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# Stress-Energy Pseudotensors and Gravitational Radiation Power



## 1 Introduction

In a curved spacetime there is, in general, no globally conserved energy-momentum. Aside from the case of scalars like electric charge, tensors defined in different tangent spaces cannot be added in nonflat spacetime.

However, if we are willing to drop the requirement that all our equations be tensor equations, then it is possible to define a globally conserved energy-momentum. Unlike a tensor equation, the form of the conservation laws we derive will change depending on the coordinate system. However, that makes them no less valid. Consider, for example, the three-dimensional equation $\underline{\nabla} \cdot \underline{B} = 0$ and its integral form, $\oint d\underline{S} \cdot \underline{B} = 0$. When written in Cartesian coordinates these equations have an entirely different form than when they are written in spherical coordinates. However, they are equally correct in either case.

The approach we will follow is to derive a conserved **pseudotensor**, a two-index object that transforms differently than the components of a tensor. Unlike a tensor, a pseudotensor can vanish at a point in one coordinate system but not in others. The connection coefficients are a good example of this, and the stress-energy pseudotensors we construct will depend explicitly on the connection coefficients in a coordinate basis.

Despite this apparent defect, pseudotensors can be quite useful. In fact, they are the only way to define an integral energy-momentum obeying an exact conservation law. Moreover, in an asymptotically flat spacetime, when tensors *can* be added from different tangent spaces, the integral energy-momentum behaves like a four-vector. Thus, we can use a pseudotensor to derive the power radiated by a localized source of gravitational radiation.

# 2 Canonical Stress-Energy Pseudotensor

The stress-energy tensor is not unique. Given any $T^{\mu\nu}$ such that $\nabla_\mu T^{\mu\nu} = 0$, one may always define other conserved stress-energy tensors by adding the divergence of another object:

$$T^{\mu\nu} \to T^{\mu\nu} + \nabla_\lambda S^{\mu\nu\lambda} \quad \text{where} \quad S^{\mu\nu\lambda} = -S^{\mu\lambda\nu} \ . \tag{1}$$

Clearly the stress-energy tensor need not even be symmetric.

Recall the equation of local stress-energy conservation:

$$\nabla_\mu T^\mu{}_\nu = (-g)^{-1/2} \partial_\mu \left( \sqrt{-g} T^\mu{}_\nu \right) - \Gamma^\lambda_{\nu\mu} T^\mu{}_\lambda = 0 \ . \tag{2}$$

Because of the $\Gamma$ terms, Gauss' theorem does not apply and the integral over a volume does not give a conserved 4-vector.

However, as we will see, it is possible to define a *pseudotensor* $\tau^\mu{}_\nu$ whose conservation law is $\partial_\mu(\sqrt{-g}\tau^\mu{}_\nu) = 0$ instead of $\nabla_\mu \tau^\mu{}_\nu = 0$. The two equations are identical in flat spacetime but the first one can be integrated by Gauss' law while the second one cannot. Moreover, there are many different conserved stress-energy pseudotensors, just as there are many different conserved stress-energy tensors.

This section will show how to construct conserved stress-energy pseudotensors and tensors, illustrating the procedure for scalar fields and for the metric. The key results are given in problem 2 of Problem Set 7.

## 2.1 Stress-energy pseudotensor for a scalar field

We begin with a simple example: a classical scalar field $\phi(x)$ with action

$$S[\phi(x)] = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x \ . \tag{3}$$

The Lagrangian density depends on $\phi$ and its derivatives but is othewise independent of the position. In this example we suppose that $\mathcal{L}$ includes no derivatives higher than first-order, but this can be easily generalized. Note that if $S$ is a scalar, then $\mathcal{L}$ must equal a scalar times the factor $\sqrt{-g}$ which is needed to convert coordinate volume to proper volume. We are not assuming flat spacetime — the treatment here is valid in curved spacetime.

Variation of the action using $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$ yields

$$\begin{aligned} \delta S &= \int \left[ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) \right] d^4x \\ &= \int \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \right\} \delta\phi(x)\, d^4x + \oint_{\text{surf}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, \delta\phi(x)\, d^3\Sigma_\mu \ . \end{aligned} \tag{4}$$

The surface term comes from integration by parts and is the counterpart of the $p\delta q$ endpoint contributions in the variation of the action of a particle. Considering arbitrary field variations $\delta\phi(x)$ that vanish on the boundary, the action principle $\delta S = 0$ gives the Euler-Lagrange equation

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \ . \tag{5}$$

Now, by assumption our Lagrangian density does not depend explicitly on the coordinates: $\partial\mathcal{L}/\partial x^\mu = 0$. This implies the existence of a conserved Hamiltonian density. To see how, recall the case of particle moving in one dimension with trajectory $q(t)$. In this case, time-indepence of the Lagrangian $L(q, \dot{q})$ implies $dH/dt = 0$ where $H = \dot{q}(\partial L/\partial \dot{q}) - L$.

