RIEMANN ZETA FUNCTIONS

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Abstract. A Riemann zeta function is a function which is analytic in the complex plane, with the possible exception of a simple pole at one, and which has characteristic Euler product and functional identity. Riemann zeta functions originate in an adelic generalization of the Laplace transformation which is defined using a theta function. Hilbert spaces, whose elements are entire functions, are obtained on application of the Mellin transformation. Maximal dissipative transformations are constructed in these spaces which have implications for zeros of zeta functions. The zeros of a Riemann zeta function in the critical strip are simple and lie on the critical line. The Euler zeta function and Dirichlet zeta functions are examples of Riemann zeta functions.

A Riemann zeta function is represented by a Dirichlet series

$$\zeta(s) = \sum \tau(n) n^{-s}$$

in the half–plane $\mathcal{R}s > 1$ with summation over the positive integers n which are relatively prime to a given positive integer ρ. A Riemann zeta function has an analytic extension to the complex plane with the possible exception of a simple pole at $s = 1$. Riemann zeta functions are divided into two classes according to Euler product and functional identity. Riemann zeta functions originate in Fourier analysis either on a plane or on a skew-plane. The Euler product for the zeta function of a plane is a product

$$\zeta(s)^{-1} = \prod (1 - \chi(p) p^{-s})$$

taken over the primes p which are not divisors of ρ. The identity

$$|\tau(p)| = 1$$

holds for every such prime p. The functional identity for the zeta function of a skew-plane states that the functions

$$(\pi/\rho)^{-\frac{1}{2}\nu - \frac{1}{2}s} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s) \zeta(s)$$

and

$$(\pi/\rho)^{-\frac{1}{2}\nu - \frac{1}{2} + \frac{1}{2}s} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}s) \zeta(1 - s^-)^-$$

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of s are linearly dependent for ν equal to zero or one. The Euler product for the zeta function of a skew–plane is a product

$$\zeta(s)^{-1} = \prod (1 - \tau(p)p^{-s} + [\tau(p)^2 - \tau(p^2)]p^{-2s})$$

taken over the primes p which are not divisors of ρ. The inequality

$$|\tau(p)| \leq 2$$

holds for every such prime p with $\tau(p)^2 - \tau(p^2)$ of absolute value one. The functional identity for the zeta function of a skew–plane states that the functions

$$(2\pi/\rho)^{-\frac{1}{2}\nu - s}\Gamma(\tfrac{1}{2}\nu + s)\zeta(s)$$

and

$$(2\pi/\rho)^{-\frac{1}{2}\nu - 1 + s}\Gamma(\tfrac{1}{2}\nu + 1 - s)\zeta(1 - s)$$

are linearly dependent for some odd positive integer ν. The Euler zeta function is a Riemann zeta function for a plane. The other Riemann zeta functions for a plane are Dirichlet zeta functions. The Riemann zeta functions for a skew-plane are examples of zeta functions for which the Ramanujan hypothesis [3], [4] is satisfied.

Zeta functions originate in Fourier analysis on locally compact rings. The locally compact field of real numbers is obtained by completion of the field of rational numbers in a topology which is compatible with additive and multiplicative structure. Other locally compact fields are constructed by completion of subrings of the rational numbers admitting topologies compatible with additive and multiplicative structure. If r is a positive integer, a corresponding subring consists of the rational numbers which have integral product with some positive integer whose prime divisors are divisors of r. The ring admits a topology for which addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. The r–adic topology is determined by its neighborhoods of the origin. Basic neighborhoods are the ideals of the integers generated by positive integers whose prime divisors are divisors of r. The r–adic line is the completion of the ring in the resulting uniform structure. Addition and multiplication have continuous extensions as transformations of the Cartesian product of the r–adic line with itself into the r–adic line. The r–adic line is a commutative ring which is canonically isomorphic to the Cartesian product of the p–adic lines taken over the prime divisors p of r. Each p–adic line is a locally compact field. An element of the r–adic line is said to be integral if it belongs to the closure of the integers in the r–adic topology. The integral elements of the r–adic line form a compact neighborhood of the origin for the r–adic topology. An invertible integral element of the r–adic line is said to be a unit if its inverse is integral. The r–adic modulus of an invertible element ξ of the r–adic line is the unique positive rational number $|\xi|_-$, which represents an element of the r–adic line, such that $|\xi|_-\xi$ is a unit. The r–adic modulus of a noninvertible element of the r–adic line is zero. Haar measure for the r–adic line is normalized so that the set of integral elements has measure one. Multiplication by an element ξ of the r–adic line multiplies Haar measure by the

r-adic modulus. The function $\exp(2\pi i\xi)$ of ξ in the r–adic line is defined by continuity from values when ξ is a rational number which represents an element of the r–adic line.

The Euclidean line is the locally compact ring of real numbers. The Euclidean modulus of an element ξ of the Euclidean line is its absolute value $|\xi|$. A unit of the Euclidean line is an element of absolute value one. Haar measure for the Euclidean line is Lebesgue measure. Multiplication by an element ξ of the Euclidean line multiplies Haar measure by a factor of the Euclidean modulus $|\xi|$. The function $\exp(2\pi i\xi)$ of ξ in the Euclidean line is continuous.

The r–adelic line is a locally compact ring which is the Cartesian product of the Euclidean line and the r–adic line. An element ξ of the r–adelic line has a Euclidean component ξ_+ and an r–adic component ξ_-. The Euclidean modulus of an element ξ of the r–adelic line is the Euclidean modulus $|\xi|_+$ of its Euclidean component ξ_+. The r–adic modulus of an element ξ of the r–adic line is the r–adic modulus $|\xi|_-$ of its r–adic component ξ_-. The r–adelic modulus of an element ξ of the r–adelic line is the product $|\xi|$ of its Euclidean modulus $|\xi|_+$ and its r–adic modulus $|\xi|_-$. An element of the r–adelic line is said to be a unit if its Euclidean modulus and its r–adic modulus are one. An element of the r–adelic line is said to be unimodular if its r–adelic modulus is one. Haar measure for the r–adelic line is the Cartesian product of Haar measure for the Euclidean line and Haar measure for the r–adic line. Multiplication by an element of the r–adelic line multiplies Haar measure by the r–adelic modulus. The function

$$\exp(2\pi i\xi) = \exp(2\pi i\xi_+)/\exp(2\pi i\xi_-)$$

of ξ in the r–adelic line is the quotient of the function $\exp(2\pi i\xi_+)$ of the Euclidean component and of the function $\exp(2\pi i\xi_-)$ of the r–adic component. A principal element of the r–adelic line is an element whose Euclidean and r–adic components are represented by equal rational numbers. The principal elements of the r–adelic line form a discrete subring of the r–adelic line. The identity

$$\exp(2\pi i\xi) = 1$$

holds for every principal element ξ of the r–adelic line. A principal element of the r–adelic line is unimodular if it is nonzero.

The adic line is a locally compact ring which is a restricted inverse limit of the r–adic lines. The ring is a completion of the field of rational numbers in a topology for which addition and multiplication are continuous as transformations of the Cartesian product of the field with itself into the field. The adic topology is determined by its neighborhoods of the origin. Basic neighborhoods are the ideals of the integers which are generated by positive integers. The adic line is the completion of the field in the resulting uniform structure. Addition and multiplication have continuous extensions as transformations of the Cartesian product of the adic line with itself into the adic line. The adic line is canonically isomorphic to a subring of the Cartesian product of the p–adic lines taken over all primes p. An element of the Cartesian product represents an element of the adic line if, and only if, its p–adic component is integral for all but a finite number of primes p. An element of the adic line is said to be integral if its p–adic component is integral for every prime p. The integral elements of the adic line form a compact neighborhood of the origin for the adic topology. An invertible integral element of the adic line is said to be a unit if its inverse is integral. The adic modulus of an invertible element ξ of the adic line is the unique positive rational number $|\xi|_-$ such that $|\xi|_-\xi$ is a unit. The adic modulus of a noninvertible element of the adic line is zero. Haar measure for the adic line is normalized so that the set of integral elements has measure one. Multiplication by an element of the adic line multiplies Haar measure by the adic modulus. The function $\exp(2\pi i\xi)$ of ξ in the adic line is defined by continuity from its values when ξ is a rational number.

The adelic line is a locally compact ring which is the Cartesian product of the Euclidean line and the adic line. An element ξ of the adelic line has a Euclidean component ξ_+ and an adic component ξ_-. The Euclidean modulus of an element ξ of the adelic line is the Euclidean modulus $|\xi|_+$ of its Euclidean component ξ_+. The adic modulus of an element ξ of the adelic line is the adic modulus $|\xi|_-$ of its adic component ξ_-. The adelic modulus of an element ξ of the adelic line is the product $|\xi|$ of its Euclidean modulus $|\xi|_+$ and its adic modulus $|\xi|_-$. An element of the adelic line is said to be a unit if its Euclidean modulus and its adic modulus are one. An element of the adelic line is said to be unimodular if its adelic modulus is one. Haar measure for the adelic line is the Cartesian product of Haar measure for the Euclidean line and Haar measure for the adic line. Multiplication by an element of the adelic line multiplies Haar measure by the adelic modulus. The function

$$\exp(2\pi i\xi) = \exp(2\pi i\xi_+)/\exp(2\pi i\xi_-)$$

of ξ in the adelic line is defined as the ratio of the function $\exp(2\pi i\xi_+)$ of ξ_+ in the Euclidean line and the function $\exp(2\pi i\xi_-)$ of ξ_- in the adic line. A principal element of the adelic line is an element whose Euclidean and adic components are represented by equal rational numbers. The principal elements of the adelic line form a discrete subring of the adelic line. An element ξ of the adelic line is a principal element if, and only if, the identity

$$\exp(2\pi i\xi\eta) = 1$$

holds for every principal element η of the adelic line. A principal element of the adelic line is unimodular if it is nonzero.

The Fourier transformation for the adelic line is an isometric transformation whose domain and range are the space of square integrable functions with respect to Haar measure for the adelic line. The transformation takes a function $f(\xi)$ of ξ in the adelic line into a function $g(\eta)$ of η in the adelic line when the identity

$$g(\eta) = \int f(\xi)\exp(2\pi i\eta\xi)d\xi$$

is formally satisfied. The integral is accepted as the definition of the transformation when the integral with respect to Haar measure for the adelic line is absolutely convergent. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

then holds with integration with respect to Haar measure for the adelic line. The identity

$$f(\eta) = \int g(\xi) \exp(-2\pi i \eta \xi) d\xi$$

holds with integration with respect to Haar measure for the adelic line when the integral is absolutely convergent. The Poisson summation formula

$$\sum f(\xi) = \sum g(\xi)$$

holds with summation over the principal elements of the adelic line when both integrals are absolutely convergent.

The Euclidean plane is the locally compact field of complex numbers. The complex conjugation of the Euclidean plane is the automorphism ξ into ξ^- of order two whose fixed field is the Euclidean line. The Euclidean modulus of an element ξ of the Euclidean plane is its absolute value $|\xi|$. An element of the Euclidean plane is said to be a unit if its Euclidean modulus is one. Haar measure for the Euclidean plane is Lebesgue measure. Multiplication by an element of the Euclidean plane multiplies Haar measure by the square of the Euclidean modulus.

The Euclidean skew–plane is a locally compact ring in which every nonzero element is invertible. The Euclidean skew–plane is an algebra over the Euclidean plane which is generated by an element j which satisfies the identity

$$j^2 = -1$$

and the identity

$$j\gamma = \gamma^- j$$

for every element γ of the Euclidean plane. The elements of the Euclidean skew–plane are of the form $\alpha + j\beta$ with α and β elements of the Euclidean plane. The conjugation of the Euclidean skew–plane is the anti–automorphism ξ into ξ^- of order two which takes

$$\alpha + j\beta$$

into

$$\alpha^- - j\beta$$

for all elements α and β of the Euclidean plane. The Euclidean plane is a subfield of the Euclidean skew–plane on which the conjugation of the Euclidean skew–plane agrees with the conjugation of the Euclidean plane. The Euclidean line is the fixed field of the conjugation of the Euclidean skew–plane. If ξ is an element of the Euclidean skew–plane, $\xi^*\xi$ is a nonnegative element of the Euclidean line which is nonzero if, and only if, ξ is nonzero. The Euclidean modulus of an element ξ of the Euclidean skew–plane is the nonnegative square root $|\xi|$ of $\xi^-\xi$. A unit of the Euclidean skew–plane is an element of Euclidean modulus one. Haar measure for the Euclidean skew–plane is the Cartesian product of the Haar measures for component Euclidean planes. Multiplication by an element of the Euclidean skew–plane multiplies Haar measure by the fourth power of the Euclidean modulus.

A theorem of Lagrange states that every positive integer is a sum of four squares of integers. If n is a positive integer, the equation

$$n = \xi^- \xi$$

has solutions

$$\xi = (\alpha + i\beta) + j(\gamma + i\delta)$$

in the elements of the Euclidean skew–plane whose components α, β, γ, and δ are all integers or all halves of odd integers. The solutions form a group of order twenty-four when n is equal to one.

The r–adic skew–plane is an algebra of dimension four over the r–adic line which is generated by the same units as i and j as the Euclidean skew–plane. The elements of the r–adic skew–plane are of the form

$$(\alpha + i\beta) + j(\gamma + i\delta)$$

for elements α, β, γ, and δ of the r–adic line. The conjugation of the r–adic skew–plane is the anti–automorphism ξ into ξ^- of order two which takes

$$(\alpha + i\beta) + j(\gamma + i\delta)$$

into

$$(\alpha - i\beta) - j(\gamma + i\delta)$$

for all elements α, β, γ, and δ of the r–adic line. The topology of the r–adic skew–plane is the Cartesian product of topologies of coordinate r–adic lines. If ξ is an element of the r–adic skew–plane, $\xi^- \xi$ is an element of the r–adic line which is invertible if, and only if, ξ is invertible. The r–adic modulus of an element ξ of the r–adic skew–plane is the nonnegative square root $|\xi|_-$ of the r–adic modulus of $\xi^- \xi$. An integral element of the r–adic skew–plane is an element ξ such that $\xi^- \xi$ is an integral element of the r–adic line. The integral elements of the r–adic skew–plane form a compact subring which is a neighborhood of the origin for the r–adic topology. A unit of the r–adic skew–plane is an invertible integral element whose inverse is integral. An element ξ of the r–adic skew–plane is a unit if, and only if, $\xi^- \xi$ is a unit of the r–adic line. Haar measure for the r–adic skew–plane is normalized so that the set of integral elements has measure one. Multiplication by an element of the r–adic skew–plane multiplies Haar measure by the fourth power of the r–adic modulus.

The r–adelic skew–plane is a locally compact ring which is the Cartesian product of the Euclidean skew–plane and the r–adic skew–plane. An element ξ of the r–adelic skew–plane has a Euclidean component ξ_+ and an r–adic component ξ_-. The conjugation of the r–adelic skew–plane is the anti–automorphism ξ into ξ^- of order two such that the Euclidean component of ξ^- is obtained from the Euclidean component of ξ under the conjugation of the Euclidean skew–plane and the r–adic component of ξ^- is obtained from the r–adic component of ξ under the conjugation of the r–adic skew–plane. The Euclidean modulus of an element ξ of the r–adelic skew–plane is the Euclidean modulus $|\xi|_+$ of its Euclidean component ξ_+. The r–adic modulus of an element ξ of the r–adelic skew–plane is the r–adic modulus $|\xi|_-$ of its r–adic component ξ_-. The r–adelic modulus of an element ξ of the r–adelic skew–plane is the product $|\xi|$ of its Euclidean modulus and its r–adic modulus. An element of the r–adelic skew–plane is said to be a unit if its Euclidean modulus and its r–adic modulus are one. An element of the r–adelic skew–plane is said to be unimodular if its r–adelic modulus is one. Haar measure for the r–adelic skew–plane is the Cartesian product of Haar measure for the Euclidean skew–plane and Haar measure for the r–adic skew–plane. Multiplication by an element of the r–adelic skew–plane multiplies Haar measure by the fourth power of the r–adelic modulus. A principal element of the r–adelic skew–plane is an element whose coordinates with respect to the canonical basis are principal elements of the r–adelic line. The principal elements of the r–adelic skew–plane form a closed subring whose nonzero elements are unimodular and invertible.

The adic skew–plane is an algebra of dimension four over the adic line which is generated by the same units i and j as the Euclidean skew–plane. The elements of the adic skew–plane are of the form

$$(\alpha + i\beta) + j(\gamma + i\delta)$$

for elements α, β, γ, and δ of the adic line. The conjugation of the adic skew–plane is the anti–automorphism ξ into ξ^- of order two which takes

$$(\alpha + i\beta) + j(\gamma + i\delta)$$

into

$$(\alpha - i\beta) - j(\gamma + i\delta)$$

for all elements α, β, γ, and δ of the adic line. The topology of the adic skew–plane is the Cartesian product of topologies of coordinate adic lines. If ξ is an element of the adic skew–plane, $\xi^-\xi$ is an element of the adic line which is invertible if, and only if, ξ is invertible. The adic modulus of an element ξ of the adic skew–plane is the nonnegative square root $|\xi|_-$ of the adic modulus of $\xi^-\xi$. An integral element of the adic skew–plane is an element ξ such that $\xi^-\xi$ is an integral element of the adic line. The integral elements of the adic skew–plane form a compact subring which is a neighborhood of the origin for the adic topology. A unit of the adic skew–plane is an invertible integral element whose inverse is integral. An element ξ of the adic skew–plane is a unit if, and only if, $\xi^-\xi$ is a unit of the adic line. Haar measure for the adic skew–plane is normalized so that the set of integral elements has measure one. Multiplication by an element of the adic skew–plane multiplies Haar measure by the fourth power of the adic modulus.

The adelic skew–plane is a locally compact ring which is the Cartesian product of the Euclidean skew–plane and the adic skew–plane. An element ξ of the adelic skew–plane has a Euclidean component ξ_+ and an adic component ξ_-. The conjugation of the adelic skew–plane is the anti–automorphism ξ into ξ^- of order two such that the Euclidean component of ξ^- is obtained from the Euclidean component of ξ under the conjugation of the Euclidean skew–plane and the adic component of ξ^- is obtained from the adic component of ξ under the conjugation of the adic skew–plane. The Euclidean modulus of an element ξ of the adelic skew–plane is the Euclidean modulus $|\xi|_+$ of its Euclidean component ξ_+. The adic modulus of an element ξ of the adelic skew–plane is the adic modulus $|\xi|_-$ of its adic component ξ_-. The adelic modulus of an element ξ of the adelic skew–plane is the product $|\xi|$ of its Euclidean modulus and its adic modulus. An element of the adelic skew–plane is said to be a unit if its Euclidean modulus and its adic modulus are one. An element of the adelic skew–plane is said to be unimodular if its adelic modulus is one. Haar measure for the adelic skew–plane is the Cartesian product of Haar measure for the Euclidean skew–plane and Haar measure for the adic skew–plane. Multiplication by an element of the adelic skew–plane multiplies Haar measure by the fourth power of the adelic modulus. A principal element of the adelic skew–plane is an element whose coordinates with respect to the canonical basis are principal elements of the adelic line. The principal elements of the adelic skew–plane form a discrete subring whose nonzero elements are unimodular and invertible.

The Fourier transformation for the adelic skew–plane is an isometric transformation whose domain and range are the space of square integrable functions with respect to Haar measure for the adelic skew–plane. The transformation takes a function $f(\xi)$ of ξ in the adelic skew–plane into a function $g(\eta)$ of η in the adelic skew–plane when the identity

$$g(\eta) = \int f(\xi) \exp(\pi i(\eta^- \xi + \xi^- \eta)) d\xi$$

is formally satisfied. The integral is accepted as the definition of the transformation when the integral with respect to Haar measure for the adelic skew–plane is absolutely convergent. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the adelic skew–plane. The identity

$$f(\eta) = \int g(\xi) \exp(-\pi i(\eta^- \xi + \xi^- \eta)) d\xi$$

holds with integration with respect to Haar measure for the adelic skew–plane when the integral is absolutely convergent. The Poisson summation formula

$$\sum f(\xi) = \sum g(\xi)$$

holds with summation over the principal elements of the adelic skew–plane when both integrals are absolutely convergent.

An r–adic plane is a maximal commutative subring of the r–adic skew–plane whose elements have rational r–adic modulus. An r–adic plane is closed under conjugation. A skew–conjugate element γ of the r–adic skew–plane, exists such that the identity

$$\gamma \xi = \xi^- \gamma$$

holds for every element ξ of the r–adic plane. It ι is an invertible skew–conjugate element of the r–adic plane, then the r–adic plane is the set of elements of the r–adic skew–plane which commute with ι. The projection of the r–adic skew–plane onto the r–adic plane takes ξ into

$$\tfrac{1}{2}\xi + \tfrac{1}{2}\iota^{-1}\xi\iota.$$

Haar measure for the r–adic plane is normalized so that the set of integral elements has measure one. Multiplication by an element of the r–adic plane multiplies Haar measure by the square of the r–adic modulus.

An r–adelic plane is the set of elements of the r–adelic skew–plane whose Euclidean component belongs to the Euclidean plane and whose r–adic component belongs to an r–adic plane. An r–adelic plane is a locally compact ring which is invariant under the conjugation of the r–adelic skew–plane. Haar measure for the r–adelic plane is the Cartesian product of Haar measure for the Euclidean plane and Haar measure for the r–adic plane. Multiplication by an element of the r–adelic plane multiplies Haar measure by the square of the r–adelic modulus.

An adic plane is the set of elements of the adic skew–plane whose p–adic component belongs to an p–adic plane for every prime p. An adic plane is a locally compact ring which is invariant under the conjugation of the adic skew–plane. Haar measure for the adic plane is normalized so that the set of integral elements has measure one. Multiplication by an element of the adic plane multiplies Haar measure by the square of the adic modulus.

An adelic plane is the set of elements of the adelic skew–plane whose Euclidean component belongs to the Euclidean plane and whose adic component belongs to an adic plane. An adelic plane is a locally compact ring which is invariant under the conjugation of the adelic skew–plane. Haar measure for the adelic plane is the Cartesian product of Haar measure for the Euclidean plane and Haar measure for the adic plane. Multiplication by an element of the adelic plane multiplies Haar measure by the square of the adelic modulus.

If ω is a unit of the Euclidean plane, an isometric transformation in the space of square integrable functions with respect to Haar measure for the Euclidean plane is defined by taking a function $f(\xi)$ of ξ in the Euclidean plane into the function $f(\omega\xi)$ of ξ on the Euclidean plane. The space is the orthogonal sum of invariant subspaces, which are indexed by the integers ν. When ν is equal to zero, a function of order ν is a function $f(\xi)$ of ξ in the Euclidean plane which satisfies the identity

$$f(\xi) = f(\omega\xi)$$

for every unit ω of the Euclidean plane. When ν is positive, a function of order ν is the product of a function of order zero and the function

$$\xi^\nu$$

of ξ in the Euclidean plane. A function of order $-\nu$ is the complex conjugate of a function of order ν.

If ω is a unit of the Euclidean skew–plane, an isometric transformation in the space of square integrable functions with respect to Haar measure for the Euclidean skew–plane is defined by taking a function $f(\xi)$ of ξ in the Euclidean skew–plane into the function $f(\omega\xi)$ of ξ in the Euclidean skew–plane. The space is the orthogonal sum of invariant subspaces, which are indexed by the integers ν. When ν is equal to zero, a function of order ν is a function $f(\xi)$ of ξ in the Euclidean skew–plane which satisfies the identity

$$f(\omega\xi) = f(\xi)$$

for every unit ω of the Euclidean skew–plane. When ν is a positive integer, a function of order ν is a finite linear combination with functions of order zero as coefficients of products

$$\prod(\tfrac{1}{2}\omega_k\xi - \tfrac{1}{2}i\omega_k\xi i)$$

with ω_k equal to one or j for every $k = 1, \ldots, \nu$. A function of order $-\nu$ is the complex conjugate of a function of order ν. The identity

$$\begin{aligned} i^\nu (i/z)^{1+\nu} \prod(\tfrac{1}{2}\omega_k\eta - \tfrac{1}{2}i\omega_k\eta i) \exp(-\pi i z^{-1}\eta^-\eta) \\ = \int \prod(\tfrac{1}{2}\omega_k\xi - \tfrac{1}{2}i\omega_k\xi i) \exp(\pi i z\xi^-\xi) \\ \times \exp(\pi i(\eta^-\xi + \xi^-\eta))d\xi \end{aligned}$$

holds when z is in the upper half–plane with integration with respect to Haar measure for the Euclidean skew–plane.

The Hankel transformation of order ν for the Euclidean plane is defined when ν is a nonnegative integer. If a function $f(\xi)$ of ξ in the Euclidean plane is square integrable with respect to Haar measure for the Euclidean plane and satisfies the identity

$$f(\omega\xi) = \omega^\nu f(\xi)$$

for every unit ω of the Euclidean plane, then its Hankel transform of order ν for the Euclidean plane is a function $g(\eta)$ of η in the Euclidean plane which is square integrable with respect to Haar measure for the Euclidean plane and which satisfies the identity

$$g(\omega\eta) = \omega^\nu g(\eta)$$

for every unit ω of the Euclidean plane. A positive parameter ρ is included in the definition of the transformation for application to zeta functions. The transformation takes the function

$$\xi^\nu \exp(\pi i\xi^-\lambda\xi/\rho)$$

of ξ in the Euclidean plane into the function

$$(i/\lambda)^{1+\nu}\xi^\nu \exp(-\pi i\xi^-\lambda^{-1}\xi/\rho)$$

of ξ in the Euclidean plane when λ is in the upper half–plane. The transformation is computed on a dense subset of its domain by the absolutely convergent integral

$$i^\nu \rho g(\eta) = \int f(\xi) \exp(\pi i(\eta^- \xi + \xi^- \eta)/\rho) d\xi$$

with respect to Haar measure for the Euclidean plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the Euclidean plane. The Hankel transformation of order ν for the Euclidean plane is its own inverse.

The Hankel transformation of order ν for the Euclidean skew–plane is defined when ν is an odd positive integer. The domain and the range of the transformation is the set of functions of ξ in the Euclidean skew–plane, which are square integrable with respect to Haar measure for the Euclidean skew–plane, which are of order ν, and which are the product of the function

$$(\tfrac{1}{2}\xi - \tfrac{1}{2}i\xi i)^\nu$$

and a function of order zero. A positive parameter ρ is included in the definition of the transformation for application to zeta functions. The transformation takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\xi)$ of ξ in the Euclidean skew–plane when the identity

$$\begin{aligned} &\int (\tfrac{1}{2}\xi^- - \tfrac{1}{2}i\xi^- i)^\nu g(\xi) \exp(2\pi i z \xi^- \xi/\rho) d\xi \\ &= (i/z)^{2+\nu} \int (\tfrac{1}{2}\xi^- - \tfrac{1}{2}i\xi^- i)^\nu f(\xi) \exp(-2\pi i z^{-1} \xi^- \xi/\rho) d\xi \end{aligned}$$

holds when z is in the upper half–plane with integration with respect to Haar measure for the Euclidean skew–plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the Euclidean skew–plane. The function $f(\xi)$ of ξ in the Euclidean skew–plane is the Hankel transform of order ν for the Euclidean skew–plane of the function $g(\xi)$ of ξ in the Euclidean skew–plane.

The Laplace transformation of order ν for the Euclidean plane permits a computation of the Hankel transformation of order ν for the Euclidean plane. The domain of the transformation is the space of functions $f(\xi)$ of ξ in the Euclidean plane which are square integrable with respect to Haar measure for the Euclidean plane and which satisfy the identity

$$f(\omega\xi) = \omega^\nu f(\xi)$$

for every unit ω of the Euclidean plane. A corresponding function $g(z)$ of z in the upper half–plane is defined by the absolutely convergent integral

$$2\pi g(z) = \int (\xi^\nu)^- f(\xi) \exp(\pi i z \xi^- \xi/\rho) d\xi$$

with respect to Haar measure for the Euclidean plane. The integral can be written

$$2\pi g(x+iy) = \pi \int_0^\infty (\xi^\nu)^- f(\xi) \exp(-\pi t y/\rho) \exp(\pi i t x/\rho) dt$$

as a Fourier integral for the Euclidean line under the constraint

$$t = \xi^- \xi.$$

The identity

$$(2/\rho) \int_{-\infty}^{+\infty} |g(x+iy)|^2 dx = \int_0^\infty |f(\xi)|^2 t^\nu \exp(-2\pi t y/\rho) dt$$

holds by the isometric property of the Fourier transformation for the Euclidean line. When ν is zero, the identity

$$(2\pi/\rho) \sup \int_{-\infty}^{+\infty} |g(x+iy)|^2 dx = \int |f(\xi)|^2 d\xi$$

holds with the least upper bound taken over all positive numbers y. The identity

$$(2\pi/\rho)^{1+\nu} \int_0^\infty \int_{-\infty}^{+\infty} |g(x+iy)|^2 y^{\nu-1} dx dy = \Gamma(\nu) \int |f(\xi)|^2 d\xi$$

holds when ν is positive. Integration on the right is with respect to Haar measure for the Euclidean plane. An analytic function $g(z)$ of z in the upper half–plane is a Laplace transform of order ν for the Euclidean plane if a finite least upper bound

$$\sup \int_{-\infty}^{+\infty} |g(x+iy)|^2 dx$$

is obtained over all positive numbers y when ν is zero and if the integral

$$\int_0^\infty \int_{-\infty}^{+\infty} |g(x+iy)|^2 y^{\nu-1} dx dy$$

is finite when ν is positive. The space of Laplace transforms of order ν for the Euclidean plane is a Hilbert space of functions analytic in the upper half–plane when it is considered with the scalar product for which the Laplace transformation of order ν for the Euclidean plane is isometric. The Hankel transformation of order ν for the Euclidean plane is unitarily equivalent under the Laplace transformation of order ν for the Euclidean plane to the isometric transformation in the space of analytic functions which takes $g(z)$ into

$$(i/z)^{1+\nu} g(-1/z).$$

The Laplace transformation of order ν for the Euclidean skew–plane permits a computation of the Hankel transformation of order ν for the Euclidean skew–plane. The domain of the transformation is the set of functions of ξ in the Euclidean skew–plane, which are square integrable with respect to Haar measure for the Euclidean skew–plane, which are of order ν, and which are the product of the function

$$(\tfrac{1}{2}\xi - \tfrac{1}{2}i\xi i)^\nu$$

and a function of order zero. The Laplace transform of order ν for the Euclidean skew–plane is the analytic function $g(z)$ of z in the upper half–plane defined by the integral

$$2\pi g(z) = \int (\tfrac{1}{2}\xi^- - \tfrac{1}{2}i\xi^- i)^\nu f(\xi) \exp(2\pi iz\xi^-\xi/\rho) d\xi$$

with respect to Haar measure for the Euclidean skew–plane. The identity

$$(2\nu + 4)(4\pi/\rho)^{2+\nu} \int_0^\infty \int_{-\infty}^{+\infty} |g(x+iy)|^2 y^\nu dxdy = 2\pi\Gamma(1+\nu) \int |f(\xi)|^2 d\xi$$

holds with integration on the right with respect to Haar measure for the Euclidean skew–plane. The space of Laplace transforms of order ν for the Euclidean skew–plane is a Hilbert space of functions analytic in the upper half–plane when it is considered with the scalar product for which the Laplace transformation of order ν for the Euclidean skew–plane is an isometry. The domain of the Laplace transformation of order ν for the Euclidean skew–plane is the domain and range of the Hankel transformation of order ν for the Euclidean skew–plane. The Hankel transformation of order ν for the Euclidean skew–plane is unitarily equivalent to the isometric transformation in the Hilbert space of analytic functions which takes $g(z)$ into

$$(i/z)^{2+\nu} g(-1/z).$$

A relation T with domain and range in a Hilbert space is said to be maximal dissipative if the relation $T - w$ has an everywhere defined inverse for some complex number w in the right half–plane and if the relation

$$(T - w)(T + w)^{-1}$$

is a contractive transformation. The condition holds for every element w of the right half–plane if it holds for some element w of the right half–plane.

The Radon transformation of order ν for the Euclidean plane is a maximal dissipative transformation in the space of functions $f(\xi)$ of ξ in the Euclidean plane which are square integrable with respect to Haar measure for the Euclidean plane and which satisfy the identity

$$f(\omega\xi) = \omega^\nu f(\xi)$$

for every unit ω of the Euclidean plane. The transformation takes a function $f(\xi)$ of ξ in the Euclidean plane into a function $g(\xi)$ of ξ in the Euclidean plane when the identity

$$\xi^{-\nu} g(\xi) = |\xi| \int_{-\infty}^{+\infty} (\xi + it\xi)^{-\nu} f(\xi + it\xi) dt$$

is formally satisfied. The integral is accepted as the definition of the transformation when

$$f(\xi) = \xi^\nu \exp(\pi i z \xi^- \xi / \rho)$$

when z is in the upper half–plane and ν is equal to zero or one. The identity,

$$g(\xi) = (i\rho/z)^{\frac{1}{2}} f(\xi)$$

then holds with the square root taken in the right half–plane. The adjoint of the Radon transformation of order ν for the Euclidean plane takes a function $f(\xi)$ of ξ in the Euclidean plane into a function $g(\xi)$ of ξ in the Euclidean plane when the identity

$$\int (\xi^\nu)^- g(\xi) \exp(\pi i z \xi^- \xi/\rho) d\xi$$
$$= (i\rho/z)^{\frac{1}{2}} \int (\xi^\nu)^- f(\xi) \exp(\pi i z \xi^- \xi/\rho) d\xi$$

holds with integration with respect to Haar measure for the Euclidean plane for ν equal to zero and one when z is in the upper half–plane. The square root is taken in the right half–plane. The Radon transformation of order ν for the Euclidean plane is the adjoint of its adjoint.

The Radon transformation of the Euclidean skew–plane is a maximal dissipative transformation in the space of square integrable functions with respect to Haar measure for the Euclidean skew–plane which are of order ν. The space of functions of order ν is the orthogonal sum of $1 + \nu$ closed subspaces, each of which is an invariant subspace in which the restriction of the Radon transformation for the Euclidean skew–plane is maximal dissipative. A subspace is determined by a product

$$\prod (\tfrac{1}{2}\omega_k \xi - \tfrac{1}{2} i \omega_k \xi i)$$

with ω_k equal to one or j for every $k = 1, \dots, \nu$. The elements of the subspace are the square integrable functions of ξ in the Euclidean skew–plane which are obtained on multiplying by a function of order zero.

