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Efimov physics beyond universality

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We provide an exact solution of the Efimov spectrum in ultracold gases within the standard two-channel model for Feshbach resonances. It is shown that the finite range in the Feshbach coupling makes the introduction of an adjustable three-body parameter obsolete. The solution explains the empirical relation between the scattering length a_- where the first Efimov state appears at the atom threshold and the van der Waals length $l_{\rm vdw}$ for open channel dominated resonances. There is a continuous crossover to the closed channel dominated limit, where the scale in the energy level diagram as a function of the inverse scattering length $1/a$ is set by the intrinsic length $r^\star$ associated with the Feshbach coupling. Our results provide a number of predictions for non-universal ratios between energies and scattering lengths that can be tested in future experiments.

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Most of the basic features that distinguish quantum from classical physics show up already at the single particle level. Genuine two-particle effects like the Hong-Ou-Mandel two-photon interference are typically a consequence of particle statistics, not of interactions [1]. Surprisingly, novel quantum effects in which statistics and interactions are combined appear at the level of three particles. As shown by Efimov in 1970 [2], three particles which interact via a resonant short-range attractive interaction exhibit an infinite sequence of three-body bound states or trimers. Remarkably, the trimers exist even in a regime where the two-body interaction does not have a bound state. Efimov trimers thus behave like Borromean rings: three of them are bound together but cutting one of the bonds makes the whole system fly apart. While theoretically predicted in a nuclear matter context, Efimov states have finally been observed with ultracold atoms [3]. The assumption of short range interactions is perfectly valid in this case and, moreover, the associated scattering lengths can be tuned by an external magnetic field, exploiting a Feshbach resonance [4]. One key signature of Efimov physics is the resonant enhancement of the three-body recombination rate when the nth Efimov state meets the atom threshold at a scattering length $a_-^{(n)}$. In most experiments, only the lowest Efimov state at $a_-=a_-^{(0)}$ can be observed because of large atom losses as the scattering length increases. An important feature of the Efimov trimers is that the binding energies exhibit universal scaling behavior. In the limit where the two-body interaction is just at the threshold to form a bound state, the ratio $E^{(n)}/E^{(n+1)}$ of consecutive binding energies approaches the universal value $e^{2\pi/s_0}\simeq 515.028$ for $n\gg 1$ with the Efimov number $s_0\approx 1.00624$. Similarly, the ratio of consecutive values of the scattering length $a_-^{(n)}$ approaches $a_-^{(n+1)}/a_-^{(n)}\to e^{\pi/s_0}\simeq 22.6942$. The origin of this universality can be understood from an effective field theory approach to the three-body problem [5, 6]. There remains, however, a non-universal aspect in the theory: Although the relative position of the trimer states is universal, their exact position in the (a,E) plane is not fixed and is determined by the so-called three-body parameter (3BP). It is presumed that the 3BP is highly sensitive to microscopic details of the underlying two-body potential as well as genuine three-body forces [7].

As more experimental data have been accumulated in recent years [8–15], a puzzling observation came to light: In most experiments, no matter which alkali atoms were used, the measured values for a_- clustered around $a_-\approx -9.45\,l_{\rm vdW}$. Where does this apparent 'universality of the three-body parameter' come from? A possible answer to this question is based on the observation that, typically, Efimov trimers that are accessible with ultracold atoms appear in a situation where the scattering length is tuned via an open channel dominated Feshbach resonance. The observed universality may thus be a simple consequence of the fact that in all cases the interactions are, up to a scale factor, almost identical. Indeed, within a single-channel description, potentials with a van der Waals tail have $l_{\rm vdw}$ as the only relevant length scale at energies much smaller than the depth of the potential well [16]. It is then plausible that $l_{\rm vdw}$ provides the characteristic scale for the 3BP. This has been confirmed in recent, independent work on this problem by Chin [17] and by Wang *et al.* [18], using single-channel potentials with a van der Waals tail.

