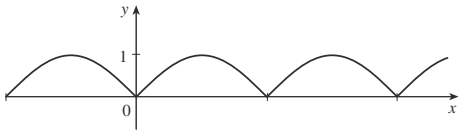
77.



79.



81.



89. 14.34457 cm^2

Exercises E page A38

1. $\overline{1}$ $\overline{2}$ $\overline{3}$ $\overline{4}$ $\overline{5}$ **3.** 3^4 3^5 3^6
5. 1 $\frac{1}{3}$ $\frac{3}{5}$ $\frac{5}{7}$ $\frac{7}{9}$ **7.** 1^{10} 2^{10} 3^{10} n^{10}
9. 1 1 1 1 1 $^{n\ 1}$ **11.** $\sum_{i\ 1}^{10} i$
13. $\sum_{i\ 1}^{19} \frac{i}{i\ 1}$ **15.** $\sum_{i\ 1}^{n} 2i$ **17.** $\sum_{i\ 0}^{5} 2^i$ **19.** $\sum_{i\ 1}^{n} x^i$
21. 80 **23.** 3276 **25.** 0 **27.** 61 **29.** $n\ n$ 1
31. $n\ n^2$ $6n$ 17 3 **33.** $n\ n^2$ $6n$ 11 3
35. $n\ n^3$ $2n^2$ n 10 4

41. (a) n^4 (b) 5^{100} 1 (c) $\frac{97}{300}$ (d) a_n a_0

43. $\frac{1}{3}$ **45.** 14 **49.** $2^{n\ 1}$ n^2 n 2

Exercises G page A56

1. 8 $4i$ **3.** 13 $18i$ **5.** 12 $7i$ **7.** $\frac{11}{13}$ $\frac{10}{13}i$

9. $\frac{1}{2}$ $\frac{1}{2}i$ **11.** i **13.** $5i$ **15.** 12 $5i$; 13

17. $4i$, 4 **19.** $\frac{3}{2}i$ **21.** 1 $2i$

23. $\frac{1}{2}$ $(\ \overline{7}\ 2)i$ **25.** 3 $\overline{2}$ cos 3 4 i sin 3 4

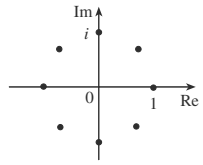
27. $5\{\cos[\tan^{\ 1}(\frac{4}{3})]$ $i\sin[\tan^{\ 1}(\frac{4}{3})]\}$

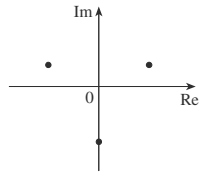
29. 4 cos 2 i sin 2 , cos 6 i sin 6 ,
$\frac{1}{2}$ cos 6 i sin 6

31. 4 $\overline{2}$ cos 7 12 i sin 7 12 ,
$(2\ \overline{2})$ cos 13 12 i sin 13 12 , $\frac{1}{4}$ cos 6 i sin 6

33. 1024 **35.** 512 $\overline{3}$ $512i$

37. 1, i, $(1\ \overline{2})$ 1 i **39.** $(\ \overline{3}\ 2)$ $\frac{1}{2}i$, i





41. i **43.** $\frac{1}{2}$ $(\ \overline{3}\ 2)i$ **45.** e^2

Algebra

Arithmetic Operations

$a\,b \quad c \quad ab \quad ac$

$\frac{a}{b} \quad \frac{c}{d} \quad \frac{ad \quad bc}{bd}$

$\frac{a \quad c}{b} \quad \frac{a}{b} \quad \frac{c}{b}$

$\frac{\frac{a}{b}}{\frac{c}{d}} \quad \frac{a}{b} \quad \frac{d}{c} \quad \frac{ad}{bc}$

