COMPLEX QUANTITIES AND THEIR USE IN ELECTRICAL ENGINEERING.

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I.—INTRODUCTION.

In the following, I shall outline a method of calculating alternate current phenomena, which, I believe, differs from former methods essentially in so far, as it allows us to represent the alternate current, the sine-function of time, by a *constant* numerical quantity, and thereby eliminates the independent variable "time" altogether from the calculation of alternate current phenomena.

Herefrom results a considerable simplification of methods. Where before we had to deal with periodic functions of an independent variable, time, we have now to add, subtract, etc., constant quantities—a matter of elementary algebra—while problems like the discussion of circuits containing distributed capacity, which before involved the integration of differential equations containing *two* independent variables: "time" and "distance," are now reduced to a differential equation with *one* independent variable only, "distance," which can easily be integrated in its most general form.

Even the restriction to sine-waves, incident to this method, is no limitation, since we can reconstruct in the usual way the complex harmonic wave from its component sine-waves; though almost always the assumption of the alternate current as a true sine-wave is warranted by practical experience, and only under rather exceptional circumstances the higher harmonics become noticeable.

In the graphical treatment of alternate current phenomena different representations have been used. It is a remarkable fact, however, that the simplest graphical representation of

periodic functions, the common, well-known *polar coordinates*; with *time* as *angle* or *amplitude*, and the *instantaneous values* of the function as *radii vectores*, which has proved its usefulness through centuries in other branches of science, and which is known to every mechanical engineer from the Zeuner diagram of valve motions of the steam engine, and should consequently be known to every electrical engineer also, it is remarkable that this polar diagram has been utterly neglected, and even where it has been used, it has been misunderstood, and the sine-wave represented—instead of by one circle—by two circles, whereby the phase of the wave becomes indefinite, and hence the diagram
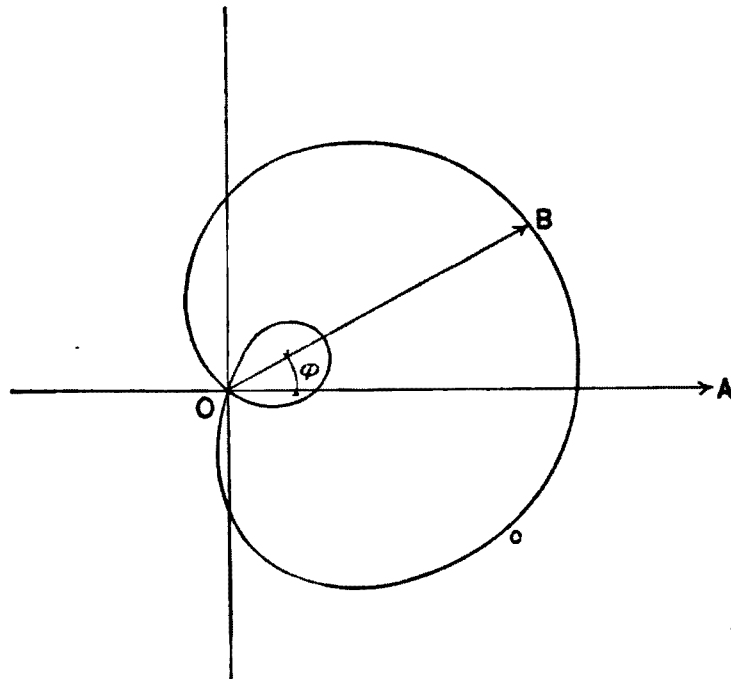

Fig. 1.

useless. In its place diagrams have been proposed, where revolving lines represent the instantaneous values by their projections upon a fixed line, etc., which diagrams evidently are not able to give as plain and intelligible a conception of the variation of instantaneous values, as a curve with the instantaneous values as radii, and the time as angle. It is easy to understand then, that graphical calculations of alternate current phenomena have found almost no entrance yet into the engineering practice.

In graphical representations of alternate currents, we shall make use, therefore, of the *Polar Coordinate System*, representing the *time* by the angle φ as amplitude, counting from an

initial radius $\overline{o\ a}$ chosen as zero time or starting point, in positive direction or counter-clockwise,* and representing the time of one complete period by one complete revolution or $360° = 2\ \pi$.

The instantaneous values of the periodic function are represented by the length of the radii vectores $\overline{o\ b} = r$, corresponding to the different angles φ or times t, and every periodic function is hereby represented by a closed curve (Fig. 1). At any time t, represented by angle or amplitude φ, the instantaneous value of the periodic function is cut out on the movable radius by its intersection $\overline{o\ b}$ with the characteristic curve c of the func-
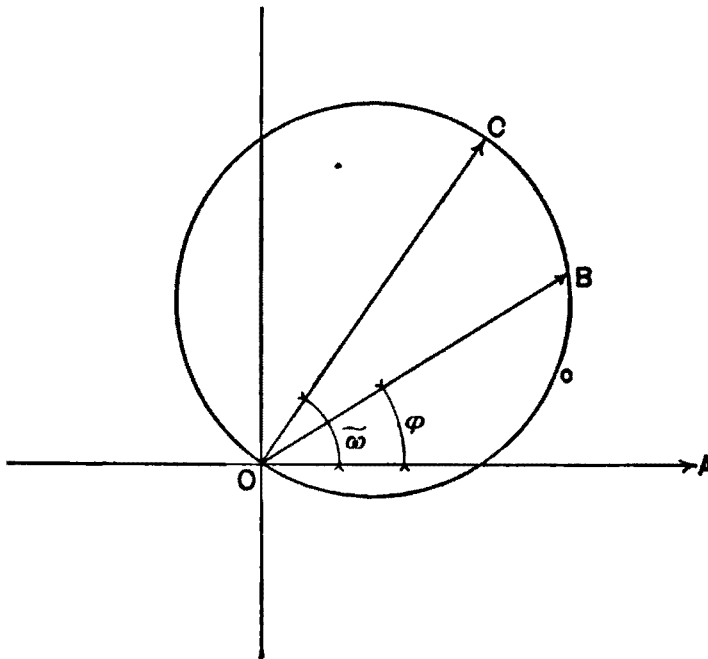

Fig. 2.

tion, and is positive, if in the direction of the radius, negative, if in opposition.

The *sine-wave* is represented by *one circle* (Fig. 2).

The diameter $\overline{o\ c}$ of the circle, which represents the sine-wave, is called the *intensity* of the sine-wave, and its amplitude, $A\ O\ B = \tilde{\omega}$, is called the *phase* of the sine-wave.

The sine-wave is completely determined and characterized by *intensity* and *phase*.

It is obvious, that the *phase* is of interest only as *difference of phase*, where several waves of different phases are under consideration.

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*This direction of rotation has been chosen as positive, since it is the direction of rotation of celestial bodies.

Where only the *integral values* of the sine-wave, and not its *instantaneous values* are required, the characteristic circle c of the sine-wave can be dropped, and its diameter $\overline{o\ c}$ considered as the representatation of the sine-wave in the polar-diagram, and in this case we can go a step further, and instead of using the *maximum value* of the wave as its representation, use the *effective value*, which in the sine wave is $= \dfrac{maximum\ value}{\sqrt{2}}$.

Where, however, the characteristic circle is drawn with the *effective value* as diameter, the instantaneous values, when taken from the diagram, have to be enlarged by $\sqrt{2}$.
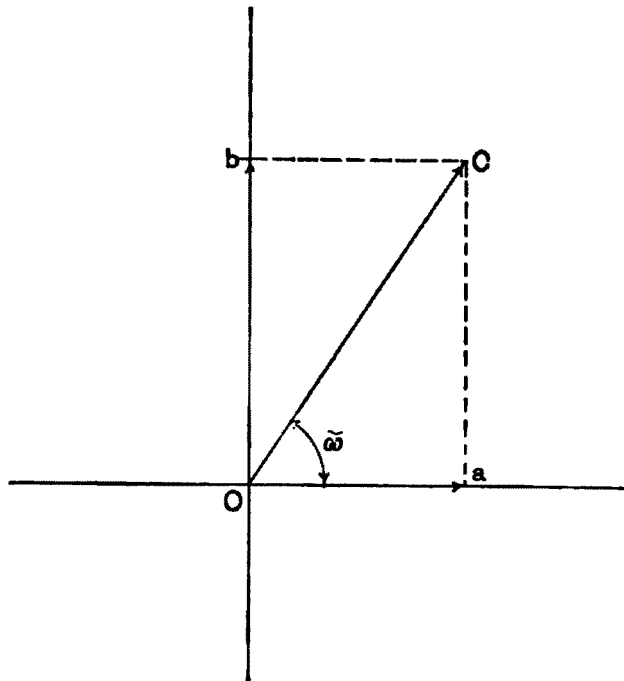

Fig. 3.

We see herefrom, that :

"*In polar coordinates, the sine-wave is represented in intensity and phase by a vector $\overline{o}$ c, and in combining or dissolving sine-waves, they are to be combined or dissolved by the parallelogram or polygon of sine-waves.*"

For the purpose of calculation, the sine-wave is represented by two constants : C, $\tilde{\omega}$, intensity and phase.

In this case the combination of sine-waves by the Law of Parallelogram, involves the use of trigonometric functions.

The sine-wave can be represented also by its *rectangular coordinates*, a and b (Fig. 3), where :

$$a = C \cos \tilde{\omega} \ \Big\}$$
$$b = C \sin \tilde{\omega} \ \Big\}$$

Here a and b are the two *rectangular components of the sine-wave.*

This representation of the sine-waves by their rectangular components a and b is very useful in so far as it avoids the use of trigonometric functions. To combine sine-waves, we have simply to add or subtract their rectangular components. For instance, if a and b are the rectangular components of one sine-wave, a^1 and b^1 those of another, the resultant or combined sine-wave has the rectangular components $a + a^1$ and $b + b^1$.

To distinguish the horizontal and the vertical components of sine-waves, so as not to mix them up in a calculation of any greater length, we may mark the ones, for instance, the vertical components, by a distinguishing index, as for instance, by the addition of the letter j, and may thus represent the sine-wave by the expression:

$$a + j\,b$$

which means, that a is the horizontal, b the vertical component of the sine-wave, and both are combined to the resultant wave:

$$C = \sqrt{a^2 + b^2}$$

which has the phase:

$$\tan \tilde{\omega} = \frac{b}{a}.$$

Analogous, $a - j\,b$ means a sine-wave with a as horizontal, and $-b$ as vertical component, etc.

For the first, j is nothing but a distinguishing index without numerical meaning.

A wave, differing in phase from the wave $a + j\,b$ by 180°, or one-half period, is represented in polar coordinates by a vector of opposite direction, hence denoted by the algebraic expression: $-a - j\,b$.

This means:

" *Multiplying the algebraic expression $a + j\,b$ of the sine-wave by* — 1, *means reversing the wave, or rotating it by* 180° = *one-half period.*

A wave of equal strength, but lagging 90° = one-quarter period behind $a + j\,b$, has the horizontal component — b, and

the vertical component a, hence is represented algebraically by the symbol:

$$j\, a - b.$$

Multiplying, however: $a + j\, b$ by j, we get:

$$j\, a + j^2\, b$$

hence, if we define the—until now meaningless—symbol j so, as to say, that:

$$j^2 = -1$$

hence: $j\,(a + j\, b) = j\, a - b,$

we have:

"*Multipling the algebraic expression $a + j\, b$ of the sine-wave by j, means rotating the wave by $90°$, or one-quarter period, that is, retarding the wave by one-quarter period.*"

In the same way:

"*Multiplying by $-j$, means advancing the wave by one-quarter period.*"

$$j^2 = -1 \text{ means}:$$

$$j = \sqrt{-1}, \text{ that is}:$$

"*j is the imaginary unit, and the sine-wave is represented by a complex imaginary quantity $a + j\, b$.*"

Herefrom we get the result :

"*In the polar diagram of time, the sine-wave is represented in intensity as well as phase by one complex quantity :*

$$a + j\, b,$$

where a is the horizontal, b the vertical component of the wave, the intensity is given by: $C = \sqrt{a^2 + b^2}$

and the phase by: $\tan \tilde{\omega} = \dfrac{b}{a},$

and it is: $a = C \cos \tilde{\omega}$

$$b = C \sin \tilde{\omega}$$

hence the wave: $a + j\, b$ can also be expressed by :

$$C\,(\cos \tilde{\omega} + j \sin \tilde{\omega})."$$

Since we have seen that sine-waves are combined by adding their rectangular components, we have :

"*Sine-waves are combined by adding their complex algebraic expressions.*"

For instance, the sine-waves:

$$a + j\, b$$

and $$a^1 + j\, b^1$$

combined give the wave:

$$A + j\,B = (a + a^1) + j\,(b + b^1).$$

As seen, the combination of sine-waves is reduced hereby to the elementary algebra of complex quantities.