Associated with the function $f(\xi)$ of ξ in the Euclidean skew–plane is a function $f(\xi, \eta)$ of ξ and η in the Euclidean skew–plane which agrees with $f(\xi)$ when η is equal to ξ. Each linear factor

$$\tfrac{1}{2}\omega_k \xi - \tfrac{1}{2} i \omega_k \xi i$$

in the product defining $f(\xi)$ either remains unchanged or is changed to

$$\tfrac{1}{2}\omega_k\eta - \tfrac{1}{2}i\omega_k\eta i$$

in the corresponding product defining $f(\xi,\eta)$. If the number of linear factors with ω_k equal to one is even, the number of these linear factors changed is equal to the number of these linear factors unchanged. If the number of linear factors with ω_k equal to one is odd, the number of these linear factors changed is one greater than the number of these linear factors unchanged. If the number of linear factors with ω_k equal to j is even, the number of these linear factors changed is equal to the number of these linear factors unchanged. If the number of linear factors with ω_k equal to j is odd, the number of these linear factors changed is one greater than the number of these linear factors unchanged. Since ν is assumed to be odd, the total number of linear factors changed is one greater than the total number of linear factors unchanged. The function of order zero which appears in $f(\xi)$ is replaced by a function of $\xi^-\xi + \eta^-\eta$ in $f(\xi,\eta)$.

The Radon transformation of order ν for the Euclidean skew–plane is defined by integration with respect to Haar measure for the hyperplane formed by the skew–conjugate elements of the skew–plane. An element ξ of the hyperplane satisfies the identity

$$\xi^- = -\xi.$$

The skew–plane is isomorphic to the Cartesian product of the hyperplane and the Euclidean line. Haar measure for the hyperplane is normalized so that Haar measure for the skew–plane is the Cartesian product of Haar measure for the hyperplane and Haar measure for the line. The Radon transformation of order ν for the Euclidean skew–plane takes a function $f(\xi)$ of ξ in the Euclidean skew–plane into a function $g(\xi)$ of ξ in the Euclidean skew–plane when the identity

$$\begin{aligned}(4\pi)^2(\tfrac{1}{2}\xi - \tfrac{1}{2}i\xi i)^{-\frac{1}{2}\nu+\frac{1}{2}}&(\tfrac{1}{2}\eta - \tfrac{1}{2}i\eta i)^{-\frac{1}{2}\nu-\frac{1}{2}}g(\xi,\eta)\\ = |\xi\eta| \iint &(\tfrac{1}{2}\xi + \tfrac{1}{2}\xi\alpha - \tfrac{1}{2}i\xi i - \tfrac{1}{2}i\xi\alpha i)^{-\frac{1}{2}\nu+\frac{1}{2}}\\ &\times(\tfrac{1}{2}\eta + \tfrac{1}{2}\eta\beta - \tfrac{1}{2}i\eta i - \tfrac{1}{2}i\eta\beta i)^{-\frac{1}{2}\nu-\frac{1}{2}}\\ &\times f(\xi+\xi\alpha, \eta+\eta\beta)|\alpha\beta|^{-2}d\alpha d\beta\end{aligned}$$

is formally satisfied with integration with respect to Haar measure for the hyperplane. The integral is accepted as the definitions when

$$f(\xi) = (\tfrac{1}{2}\xi - \tfrac{1}{2}i\xi i)^\nu \exp(2\pi iz\xi^-\xi/\rho)$$

with z in the upper half–plane, in which case

$$f(\xi,\eta) = (\tfrac{1}{2}\xi - \tfrac{1}{2}i\xi i)^{\frac{1}{2}\nu-\frac{1}{2}}(\tfrac{1}{2}\eta - \tfrac{1}{2}i\eta i)^{\frac{1}{2}\nu+\frac{1}{2}} \exp(\pi iz(\xi^-\xi + \eta^-\eta)/\rho).$$

Since the identity

$$g(\xi,\eta) = (i\rho/z)f(\xi,\eta)$$

is then satisfied, the identity

$$g(\xi) = (i\rho/z)f(\xi)$$

is satisfied. The adjoint of the Radon transformation of order ν for the Euclidean skew–plane takes a function $f(\xi)$ of ξ in the Euclidean skew–plane, which belongs to the domain of the Laplace transformation of order ν for the Euclidean skew–plane, into a function $g(\xi)$ of ξ in the Euclidean skew–plane, which belongs to the domain of the Laplace transformation of order ν for the Euclidean skew–plane, when the identity

$$\int (\tfrac{1}{2}\xi^- - \tfrac{1}{2}i\xi^- i)^\nu g(\xi) \exp(2\pi i z\xi^-\xi/\rho) d\xi$$
$$= (i\rho/z) \int (\tfrac{1}{2}\xi^- - \tfrac{1}{2}i\xi^- i)^\nu f(\xi) \exp(\pi i z\xi^-\xi/\rho) d\xi$$

holds with integration with respect to Haar measure for the Euclidean skew–plane when z is in the upper half–plane. The Radon transformation of order ν for the Euclidean skew–plane is the adjoint of its adjoint.

The domain of the Mellin transformation of order ν for the Euclidean plane is the space of functions $f(\xi)$ of ξ in the Euclidean plane which are square integrable with respect to Haar measure for the Euclidean plane, which satisfy the identity

$$f(\omega\xi) = \omega^\nu f(\xi)$$

for every unit ω of the Euclidean plane, and which vanish in a neighborhood of the origin. The Laplace transform of order ν for the Euclidean plane is the analytic function $g(z)$ of z in the upper half–plane defined by the integral

$$2\pi g(z) = \int (\xi^\nu)^- f(\xi) \exp(\pi i z\xi^-\xi/\rho) d\xi$$

with respect to Haar measure for the Euclidean plane. The Mellin transform of order ν for the Euclidean plane is an analytic function

$$F(z) = \int_0^\infty g(it) t^{\frac{1}{2}\nu - \frac{1}{2} - \frac{1}{2}iz} dt$$

of z in the upper half–plane. Since the function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu - \frac{1}{2} + \frac{1}{2}iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz)$$

admits an integral representation

$$W(z) = (\xi^-\xi)^{\frac{1}{2}\nu + \frac{1}{2} - \frac{1}{2}iz} \int_0^\infty \exp(-\pi t\xi^-\xi/\rho) t^{\frac{1}{2}\nu - \frac{1}{2} - \frac{1}{2}iz} dt$$

when z is in the upper half–plane, the identity

$$2\pi F(z)/W(z) = \int (\xi^\nu)^- f(\xi) |\xi|^{iz - \nu - 1} d\xi$$

holds when z is in the upper half–plane with integration with respect to Haar measure for the Euclidean plane. If $f(\xi)$ vanishes when $|\xi| < a$, the identity

$$\sup \int_{-\infty}^{+\infty} a^{2y} |F(x+iy)/W(x+iy)|^2 dx = \int |f(\xi)|^2 d\xi$$

holds with the least upper bound taken over all positive numbers y. Integration is with respect to Haar measure for the Euclidean plane.

The domain of the Mellin transform of order ν for the Euclidean skew–plane is the set of functions of ξ in the Euclidean skew–plane which are square integrable with respect to Haar measure for the Euclidean skew–plane, which are of order ν, and which are the product of the function

$$(\tfrac{1}{2}\xi - \tfrac{1}{2}i\xi i)^\nu$$

and a function of order zero which vanishes in a neighborhood of the origin. The Laplace transform of order ν for the Euclidean skew–plane is the analytic function $g(z)$ of z in the upper half–plane defined by the integral

$$2\pi g(z) = \int (\tfrac{1}{2}\xi^- - \tfrac{1}{2}i\xi^- i)^\nu f(\xi) \exp(2\pi i z \xi^- \xi/\rho) d\xi$$

with respect to Haar measure for the Euclidean skew–plane. The Mellin transform of order ν of the Euclidean skew–plane is an analytic function

$$F(z) = \int_0^\infty g(it) t^{\frac{1}{2}\nu - iz} dt$$

of z in the upper half–plane. Since the function

$$W(z) = (2\pi/\rho)^{-\frac{1}{2}\nu - 1 + iz} \Gamma(\tfrac{1}{2}\nu + 1 - iz)$$

admits an integral representation

$$W(z) = (\xi^- \xi)^{\frac{1}{2}\nu + 1 - iz} \int_0^\infty \exp(-2\pi t \xi^- \xi/\rho) t^{\frac{1}{2}\nu - iz} dt$$

when z is in the upper half–plane, the identity

$$2\pi F(z)/W(z) = \int (\tfrac{1}{2}\xi^- - \tfrac{1}{2}i\xi^- i)^\nu f(\xi) (\xi^- \xi)^{iz - \frac{1}{2}\nu - 1} d\xi$$

holds when z is in the upper half–plane with integration with respect to Haar measure for the Euclidean skew–plane. If $f(\xi)$ vanishes when $|\xi| < a$, the identity

$$\sup \int_{-\infty}^{+\infty} a^{2y} |F(x+iy)/W(x+iy)|^2 d\chi = \frac{\pi}{2\nu + 4} \int |f(\xi)|^2 d\xi$$

holds with the least upper bound taken over all positive numbers y. Integration on the right is with respect to Haar measure for the Euclidean skew–plane.

A characterization of Mellin transforms is made in weighted Hardy spaces. An analytic weight function is a function which is analytic and without zeros in the upper half–plane. The weighted Hardy space $\mathcal{F}(W)$ associated with an analytic weight function $W(z)$ is the Hilbert space $\mathcal{F}(W)$ whose elements are the analytic functions $F(z)$ of z in the upper half–plane such that a finite least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

is obtained over all positive numbers y. Since $F(z)/W(z)$ is of bounded type as a function of z in the upper half–plane, a boundary value function $F(x)/W(x)$ is defined almost everywhere with respect to Lebesgue measure on the real axis. The identity

$$\|F\|^2_{\mathcal{F}(W)} = \int_{-\infty}^{+\infty} |F(x)/W(x)|^2 dx$$

is satisfied. A continuous linear functional on the space is defined by taking $F(z)$ into $F(w)$ when w is in the upper half–plane. The reproducing kernel function for function values at w is

$$\frac{W(z)W(w)^-}{2\pi i(w^- - z)}.$$

The classical Hardy space for the upper half–plane is the weighted Hardy space $\mathcal{F}(W)$ when $W(z)$ is identically one. Multiplication by $W(z)$ is an isometric transformation of the classical Hardy space onto the weighted Hardy space $\mathcal{F}(W)$ whenever $W(z)$ is an analytic weight function for the upper half–plane.

The analytic weight function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu - \frac{1}{2} + \frac{1}{2}iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz)$$

appears on the characterization of Mellin transforms of order ν for the Euclidean plane. A maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space.

The analytic weight function

$$W(z) = (2\pi/\rho)^{-\frac{1}{2}\nu - 1 + iz} \Gamma(\tfrac{1}{2}\nu + 1 - iz)$$

appears in the characterization of Mellin transforms of order ν for the Euclidean skew–plane. A maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space.

Weighted Hardy spaces appear in which a maximal dissipative transformation is defined by taking $F(z)$ into $F(z+i)$ whenever the functions $F(z)$ and $F(z+i)$ of z belong to the space. The existence of a maximal dissipative shift in a weighted Hardy space $\mathcal{F}(W)$ is equivalent to properties of the weight function [7]. Since the adjoint transformation takes the reproducing kernel function

$$\frac{W(z)W(w-\frac{1}{2}i)^-}{2\pi i(w^-+\frac{1}{2}i-z)}$$

for function values at $w-\frac{1}{2}i$ in the upper half–plane into the reproducing kernel function

$$\frac{W(z)W(w+\frac{1}{2}i)}{2\pi i(w^- -\frac{1}{2}i-z)}$$

for function values at $w+\frac{1}{2}i$ in the upper half–plane, the function

$$\frac{W(z-\frac{1}{2}i)W(w+\frac{1}{2}i)^- + W(z+\frac{1}{2}i)W(w-\frac{1}{2}i)^-}{2\pi i(w^- - z)}$$

of z in the half–plane $iz^- - iz > 1$ is the reproducing kernel function for function values at w for a Hilbert space whose elements are functions analytic in the half–plane. The form of the reproducing kernel function implies that the elements of the space have analytic extensions to the upper half–plane. The weight function has an analytic extension to the half–plane such that

$$W(z)/W(z+i)$$

has nonnegative real part in the half–plane. This property of the weight function characterizes the weighted Hardy spaces which admit a maximal dissipative shift. If a weight function $W(z)$ has an analytic extension to the half–plane $-1 < iz^- - iz$ such that

$$W(z)/W(z+i)$$

has nonnegative real part in the half–plane, then a maximal dissipative transformation in the space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space.

Hilbert spaces appear whose elements are entire functions and which have these properties.

(H1) Whenever an element $F(z)$ of the space has a nonreal zero w, the function

$$F(z)(z-w^-)/(z-w)$$

belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined by taking $F(z)$ into $F(w)$ for every nonreal number w.

(H3) The function

$$F^*(z) = F(z^-)^-$$

belongs to the space whenever $F(z)$ belongs to the space, and it always has the same norm as $F(z)$.

Such spaces have simple structure. The complex numbers are treated as a coefficient Hilbert space with absolute value as norm. If w is a nonreal number, the adjoint of the transformation of the Hilbert space $\mathcal{H}$ into the coefficient space is a transformation of the coefficient space into $\mathcal{H}$ which takes c into $K(w,z)c$ for an entire function $K(w,z)$ of z. The identity

$$F(w) = \langle F(t), K(w,t)\rangle$$

reproduces the value at w of an element $F(z)$ of the space. A closed subspace consists of the functions which vanish at λ for a given nonreal number λ. The orthogonal projection in the subspace of an element $F(z)$ of the space is

$$F(z) - K(\lambda, z)K(\lambda, \lambda)^{-1}F(\lambda)$$

when the inverse of $K(\lambda, \lambda)$ exists. The properties of $K(\lambda, z)$ as a reproducing kernel function imply that $K(\lambda, \lambda)$ is a nonnegative number which vanishes only when $K(\lambda, z)$ vanishes identically. Calculations are restricted to the case in which $K(\lambda, \lambda)$ is nonzero since otherwise the space contains no nonzero element. If w is a nonreal number, the reproducing kernel function for function values at w in the subspace of functions which vanish at λ is

$$K(w,z) - K(\lambda, z)K(\lambda, \lambda)^{-1}K(w, \lambda).$$

The axiom (H1) implies that

$$[K(w,z) - K(\lambda, z)K(\lambda, \lambda)^{-1}K(w, \lambda)](z - \lambda^-)(w^- - \lambda)(z - \lambda)^{-1}(w^- - \lambda^-)^{-1}$$

is the reproducing kernel function for function values at w in the subspace of functions which vanish at λ^-. The identity

$$\begin{aligned} &(z - \lambda^-)(w^- - \lambda)[K(w,z) - K(\lambda, z)K(\lambda, \lambda)^{-1}K(w, \lambda)] \\ &= (z - \lambda)(w^- - \lambda^-)[K(w,z) - K(\lambda^-, z)K(\lambda^-, \lambda^-)^{-1}K(w, \lambda^-)] \end{aligned}$$

follows. The identity is applied in the equivalent form

$$\begin{aligned} &(\lambda - \lambda^-)(z - w^-)K(w,z) \\ &= (z - \lambda^-)K(\lambda, z)K(\lambda, \lambda)^{-1}(\lambda - w^-)K(w, \lambda) \\ &- (z - \lambda)K(\lambda^-, z)K(\lambda^-, \lambda^-)^{-1}(\lambda^- - w^-)K(w, \lambda^-). \end{aligned}$$

The axiom (H3) implies the symmetry condition

$$K(w,z) = K(w^-, z^-)^-.$$

An entire function $E(z)$ exists such that the identity

$$2\pi i(w^- - z)K(w,z) = E(z)E(w)^- - E^*(z)E(w^-)$$

holds for all complex numbers z and w. The inequality

$$|E(z^-)| < |E(z)|$$

applies when z is in the upper half–plane. Since the space is uniquely determined by the function $E(z)$, it is denoted $\mathcal{H}(E)$.

A Hilbert space $\mathcal{H}(E)$ is constructed for a given entire function $E(z)$ when the inequality

$$|E(z^-)| < |E(z)|$$

holds for z in the upper half–plane. A weighted Hardy space $\mathcal{F}(E)$ exists since $E(z)$ is an analytic weight function when considered as a function of z in the upper half–plane. The desired space $\mathcal{H}(E)$ is contained isometrically in the space $\mathcal{F}(E)$ and contains the entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(E)$. The axioms (H1), (H2), and (H3) are satisfied. If

$$E(z) = A(z) - iB(z)$$

for entire functions $A(z)$ and $B(z)$ such that

$$A(z) = A^*(z)$$

and

$$B(z) = B^*(z)$$

have real values on the real axis, the reproducing kernel function of the resulting space $\mathcal{H}(E)$ at a complex number w is

$$K(w, z) = \frac{B(z)A(w)^- - A(z)B(w)^-}{\pi(z - w^-)}.$$

If a Hilbert space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ with

$$E(z) = A(z) - iB(z)$$

for entire functions $A(z)$ and $B(z)$ which have real values on the real axis and if

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

is a matrix with real entries and determinant one, then the space is also isometrically equal to a space $\mathcal{H}(E_1)$ with

$$E_1(z) = A_1(z) - iB_1(z)$$

where the entire functions $A_1(z)$ and $B_1(z)$, which have real values on the real axis, are defined by the identities

$$A_1(z) = A(z)P + B(z)R$$

and

$$B_1(z) = A(z)Q + B(z)S.$$

A Hilbert space of entire functions is said to be symmetric about the origin if an isometric transformation of the space into itself is defined by taking $F(z)$ into $F(-z)$. The space is then the orthogonal sum of the subspace of even functions

$$F(z) = F(-z)$$

and of the subspace of odd functions

$$F(z) = -F(-z).$$

A Hilbert space $\mathcal{H}(E)$ is symmetric about the origin if the defining function $E(z)$ satisfies the symmetry condition

$$E^*(z) = E(-z).$$

The identity

$$E(z) = A(z) - iB(z)$$

then holds with $A(z)$ an even entire function and $B(z)$ an odd entire function which have real values on the real axis. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3), which is symmetric about the origin, and which contains a nonzero element, is isometrically to a space $\mathcal{H}(E)$ whose defining function $E(z)$ satisfies the symmetry condition.

If the defining function $E(z)$ of a space $\mathcal{H}(E)$ satisfies the symmetry condition, a Hilbert space $\mathcal{H}_+$ of entire functions, which satisfies the axioms (H1), (H2), and (H3), exists such that an isometric transformation of the space $\mathcal{H}_+$ onto the set of even elements of the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $F(z^2)$. If the space $\mathcal{H}_+$ contains a nonzero element, it is isometrically equal to a space $\mathcal{H}(E_+)$ for an entire function

$$E_+(z) = A_+(z) - iB_+(z)$$

defined by the identities

$$A(z) = A_+(z^2)$$

and

$$zB(z) = B_+(z^2).$$

The functions $A(z)$ and $zB(z)$ are linearly dependent when the space $\mathcal{H}_+$ contains no nonzero element. The space $\mathcal{H}(E)$ then has dimension one. A Hilbert space $\mathcal{H}_-$ of entire functions, which satisfies the axioms (H1), (H2), and (H3), exists such that an isometric transformation of the space $\mathcal{H}_-$ onto the set of odd elements of the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $zF(z^2)$. If the space $\mathcal{H}_-$ contains a nonzero element, it is isometrically equal to a space $\mathcal{H}(E_-)$ for an entire function

$$E_-(z) = A_-(z) - iB_-(z)$$

defined by the identities

$$A(z) = A_-(z^2)$$

and

$$B(z)/z = B_-(z^2).$$

The functions $A(z)$ and $B(z)/z$ are linearly dependent when the space $\mathcal{H}_-$ contains no nonzero element. The space $\mathcal{H}(E)$ then has dimension one.

An entire function $S(z)$ is said to be associated with a space $\mathcal{H}(E)$ if

$$[F(z)S(w) - S(z)F(w)]/(z - w)$$

belongs to the space for every complex number w whenever $F(z)$ belongs to the space. If a function $S(z)$ is associated with a space $\mathcal{H}(E)$, then

$$[S(z)B(w) - B(z)S(w)]/(z - w)$$

belongs to the space for every complex number w. The scalar product

$$\begin{gathered} B(\alpha)^- L(\beta, \alpha) B(\beta) \\ = (\beta - \alpha^-)\langle [S(t)B(\beta) - B(t)S(\beta)]/(t - \beta),\ [S(t)B(\alpha) - B(t)S(\alpha)]/(t - \alpha)\rangle_{\mathcal{H}(E)} \end{gathered}$$

is computable since the identities

$$L(\alpha^-, \beta^-) = -L(\beta, \alpha) = L(\alpha, \beta)^-$$

and

$$L(\beta, \gamma) - L(\alpha, \gamma) = L(\beta, \alpha^-)$$

hold for all complex numbers α, β, and γ. A function $\psi(z)$ of nonreal numbers z, which is analytic separately in the upper and lower half–planes and which satisfies the identity

$$\psi(z) + \psi^*(z) = 0,$$

exists such that

$$L(\beta, \alpha) = \pi i \psi(\beta) + \pi i \psi(\alpha)^-$$

for nonreal numbers α and β. The real part of the function is nonnegative in the upper half–plane.

If $F(z)$ is an element of the space $\mathcal{H}(E)$, a corresponding entire function $F^\sim(z)$ is defined by the identity

$$\begin{gathered} \pi B(w) F^\sim(w) + \pi i B(w) \psi(w) F(w) \\ = \langle F(t)S(w), [S(t)B(w^-) - B(t)S(w^-)]/(t - w^-)\rangle_{\mathcal{H}(E)} \end{gathered}$$

when w is not real. If $F(z)$ is an element of the space and if

$$G(z) = [F(z)S(w) - S(z)F(w)]/(z - w)$$

is the element of the space obtained for a complex number w, then the identity

$$G^\sim(z) = [F^\sim(z)S(w) - S(z)F^\sim(w)]/(z-w)$$

is satisfied. The identity for difference quotients

$$\begin{gathered}\pi G(\alpha)^- F^\sim(\beta) - \pi G^\sim(\alpha)^- F(\beta) \\ = \langle [F(t)S(\beta) - S(t)F(\beta)]/(t-\beta), G(t)S(\alpha)\rangle_{\mathcal{H}(E)} \\ -\langle F(t)S(\beta), [G(t)S(\alpha) - S(t)G(\alpha)]/(t-\alpha)\rangle_{\mathcal{H}(E)} \\ -(\beta-\alpha^-)\langle [F(t)S(\beta) - S(t)F(\beta)]/(t-\beta), [G(t)S(\alpha) - S(t)G(\alpha)]/(t-\alpha)\rangle_{\mathcal{H}(E)}\end{gathered}$$

holds for all elements $F(z)$ and $G(z)$ of the space when α and β are nonreal numbers.

The transformation which takes $F(z)$ into $F^\sim(z)$ is a generalization of the Hilbert transformation. The graph of the transformation is a Hilbert space whose elements are pairs

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions. The skew–conjugate unitary matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is treated as a generalization of the imaginary unit. The space of column vectors with complex entries is considered with the Euclidean scalar product

$$\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\rangle = \begin{pmatrix} a \\ b \end{pmatrix}^- \begin{pmatrix} a \\ b \end{pmatrix}.$$

Examples are obtained in a related theory of Hilbert spaces whose elements are pairs of entire functions. If w is a complex number, the pair

$$\begin{pmatrix} [F_+(z)S(w) - S(z)F_+(w)]/(z-w) \\ [F_-(z)S(w) - S(z)F_-(w)]/(z-w) \end{pmatrix}$$

belongs to the space whenever

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

belongs to the space. The identity for difference quotients

$$\begin{gathered}-2\pi \begin{pmatrix} G_+(\alpha) \\ G_-(\alpha) \end{pmatrix}^- I \begin{pmatrix} F_+(\beta) \\ F_-(\beta) \end{pmatrix} \\ = \langle \begin{pmatrix} [F_+(t)S(\beta) - S(t)F_+(\beta)]/(t-\beta) \\ [F_-(t)S(\beta) - S(t)F_-(\beta)]/(t-\beta) \end{pmatrix}, \begin{pmatrix} G_+(t)S(\alpha) \\ G_-(t)S(\alpha) \end{pmatrix}\rangle \\ -\langle \begin{pmatrix} F_+(t)S(\beta) \\ F_-(t)S(\beta) \end{pmatrix}, \begin{pmatrix} [G_+(t)S(\alpha) - S(t)G_+(\alpha)]/(t-\alpha) \\ [G_-(t)S(\alpha) - S(t)G_-(\alpha)]/(t-\alpha) \end{pmatrix}\rangle \\ -(\beta-\alpha^-)\langle \begin{pmatrix} [F_+(t)S(\beta) - S(t)F_+(\beta)]/(t-\beta) \\ [F_-(t)S(\beta) - S(t)F_-(\beta)]/(t-\beta) \end{pmatrix}, \begin{pmatrix} [G_+(t)S(\alpha) - S(t)G_+(\alpha)]/(t-\alpha) \\ [G_-(t)S(\alpha) - S(t)G_-(\alpha)]/(t-\alpha) \end{pmatrix}\rangle\end{gathered}$$

holds for all elements

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

and

$$\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$$

of the space when α and β are complex numbers. A continuous transformation of the space into the space of column vectors with complex entries takes

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

into

$$\begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}$$

when w is not real. The adjoint transformation takes

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

into

$$\frac{M(z)IM(w)^- - S(z)IS(w)^-}{2\pi(z - w^-)} \begin{pmatrix} u \\ v \end{pmatrix}$$

for a function

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

with matrix values which is independent of w. The entries of the matrix are entire functions which have real values on the real axis. Since the space with these properties is uniquely determined by $S(z)$ and $M(z)$, it is denoted $\mathcal{H}_S(M)$. If $M(z)$ is a given matrix of entire functions which are real on the real axis, necessary and sufficient conditions for the existence of a space $\mathcal{H}_S(M)$ are the matrix identity

$$M(z^-)IM(z)^- = S(z^-)IS(z)^-$$

and the matrix inequality

$$\frac{M(z)IM(z)^- - S(z)IS(z)^-}{z - z^-} \geq 0$$

for all complex numbers z.

An example of a space $\mathcal{H}_S(M)$ is obtained when an entire function $S(z)$ is associated with a space $\mathcal{H}(E)$. The Hilbert transformation associates an entire function $F^\sim(z)$ with every element $F(z)$ of the space in such a way that an identity for difference quotients is satisfied. The graph of the Hilbert transformation is a Hilbert space $\mathcal{H}_s(M)$ with

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

and

$$E(z) = A(z) - iB(z)$$

for entire function $C(z)$ and $D(z)$ which have real values on the real axis. The elements of the space are of the form

$$\begin{pmatrix} F(z) \\ F^\sim(z) \end{pmatrix}$$

with $F(z)$ in $\mathcal{H}(E)$. The identity

$$\left\| \begin{pmatrix} F(t) \\ F^\sim(t) \end{pmatrix} \right\|^2_{\mathcal{H}_s(M)} = 2\|F(t)\|^2_{\mathcal{H}(E)}$$

is satisfied.

The relationship between factorization and invariant subspaces is an underlying theme of the theory of Hilbert spaces of entire functions. A matrix factorization applies to entire functions $E(z)$ such that a space $\mathcal{H}(E)$ exists. When several such functions appear in factorization, it is convenient to index them with a real parameter which is treated as a new variable. When functions $E(a, z)$ and $E(b, z)$ are given, the question arises whether the space $\mathcal{H}(E(a))$ with parameter a is contained isometrically in the space $\mathcal{H}(E(b))$ with parameter b. The question is answered by answering two simpler questions. The first is whether the space $\mathcal{H}(E(a))$ is contained contractively in the space $\mathcal{H}(E(b))$. The second is whether the inclusion is isometric.

If a Hilbert space $\mathcal{P}$ is contained contractively in a Hilbert space $\mathcal{H}$, a unique Hilbert space $\mathcal{Q}$, which is contained contractively in $\mathcal{H}$, exists such that the inequality

$$\|c\|^2_{\mathcal{H}} \leq \|a\|^2_{\mathcal{P}} + \|b\|^2_{\mathcal{Q}}$$

holds whenever $c = a + b$ with a in $\mathcal{P}$ and b in $\mathcal{Q}$ and such that every element c of $\mathcal{H}$ admits some decomposition for which equality holds. The space $\mathcal{Q}$ is called the complementary space to $\mathcal{P}$ in $\mathcal{H}$. Minimal decomposition of an element c of $\mathcal{H}$ is unique. The element a of $\mathcal{P}$ is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. The element b of $\mathcal{Q}$ is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$. The intersection of $\mathcal{P}$ and $\mathcal{Q}$ is a Hilbert space $\mathcal{P} \wedge \mathcal{Q}$, which is contained contractively in $\mathcal{H}$, when considered with scalar product determined by the identity

$$\|c\|^2_{\mathcal{P}\wedge\mathcal{Q}} = \|c\|^2_{\mathcal{P}} + \|c\|^2_{\mathcal{Q}}.$$

The inclusion of $\mathcal{P}$ in $\mathcal{H}$ is isometric if, and only if, the space $\mathcal{P} \wedge \mathcal{Q}$ contains no nonzero element. The inclusion of $\mathcal{Q}$ in $\mathcal{H}$ is then isometric. A Hilbert space $\mathcal{H}$ which is so decomposed is written $P \vee Q$.

The space $\mathcal{H}_S(M)$ is denoted $\mathcal{H}(M)$ when $S(z)$ is identically one. An estimate of coefficients in the power series expansion of $M(z)$ applies when the matrix is the identity at the origin. A nonnegative matrix

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = M'(0)I$$

is constructed from derivatives at the origin. The Schmidt norm $\sigma(M)$ of a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is the nonnegative solution of the equations

$$\sigma(M)^2 = |A|^2 + |B|^2 + |C|^2 + |D|^2.$$

The coefficients in the power series expansion

$$M(z) = \sum M_n z^n$$

satisfy the inequality

$$\sigma(M_n) \leq (\alpha + \gamma)^n / n!$$

for every positive integer n.

If

$$E(a,z) = A(a,z) - iB(a,z)$$

is an entire function such that a space $\mathcal{H}(E(a))$ exists and if

$$M(a,b,z) = \begin{pmatrix} A(a,b,z) & B(a,b,z) \\ C(a,b,z) & D(a,b,z) \end{pmatrix}$$

is matrix of entire functions such that a space $\mathcal{H}(M(a,b))$ exists, then an entire function

$$E(b,z) = A(b,z) - iB(b,z)$$

such that a space $\mathcal{H}(E(b))$ exists is defined by the matrix product

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z).$$

If $F(z)$ is an element of the space $\mathcal{H}(E(a))$ and if

$$G(z) = \begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$$

is an element of the space $\mathcal{H}(M(a,b))$, then

$$H(z) = F(z) + A(a,z)G_+(z) + B(a,z)G_-(z)$$

is an element of the space $\mathcal{H}(E(b))$ which satisfies the inequality

$$\|H(z)\|^2_{\mathcal{H}(E(b))} \leq \|F(z)\|^2_{\mathcal{H}(E(a))} + \tfrac{1}{2}\|G(z)\|^2_{\mathcal{H}(M(a,b))}.$$

Every element $H(z)$ of the space $\mathcal{H}(E(b))$ admits such a decomposition for which equality holds.

The set of elements $G(z)$ of the space $\mathcal{H}(M(a,b))$ such that

$$A(a,z)G_+(z) + B(a,z)G_-(z)$$

belongs to the space $\mathcal{H}(E(a))$ is a Hilbert space $\mathcal{L}$ with scalar product determined by the identity

$$\|G(z)\|^2_{\mathcal{L}} = \|G(z)\|^2_{\mathcal{H}(M(a,b))} + 2\ \|A(a,z)G_+(z) + B(a,z)G_-(z)\|^2_{\mathcal{H}(E(a))}.$$

The pair

$$[F(z) - F(w)]/(z-w) = \begin{pmatrix} [F_+(z) - F_+(w)]/(z-w) \\ [F_-(z) - F_-(w)]/(z-w) \end{pmatrix}$$

belongs to the space for every complex number w whenever

$$F(z) = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

belongs to the space. The identity for difference quotients

$$\begin{aligned} 0 = \langle [F(t) - F(\beta)]/(t-\beta), G(t)\rangle_{\mathcal{L}} \\ -\langle F(t), [G(t) - G(\alpha)]/(t-\alpha)\rangle_{\mathcal{L}} \\ -(\beta - \alpha^-)\langle [F(t) - F(\beta)]/(t-\beta), [G(t) - G(\alpha)]/(t-\alpha)\rangle_{\mathcal{L}} \end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space when α and β are complex numbers. These properties imply that the elements of the space $\mathcal{L}$ are pairs

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

of constants which satisfy the identity

$$v^-u = u^-v.$$

The inclusion of the space $\mathcal{H}(E(a))$ in the space $\mathcal{H}(E(b))$ is isometric if, and only if, no nonzero pair of complex numbers u and v, which satisfy the identity, exists such that

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to the space $\mathcal{H}(M(a,b))$ and

$$A(a,z)u + B(a,z)v$$

belongs to the space $\mathcal{H}(E(a))$.