Exponents and Radicals

$x^m x^n \quad x^{m \quad n}$

$\frac{x^m}{x^n} \quad x^{m \quad n}$

$x^{m\,n} \quad x^{mn}$

$x^{\quad n} \quad \frac{1}{x^n}$

$xy^{\,n} \quad x^n y^n$

$\frac{x}{y}^{\,n} \quad \frac{x^n}{y^n}$

$x^{1\,n} \quad \sqrt[n]{x}$

$x^{m\,n} \quad \sqrt[n]{x^m} \quad \left(\sqrt[n]{x}\right)^m$

$\sqrt[n]{xy} \quad \sqrt[n]{x}\,\sqrt[n]{y}$

$\sqrt[n]{\frac{x}{y}} \quad \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$

Factoring Special Polynomials

$x^2 \quad y^2 \quad x \quad y\ x \quad y$

$x^3 \quad y^3 \quad x \quad y\ x^2 \quad xy \quad y^2$

$x^3 \quad y^3 \quad x \quad y\ x^2 \quad xy \quad y^2$

Binomial Theorem

$x \quad y^{\,2} \quad x^2 \quad 2xy \quad y^2 \qquad x \quad y^{\,2} \quad x^2 \quad 2xy \quad y^2$

$x \quad y^{\,3} \quad x^3 \quad 3x^2y \quad 3xy^2 \quad y^3$

$x \quad y^{\,3} \quad x^3 \quad 3x^2y \quad 3xy^2 \quad y^3$

$x \quad y^{\,n} \quad x^n \quad nx^{n\ \ 1}y \quad \frac{n\,n \quad 1}{2}x^{n\ \ 2}y^2$

$\binom{n}{k} x^{n\ \ k}y^k \qquad nxy^{n\ \ 1} \quad y^n$

where $\binom{n}{k} \quad \frac{n\,n \quad 1 \quad n \quad k \quad 1}{1 \quad 2 \quad 3 \quad k}$

Quadratic Formula

If $ax^2 \quad bx \quad c \quad 0$, then $x \quad \frac{b \quad \sqrt{b^2 \quad 4ac}}{2a}$.

Inequalities and Absolute Value

If $a \quad b$ and $b \quad c$, then $a \quad c$.

If $a \quad b$, then $a \quad c \quad b \quad c$.

If $a \quad b$ and $c \quad 0$, then $ca \quad cb$.

If $a \quad b$ and $c \quad 0$, then $ca \quad cb$.

If $a \quad 0$, then

$x \quad a$ means $x \quad a$ or $x \quad a$

$x \quad a$ means $a \quad x \quad a$

$x \quad a$ means $x \quad a$ or $x \quad a$

Geometry

Geometric Formulas

Formulas for area A, circumference C, and volume V:

Triangle

$A \quad \frac{1}{2}bh$

$\frac{1}{2}ab\sin$

Circle

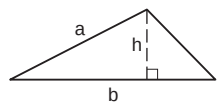
$A \quad r^2$

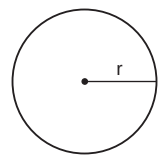
$C \quad 2\ r$

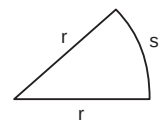
Sector of Circle

$A \quad \frac{1}{2}r^2$

$s \quad r \quad$ in radians







Sphere

$V \quad \frac{4}{3}\ r^3$

$A \quad 4\ r^2$

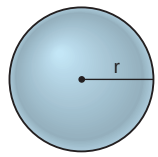
Cylinder

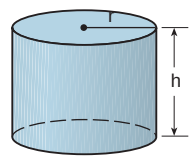
$V \quad r^2h$

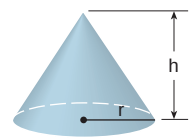
Cone

$V \quad \frac{1}{3}\ r^2h$

$A \quad r\ \sqrt{r^2 \quad h^2}$







Distance and Midpoint Formulas

Distance between $P_1\ x_1, y_1$ and $P_2\ x_2, y_2$:

$$d \quad \sqrt{x_2 \quad x_1{}^2 \quad y_2 \quad y_1{}^2}$$

Midpoint of $\overline{P_1P_2}$: $\frac{x_1 \quad x_2}{2}, \frac{y_1 \quad y_2}{2}$