If $C = c + j\,c^1$ is a sine-wave of alternate current, and r is the resistance, the E. M. F. consumed by the resistance is in phase with the current, and equal to current times resistance, hence it is:

$$r\,C = r\,c + j\,r\,c^1.$$

If L is the " coefficient of self-induction," or $s = 2\,\pi\,N\,L$ the " inductive resistance " or " ohmic inductance," which in the following shall be called the " inductance," the E. M. F. produced by the inductance (counter E. M. F. of self-induction) is equal to current times inductance, and lags 90° behind the current, hence it is represented by the algebraic expression:

$$j\,s\,C$$

and the E. M. F. required to overcome the inductance is consequently:

$$-j\,s\,C$$

that is, 90° ahead of the current (or, in the usual expression, the current lags 90° behind the E. M. F.).

Hence, the E. M. F. required to overcome the resistance r and the inductance s is:

$$(r - j\,s)\,C$$

that is:

" $I = r - j\,s$ is the expression of the impedance, in complex quantities, where r = resistance, $s = 2\,\pi\,N\,L$ = inductance."

Hence, if $C = c + j\,c^1$ is the current, the E. M. F. required to overcome the impedance $I = r - j\,s$ is:

$$E = I\,C = (r - j\,s)\,(c + j\,c^1), \text{ hence, since } j^2 = -1:$$
$$= (r\,c + s\,c^1) + j\,(r\,c^1 - s\,c)$$

or, if $E = e + j\,e^1$ is the impressed E. M. F., and $I = r - j\,s$ is the impedance, the current flowing through the circuit is:

$$C = \frac{E}{I} = \frac{e + j\,e^1}{r - j\,s}$$

or, multiplying numerator and denominator by $(r + j\,s)$, to eliminate the imaginary from the denominator:

$$C = \frac{(e + j\,e^1)\,(r + j\,s)}{r^2 + s^2} = \frac{e\,r - e^1\,s}{r^2 + s^2} + j\,\frac{e^1\,r + e\,s}{r^2 + s^2}$$

If K is the *capacity* of a condenser, connected in series into a circuit of current $C = c + j\, c^1$, the R. M. F. impressed upon the terminals of the condenser is $E = \dfrac{C}{2\,\pi\,N\,K}$, and lags 90° behind the current, hence represented by :

$$E = j\, \frac{C}{2\,\pi\,N\,K} = j\, k\, C,$$

where $k = \dfrac{1}{2\,\pi\,N\,K}$ can be called the "*capacity inductance*" or simply "*inductance*" of the condenser. Capacity inductance is of opposite sign to magnetic inductance.

That means:

"*If r = resistance*,

L = *coefficient of self-induction*, hence $s = 2\,\pi\,N\,L$ = *inductance*,

K = *capacity*, hence $k = \dfrac{1}{2\,\pi\,N\,K}$ = *capacity inductance*,

$I = r - j\, (s - k)$ *is the impedance of the circuit, and Ohm's law is re-established*:

$$E = I\, C,$$

$$C = \frac{E}{I},$$

$$I = \frac{E}{C},$$

in a more general form, however, giving not only the intensity, but also the phase of the sine-waves, by their expression in complex quantities."

In the following we shall outline the application of complex quantities to various problems of alternate and polyphase currents, and shall show that these complex quantities can be operated upon like ordinary algebraic numbers, so that for the solution of most of the problems of alternate and polyphase currents, elementary algebra is sufficient.

Algebraic operations with complex quantities:

$$j^2 = -1$$

$$a + j\, b = c\, (\cos \tilde{\omega} + j \sin \tilde{\omega})$$

$$c = \sqrt{a^2 + b^2},$$

$$\tan \tilde{\omega} = \frac{b}{a}.$$

If $a + j\,b = a^1 + j\,b^1$, it must be: $a = a^1$, $b = b^1$.

Addition and subtraction:

$$(a + j\,b) \pm (a^1 + j\,b^1) = (a \pm a^1) + j\,(b \pm b^1).$$

Multiplication:

$$(a + j\,b)\,(a^1 + j\,b^1) = (a\,a^1 - b\,b^1) + j\,(a\,b^1 + b\,a^1).$$

Division:

$$\frac{a + j\,b}{a^1 + j\,b^1} = \frac{(a + j\,b)\,(a^1 - j\,b^1)}{a^{1\,2} + b^{1\,2}} = \frac{a\,a^1 + b\,b^1}{a^{1\,2} + b^{1\,2}} + j\,\frac{a^1\,b - a\,b^1}{a^{1\,2} + b^{1\,2}}.$$

Difference of phase between:

$$a + j\,b = c\,(\cos \tilde{\omega} + j\,\sin \tilde{\omega})\ \text{and,}$$
$$a^1 + j\,b^1 = c^1\,(\cos \tilde{\omega}^1 + j\,\sin \tilde{\omega}^1):$$

$$\tan(\tilde{\omega}^1 - \tilde{\omega}) = \frac{\tan \tilde{\omega}^1 - \tan \tilde{\omega}}{1 + \tan \tilde{\omega} \tan \tilde{\omega}^1} = \frac{\dfrac{b^1}{a^1} - \dfrac{b}{a}}{1 + \dfrac{b\,b^1}{a\,a^1}} = \frac{a\,b^1 - b\,a^1}{a\,a^1 + b\,b^1}.$$

Multiplication by -1 means reversion, or rotation by $180° =$ one-half period.

Multiplication by j means rotation by $90°$, or retardation by one-quarter period.

Multiplication by $-j$ means rotation by $-90°$, or advance by one-quarter period.

Multiplication by $\cos \tilde{\omega} + j\,\sin \tilde{\omega}$ means rotation by angle $\tilde{\omega}$.

II. Circuits Containing Resistance, Inductance and Capacity.

Having now established Ohm's law as the fundamental law of alternate currents, in its complex form:

$$E = I\,C,$$

where it represents not only the *intensity*, but the *phase* of the electric quantities also, we can by simple application of Ohm's law —in the same way as in continuous current circuits, keeping in mind, however, that E, C, I are complex quantities—dissolve and calculate any alternate current circuit, or network of circuits, containing resistance, inductance, or capacity in any combination, without meeting with greater difficulties than are met with in continuous current circuits. Indeed, the continuous current distribution appears as a particular case of the general problem, characterized by the disappearance of all imaginary terms.

As an instance, we shall apply this method to an *inductive*

circuit, shunted by a condenser, and fed through inductive mains, upon which a constant alternate e. m. f. is impressed, as shown diagrammatically in Fig. 4.

Let r = resistance,

L = coefficient of self-induction, hence

$s = 2\ \pi\ N\ L$ = inductance, and :

$I = r - j\ s$ = impedance of consumer circuit.

Let r_1 = resistance of condenser leads,

K = capacity, hence

$k = \dfrac{1}{2\ \pi\ N\ K}$ = capacity inductance, and :

$I_1 = r_1 + j\ k$ = impedance of condenser circuit.

Let r_0 = resistance,

L_0 = coefficient of self-induction, hence
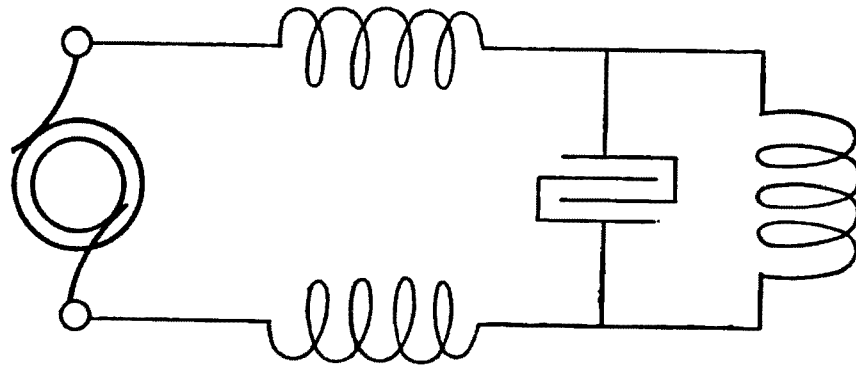
$s_0 = 2\ \pi\ N\ L_0$ = inductance, and:

Fig. 4.

$I_0 = r - j\ s_0$ = impedance of the two main leads.

Let E_0 = e. m. f. impressed upon the circuit.

We have then, if, E = e. m. f. at ends of main leads, or at terminals of consumer and condenser circuit :

Current in consumer circuit, $C = \dfrac{E}{I}$

Current in condenser circuit, $C_1 = \dfrac{E}{I_1}$

Hence, total current, $\qquad C_0 = C + C_1 = E\left(\dfrac{1}{I} + \dfrac{1}{I_1}\right)$

e. m. f. consumed in main leads $E^1 = C_0\ I_0 = E\left(\dfrac{I_0}{I} + \dfrac{I_0}{I_1}\right)$

Hence, total e. m. f. $\qquad E_0 = E + E^1 = E\left\{1 + \dfrac{I_0}{I} + \dfrac{I_0}{I_1}\right\}$

or, **E. M. F. at end of main leads,** $E = \dfrac{E_0\, I\, I_1}{I_0\, I + I_0\, I_1 + I\, I_1}$

E. M. F. consumed by main leads, $E^1 = \dfrac{E_0\, I_0\,(I + I_1)}{I_0\, I + I_0\, I_1 + I\, I_1}$

Current in consumer circuit, $C = \dfrac{E}{I} = \dfrac{E_0\, I_1}{I_0\, I + I_0\, I_1 + I\, I_1}$

Current in condenser circuit, $C_1 = \dfrac{E}{I_1} = \dfrac{E_0\, I}{I_0\, I + I_0\, I_1 + I\, I_1}$

Total current, $C_0 = C + C_1 = \dfrac{E_0\,(I + I_1)}{I_0\, I + I_0\, I_1 + I\, I_1}$

Substituting herein the values,

$$I_0 = r_0 - j\, s_0$$
$$I = r - j\, s$$
$$I_1 = r_1 + j\, k$$

and, $\qquad I_0\, I + I_0\, I_1 + I\, I_1 = a - j\, b,$

where, $\qquad a = r_0\, r + r_0\, r_1 + r\, r_1 - s_0\, s + s_0\, k + s\, k$

$\qquad\qquad b = s_0\, r + s_0\, r_1 + s\, r_1 + s\, r_0 - r_0\, k - r\, k$

we get

$$E = \frac{E_0}{a^2 + b^2}\Big\{[a(rr_1+sk)+b(r_1s-rk)]+j[b(rr_1+sk)-a(r_1s-rk)]\Big\}$$

$$C = \frac{E_0}{a^2 + b^2}\Big\{(r_1\, a - k\, b) + j\,(r_1\, b + k\, a)\Big\}$$

$$C_1 = \frac{E_0}{a_2 + b_2}\Big\{(r\, a + s\, b) + j\,(r\, b - s\, a)\Big\}$$

$$C_0 = \frac{E_0}{a^2 + b^2}\Big\{[(r + r_1)\, a + (s - k)\, b] + j[(r + r_1)b - (s - k)a]\Big\}$$

As an instance, we may consider the case:

$$E_0 = 100 \text{ volts,}$$

$r_0 = 1$ ohm	$r = 2$ ohm	$r_1 = 0$
$s_0 = 10$ ohm	$s = 10$ ohm	$k = 20$ ohm
$I_0 = 1 - 10\, j$	$I = 2 - 10\, j$	$I_1 = 20\, j$

hence $\qquad\qquad a = 302$
$\qquad\qquad\qquad b = -30$

Substituting these values, we get,

$$E_0 = 100,$$
$$E = 68.0\,(.98 + .17\, j),$$
$$E^1 = 35.1\,(.94 - .34\, j),$$
$$C = 6.6\,(.10 + .99\, j),$$

$$C_1 = 3.4\,(.17 - .98\,j),$$
$$C_0 = 3.4\,(.37 + .93\,j),$$

where the complex quantities are represented in the form $c\,(\cos \tilde{\omega} + j \sin \tilde{\omega})$, so that the numerical value in front of the parenthesis gives the *effective intensity*, the parenthesis gives the *phase* of the alternate current or E. M. F.