A converse result holds. Assume that $E(a,z)$ and $E(b,z)$ are entire functions such that spaces $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ exist and such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$. Assume that a nontrivial entire function $S(z)$ is associated with the spaces $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$. A generalization of the Hilbert transformation is defined on the space $\mathcal{H}(E(b))$, which takes an element $F(z)$ of the space $\mathcal{H}(E(b))$ into an entire function $F^\sim(z)$. The transformation takes

$$[F(z)S(w) - S(z)F(w)]/(z-w)$$

into

$$[F^\sim(z)S(w) - S(z)F^\sim(w)]/(z-w)$$

for every complex number w whenever it takes $F(z)$ into $F^\sim(z)$. An identity for difference quotients is satisfied. A generalization of the Hilbert transformation is also defined with similar properties on the space $\mathcal{H}(E(a))$. The transformation on the space $\mathcal{H}(E(a))$ is chosen as the restriction of the transformation on the space $\mathcal{H}(E(b))$. The graph of the Hilbert transformation on the space $\mathcal{H}(E(b))$ is a space $\mathcal{H}_S(M(b))$ for a matrix

$$M(b,z) = \begin{pmatrix} A(b,z) & B(b,z) \\ C(b,z) & D(b,z) \end{pmatrix}$$

of entire functions which have real values on the real axis. The matrix is chosen so that the identity

$$E(b,z) = A(b,z) - iB(b,z)$$

is satisfied. The graph of the Hilbert transformation on the space $\mathcal{H}(E(a))$ is a space $\mathcal{H}_S(M(a))$ for a matrix

$$M(a,z) = \begin{pmatrix} A(a,z) & B(a,z) \\ C(a,z) & D(a,z) \end{pmatrix}$$

of entire functions which have real values on the real axis. The matrix is chosen so that the identity

$$E(a,z) = A(a,z) - iB(a,z)$$

is satisfied. Since the space $\mathcal{H}(E(a))$ is contained isometrically in the space $\mathcal{H}(E(b))$ and since the generalized Hilbert transformation on the space $\mathcal{H}(E(a))$ is consistent with the generalized Hilbert transformation on the space $\mathcal{H}(E(b))$, the space $\mathcal{H}_S(M(a))$ is contained isometrically in the space $\mathcal{H}_S(M(b))$. A matrix

$$M(a,b,z) = \begin{pmatrix} A(a,b,z) & B(a,b,z) \\ C(a,b,z) & D(a,b,z) \end{pmatrix}$$

of entire functions is defined as the solution of the equation

$$M(b,z) = M(a,z)M(a,b,z).$$

The entries of the matrix are entire functions which have real values on the real axis. Multiplication by $M(a,z)$ acts as an isometric transformation of the desired space $\mathcal{H}(M(a,b))$ onto the orthogonal complement of the space $\mathcal{H}_S(M(a))$ in the space $\mathcal{H}_S(M(b))$. This completes the construction of a space $\mathcal{H}(M(a,b))$ which satisfies the identity

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z).$$

A simplification occurs in the theory of isometric inclusions for Hilbert spaces of entire functions [2]. Assume that $E(a,z)$ and $E(b,z)$ are entire functions, which have no real zeros, such that spaces $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ exist. If a weighted Hardy space $\mathcal{F}(W)$ exists such that the spaces $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ are contained isometrically in the space $\mathcal{F}(W)$, then either the space $\mathcal{H}(E(a))$ is contained isometrically in this space $\mathcal{H}(E(b))$ or the space $\mathcal{H}(E(b))$ is contained isometrically in the space $\mathcal{H}(E(a))$.

The hereditary nature of symmetry about the origin is an application of the ordering theorem for Hilbert spaces of entire functions. Assume that $E(a,z)$ and $E(b,z)$ are entire functions, which have no real zeros, such that spaces $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ exist. The space $\mathcal{H}(E(a))$ is symmetric about the origin if it is contained isometrically in the space $\mathcal{H}(E(b))$ and if the space $\mathcal{H}(E(b))$ is symmetric about the origin. If the symmetry conditions

$$E^*(a,z) = E(a,-z)$$

and

$$E^*(b,z) = E(b,-z)$$

are satisfied, then the identity

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

holds for a space $\mathcal{H}(M(a,b))$ whose defining matrix

$$M(a,b,z) = \begin{pmatrix} A(a,b,z) & B(a,b,z) \\ C(a,b,z) & D(a,b,z) \end{pmatrix}$$

has even entire functions on the diagonal and odd entire functions off the diagonal.

An entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half–plane, if the inequality

$$|E(x-iy)| \le |E(x+iy)|$$

holds for every real number x when y is positive, and if $|E(x+iy)|$ is a nondecreasing function of positive numbers y for every real number x. A polynomial is of Pólya class if it has no zeros in the upper half–plane. A pointwise limit of entire functions Pólya class is an entire function of Pólya class if it does not vanish identically. Every entire function of

Pólya class is a limit, uniformly on compact subsets of the complex plane, of polynomials which have no zeros in the upper half–plane. An entire function $E(z)$ of Pólya class which has no zeros is to the form

$$E(z) = E(0)\exp(-az^2 - ibz)$$

for a nonnegative number a and a complex number b whose real part is nonnegative. An entire function $E(z)$ of Pólya class is said to be determined by its zeros if it is a limit uniformly on compact subsets of the complex plane of polynomials whose zeros are contained in the zeros of $E(z)$. An entire function of Pólya class is the product of an entire function of Pólya class which has no zeros and an entire function of Pólya class which is determined by its zeros.

The pervasiveness of the Pólya class is due to its preservation under bounded type perturbations. An entire function $S(z)$ is of Pólya class if it has no zeros in the upper half–plane, if it satisfies the inequality

$$|S(x - iy)| \le |S(x + iy)|$$

for every real number x when y is positive, and if an entire function $E(z)$ of Pólya class exists such that

$$S(z)/E(z)$$

is of bounded type in the upper half–plane.

Transformations, whose domain and range are contained in Hilbert spaces of entire functions satisfying the axioms (H1), (H2), and (H3), are defined using reproducing kernel functions. Assume that the domain of the transformation is contained in a space $\mathcal{H}(E)$ and that the range of the transformation is contained in a space $\mathcal{H}(E')$. The domain of the transformation is assumed to contain the reproducing kernel functions for function values in the space $\mathcal{H}(E)$. The domain of the adjoint transformation is assumed to contain the reproducing kernel functions for function values in the space $\mathcal{H}(E')$. Define $L(w, z)$ to be the element of the space $\mathcal{H}(E)$ obtained under the adjoint transformation from the reproducing kernel function for function values at w in the space $\mathcal{H}(E')$. Then the transformation takes an element $F(z)$ of the space $\mathcal{H}(E)$ into an element $G(z)$ of the space $\mathcal{H}(E')$ if, and only if, the identity

$$G(w) = \langle F(t), L(w, t)\rangle_{\mathcal{H}(E)}$$

holds for all complex numbers w. Define $L'(w, z)$ to be the element of the space $\mathcal{H}(E')$ obtained under the transformation from the reproducing kernel function for function values at w in the space $\mathcal{H}(E)$. Then the adjoint transformation takes an element $F(z)$ of the space $\mathcal{H}(E')$ into an element $G(z)$ of the space $\mathcal{H}(E)$ if, and only if, the identity

$$G(w) = \langle F(t), L'(w, t)\rangle_{\mathcal{H}(E')}$$

holds for all complex numbers w. The identity

$$L'(w, z) = L(z, w)^-$$

is a consequence of the adjoint relationship.

The existence of reproducing kernel functions for transformations with domain and range in Hilbert spaces of entire functions is a generalization of the axiom (H2). The transformations are also assumed to satisfy a generalization of the axiom (H1).

Assume that a given transformation has domain in a space $\mathcal{H}(E)$ and range in a space $\mathcal{H}(E')$. If λ is a nonreal number, then the set of entire functions $F(z)$ such that $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) when considered with the scalar product such that multiplication by $z-\lambda$ is an isometric transformation of the space into the space $\mathcal{H}(E)$. If $F(z)$ is an entire function, then $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E)$ if, and only if, $(z-\lambda^-)F(z)$ belongs to the space $\mathcal{H}(E)$. The norm of $(z-\lambda)F(z)$ in the space $\mathcal{H}(E)$ is equal to the norm of $(z-\lambda^-)F(z)$ in the space $\mathcal{H}(E)$. The set of entire functions $F(z)$ such that $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E')$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) when considered with the scalar product such that multiplication by $z-\lambda$ is an isometric transformation of the space into the space $\mathcal{H}(E')$. If $F(z)$ is an entire function, then $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E')$ if, and only if, $(z-\lambda^-)F(z)$ belongs to the space $\mathcal{H}(E')$. The norm of $(z-\lambda)F(z)$ in the space $\mathcal{H}(E')$ is equal to the norm of $(z-\lambda^-)F(z)$ in the space $\mathcal{H}(E')$. The induced relation at λ takes an entire function $F(z)$ such that $(z-\lambda)F(z)$ belongs to the space $\mathcal{H}(E)$ into an entire function $G(z)$ such that $(z-\lambda)G(z)$ belongs to the space $\mathcal{H}(E')$ when the given transformation takes an element $H(z)$ of the space $\mathcal{H}(E)$ into the element $(z-\lambda)G(z)$ of the space $\mathcal{H}(E')$ and when $(z-\lambda)F(z)$ is the orthogonal projection of $H(z)$ into the set of elements of the space $\mathcal{H}(E)$ which vanish at λ. The given transformation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$ is said to satisfy the axiom (H1) if the induced relation at λ coincides with the induced relation at λ^- for every nonreal number λ.

An identity in reproducing kernel functions results when the given transformation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$ satisfies the generalization of the axioms (H1) and (H2) if the induced relations are transformations. The reproducing kernel function for the transformation at w is an element $L(w,z)$ of the space $\mathcal{H}(E)$ such that the identity

$$G(w) = \langle F(t), L(w,t)\rangle_{\mathcal{H}(E)}$$

holds for every complex number w. If the reproducing kernel function $L(\lambda,z)$ at λ vanishes at λ for some complex number λ, then the reproducing kernel function for the adjoint transformation at λ vanishes at λ. Since the orthogonal projection of $K(\lambda,z)$ into the subspace of elements of the space $\mathcal{H}(E)$ which vanish at λ is equal to zero, the reproducing kernel function for the adjoint transformation at λ is equal to zero if the induced relation at λ is a transformation. It follows that $L(\lambda,z)$ vanishes identically if it vanishes at λ.

If λ is a nonreal number such that $L(\lambda,z)$ does not vanish at λ, then for every complex number w, the function

$$L(w,z) - L(\lambda,z)L(\lambda,\lambda)^{-1}L(w,\lambda)$$

of z is an element of the space $\mathcal{H}(E)$ which vanishes at λ. The function

$$\frac{L(w,z) - L(\lambda,z)L(\lambda,\lambda)^{-1}L(w,\lambda)}{(z-\lambda)(w^- - \lambda^-)}$$

of z is the reproducing kernel function at w for the induced transformation at λ. If $L(\lambda^-, z)$ does not vanish at λ^-, the function

$$\frac{L(w,z) - L(\lambda^-,z)L(\lambda^-,\lambda^-)^{-1}L(w,\lambda^-)}{(z-\lambda^-)(w^- - \lambda)}$$

of z is the reproducing kernel function at w for the induced transformation at λ^-. Since these reproducing kernel functions apply to the same transformation, they are equal. The resulting identity can be written

$$L(w,z) = [Q(z)P(w^-) - P(z)Q(w^-)]/[\pi(z-w^-)]$$

for entire functions $P(z)$ and $Q(z)$ which are associated with the spaces $\mathcal{H}(E)$ and $\mathcal{H}(E')$. If the spaces are symmetric about the origin and if the transformation takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$, then the functions $P(z)$ and $Q(z)$ can be chosen to satisfy the symmetry conditions

$$P(-z) = P^*(z)$$

and

$$Q(-z) = -Q^*(z).$$

A transformation with domain in a space $\mathcal{H}(E)$ and range in a space $\mathcal{H}(E')$ is said to satisfy the axioms (H1) and (H2) if entire functions, which are associated with the spaces $\mathcal{H}(E)$ and $\mathcal{H}(E')$, exist such that the transformation takes an element $F(z)$ of $\mathcal{H}(E)$ into an element $G(z)$ of $\mathcal{H}(E')$, when and only when, the identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E)}$$

holds for all complex numbers w and if the adjoint takes an element $F(z)$ of $\mathcal{H}(E')$ into an element $G(z)$ of $\mathcal{H}(E)$ when, and only when, the identity

$$G(w) = \langle F(t), [Q^*(t)P(w)^- - P^*(t)Q(w)^-]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E')}$$

holds for all complex numbers w. The transformation is said to be symmetric about the origin if the spaces are symmetric about the origin and if the transformation takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$. If the transformation is symmetric about the origin, the defining functions $P(z)$ and $Q(z)$ can be chosen to satisfy the symmetry conditions

$$P(-z) = P^*(z)$$

and

$$Q(-z) = -Q^*(z).$$

Special Hilbert spaces of entire functions appear which admit maximal transformations of dissipative deficiency at most one. The transformation, which has domain and range in a space $\mathcal{H}(E)$, is defined by entire functions $P(z)$ and $Q(z)$ which are associated with the space. The transformation takes $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space which satisfy the identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E)}$$

for all complex numbers w. The space is symmetric about the origin and the functions satisfy the symmetry conditions

$$P(-z) = P^*(z)$$

and

$$Q(-z) = -Q^*(z)$$

when the transformation is not maximal dissipative. The reproducing kernel function for function values at $w+i$ in the space belongs to the domain of the adjoint for every complex number w. The function

$$[Q(z)P(w^-) - P(z)Q(w^-)]/[\pi(z-w^-)]$$

of z is obtained under the action of the adjoint. A Krein space of Pontryagin index at most one exists whose elements are entire functions and which admits the function

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z-w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w,

$$A(z) = P(z - \tfrac{1}{2}i)$$

and

$$B^*(z) = Q(z - \tfrac{1}{2}i).$$

The space is a Hilbert space when the transformation is maximal dissipative. The symmetry conditions

$$A(-z) = A^*(z)$$

and

$$B(-z) = -B^*(z)$$

are satisfied when the transformation is not maximal dissipative.

Hilbert spaces appear whose elements are entire functions whose structure is derived from the structure of Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3).

Theorem 1. *Assume that for some entire functions $A(z)$ and $B(z)$ a Hilbert space $\mathcal{H}$ exists whose elements are entire functions and which contains the function*

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w. Then a Hilbert space $\mathcal{P}$ exists whose elements are entire functions and which contains the function

$$\frac{[A^*(z) - iB^*(z)][A(w^-) + iB(w^-)] - [A(z) + iB(z)][A(w)^- - iB(w)^-]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. And a Hilbert space $\mathcal{Q}$ exists whose elements are entire functions and which contains the function

$$\frac{[A(z) - iB(z)][A(w)^- + iB(w)^-] - [A^*(z) + iB^*(z)][A(w^-) - iB(w^-)]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. The spaces $\mathcal{P}$ and $\mathcal{Q}$ are contained contractively in the space $\mathcal{H}$ and are complementary spaces to each other in $\mathcal{H}$.

Proof of Theorem 1. The desired conclusion is immediate when the function $A(z) - iB(z)$ vanishes identically since the space $\mathcal{P}$ is then isometrically equal to the space $\mathcal{H}$ and the space $\mathcal{Q}$ contains no nonzero element. The desired conclusion is also immediate when the function $A(z) + iB(z)$ vanishes identically since the space $\mathcal{Q}$ is then isometrically equal to the space $\mathcal{H}$ and the space $\mathcal{P}$ contains no nonzero element. When

$$S(z) = [A(z) - iB(z)][A(z) + iB(z)]$$

does not vanish identically, the determinants $S(z)$ of the matrix

$$U(z) = \begin{pmatrix} A(z) & -B(z) \\ B(z) & A(z) \end{pmatrix}$$

and $S^*(z)$ of the matrix

$$V(z) = \begin{pmatrix} A^*(z) & B^*(z) \\ -B^*(z) & A^*(z) \end{pmatrix}$$

do not vanish identically. It will be shown that a Hilbert space exists whose elements are pairs

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions and which contains the pair

$$\frac{V(z)IV(w)^- - U(z)IU(w)^-}{2\pi(z-w^-)} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

of entire functions of z as reproducing kernel function for function values at w in the direction

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

for every complex number w and for every pair of complex numbers c_+ and c_-. The resulting element of the Hilbert space represents the linear functional which takes a pair

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions of z into the number

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix}^- \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = c_+^- F_+(w) + c_-^- F_-(w).$$

The existence of the space is equivalent to the existence of a space $\mathcal{H}_S(M)$ with

$$M(z) = S(z)U(z)^{-1}V(z).$$

Since the space $\mathcal{H}_S(M)$ exists if the matrix

$$\frac{M(z)IM(w)^- - S(z)IS(w)^-}{2\pi(z-w^-)}$$

is nonnegative whenever z and w are equal, the desired Hilbert space exists if the matrix

$$\frac{V(z)IV(w)^- - U(z)IU(w)^-}{2\pi(z-w^-)}$$

is nonnegative whenever z and w are equal. Multiplication by $S(z)U(z)^{-1}$ is then an isometric transformation of the space $\mathcal{H}_S(M)$ onto the desired space. Since the matrix is diagonal whenever z and w are equal, the matrix is nonnegative if its trace

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z-w^-)}$$

is nonnegative whenever z and w are equal. Since the trace is as a function of z the reproducing kernel function for function values at w in the given space $\mathcal{H}$, the trace is nonnegative when z and w are equal. This completes the construction of the desired Hilbert space of pairs of entire functions.

Since the matrix

$$\frac{V(z)IV(w)^- - U(z)IU(w)^-}{2\pi(z - w^-)}$$

commutes with I for all complex numbers z and w, multiplication by z is an isometric transformation of the space onto itself. The space is the orthogonal sum of a subspace of eigenvectors for the eigenvalue i and a subspace of eigenvectors for the eigenvalue $-i$. The existence of the desired Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$ follows. Every element of the space is of the form

$$\begin{pmatrix} F(z) + G(z) \\ iG(z) - iF(z) \end{pmatrix}$$

with $F(z)$ in $\mathcal{P}$ and $G(z)$ in $\mathcal{Q}$. The desired properties of the spaces $\mathcal{P}$ and $\mathcal{Q}$ follow from the computation of reproducing kernel functions.

This completes the proof of the theorem.

A Hilbert space $\mathcal{P}$ exists whose elements are entire functions and which admits the function

$$\frac{[A^*(z) - iB^*(z)][A(w^-) + iB(w^-)] - [A(z) + iB(z)][A(w)^- - iB(w)^-]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w if, and only if, the entire functions $A(z)$ and $B(z)$ satisfy the inequality

$$|A(z) + iB(z)| \leq |A^*(z) - iB^*(z)|$$

when z is in the upper half–plane. The space contains no nonzero element when the entire functions

$$A^*(z) - iB^*(z)$$

and

$$A(z) + iB(z)$$

are linearly dependent. Otherwise an entire function $S(z)$, which satisfies the symmetry condition

$$S(z) = S^*(z),$$

exists such that

$$E(z) = [A^*(z) - iB^*(z)]/S(z)$$

is an entire function which satisfies the inequality

$$|E^*(z)| < |E(z)|$$

when z is in the upper half–plane. Multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E)$ onto the space $\mathcal{P}$.

A Hilbert space $\mathcal{Q}$ exists whose elements are entire functions and which admits the function

$$\frac{[A(z) - iB(z)][A(w)^- + iB(w)^-] - [A^*(z) + iB^*(z)][A(w^-) - iB(w^-)]}{2\pi i(w^- - z}$$

of z as reproducing kernel function for function values at w for every complex number w if, and only if, the entire functions $A(z)$ and $B(z)$ satisfy the inequality

$$|A^*(z) + iB^*(z)| \leq |A(z) - iB(z)|$$

when z is in the upper half–plane. The space contains no nonzero element when the entire functions

$$A^*(z) + iB^*(z)$$

and

$$A(z) - iB(z)$$

are linearly dependent. Otherwise an entire function $S(z)$, which satisfies the symmetry condition

$$S(z) = S^*(z),$$

exists such that

$$E(z) = [A(z) - iB(z)]/S(z)$$

is an entire function which satisfies the inequality

$$|E^*(z)| < |E(z)|$$

when z is in the upper half–plane. Multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E)$ onto the space $\mathcal{Q}$.

The structure theory for Hilbert spaces generalizes to Krein spaces of Pontryagin index at most one whose elements are entire functions and which admit the function

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w when the entire functions $A(z)$ and $B(z)$ satisfy the symmetry conditions

$$A(-z) = A^*(z)$$

and

$$B(-z) = -B^*(z).$$

The space is the orthogonal sum of a subspace of even functions and a subspace of odd functions, both of which are Krein spaces of Pontryagin index at most one. At least one of the subspaces is a Hilbert space.

Entire functions $A_+(z)$ and $B_+(z)$ are defined by the identities

$$A(z) + A^*(z) = A_+(z^2) + A_+^*(z^2)$$

and

$$B(z) - B^*(z) = B_+(z^2) - B_+^*(z^2)$$

and

$$zA(z) - zA^*(z) = A_+(z^2) - A_+^*(z^2)$$

and

$$zB(z) + zB^*(z) = B_+(z^2) + B_+^*(z^2).$$

A Krein space $\mathcal{H}_+$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{B_+^*(z)A_+(w^-) - A_+(z)B_+(w)^- + B_+(z)A_+(w)^- - A_+^*(z)B_+(w^-)}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w. An isometric transformation of the space $\mathcal{H}_+$ onto the subspace of even elements of the space $\mathcal{H}$ is defined by taking $F(z)$ into $F(z^2)$.

Entire functions $A_-(z)$ and $B_-(z)$ are defined by the identities

$$A(z) + A^*(z) = A_-(z^2) + A_-^*(z^2)$$

and

$$B(z) - B^*(z) = B_-(z^2) - B_-^*(z^2)$$

and

$$A(z) - A^*(z) = zA_-(z^2) - zA_-^*(z^2)$$

and

$$B(z) + B^*(z) = zB_-(z^2) + zB_-^*(z^2).$$

A Krein space $\mathcal{H}_-$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{B_-^*(z)A_-(w^-) - A_-(z)B_-(w)^- + B_-(z)A_-(w)^- - A_-^*(z)B_-(w^-)}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w. An isometric transformation of the space $\mathcal{H}_-$ onto the subspace of odd elements of $\mathcal{H}$ is defined by taking $F(z)$ into $zF(z^2)$.

A Krein space $\mathcal{P}_+$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{[A_+^*(z) - iB_+^*(z)][A_+(w^-) + iB_+(w^-)] - [A_+(z) + iB_+(z)][A_+(w)^- - iB_+(w)^-]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. A Krein space $\mathcal{Q}_+$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{[A_+(z) - iB_+(z)][A_+(w)^- + iB_+(w)^-] - [A_+^*(z) + iB_+^*(z)][A_+(w^-) - iB_+(w^-)]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for all complex numbers w. The spaces $\mathcal{P}_+$ and $\mathcal{Q}_+$ are Hilbert spaces if $\mathcal{H}_+$ is a Hilbert space.

A Krein space $\mathcal{P}_-$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{[A_-^*(z) - iB_-^*(z)][A_-(w^-) + iB_-(w^-)] - [A_-(z) + iB_-(z)][A_-(w)^- - iB(w)^-]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. A Krein space $\mathcal{Q}_-$ of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{[A_-(z) - iB_-(z)][A_-(w)^- + iB(w)^-] - [A_-^*(z) + iB_-^*(z)][A_-(w^-) - iB_-(w^-)]}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for every complex number w. The spaces $\mathcal{P}_-$ and $\mathcal{Q}_-$ are Hilbert spaces if $\mathcal{H}_-$ is a Hilbert space.

A relationship between the spaces $\mathcal{P}_+$ and $\mathcal{P}_-$ and between the spaces $\mathcal{Q}_+$ and $\mathcal{Q}_-$ results from the identities

$$A_+(z) + A_+^*(z) = A_-(z) + A_-^*(z)$$

and

$$B_+(z) - B_+^*(z) = B_-(z) - B_-^*(z)$$

and

$$A_+(z) - A_+^*(z) = zA_-(z) - zA_-^*(z)$$

and

$$B_+(z) + B_+^*(z) = zB_-(z) + zB_-^*(z).$$

The space $\mathcal{P}_+$ contains $zF(z)$ whenever $F(z)$ is an element of the space $\mathcal{P}_-$ such that $zF(z)$ belongs to $\mathcal{P}_-$. The space $\mathcal{P}_-$ contains every element of the space $\mathcal{P}_+$ such that $zF(z)$ belongs to $\mathcal{P}_+$. The identity

$$\langle tF(t), G(t)\rangle_{\mathcal{P}_+} = \langle F(t), G(t)\rangle_{\mathcal{P}_-}$$

holds whenever $F(z)$ is an element of the space $\mathcal{P}_-$ such that $zF(z)$ belongs to the space $\mathcal{P}_+$ and $G(z)$ is an element of the space $\mathcal{P}_-$ which belongs to the space $\mathcal{P}_+$. The closure in the space $\mathcal{P}_+$ of the intersection of the spaces $\mathcal{P}_+$ and $\mathcal{P}_-$ is a Hilbert space which is contained continuously and isometrically in the space $\mathcal{P}_+$ and whose orthogonal complement has dimension zero or one. The closure in the space $\mathcal{P}_-$ of the intersection of the spaces $\mathcal{P}_+$ and $\mathcal{P}_-$ is a Hilbert space which is contained continuously and isometrically in the space $\mathcal{P}_-$ and whose orthogonal complement has dimension zero or one.

The space $\mathcal{Q}_+$ contains $zF(z)$ whenever $F(z)$ is an element of the space $\mathcal{Q}_-$ such that $zF(z)$ belongs to $\mathcal{Q}_-$. The space $\mathcal{Q}_-$ contains every element $F(z)$ of the space $\mathcal{Q}_+$ such that $zF(z)$ belongs to $\mathcal{Q}_+$. The identity

$$\langle tF(t), G(t)\rangle_{\mathcal{Q}_+} = \langle F(t), G(t)\rangle_{\mathcal{Q}_-}$$

holds whenever $F(z)$ is an element of the space $\mathcal{Q}_-$ such that $zF(z)$ belongs to the space $\mathcal{Q}_+$ and $G(z)$ is an element of the space $\mathcal{Q}_-$ which belongs to the space $\mathcal{Q}_+$. The closure in the space $\mathcal{Q}_+$ of the intersection of the spaces $\mathcal{Q}_+$ and $\mathcal{Q}_-$ is a Hilbert space which is contained continuously and isometrically in the space $\mathcal{Q}_+$ and whose orthogonal complement has dimension zero or one. The closure in the space $\mathcal{Q}_-$ of the intersection of the spaces $\mathcal{Q}_+$ and $\mathcal{Q}_-$ is a Hilbert space which is contained continuously and isometrically in the space $\mathcal{Q}_-$ and whose orthogonal complement has dimension zero or one.

The Mellin transformation of order ν for the Euclidean plane gives information about the Hankel transformation of order ν for the Euclidean plane. The success of the application is due to the appearance of entire functions as Euclidean Mellin transforms of order ν of functions which vanish in a neighborhood of the origin and whose Euclidean Hankel transform of order ν also vanishes in a neighborhood of the origin. The existence of nontrivial Hankel transform pairs with these properties was observed in 1880 by Nikolai Sonine [12]. The results cited by Hardy and Titchmarsh [10] and motivate the doctoral thesis of Virginia Rovnyak [11]. If a is a positive number, a nontrivial function $f(\xi)$ of ξ in the Euclidean plane exists which is in the domain of the Hankel transformation of order ν for the Euclidean plane, which vanishes in the neighborhood $|\xi| < a$ of the origin, and whose Hankel transform of order ν for the Euclidean plane vanishes in the same neighborhood. Corresponding results follow for the Hankel transformation of order ν for the Euclidean skew–plane. If a is a positive number, a nontrivial function $f(\xi)$ of ξ in the Euclidean skew–plane exists which is in the domain of the Hankel transformation of order ν for the Euclidean skew–plane, which vanishes in the neighborhood $|\xi| < a$ of the origin, and whose Hankel transform for the Euclidean skew–plane vanishes in the same neighborhood.

Examples of Hilbert spaces of entire functions appear in the theory of the Hankel transformation of order ν for the Euclidean plane. The spaces are associated with the analytic weight function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu - \frac{1}{2} + \frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz).$$

The Sonine spaces of order ν for the Euclidean plane are Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) and which contain nonzero elements. The elements of the space of parameter a are the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$. Multiplication by a^{-iz} is an isometric transformation of the space of parameter a into the space $\mathcal{F}(W)$. The elements of the space of parameter a are the Mellin transforms of order ν for the Euclidean plane of functions $f(\xi)$ of ξ in the Euclidean plane which vanish in the neighborhood $|\xi| < a$ of the origin and whose Hankel transform of order ν for the Euclidean plane vanishes in the same neighborhood.

Examples of Hilbert spaces of entire functions appear in the theory of the Hankel transformation of order ν for the Euclidean skew–plane. The spaces are associated with the analytic weight function

$$W(z) = (2\pi/\rho)^{\frac{1}{2}\nu - 1 + iz}\Gamma(\tfrac{1}{2}\nu + 1 - iz).$$

The Sonine spaces of order ν for the Euclidean skew–plane are Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) and which contain nonzero elements. The elements of the space of parameter a are the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the space $\mathcal{F}(W)$. Multiplication by a^{-iz} is an isometric transformation of the space of parameter a into the space $\mathcal{F}(W)$. The elements of the space are the Mellin transforms of order ν for the Euclidean skew–plane of functions $f(\xi)$ of ξ in the Euclidean skew–plane which vanish in the neighborhood $|\xi| < a$ of the origin and whose Hankel transform of order ν for the Euclidean skew–plane vanishes in the same neighborhood.

Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) appear in maximal totally ordered families. If spaces $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ belong to the same family, then either a matrix factorization

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

holds for some space $\mathcal{H}(M(a,b))$ or a matrix factorization

$$(A(a,z), B(a,z)) = (A(b,z), B(b,z))M(b,a,z)$$

holds for some space $\mathcal{H}(M(b,a))$. The spaces $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ are isometrically equal when both factorizations apply. Parametrizations are made in such a way that the applicable factorization can be read from the parameters. The real numbers are applied in a parametrization which is useful for theoretical purposes. A space $\mathcal{H}(E(a))$ of the family is less than or equal to a space $\mathcal{H}(E(b))$ of the family when a is less than or equal to b. The inclusion is isometric on the domain of multiplication by z in the space $\mathcal{H}(E(a))$. The factorization

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

holds for a space $\mathcal{H}(M(a,b))$. It is convenient to choose the defining functions of the spaces so that the matrix $M(a,b,z)$ is always the identity matrix at the origin. Derivatives at the origin are then used to define matrices

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

with real entries, which are functions of parameters t, so that the identity

$$m(b) - m(a) = M'(a,b,0)I$$

holds when a is less than or equal to b. The ratio

$$E(b,z)/E(a,z)$$

is of bounded type as a function of z in the upper half–plane for all parameters a and b. The mean type of the ratio in the half–plane is of the form

$$\tau(b) - \tau(a)$$

for a function $\tau(t)$ of parameters t which is unique within an added constant. Multiplication by

$$\exp[i\tau(a)z]/\exp[i\tau(b)z]$$

is a contractive transformation of the space $\mathcal{H}(E(a))$ into the space $\mathcal{H}(E(b))$ which is isometric on the domain of multiplication by z in the space $\mathcal{H}(E(a))$. The matrix

$$\begin{pmatrix} \alpha(t) & \beta(t) + i\tau(t) \\ \beta(t) - i\tau(t) & \gamma(t) \end{pmatrix}$$

is a nondecreasing function of t. The factorization is made so that the matrix has trace t. An analytic weight function $W(z)$ may exist such that multiplication by

$$\exp[i\tau(a)z]$$

is a contractive transformation of every space $\mathcal{H}(E(a))$ into the space $\mathcal{F}(W)$, which is isometric on the domain of multiplication by z in the space $\mathcal{H}(E(a))$. A greatest parameter b exists for every parameter a such that

$$\tau(a) = \tau(b).$$

The space $\mathcal{H}(E(b))$ is the set of entire functions $F(z)$ such that

$$\exp[i\tau(b)z]F(z)$$

and

$$\exp[i\tau(b)z]F^*(z)$$

belong to the space $\mathcal{F}(W)$. Multiplication by

$$\exp[i\tau(b)z]$$

is an isometric transformation of the space $\mathcal{H}(E(b))$ into the space $\mathcal{F}(W)$. If the defining function of some space in the family is of Pólya class, then the defining function of every space in the family is of Pólya class. If the function $\tau(t)$ of parameters t has a finite lower bound, then the positive numbers can be used as parameters of the family. An entire function $S(z)$, which has real values on the real axis and which has only real zeros, then exists such that

$$E(a,z)/S(z)$$

is an entire function of exponential type for every parameter a. If the function $\tau(t)$ of parameters t has no finite lower bound, then all real numbers are required as parameters.

An example of an analytic weight function $W(z)$ is obtained when $W(z)$ is an entire function of Pólya class. A space $\mathcal{H}(W)$ exists if the functions $W(z)$ and $W^*(z)$ are linearly independent. The space is a member of a maximal totally ordered family of Hilbert spaces of entire functions associated with the analytic weight function $W(z)$. Assume that no entire function $S(z)$, which has real values on the real axis and which has only real zeros, exists such that

$$W(z)/S(z)$$

is an entire function of exponential type. Then a nonzero entire function $F(z)$ exists for every real number τ such that

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

belong to the space $\mathcal{F}(W)$. The set of entire functions $F(z)$ such that

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

belong to the space $\mathcal{F}(W)$ is a space $\mathcal{H}(E)$ such that multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of the space into the space $\mathcal{F}(W)$. The Hilbert spaces of entire functions so obtained need not be the only members of the maximal totally ordered family of spaces associated with the space $\mathcal{F}(W)$. The members of the family are parametrized by real numbers t so that the factorization

$$(A(b,z),B(b,z)) = (A(a,z),B(a,z))M(a,b,z)$$

holds for a space $\mathcal{H}(M(a,b))$ when a is less than or equal to b. The defining functions of the spaces are chosen so that the matrix $M(a,b,z)$ is always the identity matrix of the origin. The parametrization is made so that the matrix

$$\begin{pmatrix} \alpha(t) & \beta(t)+i\tau(t) \\ \beta(t)-i\tau(t) & \gamma(t) \end{pmatrix}$$

has trace t. The parameters are chosen so that the space $\mathcal{H}(W)$ is the space $\mathcal{H}(E(t))$ when t is equal to zero. The function $\tau(t)$ and the entries of $m(t)$ are chosen to have value zero at the origin.

Maximal totally ordered families of Hilbert spaces of entire functions are well behaved under approximation of weight functions. Assume that

$$W(z) = \lim W_n(z)$$

is a limit uniformly on compact subsets of the upper half–plane of analytic weight functions $W_n(z)$. A maximal totally ordered family of Hilbert spaces $\mathcal{H}(E_n(t))$ of entire functions is assumed to be associated with the analytic weight function $W_n(z)$ for every positive integer n. The parametrization is made with real numbers t so that the factorization

$$(A_n(b,z), B_n(b,z)) = (A_n(a,z), B_n(a,z))M_n(a,b,z)$$

holds for a space $\mathcal{H}(M_n(a,b))$ when a is less than or equal to b. The defining functions of the spaces are chosen so that the matrix $M_n(a,b,z)$ is always the identity matrix at the origin. The parametrization is made so that the matrix

$$\begin{pmatrix} \alpha(t) & \beta_n(t)+i\tau_n(t) \\ \beta_n(t)-i\tau_n(t) & \gamma_n(t) \end{pmatrix}$$

has trace t. Assume that the function $\tau_n(t)$ does not have a finite lower bound. All real numbers t are then parameters. Assume also that the defining functions of the spaces in the family are of Pólya class. The parametrization is made so that the space of parameter zero is contained isometrically in the space $\mathcal{F}(W_n)$ and is the set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(W_n)$. The function $\tau(t)$ and the entries of $m(t)$ are chosen to have value zero at the origin. Then for every parameter a the identity

$$\begin{aligned} &\frac{B(a,z)A(a,w)^- - A(a,z)B(a,w)^-}{\pi(z-w^-)} \\ &= \lim \frac{B_n(a,z)A_n(a,w)^- - A_n(a,z)B_n(a,w)^-}{\pi(z-w^-)} \end{aligned}$$

holds uniformly on compact subsets of the complex plane for every complex number w. The defining functions of the approximating spaces can be chosen so that the identity

$$E(a,z) = \lim E_n(a,z)$$

holds uniformly on compact subsets of the complex plane for every real number a. The identities

$$m(t) = \lim m_n(t)$$

and

$$\tau(t) = \lim \tau_n(t)$$

hold for all real numbers t.

