wellian beam. From this data it is possible to compute the response to a self luminous disk centered on the axis of the receptor and to determine how much the response is amplified by the foam cone.

It turns out that a model cone placed at the center of a Fraunhofer image and pointed at the center of the exit pupil yields about $\frac{2}{3}$ as much amplification as if it were exposed to an extended source having the same size, shape, and position as the exit pupil. It may be demonstrated in the case of an extended source in the plane of the pupil, that the amplification of the model cone increases as the size of the source decreases and this parallels what has been found in human vision in connection with the Stiles-Crawford effect.

It would be interesting to know in the case of a focused beam what would be the effect of varying the size of the exit pupil. This possibility could have been checked by varying the size of the mirror or the object and image distances, but these things have yet to be done.

It is possible that the directional sensitivity of cones might result in an improvement in resolving power when the eye is out of focus or is subject to spherical aberration. This possibility has not been tested.

The model receptor has a greater amplification for an extended Maxwellian source than for an extended Fraunhofer source, but no measurement was made of the relative radiance values of the two sources required to produce equal responses.

In the study of the response of the model cone to a focused beam the foam cone was placed at several positions in the diffraction pattern and directional sensitivity data were obtained. The direction of maximum amplification shifts as a function of the position of the receptor in the diffraction pattern.

The results obtained in this study using two wavelengths for both types of irradiation of the detector indicate that the amplification produced by the ellipsoid is more pronounced with the shorter wavelength. The results indicate that the curves used to represent the Stiles-Crawford effect must vary from wavelength to wavelength and that the amplification that takes place in a cone ought to affect its relative luminosity for the different wavelengths.

It was shown that one may treat the individual cone cell as an independent unit if its axis and the axes of neighboring cone cells are parallel. However, when the sides of two adjacent cones are nearly parallel and almost touching each other marked interaction effects occur.

JOURNAL OF THE OPTICAL SOCIETY OF AMERICA VOLUME 48, NUMBER 12 DECEMBER, 1958

Analysis of Experiments in Binocular Space Perception

ALBERT A. BLANK
Institute of Mathematical Sciences, New York University, and Mathematics Department, University of Tennessee, Knoxville, Tennessee

(Received April 18, 1958)

Experiments connected with Luneburg theory as developed by the author are analyzed with the purpose of making explicit their underlying assumptions. In particular, the role of *ad hoc* assumptions is explored in detail and minimized wherever possible. It is shown that the special assumptions under which much of the experimental work was executed may be considerably broadened thereby indicating how the theory may be more directly founded upon experiment. The principal problem is the determination of the sensory visual transformation between the geometry of the binocular perception and that of the stimulus, and, in particular, the determination of the visual radial distance function. Three principal techniques, the double circumhoropters, the Blumenfeld alleys, and the equipartitioned geodesics are discussed from this generalized point of view. The specific experimental material treated here consists of results obtained by Zajaczkowska, Shipley, and the Knapp Laboratory group at Columbia University. Some of these results appear for the first time. Theoretical material presented for the first time consists most notably of the analysis of the equipartitioned geodesics, the two-point experiments for the determination of Gaussian curvature, and the metatheoretical discussions of the several experiments.

1. INTRODUCTION

When the predictions of a formal mathematical theory are to be compared with factual experimental results, as in this analysis, one may treat the theory as closed and not subject to modification in which case the entire deductive structure stands or falls as a whole in the face of any significant experiment. We shall view the theory of binocular space perception as open in the sense that the fundamental assumptions of the theory are not sufficient to specify the system categorically but only up to a point where the completion of the system may be left to properly conducted experiments. For present purposes, we take as fundamental the axiom frame developed by the author.[1] In particular, we assume that binocular visual perception

[1] A. A. Blank, J. Opt. Soc. Am. 48, 328 (1958). (a) Sec. 5. (b) Sec. 4.

may be described in a compact convex metric space of three dimensions for which there exists a unique geodesic (visually straight) segment connecting each arbitrary pair of points. This postulation is strongly founded upon qualitative empirical observations concerned only with length orderings and alignments. Such a postulation is probably the indispensable minimum for the class of observers who can be said to have a well-defined space sense.

There are an infinity of conceivable geometries which satisfy the stated basic assumptions. The principal problems are to specify categorically for each observer the intrinsic binocular visual geometry, and to discover the relation or visual transformation between the geometry of the physical stimulus and that of the visual perception. Ultimately, it would be desirable to relate the visual geometry to anatomical and physiological parameters and the theory has already been useful and suggestive in that respect.[2]

On the basis of simple qualitative tests on a number of observers it is postulated that the binocular visual space is adequately described as a Riemannian space of constant Gaussian curvature.[1] In order to categorize the intrinsic visual geometry it is then only necessary to determine the sign of the curvature, whether positive (spherical geometry), negative (hyperbolic), or zero (Euclidean). This problem and the problem of determining the visual transformation from physical space into sensory space are left to experiment.

The greatest benefit of the systematic postulational development is that it permits analysis of experimental methods in a very general framework of weak assumptions. This is a notable advance over the initial form of the theory which was fettered by special *a priori* hypotheses. It has become easier to understand the assumptions underlying each empirical technique and to appreciate the character of the ambiguities and certainties in the inferences framed upon the resultant data. Moreover, in designing each experiment, we are freed in large measure from the obligation of deciding beforehand what are the relevant physical parameters in binocular space perception. As yet, no experimental program has made full use of this great freedom, but there can be little question of its eventual value.

For the purpose of formal simplicity only we shall here further restrict application of the theory to observers with clinically normal binocular vision. No new difficulties are introduced in principle by the study of observers with anisometropia (binocular asymmetry) and, in particular the formal analysis of aniseikonia (asymmetry due to unequal magnification) appears to be especially simple.[2(a)] However, for the purpose of determining the visual transformation, each asymmetric observer is a law unto himself, and in order to obtain some generality and simplicity in this discussion we confine our attention to the putative normal observer.

As our principal sources of empirical information we shall utilize the published researches of the Knapp Laboratory of Physiological Optics,[3] A. Zajaczkowska,[4] and T. Shipley.[5] We shall also utilize some hitherto unpublished information from the same sources.

2. SUMMARY OF THE THEORY

For the purposes of this analysis we shall touch briefly upon the central points of the theory. More detailed presentations exist elsewhere.[3,6]

It may be well to emphasize since Luneburg does not make it clear and usage differs,* that the subject matter of the theory is binocular vision in a stationary stimulus environment with freedom of fixation, but immobile erect head. Once motions of any kind are allowed, time automatically enters as a significant variable. Presumably, a completely adequate description of visual phenomena would require a space-time metric. To describe even what an observer with immovable head sees in a given stimulus configuration, it would be necessary to supply a record of his eye movements as well as the specification of the stimulus. Yet, as one might expect, if the question were not raised in such generality, the observer arrives in time at a perception which is principally dependent upon the stimulus, and not the exact history of his mode of observation. Time, in a manner of speaking, has been removed as a factor in the situation.†

Observation under the condition of a time-stabilized response is perforce the subject matter of the theory because of the obvious demand for experimental practicability. It takes time to fully develop the sense of depth in a given stimulus configuration, and the many references in Luneburg to "immediate" sensations are inappropriate from this point of view. The properties of time-stabilized binocular perception with immobile

[2] A. A. Blank, Brit. J. Physiol. Optics **14**, 154–169, 222–235 (1957). (a) Sec. 4, Horopters of Fixed Radial Distance, *et seq.* (b) Sec. 5, The Mapping of Physical into Visual Space.

[3] L. H. Hardy *et al.* and A. A. Blank *et al.*, *The Geometry of Binocular Space Perception* (Knapp Memorial Laboratories, Institute of Ophthalmology, Columbia University College of Physicians and Surgeons, New York, 1953). (a) Fig. 25, p. 56. (b) Sec. II, 2b (i), p. 43. Also similar unpublished data of the Four-Point Experiment. (c) Calculated from Table VI, p. 54 and Table VII, p. 59. d. Sec. II, 2b (ii), p. 45.

[4] A. Zajaczkowska, J. Opt. Soc. Am. **46**, 514 (1956). (a) Table IV, p. 523. (b) Fig. 11, p. 521. (c) Fig. 12 (left), p. 522. (d) Table II, p. 517.

[5] T. Shipley, J. Opt. Soc. Am. **47**, 795 (1957). (a) Figs. 10–12, pp. 810 ff. (b) Fig. 3, p. 807, ELC-OWR and BLC-OWR.

[6] A. A. Blank, J. Opt. Soc. Am. **43**, 717 (1953).

* For example, the comments of H. von Schelling [J. Opt. Soc. Am. **46**, 309 (1956)] in connection with one of Luneburg's discussions do not apply to the fully formulated theory.

† It is natural to wonder about the possibility of framing a theory of time-stabilized observations with moving head. The principal difficulty would be to determine the relevant physical parameters. A space-time theory which permitted motion of the stimulus viewed with immobile head would appear to be the simpler undertaking at this time. A second difficulty is that observations with moving head permit the observer to make judgments on the basis of motion parallax, which is primarily a monocular factor.

head can easily be described in terms of the elemental geometric relations which serve as a basis for the entire development of the theory.[1]

Since the experiments considered here have been conducted primarily in the eye level horizontal plane, some few in a plane of elevation through the interocular axis, it will be sufficient for the sequel to remain within a plane geometry.