This means: Of the 100 volts impressed, 35.1 volts are consumed by the leads, and 68.0 volts left at the end of the line.

The main current of 3.4 amperes divides into the consumer current of 6.6 amperes, and the condenser current of 3.4 amperes.

Increasing, however, the capacity K, that is reducing the capacity inductance to $k = 10$, or $I_1 = 10\,j$, we get:

$$a = 102,$$
$$b = 0.$$

Hence:
$$E_0 = 100,$$
$$E = 100\,(.98 + .20\,j),$$
$$E^1 = 19.9\,(.10 - .99\,j),$$
$$C = 9.8\,j,$$
$$C_1 = 10\,(.20 - .98\,j),$$
$$C_0 = 1.98.$$

Here, though the leads consume 19.9 volts, still the full potential of 100 volts is left at their end.

1.98 amperes in the main line divide into two branch currents, of 9.8 and of 10 amperes. We have here one of the frequent cases, where one alternate current divides into two branches, so that either branch current is larger than the undivided or total current.

Increasing the capacity still further to $k = 5$, or $I_1 = 5\,j$, gives:

$$a = 2,$$
$$b = 15.$$

Hence:
$$E_0 = 100,$$
$$E = 337\,(.32 - .95\,j),$$
$$E^1 = 318\,(- .03 + j),$$
$$C = 33.0\,(.99 + 13\,j),$$
$$C_1 = 96.3\,(1 + .06\,j).$$
$$C_0 = 63.6\,(1 + .03\,j),$$

That means, in the leads self-induction consumes an E. M. F. of 318 volts, and still 337 volts exist at the end of the line, giving

a rise of potential in the leads of 237 volts, due to the combined effect of self-induction and capacity.

The main current of 63.6 amperes divides into the two branch currents of 33.0 and 96.3 amperes.

The current which passes over the line is far larger than the current which in the absence of capacity would be permitted by the dead resistance of the line. While in this case 63.6 amperes flow over the line, a continuous E. M. F. of 100 volts would send only $\dfrac{E_0}{r_0 + r} = 33.3$ amperes over the line; and with an alternating E. M. F., but without capacity the current would be limited to 4.95 amperes only, since in this case :

$$C_0 = \frac{E_0}{(r_0 + r) - j\,(s_0 + s)} = \frac{100}{3 - 20\,j} = 4.95\,(.15 + .99\,j).$$

Even by short-circuiting the line, we get only :

$$C_0 = \frac{E_0}{r_0 - j\,s_0} = \frac{100}{1 - 10\,j} = 10\,(.1 + .99\,j),$$

or 10 amperes over the line.

Hence we have in this arrangement of a condenser shunted to the inductive circuit and fed by inductive mains, the curious result that a short-circuit at the terminals of the consumer circuit reduces the line current to about one-sixth.

As a further instance, we may consider the problem :

" What is the maximum power which can be transmitted over an inductive line into a non-inductive resistance, as lights, and how far can this output be increased by the use of shunted capacity."

Let, r_0 = resistance,
                         s_0 = inductance,
hence, $I_0 = r_0 - j\,s_0$ = impedance of the line.

Let r = resistance of the consumer circuit, which is shunted by the capacity inductance k.

r and k are to be determined as to make the power in the receiving circuit : $C^2\,r$, a maximum.

In a *continuous current circuit* the maximum output is reached, if $r = r_0$, or $E = \dfrac{E_0}{2}$, where E_0 is the E. M. F. at the beginning, E the E. M. F. at the end of the line, and $C = \dfrac{E_0}{2\,r_0}$, hence :

$$P = \frac{E_0^2}{4\,r_0}$$ the maximum output at efficiency 50 per cent.

Hence, if $E_0 = 100$, $r_0 = 1$, it is: $P = 2,500$ watts.

Very much less is the maximum output of an *alternate current circuit*. With an alternate E. M. F. E_0, but without the use of a condenser, the impedance of the whole circuit is:

$$I = r_0 + r - j\,s_0,$$

hence the current: $C = \dfrac{E_0}{I} = \dfrac{E_0\,(r_0 + r - j\,s_0)}{(r_0 + r)^2 + s_0^2}$

$$= \frac{E_0}{\sqrt{(r_0 + r)^2 + s_0^2}} \left\{ \frac{r_0 + r}{\sqrt{(r_0 + r)^2 + s_0^2}} + j\,\frac{s_0}{\sqrt{(r_0 + r)^2 + s_0^2}} \right\}$$

the E. M. F. at end of line :

$$E = Cr = \frac{E_0\,r}{\sqrt{(r_0 + r)^2 + s_0^2}} \left\{ \frac{r_0 + r}{\sqrt{(r_0 + r)^2 + s_0^2}} + j\,\frac{s_0}{\sqrt{(r_0 + r)^2 + s_0^2}} \right\},$$

hence the power: $P = EC = \dfrac{E_0^2\,r}{(r_0 + r)^2 + s_0^2}.$

The condition of maximum output is,

$$\frac{\delta P}{\delta r} = 0$$

that is,

$$0 = (r + r_0)^2 + s_0^2 - 2\,r\,(r + r_0), \text{ or,}$$
$$r_2 = r_0^2 + s_0^2,$$
$$r = \sqrt{r_0^2 + s_0^2},$$

and the maximum output is,

$$P = \frac{E_0^2}{2\,\{r_0 + \sqrt{r_0^2 + s_0^2}\}}$$

at the efficiency, $\dfrac{r}{r_0 + r} = \dfrac{\sqrt{r_0^2 + s_0^2}}{r_0 + \sqrt{r_0^2 + s_0^2}}$

In the instance, $E_0 = 100$, $r_0 = 1$, $s_0 = 10$ is:

$P = 453$ watts, against 2,500 watts with continuous currents.

If, however, we shunt the receiver circuit by capacity inductance k, we have,

Leads, $I_0 = r_0 - j\,s_0,$

Consumer circuit, $I = r, s = 0,$

Condenser circuit, $I_1 = j\,k, r_1 = 0,$

hence, by substituting in the equations derived in the first part of this chapter,

$$a = r_0\, r + s_0\, k$$

$$b = s_0\, r - k\, (r_0 + r)$$

and, $\qquad C = \dfrac{E_0\, k}{a^2 + b^2}\, (-b + j\, a),$

or, substituting,

$$\tan \tilde{\omega} = -\frac{a}{b}$$

and, $\qquad C = \dfrac{E_0\, k}{\sqrt{a^2 + b^2}}\, (\cos \tilde{\omega} + j \sin \tilde{\omega})$

$$E = C\, r = \frac{E_0\, k\, r}{\sqrt{a^2 + b^2}}\, (\cos \tilde{\omega} + j \sin \tilde{\omega})$$

hence, power,

$$P = C\, E = \frac{E_0^2\, k^2\, r}{a^2 + b^2} = \frac{E_0^2\, k^2\, r}{(r_0\, r + s_0\, k)^2 + (s_0\, r - k\, r_0 - k\, r)^2}$$

The condition of the maximum output P is,

$$\frac{\delta P}{\delta r} = 0, \frac{\delta P}{\delta k} = 0$$

that is,

$$k^2\, (r_0^2 + s_0^2) = r^2\, (r_0^2 + [s_0 - k]^2)\ k\, s_0 = r_0^2 + s_0^2$$

hence,

$$k = \frac{r_0^2 + s_0^2}{s_0}$$

$$r = \frac{r_0^2 + s_0^2}{r_0}$$

substituting this in P, we get:

$$P = \frac{E_0^2}{4\, r_0},$$

the same condition as for continuous current.

That means,

"*No matter how large the self-induction of an alternating current circuit is, by a proper use of shunted capacity the output of the circuit can always be raised to the same as for continuous currents; that is, the effect of self-induction upon the output can entirely and completely be annihilated.*"

III. The Alternate Current Transformer.

A. General Remarks.

In the coils of an alternate current transformer, E. M. F. is induced by the alternations of the magnetism, which is produced by the combined magnetizing effect of primary and secondary current.

If, M = maximum magnetism,

N = frequency (complete cycles per second),

n = number of turns,

the *effective intensity* of the induced E. M. F. is,

$$E = \sqrt{2}\, \pi\, n\, N\, M\, 10^{-8}$$
$$= 4.44\, n\, N\, M\, 10^{-8}$$

Hence, if E. M. F., frequency and number of turns are given, or chosen, this formula gives the *maximum magnetism*,

$$M = \frac{E\, 10^8}{\sqrt{2}\, \pi\, n\, N}$$

To produce the magnetism M of the transformer, a M. M. F. F is required, which is determined from the shape and the magnetic characteristic of the iron, in the usual manner.

At no load, or open secondary circuit, the M. M. F. F is furnished by the "exciting current," improperly called the "leakage current."

The energy of this current is the energy consumed by hysteresis and eddy-currents in the iron; its intensity represents the M. M. F.

This current is *not a sine-wave*, but is *distorted by hysteresis*. It reaches its maximum together with the maximum of magnetism, but passes through zero long before the magnetism.

This exciting current can be dissolved in two components: a *sine-wave C_{oo} of equal intensity and equal power with the exciting current*, and a *wattless complex higher harmonic*.

Practically this separation is made by the electro-dynamometer. Connecting ammeter, voltmeter and wattmeter into the primary of an alternate current transformer, at open secondary circuit the instrument readings give the current C_{oo} in intensity and phase, but suppress the higher harmonics.

In Fig. 5 such a wave is shown in rectangular coordinates. The sine-wave of magnetism is represented by the dotted curve M, the exciting current by the distorted curve c, which is separated into the sine-wave C_{oo} and the higher harmonic C.

As seen, the higher harmonic is small, even in a closed circuit transformer, compared with the exciting current C_{oo}, and since C_{oo} itself is only a few per cent. of the whole primary current, the higher harmonic can for all practical purposes be suppressed.

All tests made on transformers by electro-dynamometer methods suppress the higher harmonic anyway.

Representing the exciting current by a sine-wave C_{oo} of equal effective intensity and equal power with the distorted wave, the exciting current is advanced in phase against the magnetism by an angle *a*, which may be called the "*angle of hysteretic advance of phase*." This angle *a* is very small in all open circuit transformers, but may be as large as 40° to 50° in closed circuit transformers.
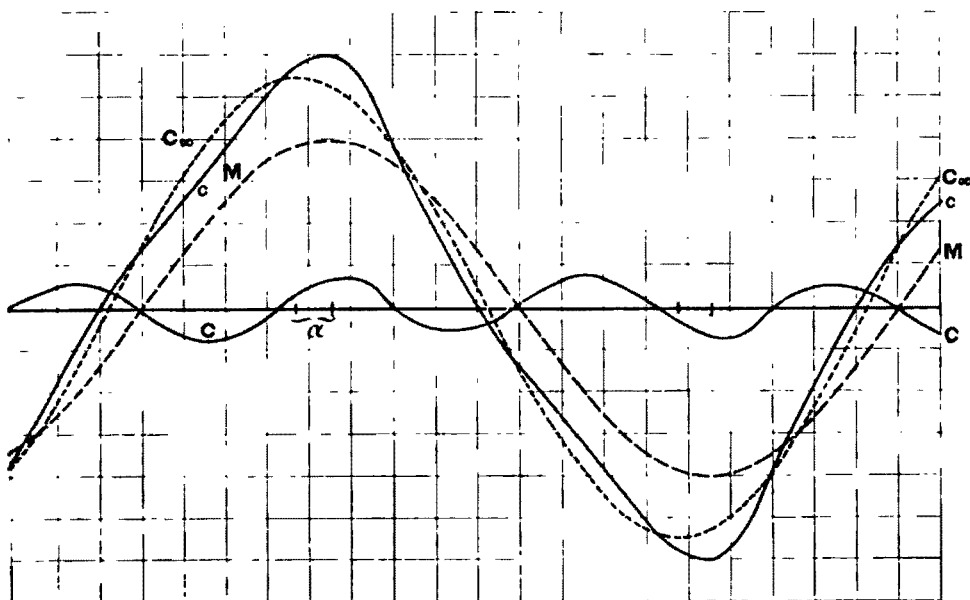

Fig. 5.