In a field theory $q(t)$ becomes $\phi(x)$ and there are $d = 4$ (for four spacetime dimensions) parameters for the field trajectories instead of just one. Therefore, instead of $dH/dt = 0$, the conservation law will read $\partial_\mu H^\mu = 0$. However, given $d$ parameters, there are $d$ conservation laws not one, so there must be a two-index Hamiltonian density $\mathcal{H}^\mu{}_\nu$ such that $\partial_\mu \mathcal{H}^\mu{}_\nu = 0$. Here, $\nu$ labels the various conserved quantities.

To construct the Hamiltonian function one must first evaluate the canonical momentum. For a single particle, $p = \partial L/\partial \dot{q}$. For a scalar field theory, the field momentum is defined similarly:

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \ . \tag{6}$$

In a simple mechanical system, the Hamiltonian is $H = p\dot{q} - L$. For a field theory the Lagrangian is replaced by the Lagrangian density, the coordinate $q$ is replaced by the field, and the momentum is the canonical momentum as in equation (6).

The **canonical stress-energy pseudotensor** is defined as the Hamiltonian density divided by $\sqrt{-g}$:

$$\tau^\mu{}_\nu \equiv (-g)^{-1/2} \left[(\partial_\nu \phi)\pi^\mu - \delta^\mu{}_\nu \mathcal{L}\right] \ . \tag{7}$$

The reader may easily check that, as a consequence of equation (5) and the chain rule $\partial_\mu f(\phi, \partial_\nu \phi) = (\partial f/\partial\phi)\partial_\mu\phi + [\partial f/\partial(\partial_\nu\phi)]\partial_\mu(\partial_\nu\phi)$, the canonical stress-energy pseudotensor obeys

$$(-g)^{-1/2}\partial_\mu \left(\sqrt{-g}\tau^\mu{}_\nu\right) = 0 \ . \tag{8}$$

## 2.2 Stress-energy pseudotensor for the metric

The results given above are easily generalized to an action that depends on a rank $(0, 2)$ tensor field $g_{\alpha\beta}$ instead of a scalar field. Let us suppose that the Lagrangian density depends only on the field and its first derivatives: $\mathcal{L} = \mathcal{L}(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta})$. (This excludes the Einstein-Hilbert action, which depends also on the second derivatives of the metric. We will return to this point in the next section.) The Euler-Lagrange equations are simply equation (5) with $\phi$ replaced by $g_{\alpha\beta}$.

Because our field has two indices, the canonical momenta have two more indices than before:

$$\pi^{\mu\alpha\beta} = \frac{\partial \mathcal{L}}{\partial_\mu (g_{\alpha\beta})} \ . \tag{9}$$

The stress-energy pseudotensor for the metric, hence for the gravitational field, is therefore

$$\tau^\mu{}_\nu \equiv (-g)^{-1/2} \left[ (\partial_\nu g_{\alpha\beta})\pi^{\mu\alpha\beta} - \delta^\mu{}_\nu \mathcal{L} \right] \ . \tag{10}$$

It obeys equation (8).

### 2.3 Covariant symmetric stress-energy tensor

Given the results presented above, it is far from obvious that there should be a conserved stress-energy *tensor*. How does one obtain a well-defined stress-energy tensor that obeys a covariant local conservation law?

Let us start from the action for the metric, with a Lagrangian density that may depend on $g_{\mu\nu}$ and on any finite number of derivatives. Then, after integration by parts, one may write the variation of the metric as the functional derivative plus surface terms:

$$\delta S[g_{\mu\nu}] = \int \frac{\delta S}{\delta g_{\mu\nu}} \, \delta g_{\mu\nu}(x) \, \sqrt{-g} \, d^4x \quad + \ \text{surface terms} \ . \tag{11}$$

(As long as we are varying only the metric, we are free to use either $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$. Variations of the two are related by $\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$.)