The categorical specification of the intrinsic visual geometry may be accomplished at one stroke by observing the qualitative outcome of a trial of the Blumenfeld Alleys experiment.[7] However, this experiment involves new assumptions and, therefore, we shall proceed to this end in an epistemologically independent fashion by first examining some of the properties of the visual transformation and then decide the issue between the euclidean, spherical, and hyperbolic geometries by a simple experiment of an entirely different kind.

An observer acts as though his perceptions originate from a single point of regard or *egocenter* and he usually takes no conscious note of the double ocular source of his visual information. The perceptual egocenter naturally suggests itself as the origin of a polar coordinate system for visual space.[8] In the sensory horizontal plane through the egocenter, we may choose polar coordinates r, φ, where r describes sensed radial distance in a properly chosen system of units, and φ denotes sensory azimuth angle, the value $\varphi=0$ being

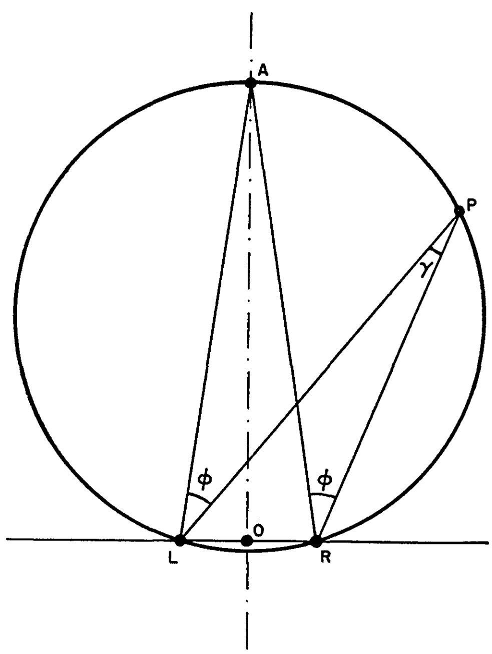

FIG. 1. Bipolar coordinates, γ, ϕ.

[7] W. Blumenfeld, Z. Psychol. **65**, 241 (1913).

[8] A. A. Blank, "The Luneburg theory of binocular space perception," in *Psychology, a Study of a Science*, edited by S. Koch, Study I, Vol. 1, (McGraw-Hill Book Company, Inc., New York, 1958); cf. Part III, Sec. A.2.

assigned to the sagittally forward direction. The metric describing the visual distance between two points $P_1=(r_1,\varphi_1)$ and $P_2=(r_2,\varphi_2)$ may then be written explicitly for each of the three possible geometries.

If the visual transformation from the physical coordinates of the stimulus to the visual coordinates (r,φ) is known, the visual metric gives a complete characterization of the geometry of the resultant impression. For the purpose of describing the visual transformation, Luneburg introduced bipolar coordinates γ, ϕ defined for any point P in the eye level horizontal plane. Consider the circle passing through P and the ocular centers L, R (Fig. 1). Such a circle is known as a Vieth-Mueller circle. The coordinate γ is the angle at which the ocular axes meet when they cross at P. The Vieth-Mueller circle, itself, satisfies the equation $\gamma=\text{const}$; it is the locus of constant convergence passing through P. If A denotes the forward intersection of the circle with the median, the angle ϕ, the *bipolar azimuth*, is the angle subtended by the arc PA at either eye. The equation $\phi=\text{const}$ describes a rectangular hyperbola (Hillebrand) passing through P and the homolateral eye with center at the point O midway between the ocular centers. We shall be somewhat inconsistent in specifying values of ϕ and γ and utilize degree measure for ϕ and radian measure for γ. The use of radian measure for γ is convenient since physical radial distance from the observer is then roughly equal to the reciprocal of γ multiplied by the distance between the ocular centers. The fundamental visual role of these coordinates in normal balanced vision (definitely not in aniseikonia[2(a)]) follows from the observation that the hyperbolas $\phi=\text{const}$ correspond very nearly to sensory loci of fixed direction from the egocenter, the so-called *radial horopters*, and that the circles $\gamma=\text{const}$ approximate the loci of fixed distance from the egocenter, the so-called *circumhoropters*. In other terms, binocular vision transforms a hyperbola $\phi=\text{const}$ into a polar ray $\varphi=\text{const}$, and a circle $\gamma=\text{const}$ into an egocentered circle $r=\text{const}$. It may be inferred therefore, that *in a fixed stimulus situation* there are two numerical functions;

$$\varphi=f(\phi), \quad r=g(\gamma), \tag{2.1}$$

which describe the sensory geometry corresponding to any given stimulus. We expressly avoid the hypothesis that the functions f and g have an invariant form independent of the particular stimulus. The basic assumptions do not guarantee the possibility of arbitrary prolongation of segments beyond the convex hull of a given visual configuration.[1,2(b)] It may well happen that the extension of a stimulus configuration may change the geometric relations among the points already present and hence, the form of the transformation (2.1). Luneburg assumed explicitly that these relations remain constant and are independent of the particular stimulus configuration. It is important to realize that no assumption of this kind is necessary. It is possible to determine the relations in various configurations by direct experiment.

A number of considerations indicate that it is possible to define $f(\phi)$, independently of the stimulus, by

$$\varphi=\phi+\phi_0, \quad (2.2)$$

where ϕ_0 is an arbitrary constant.[2,6] An over-all additive constant ϕ_0 has no effect on the metric relations and it may as well be assumed that $\phi_0=0$.‡ If the relation (2.2) is accepted as a special hypothesis, it is unnecessary to assume anything about the functional dependence of r; in particular, we need not even assume $r=g(\gamma)$. There exist techniques for the determination of r which depend only on (2.2). However, experiments do indicate that the circumhoropters, the physical loci which correspond to the perception of constant distance from the egocenter, are reasonably closely described in the central and paracentral binocular field by the equation $\gamma=$const. More peripherally, the true circumhoropters appear to be somewhat flatter than the Vieth-Mueller circles. This effect is most marked in the proximal region. However, with this reservation in mind, it is convenient to adopt the hypothesis $r=g(\gamma)$.

2a. Sign of the Gaussian Curvature

On the basis of (2.2) alone it is possible to determine the sign of the Gaussian curvature by means of extremely simple techniques which involve the use of stimuli consisting of two points. Here we discuss two such techniques.§

In one technique, the equilateral triangle experiment, the observer is instructed to set two variable lights $P_1=(\gamma_1,\phi_1)$, $P_2=(\gamma_2,\phi_2)$ so that the visual distances from the two points to himself are equal to each other and to the visual distance between the two points. In effect, the observer is asked to place the egocenter and the two perceived points at the vertices of an equilateral triangle. The instruction is given in the more complicated way in order to assure that the observer carefully compares the lengths involved and is not prejudiced by any conceptions with regard to the vertex angles of an equilateral triangle. The datum measured is the angle $\Phi=|\phi_2-\phi_1|$. If the angle is 60°, the geometry is euclidean; greater than 60°, spherical; less than 60°, hyperbolic.

In a trial of this experiment by C. J. Campbell and the author, one observer yielded a mean angular setting of 39.5°, another, 37.8°. While the observers find the subjective comparison of distances from the egocenter to other distances a difficult one, and settings are more scattered than in other experiments, the simplicity of the method has much to commend it.

‡ According to Shipley (see reference 5), Balasz and Walker give analytical indications of the reasonableness of the relation $\varphi=\phi$ (in a paper to be published).

§ Unpublished results at the Knapp Laboratory (1952).

Another simple experiment with two points is the isosceles right triangle experiment. Let a light be fixed at $P_1=(\gamma_1,\phi_1)$. The observer is instructed to set a light $P_2=(\gamma_2,\phi_2)$ on a perpendicular at P_1 to the ray OP_1, from the egocenter to P_1 (Fig. 2) in such a way that the visual separation between P_1 and P_2 is equal to that between P_1 and the egocenter O. The geometry is euclidean if the angle $\Phi=|\phi_2-\phi_1|$ is 45°; if the angle is greater, the geometry is elliptic; if less, hyperbolic. No experimental determinations have been made by this technique.

The interpretation of an angle defect in these two experiments as a hyperbolic effect is intuitively clear if it is recalled that in a Lobachevskian geometry the sum of the angles of a triangle is less than 180°, but in spherical geometry the sum is in excess.

The limited experimental findings above are cited only to show with what simple empirical means it is possible to categorize the visual geometry among the three Riemannian geometries of constant curvature. The finding here is that the Gaussian curvature is negative and this is in agreement with the great preponderance of hyperbolic results by every method.

2b. Visual Metric

Let us assume in this discussion that the visual geometry is hyperbolic.‖ In that case the metric $D(P_1,P_2)$ describing the visual distance between two

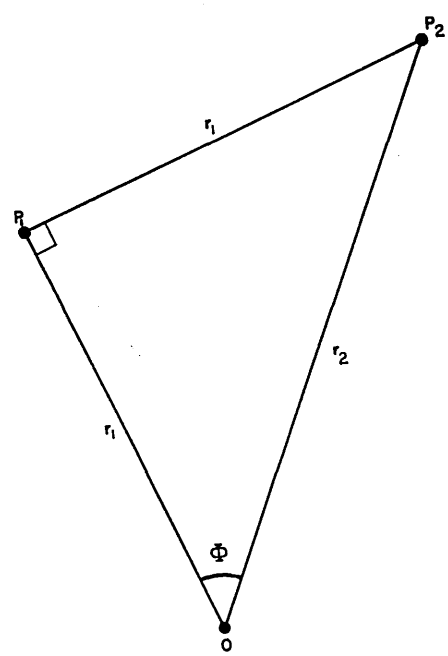

FIG. 2. Isosceles right-triangle experiment to determine the curvature of visual space.