In applications to Riemann zeta functions the spaces of a maximal totally ordered family of Hilbert spaces of entire functions are parametrized by positive numbers so that the factorization

$$(A(a,z), B(a,z)) = (A(b,z), B(b,z))M(b,a,z)$$

holds for a space $\mathcal{H}(M(b,a))$ when a is less than or equal to b. The space $\mathcal{H}(E(b))$ is contained contractively in the space $\mathcal{H}(E(a))$. The inclusion is isometric on the domain of multiplication by z in the space $\mathcal{H}(E(b))$.

The Sonine spaces of entire functions for the Euclidean plane are examples of Hilbert spaces of entire functions for which all members of the totally ordered family are known [1]. An analytic weight function $W(z)$ is given for which all members of the maximal totally ordered family are constructed. A Hilbert space of entire functions is defined for every positive number a, whose elements are the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the space $\mathcal{F}(W)$, such that multiplication by a^{-iz} is an isometric transformation of the space into the space $\mathcal{F}(W)$. The space satisfies the axioms (H1), (H2), and (H3), and contains a nonzero element. The spaces obtained are members of a maximal totally ordered family. The construction produces all members of the family. The positive numbers a in the construction are suitable as parameters of the members of the family.

A construction of Hilbert spaces of entire functions from Hilbert spaces of entire functions is made which preserves the structure of families and which is computable on weight functions. The construction is made from information about zeros of entire functions.

If $\mathcal{H}(E)$ is a given space and if a complex number λ is a zero of some nonzero element of the space, then a new Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is constructed. The elements of the new space are the entire functions $F(z)$ such that

$$(z - \lambda)F(z)$$

belongs to the given space $\mathcal{H}(E)$. A scalar product is defined in the new space so that multiplication by $z - \lambda$ is an isometric transformation of the new space into the space $\mathcal{H}(E)$. The new space is isometrically equal to a space $\mathcal{H}(E_1)$ for some entire function $E_1(z)$.

The construction of Hilbert spaces of entire functions from Hilbert spaces of entire functions is well behaved with respect to the partial ordering of spaces. Assume that $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ are given spaces such that the matrix factorization

$$(A(a,z), B(a,z)) = (A(b,z), B(b,z))M(b,a,z)$$

holds for a space $\mathcal{H}(M(b,a))$. The space $\mathcal{H}(E(b))$ is then contained contractively in the space $\mathcal{H}(E(a))$ and the inclusion is isometric on a closed subspace of $\mathcal{H}(E(b))$ of codimension zero or one. If a complex number λ is a zero of some nonzero element of the space $\mathcal{H}(E(b))$, then it is a zero of some nonzero element of the space $\mathcal{H}(E(a))$. Multiplication by $z-\lambda$ is an isometric transformation of some space $\mathcal{H}(E_1(b))$ onto the set of elements of the space $\mathcal{H}(E(b))$ which vanish at λ. Multiplication by $z-\lambda$ is an isometric transformation of some space $\mathcal{H}(E_1(a))$ onto the set of elements of the space $\mathcal{H}(E(a))$ which vanish at λ. The space $\mathcal{H}(E_1(b))$ is contained contractively in the space $\mathcal{H}(E_1(a))$ and the inclusion is isometric on a closed subspace of the space $\mathcal{H}(E_1(b))$ of codimension zero or one. The matrix factorization

$$(A_1(a,z), B_1(a,z)) = (A_1(b,z), B_1(b,z))M_1(b,a,z)$$

holds for some space $\mathcal{H}(M_1(b,a))$.

The construction of Hilbert spaces of entire functions from Hilbert spaces of entire functions is well behaved with respect to analytic weight functions when λ does not belong to the upper half–plane. Assume that an analytic weight function $W(z)$ has been used to construct a space $\mathcal{H}(E)$ using a positive number a. An entire function $F(z)$ then belongs to the space $\mathcal{H}(E)$ if, and only if,

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the space $\mathcal{F}(W)$. Multiplication by a^{-iz} is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{F}(W)$. If a complex number λ is a zero of some nonzero element of the space $\mathcal{H}(E)$, then a space $\mathcal{H}(E_1)$ exists such that multiplication by $z-\lambda$ is an isometric transformation of the space onto the set of element of the space $\mathcal{H}(E)$ which vanish at λ. If λ is not in the upper half–plane, an analytic weight function $W_1(z)$ exists such that

$$W(z) = (z-\lambda)W_1(z).$$

The space $\mathcal{H}(E_1)$ is then the set of entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the space $\mathcal{F}(W_1)$. Multiplication by a^{-iz} is an isometric transformation of the space $\mathcal{H}(E_1)$ into the space $\mathcal{F}(W_1)$.

The construction of Hilbert spaces of entire functions from Hilbert spaces of entire functions is computable. Assume that a space $\mathcal{H}(E)$ has dimension greater than one and that a complex number λ is not a zero of a given entire function $S(z)$ associated with the space. A partially isometric transformation of the space $\mathcal{H}(E)$ onto a Hilbert space of entire functions is defined by taking $F(z)$ into

$$\frac{F(z)S(\lambda) - S(z)F(\lambda)}{z - \lambda}.$$

The kernel of the transformation consists of the multiplies of $S(z)$ which belong to $\mathcal{H}(E)$. If $S(z)$ and

$$\frac{B(z)A(\lambda)^- - A(z)B(\lambda)^-}{\pi(z - \lambda^-)}$$

are linearly dependent, the range of the transformation is isometrically equal to a space $\mathcal{H}(E_1)$ with

$$E_1(z) = \frac{E(z)S(\lambda) - S(z)E(\lambda)}{z - \lambda}.$$

A construction of Hilbert spaces of entire functions from Hilbert spaces of entire functions is made using an entire function $S(z)$ of Pólya class which is determined by its zeros. Assume that a space $\mathcal{H}(E_0)$ is given. Then the set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to $\mathcal{H}(E_0)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) when considered with the unique scalar product such that multiplication by $S(z)$ is an isometric transformation of the space into the space $\mathcal{H}(E_0)$. The space is isometrically equal to a space $\mathcal{H}(E)$ if it contains a nonzero element. The space can be constructed inductively. Assume that $S(z)$ is a limit, uniformly on compact subsets of the complex plane, of polynomials $S_n(z)$ such that

$$S_{n+1}(z)/S_n(z)$$

is always a linear function of z. Then the set of entire functions $F(z)$ such that $S_n(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$ is isometrically equal to a space $\mathcal{H}(E_n)$ such that multiplication by $S_n(z)$ is an isometric transformation of the space $\mathcal{H}(E_n)$ into the space $\mathcal{H}(E_0)$. The limit

$$\begin{aligned} & S(z) \ \frac{E(z)E(w)^- - E^*(z)E(w^-)}{2\pi i(w^- - z)} \ S(w)^- \\ &= \lim S_n(z) \ \frac{E_n(z)E_n(w)^- - E_n^*(z)E_n(w^-)}{2\pi i(w^- - z)} \ S_n(w)^- \end{aligned}$$

holds in the metric topology of the space $\mathcal{H}(E_0)$ for every complex number w. The choice of defining function $E_n(z)$ can be made for every index n so that the limit

$$E(z) = \lim E_n(z)$$

holds uniformly on compact subsets of the complex plane.

The construction of Hilbert spaces of entire functions from Hilbert spaces of entire functions is well behaved with respect to the partial ordering of spaces. Assume that $\mathcal{H}(E_0(a))$ and $\mathcal{H}(E_0(b))$ are given spaces such that the matrix factorization

$$(A_0(a,z), B_0(a,z)) = (A_0(b,z), B_0(b,z))M_0(b,a,z)$$

holds for a space $\mathcal{H}(M_0(b,a))$. Assume that an entire function $S(z)$ of Pólya class is determined by its zeros. If a nonzero entire function $F(z)$ exists such that $S(z)F(z)$ belongs to $\mathcal{H}(E_0(b))$, then a nonzero entire function $F(z)$ exists such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0(a))$. The set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0(b))$ is a space $\mathcal{H}(E(b))$ such that multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E(b))$ into the space $\mathcal{H}(E_0(b))$. The set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to $\mathcal{H}(E_0(a))$ is a space $\mathcal{H}(E(a))$ such that multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E(a))$ into the space $\mathcal{H}(E_0(a))$. The matrix factorization

$$(A(a,z), B(a,z)) = (A(b,z), B(b,z))M(b,a,z)$$

holds for a space $\mathcal{H}(M(b,a))$.

The construction of Hilbert spaces of entire functions from Hilbert spaces of entire functions is well behaved with respect to analytic weight functions. Assume that an analytic weight function $W_0(z)$ has been used to construct a space $\mathcal{H}(E_0)$ using a positive number a. An entire function $F(z)$ then belongs to the space $\mathcal{H}(E_0)$ if, and only if,

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the space $\mathcal{F}(W_0)$. Multiplication by a^{-iz} is an isometric transformation of the space $\mathcal{H}(E_0)$ into the space $\mathcal{F}(W_0)$. Assume that an entire function $S(z)$ of Pólya class is determined by its zeros. If a nonzero entire function $F(z)$ exists such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$, then the set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$ is a space $\mathcal{H}(E)$ such that multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E_0)$. An analytic weight function $W(z)$ exists such that

$$S(z)W(z) = W_0(z).$$

An entire function $F(z)$ belongs to the space $\mathcal{H}(E)$ if, and only if,

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong the space $\mathcal{F}(W)$. Multiplication by a^{-iz} is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{F}(W)$.

The construction of Hilbert spaces of entire functions from Hilbert spaces of entire functions yields information about the spaces constructed from information about the given spaces when the information is propagated in the construction. When the given spaces are constructed from analytic weight functions, the information originates in weighted Hardy spaces.

Theorem 2. *Assume that $W(z)$ is an analytic weight function such that a maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space. If a space $\mathcal{H}(E)$ is contained isometrically in the space $\mathcal{F}(W)$ and contains every entire function $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(W)$, then a maximal dissipative transformation in the space $\mathcal{H}(E)$ is defined by $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space.*

Proof of Theorem 2. The dissipative property of the transformation in the space $\mathcal{H}(E)$ is immediate from the dissipative property of the transformation in the space $\mathcal{F}(W)$. The maximal dissipative property of the transformation in the space $\mathcal{H}(E)$ is verified by showing that every element of the space is of the form

$$F(z) + F(z+i)$$

for an element $F(z)$ of the space such that $F(z+i)$ belongs to the space. Since the transformation in the space $\mathcal{F}(W)$ is maximal dissipative, every element of the space is of the desired form for an element $F(z)$ of the space $\mathcal{F}(W)$ such that $F(z+i)$ belongs to the space $\mathcal{F}(W)$. It needs to be shown that $F(z)$ and $F(z+i)$ belong to the space $\mathcal{H}(E)$. Since the element of the space $\mathcal{H}(E)$ is an entire function, $F(z)$ is an entire function. Since the conjugate of an element of the space $\mathcal{H}(E)$ is an element of the space $\mathcal{H}(E)$, the entire function

$$F^*(z) + F^*(z-i)$$

is of the form

$$G(z) + G(z+i)$$

for an element $G(z)$ of the space $\mathcal{F}(W)$ such that $G(z+i)$ belongs to the space. Then $G(z)$ is an entire function which satisfies the identity

$$F^*(z) - G(z+i) = G(z) - F^*(z-i).$$

Since the transformation in the space $\mathcal{F}(W)$ is dissipative, the norm of the function

$$[F^*(z) - G(z+i)] - [F^*(z+i) + G(z)]$$

is less than or equal to the norm of the function

$$[F^*(z) - G(z+i)] + [F^*(z-i) - G(z)]$$

in the space $\mathcal{F}(W)$. Since the function

$$[F^*(z) - G(z+i)] + [F^*(z-i) - G(z)]$$

vanishes identically, the function

$$[F^*(z) - G(z+i)] - [F^*(z-i) - G(z)]$$

vanishes identically. Since the function

$$F^*(z) = G(z + i)$$

belongs to the space $\mathcal{F}(W)$, the function $F(z)$ belongs to the space $\mathcal{H}(E)$. Since the function

$$F(z) + F(z + i)$$

belongs to the space $\mathcal{H}(E)$, the function $F(z + i)$ belongs to the space $\mathcal{H}(E)$.

This completes the proof of the theorem.

Maximal transformations of dissipative deficiency at most one are constructed in related Hilbert spaces of entire functions. Assume that a space $\mathcal{H}(E)$, which is symmetric about the origin and which contains an element having a nonzero value at the origin, is given such that a maximal dissipative transformation is defined in the subspace of elements having value zero at the origin by taking $F(z)$ into

$$zF(z + i)/(z + i)$$

whenever these functions belong to the subspace. A space $\mathcal{H}(E')$, which is symmetric about the origin, is constructed so that an isometric transformation of the set of elements of the space $\mathcal{H}(E)$ having value zero at the origin onto the set of elements of the space $\mathcal{H}(E')$ having value zero at i is defined by taking $F(z)$ into

$$(z - i)F(z)/z.$$

The element $S(z)$ of the space $\mathcal{H}(E)$ defined by the identity

$$iS(z)[B(0)A(i) - A(0)B(i)] = [B(z)A(i) - A(z)B(i)]/(z - i)$$

is the constant multiple of the reproducing kernel function for function values at i which has value one at the origin. A continuous transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E')$ is defined by taking $F(z)$ into

$$G(z) = (z - i)[F(z) - S(z)F(0)]/z.$$

Since the symmetry condition

$$S^*(z) = S(-z)$$

is satisfied, the transformation takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$. Entire functions $P(z)$ and $Q(z)$, which are associated with the space $\mathcal{H}(E)$ and which have value zero at $-i$, are defined by the identities

$$\begin{aligned} &P(z)[B(0)A(i)^- - A(0)B(i)^-]/i \\ &= A(z)[B(0)A(i)^- - A(0)B(i)^-]/i - A(i)^-[B(z)A(0) - A(z)B(0)]/z \end{aligned}$$

and

$$Q(z)[B(0)A(i)^- - A(0)B(i)^-]/i$$
$$= B(z)[B(0)A(i)^- - A(0)B(i)^-]/i - B(i)^-[B(z)A(0) - A(z)B(0)]/z.$$

The symmetry conditions

$$P^*(z) = P(-z)$$

and

$$Q^*(z) = -Q(-z)$$

are satisfied. The identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t - w^-)]\rangle_{\mathcal{H}(E)}$$

holds for all complex numbers w when the transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E')$ takes $F(z)$ into $G(z)$. The entire functions $P(z)$ and $Q(z)$ are associated with the space $\mathcal{H}(E')$ since the space coincides as a set with the space $\mathcal{H}(E)$. The identity

$$G(w) = \langle F(t), [Q^*(t)P(w)^- - P^*(t)Q(w)^-]/[\pi(t - w^-)]\rangle_{\mathcal{H}(E')}$$

holds for all complex numbers w when the adjoint transformation of the space $\mathcal{H}(E')$ into the space $\mathcal{H}(E)$ takes $F(z)$ into $G(z)$. A maximal transformation of dissipative deficiency at most one is defined in the space $\mathcal{H}(E)$ by taking $F(z)$ into $G(z - i)$ whenever $F(z)$ and $G(z - i)$ are elements of the space such that the transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E')$ takes $F(z)$ into $G(z)$.

Maximal dissipative transformations are constructed in Hilbert spaces of entire functions.

Theorem 3. *Assume that a transformation with domain in the space $\mathcal{H}(E_0)$ and range in the space $\mathcal{H}(E_0')$ satisfies the axioms (H1) and (H2) and that a maximal dissipative transformation in the space $\mathcal{H}(E_0)$ is defined by taking $F(z)$ into $G(z + i)$ whenever $F(z)$ and $G(z + i)$ are elements of the space such that $G(z)$ is the image of $F(z)$ in the space $\mathcal{H}(E_0')$. Assume that $S(z)$ is a polynomial such that $S(z - i)$ has no zeros in the upper half–plane, such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$ for some nonzero entire function $F(z)$, and such that $S(z - i)G(z)$ belongs to the space $\mathcal{H}(E_0')$ for some nonzero entire function $G(z)$. Then the set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$ is a space $\mathcal{H}(E)$ which is mapped isometrically into the space $\mathcal{H}(E_0)$ under multiplication by $S(z)$. The set of entire functions $G(z)$ such that $S(z - i)G(z)$ belongs to the space $\mathcal{H}(E_0')$ is a space $\mathcal{H}(E')$ which is mapped isometrically into the space $\mathcal{H}(E_0')$ under multiplication by $S(z - i)$. A relation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$ is defined by taking $F(z)$ into $G(z)$ whenever the transformation with domain in the space $\mathcal{H}(E_0)$ and range in the space $\mathcal{H}(E_0')$ takes $H(z)$ into $S(z-i)G(z)$ and $F(z)$ is the element of the space $\mathcal{H}(E)$ such that $S(z)F(z)$ is the orthogonal projection of $H(z)$ into the image of the space $\mathcal{H}(E)$. A maximal dissipative relation in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $G(z + i)$ whenever $F(z)$ and $G(z + i)$ are elements of the space such that $G(z)$ is the image of $F(z)$ in the space $\mathcal{H}(E)$. If the maximal dissipative* *relation in the space $\mathcal{H}(E)$ is not skew–adjoint, then the relation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$ is a transformation which satisfies the axioms (H1) and (H2).*

Proof of Theorem 3. The desired conclusions are immediate when $S(z)$ is a constant since multiplication by

$$S(z) = S(z - i)$$

is then an isometric transformation of the space $\mathcal{H}(E)$ onto the space $\mathcal{H}(E_0)$ and of the space $\mathcal{H}(E')$ onto the space $\mathcal{H}(E_0')$. A proof of the theorem is first given when

$$S(z) = z - \lambda$$

is a linear function with zero λ such that $\lambda + i$ does not belong to the upper half–plane.

Since the transformation with domain in the space $\mathcal{H}(E_0)$ and range in the space $\mathcal{H}(E_0')$ satisfies the axioms (H1) and (H2), it is defined by entire functions $P_0(z)$ and $Q_0(z)$ which are associated with both spaces. The transformation takes an element $F(z)$ of the space $\mathcal{H}(E_0)$ into an element $G(z)$ of the space $\mathcal{H}(E_0')$ if, and only if, the identity

$$G(w) = \langle F(t), [Q_0(t)P_0(w^-) - P_0(t)Q_0(w^-)]/[\pi(t - w^-)]\rangle_{\mathcal{H}(E)}$$

holds for all complex numbers w. The adjoint transformation takes an element $F(z)$ of the space $\mathcal{H}(E_0')$ into an element $G(z)$ of the space $\mathcal{H}(E_0)$ if, and only if, the identity

$$G(w) = \langle F(t), [Q_0^*(t)P_0(w)^- - P_0^*(t)Q_0(w)^-]/[\pi(t - w^-)]\rangle_{\mathcal{H}(E_0')}$$

holds for all complex numbers w.

If complex numbers α, β, γ, and δ with

$$\alpha\delta - \beta\gamma = 1$$

exist such that

$$\frac{Q_0(z)\alpha - P_0(z)\beta}{z - \lambda}$$

and

$$\frac{Q_0(z)\gamma - P_0(z)\delta}{z + i - \lambda^-}$$

are entire functions, then the entire functions

$$P(z) = \frac{Q_0(z)\alpha - P_0(z)\beta}{z - \lambda}\,\gamma - \frac{Q_0(z)\gamma - P_0(z)\delta}{z + i - \lambda^-}\,\alpha$$

and

$$Q(z) = \frac{Q_0(z)\alpha - P_0(z)\beta}{z - \lambda}\,\delta - \frac{Q_0(z)\gamma - P_0(z)\delta}{z + i - \lambda^-}\,\beta$$

satisfy the identities

$$(z-\lambda)P(z) = P_0(z) - (\lambda + i - \lambda^-) \ \frac{Q_0(z)\gamma - P_0(z)\delta}{z + i - \lambda^-} \ \alpha$$

and

$$(z-\lambda)Q(z) = Q_0(z) - (\lambda + i - \lambda^-) \ \frac{Q_0(z)\gamma - P_0(z)\delta}{z + i - \lambda^-} \ \beta$$

as well as the identities

$$(z+i-\lambda^-)P(z) = P_0(z) - (\lambda + i - \lambda^-) \ \frac{Q_0(z)\alpha - P_0(z)\beta}{z - \lambda} \ \gamma$$

and

$$(z+i-\lambda^-)Q(z) = Q_0(z) - (\lambda + i - \lambda^-) \ \frac{Q_0(z)\alpha - P_0(z)\beta}{z - \lambda} \ \delta.$$

The identity

$$\begin{aligned} &\frac{Q_0(z)P_0(w^-) - P_0(z)Q_0(w^-)}{\pi(z-w^-)} \\ &+ \pi(\lambda + i - \lambda^-) \ \frac{Q_0(z)\gamma - P_0(z)\beta}{\pi(z+i-\lambda^-)} \ \frac{Q_0(w^-)\alpha - P_0(w^-)\beta}{\pi(w^- - \lambda)} \\ &= (z-\lambda) \ \frac{Q(z)P(w^-) - P(z)Q(w^-)}{\pi(z-w^-)} \ (w^- + i - \lambda^-) \end{aligned}$$

is then satisfied.

Such numbers $\alpha, \beta, \gamma,$ and δ exist when

$$Q_0(\lambda)P_0(\lambda^- - i) - P_0(\lambda)Q_0(\lambda^- - i)$$

is nonzero. The entire functions $P(z)$ and $Q(z)$ are associated with the spaces $\mathcal{H}(E)$ and $\mathcal{H}(E')$. The relation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$ is a transformation since it takes an element $F(z)$ of the space $\mathcal{H}(E)$ into an element $G(z)$ of the space $\mathcal{H}(E')$ if, and only if, the identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E)}$$

holds for all complex numbers w. The adjoint relation with domain in the space $\mathcal{H}(E')$ and range in the space $\mathcal{H}(E)$ is a transformation since it takes an element $F(z)$ of the space $\mathcal{H}(E')$ into an element $G(z)$ of the space $\mathcal{H}(E)$ if, and only if, the identity

$$G(w) = \langle F(t), [Q^*(t)P(w)^- - P^*(t)Q(w)^-]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E')}$$

holds for all complex numbers w. A maximal dissipative transformation in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space such that $G(z)$ is the image of $F(z)$ in the space $\mathcal{H}(E')$.

The existence of the desired numbers α, β, γ, and δ will now be verified when the maximal dissipative transformation in the space $\mathcal{H}(E_0)$ is not skew–adjoint. If entire functions $A(z)$ and $B(z)$ are defined by the identities

$$A(z) = P_0(z - \tfrac{1}{2}i)$$

and

$$B^*(z) = Q_0(z - \tfrac{1}{2}i),$$

then a Hilbert space exists whose elements are entire functions and which contains the function

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z - w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w. Since the maximal dissipative transformation in the space $\mathcal{H}(E_0)$ is not skew–adjoint, the space contains a nonzero element. The value of the function

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z - w^-)}$$

of z at w is positive at all but isolated points w in the complex plane. A sequence of complex numbers λ_n, which converge to λ, exits such that the function always has a positive value at w when $w = \lambda_n + \frac{1}{2}i$. Complex numbers $\alpha_n, \beta_n, \gamma_n$, and δ_n with

$$\alpha_n \delta_n - \beta_n \gamma_n = 1$$

then exist for every index n such that

$$\frac{Q_0(z)\alpha_n - P_0(z)\beta_n}{z - \lambda_n}$$

and

$$\frac{Q_0(z)\gamma_n - P_0(z)\delta_n}{z + i - \lambda_n^-}$$

are entire functions. Since the product

$$\frac{Q_0(z)\gamma_n - P_0(z)\delta_n}{\pi(z + i - \lambda_n^-)} \; \frac{Q_0(w^-)\alpha_n - P_0(w^-)\beta_n}{\pi(w^- - \lambda_n)}$$

converges for all complex numbers a and w, the choice of $\alpha_n, \beta_n, \gamma_n$, and δ_n can be made so that

$$\frac{Q_0(z)\alpha_n - P_0(z)\beta_n}{\pi(z - \lambda_n)}$$

and

$$\frac{Q_0(z)\gamma_n - P_0(z)\delta_n}{\pi(z + i - \lambda_n)}$$

converge for all complex numbers z. It follows that the limits

$$\alpha = \lim \alpha_n$$

and

$$\beta = \lim \beta_n$$

and

$$\gamma = \lim \gamma_n$$

and

$$\delta = \lim \delta_n$$

exist. Complex numbers α, β, γ, and δ with

$$\alpha\delta - \beta\gamma = 1$$

are obtained such that

$$\frac{Q_0(z)\alpha - P_0(z)\beta}{z - \lambda}$$

and

$$\frac{Q_0(z)\gamma - P_0(z)\delta}{z + i - \lambda^-}$$

are entire functions.

An inductive argument is applied when

$$S(z) = (z - \lambda_1) \dots (z - \lambda_r)$$

is a polynomial with zeros λ_n such that $\lambda_n + i$ never belongs to the upper half–plane. It can be assumed that the desired conclusion holds when any zero is omitted. No verification is needed in the remaining case since a maximal dissipative transformation appears which is skew–adjoint.

This completes the proof of the theorem.

Maximal transformations of dissipative deficiency zero or one are constructed in Hilbert spaces of entire functions which are symmetric about the origin.

Theorem 4. *Assume that a transformation with domain in a space $\mathcal{H}(E_0)$, which is symmetric about the origin, and range in a space $\mathcal{H}(E_0')$, which is symmetric about the origin, satisfies the axioms (H1) and (H2) and is symmetric about the origin and that a maximal transformation of dissipative deficiency at most one in the space $\mathcal{H}(E_0)$ is defined by taking $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space such that $G(z)$ is the image of $F(z)$ in the space $\mathcal{H}(E_0')$. Assume that $S(z)$ is a polynomial, which satisfies the symmetry condition*

$$S(-z) = S^*(z),$$

such that $S(z-i)$ has no zeros in the upper half–plane, such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$ for some nonzero entire function $F(z)$ and such that $S(z-i)G(z)$ belongs to the space $\mathcal{H}(E_0')$ for some nonzero entire function $G(z)$. Then the set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$ is a space $\mathcal{H}(E)$ which is symmetric about the origin and which is mapped isometrically into the space $\mathcal{H}(E_0)$ under multiplication by $S(z)$. The set of entire functions $G(z)$ such that $S(z-i)G(z)$ belongs to the space $\mathcal{H}(E_0')$ is a space $\mathcal{H}(E')$ which is symmetric about the origin and which is mapped isometrically into the space $\mathcal{H}(E_0')$ under multiplication by $S(z-i)$. A relation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$ is defined by taking $F(z)$ into $G(z)$ whenever the transformation with domain in the space $\mathcal{H}(E_0)$ and range in the space $\mathcal{H}(E_0')$ takes $H(z)$ into $S(z-i)G(z)$ and $F(z)$ is the element of the space $\mathcal{H}(E)$ such that $S(z)F(z)$ is the orthogonal projection of $H(z)$ into the image of the space $\mathcal{H}(E)$. A maximal relation of deficiency at most one in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are element of the space such that $G(z)$ is the image of $F(z)$ in the space $\mathcal{H}(E')$. If the maximal relation of deficiency at most one in the space $\mathcal{H}(E)$ is not skew–adjoint, then the relation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$ is a transformation which satisfies the axioms (H1) and (H2) and is symmetric about the origin.

Proof of Theorem 4. The desired conclusions are immediate when $S(z)$ is a constant since multiplication by

$$S(z) = S(z-i)$$

is then an isometric transformation of the space $\mathcal{H}(E)$ onto the space $\mathcal{H}(E_0)$ and of the space $\mathcal{H}(E')$ onto the space $\mathcal{H}(E_0')$. A proof of the theorem is first given when

$$S(z) = (z-\lambda)(z+\lambda^-)$$

is a quadratic function with zeros λ and $-\lambda^-$ such that $\lambda + i$ and $i - \lambda^-$ do not belong to the upper half–plane.

Since the spaces $\mathcal{H}(E_0)$ and $\mathcal{H}(E_0')$ are symmetric about the origin and since the transformation with domain in the space $\mathcal{H}(E_0)$ and range in the space $\mathcal{H}(E_0')$ satisfies the axioms (H1) and (H2) and is symmetric about the origin, it is defined by entire functions $P_0(z)$ and $Q_0(z)$ which are associated with both spaces and which satisfy the symmetry conditions

$$P_0(-z) = P_0^*(z)$$

and

$$Q_0(-z) = -Q_0^*(z).$$

The transformation takes an element $F(z)$ of the space $\mathcal{H}(E_0)$ into an element $G(z)$ of the space $\mathcal{H}(E_0')$ if, and only, if the identity

$$G(w) = \langle F(t), [Q_0(t)P_0(w^-) - P_0(t)Q_0(w^-)]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E_0)}$$

holds for all complex numbers w. The adjoint transformation takes an element $F(z)$ of the space $\mathcal{H}(E_0')$ into an element $G(z)$ of the space $\mathcal{H}(E_0)$ if, and only if, the identity

$$G(w) = \langle F(t), [Q_0^*(t)P_0(w)^- - P_0^*(t)Q_0(w)^-]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E_0')}$$

holds for all complex numbers w.

If complex numbers α, β, γ, and δ with

$$\alpha\delta - \beta\gamma = 1$$

exist such that

$$\frac{Q_0(z)\alpha - P_0(z)\beta}{z - \lambda}$$

and

$$\frac{Q_0(z)\gamma - P_0(z)\delta}{z + \lambda + i}$$

are entire functions, then the entire functions

$$P(z) = \frac{Q_0(z)\gamma - P_0(z)\delta}{(2\lambda + i)(z + \lambda + i)} \, a + \frac{Q_0(z)\gamma^- + P_0(z)\delta^-}{(2\lambda^- - i)(z + i - \delta^-)} \, a^-$$
$$+ \frac{Q_0(z)\alpha - P_0(z)\beta}{(2\lambda + i)(z - \lambda)} \, c + \frac{Q_0(z)\alpha^- + P_0(z)\beta^-}{(2\lambda^- - i)(z + \lambda^-)} \, c^-$$

and

$$Q(z) = \frac{Q_0(z)\gamma - P_0(z)\delta}{(2\lambda + i)(z + \lambda + i)} \, b - \frac{Q_0(z)\gamma^- + P_0(z)\delta^-}{(2\lambda^- - i)(z + i - \lambda^-)} \, b^-$$
$$+ \frac{Q_0(z)\alpha - P_0(z)\beta}{(2\lambda + i)(z - \lambda)} \, d - \frac{Q_0(z)\alpha^- + P_0(z)\beta^-}{(2\lambda^- - i)(z + \lambda^-)} \, d^-$$

satisfy the equations

$$(z - \lambda)(z + \lambda^-)P(z) = P_0(z)$$
$$+(\lambda + i - \lambda^-) \, \frac{Q_0(z)\gamma - P_0(z)\delta}{z + \lambda + i} \, a - (\lambda + i - \lambda^-) \, \frac{Q_0(z)\gamma^- + P_0(z)\delta^-}{z + i - \lambda^-} \, a^-$$

and

$$(z - \lambda)(z + \lambda^-)Q(z) = Q_0(z)$$
$$+(\lambda + i - \lambda^-) \, \frac{Q_0(z)\gamma - P_0(z)\delta}{z + \lambda + i} \, b + (\lambda + i - \lambda^-) \, \frac{Q_0(z)\gamma^- + P_0(z)\delta^-}{z + i - \lambda^-} \, b^-$$

as well as the equations

$$(z + \lambda + i)(z + i - \lambda^-)P(z) = P_0(z)$$
$$+(\lambda + i - \lambda^-) \, \frac{Q_0(z)\alpha - P_0(z)\beta}{z - \lambda} \, c - (\lambda + i - \lambda^-) \, \frac{Q_0(z)\alpha^- + P_0(z)\beta^-}{z + \lambda^-} \, c^-$$

and

$$(z + \lambda + i)(z + i - \lambda^-)Q(z) = Q_0(z)$$
$$+(\lambda + i - \lambda^-) \, \frac{Q_0(z)\alpha - P_0(z)\beta}{z - \lambda} \, d + (\lambda + i - \lambda^-) \, \frac{Q_0(z)\alpha^- + P_0(z)\beta^-}{z + \lambda^-} \, d^-$$

when a, b, c, and d are the unique solutions of the equations

$$\alpha c - \gamma a = \alpha^- c^- - \gamma^- a^-$$

and

$$\beta d - \delta b = \beta^- d^- - \delta^- b^-$$

and

$$\delta a - \beta c + \delta^- a^- - \beta^- c^- = 2$$

and

$$\alpha d - \gamma b + \alpha^- d^- - \gamma^- b^- = 2$$

as well as the equations

$$\frac{\alpha c + \gamma a}{\lambda + \frac{1}{2} i} = - \frac{\alpha^- c^- + \gamma^- a^-}{\lambda^- - \frac{1}{2} i}$$

and

$$\frac{\beta c + \delta a}{\lambda + \frac{1}{2} i} = \frac{\beta^- c^- + \delta^- a^-}{\lambda^- - \frac{1}{2} i}$$

and

$$\frac{\alpha d + \gamma b}{\lambda + \frac{1}{2} i} = \frac{\alpha^- d^- + \gamma^- b^-}{\lambda^- - \frac{1}{2} i}.$$

The identity

$$\begin{aligned} & \frac{Q_0(z) P_0(w^-) - P_0(z) Q_0(w^-)}{\pi (z - w^-)} \\ & + (ad - bc) \ \frac{(\lambda + i - \lambda^-)^2}{2\lambda + i} \ \frac{Q_0(z)\gamma - P_0(z)\delta}{z + \lambda + i} \ \frac{Q_0(w^-)\alpha - P_0(w^-)\beta}{w^- - \lambda} \\ & + (ad^- + bc^-)(\lambda + i - \lambda^-) \ \frac{Q_0(z)\gamma - P_0(z)\delta}{z + \lambda + i} \ \frac{Q_0(w^-)\alpha^- + P_0(w^-)\beta^-}{w^- + \lambda^-} \\ & + (a^- d + b^- c)(\lambda + i - \lambda^-) \ \frac{Q_0(z)\gamma^- + P_0(z)\delta^-}{z + i - \lambda^-} \ \frac{Q_0(w^-)\alpha - P_0(w^-)\beta}{w^- - \lambda} \\ & - (a^- d^- - b^- c^-) \ \frac{(\lambda + i - \lambda^-)^2}{2\lambda^- - i} \ \frac{Q_0(z)\gamma^- + P_0(z)\delta^-}{z + i - \lambda^-} \ \frac{Q_0(w^-)\alpha^- + P_0(w^-)\beta^-}{w^- + \lambda^-} \\ & = (z - \lambda)(z + \lambda^-) \ \frac{Q(z) P(w^-) - P(z) Q(w^-)}{\pi (z - w^-)} \ (w^- + \lambda + i)(w^- + i - \lambda^-) \end{aligned}$$

is then satisfied.

