THE INERTIAL MASS DEFINED IN THE GENERAL THEORY OF RELATIVITY HAS NO PHYSICAL MEANING

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It is shown that the inertial mass introduced in the general theory of relativity depends on the choice of the three-dimensional coordinate system, so that it can take arbitrary values. This means that the inertial mass in Einstein's theory is devoid of any physical meaning. In addition, the expression for the inertial mass in Einstein's theory in the general case of an arbitrary three-dimensional coordinate system does not have a classical Newtonian limit, so that the general theory of relativity does not satisfy the principle of correspondence with Newton's theory.

Introduction

It is currently assumed that in the general theory of relativity the gravitational mass of a system is equal to its inertial mass. This assertion goes back to the studies of Einstein [1], Eddington [2], Tolman [3], and Weyl [4]. Subsequently, this theorem was "proved" with various modifications by a number of other authors [5-7]. Nevertheless, we feel it is necessary to return to this apparently resolved question and make a more detailed investigation.

1. The Gravitational Mass in the General Theory of Relativity

The gravitational mass M of an arbitrary physical system in rest as a whole relative to a Schwarzschild coordinate system Galilean at infinity was defined by Einstein ([1], p. 660) as the quantity which multiplies the term $-2G/c^2r$ in the asymptotic expression $(r \to \infty)$ for the component g_{00} of the metric tensor of Riemannian space–time: $g_{00} = 1 - (2G/c^2r)M$.

A somewhat different definition of the gravitational mass was given by Tolman [3]:

$$M=\frac{c^2}{4\pi G}\int R_0{}^0\sqrt{-g}\,dV. \tag{1}$$

It follows directly from these definitions that the gravitational mass does not change under transformations of the three-dimensional coordinates, since both the component R_0^0 of the Ricci tensor and the component g_{00} of the metric tensor transform in this case as scalars.

In the case of static systems, the definitions of the gravitational mass given by Einstein and Tolman are equivalent. To see this, we write the component R_0^0 in the form

$$R_0{}^0=g^{0i}\left[\frac{\partial}{\partial x^l}\Gamma_{0i}{}^l-\frac{\partial}{\partial x^0}\Gamma_{ni}{}^n+\Gamma_{0i}{}^m\Gamma_{mn}{}^n-\Gamma_{im}{}^n\Gamma_{0n}{}^m\right].$$

After identical transformations, we obtain from this

$$R_0{}^0=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}[\sqrt{-g}\,g^{0n}\Gamma_{0n}{}^\alpha]-g^{0i}\frac{\partial}{\partial x^0}\Gamma_{ni}{}^n-\frac{1}{2}\Gamma_{ni}{}^0\frac{\partial g^{ni}}{\partial x^0}+\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^0}[\sqrt{-g}g^{0n}\Gamma_{0n}{}^0]. \tag{2}$$

Since the last three terms can be ignored for static systems, it follows from the expression (1) that

$$M=\frac{c^2}{4\pi G}\int dS_\alpha\sqrt{-g}\,g^{0n}\Gamma_{0n}{}^\alpha. \tag{3}$$

Since the metric sufficiently far from a static system must be described with given accuracy by the Schwarzschild metric, the expression (3) becomes

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$$M=-\frac{c^2}{8\pi G}\lim_{r\to\infty}\int dS^\alpha g^{00}\sqrt{-g}\frac{\partial}{\partial x^\alpha}g_{00}. \tag{4}$$

Since the integrand in (1) is a scalar under all transformations of the three-dimensional coordinate system, the gravitational mass M will also be independent of the choice of the coordinates. In Schwarzschild coordinates, we obtain from the expression (4)

$$M=\frac{c^2}{2G}\lim_{r\to\infty}\left[r^2\frac{\partial}{\partial r}g_{00}\right]=\frac{c^2}{2G}\lim_{r\to\infty}\left[r^2\frac{\partial}{\partial r}\left(1-\frac{2G}{c^2r}M\right)\right].$$

Thus, in accordance with Tolman's definition, the gravitational mass of a static system is the factor multiplying the term $-2G/c^2r$ in the asymptotic expression for the component g_{00} of the metric tensor of the Riemannian space-time. Therefore, the definitions of the gravitational mass given by Einstein and Tolman are equal for static systems.