Lines

Slope of line through $P_1\ x_1, y_1$ and $P_2\ x_2, y_2$:

$$m \quad \frac{y_2 \quad y_1}{x_2 \quad x_1}$$

Point-slope equation of line through $P_1\ x_1, y_1$ with slope m:

$$y \quad y_1 \quad m\ x \quad x_1$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y \quad mx \quad b$$

Circles

Equation of the circle with center h, k and radius r:

$$x \quad h^2 \quad y \quad k^2 \quad r^2$$

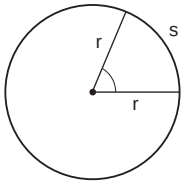
Trigonometry

radians 180

1 $\frac{}{180}$ rad 1 rad $\frac{180}{}$

s r

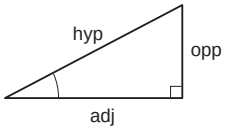
in radians



sin $\frac{opp}{hyp}$ csc $\frac{hyp}{opp}$

cos $\frac{adj}{hyp}$ sec $\frac{hyp}{adj}$

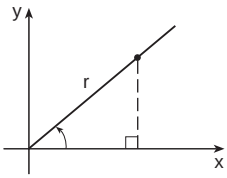
tan $\frac{opp}{adj}$ cot $\frac{adj}{opp}$

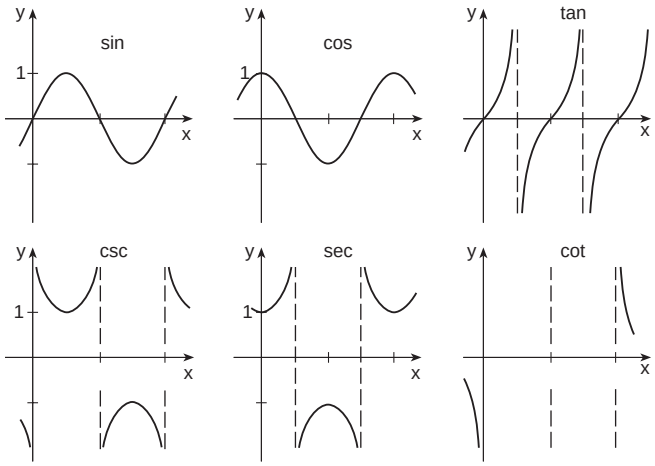


sin $\frac{y}{r}$ csc $\frac{r}{y}$

cos $\frac{x}{r}$ sec $\frac{r}{x}$

tan $\frac{y}{x}$ cot $\frac{x}{y}$





	radians	sin	cos	tan
0	0	0	1	0
30	6	1 2	$\overline{3}$ 2	$\overline{3}$ 3
45	4	$\overline{2}$ 2	$\overline{2}$ 2	1
60	3	$\overline{3}$ 2	1 2	$\overline{3}$
90	2	1	0	Ñ