‖ Although this assumption has a restrictive appearance it actually is in no way delimiting. If it is in error, the experimental results will yield either vanishing (Euclidean) or imaginary (spherical) visual lengths.

points $P_1=(r_1,\varphi_1)$, $P_2=(r_2,\varphi_2)$ is given explicitly in certain units by

$$\cosh D(P_1,P_2)=\cosh r_1 \cosh r_2 - \sinh r_1 \sinh r_2 \cos(\varphi_2-\varphi_1). \quad (2.3)$$

This formula is cognate to the Euclidean law of cosines. Luneburg used a different polar coordinate frame ρ,φ where

$$\rho=\frac{2}{(-K)^{\frac{1}{2}}}\tanh\frac{r}{2}; \quad (2.4)$$

K is a constant which is interpreted as Gaussian curvature. Luneburg's representation has the advantage that when ρ and φ are plotted on a euclidean polar frame, the resultant map is a conformal representation of the Lobachevskian plane; that is, angles are the same in the map as in the visual plane. The advantages of the coordinates r, φ lie in much greater simplicity of computation and intuitive understanding. Mathematically, the two systems are completely equivalent, the differences are purely formal. In the following, all relations will be written in the r, φ system.

It is interesting to note for the hyperbolic geometry in contrast to the Euclidean that there is an absolute unit of length and hence that it is possible to calculate the lengths of the sides in the triangle experiments of Sec. 2a from the measured angles. In the equilateral triangle experiment the side length s is given by

$$\cosh\tfrac{1}{2}s=\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}\Phi. \quad (2.5)$$

In the isosceles right triangle experiment the length r_1 of a leg is given by

$$\cosh r_1=\cot\Phi \quad (2.6a)$$

and the length r_2 of the hypotenuse by

$$\cosh r_2=\cosh^2 r_1. \quad (2.6b)$$

2c. Radial Distance Function

Under the preceding assumptions the complete geometric characterization of binocular perception for a normal individual is given by specifying the visual radial coordinate r in terms of the physical parameters. In particular, it is convenient for present purposes to accept as a good approximation the special assumption of (2.1) that the radial distance function r in a given stimulus depends only upon the convergence angle γ. Most of the experimental efforts have been devoted to the empirical determination of the function $r=g(\gamma)$, the principal methods being the Blumenfeld alleys, the Luneburg double circumhoropters, and our equipartitioned geodesics. The discussion is devoted primarily to these experiments; other experimental trials in visual mensuration such as the triangles of Sec. 2a or the Helmholtz geodesics yield some limited information concerning the radial distance function.

The possibility that the form of the function $r=g(\gamma)$ may depend upon the stimulus is an interesting one. Luneburg[9] originally considered the possibility that there exist certain transformations of the physical stimulus which do not result in changes in the visual perception, but he discarded the idea and it does not recur in his later work.[10,11] The existence of these transformations would imply that $g(\gamma)$ does change with the stimulus. Since the empirical determination of the radial distance function must automatically shed light on this question we need not assume more concerning the function $g(\gamma)$ than can be ascertained by direct experiment and this possibility may be left open.

The work at the Knapp Laboratory[3] concluded with a direct attack upon the problem of characterizing the dependency of the function $g(\gamma)$ upon the parameters of the stimulus. Zajaczkowska[4] and Shipley[5] have not yet reached that stage in their investigations but some of Zajaczkowska's results bear directly upon the problem.

3. METHODS AND RESULTS

We shall take up, in turn, each of the three principal techniques for the determination of $r(\gamma)$. In each case we shall examine the underlying assumptions as well as the experimental indications.

3a. Luneburg Double Circumhoropters¶

This experiment is performed in two independent stages, the three-point and four-point experiments. In the three-point experiment two points, $P_0=(\gamma_0,\phi_0)$ and $P_1=(\gamma_0,\phi_1)$ are fixed on the circle $\gamma=\gamma_0$. A third point, $P_2=(\gamma_1,\phi_2)$ is variable on an inner Vieth-Mueller circle $\gamma=\gamma_1$ and is set by the observer so that the distance from P_2 to P_0 is visually equated to the distance from P_1 to P_0 (Fig. 3). The observer's task is repeated for a number of different settings of $\phi_1-\phi_0$. If it is assumed that the Vieth-Mueller circles are circumhoropters in a homogeneous geometry, then the cosines

$$y=\cos(\phi_1-\phi_0), \quad x=\cos(\phi_2-\phi_0) \quad (3.1a)$$

must satisfy a linear relation

$$y=mx+b. \quad (3.1b)$$

Luneburg[11] has demonstrated the converse: among the Riemannian geometries, the homogeneous geometries

[9] R. K. Luneburg, *Mathematical Analysis of Binocular Vision* (Princeton University Press, Princeton, 1947). (a) Formula (6.892), p. 72. (b) Sec. 4.6, p. 44f.

[10] R. K. Luneburg, "Metric Methods in Binocular Visual Perception," in *Studies and Essays, Courant Anniversary Volume* (Interscience Publishers, Inc., New York, 1948).

[11] R. K. Luneburg, J. Opt. Soc. Am. 40, 627 (1950).

¶ The author prefers this name to that of double Vieth-Mueller circles since the theorerical design of the experiment is valid in principle for true circumhoropters only and these are not Vieth-Mueller circles for asymmetric observers and possibly not for balanced observers in the proximal and lateral fields.

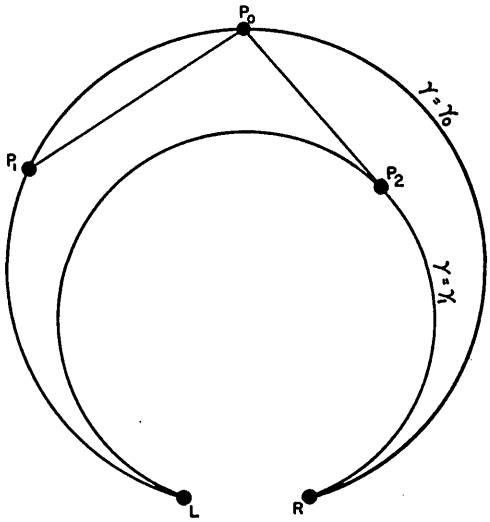

FIG. 3. Three-point double circumhoropters experiment.

are the only ones for which the general outcome of the three-point experiment is a linear relation. The experimental results do generally follow a linear pattern up to the limits of measurement (the Knapp Laboratory[3] reports angles $\phi_1-\phi_0$ out to about 24°, Zajaczkowska[4] to 21°). The values $r_0=g(\gamma_0)$ and $r_1=g(r_1)$ are determined by

$$\cosh^2 r_0=\frac{b^2}{(1-b)^2-m^2},\quad \sinh r_1=m\sinh r_0. \tag{3.1c}$$

Analogous relations hold for the other two homogeneous geometries.[6]

The physical limitations of the three-point experiment do not permit a size match if $\gamma_1-\gamma_0$ is made too large. It is generally true in these investigations that when the physical parameters are small the experiment is insensitive.[1(a)] For that reason the determination by the three-point experiments of the constants m and b is not sufficiently accurate.[3(a)]

In order to overcome the insensitivity of the three-point experiment, Luneburg devised the four-point experiment for the determination of m. Let $P_0=(\gamma_0',\phi_0)$ and $P_0^*=(\gamma_0',\phi_0^*)$ be fixed points on the circle $\gamma=\gamma_0'$. On a smaller circle $\gamma=\gamma_1'$ with $\gamma_1'>\gamma_0'$ there are two variable lights, $P_1=(\gamma_1',\phi_1)$ and $P_1^*=(\gamma_1',\phi_1^*)$. The observer's task is to equate the visual distance between P_1 and P_1^* with that between P_0 and P_0^* for several different settings of $\phi_0^*-\phi_0$. Again, a linear relation is expected, namely,

$$\sin\tfrac{1}{2}(\phi_0^*-\phi_0)=m'\sin\tfrac{1}{2}(\phi_1^*-\phi_1), \tag{3.2a}$$

where, setting $r_1'=g(\gamma_1')$, $r_0'=g(\gamma_0')$, we have

$$\sinh r_1'=m'\sinh r_0'. \tag{3.2b}$$

In the Knapp Laboratory experiments, limited departures from exact linearity were found in the extremes of the range $0\leq\phi_1^*-\phi_1\leq 50°$. (Zajaczkowska uses a more limited range and probably does not encounter this difficulty.)