Such numbers α, β, γ, and δ exist when

$$Q_0(\lambda) P_0(-\lambda - i) - P_0(\lambda) Q_0(-\lambda - i)$$

is nonzero. The entire functions $P(z)$ and $Q(z)$ are associated with the spaces $\mathcal{H}(E)$ and $\mathcal{H}(E')$ and satisfy the symmetry conditions

$$P(-z) = P^*(z)$$

and

$$Q(-z) = -Q^*(z).$$

The relation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$ is a transformation since it takes an element $F(z)$ of the space $\mathcal{H}(E)$ into an element $G(z)$ of the space $\mathcal{H}(E')$ if, and only if, the identity

$$G(w) = \langle F(t), [Q(t)P(w^-) - P(t)Q(w^-)]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E)}$$

holds for all complex numbers w. The adjoint relation with domain in the space $\mathcal{H}(E')$ and range in the space $\mathcal{H}(E)$ takes an element $F(z)$ of the space $\mathcal{H}(E')$ into an element $G(z)$ of the space $\mathcal{H}(E)$ if, and only if, the identity

$$G(w) = \langle F(t), [Q^*(t)P(w)^- - P^*(t)Q(w)^-]/[\pi(t-w^-)]\rangle_{\mathcal{H}(E')}$$

holds for all complex numbers w. A maximal dissipative transformation in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space such that $G(z)$ is the image of $F(z)$ in the space $\mathcal{H}(E')$.

The existence of the desired numbers α, β, γ, and δ will now be verified when the negative of the maximal transformation of dissipative deficiency at most one in the space $\mathcal{H}(E_0)$ is not a maximal transformation of dissipative deficiency at most one in the space. If entire functions $A(z)$ and $B(z)$ are defined by the identities

$$A(z) = P_0(z - \tfrac{1}{2}i)$$

and

$$B^*(z) = Q_0(z - \tfrac{1}{2}i),$$

then the symmetry conditions

$$A(-z) = A^*(z)$$

and

$$B(-z) = -B^*(z)$$

are satisfied. A Krein space of Pontryagin index at most one exists whose elements are entire functions and which contains the function

$$\frac{B^*(z)A(w^-) - A(z)B(w)^- + B(z)A(w)^- - A^*(z)B(w^-)}{\pi(z-w^-)}$$

of z as reproducing kernel function for function values at w for every complex number w. Since the negative of the maximal transformation of dissipative deficiency at most one in the space $\mathcal{H}(E_0)$ is not the negative of a maximal transformation of dissipative deficiency at most one in the space, the Krein space of Pontryagin index at most one contains a nonzero element and does not have dimension one. The space is the orthogonal sum of a subspace of even functions and a subspace of odd functions, each of which contains a nonzero element. The subspaces are Krein spaces of Pontryagin index at most one. At least one of these subspaces is a Hilbert space.

This completes the proof of the theorem.

The signature for the r–adic line is the homomorphism ξ into $\mathrm{sgn}(\xi)$ of the group of invertible elements of the r–adic line into the real numbers of absolute value one which has value minus one on elements whose r–adic modulus is a prime divisor of r.

The signature for the r–adic skew–plane is the homomorphism ξ into $\mathrm{sgn}(\xi)$ of the group of invertible elements of the r–adic skew–plane into the complex numbers of absolute value one which has value minus one on elements whose r–adic modulus squared is a prime divisor of r.

If ρ is a positive integer, a character modulo ρ is a homomorphism χ of the group of integers modulo ρ, which are relatively prime to ρ, into the complex numbers of absolute value one. The function is extended to the integers modulo ρ so as to vanish at integers modulo ρ which are not relatively prime to ρ. A character χ modulo ρ is said to be primitive modulo ρ if no character modulo a proper divisor of ρ exists which agrees with χ at integers which are relatively prime to ρ. If a character χ modulo ρ is primitive modulo ρ, a number $\epsilon(\chi)$ of absolute value one exists such that the identity

$$\epsilon(\chi)\chi(n)^- = \rho^{-\frac{1}{2}} \sum \chi(k) \exp(2\pi ink/\rho)$$

holds for every integer n modulo ρ with summation over the integers k modulo ρ. The principal character modulo ρ is the character modulo ρ whose only nonzero value is one. The principal character modulo ρ is a primitive character modulo ρ when, and only when, ρ is equal to one. The residue classes of integers modulo ρ are identified with the residue classes of integral elements of the r–adic line modulo ρ. A character χ modulo ρ is treated as a function of integral elements of the r–adic line which is periodic of period ρ. The character admits an extension as a homomorphism of the invertible elements of an r–adic plane into the complex numbers of absolute value one. The extended character is defined to have the value zero at noninvertible elements of the r–adic plane. Since the extension in unique within an automorphism of the r–adic plane which leaves the r–adic line fixed, it is also denoted χ. The conjugate character χ^- is defined by the identity

$$\chi^-(\xi) = \chi(\xi^-)^-$$

for every element ξ of the r–adic plane.

The domain of the Hankel transformation of order ν for the r–adic plane is the set of functions $f(\xi)$ of ξ in the r–adic plane which are square integrable with respect to Haar measure for the r–adic plane, which vanish at elements of the r–adic plane whose p–adic component is not a unit for some prime divisor p or ρ, and which satisfy the identity

$$f(\omega\xi) = \chi(\omega) f(\xi)$$

almost everywhere with respect to Haar measure for the r–adic plane when ω is a unit of the r–adic plane. Representatives are chosen in equivalence classes so that the identity holds for every element ξ of the r–adic plane. The range of the Hankel transformation of order ν for the r–adic plane is the set of functions $g(\xi)$ of ξ in the r–adic plane which are square integrable with respect to Haar measure for the r–adic plane, which vanish at elements of the r–adic plane whose p–adic component is not a unit for some prime divisor p of ρ, and which satisfy the identity

$$g(\omega\xi) = \chi^-(\omega)g(\xi)$$

for every unit ω of the r–adic plane. The Laplace kernel for the r–adic line is a function $\sigma(\eta)$ of invertible elements η of the r–adic line which vanishes when the p–adic component of η is not a unit for some prime divisor p of ρ and which is otherwise defined as an integral

$$\prod(1-p^{-1})\sigma(\eta) = \prod(1-p^{-1})^{-1}\int \exp(2\pi i\eta\xi)d\xi$$

with respect to Haar measure for the r–adic line over the set of units of the r–adic line. The product on the left is taken over the prime divisors p of r. The product on the right is taken over the prime divisors p of ρ. The function $\sigma(\eta)$ of η in the r–adic line has value zero when the p–adic component of η is not a unit for some prime divisor p of ρ or when the p–adic component of $p\eta$ is not integral for some prime divisor p of r. When the p–adic component of η is a unit for every prime divisor p of ρ and the p–adic component of $p\eta$ is integral for every prime divisor p of r, then $\sigma(\eta)$ is equal to

$$\prod(1-p^{-1})^{-1}\prod(1-p)^{-1}$$

with the product on the left taken over the prime divisors p of ρ and the product on the right taken over the prime divisors p of r such that the p–adic component of η is not integral. The integral

$$\int |\sigma(\eta)|^2 d\eta$$

with respect to Haar measure for the r–adic line is equal to the product

$$\prod(1-p^{-1})^{-1}$$

taken over the prime divisors p of r. The integral

$$\int \sigma(\alpha\eta)\sigma(\beta\eta)d\eta$$

with respect to Haar measure for the r–adic line is equal to zero when α and β are invertible elements of the r–adic line of unequal r–adic modulus. The Hankel transformation of character χ for the r–adic plane takes a function $f(\xi)$ of ξ in the r–adic plane into a function $g(\xi)$ of ξ in the r–adic plane when the identity

$$\int \chi(\xi^-)g(\xi)\sigma(\eta\xi^-\xi)d\xi = \operatorname{sgn}(\eta)|\eta|_-^{-1}\int \chi(\xi)^- f(\xi)\sigma(\eta^{-1}\xi^-\xi)d\xi$$

holds for every invertible element η of the r–adic line with integration with respect to Haar measure for the r–adic plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the r–adic plane. The Hankel transformation of character χ^- for the r–adic plane is the inverse of the Hankel transformation of character χ for the r–adic plane.

The Hankel transformation of order ν and character χ for the r–adic skew–plane is defined when ν is an odd positive integer and χ is a primitive character modulo ρ with ρ a divisor of r such that r/ρ is relatively prime to ρ and is not divisible by the square of a prime. The character is extended to a distinguished r–adic plane containing a skew–conjugate unit ι_-. The domain of the Hankel transformation of order ν and character χ for the r–adic skew–plane is a space of square integrable functions with respect to Haar measure for the r–adic skew–plane. A function of ξ in the r–adic skew–plane belongs to the space if it is a product

$$\chi(\tfrac{1}{2}\xi + \tfrac{1}{2}\iota_-^{-1}\xi\iota_-)^\nu h(\xi)$$

with $h(\xi)$ a function of ξ in the r–adic skew–plane which satisfies the identity

$$h(\xi) = h(\omega\xi)$$

for every unit ω of the r–adic skew–plane. The range of the Hankel transformation of order ν and character χ of the r–adic skew–plane is the domain of the Hankel transformation of order ν and character χ^- for the r–adic skew–plane. The Hankel transformation of order ν and character χ for the r–adic skew–plane takes a function $f(\xi)$ of ξ in the r–adic skew–plane into a function $g(\xi)$ of ξ in the r–adic skew–plane when the identity

$$\int \chi(\tfrac{1}{2}\xi^- + \tfrac{1}{2}\iota_-^{-1}\xi^-\iota_-)^\nu g(\xi)\sigma(\eta\xi^-\xi)|\xi|_-^{-1}d\xi$$
$$= \operatorname{sgn}(\eta)|\eta|_-^{-1}\int \chi^-(\tfrac{1}{2}\xi^- + \tfrac{1}{2}\iota_-^{-1}\xi^-\iota_-)^\nu f(\xi)\sigma(\eta^{-1}\xi^-\xi)|\xi|_-^{-1}d\xi$$

holds for every invertible element η of the r–adic line with integration with respect to Haar measure for the r–adic skew–plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the r–adic skew–plane. The Hankel transformation of order ν and character χ^- for the r–adic skew–plane is the inverse of the Hankel transformation of order ν and character χ for the r–adic skew–plane.

The domain of the Laplace transformation of character χ for the r–adic plane is the set of functions $f(\xi)$ of ξ in the r–adic plane which are square integrable with respect to Haar measure for the r–adic plane, which vanish at elements of the r–adic plane whose p–adic component is not a unit for some prime divisor p of ρ, and which satisfy the identity

$$f(\omega\xi) = \chi(\omega) f(\xi)$$

for every unit ω of the r–adic plane. The Laplace transform of character χ for the r–adic plane of the function $f(\xi)$ of ξ in the r–adic plane is the function $g(\eta)$ of η in the r–adic line which is defined by the integral

$$\prod (1 - p^{-2}) g(\eta) = \int \chi(\xi)^- f(\xi) \sigma(\eta \xi^- \xi) d\xi$$

with respect to Haar measure for the r–adic plane. The product is taken over the prime divisors p of r. The identity

$$\prod (1 - p^{-1})^{-1} \int |f(\xi)|^2 d\xi = \prod (1 - p^{-2}) \int |g(\eta)|^2 d\eta$$

holds with integration on the left with respect to Haar measure for the r–adic plane and with integration on the right with respect to Haar measure for the r–adic line. The products are taken over the prime divisors p of r. A function $g(\eta)$ of η in the r–adic line, which is square integrable with respect to Haar measure for the r–adic line, is the Laplace transform of character χ of a square integrable function with respect to Haar measure for the r–adic plane if, and only if, it vanishes at elements of the r–adic line whose p–adic component is not a unit for some prime divisor p of ρ, satisfies the identity

$$g(\eta) = g(\omega\eta)$$

for every unit ω of the r–adic line, and satisfies the identity

$$(1 - p) g(\eta) = g(\lambda\eta) - p g(\lambda^{-1}\eta)$$

when the p–adic modulus of η is an odd power of a prime divisor p of r, which is not a divisor of ρ, and λ is an element of the r–adic line of r–adic modulus p^{-1}.

The Laplace transformation of order ν and character χ for the r–adic skew–plane is defined when ν is an odd positive integer and χ is a primitive character modulo ρ for a divisor ρ of r such that r/ρ is relatively prime to ρ and is not divisible by the square of a prime. The character is extended to a distinguished r–adic plane containing a skew–conjugate unit ι_-. The domain of the Laplace transformation of order ν and character χ for the r–adic skew–plane is the same space of square integrable functions with respect to Haar measure for the r–adic skew–plane which is the domain of the Hankel transformation of order ν for the r–adic skew–plane. A function of ξ in the r–adic skew–plane belongs to the space if it is a product

$$\chi(\tfrac{1}{2}\xi + \tfrac{1}{2}\iota_-^{-1}\xi\iota_-)^\nu h(\xi)$$

with $h(\xi)$ a function of ξ in the r–adic skew–plane which satisfies the identity

$$h(\xi) = h(\omega\xi)$$

for every unit ω of the r–adic skew–plane. The Laplace transform of order ν and character χ for the r–adic skew–plane of a function $f(\xi)$ of ξ in the r–adic skew–plane is the function $g(\eta)$ of η in the r–adic line which is defined by the integral

$$\prod(1-p^{-1})g(\eta) = \int \chi^-(\tfrac{1}{2}\xi^- + \tfrac{1}{2}\iota_-^{-1}\xi^-\iota_-)^\nu \sigma(\eta\xi^-\xi)|\xi|_-^{-1}d\xi$$

with respect to Haar measure for the r–adic skew–plane. The product is taken over the prime divisors p of r. The identity

$$\prod(1-p^{-1})^{-1}\int |f(\xi)|^2 d\xi = \prod(1-p^{-2})\int |g(\eta)|^2 d\eta$$

holds with integration on the left with respect to Haar measure for the r–adic skew–plane and with integration on the right with respect to Haar measure for the r–adic line. The products are taken over the prime divisors p of r. A function $g(\eta)$ of η in the r–adic line, which is square integrable with respect to Haar measure for the r–adic line, is the Laplace transform of character χ for the r–adic skew–plane of a square integrable function with respect to Haar measure for the r–adic skew–plane if, and only if, it vanishes at elements of the r–adic line whose p–adic component is not a unit for some prime divisor p of ρ, satisfies the identity

$$g(\eta) = g(\omega\eta)$$

for every unit ω of the r–adic line, and satisfies the identity

$$(1-p)g(\eta) = g(\lambda\eta) - pg(\lambda^{-1}\eta)$$

when the p–adic modulus of η is an odd power of p for a prime divisor p of r, which is not a divisor of ρ, and λ is an element of the r–adic line whose r–adic modulus is p^{-1}.

The Radon transformation of character χ for the r–adic plane is a nonnegative self–adjoint transformation in the space of functions $f(\xi)$ of ξ in the r–adic plane which are square integrable with respect to Haar measure for the r–adic plane and which satisfy the identity

$$f(\xi) = \omega^\nu f(\xi)$$

for every unit ω of the r–adic plane. Haar measure for the hyperplane of skew–conjugate elements of the r–adic skew–plane is normalized so that Haar measure for the r–adic skew–plane is the Cartesian product of Haar measure for the hyperplane and Haar measure for the r–adic line. The transformation takes a function $f(\xi)$ of ξ in the r–adic plane into a function $g(\xi)$ of ξ in the r–adic plane when a function $h(\xi)$ of ξ in the r–adic skew–plane, which is square integrable with respect to Haar measure for the r–adic skew–plane and which agrees with $f(\xi)$ in the r–adic plane, exists such that the identity

$$\prod \frac{1-p^{-\frac{3}{2}}}{1-p^{-1}}\,\chi(\xi)^- g(\xi) = |\xi|_- \int \chi(\xi + \tfrac{1}{2}\xi\eta + \tfrac{1}{2}\xi\iota_-^{-1}\eta\iota_-)^- h(\xi+\xi\eta)|\eta|_-^{-2}d\eta$$

holds formally with integration with respect to Har measure for the hyperplane. The product is taken over the prime divisors p of r. The integral is accepted as the definition of the transformation when

$$f(\xi) = \chi(\xi)\sigma(\lambda\xi^-\xi)$$

for an invertible element λ of the r–adic line, in which case

$$h(\xi) = \chi(\tfrac{1}{2}\xi + \tfrac{1}{2}\iota_-^{-1}\xi\iota_-)\sigma(\lambda\xi^-\xi)$$

and

$$g(\xi) = |\lambda|_-^{-\frac{1}{2}}\chi(\xi)\sigma(\lambda\xi^-\xi).$$

The Radon transformation of order ν and character χ for the r–adic skew–plane is a nonnegative self–adjoint transformation in the space of functions $f(\xi)$ of ξ in the r–adic skew–plane which are square integrable with respect to Haar measure for the r–adic skew–plane and which are the product of the function

$$\chi(\tfrac{1}{2}\xi + \tfrac{1}{2}\iota_-^{-1}\xi\iota_-)^\nu$$

and a function of $\xi^-\xi$. Associated with the function $f(\xi)$ of ξ in the r–adic skew–plane is a function $f(\xi,\eta)$ of ξ and η in the r–adic skew–plane which agrees with $f(\xi)$ when η is equal to ξ. The function $f(\xi,\eta)$ of ξ and η in the r–adic skew–plane is the product of the function

$$\chi(\tfrac{1}{2}\xi + \tfrac{1}{2}\iota_-^{-1}\xi\iota_-)^{\frac{1}{2}\nu-\frac{1}{2}}\chi(\tfrac{1}{2}\eta + \tfrac{1}{2}\iota_-^{-1}\eta\iota_-)^{\frac{1}{2}\nu+\frac{1}{2}}$$

and a function of $\xi^-\xi + \eta^-\eta$. The Radon transformation of order ν and character χ for the r–adic skew–plane takes a function $f(\xi)$ of ξ in the r–adic skew–plane into a function $g(\xi)$ of ξ in the r–adic skew–plane when a function $h(\xi,\eta)$ of ξ and η in the r–adic skew–plane, which is square integrable with respect to the Cartesian product with itself of Haar measure for the r–adic skew–plane and which agrees with $f(\xi,\eta)$ when ξ and η are in the r–adic plane, such that the identity

$$\begin{aligned}\prod \frac{(1-p^{-\frac{3}{2}})^2}{(1-p^{-1})^2}\, &\chi(\xi)^-\chi(\eta)^-g(\xi,\eta) \\ = |\xi\eta|_- \iint &\chi(\xi + \tfrac{1}{2}\xi\alpha + \tfrac{1}{2}\xi\iota_-^{-1}\alpha\iota_-)^-\chi(\eta + \tfrac{1}{2}\eta\beta + \tfrac{1}{2}\eta\iota_-^{-1}\beta\iota_-)^- \\ &\times h(\xi+\xi\alpha, \eta+\eta\beta)|\alpha\beta|_-^{-2}d\alpha d\beta\end{aligned}$$

holds formally with integration with respect to Haar measure for the hyperplane of skew–conjugate elements of the r–adic skew–plane. The product is taken over the prime divisors p of r. The integral is accepted as the definition of the transformation when

$$f(\xi) = \chi(\tfrac{1}{2}\xi + \tfrac{1}{2}\iota_-^{-1}\xi\iota_-)^\nu\sigma(\lambda\xi^-\xi)$$

for an invertible element λ of the r–adic line in which case

$$h(\xi,\eta) = \chi(\tfrac{1}{2}\xi + \tfrac{1}{2}\iota_-^{-1}\xi\iota_-)^{\frac{1}{2}\nu-\frac{1}{2}}\chi(\tfrac{1}{2}\eta + \tfrac{1}{2}\iota_-^{-1}\eta\iota_-)^{\frac{1}{2}\nu+\frac{1}{2}}\sigma(\lambda\xi^-\xi + \lambda\eta^-\eta)$$

and

$$g(\xi) = |\lambda|_-^{-1}\chi(\tfrac{1}{2}\xi + \tfrac{1}{2}\iota_-^{-1}\xi\iota_-)^\nu \sigma(\lambda\xi^-\xi).$$

The r–adelic upper half–plane is the set of elements of the r–adelic plane whose Euclidean component belongs to the upper half–plane and whose r–adic component is an invertible element of the r–adic line. An element of the r–adelic upper half–plane, whose Euclidean component is $\tau_+ + iy$ for a real number τ_+ and a positive number y and whose r–adic component is τ_-, is written $\tau + iy$ with τ the element of the r–adelic line whose Euclidean component is τ_+ and whose r–adic component is τ_-.

A nonnegative integer ν of the same parity as χ is associated with a primitive character χ modulo ρ for the definition of the Hankel transformation of order ν and character χ for the r–adelic plane. If ω is a unimodular element of the r–adelic plane, an isometric transformation in the space of square integrable functions with respect to Haar measure for the r–adelic plane is defined by taking a function $f(\xi)$ of ξ in the r–adelic plane into the function $f(\omega\xi)$ of ξ in the r–adelic plane. A closed subspace of the space of square integrable functions with respect to Haar measure for the r–adelic plane consists of the functions $f(\xi)$ of ξ in the r–adelic plane which vanish at elements of the r–adelic plane whose p–adic component is not a unit for some prime divisor p of ρ and which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the r–adelic plane. Functions are constructed which satisfy related identities for every unimodular element ω of the r–adelic plane whose p–adic component is a unit for every prime divisor p of ρ. The set of elements of the r–adelic plane whose r–adic component is a unit is taken as a fundamental region for the r–adelic plane. A function $f(\xi)$ of ξ in the r–adelic plane, which vanishes at elements ξ of the r–adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfies the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the r–adelic plane, and which satisfies the identity

$$f(\xi) = f(\omega\xi)$$

for every nonzero principal element ω of the r–adelic line, whose p–adic component is a unit for every prime divisor p of ρ, is said to be locally square integrable if it is square integrable with respect to Haar measure for the r–adelic plane in the fundamental region. The resulting Hilbert space is the domain of the Hankel transformation of order ν and character χ for the r–adelic plane.

An odd positive integer ν is associated with a primitive character χ modulo ρ for the definition of the Hankel transformation of order ν and character χ for the r–adelic skew–plane. The character is extended to a distinguished r–adic plane. A skew–conjugate unit ι of the r–adelic plane is chosen with Euclidean component ι_+ the unit i of the Euclidean plane and with r–adic component ι_- a skew–conjugate unit of the r–adic plane.

If ω is a unit of the r–adelic skew–plane, an isometric transformation in the space of square integrable functions with respect to Haar measure for the r–adelic skew–plane is defined by taking a function $f(\xi)$ of ξ in the r–adelic skew–plane into the function $f(\omega\xi)$ of ξ in the r–adelic skew–plane. The space is the orthogonal sum of invariant subspaces indexed by the integers ν and the primitive characters χ modulo a divisor ρ of r. A function of order zero, which is defined independently of a character, is a function $f(\xi)$ of ξ in the r–adelic skew–plane which satisfies the identity

$$f(\xi) = f(\omega\xi)$$

for every unit ω of the r–adelic skew–plane. When ν is a positive integer, a function of order ν and character χ, is a finite linear combination with functions of order zero as coefficients of products

$$\prod(\tfrac{1}{2}\alpha_k\xi_+ + \tfrac{1}{2}\iota_+^{-1}\alpha_k\iota_+)\prod\chi(\tfrac{1}{2}\beta_k\xi_- + \tfrac{1}{2}\iota_-^{-1}\beta_k\iota_-)$$

with α_k equal to one or j for every $k = 1, \dots, \nu$ and for β_k equal to one or to an invertible skew–conjugate element of the r–adic skew–plane which anti–commutes with ι_- for every $k = 1, \dots, \nu$. A function of order $-\nu$ is the complex conjugate of a function of order ν. If ν is nonnegative, the identity

$$\begin{aligned} i^\nu(i/\lambda_+)^{1+\nu}|\lambda|_-^{-1-\nu}(\tfrac{1}{2}\eta_+ + \tfrac{1}{2}\iota_+^{-1}\eta_+\iota_+)^\nu\chi(\tfrac{1}{2}\eta_- + \tfrac{1}{2}\iota_-^{-1}\eta_-\iota_-)^\nu \exp(-\pi i\lambda^{-1}\eta^-\eta) \\ = \int(\tfrac{1}{2}\xi_+ + \tfrac{1}{2}\iota_+^{-1}\xi_+\iota_+)^\nu\chi(\tfrac{1}{2}\xi_- + \tfrac{1}{2}\iota_-^{-1}\xi_-\iota_-)^\nu \exp(\pi i\lambda\xi^-\xi) \\ \times\exp(\pi i(\eta^-\xi + \xi^-\eta))d\xi \end{aligned}$$

holds with integration with respect to Haar measure for the r–adelic plane when λ is in the r–adelic upper half–plane.

The invertible principal elements of the r–adelic line form a group. Elements of the group are considered equivalent if they are obtained from each other on multiplication by a unit. The equivalence classes of elements of the group are applied in an isometric summation. If a function $f(\xi)$ of ξ in the r–adelic plane is square integrable with respect to Haar measure for the r–adelic plane, vanishes outside of the fundamental region, and satisfies the identity

$$f(\omega\xi) = \omega_+^\nu\chi(\omega_-)f(\xi)$$

for every unit ω of the r–adelic plane, then a function $g(\xi)$ of ξ in the r–adelic plane, which vanishes at elements of the r–adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfies the identity

$$g(\omega\xi) = \omega_+^\nu\chi(\omega_-)g(\xi)$$

for every unit ω of the r–adelic plane, and which satisfies the identity

$$g(\xi) = g(\omega\xi)$$

for every element ω of the group whose p–adic component is a unit for every prime divisor p of ρ, is defined as a sum

$$g(\xi) = \sum f(\omega\xi)$$

over the equivalence classes of elements ω of the group whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the r–adelic plane over the fundamental region. If a locally square integrable function $h(\xi)$ of ξ in the r–adelic plane vanishes at elements of the r–adelic plane whose p–adic component is a unit for every prime divisor p of ρ, satisfies the identity

$$h(\omega\xi) = \omega_+^\nu \chi(\omega_-) h(\xi)$$

for every unit ω of the r–adelic plane, and satisfies the identity

$$h(\xi) = h(\omega\xi)$$

for every element ω of the group whose p–adic component is a unit for every prime divisor p of ρ, then $h(\xi)$ is equal to $g(\xi)$ for some such function $f(\xi)$ of ξ in the r–adelic plane. The function $f(\xi)$ of ξ in the r–adelic is equal to $h(\xi)$ in the fundamental region.

A Hilbert space is obtained as the tensor product of the range of the Laplace transformation of order ν for the Euclidean plane and the range of the Laplace transformation of character χ for the r–adic plane. An element of the space is a function $f(\eta)$ of η in the r–adelic upper half–plane which is analytic in the Euclidean component of η when the r–adic component of η is held fixed. The function vanishes at elements η of the r–adelic upper half–plane whose p–adic component is not a unit for some prime divisor p of ρ. The identity

$$f(\eta) = f(\omega\eta)$$

holds for every unit ω of the r–adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$(1 - p) f(\eta) = f(\lambda\eta) - p f(\lambda^{-1}\eta)$$

holds whenever the p–adic modulus of η is not an even power of p for some prime divisor p of r, which is not a divisor of ρ, and λ is an element of the r–adelic line whose Euclidean component is the unit of the Euclidean line and which satisfies the identity

$$p|\lambda|_- = 1.$$

When ν is zero, a finite least upper bound

$$\sup \int |f(\tau + iy)|^2 d\tau$$

is obtained over all positive numbers y. Integration is with respect to Haar measure for the r–adelic line. When ν is positive, the integral

$$\int_0^\infty \int |f(\tau + iy)|^2 y^{\nu-1} d\tau dy$$

is finite. An isometric transformation of the space into itself takes a function $f(\eta)$ of η in the r–adelic upper half–plane into the function

$$(\omega_+^- \omega_+)^\nu f(\omega^- \eta \omega)$$

of η in the r–adelic upper half–plane for every unimodular element ω of the r–adelic plane whose p–adic component is a unit for every prime divisor p of ρ. A closed subspace of the Hilbert space consists of products

$$f(\eta_+)\sigma(\eta_-)$$

with $f(\eta_+)$ a function of η_+ in the upper half–plane which is in the range of the Laplace transformation of order ν for the Euclidean plane. The Hilbert space is the orthogonal sum of closed subspaces obtained as images of the given subspace under the isometric transformations corresponding to elements ω of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ.

A Hilbert space is obtained as the tensor product of the range of the Laplace transformation of order ν for the Euclidean skew–plane and the range of the Laplace transformation of character χ for the r–adic skew–plane. An element of the space is a function $f(\eta, \gamma)$ of η and γ in the r–adelic upper half–plane which is analytic in the Euclidean components of η and γ when the r–adic components of η and γ are held fixed. The function vanishes when the p–adic component of η or the p–adic component of γ is not a unit for some prime divisor p of ρ. The identity

$$f(\lambda\eta, \gamma) = f(\eta, \lambda\gamma)$$

holds when λ is an element of the r–adelic upper half–plane, whose r–adic component is a unit of the r–adic line, such that $\lambda\eta$ and $\lambda\gamma$ as well as η and γ belong to the r–adelic upper half–plane. The identities

$$f(\omega\eta, \gamma) = f(\eta, \gamma) = f(\eta, \omega\gamma)$$

hold for every unit ω of the r–adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$(1-p)f(\eta, \gamma) = f(\lambda\eta, \gamma) - pf(\lambda^{-1}\eta, \gamma)$$

holds when the p–adic modulus of η is an odd power of p for some prime divisor p of r, which is not a divisor of ρ, and λ is an element of the r–adelic line whose Euclidean component is the unit of the Euclidean line and whose r–adic modulus is p^{-1}. The identity

$$(1-p)f(\eta, \gamma) = f(\eta, \lambda\gamma) - pf(\eta, \lambda^{-1}\gamma)$$

holds when the p–adic modulus of γ is an odd power of p for some prime divisor p of r, which is not a divisor of ρ, and λ is an element of the r–adelic line whose Euclidean component is the unit of the Euclidean line and whose r–adic component is p^{-1}.

The theta function of order ν and character χ for the r–adelic plane is a function $\theta(\eta)$ of η in the r–adelic upper half–plane which is analytic in the Euclidean component of η when the r–adic component of η is held fixed. The function vanishes at elements η of the r–adelic upper half–plane whose p–adic component is not a unit for some prime divisor p of ρ. The identity

$$\theta(\eta) = \theta(\omega\eta)$$

holds for every unit ω of the r–adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$\theta(\eta) = \omega_+^\nu \chi(\omega_-)^- \theta(\omega^2\eta)$$

holds for every element ω of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The function is defined as a sum

$$\theta(\eta) = \sum \omega_+^\nu \chi(\omega_-)^- \exp(\pi i \omega_+^2 \eta_+/\rho)\sigma(\omega_-^2\eta_-)$$

over the elements ω of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. A coefficient $\tau(n)$ is defined for every positive integer n, whose prime divisors are divisors of r but not of ρ, so that the identity

$$\prod (1-p^{-1})^{-1}\tau(n) = \chi(\omega_-)^-\sigma(\omega_-^2)$$

holds with ω the element of the principal subgroup of the r–adelic line whose r–adic component is integral and whose Euclidean component is n. The product is taken over the prime divisors p of ρ. If η_- is a unit, the identity

$$\prod (1-p^{-1})\theta(\eta) = \sum n^\nu \tau(n)\exp(\pi i n^2 \eta_+/\rho)$$

holds with summation over the positive integers n whose prime divisors are divisors of r but not of ρ. The product is taken over the prime divisors p of ρ. The identity

$$\tau(m)\tau(n) = \tau(mn)$$

holds for all positive integers m and n whose prime divisors are divisors of r but not of ρ.

The theta function of order ν and character χ for the r–adelic skew–plane is a function $\theta(\eta,\gamma)$ of elements η of the r–adelic upper half–plane and invertible elements γ of the r–adelic line which is an analytic function of the Euclidean component of η when γ and the r–adic component of η are held fixed. The function vanishes when the p–adic component of γ or the p–adic component of η is not a unit for some prime divisor p of ρ. The identities

$$\theta(\omega\eta,\gamma) = \theta(\eta,\gamma) = \theta(\eta,\omega\gamma)$$

hold for every unit ω of the r–adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$\theta(\eta,\gamma) = (\omega_+^\nu)^- \chi(\omega_-)^- \theta(\omega^- \eta\omega, \gamma)$$

holds for every representative ω of an element of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The theta function is a sum

$$\theta(\eta,\gamma) = \sum (\omega_+^\nu)^- \chi(\omega_-)^- \exp(2\pi i \omega_+^- \eta_+ \gamma_+ \omega_+/\rho)$$
$$\times \sum \lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-) \sigma(\omega_-^- \lambda_-^{-1} \eta_- \omega_-) \sigma(\lambda_- \gamma_-)$$

over the elements λ of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ and over the equivalence classes of representatives ω of elements of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. A coefficient $\tau(n)$ is defined for every positive integer n, whose prime divisors are divisors of r but not of ρ, so that the identity

$$\prod (1-p)^{-1})^{-2} \tau(n) = (\omega_+^{\nu-\frac{1}{2}})^- |\omega|_-^{\nu-\frac{1}{2}} \chi(\omega_-)^- \sum \lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-) \sigma(\omega_-^- \lambda_-^{-1} \omega_-) \sigma(\lambda_-)$$

holds with ω a representative of the element of the principal subgroup of the r–adelic line whose r–adic component is integral and whose Euclidean component is n. Summation is over the elements λ of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the prime divisors p of ρ. If γ is a unit of the r–adelic line and if the r–adic component of η is a unit of the r–adic line, the identity

$$\prod (1-p^{-1})^2 \theta(\eta,\gamma) = \sum n^{\nu-\frac{1}{2}} \tau(n) \exp(2\pi i n \eta_+/\rho)$$

holds with summation over the positive integers n whose prime divisors are divisors of r but not of ρ. The product is taken over the prime divisors p of ρ. If the Euclidean components of representatives of elements of the principal subgroup of the r–adelic line are chosen so that the identity

$$\gamma_+ = \alpha_+ \beta_+$$

holds whenever the identity

$$\gamma^* \gamma = \alpha^* \alpha \beta^* \beta$$

is satisfied, then the identity

$$\tau(m)\tau(n) = \tau(mn/k^2)$$

holds for all positive integers m and n whose prime divisors are divisors of r but not of ρ, with summation over the common divisors k of m and n.