It is an open question, however, to which extent a single-channel description captures the physics near a Feshbach resonance. In the following, we will show that a simple extension of the standard two-channel model for Feshbach resonances [4], which takes into account the finite range of the Feshbach coupling, provides a complete description of the Efimov spectrum in terms of only two experimentally accessible parameters: the van der Waals length $l_{\rm vdW}$ and the intrinsic length $r^\star$. Depending on the dimensionless resonance strength $s_{\rm res}=0.956\,l_{\rm vdw}/r^\star$ [4], there is a continuous change in the relation between the trimer energy spectrum and the scattering length, with the lowest Efimov state appearing at $a_-\approx -8.3\,l_{\rm vdW}$ as $s_{\rm res}\gg 1$ while $a_-\approx -10.3\,r^\star$ in the opposite limit

$s_{\rm res} \ll 1$. In addition, we show that for the experimentally accessible lowest Efimov states there are strong deviations from universality which have apparently been observed in recent experiments.

Specifically, we consider non-relativistic bosons described by the microscopic action (in units where $2m = \hbar = 1$)

$$S = \int_{\mathbf{r},t} \Big\{ \psi^*(\mathbf{r},t)[i\partial_t - \nabla^2]\psi(\mathbf{r},t) + \phi^*(\mathbf{r},t) P_\phi^{\rm cl} \phi(\mathbf{r},t) \Big\} + \frac{g}{2}\int_{\mathbf{r}_1,\mathbf{r}_2,t} \chi(\mathbf{r}_2 - \mathbf{r}_1) \times \Big[\phi(\frac{\mathbf{r}_1+\mathbf{r}_2}{2},t)\psi^*(\mathbf{r}_1,t)\psi^*(\mathbf{r}_2,t) + c.c.\Big], \quad (1)$$

where ψ denotes the atoms and ϕ the molecule in the closed channel. Here $P_\phi^{\rm cl} = i\partial_t - \nabla^2/2 + \nu$ with $\nu(B) = \mu(B - B_{\rm res})$ the bare detuning from the resonance and μ is the difference in the magnetic moment between the molecule and the open-channel atoms. For a description of universal features of Efimov trimers like the asymptotic ratio $a_-^{(n+1)}/a_-^{(n)} \to e^{\pi/s_0}$, the atom-molecule conversion amplitude $\sim g$ may be taken as pointlike in coordinate space [5, 6]. In reality, however, the coupling has a finite range σ which is determined by the scale of the wave function overlap between the open and closed-channel states. As has been pointed out by a number of authors [19–24], this can be accounted for by a form factor $\chi(r)$ in Eq. (1). A convenient choice is $\chi(r) \sim e^{-r/\sigma}/r$ which leads to $\chi(p) = 1/(1+\sigma^2 p^2)$ in momentum space. It is important to note that the action (1) can also be used to describe the situation where the interaction is dominated by a large background scattering length $a_{\rm bg}$. Indeed, integrating out the closed-channel field ϕ, one obtains a contribution $\sim (\psi^*\psi)^2$ that properly describes background scattering of range σ and scattering length $\sim g^2/\tilde{\nu}$, provided the Feshbach coupling $g^2 = 32\pi/r^\star \gg 1/l_{\rm vdw}$ is strong enough that the momentum dependence of $P_\phi^{\rm cl}$ can be neglected. As has been discussed in [20, 22–24], this does not work, however, in the limit of closed channel dominated resonances.

In our model the scattering of two atoms is mediated by the exchange of the closed-channel or dimer field ϕ. The two-body problem is thus solved by computing the renormalization of the inverse propagator of the dimer $\mathcal{G}_\phi^{-1}$. Evaluation of the standard ladder diagram yields

$$\mathcal{G}_\phi^{-1}(E,\mathbf{q}) = P_\phi^{\rm cl}(E,\mathbf{q}) - \frac{g^2/(32\pi)}{\sigma\left[1+\sigma\sqrt{-\frac{E}{2}+\frac{\mathbf{q}^2}{4}-i\epsilon}\right]^2} \quad (2)$$

with $P_\phi^{\rm cl}(E,\mathbf{q}) = -E + \mathbf{q}^2/2 - \nu(B) - i\epsilon$. The two-atom scattering amplitude now follows from $f(k) = g^2\chi(k)^2\mathcal{G}_\phi(2k^2,\mathbf{0})/(16\pi)$. Its standard low-energy expansion then determines the scattering length a and the effective range r_e via

$$\frac{1}{a} = \frac{1}{2\sigma} - \frac{16\pi}{g^2}\nu(B), \quad r_e = -2r^* + 3\sigma\left(1 - \frac{4\sigma}{3a}\right). \quad (3)$$

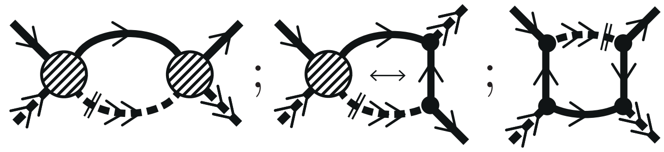

FIG. 1: Feynman diagrams contributing to the renormalization of the atom-dimer vertex $\lambda_3^{(k)}$ (large circle). The small black circle represents the atom-dimer coupling $\sim g$ and the solid (dashed) line denotes the atom (dimer) propagator.