The value of m' in (3.2b) depends on the choice of the values of γ_1' and γ_0' and may also depend upon changes in the form of the function $g(\gamma)$ due to differences in the stimulus configuration. In order to couple the two experiments it is necessary to develop some additional hypothesis about the radial distance function. From the epistemological considerations it almost seems better to deal with the insensitive three-point experiment alone and contend with variability in the results.**

It is of interest to consider the current hypotheses concerning $g(\gamma)$ in relation to the problem of coupling the two experiments. For certain reasons (discussed below in Sec. 3d) Luneburg[10] assumed that the coordinate ρ of formula (2.4) is given in the form

$$\rho=2e^{-\sigma\gamma}, \tag{3.3}$$

where σ is a personal constant of the observer. In that case the radial distance function will be determined for each observer by the two parameters σ and K. It then becomes possible to calculate the value of σ from the three-point experiment and then, without any restriction on γ_0', γ_1', to calculate K from the data of the four-point experiment.[11] We are not compelled, however, to make a categorical assumption of this type. We may entertain the more general hypothesis that $r=g(\gamma)$, where g is independent of the stimulus without postulating any special functional form. In that case, the two experiments may be matched by setting $\gamma_0'=\gamma_0$, $\gamma_1'=\gamma_1$ and it would follow that $r_0'=r_0$, $r_1'=r_1$, the slopes m and m' are equal, and hence that the Eqs. (3.1c) may be used to determine the values of r. The author's hypothesis[1,6] that visual radial distance depends upon differences in convergence rather than convergence itself would require only that $\gamma_1'-\gamma_0'=\gamma_1-\gamma_0$.

Trials of the double circumhoropters were made independently by Zajaczkowska[12] and by the Knapp Laboratory.[3] The experiments at the Knapp Laboratory cover the more extensive conditions, those of Zajaczkowska utilize only one condition (excluding pilot experiments), but for a considerable variety of observers.

The Knapp Laboratory trials were executed in two series. The earlier series was based on Luneburg's design using the assumptions of (3.3). The results have received only partial publication.[3] At the time the data were obtained no attention was paid in calculation to the matching of conditions from the two stages of the experiment. While the data almost uniformly

** It is even possible that the three-point experiment can be made adequately sensitive by using the method of "doubling back," that is, by taking P_1 and P_2 on the same side of P_0. This would increase the sensitivity by effectively doubling the range of available angles.

[12] A. Zajaczkowska, Quart. J. Exptl. Psychol. **VIII**(2), 66 (1956).

indicated negative Gaussian curvature they did not yield consistent values of σ and K under differing conditions.†† The results exhibited a significant measure of conformity with the idea that convergence disparity is the significant factor, rather than convergence (see Sec. 3d).[3(b)] In the second series of experiments, the restriction $\gamma_0=\gamma_0'$, $\gamma_1=\gamma_1'$ was imposed on each trial and the value of γ_0 was fixed throughout. As an indicator of the validity of the assumption that the function $r=g(\gamma)$ is the same in each experimental configuration we may observe whether $g(\gamma_0)$ is constant. Values of $g(\gamma_0)$ are given in Table I as calculated from the published settings of Knapp Laboratory observers. Despite the variability in the results, the difference between the two observers is evident and it is probably fair to say that $g(\gamma_0)$ is sufficiently constant for each observer to encourage the belief that the experiments are mutually consistent under the stated conditions. Zajaczkowska treats the double circumhoropter experiment by the same methods as the first Knapp Laboratory series and therefore also did not consider the problem of mutual consistency. In Zajaczkowska's work each of the DVMC experiments is performed under the following condition: the values of γ are given in the three-point experiment by $\gamma_0=0.05$, $\gamma_1=0.06$, approximately; in the four-point experiment by $\gamma_0'=0.02$, $\gamma_1'=0.06$. From the data of these experiments, values of σ and K are calculated by Luneburg's formulas. Under Luneburg's hypothesis (3.3) concerning the visual radial coordinate ρ, the quantities σ and K ought to be constants independent of the choice of γ_0, γ_1, γ_0', γ_1', but since the experiments are based on only one selection of values of these experimental variables, it is clear that the results obtained cannot verify or disprove Luneburg's hypothesis. The results of certain pilot experiments are cited as indicating that Luneburg's computation of σ rests on a satisfactory assumption.[12] In particular, Zajaczkowska reports linear plots of $\log\rho_1/\rho_0$ against $(\gamma_1-\gamma_0)$. This alone is not sufficiently specific since it includes many other possibilities such as $\rho=F(\gamma_0)e^{-\sigma\gamma}$ where $F(\gamma_0)$ is completely arbitrary. Perhaps it is also worth remarking that the exponential form is not essential since, as Shipley also notes, many other functional forms fit the data equally well. None of this is meant to preclude the possibility that other aspects of the pilot data may be more strongly indicative one way or the other.

TABLE I. Values of $r_1=g(\gamma_0)$, $\gamma_0\sim0.025$, obtained from Knapp Laboratory settings of the double circumhoropters.[a]

Observer \ $\gamma_1\sim$	0.030	0.035	0.045	0.065
G. R.	1.53	1.45	1.69	1.35
M. C. R.	0.87	1.14	0.82	0.97

[a] See reference 3c.

†† Unpublished results (1950). The data are suitable for matching, but it has not been done.

One other aspect of the matching problem may deserve mention. Some of the values of K obtained in the first series of Knapp Laboratory experiments were more negative than -1. This result is anomalous in the sense of implying the possibility of perception of sizes and distances greater than infinity. Such results are probably due to the improper matching of conditions in the two stages of the experiment and were not obtained in the matched experiments of the second series. An observer E. K. with $K=-1.06$ is also reported[12] by Zajaczkowska.‡‡

An alternative method of testing the hypothesis $\rho=2e^{-\sigma\gamma}$ is to use the calculated values of σ and K from the double circumhoropters to predict and compare with the outcomes of other experiments. For the purpose of this comparison, we shall consider in the following, the results of Zajaczkowska's experiments on the Helmholtz geodesics and the Blumenfeld alleys.

3b. Blumenfeld Alleys

The Blumenfeld alleys experiment suffers from the same fundamental weakness as the double circumhoropters in consisting of two independent stages which require the assumption that $r=g(\gamma)$ is the same function in both. Nonetheless, within the frame of this assumption, the Blumenfeld alleys present the most striking demonstration of the curvature of visual space.

The Blumenfeld alleys compare the visual perceptions of equidistance and parallelism. In the most commonly executed version of the experiment two lights are fixed on the horizon at points (γ_0,ϕ_0) and $(\gamma_0, -\phi_0)$ symmetric to the median. On each of a sequence of smaller Vieth-Mueller circles $\gamma=\gamma_i$, $(i=1, 2, 3, \cdots, n)$ with $\gamma_0<\gamma_1<\cdots<\gamma_n$, a pair of lights is placed and constrained to move on the circle.§§ To construct the *distance alley*, the observer is asked to set the lights of each adjustable pair symmetrically and so that their visual separation is equal to that between the fixed lights. In setting the *parallel alley*, the observer's task is to arrange the two rows of lights terminating at the fixed lights so that they form visually straight lines symmetric to the median and perpendicular to the sensory line running from left to right through the egocenter. This sensory line will be referred to as the *transverse axis*. If the visual geometry were Euclidean the two alley settings would be identical, but generally they are not. In fact, most settings of the alleys place

‡‡ Another observer I.M. is reported as having $\sigma=0.0$ and $K=-1.00$, but vanishing σ theoretically implies complete lack of depth perception and K becomes indeterminate. This reported value of K may have been obtained through an overlooked division by zero.

§§ The experimental practice has actually been to constrain each pair of adjustable lights to move along a line transverse to the median, rather than a Vieth-Mueller circle. This practice is more awkward theoretically, but more convenient experimentally.

the lights of the parallel alley closer to the median than the corresponding lights of the distance alley in accord with a geometry of hyperbolic character.

Another assumption implicit in the mathematical computations, aside from the hypothesis that $r=g(\gamma)$ is the same function in both stages, is connected with the meaning of parallelism. In a non-Euclidean geometry parallelism is not definable in the ordinary way by lines which do not meet no matter how far prolonged. In hyperbolic geometry there exist infinitely many such parallels to a given line through a given point; in spherical geometry there are none. The instruction for setting a parallel alley given above is unequivocal, but the use of the word, "parallel," is not the euclidean one. The Blumenfeld alleys experiment as performed hitherto has included the word, "parallel," without explanation except that the observer is asked to make sure that the parallel alley appears neither to "converge" or "diverge." Excepting a few trials of Blumenfeld's, only at the Knapp Laboratory, apparently, have the instructions been formulated to include the categorical requirement of perpendicularity to the transverse axis, and then in a different context.[3(d)] For their observers, there appeared to be no notable difference in the responses to the two different kinds of instruction.‖‖ It is worthwhile to emphasize in this respect the importance of framing perceptual criteria precisely according to the mathematical uses to which they are to be put.