Fundamental Identities

csc $\frac{1}{\sin}$ sec $\frac{1}{\cos}$

tan $\frac{\sin}{\cos}$ cot $\frac{\cos}{\sin}$

cot $\frac{1}{\tan}$ $\sin^2$ $\cos^2$ 1

1 $\tan^2$ $\sec^2$ 1 $\cot^2$ $\csc^2$

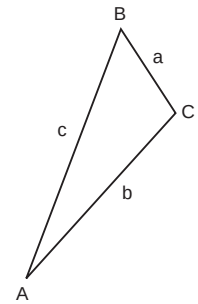
sin sin cos cos

tan tan sin $\frac{}{2}$ cos

cos $\frac{}{2}$ sin tan $\frac{}{2}$ cot

The Law of Sines

$\frac{\sin A}{a}$ $\frac{\sin B}{b}$ $\frac{\sin C}{c}$



The Law of Cosines

a^2 b^2 c^2 2bc cosA

b^2 a^2 c^2 2ac cosB

c^2 a^2 b^2 2ab cosC

Addition and Subtraction Formulas

sin x y sinx cosy cosx siny

sin x y sinx cosy cosx siny

cos x y cosx cosy sinx siny

cos x y cosx cosy sinx siny

tan x y $\frac{\tan x \quad \tan y}{1 \quad \tan x \tan y}$

tan x y $\frac{\tan x \quad \tan y}{1 \quad \tan x \tan y}$

Double-Angle Formulas

sin 2x 2 sinx cosx

cos 2x $\cos^2 x$ $\sin^2 x$ 2 $\cos^2 x$ 1 1 2 $\sin^2 x$

tan 2x $\frac{2 \tan x}{1 \quad \tan^2 x}$

Half-Angle Formulas

$\sin^2 x$ $\frac{1 \quad \cos 2x}{2}$ $\cos^2 x$ $\frac{1 \quad \cos 2x}{2}$

Differentiation Rules

1. $\frac{d}{dx} c \quad 0$

2. $\frac{d}{dx} cf\ x \quad cf\ x$

3. $\frac{d}{dx} f\ x \quad x \quad f\ x \quad x$

4. $\frac{d}{dx} f\ x \quad x \quad f\ x \quad x$

5. $\frac{d}{dx} f\ x\ x \quad f\ x\ x \quad x\ f\ x$ (Product Rule)

6. $\frac{d}{dx} \frac{f\ x}{x} \quad \frac{x\ f\ x \quad f\ x\ x}{x^{\ 2}}$ (Quotient Rule)

7. $\frac{d}{dx} f \quad x \quad f \quad x \quad x$ (Chain Rule)

8. $\frac{d}{dx} x^n \quad nx^{n\ 1}$ (Power Rule)

Exponential and Logarithmic Functions

9. $\frac{d}{dx} e^x \quad e^x$

10. $\frac{d}{dx} a^x \quad a^x \ln a$

11. $\frac{d}{dx} \ln\ x \quad \frac{1}{x}$

12. $\frac{d}{dx} \log_a x \quad \frac{1}{x \ln a}$

Trigonometric Functions

13. $\frac{d}{dx} \sin x \quad \cos x$

14. $\frac{d}{dx} \cos x \quad \sin x$

15. $\frac{d}{dx} \tan x \quad \sec^2 x$

16. $\frac{d}{dx} \csc x \quad \csc x \cot x$

17. $\frac{d}{dx} \sec x \quad \sec x \tan x$

18. $\frac{d}{dx} \cot x \quad \csc^2 x$

Inverse Trigonometric Functions

19. $\frac{d}{dx} \sin^{1} x \quad \frac{1}{\sqrt{1 \quad x^2}}$

20. $\frac{d}{dx} \cos^{1} x \quad \frac{1}{\sqrt{1 \quad x^2}}$

21. $\frac{d}{dx} \tan^{1} x \quad \frac{1}{1 \quad x^2}$

22. $\frac{d}{dx} \csc^{1} x \quad \frac{1}{x \sqrt{x^2 \quad 1}}$

23. $\frac{d}{dx} \sec^{1} x \quad \frac{1}{x \sqrt{x^2 \quad 1}}$

24. $\frac{d}{dx} \cot^{1} x \quad \frac{1}{1 \quad x^2}$

Hyperbolic Functions

25. $\frac{d}{dx} \sinh x \quad \cosh x$

26. $\frac{d}{dx} \cosh x \quad \sinh x$

27. $\frac{d}{dx} \tanh x \quad \operatorname{sech}^2 x$

28. $\frac{d}{dx} \operatorname{csch} x \quad \operatorname{csch} x \coth x$

29. $\frac{d}{dx} \operatorname{sech} x \quad \operatorname{sech} x \tanh x$

30. $\frac{d}{dx} \coth x \quad \operatorname{csch}^2 x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx} \sinh^{1} x \quad \frac{1}{\sqrt{1 \quad x^2}}$