This allows to express the bare parameters g, σ, and $\mu B_{\rm res}$ which appear in (1) in terms of fixed, experimental parameters. Close to a Feshbach resonance at magnetic field B_0, the scattering length can be written as $a(B) = -1/r^*\tilde{\nu}(B)$ where $\tilde{\nu}(B) = \mu(B - B_0)$ is the renormalized detuning in units of a wavenumber squared, while $r^\star > 0$ is the intrinsic length scale which characterizes the strength of the Feshbach coupling [4]. This fixes $g^2 = 32\pi/r^*$. Moreover, the resonance shift is given by $\mu(B_0 - B_{\rm res}) = 1/(r^\star\sigma)$, which is always positive in our model. Comparing our result for the resonance shift with the corresponding expression obtained for Feshbach resonances with interaction potentials that have a van der Waals tail [25] yields the identification $\sigma = \bar{a}$, with the so-called mean scattering length $\bar{a} = 4\pi/\Gamma(1/4)^2 l_{\rm vdw} \approx 0.956\, l_{\rm vdw}$ [26].

Based on the knowledge of the full two-body scattering amplitude, the three-body problem can be solved exactly, keeping only s-wave interactions. In particular, the three-boson scattering can be expressed in terms of an atom-dimer interaction $\sim \phi^*\psi^*\phi\psi$. The corresponding one particle irreducible atom-dimer vertex $\lambda_3(Q_1,Q_2,Q_3)$ $[Q_i = (E_i,\mathbf{q}_i)]$ develops a complicated energy and momentum dependence which determines the full Efimov spectrum for arbitrary values of the scattering length. The derivation becomes particularly simple using the functional renormalization group (fRG) [27]. The central quantity of the fRG is an RG scale k dependent effective action Γ_k which interpolates between the microscopic action $S = \Gamma_{k=\Lambda}$ and the full quantum effective action $\Gamma = \Gamma_{k=0}$ by successively including quantum fluctuations on momentum scales $q \gtrsim k$. Here, we adopt an RG strategy adjusted to the few-body problem as discussed in [28, 29], where the flowing action Γ_k is of the form of S in (1) but with $P_\phi^{\rm cl}$ replaced by $1/\mathcal{G}_\phi$ from (2) and an additional term

$$\Gamma_k^{\rm 3B} = -\int_{Q_1,Q_2,Q_3} \lambda_3^{(k)}(Q_1,Q_2,Q_3) \times \phi^*(Q_1)\psi^*(Q_2)\phi(Q_3)\psi(Q_1+Q_2-Q_3). \quad (4)$$

Since we do not consider a microscopic three-body force here, we have $\lambda_3^{(\Lambda)} = 0$ at the UV scale Λ. The atom-dimer vertex $\lambda_3^{(k)}$ is then the only running coupling in Γ_k. It is important to note that the truncation of Γ_k

is complete for the solution of the three-body problem as no additional couplings can be generated in the RG flow [28, 29]. In our scheme the propagator of the bosons ψ is not regularized and the dimer ϕ is supplemented with a sharp momentum regulator. In Fig. 1 we show the Feynman diagrams contributing to the flow of $\lambda_3^{(k)}$. The number of independent energies and momenta is reduced by working in the center-of-mass frame and by noting that the loop frequency integration puts one internal atom on mass-shell [6]. After performing the s-wave projection $\lambda_3^{(k)}(q_1, q_2; E) = 1/(2g)\int d\cos\theta \lambda_3^{(k)}(\mathbf{q}_1, \mathbf{q}_2; E)$, $\theta = \angle(\mathbf{q}_1, \mathbf{q}_2)$, one finds the RG equation