Two methods have been given for determining the function $r=g(\gamma)$ from the alley settings, one by Luneburg[11] on the assumption that σ and K are constants and one by the author.[6] Either method can be used to calculate $r_0=g(\gamma_0)$ simply. The distance and parallel alley settings may be fitted by curves described by equations $\phi=\phi_d(\gamma)$ and $\phi=\phi_p(\gamma)$, respectively. The equation of the distance alleys is given by

$$\sinh r \sin\varphi_d=\sinh r_0 \sin\varphi_0, \tag{3.4}$$

the parallel alley, by

$$\tanh r \sin\varphi_p=\tanh r_0 \sin\varphi_0. \tag{3.5}$$

Luneburg's method consists of calculating values of σ and K from the slopes of the tangents at the fixed far points (γ_0,ϕ_0), $(\gamma_0,-\phi_0)$. This method has the disadvantage of failing to utilize the settings of the more proximal points of the alleys. Luneburg uses certain mathematical approximations which are avoided by the following. Taking $\varphi_p=\phi_p$, $\varphi_d=\phi_d$, and

$$\rho=2e^{-\sigma\gamma}=\frac{2}{(-K)^{\frac{1}{2}}}\tanh\frac{r}{2}, \tag{3.6}$$

we differentiate in both equations with respect to γ. Since the derivatives

$$a_p=(d\phi_p/d\gamma)_{\gamma=\gamma_0},\quad a_d=(d\phi_d/d\gamma)_{\gamma=\gamma_0}$$

are given by the experimental slopes of the alleys, we then may calculate σ and K. From the relation (3.6) we obtain on differentiation,

$$(dr/d\gamma)_{\gamma=\gamma_0}=-\sigma\sinh r_0 \tag{3.7}$$

and hence,

$$\begin{aligned}\sigma\tan\phi_0\cosh r_0&=a_d\\ \sigma\tan\phi_0&=a_p\cosh r_0.\end{aligned} \tag{3.8}$$

Upon eliminating $\cosh r_0$ from (3.8) we obtain

$$\sigma=\frac{[a_pa_d]^{\frac{1}{2}}}{\tan\phi_0}. \tag{3.9}$$

Upon eliminating σ from the equations we obtain r_0 by

$$\cosh r_0=[a_d/a_p]^{\frac{1}{2}}. \tag{3.10}$$

From the value of r_0, the other values of r may be calculated using (3.4) and (3.5). Equation (3.10) can be used to obtain K by employing the identity

$$\cosh r_0=\frac{1+\tanh^2 r/2}{1-\tanh^2 r/2}$$

to yield

$$K=-e^{2\sigma\gamma}\frac{[a_d/a_p]^{\frac{1}{2}}-1}{[a_d/a_p]^{\frac{1}{2}}+1}. \tag{3.11}$$

The author's method consists of eliminating r in Eqs. (3.4) and (3.5) to obtain r_0 by

$$\cosh^2 r_0=\frac{\sin^2\phi_d-\sin^2\phi_0}{\sin^2\phi_p-\sin^2\phi_0}=C(\gamma). \tag{3.12}$$

The values of r for other values of γ may then be found from (3.4) and (3.5). If all the other assumptions of the experiment are correct, the result (3.12) constitutes a check on the constancy of curvature. The right-hand side of the equation is given by a function of γ which can be a constant, in general, if and only if the geometry has constant Gaussian curvature.

In practice, of course, $C(\gamma)$ will never be exactly constant. At best, there will be small random variations about some mean value. The question arises as to the kind of average to be taken as the mean of the empirical function. Shipley does not agree with the author's use of an average weighted preferentially for larger values of γ. The choice of average can make little difference if the values of $C(\gamma)$ stay within a sufficiently small interval. The author's choice was suggested by the fact that the alleys were automatically fitted at the distal end, and by the observation that stress factors have more influence at the distal end. However, Shipley is probably right in suggesting that, in the absence of any overriding consideration, it is best to use the arithmetic mean.

‖‖ Blumenfeld,[7] on the other hand, usually employed the criterion of perpendicularity only after the observer made atypical settings of the parallel alley.

After Blumenfeld,[7] trials of the Blumenfeld alleys were made independently by the Knapp Laboratory,[3,13] Zajaczkowska,[4] and Shipley.[5] The Knapp Laboratory reports results under only one condition. Zajaczkowska uses three different settings of the fixed lights. Shipley, reporting on two observers only, has tried also alleys set obliquely, although the results of the oblique settings are not yet reported. Shipley also reports results in planes of elevation and depression.

In the Knapp Laboratory experiments with $\gamma_0 \sim 0.013$ it was noted that values of $g(\gamma_0)$ were found which were comparable with the double circumhoropter results of Table I, although the values of γ_0 differ appreciably (corresponding to sagittal distances of roughly 2.5 and 5.0 meters).[3] For observer G. R. the alleys yield $g(\gamma_0)=1.47$, for M. C. R. $g(\gamma_0)=0.93$.

Zajaczkowska executes experiments under the conditions

1. "classic alleys": $\gamma_0 \sim 0.015$, $\phi_0 = 4.1°$,
2. "intermediate alleys": $\gamma_0 \sim 0.021$, $\phi_0 = 5.75°$,
3. "broad alleys": $\gamma_0 \sim 0.044$, $\phi_0 = 11.4°$.

The results indicate that σ decreases in going from the classic alleys to the broad alleys, and the absolute value of K increases. The alley experiments therefore do not support the constancy of σ or K. From Zajaczkowska's given values of σ and K the value of r_0 may be calculated by means of Eq. (3.10) to yield

$$r_0 = 2 \operatorname{arctanh}[(-K)^{\frac{1}{2}} e^{-\sigma\gamma_0}]. \tag{3.13}$$

Zajaczkowska calculates r_0 in this way, and also by formula (3.12). However, for the purposes of comparison Zajaczkowska in (3.13) uses σ and K from the double circumhoropters rather than from the alley data. The author has computed r_0 directly from the alley information[4(a)] and the values are compared in Table II with values obtained from her raw data by Shipley[5(a)] using the method of (3.12).

The values of r_0 in Table II obtained by any single method seem to be broadly comparable in the first two columns, but generally of a different magnitude in the third. If the results are taken at face value they give conflicting testimony as to the merits of the various hypotheses concerning $g(\gamma)$ cited in Sec. 3a. The author believes that this difficulty will be resolved when more is known about the empirical visual transformation in the lateral periphery and proximal regions of the binocular field. In particular, the hypothesis $r=g(\gamma)$ may not be sufficiently precise in those areas.

TABLE II. Values of r_0 calculated by the methods of Blank (3.12) and Luneburg (3.13) for observers of Zajaczkowska.

Observer	Classic alleys (B)	Classic alleys (L)	Intermediate alleys (B)	Intermediate alleys (L)	Broad alleys (B)	Broad alleys (L)
B. A.	...	...	0.93	1.06	0.82	0.71
S. V. S.	0.64	0.46	0.33	0.37	0.89	0.71
H. S.	2.00	1.61	2.11	1.53	1.30	1.27
K. G.	1.12	0.83	1.52	1.14	0.92	0.75
W. K.	0.93	0.79	0.89	0.87	0.74	0.78

[13] Hardy, Rand, and Rittler, A. M. A. Arch. Ophthalmol. 45, 53 (1951). (a) Chart 2A. (b) Chart 3.

A new contribution to the literature on the alleys consists of Shipley's experiments on two observers who construct the Blumenfeld alleys in the planes of elevation and depression as well as at eye level. In addition, other experiments concerned with oblique alleys are mentioned but not reported in detail. For one observer, Shipley reports typical settings with the distance alleys broader than the parallel alleys. For the other observer, the two kinds of alleys are superimposed at eye level and below, but the setting above eye level is typical.

Shipley calculates the radial distance function by the author's method for observers of Zajczkowska and other experimenters in addition to his own. The principal datum computed is the ratio $C(\gamma)$ of (3.12). The theoretical prediction is that $C(\gamma)$ is constant and Shipley justifiably devotes considerable attention to how well this prediction is met.

On the assumption that the mapping of physical into visual space is factually of the form $\varphi=\phi$, $r=g(\gamma)$, the constancy of $C(\gamma)$ is itself a demonstration of the constant curvature of visual space. It is therefore significant if the function $C(\gamma)$ exhibits a continuing trend rather than a more or less random variation about some mean value. In the majority of cases Shipley reports essential constancy or only moderate departures from constancy. However, a substantial fraction of observers do exhibit significant continuing variation in one direction. A fact of importance is that the strongest deviations are uniformly of the same kind; they show an increase of $C(\gamma)$ with γ. Such an effect may be caused in a number of ways and several of these causes may be simultaneously operative in any given setting of the alleys. There seems to be no systematic discussion of the matter in the literature and therefore we now enumerate and discuss what seem to be the most important causes.

1. *Direction of the Parallel Alleys*

It will be recalled that for the purposes of Luneburg's mathematical analysis, the parallel alley is assumed to be perpendicular to the sensory transverse axis through the egocenter. However, in almost all performances of the Blumenfeld alleys experiment reported to date, this definition is conspicuously absent in the instructions to the observers. It is natural therefore to ask whether the parallel alley is actually orthogonal to the transverse axis. By use of the partitioning technique, it is, of course, possible to check the question of perpendicularity directly (see Sec. 3c).

Shipley[5] has put primary emphasis on this question. He suggested that the parallel alleys are actually perpendiculars to the transverse line joining the two distant fixed points. This would imply positive gaussian curvature in the typical mode of setting the Blumenfeld alleys. At the time of publication he had not yet computed the angle made by a side of the parallel alley with the transverse axis, but now has obtained an interior angle of about 87° in at least one case (communicated January 31, 1958). This deviation is not large but, if significant, it would mean that the parallel alleys would have to be set farther inwards to meet the criterion of perpendicularity. This would reinforce the presumption of negative curvature. It is worth remarking that if the parallel alley were perpendicular to the transversal between the fixed points (spherical geometry) then there would have to be an angle excess instead of the angle defect observed.

2. *Visual Transformation*

The mapping of physical into visual space may not be governed exactly by relations of the form $\varphi=\phi$, $r=g(\gamma)$. For example, it has been noted that the empirical circumhoropters $r=$const are not precisely Vieth-Mueller circles, but tend to flatten at the sides of the binocular field. The effect is most pronounced in the proximal region. For observers who wear glasses there will be refractive effects at oblique angles of gaze, but these are computable. Observers with uncorrected asymmetries also cannot be expected to have the stated form of visual transformation.[2(a)] This consideration may be important for certain observers of Zajaczkowska who were reported to have uncorrected asymmetries.