We can now in the usual manner dissolve the sine-wave of exciting current C_{oo} into its two rectangular components:

h, the "*hysteretic energy current*" at right angles with the magnetism, hence in phase with the induced E. M. F., and, therefore *representing consumption of energy;* and,

g, the "*magnetizing current*" in phase with the magnetism, hence at right angles with the induced E. M. F., and, therefore, *wattless.*

$h = C_{oo} \sin a$, and can be calculated from the loss of energy by hysteresis (and eddies), for it is:

$$h = \frac{\text{energy consumed by hysteresis}}{\text{primary E. M. F.}},$$

And since C_{oo} can be calculated from shape and characteristic of the iron, the angle of hysteretic advance of phase a is given by:

$$\sin a = \frac{h}{C_{oo}}.$$

The magnetizing current $g = C_{oo} \cos a$ does not consume energy (except by resistance), and can be supplied by a condenser of suitable capacity shunted to the transformer.

Since in the closed circuit transformer h, which cannot be supplied by a condenser, is not much smaller than C_{oo}, there is no advantage in using a condenser on a closed circuit transformer. In an open circuit transformer, however, or transformer motor, C_{oo} is very much larger than h, and a condenser may be of advantage in reducing the exciting current from C_{oo} to h.

B.—The Closed Circuit Lighting Transformer.

The alternate current transformer with closed magnetic circuit, when feeding into a non-inductive resistance, as lights, can be characterized by four constants :

ρ = resistance loss as fraction of the total transformed power:

$$\rho = \frac{\text{resistance loss}}{\text{total power}} \text{ at full load.}$$

ε = hysteretic loss as fraction of the total transformed power:

$$\varepsilon = \frac{\text{hysteretic loss}}{\text{total power}} \text{ at full load.}$$

σ = E. M. F. of self-induction as fraction of total E. M. F.:

$$\sigma = \frac{\text{self-induction}}{\text{total E. M. F.}} \text{ at full load.}$$

τ = magnetizing current as fraction of total current:

$$\tau = \frac{\text{magnetizing current}}{\text{total current}} \text{ at full load.}$$

Denoting

In primary:		In secondary :	
n_o	and	n_1	= number of turns.
r_o	"	r_1	= resistances.
C_o	"	C_1	= currents.
E_o	"	E_1	= induced E. M. F.'s.
E	"	E_t	= E. M. F.'s at terminals.
C_o^1	"	C_1^1	= currents at full load.

we have then :

$$\rho = \frac{C_1^1 \, r_1}{E_1} + \frac{C_o^1 \, r_o}{E_o},$$

$$\sigma = \frac{S}{M}, \text{ where}$$

S = magnetism leaking between primary and secondary.
M = magnetism surrounding both primary and secondary.

$$\varepsilon = \frac{h}{C_0{}^1},$$

$$\tau = \frac{g}{C_0{}^1}.$$

Hence at the fraction ϑ of full lead.

$$\vartheta = \frac{C_1}{C_1{}^1}$$

choosing the induced E. M. F. as the real axis of coordinates,
the magnetism as the imaginary axis of coordinates), we have,

Primary exciting current, $\quad C_{00} = h + j\,g$

Primary current corresponding to secondary current C_1, $\quad\Big\}\ C\ = \frac{n_1}{n_0}\,C_1$

hence, total primary current, $\quad C_0 = C + C_{00} = \frac{n_1}{n_0}C_1 + h + j\,g$

and, ratio of currents, $\quad \dfrac{C_0}{C_1} = \dfrac{n_1}{n_0} + \dfrac{h + j\,g}{C_1}$

Since, however,

$$C_{00} = h + j\,g = C_0{}^1\,(\varepsilon + j\,\tau) = \frac{n_1}{n_0}\,C_1{}^1\,(\varepsilon + j\,\tau) = \frac{n_1}{n_0}\,C_1\,\frac{\varepsilon + j\,\tau}{\vartheta},$$

we have, substituted,

Ratio of Currents.

$$\frac{C_0}{C_1} = \frac{n_1}{n_0}\left\{ 1 + \frac{\varepsilon}{\vartheta} + j\,\frac{\tau}{\vartheta}\right\}$$

$$= \frac{n_1}{n_0}\sqrt{\left(1 + \frac{\varepsilon}{\vartheta}\right)^2 + \left(\frac{\tau}{\vartheta}\right)^2}\ \text{ or, for medium and large load,}$$

$$= \frac{n_1}{n_0}\left\{ 1 + \frac{\varepsilon}{\vartheta} + \frac{\tau^2}{2\,\vartheta^2}\right\}$$

The E. M. F. at the secondary terminals is,

$$E_t = E_1 - C_1\,r_1 = E_1\left\{ 1 - \rho\,\frac{\vartheta}{2}\right\}$$

at the primary terminals,

$$E = E_0 + C_0\,(r_0 - j\,s_0) = E_0\left\{ 1 + \rho\,\frac{\vartheta}{2} - j\,\sigma\,\vartheta\right\}$$

hence, since $\qquad\qquad E_0 = \dfrac{\vartheta_0}{n_1}\, E_1$,

Ratio of e. m. f.'s *at terminals.*

$$\frac{E}{E_t} = \frac{n_0}{n_1} \left\{ 1 + \rho\,\vartheta - j\,\sigma\,\vartheta \right\}$$

$$= \frac{n_0}{n_1} \left\{ 1 + \rho\,\vartheta + \frac{\sigma_2\,\vartheta^2}{2} \right\}$$

Difference of phase $\bar{\omega}$ between e. m. f. at primary terminals and primary currents.

Since we have seen, that multiplying a complex quantity by $(\cos \bar{\omega} + j \sin \bar{\omega})$, means rotating its vector by angle $\bar{\omega}$, the difference of phase between primary current and e. m. f., $\bar{\omega}$ is, given by,

$$C_0 = a\,(\cos \bar{\omega} + j \sin \bar{\omega})\, E$$

or, $\qquad\qquad a\,(\cos \bar{\omega} + j \sin \bar{\omega}) = \dfrac{C_0}{E}$

where $\bar{\omega}$ is the difference of phase, and a a constant.

Since in the present case the secondary current is in phase with the secondary e. m. f., it is, $b = \dfrac{E_t}{C_1}$

combining this with the foregoing, we have,

$$a\,b\,(\cos \bar{\omega} + j \sin \bar{\omega}) = \frac{C_0}{C_1}\,\frac{E_t}{E}$$

$$= \left(\frac{n_1}{n_0}\right)^2 \frac{1 + \dfrac{\varepsilon}{\vartheta} + j\,\dfrac{\tau}{\vartheta}}{1 + \rho\,\vartheta - j\,\sigma\,\vartheta},$$

$$= \left(\frac{n_1}{n_0}\right)^2 \left\{ 1 - \rho\,\vartheta + \frac{\varepsilon}{\vartheta} + j\,\sigma\,\vartheta + j\,\frac{\tau}{\vartheta} \right\}$$

hence,

$$a\,b\,\cos \bar{\omega} = \left(\frac{n_1}{n_0}\right)^2 \left(1 - \rho\,\vartheta + \frac{\varepsilon}{\vartheta}\right) = \left(\frac{n_1}{n_0}\right)^2$$

$$a\,b\,\sin \bar{\omega} = \left(\frac{n_1}{n_0}\right)^2 \left(\sigma\,\vartheta + \frac{\tau}{\vartheta}\right)$$

and,

Difference of phase between primary current and e. m. f. *at terminals.*

$$\tan \omega = \sigma\,\vartheta + \frac{\tau}{\vartheta}$$

hence:

" *With varying load ϑ, the difference of phase $\bar{\omega}$ or the lag, first decreases, reaches a minimum at $\sigma\,\vartheta = \dfrac{\tau}{\vartheta}$ or $\vartheta = \sqrt{\dfrac{\tau}{\sigma}}$, and afterwards increases again.*"

At light loads it is mainly the magnetizing current τ, at large load the self-induction σ, which determine the lag.

The formula, $\tan \bar{\omega} = \sigma\,\vartheta + \dfrac{\tau}{\vartheta}$ is only an approximation, and ceases to hold for any light load, where we have to use the complete expression.

$$\tan \omega = \frac{\sigma\,\vartheta + \dfrac{\tau}{\vartheta}}{1 - \rho_1\,\vartheta + \dfrac{\varepsilon}{\vartheta}}$$

The *efficiency* is, $1 - \left(\rho\,\vartheta + \dfrac{\varepsilon}{\vartheta}\right)$, and the

Loss coefficient, $\rho\,\vartheta + \dfrac{\varepsilon}{\vartheta}$

hence a minimum at, $\vartheta = \sqrt{\dfrac{\varepsilon}{\rho}}$, the point of maximum efficiency.

Let, as an instance, be:

$$\frac{n_0}{n_1} = 10 \qquad \rho = .02 \qquad \varepsilon = .03$$
$$\sigma = .06 \qquad \tau = .08$$

hence,
at full load, $\vartheta = 1$,

$$\frac{C_0}{C_1} = .1\,(1 + .03 + .0032) = .1033$$

$$\frac{E_0}{E_t} = 10\,(1 + .02 + .0018) = 10.22$$

$\tan \omega = .06 + .08 = .14$, or, $\omega = 8°$,

energy factor, $\cos \omega = .99$

at 100% overload, $\vartheta = 2$,

$$\frac{C_0}{C_1} = .1\,(1 + .015 + .0008) = .1016$$

$$\frac{E_0}{E_t} = 10\,(1 + .04 + .0072) = 10.47$$

$$\tan \omega = .12 + .04 = .16, \text{ or, } \omega \; 9°,$$

energy factor, $\cos \omega = .99$

at one-half load:

$$\vartheta = .5: \quad \frac{C_0}{C_1} = .1 \; (1 + .06 + .0128) = .1073.$$

$$\frac{E_0}{E_t} = 10 \; (1 + .01 + .0005) = 10.11.$$

$\tan \tilde{\omega} = .03 + .16 = .19$, or $\tilde{\omega} = 11°$, energy factor: $\cos \tilde{\omega} = .98$.
at one-tenth load:

$$\frac{C_0}{C_1} = 1 \; (1 + .3 + .32) = .162.$$

or more exactly,

$$= .1 \; \sqrt{(1 + .3)^2 + .8^2} = .153.$$

$$\frac{E_0}{E_t} = 10 \; (1 + .002 + .0000) = 10.02.$$

$$\tan \dot{\tilde{\omega}} = .006 + .8 = .806.$$

or more exactly,

$$= \frac{.006 + .8}{1 - .002 + .3} = .62, \text{ or } \tilde{\omega} = 32°, \text{ energy factor: } \cos \tilde{\omega} = .85,$$

at open secondary:

$$\tan \tilde{\omega} = \frac{.08}{.03} = 2.67, \text{ or } \tilde{\omega} = 70°, \text{ energy factor: } \cos \tilde{\omega} = .34,$$

the minimum lag takes place at:

$$\vartheta = \sqrt{\frac{.08}{.06}} = 1.155,$$

or $15\frac{1}{2}$ per cent. overload, and is:
$\tan \tilde{\omega} = .0693 + .0693 = .1386$, or $\tilde{\omega} = 7.9°$, energy factor: $\cos \tilde{\omega} = .99$,
the efficiency is a maximum at:

$$\dot{\vartheta} = \sqrt{\frac{.03}{.02}} = 1.225.$$

or $22\frac{1}{2}$ per cent. overload, and is:

$$1 - .0245 - .0245 = .951, \text{ or } 95.1 \text{ per cent.}$$

C.—General Equations of Alternate Current Transformer.

The foregoing considerations will apply strictly only to the closed circuit transformer, where ρ, σ^2, ϵ, τ^2 are so small that their

products and higher powers may be neglected when feeding into a non-inductive resistance.