$$\partial_k \lambda_3^{(k)}(q_1, q_2; E) = -\frac{g^2 k^2 \mathcal{G}_\phi(E - k^2, k)}{2\pi^2} \times \Big[\lambda_3^{(k)}(q_1, k; E)\lambda_3^{(k)}(k, q_2; E) + \lambda_3^{(k)}(q_1, k; E) G_E(k, q_2) + G_E(q_1, k)\lambda_3^{(k)}(k, q_2; E) + G_E(q_1, k) G_E(k, q_2)\Big], \quad (5)$$

where

$$G_E(p, q) \equiv \frac{1}{2}\int_{-1}^{1} d\cos\theta \frac{\chi(|\mathbf{p} + \frac{\mathbf{q}}{2}|)\chi(|\mathbf{q} + \frac{\mathbf{p}}{2}|)}{-E + \mathbf{p}^2 + \mathbf{q}^2 + (\mathbf{p} + \mathbf{q})^2 - i\epsilon}. \quad (6)$$

Making use of the binomial form of Eq. (5) the flow can be integrated analytically and yields

$$f_E(q_1, q_2) = g_E(q_1, q_2) - \int_0^\Lambda dl\, g_E(q_1, l)\, \zeta_E(l)\, f_E(l, q_2), \quad (7)$$

which is a modified form of the well-known STM equation [30] with $f_E(q_1, q_2) = g_E(q_1, q_2) + \tilde{\lambda}_E(q_1, q_2)$, $g_E(q_1, q_2) = 16 q_1 q_2 G_E(q_1, q_2)$, $\tilde{\lambda}_E(q_1, q_2) = 16 q_1 q_2\, \lambda_3(q_1, q_2; E)$, and $\zeta_E(l) = -g^2 \xi \mathcal{G}_\phi(E - l^2, l)/(32\pi)$. Due to the presence of the form factor χ in g_E and the finite range corrections in Eq. (2) the UV limit $\Lambda \to \infty$ can safely be taken. This explicitly demonstrates the independence of the cutoff and leads to the disappearance of the 3BP.

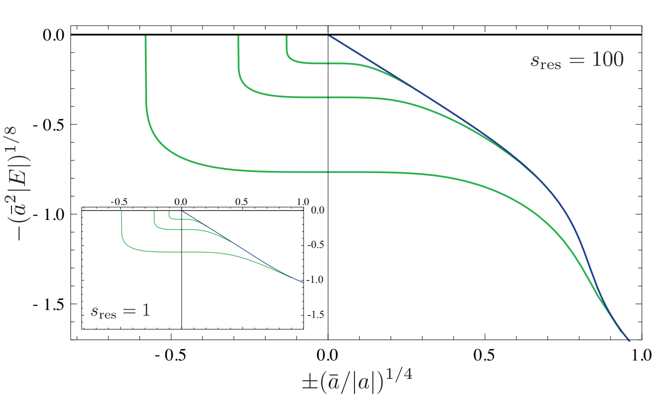

FIG. 2: (color online). The Efimov spectrum in dimensionless units for a broad Feshbach resonance of strength $s_{\rm res} = 100$. The inset shows the spectrum for a resonance of intermediate strength $s_{\rm res} = 1$. The dimer binding energy is shown in blue.

The knowledge of the full vertex λ_3 gives all information about the scattering of three bosons, such as bound states, recombination rates, and lifetimes, by evaluating the corresponding tree-level diagrams [6]. In the following we compute the trimer bound state spectrum by identifying the poles of λ_3 as a function of the energy E. In the vicinity of a bound state pole the atom-dimer vertex can be parametrized as $\lambda_3(q_1, q_2; E) \approx \mathcal{B}(q_1, q_2)/[E + E_T^{(n)} + i\Gamma_T^{(n)}]$. When inserted into Eq. (7) an integral equation for $\mathcal{B}$ is obtained which is solved by discretization and amounts to evaluating the determinant $\det[\mathcal{C} - \mathbb{I}] = 0$ with $\mathcal{C}(q_1, q_2) = g^2 g_E(q_1, q_2)\mathcal{G}_\phi(E - q_2^2, q_2)/(32\pi)$. $\mathcal{C}$ has a log-periodic structure where low-momentum modes are suppressed by any finite $1/a \neq 0$ and energy $E < 0$ below the atom-dimer threshold. High-momentum modes are suppressed due to the finite range potential of our model.