2. Inertial Mass in the General Relativity

The concept of the inertial mass of a physical system in the general theory of relativity was intimately related by Einstein to the concept of energy of the system ([1], p.660): "...the quantity that we have interpreted as the energy also plays the role of inertial mass in accordance with the special theory of relativity." However, in the general theory of relativity it is not possible to introduce the concept of the energy of a system consisting of matter and the gravitational field, since in Einstein's theory matter and the gravitational field are characterized by quantities of different dimension: the physical characteristic of the gravitational field is the curvature tensor, i.e., a tensor of fourth rank, while the matter is characterized by the energy-momentum tensor, i.e., a tensor of second rank. Because of the difference between the ranks, general relativity does not in principle contain any conservation laws (apart from the Einstein equations themselves) linking the matter and the gravitational field [8]. Thus, the general theory of relativity was constructed at the price of giving up the energy–momentum conservation laws for the matter and the gravitational field taken together.

Following the book of Landau and Lifshitz [5], who use more modern notation, let us consider the manner in which the concept of the energy of a system was introduced in the general theory of relativity by Einstein ([1], p.528) and other authors [2-7, 9-11].

If Einstein's equations ([5], §96) are written in the form

$$-\frac{c^4}{8\pi G}g\left[R^{ik}-\frac{1}{2}g^{ik}R\right]=-gT^{ik}, \tag{5}$$

then the left-hand side can be split in a noncovariant manner into two terms:

$$-\frac{c^4}{8\pi G}g\left[R^{ik}-\frac{1}{2}g^{ik}R\right]=\frac{\partial}{\partial x^l}h^{ikl}+g\tau^{ik}, \tag{6}$$

where $\tau^{ik}=\tau^{ki}$ is the energy–momentum pseudotensor of the gravitational field, and $h^{ikl}=h^{kil}=-h^{ilk}$ is the spin pseudotensor:

$$h^{ikl}=\frac{c^4}{16\pi G}\frac{\partial}{\partial x^m}\left[-g\left(g^{ik}g^{ml}-g^{il}g^{km}\right)\right]. \tag{7}$$

Substituting (6) in (5), we obtain

$$-g\left[T^{ik}+\tau^{ik}\right]=\frac{\partial}{\partial x^l}h^{ikl}. \tag{8}$$

By virtue of the identity $\partial^2h^{ikl}/\partial x^k\partial x^l=0$, the Einstein equations (8) yield the differential conservation law

$$\frac{\partial}{\partial x^k}\left[-g\left(T^{ik}+\tau^{ik}\right)\right]=0. \tag{9}$$

Integrating this relation over a sufficiently large volume and assuming there are no "energy" fluxes through the surface bounding the volume of integration, an integral "energy–momentum conservation law for the system" is usually obtained from the expression (9):

$$\frac{d}{dt}\int(-g)\left[T^{i0}+\tau^{i0}\right]dV=0. \tag{10}$$

From this there follow four quantities that do not depend on the time:

$$P^i = \frac{1}{c}\int (-g)\,[T^{i0}+\tau^{i0}]\,dV. \tag{11}$$

By means of the Einstein equations (8), Eq. (11) can be rewritten in the form

$$P^i = \frac{1}{c}\oint h^{i0\alpha}\,dS_\alpha. \tag{12}$$

In Einstein's opinion ([1], p. 652), the four quantities P^i represent the energy ($i = 0$) and momentum ($i = 1, 2, 3$) of the physical system. It is usually asserted (see [5], p. 283): "The quantities P^i (the 4-momentum of field plus matter) have a completely definite meaning and are independent of the choice of the reference system to just the extent that is necessary on the basis of physical considerations."

On the basis of such a definition of the "energy and momentum" of the system consisting of matter and the gravitational field, the concept of the inertial mass m of the system is introduced in the general theory of relativity:

$$m = \frac{1}{c}P^0 = \frac{1}{c^2}\int (-g)\,[T^{00}+\tau^{00}]\,dV. \tag{13}$$

To calculate the inertial mass of the system, the Schwarzschild solution is generally used.