The open circuit transformer, and in general the transformer feeding into an inductive circuit—in which case σ and τ become of greatly increased importance—requires a fuller consideration. Let:

n_0 and n_1 = number of turns,

r_0 " r_1 = resistance,

$s_0 = 2\pi\, N\, L_0$ and $s_1 = 2\,\pi\, N\, L_1$ = self-inductances, hence:

$I_0 = r_0 - j\, s_0$ and $I_1 = r_1 - j\, s_1$ = impedances of the two transformer coils.

The secondary terminals may be connected to a circuit of resistance R and inductance S, hence of impedance $I = R - j\, S$.

Then we have:

Magnetism: $j\, M.$

Secondary induced **E. M. F.**: $E_1 = \sqrt{2}\ \pi\, n_1\, N\, M\, 10^{-\cdot}.$

Primary induced **E. M. F.**: $E_0 = \sqrt{2}\ \pi\, n_0\, N\, M\, 10^{-8}\, \dfrac{n_0}{n_1}\, E_1.$

Secondary current: $\quad C_1 = \dfrac{E_1}{I + I_1} = \dfrac{E_1}{(R + r_1) - j(S + s_1)}$

or: $\quad C_1 = a + j\, b,$

where:

$$a = \frac{E_1(R + r_1)}{(R + r_1)^2 + (S + s_1)^2}, \qquad b = \frac{E_1(S + s_1)}{(R + r_1)^2 + (S + s_2)^2}.$$

Primary current corresponding hereto:

$$C = \frac{n_1}{n_0}\, C_1 = \frac{n_1}{n_0}\, a + j\, \frac{n_1}{n_0}\, b.$$

Primary exciting current:

$$C_{00} = h + j\, g,$$

hence, total primary current:

$$C_0 = C + C_{00} = \left(\frac{n_1}{n_0}\, a + h\right) + j\left(\frac{n_1}{n_0}\, b + g\right)$$

or: $\quad C_0 = c + j\, d,$

where:

$$c = \frac{\frac{n_1}{n_0}\, E_1(R + r_1)}{(R + r_1)^2 + (S + s_1)^2} + h, \quad d = \frac{\frac{n_1}{n_0}\, E_1(S + s_1)}{(R + r_1)^2 + (S + s_1)^2} + g,$$

herefrom we get:

E. M. F. consumed by primary impedance:

$$C_0 \, I_0 = (c + j \, d) \, (r_0 - j \, s_0)$$
$$= (c \, r_0 + d \, s_0) + j \, (d \, r_0 - c \, s_0).$$

E. M. F. consumed by secondary impedance:

$$C_1 \, I_1 = (a + j \, b) \, (r_1 - j \, s_1)$$
$$= (a \, r_1 + b \, s_1) + j \, (b \, r_1 - a \, s_1).$$

hence, E. M. F. at secondary terminals:

$$E_t = E_1 - C_1 \, I_1 = E_1 \left\{ 1 - \frac{(a \, r_1 + b \, s_1) + j \, (b \, r_1 - a \, s_1)}{E_1} \right\}.$$

E. M. F. at primary terminals:

$$E = E_0 + C_0 \, I_0 = E \left\{ 1 + \frac{(c \, r_0 + d \, s_0) + j \, (d \, r_0 - c \, s_0)}{E} \right\}.$$

Substituting now in C_1, C_0, E_t, E the values of $a, b, c, d,$ we get:

Secondary current:

$$C_1 = \frac{E_1 \, (R + r_1)}{(R + r_1)^2 + (S + s_1)^2} + j \, \frac{E_1 \, (S + s_1)}{(R + r_1)^2 + (S + s_1)^2}.$$

Primary current:

$$C_0 = \left\{ \frac{\frac{n_1}{n_0} E_1 (R + r_1)}{(R + r_1)^2 + (S + s_1)} + h \right\} + j \left\{ \frac{\frac{n_1}{n_0} E_1 (S + s_1)}{(R + r_1)^2 + S + s_1)^2} + g \right\}.$$

E. M. F. *at secondary terminals,*

$$E_t = E_1 \left\{ 1 - \frac{r_1 (R + r_1) + s_1 (S + s_1)}{(R + r_1)^2 + (S + s_1)^2} \right\} - j E_1 \left\{ \frac{S r_1 - R s_1}{(R + r_1)^2 + (S + s_1)^2} \right\}$$

E. M. F. *at primary terminals,*

$$E = \left[\frac{n_0}{n_1} E_1 \left\{ 1 + \left(\frac{n_1}{n_0} \right)^2 \frac{r_0 (R + r_1) + s_0 (S + s_1)}{(R + r_1)^2 + (S + s_1)^2} \right\} + (r_0 h + s_0 g) \right]$$

$$+ j \left[\frac{n_1}{n_0} E_1 \left\{ \frac{r_0 (S + s_1) - s_0 (R + r_1)}{(R + r_1)^2 + (S + s_1)^2} \right\} + (r_0 \, g - s_0 \, h) \right]$$

the general equations of the alternate current transformer, representing the currents and E. M. F.*'s in intensity and phase.*

In general, the percentage of resistance in inductance will be the same, or can without noticeable error be assumed the same in primary as in secondary circuit.

That means,

$$r_0 = \left(\frac{n_0}{n_1} \right)^2 r_1 \ , \ s_0 = \left(\frac{n_0}{n_1} \right)^2 s_1$$

substituting this, we get,

R. M. F. *at secondary terminals,*

$$E_t = E_1 [1 - A - j B]$$

E. M. F. *at primary terminals,*

$$E = \frac{n_0}{n_1} E_1 \left\{ 1 + A + j B \right\} + (r_0 h + s_0 g) + j (r_0 g - s_0 h)$$

where,

$$A = \frac{r_1 (R + r_1) + s_1 (S + s_1)}{(R + r_1)^2 + (S + s_1)^2}$$

$$B = \frac{r_1 S - s_1 R}{(R + r_1)^2 + (S + s_1)^2}$$

Therefore we get for the closed circuit transformer, feeding into a non-inductive resistance, $S = O$.

$$\frac{E}{E_t} = \frac{n_0}{n_1} \left\{ 1 + \rho + \frac{\sigma^2}{2} \right\}$$

$$\frac{C_0}{C_1} = \frac{n_1}{n} \left\{ 1 + \varepsilon + \frac{\tau^2}{2} + \frac{\tau \sigma}{2} \right\}$$

at full load.

IV. Distributed Capacity, Inductance, Leakage and Resistance.

In many cases, especially in long circuits, as lines conveying alternate power currents at high potentials over long distances by overhead conductors or underground cables, or very feeble currents at extremely high frequency, as telephone currents, the consideration of the *resistance*—which consumes E. M. F. in phase with the current—and of the *inductance* —which consumes E. M. F. in quadrature with the current—is not sufficient for the explanation of the phenomena taking place in the line, but several other factors have to be taken into account.

In long lines, especially at high potentials, the *electrostatic capacity* of the line is sufficient to consume noticeable currents. The charging current of the line-condenser is proportional to the difference of potential, and one-quarter period ahead of the E. M. F. Hence it will either increase or decrease the main current, according to the relative phase of the main current and the E. M. F.

In consequence hereof, the current will change in the line ·from point to point, in intensity as well as phase, and the E. M.

F.'s consumed by resistance and inductance will, therefore, change also from point to point, being dependent upon the current.

In considering the effect of capacity, it is not permissible, however, to neglect the inductance, since in overhead lines the inductance is usually at least of the same magnitude as the condenser effect, and is not negligible in concentric cables even. In the latter, however, and to a lesser extent everywhere else, still other factors have to be considered.

The line consumes not only *currents in quadrature* with the E. M. F., but also *currents in phase* with the E. M. F.

Since no insulator has an infinite resistance, and at higher potentials not only *leakage*, but even direct *escape of electricity* into the air takes place by "silent discharge," we have to recognize the existence of a current approximately proportional, and in phase with the E. M. F. of the line. This current represents ·consumption of energy, and is therefore analogous to the E. M. F. consumed by resistance, while the condenser current, and the E. M. F. of inductance are wattless.

Furthermore, the alternate current passing over the line induces in all neighboring conductors secondary currents, which react upon the primary current and thereby introduce E. M. F.'s of *mutual inductance* into the primary circuit.

Mutual inductance is neither in phase nor in quadrature with the current, and can, therefore, be dissolved into an *energy component* of mutual inductance—which acts like an increase of resistance—in phase with the current, and a *wattless component*, in quadrature with the current—which decreases the self-inductance.

The mutual inductance is by no means negligible, as for instance, its disturbing influence in telephone circuits shows.

The alternate potential of the line induces by *electrostatic influence* electric charges in neighboring conductors outside of the circuit, which retain corresponding opposite charges in the line wires. This electrostatic influence requires the expenditure of a current, proportional to the E. M. F., and consisting of an *energy component*, in phase with the E. M. F., and a *wattless component*, in quadrature thereto.

The alternate electro-magnetic field of force, set up by the line current, causes in some materials a loss of energy by *electro-magnetic hysteresis*, requiring the expenditure of an E. M. F. in phase with the current, which acts like an increase of resis-.

tance. The wattless component of this ·E. M. F. disappears under "inductance," or rather we must say, that the *hysteretic* E. M. F. is the *energy component of inductance.* This magnetic hysteresis loss may take place in the conductor proper, if iron wires are used, and will then be very serious at high frequencies, as with telephone currents, or it may take place in the iron armor of the cable, etc.

The effect of the "*eddy currents*" is referred to already under " mutual inductance," whose energy component it is.

The alternating electrostatic field of force, expends energy in dielectrics by what I called "*dielectric hysteresis.*" In concentric cables, where the electrostatic gradient in the dielectric is comparatively large, the dielectric hysteresis may at high potentials even consume more energy than the ohmic resistance.

The dielectric hysteresis appears in the circuit as consumption of a current, whose component in phase with the E. M. F. is the " *dielectric energy current* "—the component in quadrature with the E. M. F. disappears in the "*condenser current,*" whose energy component the dielectric energy current is.

Besides this, there is the increase of ohmic resistance due to *unequal distribution of current,* which, however, is practically never large enough to be noticeable.

Hence we have the phenomena:

Resistance—consumes E. M. F. in phase with current.

Self-inductance, and its energy component *electro-magnetic hysteresis.*

Mutual inductance, and its energy component *eddy currents.*

Leakage—consumes current in phase with E. M. F.

Capacity, and its energy component *dielectric hysteresis.*

Influence.

This gives, as the most general case, per unit length of line; E. M. F.'s *consumed in phase with the current* C, *and* $= r\ C$, representing consumption of energy and due to :

> *Resistance,* and its increase by unequal current distribution.
>
> Energy component of *self-induction,* or *electro-magnetic hysteresis.*
>
> Energy component of *mutual inductance,* or *induced currents.*

E. M. F.'s *consumed in quadrature with the current* C, *and* $=$ $s\ C$, being wattless, and due to:

> *Self-inductance.*

Mutual inductance.

Currents consumed in phase with the E. M. F. *E and* $= \vartheta E$, representing consumption of energy, and due to :

> *Leakage* through the insulation, including silent discharge.
> Energy component of *capacity*, or *dielectric hysteresis.*
> Energy component of *electrostatic influence.*

Currents consumed in quadrature with the E. M. F. *E and* $=$ $x E$, being wattless, and due to :

> *Capacity.*
> *Electrostatic influence.*

Hence we get four constants:

> r, s, ϑ, x.

representing the coefficient, per unit length of line, of :

> E. M. F.'s consumed in phase with current, r.
> E. M. F.'s consumed in quadrature with current, s.
> Currents consumed in phase with E. M. F., ϑ.
> Currents consumed in quadrature with E. M. F., x.

This line we may assume now as feeding into a *receiver circuit of any description*, and determine current and E. M. F. at any point of the circuit:

That is :

E. M. F. and current (differing in phase by any desired angle) may be given at the terminals of the receiver circuit. To be determined is the E. M. F. and the current at any point of the line, for instance at the generator terminals.

Or:

Impedance $I = R - j S$ of receiver circuit, and E. M. F. E_0 at generator terminals are given. Current and E. M. F. at any point of circuit are to be determined, etc.

The cases, which are usually and solely treated:

> 1. Current $= O$ at end of line, that is open circuit.
> 2. E. M. F. $= O$ at end of line, that is line grounded, and
> 3. Line of infinite length

are evidently of little practical interest, but of importance is only the case of a line feeding into an inductive or non-inductive receiver circuit.