3. *Curvature of the Visual Space*

If $C(\gamma)$ is not constant, it is conceivable that the curvature of the observer's space is not constant. If true, this may lead to severe mathematical difficulties. It is therefore worthwhile to check this point directly by testing the desarguesian hypothesis[1(b)] for any problem observer. For example, Shipley's observer OWR appears to have negative curvature in some parts of his space and zero curvature in others. In order to obviate such causes as inaptitude or misinterpretation on the part of the observer, it is desirable to perform this simple check.

4. *Semantic Factors*

There frequently appear to be difficulties connected with the distinction in meaning between visual and physical mensuration. The problem is more serious with adults. Children, evidently, have fewer difficulties in acting directly upon the basis of their perceptions rather than upon any educated geometrical conceptions. Both Luneburg and Zajaczkowska remark upon the quality of children as observers.

The criterion for setting the distance alleys generally seems to be well understood. The primary semantic difficulties lie in the parallel alleys. Observers have been known to assert that physical parallel lines such as railroad tracks appear parallel to them. One observer reported of a picture of railroad tracks meeting at a vanishing point, that the tracks appeared parallel; yet two intersecting lines drawn in the same manner did not appear so. It is clear that the difficulty in this case is semantic. It seems probable that most of the reported strongly elliptic settings[13(a)] are due to this difficulty. Strangely enough, this difficulty does not carry over to the distance alleys. Most of the observers who do not distinguish between physical and sensory parallelism, do distinguish between the criteria for the parallel and distance alleys and set the two kinds of alley differently.

Another kind of observer sets the two types of alley in almost exactly the same way.[13(b),4(b),5(b)] In some of these cases there have been verbal indications that the observer does not find any distinction in meaning between the two criteria.

For a third class of observers, each side of the parallel alley is set very nearly as a physically straight line through the homolateral eye.[13(b),4(c)] The possibility that these settings are made by a monocular criterion must be kept in mind. An observer whose performance falls in the latter categories may, of course, have weak binocular function and should be tested for this by standard clinical procedures.

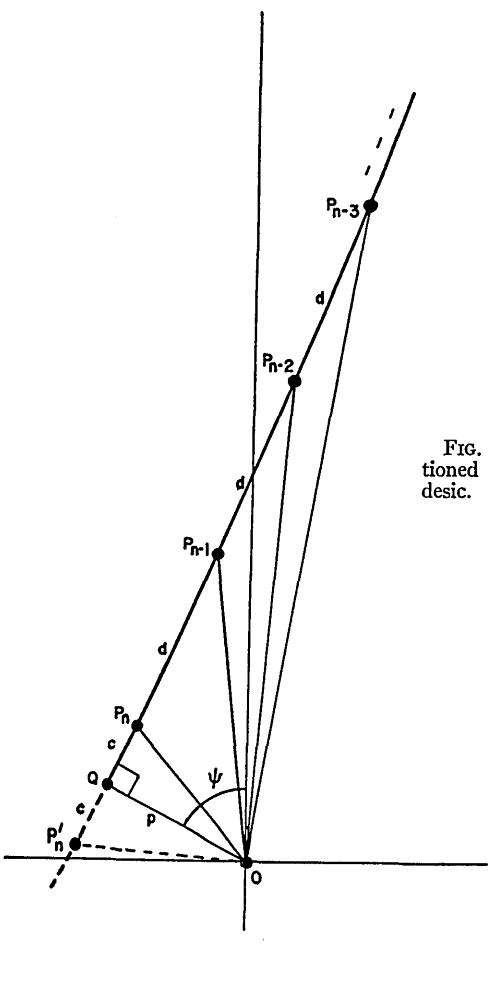

FIG. 4. Equipartitioned visual geodesic.

5. *Stress*

Whenever the azimuth angle between any two points is very small, for example, when one light is visually almost directly in front of another, there are definite indications of stress. Observers frequently complain about the effort of concentration necessary to properly appreciate the spatial relations among such points. This situation is marked at the distal end of the alleys and very clearly so in the parallel alley. Zajaczkowska puts considerable emphasis on this matter. She points out that the observer may tend to bend the more distal regions of the parallel alley away from the median line in automatic avoidance of stress.

6. *Training*

The Blumenfeld alleys require a painstaking and time consuming effort on the part of the observer. Observers become easily fatigued and rarely set alleys to their own satisfaction at the first try. The usual trend of events for those observers in which training has an effect is that successive parallel alleys tend to be brought closer and closer together until a stable situation is reached. It is probably best to postpone the taking of alley data until an observer has a measure of confidence in his skill.

3c. Equipartitioned Geodesics

This technique is in a sense the purest of the three. The assumption made is that of (2.2) alone, namely that visual azimuth differs from bipolar azimuth in an additive constant only. In practice the experiment has been performed by the Knapp Laboratory and by Shipley only in connection with settings of parallel alleys, but this is by no means requisite. The general mathematical analysis of this technique has been in existence for some time¶¶ but prior descriptions have been incomplete and therefore the analysis is given here in detail.

There are several variations of the method of equipartitioned geodesics. Of these the simplest is the following: Two points $P_0=(\gamma_0,\phi_0)$ and $P_n=(\gamma_n,\phi_n)$ are fixed. The point P_0 is the more distal, $(\gamma_0\leq\gamma_n)$. Some $n-1$ points $P_k=(\gamma_k,\phi_k)$, $(k=1, 2, \cdots, n-1)$, are set by the observer in alignment with P_0 and P_k and in the order of the indices. The visual lengths between the successive points are made equal so that the segment P_0P_n is divided into n visually equal parts.

The most convenient form of the equation of a visual geodesic in polar coordinates is the so-called *normal* form in which the defining parameters are the length p of the perpendicular from the egocenter to geodesic, and the angle ψ made by this perpendicular with the sensory sagittal axis, Fig. 4. The method cannot be applied if $p=0$, that is, if the line is directed through the egocenter. If $p\neq 0$, we then have, for each point P_k of the geodesic,

$$\tanh r_k \cos(\varphi_k-\psi)=\tanh p, \tag{3.14}$$

where (r_k,φ_k) are visual coordinates for P_k.

If ψ and one value of r are both known it is possible to determine all other values of r by (3.14). If two values r_i and r_j are known as from the double circumhoropters, the angle ψ may be determined by

$$\tan\psi=-\frac{\tanh r_i \cos\varphi_i-\tanh r_j \cos\varphi_j}{\tanh r_i \sin\varphi_i-\tanh r_j \sin\varphi_j}. \tag{3.15}$$

If the equipartitioned geogesic is to be used for an independent determination of r, the angle ψ must be determined, either empirically or from the given data. The simpler procedure is empirical, to construct the geodesic normal to a known ray from the egocenter. The method will be insensitive unless a large range of values of r is covered. Equipartitioned Helmholtz geodesics will generally not provide adequate information, for example. However, geodesics turned far from a transverse orientation will not have normals directed within the range of values of ϕ limited by the binocular field, and therefore ψ will have to be determined by calculation with the sole exception of parallel alleys where it is assumed $\psi=90°$.

Let us assume that ψ is known. Every unknown in the problem may then be considered to be determined by three quantities; p, the distance QP_k from the foot Q of the normal to any other point P_k on the geodesic, and the distance r_k of P_k from 0. Every other quantity is determined by the hyperbolic law of sines. It follows that only three conditions or three points on the geodesic are needed for the solution of the problem. Let P_1, P_2, P_3 denote any three points which appear in that order on the geodesic with P_2 visually midway between P_1 and P_3. If D_i denotes the distance QP_i, $(i=1, 2, 3)$ we have

$$2D_2=D_1+D_3. \tag{3.16}$$

The values D_i may be eliminated from (3.16) and the equation

$$\tanh D_i=\sinh p \tan(\phi_i-\psi), \quad (i=1, 2, 3), \tag{3.17}$$

to yield

$$\sinh^2 p=\frac{1}{\tan^2(\phi_2-\psi)}\left[\frac{2-(S+T)}{(S+T)-2ST}\right], \tag{3.18}$$

where

$$S=\frac{\tan(\phi_1-\psi)}{\tan(\phi_2-\psi)}, \quad T=\frac{\tan(\phi_3-\psi)}{\tan(\phi_2-\psi)}. \tag{3.18a}$$

From the value of p obtained through (3.18) and the known value of ψ, the other values of r are given by Eq. (3.14).