Now we use the fact that the action is a scalar, hence invariant under arbitrary coordinate transformations. We make an infinitesimal coordinate transformation $x^\mu \to x^\mu - \xi^\mu(x)$, which transforms the metric components $g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$. Other tensor fields are also modified by this coordinate transformation: $\psi \to \psi + \mathcal{L}_\xi \psi$ where $\mathcal{L}_\xi$ is the Lie derivative and $\psi$ is any other field (e.g. the electromagnetic potential $A_\mu$). However, we assume that other fields obey their equations of motion so that $\delta S/\delta\psi = 0$ and therefore $\delta\psi = \mathcal{L}_\xi \psi$ makes no change to the action. Thus, we obtain

$$0 = \delta S = \int 2 \frac{\delta S}{\delta g_{\mu\nu}} \, \nabla_\mu \xi_\nu \, \sqrt{-g} \, d^4x = - \int \nabla_\mu \left( 2 \frac{\delta S}{\delta g_{\mu\nu}} \right) \xi_\nu \, \sqrt{-g} \, d^4x \ . \tag{12}$$

Since this must hold for any $\xi_\nu(x)$, we obtain $\nabla_\mu T^{\mu\nu} = 0$ with $T^{\mu\nu} \equiv 2\delta S/\delta g_{\mu\nu} = -2g^{\mu\alpha} g^{\nu\beta} \delta S/\delta g^{\alpha\beta}$.

The same derivation, without the requirement that other fields obey their equations of motion, when applied to the Einstein-Hilbert action implies $\nabla_\mu G^{\mu\nu} = 0$.

Note that the derivation of a conserved stress-energy tensor is quite different from the derivation of a conserved pseudotensor. However, both rely on the fact of coordinate-invariance. In the pseudotensor case this arises in the assuption that $\mathcal{L}$ depends on position only through the fields; $\partial_\mu \mathcal{L} = 0$ at fixed values of the fields and their derivatives. This led to a set of conserved Hamiltonian densities. In the tensor case, coordinate-invariance was used to demand that $\delta S = 0$ under a diffeomorphism $g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$, with the implication that $\nabla_\mu(\delta S/\delta g_{\mu\nu}) = 0$.

One may ask, are the two approaches more directly related? Does one follow from the other?

Because there are many possible stress-energy tensors and pseudotensors, I am not sure whether a general relationship holds between them all, except to say that all stress-energy conservation laws are implied by the field equations. However, in particular cases the relationship is more straightforward. In the usual derivation of the electromagnetic stress-energy tensor, for example, one starts with the canonical approach and obtains a $\tau^\mu{}_\nu$ that is not symmetric (e.g. Jackson 1975, section 12.10). A term is then added like that of equation (1) so as to symmetrize the stress-energy. The result is the same as the covariant stress-energy $-2\delta S/\delta g^{\mu\nu}$. We will consider the case of the gravitational action in the next section.

# 3 Schrödinger Action and Stress-Energy Pseudotensor

The Einstein-Hilbert action is linear in the Ricci scalar, which is linear in the second derivatives of the metric:

$$\begin{aligned} 16\pi G\, S_{\mathrm{EH}}[g^{\mu\nu}] &= \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu} \\ &= \int d^4x \sqrt{-g}\, g^{\mu\nu} \left(\partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\mu \Gamma^\alpha{}_{\alpha\nu} - \gamma_{\mu\nu}\right) , \end{aligned} \tag{13}$$

where

$$\gamma_{\mu\nu} \equiv \Gamma^\alpha{}_{\beta\mu}\Gamma^\beta{}_{\alpha\nu} - \Gamma^\alpha{}_{\mu\nu}\gamma^\beta{}_{\alpha\beta} . \tag{14}$$

For convenience we also define $\gamma \equiv \sqrt{-g}\, g^{\mu\nu}\gamma_{\mu\nu}$. Then, following Dirac (1975), we rewrite the integrand of the Einstein-Hilbert action:

$$\begin{aligned} \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu} &= \partial_\alpha \left[\sqrt{-g}\, \left(g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} - g^{\alpha\nu}\Gamma^\mu{}_{\mu\nu}\right)\right] + \Gamma^\alpha{}_{\alpha\nu}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}) - \Gamma^\alpha{}_{\mu\nu}\partial_\alpha(\sqrt{-g}\, g^{\mu\nu}) - \gamma \\ &= \partial_\alpha w^\alpha + \gamma\ , \quad w^\alpha \equiv \sqrt{-g}\, \left(g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} - g^{\alpha\nu}\Gamma^\mu{}_{\mu\nu}\right) . \end{aligned} \tag{15}$$