Of the four line constants, r, s, ϑ, x, usually :

r is mainly the resistance, per unit length of line.

s is mainly $= 2 \pi N L$, where $L =$ coefficient of self-induction, per unit length of line.

ϑ is mainly $= \dfrac{1}{i}$, where i the insulation resistance, per unit length of line.

x is mainly $= 2 \pi N K$, where $K =$ the capacity, per unit length of line.

Counting now the distance x from a point O of the line, which has the E. M. F., $\qquad E_1 = e_1 + j e_1{}^1$

the current, $\qquad C_1 = c_1 + j c_1{}^1$

and counting x positive in the direction of rising energy,

counting x negative in the direction of decreasing energy,

we have at any point x, in the line differential $d x$:

Leakage current, $\qquad E \vartheta d x$

Capacity current, $\qquad - j E x d x$

hence, total current consumed by $d x$:

$$d C = E (\vartheta - j x) d x, \text{ or:}$$

$$\frac{d C}{d x} = E (\vartheta - j x) \qquad (1.)$$

E. M. F. consumed by resistance, $\qquad C r d x$

E. M. F. consumed by inductance, $\qquad - j C s d x$

hence, total E. M. F. consumed by $d x$:

$$d E = C (r - j s) d x, \text{ or:}$$

$$\frac{d E}{d x} = C (r j s) \qquad (2.)$$

These *Fundamental Differential Equations* (1.) and (2.) are symmetrical in C and E.

Differentiating these equations:

$$\left.\begin{aligned} \frac{d^2 C}{d x^2} &= \frac{d E}{d x} (\vartheta - j x) \\[2mm] \frac{d^2 E}{d x^2} &= \frac{d C}{d x} (r - j s) \end{aligned}\right\} \qquad (3.)$$

and substituting (3.) in (1.) and (2.), gives:

$$\frac{d^2 E}{d x^2} = E (\vartheta - j x) (r - j s) \qquad (4.)$$

$$\frac{d^2 C}{d x^2} = C (\vartheta - j x) (r - j s) \qquad (5.)$$

The Differential Equation of C and of E.

These Differential Equations are identical, and consequently C and E are functions differing by their limiting conditions only.

These equations (4.) and (5.) are of the form:

$$\frac{d^2 w}{d x^2} = w (\vartheta - j x) (r - j s) \tag{6.}$$

and are integrated by:

$$w = a \, \epsilon^{v x}$$

where ϵ is the base of natural logarithms.

For, differentiating this, we get:

$$\frac{d^2 w}{d x^2} = v^2 a \, \epsilon^{v x} = v^2 w$$

hence:

$$v^2 = (\vartheta - j x) (r - j s) \tag{7.}$$

or:

$$v = \pm \sqrt{(\vartheta - j x) (r - j s)}$$

hence, the *complete integral* is;

$$w = a \, \epsilon^{+v x} + b \, \epsilon^{-v x} \tag{8.}$$

where a and b are the two constants of integration.

Substituting:

$$v = a - j \beta \tag{9.}$$

in (7.), we have:

$$(a - j \beta)^2 = (\vartheta - j x) (r - j s), \text{ or}:$$

$$\left.\begin{aligned} a^2 - \beta^2 &= \vartheta \, r - x \, s \\ 2 \, a \, \beta &= \vartheta \, s + x \, r \end{aligned}\right\} \tag{10.}$$

herefrom: $\quad a^2 + \beta^2 \;=\; \sqrt{(\vartheta^2 + x^2)(r^2 + s^2)}$

and: $\quad a = \sqrt{\tfrac{1}{2} \{ \sqrt{(\vartheta^2 + x^2)(r^2 + s^2)} + (\vartheta \, r - x \, s)}$

$$\beta = \sqrt{\tfrac{1}{2} \{ \sqrt{(\vartheta^2 + x^2)(r^2 + s^2)} - (\vartheta \, r - x \, s)} \tag{11.}$$

substituting (9.) in (8.):

$$w = a \, \epsilon^{(a-j\beta)x} + b \, \epsilon^{-(a-jb)x}$$

$$= a \, \epsilon^{ax} (\cos \beta x - j \sin \beta x) + b \, \epsilon^{-ax} (\cos \beta x + j \sin \beta x)$$

$$w = (a \, \epsilon^{ax} + b \, \epsilon^{-ax}) \cos \beta x - j (a \, \epsilon^{ax} - b \, \epsilon^{-ax}) \sin \beta x \tag{12.}$$

the general solution of differential equations (4.) and (5.)

Differentiating (8.) gives:

$$\frac{d w}{d x} = v (a \, \epsilon^{v x} - b \, \epsilon^{-v x})$$

hence, substituting (9.):

$$\frac{dw}{dx} = (a - j\beta)\{(a\varepsilon^{ax} - b\varepsilon^{-ax})\cos\beta x - j(a\varepsilon^{ax} + b\varepsilon^{-ax})\sin\beta x\} \text{ (13.)}$$

substituting now C for w, and substituting (13.) in (1.), and writing:

$$(a - j\beta)\, a = A$$
$$(a - j\beta)\, b = B$$

we get the

General Integral Equations of the Problem.

$$C = \frac{1}{a-j\beta}\{(A\varepsilon^{ax} + B\varepsilon^{-ax})\cos\beta x - j(A\varepsilon^{ax} - B\varepsilon^{-ax})\sin\beta x\}$$

$$E = \frac{1}{\vartheta - jx}\{(A\varepsilon^{ax} - B\varepsilon^{-ax})\cos\beta x - j(A\varepsilon^{ax} + B\varepsilon^{-ax})\sin\beta x\}$$

(14.)

where A and B are the Constants of Integration.

If: $\qquad C_1 = c_1 + j\, c_1{}^1$ is the current,

$\qquad\qquad E_1 = e_1 + j\, e_1{}^1$ is the E. M. F., $\qquad$ (15.)

at the point: $\qquad x = O$,

We get, substituting (15.) in (14.)

$$2A = \{(ac_1 + \beta c_1{}^1) + (\vartheta e_1 + xe_1{}^1)\} + j\{(ac_1{}^1 - \beta c_1) + (\vartheta e_1{}^1 - xe_1)\}$$
$$2B = \{(ac_1 + \beta c_1{}^1) - (\vartheta e_1 + xe_1{}^1)\} + j\{(ac_1{}^1 - \beta c_1) - (\vartheta e_1{}^1 - xe_1)\}$$

(16.)

If: $I = R - j\, S$ is the impedance of the receiver circuit, and

$$E_0 = e_0 + j\, e_0{}^1 \qquad (17.)$$

is the E. M. F. at the dynamo terminals, and

$l = $ length of line, we get at: $x = O$:

$$C = \frac{A + B}{a - j\beta}.$$

$$E = \frac{A - B}{\beta - jx}$$

hence:

$$I = \frac{E}{C} = \frac{A - B}{A + B}\frac{a - j\beta}{\beta - jx}, \text{ or:}$$

$$\frac{A - B}{A + B} = I\frac{\beta - jx}{a - j\beta} \qquad (18.)$$

and at: $x = l$:

$$E_0 = \frac{1}{\beta - jx}\left\{(A\varepsilon^{al} - B\varepsilon^{-al})\cos\beta l - j(A\varepsilon^{al} + B\varepsilon^{-al})\sin\beta l\right\} \text{ (19.)}$$

Equations (18.) and (19.) determine the constants A and B, which, substituted in (14.), give the final integral equations.

The length: $x_0 = \dfrac{2\pi}{\beta}$ is a complete wave length, that means,

in the distance $x_0 = \dfrac{2\pi}{\beta}$ the phase of current and E. M. F. repeat,

in half this distance they are just opposite. Hence the remarkable condition exists in a very long line, that at different points at the same time the currents flow in opposite directions, and the E. M. F.'s are opposite.

The *Difference of phase* between current and E. M. F. at any point of the line is determined by :

$$I\,(\cos \omega + j \sin \omega) = \frac{C}{E},$$

where I is a constant.

Hence, ω varies from point to point, oscillating around a medium position ω_∞, which it approaches at infinity.

This difference of phase, towards which current and E. M. F. tend at infinity, is determined by :

$$1\,(\cos \omega_\infty + j \sin \omega_\infty) = \frac{C_\infty}{E_\infty}$$

or, substituting for C_∞ and E_∞ their values, sine $\epsilon^{-\alpha x} = 0$, and $A\,\epsilon^{\alpha x}\,(\cos \beta x - \sin \beta x)$ cancels :

$$1\,(\cos \omega_\infty + j \sin \omega_\infty) = \frac{\vartheta - jx}{\alpha - j\beta}$$

$$= \frac{(\alpha\,\vartheta + \beta\,x) - j\,(\alpha\,x - \beta\,\vartheta)}{\alpha^2 + \beta^2}$$

$$\tan \omega_\infty = \frac{\beta\,\vartheta - \alpha\,x}{\alpha\,\vartheta + \beta\,x} \qquad (20.)$$

This angle $\omega_\infty = 0$, that is, current and E. M. F. come more and more in phase with each other, if: $\beta\,\vartheta - \alpha\,x = 0$, that is :

$$\alpha \div \beta = \vartheta \div x, \text{ or :}$$

$$\frac{\alpha^2 - \beta^2}{2\,\alpha\,\beta} = \frac{\vartheta^2 - x^2}{2\,\vartheta\,x}$$

substituting (10.), gives: $\dfrac{\vartheta\,r - x\,s}{\vartheta\,s + x\,r} = \dfrac{\vartheta^2 - x^2}{2\,\vartheta\,x}$,

hence, expanded :

$$r \div s = \vartheta \div x. \tag{21.}$$

that is :

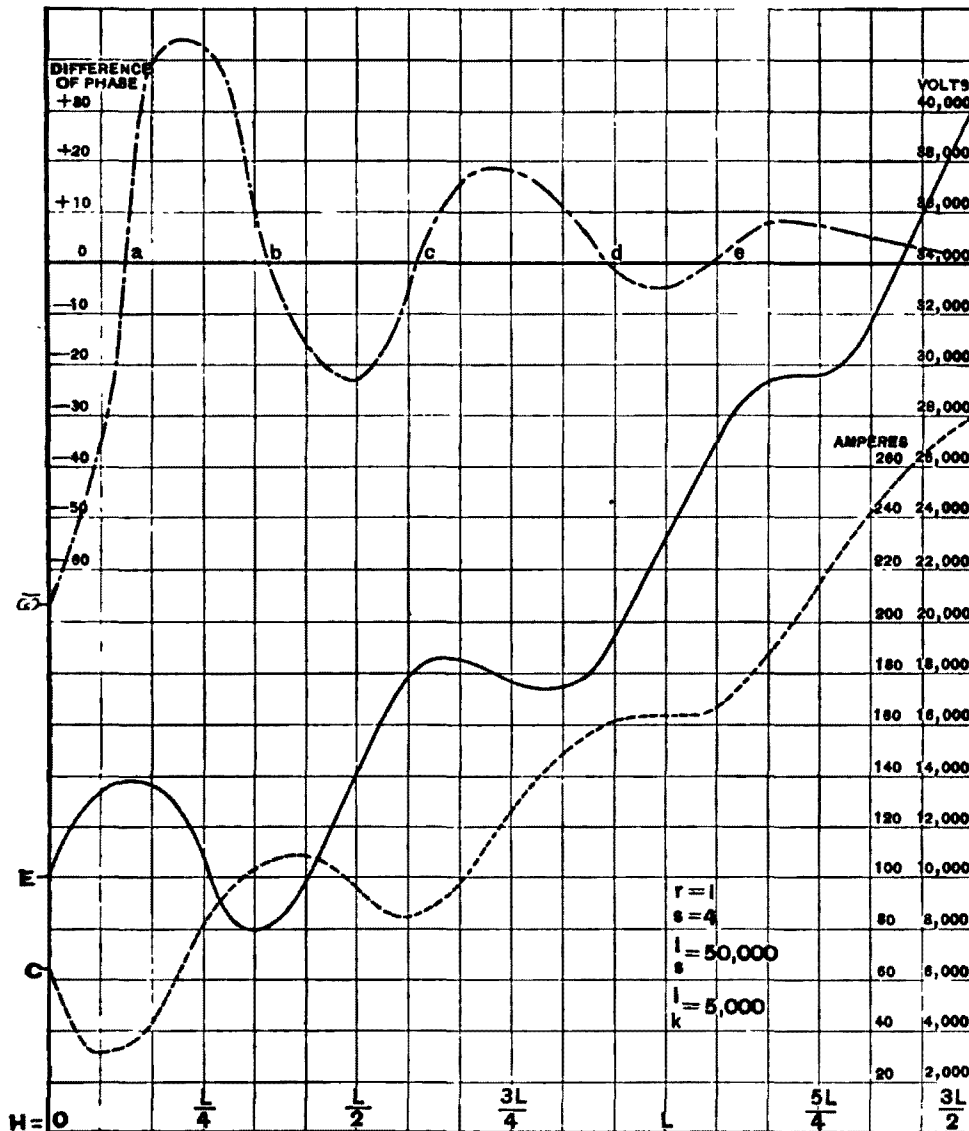"*The ratio of resistance to inductance equals the ratio of leakage to capacity.*"

FIG. 6.