32. $\frac{d}{dx} \cosh^{1} x \quad \frac{1}{\sqrt{x^2 \quad 1}}$

33. $\frac{d}{dx} \tanh^{1} x \quad \frac{1}{1 \quad x^2}$

34. $\frac{d}{dx} \operatorname{csch}^{1} x \quad \frac{1}{x \sqrt{x^2 \quad 1}}$

35. $\frac{d}{dx} \operatorname{sech}^{1} x \quad \frac{1}{x \sqrt{1 \quad x^2}}$

36. $\frac{d}{dx} \coth^{1} x \quad \frac{1}{1 \quad x^2}$

Table of Integrals

1. $u\,dv = uv - \int v\,du$

2. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \frac{du}{u} = \ln|u| + C$

4. $\int e^u\,du = e^u + C$

5. $\int a^u\,du = \frac{a^u}{\ln a} + C$

6. $\int \sin u\,du = -\cos u + C$

7. $\int \cos u\,du = \sin u + C$

8. $\int \sec^2 u\,du = \tan u + C$

9. $\int \csc^2 u\,du = -\cot u + C$

10. $\int \sec u \tan u\,du = \sec u + C$

11. $\int \csc u \cot u\,du = -\csc u + C$

12. $\int \tan u\,du = \ln|\sec u| + C$

13. $\int \cot u\,du = \ln|\sin u| + C$

14. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

15. $\int \csc u\,du = \ln|\csc u - \cot u| + C$

16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

17. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

19. $\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

20. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

Forms Involving $\sqrt{a^2 - u^2}$, $a > 0$

21. $\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2 - u^2}\right) + C$

22. $\int u^2\sqrt{a^2 - u^2}\,du = \frac{u}{8}(a^2 - 2u^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\ln\left(u + \sqrt{a^2 - u^2}\right) + C$

23. $\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

24. $\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{\sqrt{a^2 - u^2}}{u} - \ln\left(u + \sqrt{a^2 - u^2}\right) + C$

25. $\int \frac{du}{\sqrt{a^2 - u^2}} = \ln\left(u + \sqrt{a^2 - u^2}\right) + C$

26. $\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = \frac{u}{2}\sqrt{a^2 - u^2} - \frac{a^2}{2}\ln\left(u + \sqrt{a^2 - u^2}\right) + C$

27. $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2 - u^2} + a}{u}\right| + C$

28. $\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u} + C$

29. $\int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Forms Involving $\sqrt{a^2 - u^2}$, $a > 0$

30. $\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

31. $\int u^2\sqrt{a^2 - u^2}\,du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

32. $\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

33. $\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

34. $\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

35. $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

36. $\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

37. $\int (a^2 - u^2)^{3/2}\,du = \frac{u}{8}(2u^2 - 5a^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

38. $\int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Forms Involving $\sqrt{u^2 - a^2}$, $a > 0$

39. $\int \sqrt{u^2 - a^2}\,du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

40. $\int u^2\sqrt{u^2 - a^2}\,du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

41. $\int \frac{\sqrt{u^2 - a^2}}{u}\,du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{u} + C$

42. $\int \frac{\sqrt{u^2 - a^2}}{u^2}\,du = -\frac{\sqrt{u^2 - a^2}}{u} - \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

43. $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

44. $\int \frac{u^2\,du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

45. $\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

46. $\int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

Forms Involving $a + bu$

47. $$\frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a+bu - a\ln|a+bu|\right) + C$$

48. $$\frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$$

49. $$\frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

50. $$\frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

51. $$\frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$$

52. $$\frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

53. $$\frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left[a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right] + C$$

54. $$u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a+bu)^{3/2} + C$$

55. $$\frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a+bu} + C$$

56. $$\frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(8a^2 + 3b^2u^2 - 4abu)\sqrt{a+bu} + C$$

57. $$\frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$$

$$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$$

58. $$\frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int\frac{du}{u\sqrt{a+bu}}$$

59. $$\frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int\frac{du}{u\sqrt{a+bu}}$$

60. $$u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

61. $$\frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int\frac{u^{n-1}\,du}{\sqrt{a+bu}}$$