In isotropic Cartesian coordinates, the metric of the Riemannian space–time can be written in this case in the form

$$g_{\alpha\beta} = -\delta_{\alpha\beta}\left[1+\frac{r_g}{4r}\right]^4; \quad g_{00} = \left[1-\frac{r_g}{4r}\right]^2 \Big/ \left[1+\frac{r_g}{4r}\right]^2, \tag{14}$$

where $r = \sqrt{x^2+y^2+z^2}$, $r_g = 2GM/c^2$. These coordinates are asymptotically Galilean, since in the limit $r \to \infty$

$$g_{00} = 1+O\left(\frac{1}{r}\right); \quad g_{\alpha\beta} = -\delta_{\alpha\beta}\left[1+O\left(\frac{1}{r}\right)\right]. \tag{15}$$

Using the covariant components (14) of the metric, we obtain from the expression (7)

$$h^{00\alpha} = -\frac{c^4}{16\pi G}\frac{\partial}{\partial x^\beta}\left[g_{11}g_{22}g_{33}g^{\alpha\beta}\right].$$

Substituting this expression in (12), noting that

$$dS_\alpha = -\frac{x_\alpha}{r}r^2\sin\theta\,d\theta\,d\varphi, \tag{16}$$

and integrating over an infinitely distant surface, we obtain

$$P^0 = \frac{c^3}{16\pi G}\lim_{r\to\infty} r^2\int \frac{x_\alpha}{r}\frac{\partial}{\partial x^\beta}\left[-g_{11}g_{22}g_{33}g^{\alpha\beta}\right]\sin\theta\,d\theta\,d\varphi. \tag{17}$$

Thus, the component P^0 does not depend on the component g_{00} of the metric tensor of the Riemannian space–time. Substituting the expressions (14) in Eq. (17) and noting that

$$\frac{\partial}{\partial x^\beta}f(r) = -\frac{x_\alpha}{r}\frac{\partial}{\partial r}f(r), \quad x_\alpha x^\alpha = -r^2, \tag{18}$$

we obtain

$$P^0 = c^3 r_g/2G = Mc. \tag{19}$$

It was this equality of the "inertial mass" to the gravitational mass which led to the assertion that they are equal in the general theory of relativity ([5], p. 334): "... $P^\alpha = 0$, $P^0 = Mc$, a result which was naturally to be expected. It is an expression of the equality of "gravitational" and "inertial" mass ("gravitational" mass is the mass that determines the gravitational field produced by the body, the same mass that appears in the metric tensor in a gravitational field, or, in particular, in Newton's law; "inertial" mass is the mass that determines the ratio of energy and momentum of the body; in particular, the rest energy of the body is equal to this mass multiplied by c^2)."

However, this assertion of Einstein ([1], p. 660) and other authors [2-7, 9-11] is incorrect. As will be shown below, the "energy" (11) of the system and, therefore, its "inertial mass" have no physical meaning, since their value depends even on the choice of the three-dimensional coordinate system.

The general theory of relativity does not in principle admit the introduction of a concept of inertial mass, since in Einstein's theory there are no integrals of the motion linking the matter and the gravitational field (characterized by the curvature tensor). The only formal conservation law [8] in the general theory of relativity is provided by the Einstein equations themselves, which lead to integrals of the motion identically equal to zero, which precludes the introduction of an inertial mass.

3. The Concept of Inertial Mass is Meaningless in the General Theory of Relativity

An elementary requirement which a definition of inertial mass must satisfy is the condition that its value should be independent of the choice of the three-dimensional coordinate system, which is the case in any physical theory. However, in the general theory of relativity the definition (13) of the inertial mass does not satisfy this requirement.

We show, for example, that in the case of the Schwarzschild solution the inertial mass (13) may take all values depending on the choice of the system of spatial coordinates. For this, we go over from the three-dimensional Cartesian coordinates x_C^α to other coordinates x_H^α, which are related to the old coordinates by

$$x_c^\alpha = x_H^\alpha [1 + f(r_H)], \tag{20}$$

where $r_H = \sqrt{x_H^2 + y_H^2 + z_H^2}$, $f(r_H)$ is an arbitrary nonsingular function satisfying the conditions

$$f(r_H) \geqslant 0, \quad \lim_{r_H \to \infty} f(r_H) = 0, \quad \lim_{r_H \to \infty} r_H \frac{\partial}{\partial r_H} f(r_H) = 0. \tag{21}$$