This angle $\omega_\infty = 45°$, that is, current and E. M. F. differ by one-eighth period, if : $\beta \vartheta - a x = a \vartheta + \beta x$, that is :

$$\frac{a}{\beta} = \frac{\vartheta + x}{\vartheta - x}$$

or :

$$r \vartheta + s x = 0 \tag{22.}$$

that is:

"Two of the four line constants must be $= O$, either ϑ and x or ϑ and s."

As an instance, in Fig. 6 a line diagram is shown, with the distances from the receiver end as abscissae. Figure 6 represents one and a half complete waves, and gives total effective current, total E. M. F., and difference of phase between both, as functions of the distance from receiver circuit, under the conditions:

E. M. F. at receiving end: 10,000 volts, hence: $E_1 = e_1 = 10,000$.

Current at receiving end: 65 amperes at .385 energy coefficient, that is:

$$C_1 = c_1 + j\, c_1{}^1 = 25 + 60\,j.$$

Line constants per unit length:

$$r = 1 \qquad \vartheta = 2 \times 10^{-5}$$
$$s = 4 \qquad x = 20 \times 10^{-5}$$

hence:

$$a = 4.95 \times 10^{-3}$$
$$\beta = 28.36 \times 10^{-3}$$
$$a^2 + \beta^2 = .829 \times 10^{-3}$$

$$x_0 = L = \frac{2\pi}{\beta} = 221.5 = \text{length of line, corresponding to}$$

one complete period of wave propagation.

$$A = 1.012 - 1.206\,j$$
$$B = .812 + .794\,j$$

These values substituted give:

$$C = \left\{ \epsilon^{ax}(47.3\cos\beta x + 27.4\sin\beta x) - \epsilon^{-ax}(22.3\cos\beta x + 32.6\sin\beta x) \right\}$$

$$+ j\left\{ \epsilon^{ax}(27.4\cos\beta x - 47.3\sin\beta x) + \epsilon^{-ax}(32.6\cos\beta x - 22.3\sin\beta x) \right\}$$

$$E = \left\{ \epsilon^{ax}(6450\cos\beta x + 4410\sin\beta x) + \epsilon^{-ax}(3530\cos\beta x - 4410\sin\beta x) \right\}$$

$$+ j\left\{ \epsilon^{ax}(4410\cos\beta x - 6450\sin\beta x) - \epsilon^{-ax}(4410\cos\beta x - 3530\sin\beta x) \right\}$$

$$\tan\omega_\infty = \frac{\beta\,\vartheta - a\,x}{a\,\vartheta + \beta\,x} = -.072, \qquad \omega_\infty = -4.2°.$$

Some Particular Cases.

.1. *Open Circuit at End of Line.*

$$x = 0. \qquad C_1 = 0.$$

$$A = (\vartheta\, e_1 + x\, e_1{}^1) + j\, (\vartheta\, e_1{}^1 - x\, e_1) = - B$$

hence:

$$E = \frac{1}{\vartheta - j\,x}\, A\, \left\{ (\varepsilon^{ax} + \varepsilon^{-ax})\cos \beta\, x - j\, (\varepsilon^{ax} - \varepsilon^{-ax})\sin \beta\, x \right\}$$

$$C = \frac{1}{a - j\,\beta}\, A\, \left\{ (\varepsilon^{ax} - \varepsilon^{-ax})\cos \beta\, x - j\, (\varepsilon^{ax} + \varepsilon^{-ax})\sin \beta\, x \right\}$$

B. Line Grounded at End.

$$x = 0. \qquad E_1 = 0.$$

$$A = (a\, c_1 + \beta\, c_1{}^1) + j\, (a\, c_1{}^1 - \beta\, c_1) = B,$$

hence:

$$E = \frac{1}{\vartheta - j\,x}\, A\, \left\{ (\varepsilon^{ax} - \varepsilon^{-ax})\cos \beta\, x - j\, (\varepsilon^{ax} + \varepsilon^{-ax})\sin \beta\, x \right\}$$

$$C = \frac{1}{a - j\,\beta}\, A\, \left\{ (\varepsilon^{ax} + \varepsilon^{-ax})\cos \beta\, x - j\, (\varepsilon^{ax} - \varepsilon^{-ax})\sin \beta\, x \right\}$$

C. Infinitely Long Conductors.

Replacing x by $- x$, that is, counting distance positive in the direction of decreasing energy, we have:

$$x = \infty : \qquad C = 0, \qquad E = 0 :$$

hence: $\qquad B = 0$

and: $\qquad E = \dfrac{1}{\vartheta - j\,x}\, A\, \varepsilon^{-ax}\, (\cos \beta\, x + j\, \sin \beta\, x)$

$$C = \frac{1}{a - j\,\beta}\, A\, \varepsilon^{-ax}\, (\cos \beta\, x + j\, \sin \beta\, x)$$

revolving decay of the wave.

The *total impedance* of the infinite circuit is:

$$I = \frac{E}{C}$$

$$= \frac{a - j\,\beta}{\vartheta - j\,x}$$

$$= \frac{(a\, \vartheta + \beta\, x) - j\, (\beta\, \vartheta - a\, x)}{\vartheta^2 + x^2}$$

" The infinitely long conductor acts like an impedance

$$I = \frac{\alpha\,\vartheta + \beta\,x}{\vartheta^2 + x^2} - j\,\frac{\beta\,\vartheta - \alpha\,x}{\vartheta^2 + x^2},\ \text{that is,}\ \text{like}\ \text{a}\ \text{resistance}$$

$$R = \frac{\alpha\,\vartheta + \beta\,x}{\vartheta^2 + x^2},\ \text{combined with an inductance}\ S = \frac{\beta\,\vartheta - \alpha\,x}{\vartheta^2 + x^2}.$$

Herefrom we get the difference of phase between E. M. F. and

current: $$\tan \omega = \frac{S}{R} = \frac{\beta\,\vartheta - \alpha\,x}{\alpha\,\vartheta + \beta\,x}$$

which is constant at all points of the line:

If: $\vartheta = O$, $s = O$, we have: $\alpha = \beta = \sqrt{\dfrac{x\,r}{2}}$, hence:

$$\tan \omega = 1,\ \text{or:}\qquad \omega = 45°.$$

that is, current and E. M. F. differ by one-eighth period.
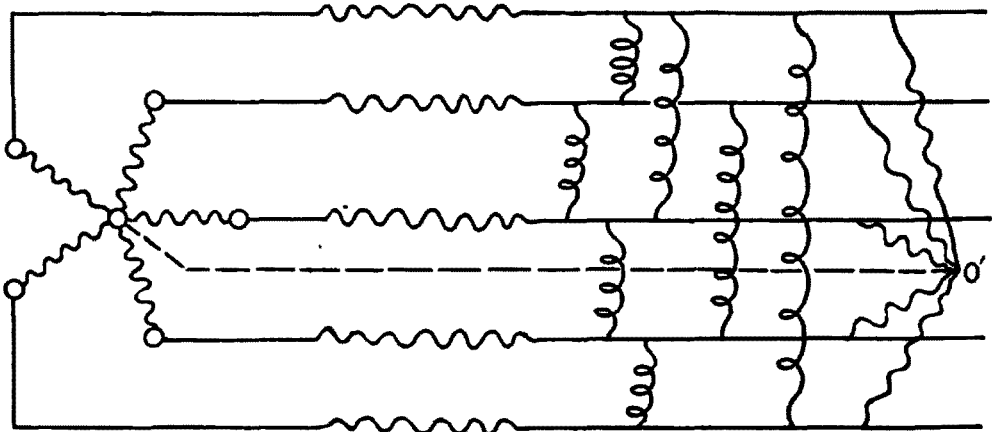

FIG. 7.

D. Generator Feeding into Closed Circuit:

Let $x = O$ be the center of the cable. It is then:

$$C_x = C_{-x},$$

$$E_x = -E_{-x},$$

hence: $E = O$ at $x = O$,

that means, the equations are the same as in *B.*, where the line is grounded at $r = O$.

V. POLYPHASE SYSTEMS.

In polyphase systems, we have two ways of connecting the n circuits of an n-phase generator with each other and with the line.

1. The *star connection*, represented diagrammatically in Fig. 7, where the n-circuits, containing E. M. F.'s differing from each

other by $\dfrac{1}{n}$ of a period, are connected together at one end into a *neutral point O*—which may either be grounded or not—while the other ends of the circuit are connected to the line-wires, and:

2. The *ring connection*, represented by Fig. 8, where the n generator circuits are connected in closed circuit, and the n line wires connected to the points of contact of adjacent circuits.

Outside of the generator the two systems are identical.

The consumer circuits may now either be connected between any pairs or sets of line-wires, or between the wires and a neutral point O', which may be grounded, or connected to the neutral point of the generator O.

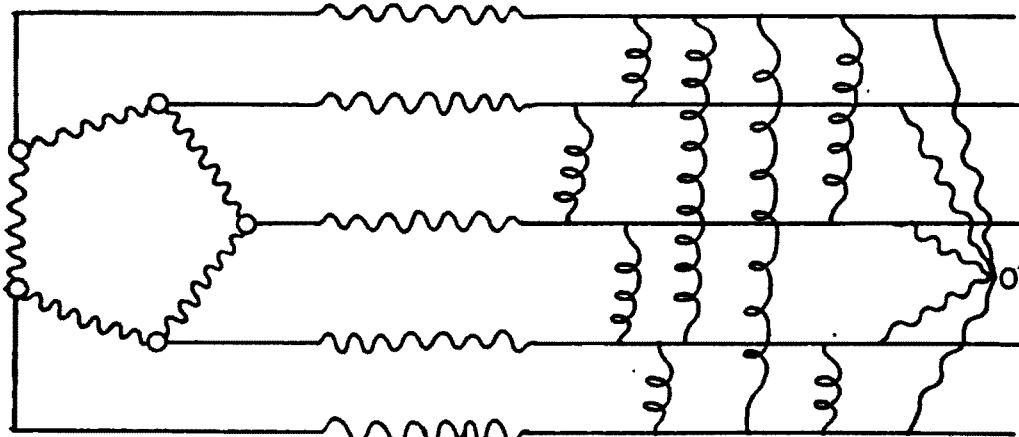1. Let now, in the *star connection* of generator, E be the E.

FIG. 8.