If ψ is not known an additional defining condition is required and we must employ a minimum of four

¶¶ Unpublished work at the Knapp Laboratory (1951).

equally spaced points, P_1, P_2, P_3, P_4. We may then determine ψ and proceed by the method above to obtain the values of r. For notational simplicity we set

$$\beta=2(\phi_4-\psi),\quad \alpha_1=\phi_3-\phi_4,\quad \alpha_2=\phi_2-\phi_4,\quad \alpha_3=\phi_1-\phi_4. \quad (3.19)$$

The visual length of the three equal segments is denoted by d, and the distance from Q to P_4 by c (Fig. 4). For computational convenience we introduce the auxiliary point P_4' on the other side of Q from P_4 and at the same distance c. By employing the hyperbolic law of sines successively to the triangles $P_4'OP_3$, $P_4'OP_2$, $P_4'OP_1$, we obtain three relations in the three unknowns β, c, d,

$$\frac{\sinh(2c+kd)}{\sin(\beta+\alpha_k)}=\frac{\sinh kd}{\sin\alpha_k},\quad (k=1,2,3).$$

The value of β and hence of ψ may be determined by eliminating c and d from these equations. For this purpose the equations are more conveniently written in the form

$$\sinh 2c\, \coth kd+\cosh 2c=\sin\beta\,\cot\alpha_k+\cos\beta,\quad (k=1,2,3). \quad (3.20)$$

These equations are easily solved for d to yield

$$4\cosh^2 d=\frac{\sin\alpha_2\sin(\alpha_3-\alpha_1)}{\sin\alpha_1\sin(\alpha_3-\alpha_2)}=1-\frac{\sin\alpha_2\sin(\alpha_2-\alpha_1)}{\sin\alpha_1\sin(\alpha_2-\alpha_3)}. \quad (3.21)$$

From the first two equations in (3.20) we then obtain

$$\sinh 2c=(\cot\alpha_1-\cot\alpha_2)\sinh 2d\,\sin\beta. \quad (3.22)$$

Taking the value of sinh2c from (3.22) and using $\cosh 2c=(1+\sinh^2 2c)^{\frac{1}{2}}$ in the first equation of (3.20), we obtain β explicitly in the form,

$$\tan\beta=\frac{4\lambda\cosh^2 d-2\mu}{1+\mu^2+4\lambda(\lambda-\mu)\cosh^2 d}, \quad (3.23)$$

where

$$\lambda=\cot\alpha_1-\cot\alpha_2,\quad \mu=\cot\alpha_1, \quad (3.23a)$$

and 4 $\cosh^2 d$ is given by (3.21).

The principal empirical caution in the use of these methods is that the partition points must be well separated. If too small a segment is partitioned the curvature of the space will not be made strongly manifest and the experiment will be rendered insensitive.[1(a)] That this is not an empty caution is made clear by certain remarks of Shipley.[5] He observes that an equipartition of a parallel alley with five points may yield negative values on the right side of Eq. (3.18). Observations at the Knapp Laboratory (unpublished) have shown that this occurs with considerable frequency only if the three most distal points are used in the calculation. For a given observer the sign in (3.18) is extremely critical with respect to the setting of the second most distal light. In the parallel alley the three most distal lights are separated by small azimuth angles and the two farthest lights are quite close in azimuth. The situation, therefore, is one in which the stress factors discussed in Sec. 3b above may easily affect the setting. Since the region of visual space involved is not extensive, formula (3.18) will be insensitive and slight changes in the setting will easily affect the sign of the apparent curvature.

It was found above that four conditions are sufficient to determine the mapping of a visual geodesic. If more conditions are employed it is possible to eliminate the four unknowns and obtain nontrivial relations among the azimuth angles. Let $P_k=(r_k,\phi_k)$, $(k=0,1,2,3,4)$, denote five visually collinear, equally spaced points. From the law of sines it is easily demonstrated that

$$\frac{\sin(\phi_4-\phi_2)\sin(\phi_3-\phi_2)\sin(\phi_1-\phi_0)}{\sin(\phi_0-\phi_2)\sin(\phi_1-\phi_2)\sin(\phi_4-\phi_3)}=1. \quad (3.24)$$

This relation is equivalent to that obtained by Shipley.[14]

In Shipley's paper,[5] although the partition technique was used, the data had not yet been analyzed by the above method or an equivalent except for an attempt to apply (3.24). At that time the analysis of the Blumenfeld alleys was employed.

At the Knapp Laboratory, trials of the equipartitioned parallel alleys were run with three points on a side. The most distal point $P_0=(\gamma_0,\phi_0)$ was kept at fixed azimuth ϕ_0, but γ_0 was given a sequence of different values. The most proximal point $P_2=(\gamma_2,\phi_2)$ was constrained so that the difference $\gamma_2-\gamma_0$ was kept very nearly constant throughout the series. The values of $r_0=g(\gamma_0)$ obtained for each observer was quite constant throughout the series independently of γ_0 (see Fig. 6). The mean values obtained for observers G. R., $r_0=1.39$, and M. C. R., $r_0=1.03$, are comparable to the values cited previously obtained by the other techniques. (Only these observers are utilized for these comparisons because they were the only ones to complete the full course of experiments by the three methods.)

3d. Iseikonic Transformations

It will have been appreciated from the foregoing that generalized treatment of the visual transformation and, in particular, the radial distance function is compelled by the experimental results. If it is assumed that $r=g(\gamma)$ is an adequate hypothesis over much of the binocular field, then the above indications that $r_0=g(\gamma_0)$ (where γ_0 is the minimum value of γ in each configuration) is insensitive to changes in the value of γ_0 virtually force consideration of the possibility that

[14] See reference 5, page 819.

the form of $g(\gamma)$ changes from configuration to configuration.

Ames was the first to suggest the existence of entire classes of stimulus configurations which are binocularly indistinguishable. Further, he was able to construct empirically a sequence of distorted rooms which yield the same binocular impression as a commonplace rectangular room.[15] Luneburg assumed that the binocularly equivalent rooms were related to one another by rigid translatory motions in the visual space. This hypothesis does not seem to be sustained by the empirical evidence; one direct indication being the fact that the sensed relative position of the observer and the room remains constant in the Ames demonstrations.

The Ames rooms lead to the supposition of a regular transformation law which connects any pair of binocularly equivalent stimuli. In particular, much of the experimental evidence in the horizontal plane indicates that an iseikonic transformation of the form

$$\gamma \rightarrow \gamma + \lambda, \tag{3.25}$$

i.e., the addition of an over-all constant λ to the coordinate γ, does not affect visual relations. In other words, if the entire physical stimulus is transformed by adding a fixed constant to the γ coordinate of each of its points, the visual impression given by the new stimulus is metrically identical with that given by the old. This mapping explains certain experimental observations such as the Helmholtz geodesics and the Ames binocularly distorted rooms without any explicit knowledge of metric properties.

The experimental material cited by the Knapp Laboratory workers in connection with (3.25) consists of trials of the three-point double circumhoropters and equipartitioned parallel alleys.[3] Similar results (unpublished) were also obtained from the four-point double circumhoropters. These results were generally quite favorable to the hypothesis (as in Fig. 6).

Zajaczkowska did not attempt to test (3.25) directly but some of her data for the Helmholtz geodesics are relevant*** and will be treated here.

The Helmholtz geodesics[16] are simply the visual geodesics perpendicular to the median line of the horizontal plane. These geodesics form a one-parameter set of curves satisfying the equation

$$\tanh r \cos\varphi = \text{const.} \tag{3.26}$$

In the present variant of the experiment, two lights at the points (γ_0,ϕ_0), $(\gamma_0 - \phi_0)$ are placed symmetrically

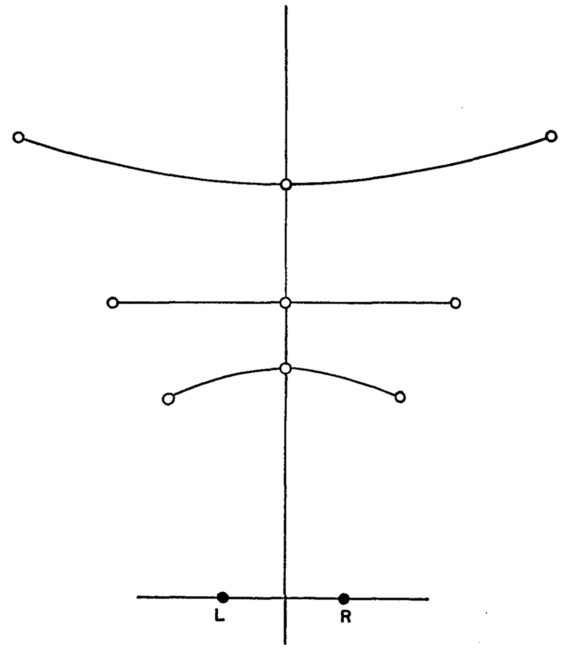

FIG. 5. Helmholtz geodesics.

with respect to the median. The observer's task is to set a light $(\gamma_1,0)$, variable on the median, in visual alignment with the two fixed points. Characteristically, the observer's settings are not in physical alignment, but, at near the setting arches away from the observer, while at far it arches toward him (Fig. 5). At some intermediate distance the three lights will also be in physical alignment. This distance is computed by Zajaczkowska using Luneburg's method[9(a)] which is applicable for small angles ϕ_0. The agreement between the experimental values and the calculated values for the distance of the flat geodesic is remarkably good in six of the nine cases reported.[4] The Helmholtz settings are also given with the fixed lights placed three meters forward from the eyes at $\phi = \pm 5.71°$. Here the calculated values generally predict much greater curvature in the geodesics than is found experimentally. For these observers, settings were obtained for eight forward positions of the fixed lights varying from 50 cm to 300 cm frontally. In five of these settings, varying from 50 cm to 139 cm frontally, the angle ϕ_0 is kept the same ($\phi_0 = 10°$) so that the fixed conditions in these settings are connected by iseikonic transformations. The remaining settings can also be compared if we entertain an additional hypothesis; that the form of $g(\gamma)$ depends only on the parameter $\gamma_0 = \min\gamma$ and that $g(\gamma_0)$ is a constant of the observer; hence from (3.25) that the functional dependence of the visual radial coordinate is in the form $r = r(\gamma - \gamma_0)$.