The theta function of order ν and character χ for the r–adelic plane is used to define the Laplace transformation of order ν and character χ for the r–adelic plane. The domain of the transformation is the space of locally square integrable functions $f(\xi)$ of ξ in the

r–adelic plane which vanish at elements ξ of the r–adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the r–adelic plane, and which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every element ω of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The Laplace transform of order ν and character χ for the r–adelic plane is the function $g(\eta)$ of n in the r–adelic upper half–plane defined by the integral

$$\prod(1 - p^{-2}) 2\pi g(\eta) = \int (\xi_+^\nu)^- \chi(\xi_-)^- f(\xi) \theta(\xi^- \eta \xi) d\xi$$

with respect to Haar measure for the r–adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the prime divisors p of r. The function $g(\eta)$ of η in the r–adelic upper half–plane is an analytic function of the Euclidean component of η when the r–adic component of η is held fixed. The function vanishes at elements of the r–adelic upper half–plane whose p–adic component is not a unit for some prime divisor p of ρ. The identity

$$g(\eta) = g(\omega\eta)$$

holds for every unit ω of the r–adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$g(\eta) = \omega_+^\nu \chi(\omega_-)^- g(\omega^2 \eta)$$

holds for every element ω of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$(1 - p) g(\eta) = g(\lambda\eta) - p g(\lambda^{-1}\eta)$$

holds when the p–adic modulus of η is not an even power of p for some prime divisor p of r, which is not a divisor of ρ, and when λ is an element of the r–adelic line whose Euclidean component is the unit of the Euclidean line and which satisfies the identity

$$p|\lambda|_- = 1.$$

When ν is zero, the identity

$$\prod(1 - p^{-1})^{-1} (2\pi/\rho) \int |f(\xi)|^2 d\xi = \prod(1 - p^{-2}) \sup \int |g(\tau + iy)|^2 d\tau$$

holds with the least upper bound taken over all positive numbers y. Integration on the left is with respect to Haar measure for the r–adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. Integration on the right is with respect to Haar measure for the r–adelic line over a fundamental regions containing an element whose p–adic component is a unit for every prime divisor p of ρ. The products are taken over the prime divisors p of r. When ν is positive, the identity

$$\prod(1-p^{-1})^{-1}(2\pi/\rho)^{\nu}\Gamma(\nu)\int|f(\xi)|^2d\xi=\prod(1-p^{-2})\int_0^{\infty}\int|g(\tau+iy)|^2y^{\nu-1}d\tau dy$$

is satisfied.

The theta function of order ν and character χ for the r–adelic skew–plane is used to define the Laplace transformation of order ν and character χ for the r–adelic skew–plane. The domain of the transformation is the set of locally square integrable functions $f(\xi)$ of ξ in the r–adelic skew–plane which vanish at elements of the r–adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi)=\omega_+^{\nu}\chi(\omega_-)f(\xi)$$

for every unit ω of the r–adelic skew–plane, and which satisfy the identity

$$f(\xi)=f(\omega\xi)$$

for every representative ω of an element of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The Laplace transform of order ν and character χ for the r–adelic skew–plane is the function $g(\eta,\gamma)$ of elements η of the r–adelic upper half–plane and invertible elements γ of the r–adic line defined by the integral

$$\prod(1-p^{-2})2\pi g(\eta,\gamma)=\int(\xi_+^{\nu})^-\chi(\xi_-)^-f(\xi)\theta(\xi^-\eta\xi,\gamma)d\xi$$

with respect to Haar measure for the r–adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the prime divisors p of r. The function $g(\eta,\gamma)$ of elements η of the r–adelic upper half–plane and invertible elements γ of the r–adelic line is an analytic function of the Euclidean component of η when γ and the r–adic component of η are held fixed. The function $g(\eta,\gamma)$ vanishes when the p–adic component of η or the p–adic component of γ is not a unit for some prime divisor p of ρ. The identities

$$g(\omega\eta,\gamma)=g(\eta,\gamma)=g(\eta,\omega\gamma)$$

hold for every unit ω of the r–adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$g(\eta,\gamma)=(\omega^{\nu})^-\chi(\omega_-)^-g(\omega^-\eta\omega,\gamma)$$

holds for every representative ω of an element of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$g(\eta,\gamma)=\lambda_+^{2\nu-1}|\lambda|_-^{2\nu-1}\chi(\lambda_-)g(\lambda^{-1}\eta,\lambda\gamma)$$

holds for every invertible element λ of the r–adelic line whose r–adic modulus is rational and whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$(1-p)g(\eta,\gamma) = g(\lambda\eta,\gamma) - pg(\lambda^{-1}\eta,\gamma)$$

holds when the p–adic modulus of η is not an integral power of p for some prime divisor p of r, which is not a divisor of ρ, and when λ is an element of the r–adelic line whose Euclidean component is the unit of the Euclidean diline and which satisfies the identity

$$p|\lambda|_- = 1.$$

The identity

$$(1-p)g(\eta,\gamma) = g(\eta,\lambda\gamma) - pg(\eta,\lambda^{-1}\gamma)$$

holds when the p–adic modulus of γ is not an integral power of p for some prime divisor p of r, which is not a divisor of ρ, and when λ is an element of the r–adelic line whose Euclidean component is the unit of the Euclidean line and which satisfies the identity

$$p|\lambda|_- = 1.$$

The theta function of order ν and character χ^- for the r–adelic plane is computable from the theta function $\theta(\eta)$ of order ν and character χ for the r–adelic plane as the function

$$\theta(-\eta^-)^-$$

of η in the r–adelic upper half–plane. The domain of the Hankel transformation of order ν and character χ for the r–adelic plane is the set of locally square integrable functions $f(\xi)$ of ξ in the r–adelic plane which vanish at elements of the r–adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the r–adelic plane, and which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every element ω of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The range of the Hankel transformation of order ν and character χ for the r–adelic plane is the set of locally square integrable functions $g(\xi)$ of ξ in the r–adelic plane which vanish at elements of the r–adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$g(\omega\xi) = \omega_+^\nu \chi^-(\omega_-) g(\xi)$$

for every unit ω of the r–adelic plane, and which satisfy the identity

$$g(\xi) = g(\omega\xi)$$

for every element ω of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The transformation takes a function $f(\xi)$ of ξ in the r–adelic plane into a function $g(\xi)$ of ξ in the r–adelic plane when the identity

$$(i/\eta_+)^{1+\nu}\operatorname{sgn}(\eta_-)|\eta|_-^{-1}\int(\xi_+^\nu)^-\chi(\xi_-)^-f(\xi)\theta(-\xi^-\eta^{-1}\xi)d\xi$$
$$=\int(\xi_+^\nu)^-\chi(\xi_-^-)g(\xi)\theta(-\xi^-\eta^-\xi)^-d\xi$$

holds for η in the r–adelic upper half–plane. Integration is with respect to Haar measure for the r–adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$\int|f(\xi)|^2d\xi=\int|g(\xi)|^2d\xi$$

holds with integration with respect to Haar measure for the r–adelic plane over such a region. The inverse of the Hankel transformation of order ν and character χ for the r–adelic plane is the Hankel transformation of order ν and character χ^- for the r–adelic plane.

The theta function of order ν and character χ^- for the r–adelic skew–plane is computable from the theta function $\theta(\eta,\gamma)$ of order ν and character χ for the r–adelic skew–plane as the function

$$\theta(-\eta^-,\gamma)^-$$

of elements η of the r–adelic upper half–plane and invertible elements γ of the r–adelic line. The domain of the Hankel transformation of order ν and character χ for the r–adelic skew–plane is the set of locally square integrable functions $f(\xi)$ of ξ in the r–adelic skew–plane which vanish at elements of the r–adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi)=\omega_+^\nu\chi(\omega_-)f(\xi)$$

for every unit ω of the r–adelic skew–plane, and which satisfy the identity

$$f(\xi)=f(\omega\xi)$$

for every representative ω of an element of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The range of the Hankel transformation of order ν and character χ for the r–adelic skew–plane is the set of locally square integrable functions $g(\xi)$ of ξ in the r–adelic skew–plane which vanish at elements of the r–adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$g(\omega\xi)=\omega_+^\nu\chi^-(\omega_-)g(\xi)$$

for every unit ω of the r–adelic skew–plane, and which satisfy the identity

$$g(\xi) = g(\omega\xi)$$

for every representative ω of an element of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The transformation takes a function $f(\xi)$ of ξ in the r–adelic skew–plane into a function $g(\xi)$ of ξ in the r–adelic skew–plane when the identity

$$(i/\eta_+)^\nu \operatorname{sgn}(\eta_-\gamma_-)|\eta\gamma|_-^{-2} \int (\xi_+^\nu)^- \chi(\xi_-)^- f(\xi)\theta(-\xi^-\eta^{-1}\xi, \gamma^{-1}) d\xi$$
$$= \int (\xi_+^\nu)^- \chi(\xi_-^-) g(\xi)\theta(-\xi^-\eta^-\xi, \gamma)^- d\xi$$

holds for η in the r–adelic upper half–plane and γ an invertible element of the r–adelic line. Integration is with respect to Haar measure for the r–adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the r–adelic skew–plane over such a region. The inverse of the Hankel transformation of order ν and character χ for the r–adelic skew–plane is the Hankel transformation of order ν and character χ^- for the r–adelic skew–plane.

Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) are associated with Riemann zeta functions. The spaces originate in the spectral theory for the Laplace–Beltrami operator for the hyperbolic geometry of the upper half–plane as it acts on functions which are invariant under a Hecke subgroup of the modular group [5].

The Mellin transformation of order ν and character χ for the r–adelic plane is a spectral theory for the Laplace transformation of order ν and character χ for the r–adelic plane. The domain of the Laplace transformation of order ν and character χ for the r–adelic plane is the set of locally square integrable functions $f(\xi)$ of ξ in the r–adelic plane which vanish at elements ξ of the r–adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the r–adelic plane, and which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every element ω of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The Laplace transform of order ν and character χ

for the r–adelic plane is the function $g(\eta)$ of η in the r–adelic upper half–plane defined by the integral

$$\prod(1-p^{-2})2\pi g(\eta) = \int (\xi_+^\nu)^- \chi(\xi_-)^- f(\xi)\theta(\xi^-\eta\xi)d\xi$$

with respect to Haar measure for the r–adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The Mellin transform of order ν and character χ for the r–adelic plane is an analytic function $F(z)$ of z in the upper half–plane defined by the integral

$$\prod(1-p^{-1})^{-1}F(z) = \int_0^\infty g(\eta)t^{\frac{1}{2}\nu-\frac{1}{2}-\frac{1}{2}iz}dt$$

under the constraint

$$\eta_+ = it$$

when η_- is a unit of the r–adic line if the function $f(\xi)$ of ξ in the r–adelic plane vanishes in a neighborhood $|\xi| < a$ of the origin. The product is taken over the prime divisors p of ρ. The integral can be evaluated under the same constraint when η_- is an element of the r–adic line whose r–adic modulus is integral and is a divisor of r which is relatively prime to ρ. When this change is made in the right side of the identity, the left side is multiplied by the product

$$\prod \frac{1-\chi(\lambda_-)^-\lambda_+^{iz}}{1-\lambda_+}$$

taken over the elements λ of the principal subgroup of the r–adelic line whose r–adic component is integral and whose Euclidean component is a prime divisor of $|\eta|_-$. A computation of the integral is made from the zeta function

$$\zeta(s) = \sum \tau(n)n^{-s}$$

of order ν and character χ for the r–adelic plane, which is defined in the half–plane $\mathcal{R}s > 1$ as a sum over the positive integers n whose prime divisors are divisors of r but not of ρ. The identity

$$\tau(n) = \chi(p)^{-k}$$

holds when a positive integer

$$n = p^k$$

is a power of a prime p which is a divisor of r but not of ρ. The zeta function is represented in the complex plane by the Euler product

$$\zeta(s)^{-1} = \prod(1-\tau(p)p^{-s})$$

taken over the prime divisors p of r which are not divisors of ρ. The analytic weight function

$$W(z) = (\rho/\pi)^{\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}-\tfrac{1}{2}iz)\zeta(1-iz)$$

is applied in the characterization of Mellin transforms of order ν and character χ for the r–adelic plane. The weight function is represented in the upper half–plane by the integral

$$\prod(1-p^{-1})^{-1}W(z)=\int_0^\infty \theta(\eta)t^{\frac{1}{2}\nu-\frac{1}{2}-\frac{1}{2}iz}dt$$

under the constraint

$$\eta_+ = it$$

when η_- is a unit of the r–adic line. The product is taken over the prime divisors p of ρ. The identity

$$\prod(1-p^{-2})2\pi F(z)/W(z)=\int(\xi_+^\nu)^-\chi(\xi_-)^-f(\xi)|\xi|^{iz-\nu-1}d\xi$$

holds when z is in the upper half–plane. Integration is with respect to Haar measure for the r–adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product on the left is taken over the prime divisors p of r. The function

$$a^{-iz}F(z)$$

of z in the upper half–plane is characterized as an element of the weighted Hardy space $\mathcal{F}(W)$ which satisfies the identity

$$\prod(1-p^{-2})\int_{-\infty}^{+\infty}|F(t)/W(t)|^2dt=\int|f(\xi)|^2d\xi.$$

Integration on the right is with respect to Haar measure for the r–adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the prime divisors p of r. The identity

$$\prod(1-p^{-2})\int_{-\infty}^{+\infty}|F(t)/W(t)|^2dt$$
$$=\sum\alpha^{-1}\int_{-\infty}^{+\infty}\left|\prod\frac{1-\chi(\lambda_-)^-\lambda_+^{it}}{1-\lambda_+^{-1}}\;\frac{\zeta(1-it)F(t)}{W(t)}\right|^2dt$$

holds with summation over the divisors α of r, which are relatively prime to ρ, with the product on the left taken over the prime divisors p of r, which are not divisors of ρ, and with the product on the right taken over the elements λ of the principal subgroup of the r–adelic line whose r–adic component is integral and whose Euclidean component is a prime divisor p of α. If the Hankel transform of order ν and character χ for the r–adelic plane of the function $f(\xi)$ of ξ in the r–adelic plane vanishes when $|\xi|<a$, then its Mellin transform of order ν and character χ^- for the r–adelic plane is an entire function which is the analytic extension of the function

$$F(-z)$$

of z to the complex plane.

The Mellin transformation of order ν and character χ for the r–adelic skew–plane is a spectral theory for the Laplace transformation of order ν and character χ for the r–adelic skew–plane. The domain of the Laplace transformation of order ν and character χ for the r–adelic skew–plane is the set of locally square integrable functions $f(\xi)$ of ξ in the r–adelic skew–plane which vanish at elements ξ of the r–adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the r–adelic skew–plane, and which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every representative ω of an element of the principal subgroup of the r–adelic line whose p–adic component is a unit for every prime divisor p of ρ. The Laplace transform of order ν and character χ for the r–adelic diplane is the function $g(\eta, \gamma)$ of elements η of the r–adelic upper half–plane and invertible elements γ of the r–adelic line defined by the integral

$$\prod (1 - p^{-2}) 2\pi g(\eta, \gamma) = \int (\xi_+^\nu)^- \chi(\xi_-)^- f(\xi) \theta(\xi^- \eta \xi, \gamma) d\xi$$

with respect to Haar measure for the r–adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the prime divisors p of r. The Mellin transform of order ν and character χ for the r–adelic diplane is an analytic function $F(z)$ of z in the upper half–plane defined by the integral

$$\prod (1 - p^{-1})^{-2} F(z) = \int_0^\infty g(\eta, \gamma) t^{\nu - \frac{1}{2} - iz} dt$$

under the constraint

$$\eta_+ = it$$

when γ is a unit of the r–adelic line and η_- is a unit of the r–adic line if the function $f(\xi)$ of ξ in the r–adelic skew–plane vanishes in a neighborhood $|\xi| < a$ of the origin. The product is taken over the prime divisors p of ρ. The integral can be evaluated under the same constraint when γ and η_- are elements of the r–adic line the square of whose r–adic modulus is integral and is a divisor of r which is relatively prime to ρ. When this change is made in the right side of the identity, the left side is multiplied by the product

$$\prod \frac{1 - (\lambda_+^{\nu - \frac{1}{2}})^- |\lambda|_-^{\nu - \frac{1}{2}} \chi(\lambda_-)^- \lambda_+^{2iz}}{1 - \lambda_+^2}$$

taken over the equivalence classes of representatives λ of elements of the principal subgroup of the r–adelic line whose r–adic component is integral and whose Euclidean component is a divisor of $|\eta|_-^2$, and is also multiplied by the product

$$\prod \frac{1 - \lambda_+^{\nu - \frac{1}{2}} |\lambda|_-^{\nu - \frac{1}{2}} \chi(\lambda_-) \lambda_+^{2iz}}{1 - \lambda_+^2}$$

taken over the equivalence classes of representatives λ of elements of the principal subgroup of the r–adelic line whose r–adic component is integral and whose Euclidean component is a divisor of $|\gamma|_-^2$. A computation of the integral is made using the zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

of order ν and character χ for the r–adelic skew–plane, which is defined in the half–plane $\mathcal{R}s > 1$ as a sum over the positive integers n whose prime divisors are divisors of r but not of ρ. If a positive integer

$$n = p^k$$

is a power of prime p which is a divisor of r but not of ρ, then the identity

$$\tau(n) = \frac{[\lambda_+^{\nu-\frac{1}{2}}|\lambda|_-^{\nu-\frac{1}{2}}\chi(\lambda_-)]^{1+k} - [\lambda_+^{\nu-\frac{1}{2}}|\lambda|_-^{\nu-\frac{1}{2}}\chi(\lambda_-)]^{-1-k}}{[\lambda_+^{\nu-\frac{1}{2}}|\lambda|_-^{\nu-\frac{1}{2}}\chi(\lambda_-)] - [\lambda_+^{\nu-\frac{1}{2}}|\lambda|_-^{\nu-\frac{1}{2}}\chi(\lambda_-)]^{-1}}$$

holds with λ a representative of the element of the principal subgroup of the r–adelic line whose r–adic component is integral and whose Euclidean component is p. The fraction is defined by continuity when the denominator is equal to zero. The function is defined in the complex plane by the Euler product

$$\zeta(s)^{-1} = \prod (1 - \tau(p)p^{-s} + p^{-2s})$$

taken over the prime divisors p of r which are not divisors of ρ. The analytic weight function

$$W(z) = (2\pi/\rho)^{-\frac{1}{2}\nu - 1 + iz}\Gamma(\tfrac{1}{2}\nu + 1 - iz)\zeta(1 - iz)$$

is applied in the characterization of Mellin transforms of order ν and character χ for the r–adelic skew–plane. The weight function is represented in the upper half–plane by the integral

$$\prod (1 - p^{-1})^2 W(z) = \int_0^\infty \theta(\eta, \gamma) t^{\nu - \frac{1}{2} - iz} dt$$

under the constraint

$$\eta_+ = it$$

when γ is a unit of the r–adelic line and η_- is a unit of the r–adelic line. The identity

$$\prod (1 - p^{-2})^2 2\pi F(z)/W(z) = \int (\xi_+^\nu)^- \chi(\xi_-)^- f(\xi) |\xi|^{2iz - 2\nu - 3} d\xi$$

holds when z is in the upper half–plane. Integration is with respect to Haar measure for the r–adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the prime divisors p of r. The function

$$a^{-iz} F(z)$$

of z in the upper half–plane is characterized as an element of the weighted Hardy space $\mathcal{F}(W)$ which satisfies the identity

$$\prod (1 - p^{-2}) \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \int |f(\xi)|^2 d\xi.$$

Integration on the right is with respect to Haar measure for the r–adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the prime divisors p of r. The identity

$$\prod (1 - p^{-1})^4 \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt$$
$$= \sum \alpha^{-1} \beta^{-1} \int \left| \frac{1 - \lambda_+^{\nu - \frac{1}{2}} |\lambda|_-^{\nu - \frac{1}{2}} |\lambda|_-^{\nu - \frac{1}{2}} \chi(\lambda_-) \lambda_+^{2it}}{1 - \lambda_+^{-2}} \right.$$
$$\times \prod \frac{1 - (\lambda_+^{\nu - \frac{1}{2}})^- |\lambda|_-^{\nu - \frac{1}{2}} \chi(\lambda_-)^- \lambda_+^{2it}}{1 - \lambda_+^{-2}}$$
$$\left. \times \frac{\zeta(1 - it) F(t)}{W(t)} \right|^2 dt$$

holds with summation over the divisors α and β or r, which are relatively prime to ρ, and with the product on the left taken over the prime divisors p of r which are not divisors of ρ. The first product on the right is taken over the equivalence classes of representatives λ of elements of the principal subgroup of the r–adelic line whose r–adic component is integral and whose Euclidean component is a prime divisor of α. The second product on the right is taken over the equivalence classes of representatives λ of elements of the principal subgroup of the r–adelic line whose r–adic component is integral and whose Euclidean component is a divisor of β. If the Hankel transform of order ν and character χ for the r–adelic skew–plane of the function $f(\xi)$ of ξ in the r–adelic diplane vanishes when $|\xi| < a$, then its Mellin transform of order ν and character χ for the r–adelic skew–plane is an entire function which is the analytic continuation of the function

$$F(-z)$$

to the complex plane.

The Sonine spaces of order ν and character χ for the r–adelic plane are defined using the analytic weight function

$$W(z) = (\pi/\rho)^{-\frac{1}{2}\nu - \frac{1}{2} + \frac{1}{2}iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz) \zeta(1 - iz)$$

constructed from the zeta function of order ν and character χ for the r–adelic plane. The space of parameter a contains the entire functions $F(z)$ such that

$$a^{-iz} F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is obtained when the space is considered with the scalar product such that multiplication by a^{-iz} is an isometric transformation of the space into the space $\mathcal{F}(W)$. The space is a space $\mathcal{H}(E)$ which coincides as a set with the Sonine space of order ν and parameter a for the Euclidean plane. The Sonine spaces of order ν for the Euclidean plane are constructed from the analytic weight function

$$W_0(z) = (\pi/\rho)^{-\frac{1}{2}\nu-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz).$$

The identity

$$S(z)W(z) = (r/\rho)^{-\frac{1}{2}iz}W_0(z)$$

holds with

$$S(z) = (r/\rho)^{-\frac{1}{2}iz}\zeta(1-iz)^{-1}$$

an entire function of Pólya class, which is determined by its zeros, such that $S(z-i)$ is of Pólya class. The Sonine space of order ν and parameter $ar^{-\frac{1}{2}}$ for the Euclidean plane is the set of entire functions $F(z)$ such that

$$r^{\frac{1}{2}iz}a^{-iz}F(z)$$

and

$$r^{\frac{1}{2}iz}a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W_0)$. The space is a space $\mathcal{H}(E_0)$ such that multiplication by $r^{\frac{1}{2}iz}a^{-iz}$ is an isometric transformation of the space into the space $\mathcal{F}(W_0)$. The space $\mathcal{H}(E)$ is then the set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$. A maximal dissipative transformation in the space $\mathcal{F}(W_0)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space. A maximal dissipative transformation in the space $\mathcal{H}(E_0)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space. The set of entire functions $F(z)$ such that $S(z-i)F(z)$ belongs to the space $\mathcal{H}(E_0)$ is a space $\mathcal{H}(E')$ such that multiplication by $S(z-i)$ is an isometric transformation of the space into the space $\mathcal{H}(E_0)$. The space $\mathcal{H}(E)$ is contained contractively in the space $\mathcal{H}(E')$. A relation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$, which satisfies the axiom (H1), exists which takes $F(z)$ into $G(z)$ when $S(z)F(z)$ is the orthogonal projection into the image of the space $\mathcal{H}(E)$ of the element $S(z-i)G(z)$ of the space $\mathcal{H}(E')$. A maximal dissipative relation in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space such that $G(z)$ is the image of $F(z)$ in the space $\mathcal{H}(E')$.

The augmented Sonine spaces of zero order and principal character for the r–adelic plane are defined using the analytic weight function

$$W(z) = iz(iz-1)\pi^{-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2} - \tfrac{1}{2}iz)\zeta(1-iz)$$

constructed from the zeta function of zero order and principal character for the r–adelic plane. The space of parameter a contains the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$. A Hilbert space of entire functions, which satisfies the axioms (H1), (H2), and (H3) and which is symmetric about the origin, is obtained when the space is considered with the scalar product such that multiplication by a^{-iz} is an isometric transformation of the space into the space $\mathcal{F}(W)$. The space is a space $\mathcal{H}(E)$ which coincides as a set with the augmented Sonine space of zero order and parameter a for the Euclidean plane. The augmented Sonine spaces of zero order for the Euclidean plane are constructed from the analytic weight function

$$W_0(z) = iz(iz-1)\pi^{-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}iz).$$

The identity

$$S(z)W(z) = r^{-\frac{1}{2}iz}W_0(z)$$

holds with

$$S(z) = r^{-\frac{1}{2}iz}\zeta(1-iz)^{-1}$$

an entire function of Pólya class, which is determined by its zeros and which satisfies the symmetry condition

$$S(-z) = S^*(z),$$

such that $S(z-i)$ is of Pólya class. The augmented Sonine space of zero order and parameter $ar^{-\frac{1}{2}}$ for the Euclidean plane is the set of entire functions $F(z)$ such that

$$r^{\frac{1}{2}iz}a^{-iz}F(z)$$

and

$$r^{\frac{1}{2}iz}a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W_0)$. The space is a space $\mathcal{H}(E_0)$, which is symmetric about the origin, such that multiplication by $r^{\frac{1}{2}iz}a^{-iz}$ is an isometric transformation of the space into the space $\mathcal{F}(W_0)$. The space $\mathcal{H}(E)$ is then the set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$. Multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E_0)$. A space $\mathcal{H}(E_0')$, which is symmetric about the origin, is constructed so that an isometric transformation of the set of elements of the space $\mathcal{H}(E_0)$ having value zero at the origin onto the set of elements of the space $\mathcal{H}(E_0')$ having value zero at the origin onto the set of elements of the space $\mathcal{H}(E_0')$ having value zero at i is defined by taking $F(z)$ into

$$(z-i)F(z)/z.$$

A continuous transformation of the space $\mathcal{H}(E_0)$ into the space $\mathcal{H}(E_0')$ is obtained as the unique extension which annihilates the reproducing kernel function for function values at $-i$. Entire functions $P_0(z)$ and $Q_0(z)$, which are associated with the spaces $\mathcal{H}(E_0)$ and $\mathcal{H}(E_0')$ and which satisfy the symmetry conditions

$$P_0(-z) = P_0^*(z)$$

and

$$Q_0(-z) = -Q_0^*(z),$$

exist such that the transformation of the space $\mathcal{H}(E_0)$ into the space $\mathcal{H}(E_0')$ takes an element $F(z)$ of the space $\mathcal{H}(E_0)$ into an element $G(z)$ of the space $\mathcal{H}(E_0')$ whenever the identity

$$G(w) = \langle F(t), [Q_0(t)P_0(w^-) - P_0(t)Q_0(w^-)]/[\pi(t - w^-)]\rangle_{\mathcal{H}(E_0)}$$

holds for all complex numbers w and such that the adjoint transformation of the space $\mathcal{H}(E_0')$ into the space $\mathcal{H}(E_0)$ takes an element $F(z)$ of the space $\mathcal{H}(E_0')$ into an element $G(z)$ of the space $\mathcal{H}(E_0)$ when the identity

$$G(w) = \langle F(t), [Q_0^*(t)P_0(w)^- - P_0^*(t)Q_0(w)^-]/[\pi(t - w^-)]\rangle_{\mathcal{H}(E_0')}$$

holds for all complex numbers w. A maximal transformation of dissipative deficiency at most one in the space $\mathcal{H}(E_0)$ is defined by taking $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space such that the transformation of the space $\mathcal{H}(E_0)$ into the space $\mathcal{H}(E_0')$ takes $F(z)$ into $G(z)$. The set of entire functions $F(z)$ such that $S(z-i)F(z)$ belongs to the space $\mathcal{H}(E_0')$ is a space $\mathcal{H}(E')$, which is symmetric about the origin, such that multiplication by $S(z-i)$ is an isometric transformation of the space $\mathcal{H}(E')$ into the space $\mathcal{H}(E_0')$. The space $\mathcal{H}(E)$ is contained contractively in the space $\mathcal{H}(E')$. A relation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$, which takes $F^*(-z)$ into $G^*(-z)$ whenever it takes $F(z)$ into $G(z)$ and which takes $F(z)$ into $G(z)$ when $S(z)F(z)$ is the orthogonal projection into the image of the space $\mathcal{H}(E)$ of an element $H(z)$ of the space $\mathcal{H}(E_0)$ whose image in the space $\mathcal{H}(E_0')$ is $S(z-i)G(z)$. A maximal relation of dissipative deficiency at most one in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space such that $G(z)$ is the image of $F(z)$ in the space $\mathcal{H}(E')$.

The Sonine spaces of order ν and character χ for the r–adelic skew–plane are defined using the analytic weight function

$$W(z) = (2\pi/\rho)^{-\frac{1}{2}\nu-1+iz}\Gamma(\tfrac{1}{2}\nu + 1 - iz)\zeta(1 - iz)$$

constructed from the zeta function of order ν and character χ for the r–adelic skew–plane. The space of parameter a contains the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is obtained when the space is considered with the scalar product such that multiplication by a^{-iz} is an isometric transformation of the space into the space $\mathcal{F}(W)$. The space is a space $\mathcal{H}(E)$ which coincides as a set with the Sonine space of order ν and parameter a for the Euclidean skew–plane. The Sonine spaces of order ν for the Euclidean skew–plane are constructed from the analytic weight function

$$W_0(z) = (2\pi/\rho)^{-\frac{1}{2}\nu-1+iz}\Gamma(\tfrac{1}{2}\nu + 1 - iz).$$

The identity

$$S(z)W(z) = (r/\rho)^{-iz}W_0(z)$$

holds with

$$S(z) = (r/\rho)^{-iz}\zeta(1 - iz)^{-1}$$

an entire function of Pólya class, which is determined by its zeros, such that $S(z - i)$ is of Pólya class. The Sonine space of order ν and parameter $a\rho/r$ for the Euclidean diplane is the set of entire functions $F(z)$ such that

$$(r/\rho)^{iz}a^{-iz}F(z)$$

and

$$(r/\rho)^{iz}a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W_0)$. The space is a space $\mathcal{H}(E_0)$ such that multiplication by $(a\rho/r)^{-iz}$ is an isometric transformation of the space into the space $\mathcal{F}(W_0)$. The space $\mathcal{H}(E)$ is then the set of entire functions $F(z)$ such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$. Multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{H}(E_0)$. A maximal dissipative transformation in the space $\mathcal{F}(W_0)$ is defined by taking $F(z)$ into $F(z + i)$ whenever $F(z)$ and $F(z + i)$ belong to the space. A maximal dissipative transformation in the space $\mathcal{H}(E_0)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z + i)$ belong to the space. The set of entire functions $F(z)$ such that $S(z - i)F(z)$ belongs to the space $\mathcal{H}(E_0)$ is a space $\mathcal{H}(E')$ such that multiplication by $S(z - i)$ is an isometric transformation of the space into the space $\mathcal{H}(E_0)$. The space $\mathcal{H}(E)$ is contained contractively in the space $\mathcal{H}(E')$. A relation with domain in the space $\mathcal{H}(E)$ and range in the space $\mathcal{H}(E')$, which satisfies the axiom (H1), exists which takes $F(z)$ into $G(z)$ when $S(z)F(z)$ is the orthogonal projection into the image of the space $\mathcal{H}(E)$ of the element $S(z - i)G(z)$ of the space $\mathcal{H}(E')$. A maximal dissipative relation in the space $\mathcal{H}(E)$ is defined by taking $F(z)$ into $G(z + i)$ whenever $F(z)$ and $G(z + i)$ are elements of the space such that $G(z)$ is the image of $F(z)$ in the space $\mathcal{H}(E')$.

A renormalization of Haar measure is made for the adic line. A dual Haar measure exists for every renormalization of Haar measure. The renormalization of Haar measure for the adic line is made so that the set of units of the adic line has measure one with respect to the dual Haar measure. Renormalized Haar measure for the p–adic line is the renormalization of Haar measure for which the set of integral elements has measure

$$1 - p^{-1}.$$

Renormalized Haar measure for the adic line is the Cartesian product of the renormalized Haar measures for the p–adic lines. Renormalized Haar measure for the adic line is singular with respect to Haar measure for the adic line.

A renormalization of Haar measure is made for the adic skew–plane. A dual Haar measure exists for every renormalization of Haar measure. The renormalization of Haar measure for the adic skew–plane is made so that the set of units of the adic skew–plane has measure one with respect to the dual Haar measure. Renormalized Haar measure for the p–adic skew–plane is the normalization of Haar measure for which the set of integral elements has measure

$$1 - p^{-1}.$$

Renormalized Haar measure for the adic skew–plane is the Cartesian product of the renormalized Haar measures for the p–adic skew–planes. Renormalized Haar measure for the adic skew–plane is singular with respect to Haar measure for the adic skew–plane.