It is readily seen that the transformation (20) corresponds to a change in the arithmetization of the points of three-dimensional space along the radius: $r_c = r_H[1 + f(r_H)]$. If the transformation (20) is to have an inverse and be a one-to-one transformation, it is necessary and sufficient that the condition $\partial r_c/\partial r_H = 1 + f + r_H f' > 0$, where $f' = \partial f(r_H)/\partial r_H$, hold. Then the Jacobian of the transformation is nonvanishing:

$$J = \det \left\| \frac{\partial x_c}{\partial x_H} \right\| = (1+f)^2 \frac{\partial r_c}{\partial r_H} \neq 0.$$

In particular, all the imposed requirements are satisfied by the function

$$f(r_H) = \alpha^2 \sqrt{\frac{8GM}{c^2 r_H}} [1 - \exp(-\varepsilon^2 r_H)], \tag{22}$$

where α and ε are arbitrary nonvanishing numbers.

Since in the given case

$$\frac{\partial r_c}{\partial r_H} = 1 + \alpha^2 \sqrt{\frac{8GM}{c^2 r_H}} \left[\frac{1}{2} + \left(\varepsilon^2 r_H - \frac{1}{2} \right) \exp(-\varepsilon^2 r_H) \right],$$

$f(r_H)$ is a monotonic function of r_H. It is readily seen that $f(r_H)$ is a non-negative nonsingular function in the whole of space. The Jacobian of the transformation in this case is strictly greater than unity: $J = (1+f)^2 \partial r_c / \partial r_H > 1$. Therefore, the transformation (20) with the function $f(r_H)$ defined by the expression (22) has an inverse and is one-to-one.

It is obvious that under the transformation (20) the value of the gravitational mass (1) does not change.

We now calculate the value of the "inertial mass" (13) in the new coordinates x_H^α. Using the transformation law of the metric tensor,

$$g_{ik}^H = \frac{\partial x_c^l}{\partial x_H^i} \frac{\partial x_c^m}{\partial x_H^k} g_{lm}^c (x_c(x_H)), \tag{23}$$

we find the components of the Schwarzschild metric in the new coordinates. As a result, we obtain

$$g_{00} = \left[1 - \frac{r_g}{4r_H(1+f)} \right]^2 \Big/ \left[1 + \frac{r_g}{4r_H(1+f)} \right]^2; \quad g_{\alpha\beta} = \left[1 + \frac{r_g}{4r_H(1+f)} \right]^4 \left\{ -\delta_{\alpha\beta}(1+f)^2 - x_\alpha^H x_\beta^H \left[(f')^2 + \frac{2}{r_H} f'(1+f) \right] \right\}. \tag{24}$$

The determinant of the metric tensor (24) has the form

$$g=-g_{00}\left[1+\frac{r_g}{4r_H(1+f)}\right]^{12}(1+f)^4[(1+f)^2+r_H^2(f')^2+2r_Hf'(1+f)]. \quad (25)$$

It should be noted especially that the metric (24) is asymptotically Galilean:

$$\lim_{r_H\to\infty} g_{00}=1; \quad \lim_{r_H\to\infty} g_{\alpha\beta}=-\delta_{\alpha\beta}.$$

In the special case when the function $f(r_H)$ is defined by (22) and $r_H \to \infty$, the metric of the Riemannian space–time will have the asymptotic behavior

$$g_{00}=1+O\left(\frac{1}{r_H}\right); \quad g_{\alpha\beta}=-\delta_{\alpha\beta}+O\left(\frac{1}{\sqrt{r_H}}\right). \quad (26)$$

For the contravariant components of the metric (24), we have

$$g^{00}=1/g_{00}; \quad g^{\alpha\beta}=-\delta^{\alpha\beta}A+x_H^\alpha x_H^\beta B, \quad (27)$$

where we have introduced the notation

$$A=(1+f)^{-2}\left[1+\frac{r_g}{4r_H(1+f)}\right]^{-4};$$

$$B=[r_H(f')^2+2f'(1+f)]\left\{r_H\left[1+\frac{r_g}{4r_H(1+f)}\right]^4(1+f)^2[(1+f)^2+r_H^2(f')^2+2r_Hf'(1+f)]\right\}^{-1}.$$