62. $$\frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int\frac{du}{u^{n-1}\sqrt{a+bu}}$$

Table of Integrals

63. $\sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

64. $\cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

65. $\tan^2 u\,du = \tan u - u + C$

66. $\cot^2 u\,du = -\cot u - u + C$

67. $\sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$

68. $\cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$

69. $\tan^3 u\,du = \frac{1}{2}\tan^2 u - \ln|\cos u| + C$

70. $\cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

71. $\sec^3 u\,du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

72. $\csc^3 u\,du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

73. $\sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

74. $\cos^n u\,du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

75. $\tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

76. $\cot^n u\,du = -\frac{1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

77. $\sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

78. $\csc^n u\,du = -\frac{1}{n-1}\cot u\csc^{n-2} u - \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

79. $\sin au\sin bu\,du = -\frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

80. $\cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

81. $\sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

82. $\int u\sin u\,du = \sin u - u\cos u + C$

83. $\int u\cos u\,du = \cos u + u\sin u + C$

84. $\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

85. $\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

86. $\int \sin^n u\cos^m u\,du = -\frac{\sin^{n-1} u\cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u\cos^m u\,du$

$= \frac{\sin^{n+1} u\cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u\cos^{m-2} u\,du$

Inverse Trigonometric Forms

87. $\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

88. $\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

89. $\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

90. $\int u\sin^{-1} u\,du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

91. $\int u\cos^{-1} u\,du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

92. $\int u\tan^{-1} u\,du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

93. $\int u^n\sin^{-1} u\,du = \frac{1}{n+1}u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}, \quad n \neq -1$

94. $\int u^n\cos^{-1} u\,du = \frac{1}{n+1}u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}, \quad n \neq -1$

95. $\int u^n\tan^{-1} u\,du = \frac{1}{n+1}u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}, \quad n \neq -1$

Table of Integrals

Exponential and Logarithmic Forms

96. $ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$

97. $u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

98. $e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

99. $e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

100. $\ln u\,du = u\ln u - u + C$

101. $u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}(n+1)\ln u - 1) + C$

102. $\frac{1}{u\ln u}\,du = \ln(\ln u) + C$

Hyperbolic Forms

103. $\sinh u\,du = \cosh u + C$

104. $\cosh u\,du = \sinh u + C$

105. $\tanh u\,du = \ln\cosh u + C$

106. $\coth u\,du = \ln|\sinh u| + C$

107. $\operatorname{sech} u\,du = \tan^{-1}(\sinh u) + C$

108. $\operatorname{csch} u\,du = \ln\left|\tanh\tfrac{1}{2}u\right| + C$

109. $\operatorname{sech}^2 u\,du = \tanh u + C$

110. $\operatorname{csch}^2 u\,du = -\coth u + C$

111. $\operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$

112. $\operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

Forms Involving $\sqrt{2au - u^2}$, $a > 0$

113. $\sqrt{2au - u^2}\,du = \frac{(u-a)}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\frac{(a-u)}{a} + C$

114. $u\sqrt{2au - u^2}\,du = \frac{(2u^2 + au - 3a^2)}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\frac{(a-u)}{a} + C$

115. $\frac{\sqrt{2au - u^2}}{u}\,du = \sqrt{2au - u^2} + a\cos^{-1}\frac{(a-u)}{a} + C$

116. $\frac{\sqrt{2au - u^2}}{u^2}\,du = -\frac{2\sqrt{2au - u^2}}{u} - \cos^{-1}\frac{(a-u)}{a} + C$

117. $\frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\frac{(a-u)}{a} + C$

118. $\frac{u\,du}{\sqrt{2au - u^2}} = \sqrt{2au - u^2} + a\cos^{-1}\frac{(a-u)}{a} + C$

119. $\frac{u^2\,du}{\sqrt{2au - u^2}} = \frac{(u+3a)}{2}\sqrt{2au - u^2} - \frac{3a^2}{2}\cos^{-1}\frac{(a-u)}{a} + C$

120. $\frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$