In Fig. 2 we show the Efimov spectrum including the atom-dimer threshold for a broad and a Feshbach resonance of intermediate strength in dimensionless units. The position of the trimer states in the $(1/a, E)$ plane is completely fixed by our calculation. The overall appearance of the spectrum remains similar as the strength of the resonance is varied. For narrow resonances it gets pushed towards the unitarity point $E = 1/a = 0$ while for open channel dominated resonances it reaches a maximal extent in the $(1/a, E)$ plane. The detailed position of the lowest energy levels depends on both the value of the van der Waals length and the resonance strength $s_{\rm res}$. Universality is only reached in the experimentally hardly accessible limit $n \gg 1$ where the ratios of $a_-^{(n)}$, $a_*^{(n)}$ (the scattering length for which the trimer meets the atom-dimer threshold), and $E^{(n)}$ for consecutive lev-

$s_{\rm res}$	n	0	1	2	UT
100	$E^{(n)}/E^{(n+1)}$	530.871	515.206	515.035	515.028
	$a_-^{(n+1)}/a_-^{(n)}$	17.083	21.827	22.654	22.694
	$a_*^{(n+1)}/a_*^{(n)}$	3.980	40.033	23.345	22.694
	$\kappa_*^{(n)} a_-^{(n)}$	2.121	1.573	1.512	1.5076
1	$E^{(n)}/E^{(n+1)}$	515.830	515.039	515.035	515.028
	$a_-^{(n+1)}/a_-^{(n)}$	22.869	22.650	22.690	22.694
	$a_*^{(n+1)}/a_*^{(n)}$	17.183	22.303	22.716	22.694
	$\kappa_*^{(n)} a_-^{(n)}$	1.500	1.511	1.508	1.5076
0.1	$E^{(n)}/E^{(n+1)}$	521.273	515.059	515.010	515.028
	$a_-^{(n+1)}/a_-^{(n)}$	26.230	22.964	22.71	22.694
	$a_*^{(n+1)}/a_*^{(n)}$	26.965	21.286	22.48	22.694
	$\kappa_*^{(n)} a_-^{(n)}$	1.296	1.489	1.506	1.5076

TABLE I: The ratio between consecutive energies $E^{(n)} = \hbar^2(\kappa_*^{(n)})^2/m$ and threshold scattering lengths $(a_{-,*}^{(n)})$ as well as the product $a_-^{(n)}\kappa_*^{(n)}$ for a broad ($s_{\rm res} = 100$), intermediate ($s_{\rm res} = 1$) and narrow ($s_{\rm res} = 0.1$) Feshbach resonance and the three lowest-lying Efimov states $n = 0, 1, 2$. The rightmost column shows the predictions of universal theory.

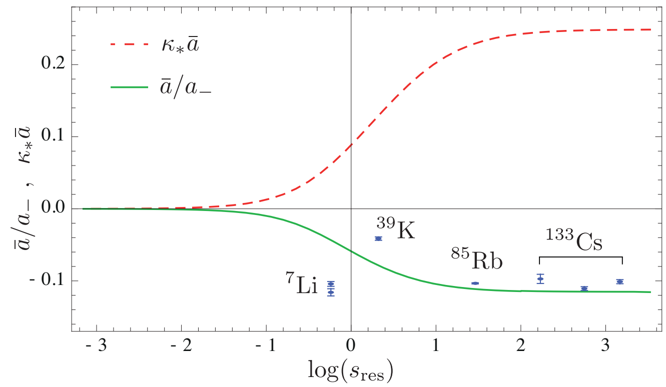

FIG. 3: (color online). Inverse threshold scattering length a_- (solid line) and wavenumber κ_* (dashed line) in units of $\bar{a}$ as functions of the resonance strength $s_{\rm res}$. The dots with error bars show the experimental results for ^{7}Li [10, 13], ^{39}K [11], ^{85}Rb [15] and ^{133}Cs [14].