The kernel for the Laplace transformation of character χ for the adic plane is a function $\sigma(\eta)$ of invertible elements η of the adic line which vanishes when the p–adic component of η is not a unit for some prime divisor p of ρ and which is otherwise defined as an integral

$$\sigma(\eta) = \prod(1 - p^{-1})^{-1} \int \exp(2\pi i\eta\xi) d\xi$$

with respect to the dual Haar measure for the adic line over the set of units for the adic line. The product is taken over the prime divisors p of ρ. The integral

$$\int |\sigma(\eta)|^2 d\eta$$

with respect to renormalized Haar measure for the adic line is equal to one. The integral

$$\int \sigma(\beta\eta)^- \sigma(\alpha\eta) d\eta$$

with respect to renormalized Haar measure for the adic line is equal to zero when α and β are invertible elements of the adic line of unequal adic modulus whose p–adic component is a unit for every prime divisor p of ρ. The function $\sigma(\eta)$ of η in the adic line has the value zero when the p–adic component of η is not a unit for some prime divisor p of ρ or when the p–adic component of $p\eta$ is not integral for some prime p. When the p–adic component of η is a unit for every prime divisor p of ρ and when the p–adic component of $p\eta$ is integral for every prime p, then $\sigma(\eta)$ is equal to

$$\prod(1 - p^{-1})^{-1} \prod(1 - p)^{-1}$$

with the product on the left taken over the prime divisors p of ρ and the product on the right taken over the primes p such that the p–adic component of η is not integral.

The kernel for the Laplace transformation of character χ for the adic skew–plane is a function $\sigma(\eta)$ of invertible elements η of the adic skew–plane has value zero when the

p–adic component of η is not a unit for some prime divisor p of ρ and is otherwise defined as an integral

$$\sigma(\eta) = \prod (1 - p^{-1})^{-1} \int \exp(\pi i(\eta^- \xi + \xi^- \eta)) d\xi$$

with respect to the dual Haar measure for the adic skew–plane over the set of units. The product is taken over the prime divisors p of ρ. The integral

$$\int |\sigma(\eta)|^2 d\eta$$

with respect to renormalized Haar measure for the adic skew–plane is equal to one. The integral

$$\int \sigma(\beta\eta)^- \sigma(\alpha\eta) d\eta$$

with respect to renormalized Haar measure for the adic skew–plane is equal to zero when α and β are invertible elements of the adic skew–plane of unequal adic modulus whose p–adic component is a unit for every prime divisor p of ρ. The function $\sigma(\eta)$ of invertible elements η of the adic skew–plane vanishes when the p–adic component of η is not a unit for some prime divisor p of ρ or when the p–adic component of $p\eta^*\eta$ is not integral for some prime p. When the p–adic component of η is a unit for every prime divisor p of ρ and the p–adic component of $p\eta^*\eta$ is integral for every prime p, then $\sigma(\eta)$ is equal to

$$\prod (1 - p^{-1})^{-1} \prod (1 - p)^{-1}$$

with the product on the left taken over the prime divisors p of ρ and the product on the right taken over the primes p such that the p–adic component of η is not integral.

A character χ for the adic plane is used in the definition of the Laplace transformation of character χ for the adic plane. The transformation is defined on the set of functions $f(\xi)$ of ξ in the adic plane which are square integrable with respect to Haar measure for the adic plane, which vanish at elements of the adic plane whose p–adic component is not a unit for a prime divisor p of ρ, and which satisfy the identity

$$f(\omega\xi) = \chi(\omega) f(\xi)$$

for every unit ω of the adic plane. The Laplace transform of character χ of the function $f(\xi)$ of ξ in the adic plane is the function $g(\eta)$ of η in the adic line which is defined by the integral

$$\prod (1 - p^{-2}) g(\eta) = \int \chi(\xi)^- f(\xi) \sigma(\xi^- \eta \xi) d\xi$$

with respect to Haar measure for the adic plane using the Laplace kernel for the adic plane. The product is taken over the primes p. The identity

$$\int |f(\xi)|^2 d\xi = \prod (1 - p^{-2}) \int |g(\eta)|^2 d\eta$$

holds with integration on the left with respect to Haar measure for the adic plane and with integration on the right with respect to renormalized Haar measure for the adic line. The product is taken over the primes p. A function $g(\eta)$ of η in the adic line, which is square integrable with respect to renormalized Haar measure for the adic line, is the Laplace transform of order χ of a square integrable function with respect to Haar measure for the adic plane if, and only if, it vanishes at elements of the adic line whose p–adic component is not a unit for some prime divisor p of ρ, satisfies the identity

$$g(\eta) = g(\omega\eta)$$

for every unit ω of the adic line, and satisfies the identity

$$(1-p)g(\eta) = g(\lambda\eta) - pg(\lambda^{-1}\eta)$$

when the p–adic modulus of η is an odd power of p for some prime p, which is not a divisor of ρ, and when λ is an element of the adic line such that

$$p|\lambda|_- = 1.$$

A character χ for the adic skew–plane is used in the definition of the Laplace transformation of character χ for the adic skew–plane. The transformation is defined on the set of functions $f(\xi)$ of ξ in the adic skew–plane which are square integrable with respect to Haar measure for the adic skew–plane, which vanish at elements of the adic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, and which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the adic skew–plane. The Laplace transform of character χ for the adic skew–plane of the function $f(\xi)$ of ξ in the adic skew–plane is the function $g(\eta)$ of invertible elements η of the adic line defined by the integral

$$\prod(1-p^{-2})g(\eta) = \int \chi(\xi)^- f(\xi)\theta(\xi^-\eta\xi)d\xi$$

with respect to Haar measure for the adic skew–plane. The product is taken over the primes p. The identity

$$\int |f(\xi)|^2 d\xi = \prod(1-p^{-2}) \int |g(\eta)|^2 d\eta$$

holds with integration on the left with respect to Haar measure for the adic skew–plane and with integration on the right with respect to renormalized Haar measure for the adic line. The product is taken over the primes p. A function $g(\eta)$ of invertible elements η of the adic line, which is square integrable with respect to renormalized Haar measure for the adic line, is a Laplace transform of character χ for the adic skew–plane if, and only if, it vanishes when the p–adic component of η is not a unit for some prime divisor p of ρ, the identity

$$g(\eta) = g(\omega\eta)$$

holds for every unit ω of the adic line, and satisfies the identities

$$(1-p)g(\eta,\gamma) = g(\lambda\eta,\gamma) - pg(\lambda^{-1}\eta,\gamma)$$

when η is an invertible element of the adic line whose p–adic modulus is not an integral power of p for some prime p, which is not a divisor of ρ, and λ is an element of the adic diline such that

$$p|\lambda|_-^2 = 1.$$

A character χ for the adic plane and its conjugate character χ^- are used in the definition of the Hankel transformation of character χ for the adic plane. The domain of the transformation is the set of functions $f(\xi)$ of ξ in the adic plane which are square integrable with respect to Haar measure for the adic plane, which vanish at elements of the adic plane whose p–adic component is not a unit for some prime divisor p of ρ, and which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the adic plane. The range of the transformation is the set of functions $g(\xi)$ of ξ in the adic plane which are square integrable with respect to Haar measure for the adic plane, which vanish at elements of the adic plane whose p–adic component is not a unit for some prime divisor p of ρ, and which satisfy the identity

$$g(\omega\xi) = \chi^-(\omega)g(\xi)$$

for every unit ω of the adic plane. The transformation takes a function $f(\xi)$ of ξ in the adic plane into a function $g(\xi)$ of ξ in the adic plane when the identity

$$\int \chi(\xi^-)g(\xi)\sigma(\xi^-\eta\xi)d\xi = \operatorname{sgn}(\eta)|\eta|_-^{-1}\int \chi(\xi)^-f(\xi)\sigma(\xi^-\eta^{-1}\xi)d\xi$$

holds for every invertible element η of the adic line with integration with respect to Haar measure for the adic plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the adic plane. The Hankel transformation of character χ^- for the adic plane is the inverse of the Hankel transformation of character χ for the adic plane.

A character χ for the adic skew–plane and its conjugate character χ^- for the adic skew–plane are used in the definition of the Hankel transformation of character χ for the adic skew–plane. The domain of the transformation is the set of functions $f(\xi)$ of ξ in the adic skew–plane which are square integrable with respect to Haar measure for the adic skew–plane, which vanish at elements of the adic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, and which satisfy the identity

$$f(\omega\xi) = \chi(\omega)f(\xi)$$

for every unit ω of the adic skew–plane. The range of the transformation is the set of functions $g(\xi)$ of ξ in the adic skew–plane which are square integrable with respect to Haar measure for the adic skew–plane, which vanish at elements of the adic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, and which satisfy the identity

$$g(\omega\xi) = \chi^-(\omega)g(\xi)$$

for every unit ω of the adic skew–plane. The transformation takes a function $f(\xi)$ of ξ in the adic skew–plane into a function $g(\xi)$ of ξ in the adic skew–plane when the identity

$$\int \chi(\xi^-)g(\xi)\sigma(\xi^-\eta\xi)^- d\xi = \operatorname{sgn}(\eta)|\eta|_-^{-2} \int \chi(\xi)^- f(\xi)\sigma(\xi^-\eta^{-1}\xi)d\xi$$

holds for all invertible elements η of the adic line with integration with respect to Haar measure for the adic skew–plane. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the adic skew–plane. The Hankel transformation of order χ^- for the adic skew–plane is the inverse of the Hankel transformation of order χ for the adic skew–plane.

The principal subgroup of the adelic line is the set of elements of the adelic line whose Euclidean and adic components are represented by equal positive rational numbers. An element of the principal subgroup of the adelic line admits a representative $\lambda^*\lambda$ with λ a unimodular element of the adelic diline whose Euclidean component is determined by the requirements of the functional identity. Representatives λ are considered equivalent if they represent the same element of the adelic line.

Renormalized Haar measure for the adelic line is the Cartesian product of Haar measure for the Euclidean line and renormalized Haar measure for the adic line. Renormalized Haar measure for the adelic diline is the Cartesian product of Haar measure for the Euclidean diline and renormalized Haar measure for the adic diline.

The adelic upper half–plane is the set of elements of the adelic plane whose Euclidean component belongs to the upper half–plane and whose adic component is an invertible element of the adic line. An element of the adelic upper half–plane, whose Euclidean component is $\tau_+ + iy$ for a real number τ_+ and a positive number y and whose adic component is τ_-, is written $\tau + iy$ with τ the element of the adelic line whose Euclidean component is τ_+ and whose adic component is τ_-.

A nonnegative integer ν of the same parity as χ is associated with a character χ for the adic plane for the definition of the Laplace transformation of order ν and character

χ for the adelic plane. If ω is a unimodular element of the adelic plane, an isometric transformation in the space of square integrable functions with respect to Haar measure for the adelic plane is defined by taking a function $f(\xi)$ of ξ in the adelic plane into the function $f(\omega\xi)$ of ξ in the adelic plane. A closed subspace of the space of square integrable functions with respect to Haar measure for the adelic plane consists of the functions $f(\xi)$ of ξ in the adelic plane which vanish at elements ξ of the adelic plane whose p–adic component is not a unit for some prime divisor p of ρ and which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic plane. Functions are constructed which satisfy related identities for every unimodular element ω of the adelic plane whose p–adic component is a unit for every prime divisor p of ρ. The noninvertible elements of the adelic plane form a set of zero Haar measure. The set of invertible elements of the adelic plane is a union of disjoint open sets, called fundamental regions, which are invariant under multiplication by elements of the adelic plane whose adic component is a unit. Invertible elements of the adelic plane belong to the same fundamental region if, and only if they have equal adic modulus. A function $f(\xi)$ of ξ in the adelic plane, which vanishes at elements ξ of the adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfies the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic plane, and which satisfies the identity

$$f(\xi) = f(\omega\xi)$$

for every element of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, is said to be locally square integrable if it is square integrable with respect to Haar measure for the adelic plane in a fundamental region which contains an element whose p–adic component is a unit for every prime divisor p of ρ. The integral

$$\int |f(\xi)|^2 d\xi$$

with respect to Haar measure for the adelic plane over such a region is independent of the choice of region. The resulting Hilbert space is the domain of the Laplace transformation of order ν and character χ for the adelic plane.

A positive integer ν of the same parity as χ is associated with a character χ for the adic skew–plane for the definition of the Laplace transformation of order ν and character χ for the adelic skew–plane. If ω is a unimodular element of the adelic skew–plane, an isometric transformation in the space of square integrable functions with respect to Haar measure for the adelic skew–plane is defined by taking a function $f(\xi)$ of ξ in the adelic skew–plane into the function $f(\omega\xi)$ of ξ in the adelic skew–plane. A closed subspace of the space of square integrable functions with respect to Haar measure for the adelic skew–plane consists of the functions $f(\xi)$ of ξ in the adelic skew–plane which vanish at elements of the adelic diplane whose p–adic component is not a unit for some prime divisor p of ρ and which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic skew–plane. Functions are constructed which satisfy related identities for every unimodular element ω of the adelic skew–plane whose p–adic component is a unit for every prime divisor p of ρ. The set of noninvertible elements of the adelic skew–plane is a union of disjoint open subsets, called fundamental regions, which are invariant under multiplication by invertible elements of the adelic skew–plane whose adic component is a unit. Invertible elements of the adelic skew–plane belong to the same fundamental region if, and only if, they have equal adic modulus. A function $f(\xi)$ of ξ in the adelic skew–plane, which vanishes at elements of the adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfies the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic skew–plane, and which satisfies the identity

$$f(\xi) = f(\omega\xi)$$

for every representative ω of an element of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, is said to be locally square integrable if the integral

$$\int |f(\xi)|^2 d\xi$$

with respect to Haar measure for the adelic skew–plane is finite over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The integral is independent of the choice of region. The resulting Hilbert is the domain of the Laplace transformation of order ν and character χ for the adelic skew–plane.

If a function $f(\xi)$ of ξ in the adelic plane is square integrable with respect to Haar measure for the adelic plane, vanishes at elements of the adelic plane whose adic component is not a unit, and satisfies the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic plane, then a function $g(\xi)$ of ξ of the adelic plane, which vanishes at elements of the adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfies the identity

$$g(\omega\xi) = \omega_+^\nu \chi(\omega_-) g(\xi)$$

for every unit ω of the adelic plane, and which satisfies the identity

$$g(\xi) = g(\omega\xi)$$

for every element ω of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, is defined as a sum

$$g(\xi) = \sum f(\omega\xi)$$

over the elements ω of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$\int |g(\xi)|^2 d\xi = \int |f(\xi)|^2 d\xi$$

is satisfied with integration on the left with respect to Haar measure for the adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ and with integration on the right with respect to Haar measure for the adelic plane over the whole plane. If a locally square integrable function $h(\xi)$ of ξ in the adelic plane vanishes at elements of the adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, satisfies the identity

$$h(\omega\xi) = \omega_+^\nu \chi(\omega_-) h(\xi)$$

for every unit ω of the adelic plane, and satisfies the identity

$$h(\xi) = h(\omega\xi)$$

for every element ω of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, then

$$h(\xi) = g(\xi)$$

almost everywhere with respect to Haar measure for the adelic plane for some such choice of function $f(\xi)$ of ξ in the adelic plane. The function $f(\xi)$ of ξ in the adelic plane is chosen so that the identity

$$h(\xi) = f(\xi)$$

holds almost everywhere with respect to Haar measure for the adelic plane on the set of elements ξ of the adelic plane whose adic component is a unit.

If a function $f(\xi)$ of ξ in the adelic skew–plane is square integrable with respect to Haar measure for the adelic skew–plane, vanishes at elements of the adelic skew–plane whose adic component is not a unit, and satisfies the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic skew–plane, then a function $g(\xi)$ of ξ in the adelic skew–plane, which vanishes at elements of the adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfies the identity

$$g(\omega\xi) = \omega_+^\nu \chi(\omega_-) g(\xi)$$

for every unit ω of the adelic skew–plane, and which satisfies the identity

$$g(\xi) = g(\omega\xi)$$

for every distinguished representative ω of an element of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, is defined as a sum

$$g(\xi) = \sum f(\omega\xi)$$

over the distinguished representatives ω of elements of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$\int |g(\xi)|^2 d\xi = \int |f(\xi)|^2 d\xi$$

holds with integration on the left with respect to Haar measure for the adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ and with integration on the right with respect to Haar measure for the adelic skew–plane over the whole skew–plane. If a locally square integrable function $h(\xi)$ of ξ in the adelic skew–plane vanishes at elements of the adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, satisfies the identity

$$h(\omega\xi) = \omega_+^\nu \chi(\omega_-) h(\xi)$$

for every unit ω of the adelic skew–plane, and satisfies the identity

$$h(\xi) = h(\omega\xi)$$

for every distinguished representative ω of an element of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ, then

$$h(\xi) = g(\xi)$$

almost everywhere with respect to Haar measure for the adelic skew–plane for some such function $f(\xi)$ of ξ in the adelic skew–plane. The function $f(\xi)$ of ξ in the adelic skew–plane is chosen so that the identity

$$h(\xi) = f(\xi)$$

holds almost everywhere with respect to Haar measure for the adelic skew–plane on the set of elements ξ of the adelic skew–plane whose adic component is a unit.

The noninvertible elements of the adelic line form a set of zero Haar measure. The set of invertible elements of the adelic line is the union of fundamental subregions. Invertible elements of the adelic line belong to the same subregion if they have the same p–adic modulus for every prime p. Subregions are said to be mated with respect to a prime p if the ratio of the p–adic modulus of the elements of one subregion to the p–adic modulus of the elements of the other subregion is an odd power of p. A fundamental region for the adelic line is a maximal union of subregions such that any two subregions are mated with respect to a prime p.

A Hilbert space is obtained as the tensor product of the range of the Laplace transformation of order ν for the Euclidean plane and the Laplace transformation of character χ for the adic plane. An element of the space is a function $f(\eta)$ of η in the adelic upper half–plane which is analytic in the Euclidean component of η when the adic component of η is held fixed. The function vanishes at elements of the adelic upper half–plane whose p–adic component is not a unit for some prime divisor p of ρ. The identity

$$f(\eta) = f(\omega\eta)$$

holds for every unit ω of the adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$(1-p)f(\eta) = f(\lambda\eta) - pf(\lambda^{-1}\eta)$$

holds whenever the p–adic modulus of p is not an even power of p for some prime p, which is not a divisor of ρ, and λ is an element of the adelic line whose Euclidean component is the unit of the Euclidean line and which satisfies the identity

$$p|\lambda|_- = 1.$$

When ν is zero, a finite least upper bound

$$\sup \int |f(\tau+iy)^2 d\tau$$

is obtained over all positive numbers y. Integration is with respect to renormalized Haar measure for the adelic line. When ν is positive, the integral

$$\int_0^\infty \int |f(\tau+iy)|^2 y^{\nu-1} d\tau dy$$

is finite. An isometric transformation of the space into itself takes a function $f(\eta)$ of η in the adelic upper half–plane into the function

$$(\omega_+^- \omega_+)^\nu f(\omega^- \eta\omega)$$

of η in the adelic upper half–plane for every unimodular element ω of the adelic plane whose p–adic component is a unit for every prime divisor p of ρ. A closed subspace of the Hilbert space consists of products

$$f(\eta_+)\sigma(\eta_-)$$

with $f(\eta_+)$ a function of η_+ in the upper half–plane which is in the range of the Laplace transformation of order ν for the Euclidean plane and with $\sigma(\eta)$ the kernel for the Laplace transformation of character χ for the adic plane. The Hilbert space is the orthogonal sum of closed subspaces obtained as images of the given subspace under the isometric transformations corresponding to elements ω of the principal subgroup of the adelic line whose p-adic component is a unit for every prime divisor p of ρ.

A Hilbert space is constructed from the tensor product of the range of the Laplace transformation of order ν for the Euclidean skew–plane and the range of the Laplace transformation of character χ for the adic skew–plane. An element of the space is a function $f(\eta, \gamma)$ of elements η of elements η of the adelic upper half–plane and invertible elements γ of the adelic line which is analytic in the Euclidean component of η when γ and the adic component of η are held fixed. The function $f(\eta, \gamma)$ vanishes when the p–adic component of η or the p–adic component of γ is not a unit for some prime divisor p of ρ. The identities

$$f(\omega\eta, \gamma) = f(\eta, \gamma) = f(\eta, \omega\gamma)$$

hold for every unit ω of the adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$f(\eta, \gamma) = \lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-) f(\lambda^{-1}\eta, \lambda\gamma)$$

holds for every invertible element λ of the adelic line whose adic modulus is rational and whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$(1-p)f(\eta, \gamma) = f(\lambda\eta, \gamma) - pf(\lambda^{-1}\eta, \gamma)$$

holds when the p–adic component of η is not an integral power of p for some prime p, which is not a divisor of ρ, and when λ is an element of the adelic line whose Euclidean component is the unit of the Euclidean line and which satisfies the identity

$$p|\lambda|_-^2 = 1.$$

The identity

$$(1-p)f(\eta, \gamma) = f(\eta, \lambda\gamma) - pf(\eta, \lambda^{-1}\gamma)$$

holds when the p–adic component of γ is not an integral power of p for some prime p, which is not a divisor of ρ, and when λ is an element of the adelic line whose Euclidean component is the unit of the Euclidean line and which satisfies the identity

$$p|\lambda|_-^2 = 1.$$

The function $f(\eta, \gamma)$ of η in the adelic upper half–plane and γ in the adelic line is determined by its values when the Euclidean component of γ is a unit and the adic modulus of γ is an integer which is relatively prime to ρ and is not divisible by the square of a prime. For fixed γ the function $f(\eta, \gamma)$ of η belongs to the tensor product of the range of the Laplace transformation of order ν for the Euclidean skew–plane and the range of the Laplace transformation of character χ for the adic skew–plane. The scalar self–product of the function $f(\eta, \gamma)$ of η and γ is defined as the sum of the scalar self–products of the functions of η in the tensor product space taken over the positive integers which are relatively prime to ρ and which are not divisible by the square of a prime. An isometric transformation of the resulting Hilbert space onto itself is defined by taking a function $f(\eta, \gamma)$ of elements η of the adelic upper half–plane and invertible elements γ of the adelic line into the function

$$(\omega_+^- \omega_+)^\nu f(\omega^- \eta \omega, \gamma)$$

of elements of the adelic upper half–plane and invertible elements of the adelic line for every unimodular element ω of the adelic skew–plane whose p–adic component is a unit for every prime divisor p of ρ. A closed subspace of the Hilbert space consists of products

$$f(\eta_+ \gamma_+) \sum \lambda_+^{\nu - \frac{1}{2}} |\lambda|_-^{\nu - \frac{1}{2}} \chi(\lambda_-) \sigma(\lambda_-^{-1} \eta_-) \sigma(\lambda_- \gamma_-)$$

with $f(\eta_+)$ a function of η_+ in the range of the Laplace transformation of order ν for the Euclidean skew–plane. Summation is over the equivalence classes of representatives λ of elements of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The Hilbert space is the orthogonal sum of closed subspaces obtained as images of the given subspace under the isometric transformations corresponding to the equivalence classes of representatives ω of elements of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ.

The theta function of order ν and character χ for the adelic plane is a function $\theta(\eta)$ of η in the adelic upper half–plane which is analytic in the Euclidean component of η when the adic component of η is held fixed. The function vanishes at elements of the adelic upper half–plane whose adic component is not a unit for some prime divisor p of ρ. The identity

$$\theta(\eta) = \theta(\omega \eta)$$

holds for every unit ω of the adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$\theta(\eta) = \omega_+^\nu \chi(\omega_-)^- \theta(\omega^2 \eta)$$

holds for every element ω of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The function is defined as a sum

$$\theta(\eta) = \sum \omega_+^\nu \chi(\omega_-)^- \exp(\pi i \omega_+^2 \eta_+ / \rho) \sigma(\omega_-^2 \eta_-)$$

over the elements ω of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. A coefficient $\tau(n)$ is defined for every positive integer n, which is relatively prime to ρ, so that the identity

$$\prod (1 - p^{-1})^{-1} \tau(n) = \chi(\omega_-)^- \sigma(\omega_-^2)$$

holds with ω the unique element of the principal subgroup of the adelic line whose adic component is integral and whose Euclidean component is n. The product is taken over the prime divisors p of ρ. If η_- is a unit, the identity

$$\theta(\eta) = \sum n^\nu \tau(n) \exp(\pi i n^2 \eta_+ / \rho)$$

holds with summation over the positive integers n which are relatively prime to ρ. The identity

$$\tau(m)\tau(n) = \tau(mn)$$

holds for all positive integers m and n which are not divisors of ρ.

The theta function of order ν and character χ for the adelic skew–plane is a function $\theta(\eta, \gamma)$ of elements η of the adelic upper half–plane and invertible elements γ of the adelic line which is an analytic function of the Euclidean component of η when γ and the adic component of η are held fixed. The function vanishes when the p–adic component of γ or the p–adic component of η is not a unit for some prime divisor p of ρ. The identities

$$\theta(\omega\eta, \gamma) = \theta(\eta, \gamma) = \theta(\eta, \omega\gamma)$$

hold for every unit ω of the adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$\theta(\eta, \gamma) = (\omega_+^\nu)^- \chi(\omega_-)^- \theta(\omega^- \eta\omega, \gamma)$$

holds for every representative ω of an element of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The theta function is a sum

$$\theta(\eta, \gamma) = \sum (\omega_+^{2\nu-1})^- \chi(\omega_-)^- \exp(2\pi i \omega_+^- \eta_+ \gamma_+ \omega_+ / \rho)$$
$$\times \sum \lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-) \sigma(\omega_-^- \lambda_-^{-1} \eta_- \omega_-) \sigma(\lambda_- \gamma_-)$$

over the elements λ of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ and over the equivalence classes of representatives ω of elements of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. A coefficient $\tau(n)$ is defined for every positive integer n, which is relatively prime to ρ, so that the identity

$$\prod (1 - p^{-1})^{-2} \tau(n) = (\omega_+^{\nu-\frac{1}{2}})^- |\omega|_-^{\nu-\frac{1}{2}} \chi(\omega_-)^- \sum \lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-) \sigma(\omega_-^- \lambda_-^{-1} \omega_-) \sigma(\lambda_-)$$

holds with ω a representative of the element of the principal subgroup of the adelic line whose adic component is integral and whose Euclidean component is n. Summation is over the elements λ of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the prime divisors p of ρ. If γ is a unit of the adelic line and if the adic component of η is a unit of the adic line, the identity

$$\prod (1 - p^{-1})^{-2} \theta(\eta, \gamma) = \sum n^{\nu-\frac{1}{2}} \tau(n) \exp(2\pi i n \eta_+ / \rho)$$

holds with summation over the positive integers n which are relatively prime to ρ. The product is taken over the prime divisors p of ρ. If the Euclidean components of representatives of elements of the principal subgroup of the adelic line are chosen so that the identity

$$\gamma_+ = \alpha_+ \beta_+$$

holds whenever the identity

$$\gamma^*\gamma = \alpha^*\alpha\beta^*\beta$$

is satisfied, then the identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

holds for all positive integers m and n, which are relatively prime to ρ, with summation over the common divisors k of m and n.

The theta function of order ν and character χ for the adelic plane is used to define the Laplace transformation of order ν and character χ for the adelic plane. The domain of the transformation is the space of locally square integrable functions $f(\xi)$ of ξ in the adelic plane which vanish at elements of the adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic plane, and which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every element ω of the principal subgroup of the adelic line. The Laplace transform of order ν and character χ for the adelic plane is the function $g(\eta)$ of η in the adelic upper half–plane defined by the integral

$$\prod (1-p^{-2}) 2\pi g(\eta) = \int (\xi_+^\nu)^- \chi(\omega_-)^- f(\xi) \theta(\xi^- \eta \xi) d\xi$$

with respect to Haar measure for the adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the primes p. The function $g(\eta)$ of η in the adelic upper half–plane is an analytic function of the Euclidean component of η when the adic component of η is held fixed. The function vanishes at elements of the adelic upper half–plane whose p–adic component is not a unit for some prime divisor p of ρ. The identity

$$g(\eta) = g(\omega\eta)$$

holds for every unit ω of the adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$g(\eta) = \omega_+^\nu \chi(\omega_-)^- g(\omega^2\eta)$$

holds for every element ω of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$(1-p)g(\eta) = g(\lambda\eta) - pg(\lambda^{-1}\eta)$$

holds when the p–adic modulus of η is not an even power of p for some prime p, which is not a divisor of ρ, and when λ is an element of the adelic line whose Euclidean component is the unit of the Euclidean line and which satisfies the identity

$$p|\lambda|_- = 1.$$

When ν is zero, the identity

$$(2\pi/\rho)\int |f(\xi)|^2 d\xi = \prod(1-p^{-2}) \sup \int |g(\tau+iy)|^2 d\tau$$

holds with the least upper bound taken over all positive numbers y. Integration on the left is with respect to Haar measure for the adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. Integration on the right is with respect to renormalized Haar measure for the adelic line over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the primes p. When ν is positive, the identity

$$(2\pi/\rho)^\nu \Gamma(\nu)\int |f(\xi)|^2 d\xi = \prod(1-p^{-2}) \int_0^\infty \int |g(\tau+iy)|^2 y^{\nu-1} d\tau dy$$

is satisfied.

The theta function of order ν and character χ for the adelic skew–plane is used to define the Laplace transformation of order ν and character χ for the adelic skew–plane. The domain of the transformation is the set of locally square integrable functions $f(\xi)$ of ξ in the adelic skew–plane which vanish at elements of the adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic plane, and which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every representative ω of an element of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The Laplace transform of order ν and character χ for the adelic skew–plane is the function $g(\eta,\gamma)$ of elements η of the adelic upper half–plane and invertible elements γ of the adic line defined by the integral

$$\prod(1-p^{-2})2\pi g(\eta,\gamma) = \int (\xi_+^{2\nu+1})^- \chi(\xi_-)^- f(\xi)\theta(\xi^-\eta\xi,\gamma)d\xi$$

with respect to Haar measure for the adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the primes p. The function $g(\eta,\gamma)$ of elements η of the adelic upper half–plane and invertible elements γ of the adelic line is an analytic function of the Euclidean component of η when γ and the adic component of η are held fixed. The function

$g(\eta, \gamma)$ vanishes when the p–adic component of η or the p–adic component of γ is not a unit for some prime divisor p of ρ. The identities

$$g(\omega\eta, \gamma) = g(\eta, \gamma) = g(\eta, \omega\gamma)$$

hold for every unit ω of the adelic line whose Euclidean component is the unit of the Euclidean line. The identity

$$g(\eta, \gamma) = (\omega_+^\nu)^- \chi(\omega_-)^- g(\omega^- \eta\omega, \gamma)$$

holds for every representative ω of an element of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$g(\eta, \gamma) = \lambda_+^{2\nu-1} |\lambda|_-^{2\nu-1} \chi(\lambda_-) g(\lambda^{-1}\eta, \lambda\gamma)$$

holds for every invertible element λ of the adelic line whose adic modulus is rational and whose adic component is an element of the principal subgroup of the adic line whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$(1 - p) g(\eta, \gamma) = g(\lambda\eta, \gamma) - pg(\lambda^{-1}\eta, \gamma)$$

holds when the p–adic modulus of η is not an integral power of p for some prime p, which is not a divisor of ρ, and when λ is an element of the adelic line whose Euclidean component is the unit of the Euclidean line and which satisfies the identity

$$p|\lambda|_-^2 = 1.$$

The identity

$$(1 - p) g(\eta, \gamma) = g(\eta, \lambda\gamma) - pg(\eta, \lambda^{-1}\gamma)$$

holds when the p–adic modulus of γ is not an integral power of p for some prime p, which is not a divisor of ρ, and λ is an element of the adelic line whose Euclidean component is the unit of the Euclidean line and which satisfies the identity

$$p|\lambda|_-^2 = 1.$$

The identity

$$(1 - p) g(\eta, \gamma) = g(\eta, \lambda\gamma) - pg(\eta, \lambda^{-1}\gamma)$$

holds when the p–adic modulus of γ is not an integral power of p for some prime p, which is not a divisor of ρ, and when λ is an element of the adelic line whose Euclidean component is the unit of the Euclidean line and which satisfies the identity

$$p|\lambda|_-^2 = 1.$$

The theta function of order ν and character χ^- for the adelic plane is computable from the theta function $\theta(\eta)$ of order ν and character χ for the adelic plane as the function

$$\theta(-\eta^-)^-$$

of η in the adelic upper half–plane. The domain of the Hankel transformation of order ν and character χ for the adelic plane is the set of locally square integrable functions $f(\xi)$ of ξ in the adelic plane which vanish at elements of the adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic plane, and which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every element ω of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The range of the Hankel transformation of order ν and character χ for the adelic plane is the set of locally square integrable functions $g(\xi)$ of ξ in the adelic plane which vanish at elements of the adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$g(\omega\xi) = \omega_+^\nu \chi^-(\omega_-) g(\xi)$$

for every unit ω of the adelic plane, and which satisfy the identity

$$g(\xi) = g(\omega\xi)$$

for every element ω of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The transformation takes a function $f(\xi)$ of ξ in the adelic plane into a function $g(\xi)$ of ξ in the adelic plane when the identity

$$\begin{aligned}(i/\eta_+)^{1+\nu}\,\mathrm{sgn}(\eta_-)|\eta|_-^{-1} &\int (\xi_+^\nu)^- \chi(\xi_-)^- f(\xi)\theta(-\xi^-\eta^{-1}\xi)d\xi \\ &= \int (\xi_+^\nu)^- \chi(\xi_-^-) g(\xi)\theta(-\xi^-\eta^-\xi)^- d\xi\end{aligned}$$

holds when η is in the adelic upper half–plane. Integration is with respect to Haar measure for the adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the adelic plane over such a region. The inverse of the Hankel transformation of order ν and character χ for the adelic plane is the Hankel transformation of order ν and character χ^- for the adelic plane.

The theta function of order ν and character χ^- for the adelic skew–plane is computable from the theta function $\theta(\eta, \gamma)$ of order ν and character χ for the adelic skew–plane as the function

$$\theta(-\eta^-, \gamma)^-$$

of elements η of the adelic upper half–plane and invertible elements γ of the adelic line. The domain of the Hankel transformation of order ν and character χ for the adelic skew–plane is the set of locally square integrable functions $f(\xi)$ of ξ in the adelic skew–plane which vanish at elements of the adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic skew–plane, and which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every representative ω of an element of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The range of the Hankel transformation of order ν and character χ for the adelic skew–plane is the set of locally square integrable functions $g(\xi)$ of ξ in the adelic skew–plane which vanish at elements of the adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$g(\omega\xi) = \omega_+^\nu \chi^-(\omega_-) g(\xi)$$

for every unit ω of the adelic skew–plane, and which satisfy the identity

$$g(\xi) = g(\omega\xi)$$

for every representative ω of an element of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The transformation takes a function $f(\xi)$ of ξ in the adelic skew–plane into a function $g(\xi)$ of ξ in the adelic skew–plane when the identity

$$(i/\eta_+)^{1+2\nu} \operatorname{sgn}(\eta_-\gamma_-) |\eta\gamma|_-^{-2} \int (\xi_+^{2\nu+1})^- \chi(\xi_-)^- f(\xi) \theta(-\xi^-\eta^{-1}\xi, \gamma^{-1}) d\xi$$
$$= \int (\xi_+^{2\nu+1})^- \chi(\xi_-^-) g(\xi) \theta(-\xi^-\eta^-\xi, \gamma)^- d\xi$$

holds when η is in the adelic upper half–plane and γ is an invertible element of the adic line. Integration is with respect to Haar measure for the adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The identity

$$\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi$$

holds with integration with respect to Haar measure for the adelic skew–plane over such a region. The inverse of the Hankel transformation of order ν and character χ for the adelic skew–plane is the Hankel transformation of order ν and character χ^- for the adelic skew–plane.