The method of analysis of the data is simple. Setting $r_0 = g(\gamma_0)$, $r_1 = g(\gamma_1)$, we have from (3.26),

$$\tanh r_0 \cos\phi_0 = \tanh r_1.$$

[15] W. H. Ittelson, "Binocular distorted rooms," *The Ames Demonstrations in Perception* (Princeton University Press, Princeton, New Jersey, 1952), p. 50–52.

*** Zajaczkowska has kindly supplied her complete data from these experiments for this analysis.

[16] V. Kries Helmholtz and Southall (translator,) *Physiological Optics* (Optical Society of America, Rochester, New York, 1925), Vol. 3, p. 318.

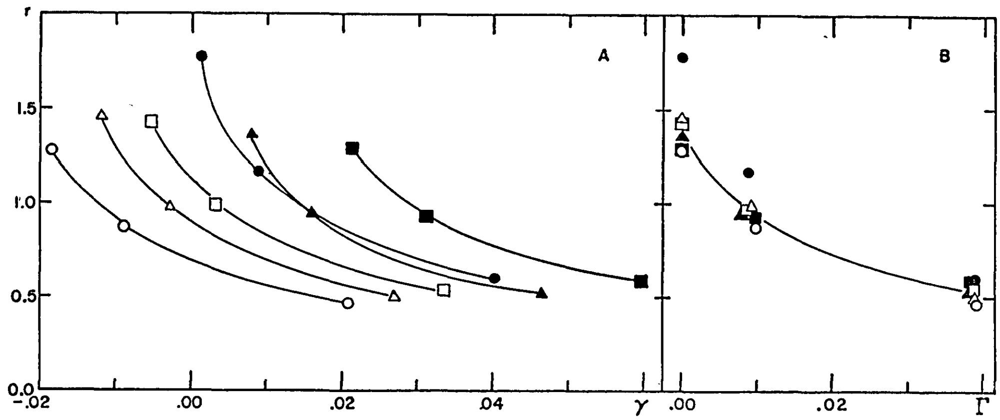

FIG. 6. Visual radial coordinate r as obtained from equipartitioned parallel alleys. (a) r *vs* convergence γ. (b) r *vs* convergence disparity, Γ. (Negative values of γ indicate divergence.)

To first order, it follows that

$$\tanh r_1 \sim \tanh r_0 + \frac{\gamma_1-\gamma_0}{\cosh^2 r_0}\left(\frac{dr}{d\gamma}\right)_{\gamma=\gamma_0}, \qquad (3.27)$$

and we obtain

$$\gamma_1-\gamma_0 \sim -\tfrac{1}{2}\sinh 2r_0(1-\cos\phi_0)\Big/\left(\frac{dr}{d\gamma}\right)_{\gamma=\gamma_0}. \qquad (3.28a)$$

If ϕ_0 is kept fixed then the hypothesis of (3.25) implies that $\gamma_1-\gamma_0$ remains constant. If $\rho=2e^{-\sigma\gamma}$, then the derivative in (3.27) may be obtained from (3.7). From (3.26) it then follows that the convergence disparity is given by

$$\gamma_1-\gamma_0 \sim \frac{1-\cos\phi_0}{\sigma}\cosh r_0. \qquad (3.28b)$$

Since under the Luneburg hypothesis r_0 increases with distance from the observer, σ is assumed constant, then $\gamma_1-\gamma_0$ should increase with distance when ϕ_0 is fixed.

In order to compare all eight settings we may examine the datum $(\gamma_1-\gamma_0)/(1-\cos\phi_0)$. This quantity should increase distally if $\rho=2e^{-\sigma\gamma}$ and should remain constant if $r=r(\gamma-\gamma_0)$. The results given in Table III show that as distance from the observer is increased the values first decrease in the proximal region and then maintain a fairly constant level. This performance runs directly counter to the hypothesis $\rho=2e^{-\sigma\gamma}$ and agrees with the suggested alternate only if the proximal region is omitted.

TABLE III. Values of $(\gamma_1-\gamma_0)/(1-\cos\phi_0)$ obtained from settings of Helmholtz geodesics by nine observers of Zajaczkowska.

	$\varphi_0=10°$					9.46°	7.37°	5.70°
x cm	50	65	83	108	139	180	232	300
A. J.	0.24	0.11	0.16	0.14	0.09	0.09	0.13	0.18
T. K.	0.24	0.19	0.14	0.14	0.11	0.09	0.09	0.08
M. K.	0.20	0.18	0.14	0.12	0.10	0.09	0.10	0.12
M. V. S.	0.21	0.14	0.13	0.11	0.09	0.09	0.09	0.09
B. A.	0.21	0.15	0.16	0.09	0.11	0.08	0.08	0.07
S. V. S.	0.29	0.23	0.19	0.22	0.16	0.14	0.14	0.15
H. S.	0.24	0.18	0.14	0.14	0.13	0.12	0.15	0.17
K. G.	0.23	0.15	0.18	0.14	0.11	0.11	0.12	0.11
W. K.	0.07	0.07	0.10	0.09	0.07	0.07	0.10	0.09

A clear showing in favor of the iseikonic transformations was first obtained at the Knapp Laboratory as the surprising consequence of a routine series of preliminary trials of a new experiment and a new piece of apparatus. These were trials of the equipartitioned parallel alleys conducted by C. J. Campbell and the author. In this experiment, parallel alleys consisting of three lights on a side are constructed. The middle light on each side is not constrained to move on a Vieth-Mueller circle but is set by the observer so that it appears to divide the segment between the near and far point into two equal parts. The trials were performed with approximately fixed convergence disparity between the near and far lights but with varying choices of the far point. The values of $r=g(\gamma)$ can be calculated by the same method as for the equipartitioned geodesics. The results are plotted in Fig. 6(a). Note that the data of the several experiments,††† with one exception, are fitted by curves of roughly the same length and shape which look very much like the same curve translated laterally. In fact by plotting each curve in terms of convergence disparity $\Gamma=\gamma-\gamma_0$ [Fig. 6(b)], a remarkable degree of correspondence is revealed. This initial observation remains a striking exhibition of the point in question.

It is of some interest to observe that Luneburg did not completely disregard the iseikonic transformation (3.25). His interpretation[10] of certain experiments[9(b)] was that the transformation (3.25) is conformal, that is, leaves visual angles unchanged. If this is coupled with the assumption that visual radial distance r is a function of convergence angle alone then Luneburg's conclusion of (3.3) is inescapable. If on the other hand, one accepts the principle that these transformations leave *all* visual relations unchanged, then all we may conclude is that convergence disparities are effective in depth perception rather than convergence itself and we are left with the empirical problem of determining the exact character of the radial distance function.

††† From a letter to Paul Boeder (December 20, 1951).

4. CONCLUSION

We have attempted here to present the problem of determining the visual transformation in such a way that the assumptions underlying the experimental techniques were made explicit. In examining various hypotheses concerning the visual transformation the experimental results themselves compel us to retain a generalized approach toward the problem of determining the visual transformation rather than to adopt, too early, some overly special formulation. In fact, the evidence we have examined here cannot be made to conform to the most commonly held hypothesis that visual radius is a function of convergence alone. On the other hand, none of the other specific hypotheses considered can quite account for all observations. However, it would be disappointing not to have some synthesis and the author, having shown that the mathematics leaves one free not to do so, will now make bold to iterate his own.

On the whole, the body of the evidence supports the belief that convergence disparity rather than convergence is the more significant parameter in the perception of depth. This is to be coupled with the observation for certain observers that the maximal visual radial distance in each visual configuration appears to have approximately the same value in a number of different experiments. These statements suggest that we may be able to utilize the functional form $r=r(\Gamma)$ where $\Gamma=\gamma-\gamma_0$ is the convergence disparity with respect to the minimum value, γ_0, of convergence angle among the points of the configuration.

Let us see how well this hypothesis is supported by the known results and what the discrepancies are.

The bulk of the empirical support for the hypothesis comes from the work of the Knapp Laboratory[3] which represents the most detailed investigation of the subject to date. (It is to be regretted that much of that work is unpublished.) We also find some further support in Zajaczkowska's work, and some discrepancies. We have seen that the "classic" and "intermediate" alleys yield approximately the same maximum value of r, that the Helmholtz geodesics experiments conform to the hypothesis over a considerable region. In the same work, however, we note that the "broad" alleys do not conform, that the results of the Helmholtz experiment do not follow the same pattern in the proximal region. It is no doubt significant that the discrepancies mentioned in this context and elsewhere seem to originate primarily in the proximal region and the lateral periphery of the binocular field. Furthermore all three, Zajaczkowska, Shipley, and the Knapp Laboratory have indicated their impressions of unresolved difficulties in these regions. The author believes that the proposed general form of the visual transformation (2.1) is not valid in those regions and, in particular, that γ may not be the appropriate parameter in those parts of the visual field. There is little question but that further exploratory work in the lateral periphery and the proximal region is needed for the complete study of the relevant factors. Aside from modifications involving "distortions" in those regions, however, we believe that the hypothesis $r=r(\Gamma)$ will bear the weight of the experimental evidence.

5. ACKNOWLEDGMENTS

I am grateful to Dr. Zajaczkowska and Dr. Shipley for reading an early version of this paper and giving me the benefit of their comments and certain unpublished data. I am also indebted to Arthur Linksz for calling my attention to their work and its implications.