els approach their universal values. It is instructive to quantify to which extent the lowest states deviate from the universal prediction. In Table I our results for various quantities for a broad ($s_{\rm res} = 100$), an intermediate ($s_{\rm res} = 1$), and a narrow ($s_{\rm res} = 0.1$) Feshbach resonance are shown (for details cf. [31]). Although the third state already follows almost universal behavior regardless of the value of $s_{\rm res}$, the experimentally most relevant lowest states exhibit large deviations. In fact, our prediction for open channel dominated resonances is supported by recent measurements of the position of the second Efimov trimer in ^{6}Li who find a ratio near 19.7 [32, 33]. Remarkably, for intermediate Feshbach resonances ($s_{\rm res} \approx 1$), the interplay between the scales r^* and σ leads to ratios close to their universal ones even for the lowest states. Note that the values of $a_*^{(n)}$ for small n are highly sensitive to the precise form of the two-body bound state spectrum which is strongly non-universal [37]. The ratios between the lowest $a_*^{(n)}$ are therefore in general not suitable for a measurement of universal ratios. By contrast, this is not the case for the Efimov spectrum in the regime $a \leq 0$, which is determined by energies on the order of or below the scale set by the van der Waals length.

We finally study the dependence of $a_-^{(n)}$ and $\kappa_*^{(n)}$ on the strength of the Feshbach resonance. In Fig. 3 the behavior for the lowest, experimentally accessible, state is shown. For open channel dominated resonances $a_-/\bar{a}$ and $\bar{a}\kappa_*$ become independent of $s_{\rm res}$ and thus of r^* and we find $a_- \approx -8.27\, l_{\rm vdw}$ and $\kappa_* l_{\rm vdw} = 0.26$. In the limit of closed channel dominated resonances, the van der Waals length becomes irrelevant and the scale for the full Efimov spectrum is set by r^* only. Specifically, we find $a_-^{(n)} = \xi^{(n)} r^*$ and $\kappa_*^{(n)} r^* = \eta^{(n)}$ with numbers $\xi^{(n)}$ and $\eta^{(n)}$ which approach universal values as $n \to \infty$. In fact, as $n \gg 1$, we accurately reproduce the predictions for narrow Feshbach resonances which were derived previously within a zero range model where $\sigma = 0$ [34, 35]. Note, however, that the low-lying Efimov states deviate from these universal predictions. While universal theory predicts for example $a_- = -12.90\, r^*$ and $\kappa_* r^* = 0.117$ [34, 35], we find $a_- = -10.3\, r^*$ and $\kappa_* r^* = 0.125$. Comparing to the experimental data, we find that the open channel dominated resonances in ^{85}Rb [15] and in ^{133}Cs [14], which also exhibit a large background scattering length, fit well into our prediction, apart from a 12 % deviation. The regime of Feshbach resonances of intermediate strength $s_{\rm res} \approx 1$ is particularly interesting, since both scales r^* and σ are relevant. Our approach equally applies to this regime, which is realized, e.g., in the case of ^{39}K, where $s_{\rm res} \simeq 2.1$ [11]. As shown in Fig. 3, the observation [11] of a considerable deviation from the apparent 'universal' result $a_- \approx -9.45\, l_{\rm vdW}$ in this case is qualitatively explained by our theory. Unfortunately, the case of ^{7}Li, which seems to follow nicely the result $a_- \approx -9.45\, l_{\rm vdW}$ [10, 13] despite the even smaller value $s_{\rm res} \simeq 0.58$ [13] of the resonance strength is not consistent with our prediction. A possible origin of this discrepancy might be that the identification of the scale σ of the Feshbach coupling with the van der Waals length does not apply in this case [38]. Note, however, that the proportionality between a_- and $l_{\rm vdW}$ is certainly invalid in the limit $s_{\rm res} \ll 1$.

In conclusion, we have presented a simple model containing only r^* and $l_{\rm vdw}$ as experimentally accessible parameters, which allows to predict the full Efimov spectrum in quantitative terms for Feshbach resonances of arbitrary strength without an adjustable 3BP. Our results provide an explanation for why the 3BP appears to be a 'universal' number in terms of the van der Waals length, which applies for open channel dominated resonances. It is important to stress, however, that for the lowest Efimov states there is no universality in its standard meaning. Beyond the dependence on the resonance strength, the value of the ratio $a_-/l_{\rm vdW}$ exhibits a weak dependence on the precise form of the cutoff function $\chi(r)$ in our model (1). Moreover, these numbers will be changed by taking into account three-body forces, e.g., of the Axilrod-Teller type, that are present for neutral atoms with van der Waals interactions [36]. Still, our exact solution of the standard two-channel model for Feshbach resonances of ultracold atoms captures the qualitative features necessary for an understanding of Efimov physics beyond the universal regime.