The Mellin transformation of order ν and character χ for the adelic plane is a spectral theory for the Laplace transformation of order ν and character χ for the adelic plane. The domain of the Laplace transformation of order ν and character χ for the adelic plane is the set of locally square integrable functions $f(\xi)$ of ξ in the adelic plane which vanish at elements of the adelic plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic plane, and which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every element ω of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The Laplace transform of order ν and character χ for the adelic plane is the function $g(\eta)$ of η in the adelic upper half–plane defined by the integral

$$\prod(1-p^{-2})2\pi g(\eta) = \int (\xi_+^\nu)^- \chi(\xi_-)^- f(\xi)\theta(\xi^-\eta\xi)d\xi$$

with respect to Haar measure for the adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The Mellin transform of order ν and character χ for the adelic plane is an analytic function $F(z)$ of z in the upper half–plane defined by the integral

$$\prod(1-p^{-1})^{-1} F(z) = \int_0^\infty g(\eta) t^{\frac{1}{2}\nu - \frac{1}{2} - \frac{1}{2}iz} dt$$

under the constraint

$$\eta_+ = it$$

when η_- is a unit of the adic line if the function $f(\xi)$ of ξ in the adelic plane vanishes in a neighborhood $|\xi| < a$ of the origin. The product is taken over the prime divisors p of ρ. The integral can also be evaluated under the same constraint when η_- is an element of the adic line whose adic modulus is integral, is not divisible by the square of a prime, and is relatively prime to ρ. When this change is made in the right side of the identity, the left side is multiplied by the product

$$\prod \frac{1-\chi(\lambda_-)^-\lambda_+^{iz}}{1-\lambda_+}$$

taken over the elements λ of the principal subgroup of the adelic line whose adic component is integral and whose Euclidean component is a prime divisor of $|\eta|_-$. A computation of the integral is made from the zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

of order ν and character χ for the adelic plane, which is defined in the half–plane $\mathcal{R}s > 1$ as a sum over the positive integers n which are relatively prime to ρ. The identity

$$\tau(n) = \chi(p)^{-k}$$

holds when a positive integer

$$n = p^k$$

is a power of a prime p which is not a divisor of ρ. The zeta function is represented in the half–plane by the Euler product

$$\zeta(s)^{-1} = \prod (1 - \tau(p) p^{-s})$$

taken over the primes p which are not divisors of ρ. The analytic weight function

$$W(z) = (\rho/\pi)^{\frac{1}{2}\nu + \frac{1}{2} - \frac{1}{2}iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}iz) \zeta(1 - iz)$$

is applied in the characterization of Mellin transforms of order ν and character χ for the adelic plane. The weight function is represented in the upper half–plane by the integral

$$\prod (1 - p^{-1})^{-1} W(z) = \int_0^\infty \theta(\eta) t^{\frac{1}{2}\nu - \frac{1}{2} - \frac{1}{2}iz} dt$$

under the constraint

$$\eta_+ = it$$

when η_- is a unit of the adic line. The product is taken over the prime divisors p of ρ. The identity

$$\prod (1 - p^{-2}) 2\pi F(z)/W(z) = \int (\xi_+^\nu)^- \chi(\xi_-)^- f(\xi) |\xi|^{iz - \nu - 1} d\xi$$

holds when z is in the upper half–plane. Integration is with respect to Haar measure for the adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the primes p. The function

$$a^{-iz} F(z)$$

of z in the upper half–plane is characterized as an element of the weighted Hardy space $\mathcal{F}(W)$ which satisfies the identity

$$\prod (1 - p^{-2}) \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \int |f(\xi)|^2 d\xi.$$

Integration on the right is with respect to Haar measure for the adelic plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the primes p. The identity

$$\prod (1 - p^{-2} \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \sum \alpha^{-1} \int_{-\infty}^{+\infty} \left| \prod \frac{1 - \chi(\lambda_-)^- \lambda_+^{it}}{1 - \lambda_+^{-1}} \, \frac{\zeta(1 - it) F(t)}{W(t)} \right|^2 dt$$

holds with summation over the positive integers α which are relatively prime to ρ and which are not divisible by the square of a prime, with the product on the left taken over the primes p which are not divisors of ρ, and with the product on the right taken over the elements λ of the principal subgroup of the adelic line whose adic component is integral and whose Euclidean component is a prime divisor of α. If the Hankel transform of order ν and character χ for the adelic plane vanishes when $|\xi| < a$, then the Mellin transform of order ν and character χ^- for the adelic plane is an entire function which is the analytic extension of the function

$$F(-z)$$

to the complex plane.

The Mellin transformation of order ν and character χ for the adelic skew–plane is a spectral theory for the Laplace transformation of order ν and character χ for the adelic skew–plane. The domain of the Laplace transformation of order ν and character χ for the adelic skew–plane is the set of locally square integrable functions $f(\xi)$ of ξ in the adelic skew–plane which vanish at elements ξ of the adelic skew–plane whose p–adic component is not a unit for some prime divisor p of ρ, which satisfy the identity

$$f(\omega\xi) = \omega_+^\nu \chi(\omega_-) f(\xi)$$

for every unit ω of the adelic skew–plane, and which satisfy the identity

$$f(\xi) = f(\omega\xi)$$

for every representative ω of an element of the principal subgroup of the adelic line whose p–adic component is a unit for every prime divisor p of ρ. The Laplace transform of order ν and character χ for the adelic skew–plane is the function $g(\eta, \gamma)$ of elements η of the adelic upper half–plane and invertible elements γ of the adelic line defined by the integral

$$\prod (1 - p^{-2}) 2\pi g(\eta, \gamma) = \int (\xi_+^{2\nu-1})^- \chi(\xi_-)^- f(\xi) \theta(\xi^- \eta \xi, \gamma) d\xi$$

with respect to Haar measure for the adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the primes p. The Mellin transform of order ν and character χ for the adelic skew–plane is an analytic function $F(z)$ of z in the upper half–plane defined by the integral

$$\prod (1 - p^{-1})^{-2} F(z) = \int_0^\infty g(\eta, \gamma) t^{\nu - \frac{1}{2} - iz} dt$$

under the constraint

$$\eta_+ = it$$

when γ is a unit of the adelic line and η_- is a unit of the adic line if the function $f(\xi)$ of ξ in the adelic skew–plane vanishes in a neighborhood $|\xi| < a$ of the origin. The product is taken over the prime divisors p of ρ. The integral can be evaluated under the same constraint when γ and η_- are elements of the adic line whose adic modulus is integral, is not divisible by the square of a prime, and is relatively prime to ρ. When this change is made in the right side of the identity, the left side is multiplied by the product

$$\prod \frac{1 - (\lambda_+^{\nu-\frac{1}{2}})^- |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-)^- \lambda_+^{2iz}}{1 - \lambda_+^2}$$

taken over the equivalence classes of representatives λ of elements of the principal subgroup of the adelic line whose adic component is integral and whose Euclidean component is a prime divisor of $|\eta|_-^2$, and also by the product

$$\prod \frac{1 - \lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-) \lambda_+^{2iz}}{1 - \lambda_+^2}$$

taken over the equivalence classes of representatives λ of elements of the principal subgroup of the adelic line whose adic component is integral and whose Euclidean component is a prime divisor of $|\lambda|_-^2$. A computation of the integral is made using the zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

of order ν and character χ for the adelic skew–plane, which is defined in the half–plane $\mathcal{R} > 1$ as a sum over the positive integers n which are relatively prime to ρ. If a positive integer

$$n = p^k$$

is a power of a prime p which is not a divisor of ρ, then the identity

$$\tau(n) = \frac{[\lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-)]^{1+k} - [\lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-)]^{-1-k}}{[\lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-)] - [\lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-)]^{-1}}$$

holds with λ a representative of an element of the principal subgroup of the adelic line whose adic component is integral and whose Euclidean component is p. The fraction is defined by continuity when the denominator is equal to zero. The function is represented in the half–plane by the Euler product

$$\zeta(s)^{-1} = \prod (1 - \tau(p) p^{-s} + p^{-2s})$$

taken over the primes p which are not divisors of ρ. The analytic weight function

$$W(z) = (2\pi/\rho)^{-\frac{1}{2}\nu - \frac{1}{2} + iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz) \zeta(1 - iz)$$

is applied in the characterization of Mellin transforms of order ν and character χ for the adelic skew–plane. The weight function is represented in the upper half–plane by the integral

$$\prod (1 - p^{-1})^{-2} W(z) = \int_0^\infty \theta(\eta, \gamma) t^{\nu - \frac{1}{2} - iz} dt$$

under the constraint

$$\eta_+ = it$$

when γ is a unit of the adelic diline and η_- is a unit of the adic diline. The product is taken over the prime divisors p of ρ. The identity

$$\prod (1-p^{-2}) 2\pi F(z)/W(z) = \int (\xi_+^\nu)^- \chi(\xi_-)^- f(\xi) |\xi|^{2iz-2\nu-3} d\xi$$

holds when z is in the upper half–plane. Integration is with respect to Haar measure for the adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the primes p. The function

$$a^{-iz} F(z)$$

of z in the upper half–plane is characterized as an element of the weighted Hardy space $\mathcal{F}(W)$ which satisfies the identity

$$\prod (1-p^{-2}) \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \int |f(\xi)|^2 d\xi.$$

Integration on the right is with respect to Haar measure for the adelic skew–plane over a fundamental region containing an element whose p–adic component is a unit for every prime divisor p of ρ. The product is taken over the primes p. The identity

$$\prod (1-p^{-1})^4 \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \sum \alpha^{-1}\beta^{-1} \int \Bigg| \prod \frac{1-\lambda_+^{\nu-\frac{1}{2}} |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-) \lambda_+^{2it}}{1-\lambda_+^{-2}}$$
$$\times \prod \frac{1-(\lambda_+^{\nu-\frac{1}{2}})^- |\lambda|_-^{\nu-\frac{1}{2}} \chi(\lambda_-)^- \lambda_+^{2it}}{1-\lambda_+^{-2}} \frac{\zeta(1-it)F(t)}{W(t)} \Bigg|^2 dt$$

holds with summation over the positive integers α and β, which are relatively prime to ρ and are not divisible by the square of a prime, and with the product on the left taken over the primes p which are not divisors of ρ. The first product on the right is taken over the equivalence classes of representatives λ of elements of the principal subgroup of the adelic line whose adic component is integral and whose Euclidean component is a prime divisor of α. The second product on the right is taken over the equivalence classes of representatives λ of elements of the principal subgroup of the adelic line whose adic component is integral and whose Euclidean component is a prime divisor of β. If the Hankel transform of order ν and character χ for the adelic skew–plane vanishes when $|\xi| < a$, then its Mellin transform of order ν and character χ^- for the adelic skew–plane is an entire function which is the analytic extension of the function

$$F(-z)$$

to the complex plane.

The functional identity for the zeta function of order ν and character χ for the adelic plane is applied for the construction of the Sonine spaces of order ν and character χ for the adelic plane. When χ is not the principal character, the functional identity states that the entire functions

$$(\pi/\rho)^{-\frac{1}{2}\nu-\frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s)\zeta(s)$$

and

$$(\pi/\rho)^{-\frac{1}{2}\nu-\frac{1}{2}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}-\tfrac{1}{2}s)\zeta(1-s^-)^-$$

of s are linearly dependent. The analytic weight function

$$W(z)=(\pi/\rho)^{-\frac{1}{2}\nu-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}-\tfrac{1}{2}iz)\zeta(1-iz)$$

constructed from the zeta function of order ν and character χ for the adelic plane is used to define the Sonine spaces of order ν and character χ for the adelic plane. The weight function is an entire function of Pólya class, which is determined by its zeros, such that $W(z-i)$ and $W^*(z)$ are linearly dependent. The Sonine space of parameter a of order ν and character χ for the adelic plane contains the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is obtained when the space is considered with the scalar product such that multiplication by a^{-iz} is an isometric transformation of the space into the space $\mathcal{F}(W)$. The space is isometrically equal to a space $\mathcal{H}(E)$ with

$$E(z)=a^{iz}W(z)$$

when a is less than or equal to one. The space is a space $\mathcal{H}(E)$ for every positive number a by the existence theorem for Hilbert spaces of entire functions which are contained isometrically in a given space $\mathcal{H}(E)$.

The functional identity for the zeta function of zero order and principal character for the adelic plane is applied for the construction of the augmented Sonine spaces of zero order and principal character for the adelic plane. The functional identity states that the entire functions

$$s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)$$

and

$$s(s-1)\pi^{-\frac{1}{2}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\zeta(1-s)$$

of s are equal. The analytic weight function

$$W(z)=iz(iz-1)\pi^{-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}iz)\zeta(1-iz)$$

constructed from the zeta function of zero order and principal character for the adelic plane is used to define the augmented Sonine spaces of zero order and principal character for the adelic plane. The weight function is an entire function of Pólya class, which is determined by its zeros, such that $W(z-i)$ and $W^*(z)$ are equal. The augmented Sonine space of parameter a of zero order and principal character for the adelic plane contains the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$. A Hilbert space of entire functions which satisfied the axioms (H1), (H2), and (H3) is obtained when the space is considered with the scalar product such that multiplication by a^{-iz} is an isometric transformation of the space into the space $\mathcal{F}(W)$. The space is isometrically equal to a space $\mathcal{H}(E)$ with

$$E(z) = a^{iz}W(z)$$

when a is less than or equal to one. The space is a space $\mathcal{H}(E)$ for every positive number a by the existence theorem for Hilbert spaces of entire functions which are contained isometrically in a given space $\mathcal{H}(E)$.

The functional identity for the zeta function of order ν and character χ for the adelic skew–plane is applied for the construction of the Sonine spaces of order ν and character χ for the adelic skew–plane. The functional identity states that the entire functions

$$(2\pi/\rho)^{-\frac{1}{2}\nu-s}\Gamma(\tfrac{1}{2}\nu+s)\zeta(s)$$

and

$$(2\pi/\rho)^{-\frac{1}{2}\nu-1+s}\Gamma(\tfrac{1}{2}\nu+1-s)\zeta(1-s)$$

are equal. The analytic weight function

$$W(z) = (2\pi/\rho)^{-\frac{1}{2}\nu-1+iz}\Gamma(\tfrac{1}{2}\nu+1-iz)\zeta(1-iz)$$

constructed from the zeta function of order ν and character χ for the adelic skew–plane is used to define the Sonine spaces of order ν and character χ for the adelic skew–plane. The weight function is an entire function of Pólya class, which is determined by its zeros, such that $W(z-i)$ and $W^*(z)$ are equal. The Sonine space of parameter a of order ν and character χ for the adelic skew–plane contains the entire functions $F(z)$ such that

$$a^{-iz}F(z)$$

and

$$a^{-iz}F^*(z)$$

belong to the weighted Hardy space $\mathcal{F}(W)$. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is obtained when the space is considered with the scalar product such that multiplication by a^{-iz} is an isometric transformation of the space into the space $\mathcal{F}(W)$. The space is isometrically equal to a space $\mathcal{H}(E)$ with

$$E(z) = a^{iz}W(z)$$

when a is less than or equal to one. The space is a space $\mathcal{H}(E)$ for every positive number a by the existence theorem for Hilbert spaces of entire functions contained isometrically in a space $\mathcal{H}(E)$.

The Sonine space of parameter a of order ν and character χ for the adelic plane is contained contractively in the Sonine space of parameter a of order ν for the Euclidean plane when χ is a nonprincipal character. The Sonine space of parameter a of order ν and character χ for the r–adelic plane coincides with the Sonine space of parameter a of order ν for the Euclidean plane. The Sonine space of parameter a of order ν and character χ for the r–adelic plane is contained contractively in the Sonine space of parameter a of order ν and character χ for the r'–adelic plane when r is divisible by r'. An element of the Sonine space of parameter a of order ν for the Euclidean plane belongs to the Sonine space of parameter a of order ν and character χ for the adelic plane if, and only if, its norm as an element of the Sonine space of parameter a of order ν and character χ for the r–adelic plane is a bounded function of r. The norm of an element of the Sonine space of parameter a of order ν and character χ for the adelic plane is the least upper bound of its norms in the Sonine spaces of parameter a of order ν and character χ for the r–adelic planes. The reproducing kernel function for function values at w in the Sonine space of parameter a of order ν and character χ for the adelic plane is the limit uniformly on compact subsets of the complex plane of the reproducing kernel functions for function values at w in the Sonine spaces of parameter a or order ν and character χ for the r–adelic planes for every complex number w.

A proof of the Riemann hypothesis for the zeta function of order ν and character χ for the adelic plane results from the construction of a maximal dissipative transformation in a Sonine space of order ν and character χ for the adelic plane when χ is not the principal character.

Theorem 5. *If the entire function*

$$E(a,z) = a^{iz}(\pi/\rho)^{-\frac{1}{2}\nu-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}-\tfrac{1}{2}iz)\zeta(1-iz)$$

is defined using the zeta function of order ν and character χ for the adelic plane when $0 < a \le 1$ and χ is a primitive character modulo ρ, ρ not one, then a maximal dissipative transformation in the space $\mathcal{H}(E(a))$ is defined, using a function $\kappa(\lambda)$ of zeros λ of $E(a,z)$, by taking $F(z)$ into $G(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space such that the identity

$$G(\lambda) = \kappa(\lambda)F(\lambda)$$

holds for every zero λ of $E(a,z)$. Every zero λ of $E(a,z)$ is simple and lies on the line

$$\lambda^- - \lambda = i.$$

The Sonine space $\mathcal{H}(E(a))$ of parameter a of order ν and character χ for the adelic plane is approximated by spaces $\mathcal{H}(E_n(a))$ constructed from the Sonine spaces of order ν for the Euclidean plane. The spaces $\mathcal{H}(E_n(a))$ are constructed using polynomials $S_n(z)$, such that $S_n(z-i)$ has no zeros in the upper half–plane, from the Sonine spaces of order ν for the Euclidean plane. The space $\mathcal{H}(E_n(a))$ contains the entire functions $F(z)$ such that $S_n(z)F(z)$ belongs to the Sonine space of parameter a of order ν for the Euclidean plane. Multiplication by $S_n(z)$ is an isometric transformation of the space $\mathcal{H}(E_n(a))$ into the Sonine space of parameter a of order ν for the Euclidean plane.

The approximation is made in the weighted Hardy space $\mathcal{F}(W)$ associated with the analytic weight function

$$W(z) = (r/\pi)^{-\frac{1}{2}\nu-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}-\tfrac{1}{2}iz)$$

for the Sonine spaces of order ν for the Euclidean plane. The polynomials $S_n(z)$ and corresponding numbers τ_n, $0<\tau_n\leq 1$, are chosen so that the function

$$\frac{E(a,z)E(a,w)^- - E^*(a,z)E(a,w^-)}{2\pi i(w^- - z)\zeta(1-iz)}$$

of z is the limit in the metric topology of the space $\mathcal{F}(W)$ of the functions

$$\tau_n^{-iz}S_n(z)\ \frac{E_n(a\tau_n,z)E_n(a\tau_n,w)^- - E_n^*(a\tau_n,z)E_n(a\tau_n,w^-)}{2\pi i(w^- - z)}$$

of z for every complex number w when $0<a\leq 1$ and so that the function

$$\frac{E(a,z)E(a,w)^- - E^*(a,z)E(a,w^-)}{2\pi i(w^- - z)}$$

of z is the limit of the functions

$$\frac{E_n(a\tau_n,z)E_n(a\tau_n,w)^- - E_n^*(a\tau_n,z)E_n(a\tau_n,w^-)}{2\pi i(w^- - z)}$$

of z uniformly on compact subsets of the complex plane for every complex number w. The function

$$\zeta(1-iz)^{-1}$$

of z is the limit of the functions

$$\tau_n^{-iz}S_n(z)$$

of z uniformly on compact subsets of the upper half–plane.

A maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space. A maximal dissipative transformation in the Sonine space of parameter a of order ν for the Euclidean plane is defined by taking $F(z)$ into $aF(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space. A maximal dissipative transformation in the space $\mathcal{H}(E_n(a))$ is defined by taking $F(z)$ into $aG(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space such that the identity

$$H(z+i) = S_n(z)G(z+i)$$

holds for an element $H(z)$ of the Sonine space of parameter a of order ν for the Euclidean plane whose orthogonal projection into the image of the space $\mathcal{H}(E_n(a))$ is $S_n(z)F(z)$.

A maximal dissipative relation in the space $\mathcal{H}(E(a))$ is defined as a limit of the maximal dissipative transformations in the spaces $\mathcal{H}(E_n(a\tau_n))$. The relation takes $F(z)$ into $aG(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space which are obtained as limits

$$F(z)/\zeta(1-iz) = \lim \tau_n^{-iz} S_n(z) F_n(z)$$

and

$$G(z+i)/\zeta(1-iz) = \lim \tau_n^{-iz} S_n(z) G_n(z+i)$$

in the metric topology of the space $\mathcal{F}(W)$ from elements $F_n(z)$ and $G_n(z+i)$ of the space $\mathcal{H}(E_n(a\tau_n))$ such that the maximal dissipative transformation in the space takes $F_n(z)$ into $a\tau_n G_n(z+i)$ for every n. The dissipative property of the relation is immediate since the identity

$$\langle F(t), G(t+i)\rangle_{\mathcal{H}(E(a))} = \lim \langle F_n(t), G_n(t+i)\rangle_{\mathcal{H}(E_n(a\tau_n))}$$

is satisfied with

$$\langle F_n(t), G_n(t+i)\rangle_{\mathcal{H}(E_n(a))} + \langle G_n(t+i), F_n(t)\rangle_{\mathcal{H}(E_n(a\tau_n))}$$

nonnegative for every n.

The maximal dissipative property of the relation in the space $\mathcal{H}(E(a))$ is proved by showing that every element of the space is of the form

$$F(z) + G(z+i)$$

with $F(z)$ and $G(z+i)$ elements of the space such that the relation takes $F(z)$ into $aG(z+i)$. An element $H(z)$ of the space is obtained as a limit

$$H(z)/\zeta(1-iz) = \lim \tau_n^{-iz} S_n(z) H_n(z)$$

in the metric topology of the space $\mathcal{F}(W)$ from elements $H_n(z)$ of the space $\mathcal{H}(E_n(a\tau_n))$ for every n. Since the transformation in the space $\mathcal{H}(E_n(a\tau_n))$ is maximal dissipative, the identity

$$H_n(z) = F_n(z) + G_n(z+i)$$

holds with $F_n(z)$ and $G_n(z+i)$ elements of the space such that the transformation takes $F_n(z)$ into $a\tau_n G_n(z+i)$. Since the norm of

$$\tau_n^{-iz} S_n(z)[F_n(z) - G_n(z+i)]$$

in the space $\mathcal{F}(W)$ is less than or equal to the norm of

$$\tau_n^{-iz} S_n(z)[F_n(z) + G_n(z+i)]$$

in the space, elements $F(z)$ and $G(z+i)$ of the space $\mathcal{H}(E(a))$ are obtained as limits

$$F(z)/\zeta(1-iz) = \lim \tau_n^{-iz} S_n(z) F_n(z)$$

and

$$G(z+i)/\zeta(1-iz) = \lim \tau_n^{-iz} S_n(z) G_n(z+i)$$

in the metric topology of the space $\mathcal{F}(W)$. The identity

$$H(z) = F(z) + G(z+i)$$

holds with elements $F(z)$ and $G(z+i)$ of the space $\mathcal{H}(E(a))$ such that the relation in the space takes $F(z)$ into $aG(z+i)$. This completes the verification that the relation is maximal dissipative.

A maximal dissipative relation is constructed in a related Hilbert spaces of entire functions for every positive integer r, which is divisible by ρ, such that r/ρ is relatively prime to ρ and is not divisible by the square of a prime. The space $\mathcal{H}(E'(a))$ is defined when the parameter

$$a \leq (\rho/r)^{\frac{1}{2}}$$

is sufficiently small. The function $E'(a,z)$ is defined as the product of

$$E(a(r/\rho)^{\frac{1}{2}}, z)$$

and the entire function

$$(r/\rho)^{-\frac{1}{2}iz} \zeta_r(1-iz)^{-1}$$

of Pólya class constructed from the zeta function of character χ for the r–adic plane. An entire function of Pólya class, which is determined by its zeros, is obtained when z is replaced by $z-i$. Multiplication by

$$a^{-iz} \zeta_r(1-iz)/\zeta(1-iz)$$

is an isometric transformation of the space $\mathcal{H}(E'(a))$ into the space $\mathcal{F}(W)$.

A maximal dissipative relation in the space $\mathcal{H}(E'(a))$ is constructed from the maximal dissipative transformations in the Sonine spaces of order ν for the Euclidean plane. The space $\mathcal{H}(E'_n(a))$ is defined using a polynomial divisor $S'_n(z)$ of $S_n(z)$ from the Sonine space of parameter a of order ν for the Euclidean plane. The space $\mathcal{H}(E'_n(a))$ contains the entire functions $F(z)$ such that $S'_n(z)F(z)$ belongs to the Sonine space of parameter a of order ν for the Euclidean plane. Multiplication by $S'_n(z)$ is an isometric transformation of the space $\mathcal{H}(E'_n(a))$ into the Sonine space of parameter a of order ν for the Euclidean plane.

The polynomials $S'_n(z)$ and corresponding positive numbers τ'_n are chosen so that the spaces $\mathcal{H}(E'(a\tau'_n))$ converge to the space $\mathcal{H}(E'(a))$. The reproducing kernel function for function values at w in the space $\mathcal{H}(E'_n(a\tau'_n))$ converges to the reproducing kernel function for function values at w in the space $\mathcal{H}(E'(a))$ uniformly on compact subsets of the complex plane for every complex number w. The image in the space $\mathcal{F}(W)$ of the reproducing kernel function for function values at w in the space $\mathcal{H}(E'(a\tau_n'))$ converges in the metric topology of the space $\mathcal{F}(W)$ to the image in the space $\mathcal{F}(W)$ of the reproducing kernel function for function values at w in the space $\mathcal{H}(E'(a))$ for every complex number w. The function

$$\zeta_r(1-iz)/\zeta(1-iz)$$

of z is the limit of the functions

$$(\tau_n')^{-iz}S_n'(z)$$

uniformly on compact subsets of the upper half–plane. The maximal dissipative relation in the space $\mathcal{H}(E'(a))$ takes $F(z)$ into $aG(z+i)$ whenever $F(z)$ and $G(z+i)$ are elements of the space which are obtained as limits

$$F(z)\zeta_r(1-iz)/\zeta(1-iz) = \lim(\tau_n')^{-iz}S_n'(z)F_n(z)$$

and

$$G(z+i)\zeta_r(1-iz)/\zeta(1-iz) = \lim(\tau_n')^{-iz}S_n'(z)G(z+i)$$

in the metric topology of the space $\mathcal{F}(W)$ from elements $F_n(z)$ and $G_n(z+i)$ of the space $\mathcal{H}(E_n'(a\tau_n'))$ such that the maximal dissipative transformation in the space takes $F(z)$ into $a\tau_n'G(z+i)$ for every n.

The approximating polynomials $S_n'(z)$ are chosen so that the quotient polynomials

$$S_n(z)/S_n'(z)$$

converge uniformly on compact subsets of the complex plane to the entire function

$$(r/\rho)^{-\frac{1}{2}iz}\zeta_r(1-iz)^{-1}$$

of Pólya class with the zeros of the quotient polynomials always chosen as zeros of the limit entire function so that

$$(r/\rho)^{-\frac{1}{2}iz}\zeta_r(1-iz)^{-1}S_n'(z)/S_n(z)$$

is an entire function. The numbers τ_n' are chosen to satisfy the inequality

$$\tau_n' \le \tau_n.$$

The identity

$$(r/\rho)^{\frac{1}{2}} = \lim \tau_n/\tau_n'$$

is then satisfied.

This information permits a construction of the maximal dissipative relation in the space $\mathcal{H}(E(a))$ from the maximal dissipative relation in the space $\mathcal{H}(E'(a(\rho/r)^{\frac{1}{2}})$. The maximal dissipative relation in the space $\mathcal{H}(E(a))$ takes an element $F(z)$ of the space into an element

$aG(z+i)$ of the space if, and only if, sequences of element $F_n(z)$ and $G_n(z+i)$ of the space $\mathcal{H}(E'(a(\rho/r)^{\frac{1}{2}})$ exist such that

$$a^{-iz}G(z+i)/\zeta(1-iz)$$

is a limit in the metric topology of the space $\mathcal{F}(W)$ of the elements

$$a^{-iz}(\rho/r)^{-\frac{1}{2}iz}G_n(z+i)\zeta_r(1-iz)/\zeta(1-iz)$$

of the space and such that

$$a^{-iz}F(z)/\zeta(1-iz)$$

is the limit in the metric topology of the space $\mathcal{F}(W)$ of the orthogonal projection of the elements

$$a^{-iz}(\rho/r)^{-\frac{1}{2}iz}F_n(z)\zeta_r(1-iz)/\zeta(1-iz)$$

into the image of the space $\mathcal{H}(E(a))$ in the space $\mathcal{F}(W)$.

The augmented Sonine space of parameter a of zero order and principal character for the adelic plane is contained contractively in the augmented Sonine space of parameter a of zero order for the Euclidean plane. The augmented Sonine space of parameter a of zero order and principal character for the r–adelic plane coincides with the augmented Sonine space of parameter a of zero order for the Euclidean plane. The augmented Sonine space of parameter a of zero order and principal character for the r–adelic plane is contained contractively in the augmented Sonine space of parameter a of zero order and principal character for the r'–adelic plane when r is divisible by r'. An element of the augmented Sonine space of parameter a of zero order for the Euclidean plane belongs to the augmented Sonine space of parameter a of zero order and principal character for the adelic plane if, and only if, its norm as an element of the augmented Sonine space of parameter a of zero order and principal character for the r–adelic plane is a bounded function of r. The norm of an element of the augmented Sonine space of parameter a of zero order and principal character for the adelic plane is the least upper bound of its norms in the augmented Sonine spaces of parameter a of zero order and principal character for the r–adelic planes. The reproducing kernel function for function values at w in the augmented Sonine space of parameter a of zero order and principal character for the adelic plane is the limit uniformly on compact subsets of the complex plane of the reproducing kernel functions for function values at w in the augmented Sonine spaces of parameter a of zero order and principal character for the r–adelic planes for every complex number w.

A proof of the Riemann hypothesis for the zeta function of zero order and principal character for the adelic plane results from the construction of a maximal transformation of dissipative deficiency at most one in an augmented Sonine space of zero order and principal character for the adelic plane.

Theorem 6. *If the entire function*

$$E(a,z) = a^{iz}\pi^{-\frac{1}{2}+\frac{1}{2}iz}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}iz)\zeta(1-iz)$$

is defined using the zeta function of zero order and principal character for the adelic plane when $0 < a \leq 1$, then a maximal transformation of dissipative deficiency at most one in the space $\mathcal{H}(E(a))$ is defined, using a function $\kappa(\lambda)$ of zeros λ of $E(a, z)$, by taking $F(z)$ into $G(z + i)$ whenever $F(z)$ and $G(z + i)$ are elements of the space such that the identity

$$G(\lambda) = \kappa(\lambda)F(\lambda)$$

holds for every zero λ of $E(a, z)$. Every zero λ of $E(a, z)$ is simple and lies on the line

$$\lambda^{-} - \lambda = i.$$

The Sonine space of parameter a of order ν and character χ for the adelic skew–plane is contained contractively in the Sonine space of parameter a of order ν for the Euclidean skew–plane. The Sonine space of parameter a of order ν and character χ for the r–adelic skew–plane coincides with the Sonine space of parameter a of order ν for the Euclidean skew–plane. The Sonine space of parameter a of order ν and character χ for the r–adelic skew–plane is contained contractively in the Sonine space of parameter a of order and character χ for the r'–adelic skew–plane when r is divisible by r'. An element of the Sonine space of parameter a of order ν for the Euclidean skew–plane belongs to the Sonine space of parameter a of order ν and character χ for the adelic skew–plane if, and only if, its norm as an element of the Sonine space of parameter a of order ν and character χ for the r–adelic skew–plane is a bounded function of r. The norm of an element of the Sonine space of parameter a of order ν and character χ for the adelic skew–plane is the least upper bound of its norms in the Sonine spaces of parameter a of order ν and character χ for the r–adelic skew–planes. The reproducing kernel function for function values at w in the Sonine space of parameter a of order ν and character χ for the adelic skew–plane is the limit uniformly on compact subsets of the complex plane of the reproducing kernel functions for function values at w in the Sonine spaces of parameter a of order ν and character χ for the r–adelic skew–planes for all complex numbers w.

A proof of the Riemann hypothesis for the zeta function of order ν and character χ for the adelic skew–plane results from the construction of a maximal dissipative transformation in a Sonine space of order ν and character χ for the adelic skew–plane.

Theorem 7. *If the entire function*

$$E(a, z) = a^{iz}(2\pi/\rho)^{-\frac{1}{2}\nu - 1 + iz}\Gamma(\tfrac{1}{2}\nu + 1 - iz)\zeta(1 - iz)$$

is defined using the zeta function of order ν and character χ for the adelic skew–plane when $0 < a \leq 1$, then a maximal dissipative transformation in the space $\mathcal{H}(E(a))$ is defined, using a function $\kappa(\lambda)$ of zeros λ of $E(a, z)$, by taking $F(z)$ into $G(z + i)$ whenever $F(z)$ and $G(z + i)$ are elements of the space such that the identity

$$G(\lambda) = \kappa(\lambda)F(\lambda)$$

holds for every zero λ *of* $E(a,z)$. *Every zero* λ *of* $E(a,z)$ *is simple and lies on the line*

$$\lambda^- - \lambda = i.$$

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