The divergence term contributes only a surface term that does not affect the functional derivative. Thus, the Einstein equations follow from a modified action known as the **Schrödinger action** (Schrödinger 1950):

$$S_{\mathrm{GS}}[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, g^{\mu\nu}\gamma_{\mu\nu} = \int d^4x\, \mathcal{L}_{\mathrm{GS}}(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta}) \tag{16}$$

where $\gamma_{\mu\nu}$ is defined in equation (14). Note that $\gamma_{\mu\nu}$ is not a tensor, and therefore $S_{\rm GS}$ is not a scalar. It differs from the Einstein-Hilbert action by a surface term which is also not a scalar but which makes no contribution to the equations of motion. Thus, the Schrödinger action has the same functional derivative as the Einstein-Hilbert action:

$$\frac{\delta S_{\rm GS}}{\delta g^{\mu\nu}} = G_{\mu\nu} \ . \tag{17}$$

One may regard the Schrödinger action as the Einstein-Hilbert action minus the second-derivative terms. From the viewpoint of tensors, these terms are crucial because one can always choose coordinates so that the connection coefficients vanish at a point, hence $\gamma_{\mu\nu} = 0$ at a point. However, one cannot do this everywhere (unless the spacetime is globally flat) and therefore one cannot transform $S_{\rm GS}$ away. At most, one can add boundary terms that have no effect on the equations of motion.

Given our strong emphasis on tensors in general relativity, one may well ask whether it is valid to use an action that is not a scalar under arbitrary coordinate transformations. The answer at the classical level is yes, of course, as long as it gives the correct equations of motion. From this perspective, the Schrödinger action is just as good as the Einstein-Hilbert action. Moreover, it depends only on the metric and its first derivatives (through the definition of the connection coefficients in a coordinate basis). As such, it enables us to construct a stress-energy psuedotensor.

To derive the conserved stress-energy pseudotensor, we follow the approach of Section 2. After some algebra we get the canonical field momentum,

$$16\pi G \frac{\pi^{\mu\alpha\beta}}{\sqrt{-g}} = \Gamma^{\mu\alpha\beta} - g^{\mu(\alpha}g^{\beta)\kappa}\Gamma^{\lambda}{}_{\kappa\lambda} - g^{\alpha\beta}g^{\mu\kappa}g^{\lambda\sigma}\Gamma_{[\kappa\lambda]\sigma} \ , \tag{18}$$

where the metric is used to raise and lower indices on the connection coefficients: $\Gamma^{\alpha\beta\mu} = g^{\alpha\kappa}g^{\beta\lambda}\Gamma_{\kappa\lambda\mu}$, etc.

Before giving the stress-energy pseudotensor, we note another relation between the Einstein-Hilbert and Schrödinger Lagrangians. From equations (15) and (18), we find $16\pi G\, g_{\alpha\beta}\, \pi^{\mu\alpha\beta} = -w^{\mu}$ and therefore

$$\mathcal{L}_{\rm EH} = \mathcal{L}_{\rm GS} - \partial_{\mu}\left(g_{\alpha\beta}\, \pi^{\mu\alpha\beta}\right) \ . \tag{19}$$

Thus, the two Lagrangians are related by a simple transformation equivalent to $L \to L - (d/dt)(pq)$ for the elementary mechanics of a particle in one dimension. This extra term obviously contributes nothing but boundary terms to the variation of the action.

Now we give the Schrödinger stress-energy pseudotensor, which follows from equation (10):

$$16\pi G\, \tau^{\mu}{}_{\nu} = 2g^{\alpha\beta}\Gamma^{\mu}{}_{\alpha\kappa}\Gamma^{\kappa}{}_{\beta\nu} - 2g^{\alpha(\mu}\Gamma^{\lambda)}{}_{\alpha\nu}\Gamma^{\kappa}{}_{\lambda\kappa} + 2g^{\alpha[\mu}\Gamma^{\lambda]}{}_{\alpha\lambda}\Gamma^{\kappa}{}_{\nu\kappa} - \delta^{\mu}{}_{\nu}g^{\alpha\beta}\left[\Gamma^{\kappa}{}_{\lambda\alpha}\Gamma^{\lambda}{}_{\kappa\beta} - \Gamma^{\kappa}{}_{\alpha\beta}\Gamma^{\lambda}{}_{\kappa\lambda}\right] \ . \tag{20}$$

Including all terms in the action (matter plus gravitational), the equations of motion imply (problem 2 of Problem Set 7)

$$(-g)^{-1/2}\partial_\mu\left[\sqrt{-g}\,\left(\tau^\mu{}_\nu + T^\mu{}_\nu\right)\right] = 0 \tag{21}$$

where $T^\mu{}_\nu$ is the stress-energy tensor of the matter. Equation (21) may be regarded as a statement of energy conservation for gravitation and matter, since it can be integrated over a 4-volume bounded by surfaces of constant $x^0$ to give

$$\frac{d\mathcal{P}_\nu}{dx^0} = -\oint dS\, n_i \sqrt{-g}\left(\tau^i{}_\nu + T^i{}_\nu\right)\ ,\quad \mathcal{P}_\nu(x^0) \equiv \int_V d^3x\,\sqrt{-g}\left(\tau^0{}_\nu + T^0{}_\nu\right) \tag{22}$$

where the surface integral is taken at fixed $x^0$ over the surface (with normal one-form $n_i$) bounding the 3-volume $V$. Although equations (21) and (22) are not tensor equations, they are exact in every coordinate system.