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[37] For example, a positive background scattering length a_{bg} leads to a open-channel bound state which is not included in our model. Similarily, due to the absence of a finite background scattering length, the lowest trimer eventually always meets the atom-dimer threshold.
[38] Note that for $s_{\mathrm{res}} \lesssim 1$ and a finite background scattering length, a_{bg} also becomes a relevant scale as shown by Pricoupenko [22].

Supplementary material

In this supplementary material we give results for the dependence of various 'universal' ratios on the continuous varied strength $s_{\rm res}$ of the Feshbach resonance.

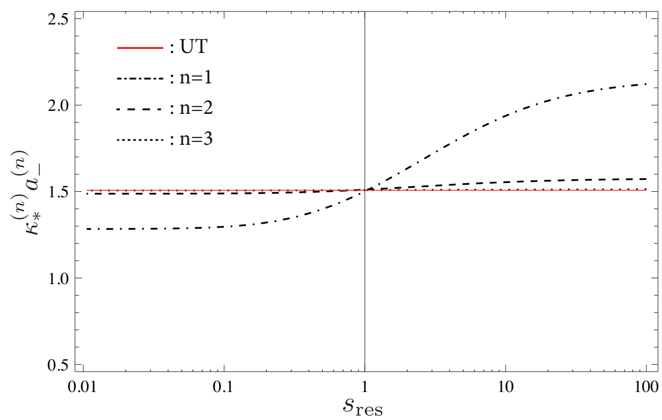

FIG. 4: (color online). The ratio $\kappa_*^{(n)} a_-^{(n)}$, which can be viewed as a measure of the distortion of the trimer levels from their universal shape in the (a, E) plane, as function of the resonance strength $s_{\rm res}$. Shown are the results for the lowest three levels (black). The universal result is shown in red. While the third state almost behaves universally the lowest state shows strong deviations from the universal prediction.

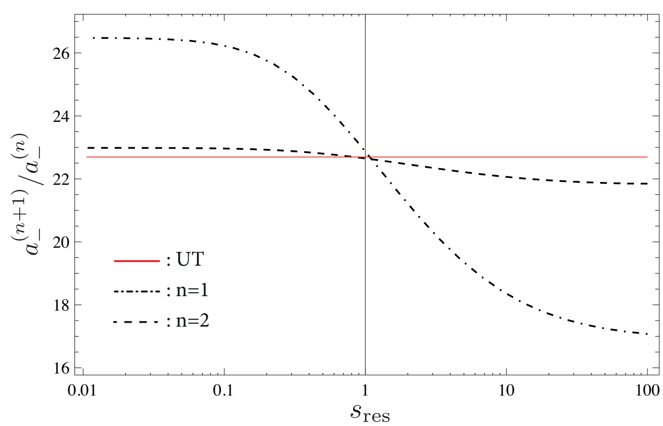

FIG. 5: (color online). The ratio of scattering lengths $a_-^{(n+1)}/a_-^{(n)}$ where the consecutive trimer states meet the atom threshold at $E = 0$ as function of the resonance strength $s_{\rm res}$ for the first two states (black). The universal prediction is shown in red.

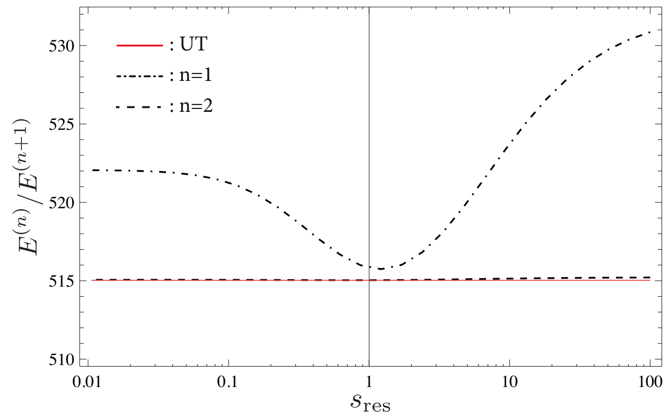

FIG. 6: (color online). The ratio $E^{(n)}/E^{(n+1)}$ of the consecutive trimer energies at unitarity $1/a = 0$ as function of the resonance strength $s_{\rm res}$ for the first two states (black). The universal prediction is shown in red.