Substituting the expressions (27) and (25) in (12), we obtain

$$P^0=\frac{c^3}{16\pi G}\lim_{r_H\to\infty} r_H^2\int\frac{x_\alpha^H}{r_H}\frac{\partial}{\partial x_H^\beta}\left\{-\delta^{\alpha\beta}(1+f)^2\left[1+\frac{r_g}{4r_H(1+f)}\right]^8[(1+f)^2+r_H^2(f')^2+2r_Hf'(1+f)]+\right.$$
$$\left.\frac{x_H^\alpha x_H^\beta}{r_H^2}[1+f]^2\left[1+\frac{r_g}{4r_H(1+f)}\right]^8[r_H^2(f')^2+2r_Hf'(1+f)]\right\}.$$

By virtue of the relations (18),

$$P^0=\frac{c^3}{2G}\lim_{r_H\to\infty}\left\{r_H^3(f')^2(1+f)^2\left[1+\frac{r_g}{4r_H(1+f)}\right]^8+r_g(1+f)^2(1+f+r_Hf')\left[1+\frac{r_g}{4r_H(1+f)}\right]^7\right\}. \quad (28)$$

Using the asymptotic expression (21) for f, we finally obtain

$$P^0=\frac{c^3}{2G}\lim_{H\to\infty}\{r_g+r_H^3(f')^2\}. \quad (29)$$

Thus, the "inertial mass" depends on the rate at which f' tends to zero as $r_H \to \infty$. In particular, choosing the function $f(r_H)$ in the form (22), we obtain for the "inertial mass" from the expression (29)

$$m=M(1+\alpha^4). \quad (30)$$

It follows that for the "inertial mass" (13) of the system consisting of matter and the gravitational field in the general theory of relativity we can, because the value of α is arbitrary, obtain any preassigned number $m \geq M$ depending on the choice of the spatial coordinates, although the gravitational mass M (1) of this system and, therefore, all three effects in general relativity remain unchanged. We note also that under more general transformations of the spatial coordinates that leave the metric asymptotically Galilean the "inertial mass" (13) of the system may take all preassigned values, both positive and negative.

Thus, we see that in the general theory of relativity the "inertial mass," which was first introduced by Einstein and subsequently taken over by many authors [2-7, 9-11], depends on the choice of the three-dimensional coordinate system, and it therefore has no physical meaning. Therefore, the assertion that the inertial and gravitational masses are equal in Einstein's theory is also devoid of physical meaning. Such equality holds only in a small class of three-dimensional coordinate systems, and since the "inertial" (13) and gravitational (1) masses have different transformation laws, they are no longer equal after transition to different three-dimensional coordinate systems. In addition, the definition (13) of the inertial mass in the general theory of relativity does not satisfy the principle of correspondence with Newton's theory. Indeed, since the inertial mass m in Einstein's theory depends on the choice of the three-dimensional coordinate system, its expression in the general case of an arbitrary three-dimensional coordinate system does not go over into the corresponding expression of Newton's theory, in which the inertial mass does not depend on the choice of the spatial coordinates. Thus, in the general theory of relativity there is no classical Newtonian limit and, therefore, it does not satisfy the correspondence principle.

This leads us to ask why the meaninglessness of the definition (11) of the "energy and momentum" of a system and its "inertial" mass in the general theory of relativity has remained obscured until now.

This can only be explained by the fact that usually all calculations of the "energy, momentum, and inertial mass" have been made in a small class of three-dimensional coordinate systems in which the "inertial" and the gravitational mass are equal.*

In the same class of coordinate systems, the expression (13) for the inertial mass in the Newtonian approximation is equal to the corresponding expression in Newton's theory, which created the illusion that the general theory of relativity has a classical Newtonian limit. It was then apparently regarded as superfluous to consider the physical meaning of the inertial mass (13) introduced in the general theory of relativity.

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* Some aspects of this were made by Møller [12], but he did not succeed in understanding the essence of the matter and therefore failed to draw the appropriate conclusions, namely, that the inertial mass has no physical meaning in the general theory of relativity and, therefore, that theory has no Newtonian limit.