# 4 Gravitational Radiation Emitted Power

We can use the stress-energy pseudotensor to determine the energy flux density of gravitational radiation in a calculation similar to the derivation of the Poynting flux for electromagnetic radiation. The full calculation is not presented here, although we set it up. We will assume that the gravity waves are weak and the spacetime is nearly Minkowski. We ignore the non-radiative scalar and vector parts of the linearized metric and consider only the transverse-traceless part due to gravitational radiation:

$$ds^2 = -dt^2 + (\delta_{ij} + 2s_{ij})dx^i dx^j\ ,\quad s^i{}_i = 0\ ,\quad \partial_i s^i{}_j = 0\ . \tag{23}$$

Note that MTW and most other authors write $h_{ij} = 2s_{ij}$; I've inserted the factor of 2 so that $s_{ij}$ is the strain matrix and not twice the strain. The Minkowksi metric is used to raise and lower all indices. (See the 8.962 notes *Gravitation in the Weak-Field Limit.*)

With this metric, the nonzero connection coefficients are

$$\Gamma^0{}_{ij} = \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \partial_t s_{ij}\ ,\quad \Gamma^k{}_{ij} = \frac{1}{2}(\partial_i s_{jk} + \partial_j s_{ik} - \partial_k s_{ij})\ . \tag{24}$$

Substituting this into equation (20) gives the energy density and energy flux density

$$\begin{aligned} 16\pi G\tau^0{}_0 &= (\partial_t s_{ij})^2 + (\partial_k s_{ij})^2 - 2(\partial_i s_{jk})(\partial_j s_{ik})\ , \\ 16\pi G\tau^i{}_0 &= 4(\partial_t s_{jk})(\partial_j s_{ik}) - 2(\partial_t s_{jk})(\partial_i s_{jk})\ . \end{aligned} \tag{25}$$

The reader may check that, in vacuum where $(-\partial_t^2 + \partial^2)s_{ij} = 0$, $\partial_t \tau^0{}_0 + \partial_i \tau^i{}_0 = 0$. In applying quation (21), we use $\sqrt{-g} \approx 1$ in linear theory.

The stress-energy pseudotensor we have derived is not symmetric. It may be symmetrized by adding a derivative term (cf. Jackson 1975, section 12.10):

$$t^{\mu}{}_{\nu} = \tau^{\mu}{}_{\nu} + \frac{1}{\sqrt{-g}}\,\partial_\lambda \left(\sqrt{-g}\,S^{\lambda\mu\nu}\right) \quad \text{where} \quad S^{\lambda\mu\nu} = -S^{\mu\lambda\nu} \ . \tag{26}$$

Landau and Lifshitz (1975, section 96) give an alternative procedure for deriving a symmetry stress-energy pseudotensor. Their tensor is quite complicated and I don't know the $S^{\lambda\mu\nu}$ that transforms it to the Schrödinger pseudotensor. A symmetric stress-energy pseudotensor is useful because it allows one to formulate a conservation law for angular momentum. I will not go into that here (see MTW chapters 19 and 20).

Another advantage of the Landau-Liftshitz pseudotensor is the simple form it takes for a plane gravitational wave:

$$t_{\mu\nu} = \frac{1}{32\pi G}\,(\partial_\mu h^{\alpha}{}_{\beta})(\partial_\nu h^{\beta}{}_{\alpha}) \ . \tag{27}$$

Consider a plane wave propagating in the 1-direction. The nonzero strain components are $s_{22} = -s_{33}$ and $s_{23}$. In vacuum, these components are functions of $t - x^1$ and therefore $\partial_1 s_{ij} = -\partial_t s_{ij}$. It follows immediately that, in the transverse-traceless gauge of equation (23),

$$t^{00} = -t^{10} = t^{11} = \frac{1}{8\pi G}\,\dot{s}^2_{ij} \tag{28}$$

where a dot denotes $\partial_t$. In section 36.7 of the text, MTW use this together with the solution of the wave equation for $s_{ij}$ (including the source) to derive the gravitational radiation power crossing a sphere of radius $r$. For a nonrelativistically moving source the results is the famous quadrupole formula, equation (36.23) of MTW.

# References


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