M. F. of one branch of the generator, and let 1, 2, . . . n be the generator circuits.

Since the E. M. F.'s of adjacent circuits differ by $\dfrac{1}{n}$ of a period,

$= \dfrac{2\,\pi}{n}$, and rotation by $\dfrac{2\,\pi}{n}$ is represented algebraically by multiplication with:

$$\epsilon = \cos\frac{2\,\pi}{n} + j\sin\frac{2\,\pi}{n} = \sqrt[n]{1} \qquad (1.)$$

The E. M. F. in any circuit i is:

$$E_i = \epsilon^i E \qquad (2.)$$

Hence, if C_i is the current in circuit i, and I is the impedance per generator circuit, we have:

E. M. F. at terminal i of generator:

$$E'_i = E_i - C_i I = \varepsilon^i E - C_i I \qquad (3.)$$

And the E. M. F. at the end of a line of impedance I_i, connected to terminal i:

$$E''_i = E_i - C_i (I + I_i) = \varepsilon^i E - C_i (I + I_i) \qquad (4.)$$

Let now E_{ix} denote the difference of potential between any pair of terminals i and x,

where: $E_{ix} = - E_{xi}$ (5.)

we have:

E. M. F. of generator, acting between terminals i and x:

$$E_{ix} = (\varepsilon_i - \varepsilon_x) E \qquad (6.)$$

Difference of potential between generator terminals i and x:

$$E'_{ix} = (\varepsilon^i - \varepsilon^x) E - I(C_i - C_x) \qquad (7.)$$

Difference of potential between lines i and x:

$$E''_{ix} = (\varepsilon^i - \varepsilon^x) E - I(C_i - C_x) - (I_i C_i - I_x C_x) \quad (8.)$$

If now C_{ix} represents the current, which passes from line i to x (and which is determined by the impedance I_{ix} of the apparatus connected between i and x:

$$C_{ix} = \frac{E_{ix}}{I_{ix}}$$

and if C_{io} denotes the current passing from line i to neutral point O', we have:

$$C_i = \sum_x^n C_{ix} \qquad (9.)$$

Furthermore, if the neutral points O and O' are insulated.

$$\left.\begin{array}{c} \sum_i^n C_i = O \\[2mm] \sum_i^n C_{io} = O \end{array}\right\} \qquad (10.)$$

If, however, the neutral point O and O' are grounded, or connected together:

$$\sum_i^n C_i = \sum_i^n C_{io} \qquad (11.)$$

2. In the case of the *ring connected* generator, the generator E. M. F.'s:

$$\varepsilon^i E$$

Take the place of the E. M. F.'s

$$E_{i\,i+1}$$

of the star connection, hence the E. M. F. between any pair of terminals i and x is:

$$\sum (E_i + E_{i+1} + \ldots + E_x) = E \sum_i^n \vartheta\ \varepsilon^\vartheta \qquad (12.)$$

All the other considerations remain essentially the same, so that:

"Any polyphase system of the E. M. F.'s:

$$E_i = \varepsilon^i\,E, \qquad i = 1, 2, \ldots n, \qquad \varepsilon = \sqrt[n]{1} \qquad (13.)$$

can be dissolved by Ohm's law:
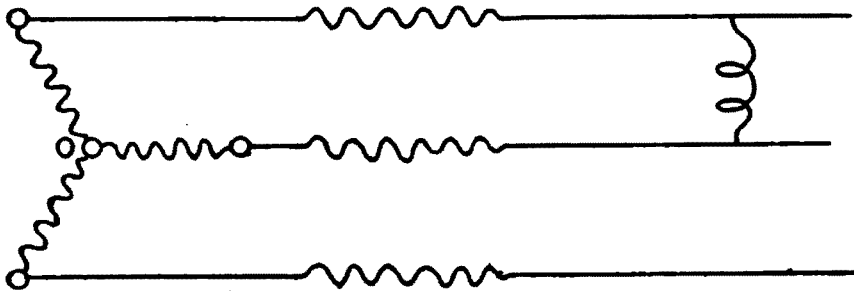
$$E = C\,I \qquad (14.)$$

FIG. 9.

and Kirchhoff's laws:

$$\sum E = O \text{ in any closed circuit,} \qquad (15.)$$

$$\sum C = O \text{ at any point of distribution."} \qquad (16.)$$

It would carry me too far for the scope of this paper, to enter further into the general theory of the polyphase systems, and it may be sufficient therefore, to show in a particular instance, taken from the threephase system, what remarkable phenomena can be expected in polyphase systems.

Unbalanced Threephase System.

Let, in a threephase system, Fig. 9, with star connected generator,

$$E,\ \varepsilon\,E,\ \varepsilon^2\,E$$

be the E. M. F.'s of the three generator branches, where:

$$\varepsilon = \sqrt[3]{1} = \frac{-1 + j\sqrt{3}}{2}$$

$$\varepsilon^2 \qquad = \frac{-1 - j\sqrt{3}}{2}$$

Let I = impedance per generator branch,

I_1 = impedance per line,

and let one pair of lines be connected by an impedance I_2.

We have then, if C = the current flowing in this loaded branch—the two other branches being unloaded, or open—that is, the system "unbalanced."

E. M. F. in generator circuits:

$$\left. \begin{array}{c} E \\ \varepsilon E \\ \varepsilon^2 E \end{array} \right\} \quad (17.)$$

Potentials at generator terminals:

$$\left. \begin{array}{c} E - C I \\ \varepsilon E + C I \\ \varepsilon^2 E \end{array} \right\} \quad (18.)$$

Potentials at end of lines:

$$\left. \begin{array}{c} E - C(I + I_1) \\ \varepsilon E + C(I + I_1) \\ \varepsilon_2 E \end{array} \right\} \quad (19.)$$

Hence, differences of potential at generator terminals:

$$\left. \begin{array}{l} E(1 - \varepsilon) - 2\,C I \quad \text{—loaded branch.} \\ \varepsilon E(1 - \varepsilon) + C I \\ \varepsilon_2 E(1 - \varepsilon) + C I \end{array} \right\} \quad (20.)$$

with the lower two "—unloaded branches."

Difference of potential at ends of line:

$$\left. \begin{array}{l} E(1 - \varepsilon) - 2\,C(I + I_1) \quad \text{—loaded branch.} \\ \varepsilon E(1 - \varepsilon) + C(I + I_1) \\ \varepsilon^2 E(1 - \varepsilon) + C(I + I_1) \end{array} \right\} \quad (21.)$$

with the lower two "—unloaded branches."

Hence, current in loaded branch:

$$C = \frac{E(1 - \varepsilon) - 2\,C(I + I_1)}{I_2}$$

or, expanded:

$$C = \frac{E(1 - \varepsilon)}{I_2 + 2(I + I_1)}, \text{ as was to be expected, since} \quad (22.)$$

$I_2 + 2(I + I_1)$ is the total impedance, $E(1 - \varepsilon)$ the E. M. F. of this circuit.

Substituting (22.) in (20.) and (21.), we get:

STEINMETZ ON COMPLEX QUANTITIES.

Difference of potential at generator terminals:

$$E(1-\varepsilon)\left(1 - \frac{2\,I}{I_2 + 2(I + I_1)}\right) \quad \text{—loaded branch.}$$

$$E(1-\varepsilon)\left(\varepsilon + \frac{I}{I_2 + 2(I + I_1)}\right)$$

$$\left. \begin{array}{c} \\ \text{—unloaded branches.} \\ \\ \end{array} \right\} \quad (23.)$$

$$E(1-\varepsilon)\left(\varepsilon^2 + \frac{I}{I_1 + 2(I + I_1)}\right)$$

Difference of potential at ends of line:

$$E(1-\varepsilon)\left(1 - \frac{2(I + I_1)}{I_2 + 2(I + I_1)}\right) \quad \text{—loaded branch.}$$

$$E(1-\varepsilon)\left(\varepsilon + \frac{I + I_1}{I_2 + 2(I + I_1)}\right)$$

$$\left. \begin{array}{c} \\ \text{—unloaded branches.} \\ \\ \end{array} \right\} \quad (24.)$$

$$E(1-\varepsilon)\left(\varepsilon^2 + \frac{I + I_1}{I_2 + 2(I + I_1)}\right)$$

These are three different values. That means:

" Loading in a three phase system one branch only, the potentials of the two unloaded branches become unequal also."

It is self evident, that this phenomenon of unbalancing does not take place in the three phase system only, but just as well in any other polyphase system, and that the amount of unbalancing depends upon the constants of the circuit, hence, can by a proper arrangement be reduced to almost nil, or can be exaggerated greatly by an improper choice of circuit constant.

As an instance, we may consider the numerical example:

Generator E. M. F. 100 volts between terminals, hence:

$$E(1-\varepsilon) = 100$$

Resistance per generator branch,   .01 ohms.   ·

Inductance per generator branch,   .05 ohms.

Hence, impedance per generator branch, .01 — .05 j.

Case 1. Non-inductive line of   .1   ohms.

         Non-inductive load of   .1   ohms.

Case 2. Non-inductive line of   .1   ohms.

         Inductive   load of — j ohms.

Case 3. Inductive   line of — .1j ohms.

         Inductive   load of — j ohms.

Case 4. Inductive   line of — .1j ohms.

         Non-inductive load of   .1   ohms.

Substituting these values in equations (22.), (23.) and (24.) we get (writing all the quantities in the form, c ($\cos \omega + j \sin \omega$):

1. Non-inductive line and non-inductive load; $I_1 = .1, I_2 = 1$:

$C = 81.6(.99 + .08j)$

$$E_1 = 98.0(\ .99 + .08j) \qquad E_1 = 81.6(\ .99 + .08j)$$
$$E_2 = 95.9(-.51 + .86j) \qquad E_2 = 92.6(-.44 + .90j)$$
$$E_3 = 102.9(-.47 - .88j) \qquad E_3 = 98.7(-.41 - .91j)$$

2. Non-inductive line, inductive load: $I_1 = .1, I_2 = -j$:

$C = 89.0(.20 + .98j)$

$$E_1 90.9 \qquad E_1 = 89.1(\ .98 - .20j)$$
$$E_2 = 97.8(-.47 + .88j) \qquad E_2 = 104.9(-.42 + .92j)$$
$$E_3 = 97.8(-.47 - .88j) \qquad E_3 = 89.3(-.49 - .86j)$$

3. Inductive line and inductive load: $I_1 = -.1j, I_2 = -j$:

$C = 72.0(.01 + j)$

$$E_1 = 92.8(1 - \ .01j) \qquad E_1 = 78.4(1 - \ .01j)$$
$$E_2 = 98.8(\ -.47 + .88j) \qquad E_2 = 95.6(\ -.41 + .91j)$$
$$E_3 = 98.7(\ -.48 - .88j) \qquad E_3 = 94.6(\ -.41 - .91j)$$

4. Inductive line, non-inductive load: $I_1 = -.1j, I_2 = 1$:

$C = 94.0(.96 + .28j)$

$$E_1 = 95.9(\ .99 + .09j) \qquad E_1 = 94.1(\ .96 + .28j)$$
$$E_2 = 95.2(-.50 + .86j) \qquad E_2 = 86.1(-.52 + .85j)$$
$$E_3 = 102.7(-.47 - .88j) \qquad E_3 = 109.6(-.41 - .91j)$$

Remarkable is in 1. and in 4. the *rise of potential* in the line in the branch E_2.

Apparently these values look rather irregular, sometimes the one, sometimes the other unloaded branch being higher. Looking closer into it, however, we can not fail to see the regularity displayed in the variation of potential, which makes it possible to control this phenomenon.

Lynn, Mass., July, 1893.

DISCUSSION.

PROF. MACFARLANE:—I wish to make a remark in regard to the fundamental principle of the use of complex quantities. The letter j was first introduced as a distinguishing index without mathematical meaning, and afterwards defined by the equation $j^2 = -1$. Such definition is ambiguous, for it refers to orthogonal projection of a straight line upon another straight line, and the right angle may be at the former straight line or at the latter. The latter case is the ordinary meaning.

It is not true that algebra is limited or bounded by the ordinary complex quantity. There is a more general complex quantity which applies to space, and of which the complex quantity in a plane is only a special case.

MR. STEINMETZ:—In introducing j, first as distinguishing index and then defining it as $\sqrt{-1}$ my object was to introduce the complex quantity in an elementary and graphical manner, without reference to higher mathematics. To make the reasoning more complete, I might have added that the definition $j = \sqrt{-1}$ does not contradict the original definition of j as index without numerical meaning, since in the range of ordinary numbers $\sqrt{-1}$ is meaningless.

From the mathematical standpoint the complex quantity can directly be introduced without further explanation, since in pure mathematics, for instance, the theory of functions, the plane, is known as the standard representation of the complex quantity.

Referring to Prof. Macfarlane's last remark, my meaning is that the complex quantity is the last and most general *algebraic* number. No further generalization of numbers exists which fulfills the fundamental condition of algebraic numbers, that if a product is zero one of the factors must be zero. This is the reason why the complex quantity of the higher order does not prove as useful in space as the algebraic complex quantity in the plane.

The following paper was then read: