GLOBULAR CELESTIAL NAVIGATION BY MEANS OF TRANSFORMATION OF GEOCENTRIC PLANAR MEASUREMENTS TO A SPHERICAL COORDINATE SYSTEM

WHAT IS A TRANSFORMATION

One common transformation equation is the 2D translation equation, which allows you to shift a point in a Cartesian coordinate system by a certain amount in the x and y directions. The equation is as follows:

NewX = OldX + dx
NewY = OldY + dy

In this equation, (OldX, OldY) represents the original coordinates of the point, and (NewX, NewY) represents the transformed coordinates after the translation. dx and dy represent the amount of shift in the x and y directions, respectively.

The purpose of a transformation equation is to convert a set of coordinates from one coordinate system to another. A coordinate system is a reference frame used to locate points in space. It consists of an origin (a fixed point) and a set of axes (lines) that define the directions and scales of measurement.

2. Symbols for heliocentric and geocentric coordinates

Heliocentric:

spherical ecliptic	l, b, r	with appropriate subscripts
rectangular equatorial	x, y, z	
rectangular ecliptic, geometric for mean equinox of date	x_0, y_0, z_0	

Geocentric:

spherical ecliptic	λ, β, Δ	with appropriate subscripts
spherical equatorial	α, δ, Δ	
rectangular equatorial	ξ, η, ζ	
rectangular equatorial (Sun)	X, Y, Z	

6. Figure of the Earth

ϕ = geographic, or geodetic, latitude—see special note in section 2F

ϕ' = geocentric latitude $\tan\phi' = (1 - e^2)\tan\phi$

ϕ_1 = parametric latitude $\tan\phi_1 = (1 - f)\tan\phi$

e = ellipticity, or eccentricity, of the Earth's meridian

f = flattening $1 - f = (1 - e^2)^{\frac{1}{2}}$

ρ = geocentric distance in units of the Earth's equatorial radius

S, C = auxiliary functions such that $\rho\sin\phi' = S\sin\phi$

$\rho\cos\phi' = C\cos\phi = \cos\phi_1$

2. COORDINATE AND REFERENCE SYSTEMS

A. COORDINATE SYSTEMS

The fundamental astronomical reference systems are based on the celestial equator, coplanar with the Earth's equator, and the ecliptic, the plane* of the Earth's orbit round the Sun. The angular coordinates in these planes are measured from the ascending node of the ecliptic on the equator, or the point at which the Sun in its annual apparent path round the Earth crosses the equator from south to north; and they are measured positively to the east, that is in the direction of the Sun's motion with respect to the stars. The ascending node of the ecliptic on the equator is referred to as " the vernal equinox ", " the first point of Aries ", or simply as " the equinox ". The axes of the corresponding rectangular coordinate systems are right-handed, i.e. the x-axis is directed towards the equinox, the y-axis to a point 90° to the east, while the z-axis is positive to the north.

The position of a point in space may be specified astronomically by reference to a wide variety of coordinate systems; and it may be given by means of (among other less usual systems) either spherical coordinates, consisting of a direction and a distance, or rectangular coordinates, consisting of the projections of the distance on three rectangular axes. The systems are determined by the two following characteristics:

(a) *Origin of coordinates*—and designation.

(i) The observer—topocentric.

(ii) The centre of the Earth—geocentric.

(iii) The centre of the Sun—heliocentric.

(iv) The centre of mass of the solar system—barycentric.

(b) *Reference planes and directions*—and designation of spherical coordinates.

(i) The horizon and the local meridian—azimuth and altitude.

(ii) The equator and the local meridian—hour angle and declination.

(iii) The equator and the equinox—(equatorial) right ascension and declination.

(iv) The ecliptic and the equinox—(ecliptic or celestial) longitude and latitude.

(v) The plane of an orbit and its equatorial or ecliptic node—orbital longitude and latitude.

*More strictly, the mean plane of the orbital motion, ignoring periodic perturbations.

Barycentric coordinates are often referred to the centre of mass of the Sun and the inner planets, and less often to other combinations. The equator, the ecliptic, and the equinox are constantly in motion due to the effects of precession and nutation, and must be further specified; this is done in sub-sections B and C. A notation to distinguish the various systems in current use is introduced in section 1G.

The reduction from geocentric to topocentric coordinates depends on the figure of the Earth, and is considered in detail in sub-section F. In most cases of astronomical interest, the differences are so small that they can be applied as first-order differential corrections.

Positions may be of several kinds, including: the *geometric* position derived from the actual position at the time of observation; the *apparent* position in which an observer, situated at the origin of coordinates, would theoretically see the object; and the *astrometric* position, in which corrections have been made for some small terms of aberration in order that it may be directly comparable with the tabulated catalogue positions of stars. The apparent position is derived from the geometric position by the application of corrections for aberration, and where relevant for refraction. However, refraction is dependent on the observer's local reference system and is invariably treated as a correction to the observation rather than to the ephemeris position; exceptions only occur for phenomena that are essentially topocentric, such as rising and setting and (in principle, though the correction is neglected in practice) for eclipses and occultations. For geocentric coordinates the apparent position is the direction in which an observer at the centre of the Earth would see the object, and refraction does not enter. Aberration is dealt with in sub-section D and refraction briefly in sub-section E.

In the present sub-section the effects of precession, nutation, aberration, refraction, and parallax are ignored in order to present the relationships between the coordinate systems. The general notation used is restricted to this purpose and should not be confused with the more detailed notation in section 1G necessary to distinguish between the different kinds of position.

Not all combinations of (a) and (b) occur and many are not used in the Ephemeris; (a) (iv), in particular, is therefore not referred to again. Moreover, if corrections for parallax be deferred, there is no difference between (a) (i) and (a) (ii), which can be treated together.

For *geocentric* spherical coordinates there are thus the four practical reference systems of:

(i) azimuth (A) measured from the north through east in the plane of the horizon, and altitude (a) measured perpendicular to the horizon; in astronomy the zenith distance ($z = 90° - a$) is more generally used, but the altitude is retained in the formulae for reasons of symmetry;

(ii) hour angle (h) measured westwards in the plane of the equator from the meridian, and declination (δ) measured perpendicular to the equator, positive to the north;

(iii) right ascension (α) measured from the equinox eastwards in the plane of the equator, and declination (δ);

(iv) longitude (λ) measured from the equinox eastwards in the plane of the ecliptic, and latitude (β) measured perpendicular to the ecliptic, positive to the north.

The formulae connecting these coordinates are:

Azimuth/altitude : Hour angle/declination

$$\begin{aligned} \cos a \sin A &= -\cos\delta \sin h \\ \cos a \cos A &= \sin\delta \cos\phi - \cos\delta \cos h \sin\phi \\ \sin a &= \sin\delta \sin\phi + \cos\delta \cos h \cos\phi \end{aligned}$$

$$\begin{aligned} \cos\delta \sin h &= -\cos a \sin A \\ \cos\delta \cos h &= \sin a \cos\phi - \cos a \cos A \sin\phi \\ \sin\delta &= \sin a \sin\phi + \cos a \cos A \cos\phi \end{aligned}$$

where ϕ is the latitude of the observer. Note that the conversion corresponds to a simple rotation of the frame of reference through an angle $90° - \phi$ in the plane of the meridian.

Hour angle/declination : Right ascension/declination

The two systems are identical apart from the origin, and direction, of measurement of hour angle and right ascension, which are connected by the relation:

$$h = \text{local sidereal time} - \alpha$$

since local sidereal time is the hour angle of the equinox.

Right ascension/declination : Longitude/latitude

$$\begin{aligned} \cos\delta \cos\alpha &= \cos\beta \cos\lambda \\ \cos\delta \sin\alpha &= \cos\beta \sin\lambda \cos\epsilon - \sin\beta \sin\epsilon \\ \sin\delta &= \cos\beta \sin\lambda \sin\epsilon + \sin\beta \cos\epsilon \end{aligned}$$

$$\begin{aligned} \cos\beta \cos\lambda &= \cos\delta \cos\alpha \\ \cos\beta \sin\lambda &= \cos\delta \sin\alpha \cos\epsilon + \sin\delta \sin\epsilon \\ \sin\beta &= -\cos\delta \sin\alpha \sin\epsilon + \sin\delta \cos\epsilon \end{aligned}$$

where ϵ is the obliquity of the ecliptic (corresponding to the particular equator and ecliptic used). Geocentric longitude and latitude are used now only for the Sun and Moon. Note that the conversions correspond to a simple rotation round the x-axis through an angle ϵ.

The corresponding equatorial rectangular coordinates and distance are denoted by X, Y, Z, and R for the Sun and by ξ, η, ζ, and Δ for the planets; they are derived from the spherical coordinates by the formulae:

$$\begin{aligned} X/R \text{ or } \xi/\Delta &= \cos\delta \cos\alpha \\ Y/R \text{ or } \eta/\Delta &= \cos\delta \sin\alpha \\ Z/R \text{ or } \zeta/\Delta &= \sin\delta \end{aligned}$$

Geocentric ecliptic rectangular coordinates are rarely (if ever) used.

For *heliocentric* coordinates there are only the two practical reference systems—the equatorial and the ecliptic; and in the equatorial system only rectangular coordinates are used. The relationships between the ecliptic rectangular coordinates (x_c, y_c, z_c), the ecliptic longitude, latitude, and distance (l, b, r), and the equatorial rectangular coordinates (x, y, z) are:

$$
\begin{aligned}
x_c &= r \cos b \cos l = x \\
y_c &= r \cos b \sin l = +y \cos \epsilon + z \sin \epsilon \\
z_c &= r \sin b = -y \sin \epsilon + z \cos \epsilon \\
x &= x_c = r(\cos b \cos l) \\
y &= y_c \cos \epsilon - z_c \sin \epsilon = r(\cos b \sin l \cos \epsilon - \sin b \sin \epsilon) \\
z &= y_c \sin \epsilon + z_c \cos \epsilon = r(\cos b \sin l \sin \epsilon + \sin b \cos \epsilon)
\end{aligned}
$$

The conversion from *heliocentric* to *geocentric* coordinates is performed in terms of equatorial rectangular coordinates through:

$$
\begin{aligned}
\xi &= x + X \\
\eta &= y + Y \\
\zeta &= z + Z
\end{aligned}
$$

where X, Y, Z are the geocentric coordinates of the Sun.

The calculation of the spherical coordinates from the rectangular coordinates, or from the known direction cosines, typified by:

$$
\begin{aligned}
\Delta \cos \delta \cos \alpha &= \xi \\
\Delta \cos \delta \sin \alpha &= \eta \\
\Delta \sin \delta &= \zeta
\end{aligned}
$$

is performed by:

$$
\tan \alpha = \eta/\xi \qquad \cot \alpha = \xi/\eta
$$
$$
\Delta = (\xi^2 + \eta^2 + \zeta^2)^{\frac{1}{2}} \qquad \sin \delta = \zeta/\Delta
$$

The quadrant of α is determined by the signs of ξ and η, and that of δ by the sign of ζ; Δ and $\Delta \cos \delta$ are always positive. The formulae for α and δ may be written:

$$
\begin{aligned}
\alpha &= \tan^{-1}\eta/\xi \quad \text{or arctan } \eta/\xi \\
&= \cot^{-1}\xi/\eta \quad \text{or arccot } \xi/\eta \\
\delta &= \sin^{-1}\zeta/\Delta \quad \text{or arcsin } \zeta/\Delta
\end{aligned}
$$

provided that the appropriate values, and not necessarily the principal values, of the multi-valued functions are taken.

Notes on the technique of practical calculation using these formulae, and on the most suitable trigonometric tables to use, are given in section 16A.

Many of the conversions above correspond to a simple rotation of the frame of reference about one of its axes. These are special cases of the general conversion from a set of axes designated by x, y, z to a set designated by x', y', z'; the two systems are connected by the formulae:

$$
\begin{aligned}
x &= l_1 x' + l_2 y' + l_3 z' \qquad & x' &= l_1 x + m_1 y + n_1 z \\
y &= m_1 x' + m_2 y' + m_3 z' \qquad & y' &= l_2 x + m_2 y + n_2 z \\
z &= n_1 x' + n_2 y' + n_3 z' \qquad & z' &= l_3 x + m_3 y + n_3 z
\end{aligned}
$$

where l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of x', y', z' referred to the system x, y, z. The direction cosines satisfy the relations typified by:

$$
l_1^2 + m_1^2 + n_1^2 = 1 \qquad l_1^2 + l_2^2 + l_3^2 = 1
$$
$$
l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \qquad m_1 n_1 + m_2 n_2 + m_3 n_3 = 0
$$

These nine quantities can be expressed in terms of the Eulerian angles θ, ϕ, ψ by:

$$
\begin{aligned}
l_1 &= +\cos \phi \cos \theta \cos \psi - \sin \phi \sin \psi \\
l_2 &= -\cos \phi \cos \theta \sin \psi - \sin \phi \cos \psi \\
l_3 &= +\cos \phi \sin \theta
\end{aligned}
$$

$$
\begin{aligned}
m_1 &= +\sin \phi \cos \theta \cos \psi + \cos \phi \sin \psi \\
m_2 &= -\sin \phi \cos \theta \sin \psi + \cos \phi \cos \psi \\
m_3 &= +\sin \phi \sin \theta
\end{aligned}
$$

$$
\begin{aligned}
n_1 &= -\sin \theta \cos \psi \\
n_2 &= +\sin \theta \sin \psi \\
n_3 &= +\cos \theta
\end{aligned}
$$

In this case the conversion corresponds to a rotation ϕ about the z-axis, θ about the new position of the y-axis, and ψ about the new (and final) position of the z-axis. The transformation is equivalent to a single rotation about some line not in general coincident with one of the axes; but such single rotations are not frequently encountered in astronomical practice.

Example 9.2. Test for occurrence of eclipse

q = ratio of daily motions (1961 February 15–16) of Moon and Sun in longitude $= 54143''/3636''.8 = 14{\cdot}9$

I (*A.E.* 1961, p. 51) $= 5^\circ{\cdot}1$ $\tan I = 0{\cdot}089$ $q/(q-1) = 1{\cdot}072$

$\tan I' = 0{\cdot}095$ $\sec I' = 1{\cdot}0045$

By interpolation in *A.E.* 1961 to 1961 February $15^d\ 08^h\ 11^m$:

$\pi_{\leftmoon}\ 61'\ 05''{\cdot}7$ $\sin s_{\leftmoon} = 0{\cdot}2722\ 74 \sin \pi_{\leftmoon}$

$s_{\leftmoon}\ 16\ 38{\cdot}0$ $s_{\leftmoon} = 0''{\cdot}08 + 0{\cdot}2722\ 39\ \pi_{\leftmoon}$

$\pi_{\odot}\ 8{\cdot}9$

$s_{\odot}\ 16\ 11{\cdot}4$ $s_{\odot} = 16'\ 12''{\cdot}97$ (*A.E.*, p. 21) $- 1''{\cdot}55$ (irradiation)

Thus $\sec I'\ (\pi_{\leftmoon} - \pi_{\odot} + s_{\leftmoon} + s_{\odot}) = 1{\cdot}0045 \times 1^\circ\ 33'\ 46'' = 1^\circ\ 34'\ 11''$

$\beta_{\leftmoon}$ is $0^\circ\ 54'$ so that an eclipse is certain.

Besselian elements

The calculation of eclipses is carried out in accordance with Bessel's method. In solar eclipses the Besselian elements describe the geometric position of the shadow of the Moon relative to the Earth. The exterior tangents to the surfaces of the Sun and the Moon form the umbral cone, the interior tangents the penumbral cone. The common axis of the two cones is the axis of the shadow. The geocentric plane perpendicular to the axis of the shadow is called the fundamental plane, and is taken as the *xy*-plane of a system of geocentric rectangular coordinates. The *x*-axis is the intersection of the fundamental plane with the plane of the equator and is directed positively towards the east; the *y*-axis is directed positively towards the north. The *z*-axis is parallel to the axis of the shadow and is positive towards the Moon. See figure 9.2, which shows the projection of the observer and the shadow on the fundamental plane.

Let a and d designate the right ascension and declination of the point Z on the celestial sphere towards which the axis of the shadow is directed, and G the distance between the centres of the Sun and Moon; then:

$$G \cos d \cos a = R \cos \delta_{\odot} \cos \alpha_{\odot} - r_{\leftmoon} \cos \delta_{\leftmoon} \cos \alpha_{\leftmoon}$$
$$G \cos d \sin a = R \cos \delta_{\odot} \sin \alpha_{\odot} - r_{\leftmoon} \cos \delta_{\leftmoon} \sin \alpha_{\leftmoon}$$
$$G \sin d = R \sin \delta_{\odot} - r_{\leftmoon} \sin \delta_{\leftmoon}$$

In practice, it is convenient to set:

$$g = G/R \qquad b = r_{\leftmoon}/R = \sin \pi_{\odot}/\sin \pi_{\leftmoon}$$

which yields:

$$g \cos d \cos a = \cos \delta_{\odot} \cos \alpha_{\odot} - b \cos \delta_{\leftmoon} \cos \alpha_{\leftmoon}$$
$$g \cos d \sin a = \cos \delta_{\odot} \sin \alpha_{\odot} - b \cos \delta_{\leftmoon} \sin \alpha_{\leftmoon}$$
$$g \sin d = \sin \delta_{\odot} - b \sin \delta_{\leftmoon}$$

In numerical calculations b is evaluated from:

$$b = \sin \pi_{\odot}/R \sin \pi_{\leftmoon}$$

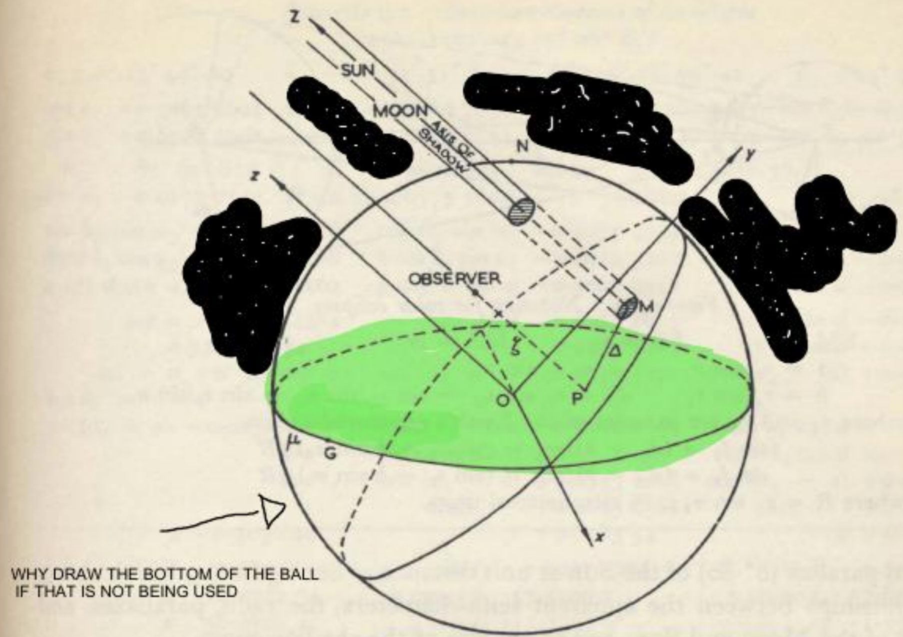

Figure 9.2. Projection of the observer and shadow on the fundamental plane

P Projection of observer (ξ, η)
M Projection of axis of shadow (x, y)

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fundamental plane, in units of the equatorial radius of the Earth, from:

$$x = r_{\leftmoon} \{\cos \delta_{\leftmoon} \sin (\alpha_{\leftmoon} - a)\}$$
$$y = r_{\leftmoon} \{\sin \delta_{\leftmoon} \cos d - \cos \delta_{\leftmoon} \sin d \cos (\alpha_{\leftmoon} - a)\}$$
$$z = r_{\leftmoon} \{\sin \delta_{\leftmoon} \sin d + \cos \delta_{\leftmoon} \cos d \cos (\alpha_{\leftmoon} - a)\}$$

in which:

$$r_{\leftmoon} = 1/\sin \pi_{\leftmoon}$$

The coordinates x, y are also those of the intersection of the axis of shadow with the fundamental plane.

In the tabulation of Besselian elements of eclipses, the right ascension a of the point Z is conventionally replaced for practical use by the ephemeris hour angle μ of that point, given by:

$$\mu = \text{ephemeris sidereal time} - a$$

The angles f_1, f_2 which the generators of the penumbral (subscript 1) and

$b \cos \delta_{\mathbb{C}} \cos a_{\mathbb{C}}$	+0·0020 2166	$b \cos \delta_{\mathbb{C}} \sin a_{\mathbb{C}}$	−0·0012 5207	$b \sin \delta_{\mathbb{C}}$	−0·0005 0078
$g \cos d \cos a$	+0·8310 2162	$g \cos d \sin a$	−0·5062 9547	$g \sin d$	−0·2195 7735
$\tan a$	−0·6092 4464	g^2	0·9951 4625	$\sin d$	−0·2201 1219
a	328° 38′ 54″·10	g	0·9975 7017	$\cos d$	+0·9754 7456
$a_{\mathbb{C}} - a$	−0° 25′ 09″·81	$\sin (a_{\mathbb{C}} - a)$	−0·0073 1970	$\cos (a_{\mathbb{C}} - a)$	+0·9999 7321
$\cos \delta_{\mathbb{C}} \times$ $\sin (a_{\mathbb{C}} - a)$	−0·0071 6260	$\sin \delta_{\mathbb{C}} \cos d$	−0·2010 1657	$\sin \delta_{\mathbb{C}} \sin d$	+0·0453 5864
		$-\cos \delta_{\mathbb{C}} \sin d \times$ $\cos (a_{\mathbb{C}} - a)$	+0·2153 8218	$+\cos \delta_{\mathbb{C}} \cos d \times$ $\cos (a_{\mathbb{C}} - a)$	+0·9545 1252
		sum	+0·0143 6561	sum	+0·9998 7116
x	−0·4030 40	y	+0·8083 54	z	+56·2628 4
gR	0·9854 801	$\sin f_1$	0·0047 3273 5	$\sin f_2$	0·0047 0916 1
k	0·2722 74	$k \operatorname{cosec} f_1$	57·5299 5	$k \operatorname{cosec} f_2$	57·8179 4
z	56·2628 4	c_1	113·7927 9	c_2	−1·5551 0
		$\tan f_1$	0·0047 3278 8	$\tan f_2$	0·0047 0921 3
		l_1	0·5385 57	l_2	−0·0073 23

" Apparent sidereal time " (*A.E.*, p. 11)	17^h 40^m 21^s·093
= Ephemeris sidereal time at 8^h E.T.	265° 05′ 16″·40
a	328 38 54·10
Ephemeris hour angle, μ	296 26 22·3

From a similar calculation for February 15^d 09^h

μ =	311° 26′ 29″·2	$\sin d$ =	−0·2198 7475
hourly change =	15° 00′ 06″·9	hourly change =	+0·0002 3744
μ' =	0·2618 328	d' =	+0·0002 4341

Coordinates of the observer

For an observer located on the surface of the Earth in ephemeris longitude λ^*, geocentric latitude ϕ', at a distance ρ from the centre of the terrestrial spheroid, his geocentric rectangular coordinates ξ, η, ζ, referred to the x, y, z system of axes in units of the Earth's equatorial radius, are found in terms of the Besselian elements from:

$$\xi = \rho \cos \phi' \sin \theta$$
$$\eta = \rho \sin \phi' \cos d - \rho \cos \phi' \sin d \cos \theta$$
$$\zeta = \rho \sin \phi' \sin d + \rho \cos \phi' \cos d \cos \theta$$

in which:†

$$\theta = \mu - \lambda^*$$

Their hourly variations are found from:

$$\xi' = +\mu' \rho \cos \phi' \cos \theta$$
$$\eta' = +\mu' \rho \cos \phi' \sin d \sin \theta - d' (\rho \sin \phi' \sin d + \rho \cos \phi' \cos d \cos \theta)$$
$$\zeta' = -\mu' \rho \cos \phi' \cos d \sin \theta + d' (\rho \sin \phi' \cos d - \rho \cos \phi' \sin d \cos \theta)$$

†Longitude is here measured positively to the west.

each point from the centre of the Earth. This distance is not known a priori, because it is a function of the latitude of the point. The problem of determining ζ could be solved by successive approximations, but the procedure outlined below was devised by Bessel to provide a direct computation.

If ϕ is the geodetic latitude of a point on the Earth's surface and ϕ' its geocentric latitude, then:

$$\rho \sin \phi' = (1 - e^2) \sin \phi \, (1 - e^2 \sin^2 \phi)^{-\frac{1}{2}} = S \sin \phi$$
$$\rho \cos \phi' = \cos \phi \, (1 - e^2 \sin^2 \phi)^{-\frac{1}{2}} = C \cos \phi$$

in which e is the ellipticity of the Earth's spheroid. For Hayford's spheroid, * e^2 is equal to 0·0067 2267. S and C are tabulated in table 2.8.

Let ϕ_1 be the parametric latitude, such that:

$$\sin \phi_1 = \rho \sin \phi' (1 - e^2)^{-\frac{1}{2}} \qquad \cos \phi_1 = \rho \cos \phi' \qquad \sin^2 \phi_1 + \cos^2 \phi_1 = 1$$

and set:

$$\rho_1 \sin d_1 = \sin d \qquad \rho_2 \sin d_2 = \sin d \, (1 - e^2)^{\frac{1}{2}}$$
$$\rho_1 \cos d_1 = \cos d \, (1 - e^2)^{\frac{1}{2}} \qquad \rho_2 \cos d_2 = \cos d$$
$$\eta_1 = \eta / \rho_1 \qquad \zeta_1^2 = 1 - \xi^2 - \eta_1^2$$

By means of these new quantities, the equations for ξ, η, ζ above may be transformed into:

$$\xi = \cos \phi_1 \sin \theta$$
$$\eta_1 = \sin \phi_1 \cos d_1 - \cos \phi_1 \sin d_1 \cos \theta$$
$$\zeta_1 = \sin \phi_1 \sin d_1 + \cos \phi_1 \cos d_1 \cos \theta$$

and it may be noted that:

$$\zeta = \rho_2 \sin \phi_1 \sin d_2 + \rho_2 \cos \phi_1 \cos d_2 \cos \theta$$

or

$$\zeta = \rho_2 \{ \zeta_1 \cos (d_1 - d_2) - \eta_1 \sin (d_1 - d_2) \}$$

According to the above relations, it is seen that the quantities required for the introduction of the flattening of the Earth into eclipse calculations are ρ_1, ρ_2, $\sin d_1$, $\cos d_1$, $\sin (d_1 - d_2)$, and $\cos (d_1 - d_2)$. With these quantities, ξ, η_1, ζ_1, may be calculated from given values of ξ, η to enable ϕ_1 (and thus ϕ) and θ to be deduced; and ζ can be calculated directly from values of η_1, ζ_1 and indirectly from ξ, η. They may be obtained from the following formulae:

$$\rho_1 = (1 - e^2 \cos^2 d)^{\frac{1}{2}} \qquad \rho_2 = (1 - e^2 \sin^2 d)^{\frac{1}{2}}$$
$$\rho_1 \sin d_1 = \sin d \qquad \rho_1 \cos d_1 = (1 - e^2)^{\frac{1}{2}} \cos d$$
$$\rho_1 \rho_2 \sin (d_1 - d_2) = e^2 \sin d \cos d \qquad \rho_1 \rho_2 \cos (d_1 - d_2) = (1 - e^2)^{\frac{1}{2}}$$

*e^2 = 0·0066 9454 for 1968 onwards; corresponding values of S and C are tabulated in *A.E.* Table *VII*.

WTF DID I JUST READ?

Here's the TL;DR: Using planar angle measurements of the sky, a 2D lat/long coordinate system can be made to fit a 3d spherical coordinate system using the provided transformations.

Long version:

1. Coordinate Systems: The fundamental reference systems in astronomy are based on the celestial equator and the ecliptic. Angular coordinates are measured from the ascending node of the ecliptic on the equator, known as the vernal equinox or the first point of Aries.

2. Rectangular Coordinate Systems: Coordinate systems can be specified using either spherical coordinates (direction and distance) or rectangular coordinates (projections on three axes). The axes of the rectangular coordinate systems are right-handed, with the x-axis directed towards the equinox, the y-axis to a point 90° east, and the z-axis positive to the north.

3. Origin of Coordinates: Coordinates can be based on different reference points: observer-topocentric, center of the Earth-geocentric, center of the Sun-heliocentric, or center of mass of the solar system-barycentric.

4. Reference Planes and Directions: Spherical coordinates are determined by reference planes and directions, such as the horizon and local meridian (azimuth and altitude), the equator and local meridian (hour angle and declination), the equator and equinox (right ascension and declination), the ecliptic and equinox (ecliptic or celestial longitude and latitude), and the plane of an orbit and its node (orbital longitude and latitude).

5. Reduction to Topocentric Coordinates: The conversion from geocentric to topocentric coordinates depends on the figure of the Earth and involves small differential corrections.

6. Position Types: Positions in astronomy can be geometric (actual position at the time of observation), apparent (corrected for aberration and refraction), or astrometric (directly comparable with catalog positions of stars).

7. Coordinate Conversions: The passage provides formulas for converting between different coordinate systems, such as azimuth/altitude, hour angle/declination, right ascension/declination, and longitude/latitude.

The geocentric rectangular coordinates (x, y, z) can be converted to heliocentric coordinates (xc, yc, zc) using the following equations:

$$xc = x + X$$
$$yc = y + Y$$
$$zc = z + Z$$

In these equations, (X, Y, Z) represents the geocentric coordinates of the Sun.

The heliocentric coordinate system is derived from the geocentric coordinate system, specifically using the geocentric coordinates of the Sun. The geocentric coordinates serve as a reference point for determining the position of celestial objects relative to the Earth. From the geocentric coordinates, the heliocentric coordinates can be derived by adding the respective geocentric coordinates of the Sun. This conversion allows for a different perspective where the positions of celestial objects are described relative to the Sun instead of the Earth.

EQUATION & VARIABLES DEFINED

- ϕ represents the geographic or geodetic latitude, which is the angle measured from the equatorial plane to a point on the Earth's surface.
- ϕ' represents the geocentric latitude, which is the angle between the equatorial plane and the line joining the Earth's center to a point on the Earth's surface.
- $\tan \phi' = (l - e^2) \tan \phi$ is an equation relating the geocentric latitude (ϕ') to the geographic latitude (ϕ) and the ellipticity (e) of the Earth's meridian. It states that the tangent of the geocentric latitude is equal to the product of the tangent of the geographic latitude and the factor ($l - e^2$), where l represents the spherical radius of the Earth.
- ϕ_1 represents the parametric latitude, which is another way of expressing the relationship between the geographic and geocentric latitudes.
- $\tan \phi_1 = (l - f) \tan \phi$ is an equation relating the parametric latitude (ϕ_1) to the geographic latitude (ϕ) and the flattening (f) of the Earth. It states that the tangent of the parametric latitude is equal to the product of the tangent of the geographic latitude and the factor ($l - f$), where l represents the spherical radius of the Earth.
- e represents the ellipticity or eccentricity of the Earth's meridian. It is a measure of the departure of the Earth's shape from a perfect sphere.
- f represents the flattening factor, which quantifies the oblateness or departure from a perfect sphere. It is defined as $l - f = (l - e^2)^{1/2}$, where l represents the spherical radius of the Earth.
- p represents the geocentric distance, which is the distance from the center of the Earth to a point on the Earth's surface, measured in units of the Earth's equatorial radius.
- S and C are auxiliary functions used in calculations. They are defined such that $p \sin \phi' = S \sin \phi$ and $p \cos \phi' = C \sin \phi = \cos p \sin \phi' = S \sin \phi_1$. These functions are related to the trigonometric values of the latitudes and the geocentric distance.

TRANSFORMING A GEOCENTRIC RECTANGULAR COORDINATE SYSTEM INTO A SPHERICAL COORDINATE SYSTEM

(x) $\xi = \cos \phi_1 \sin \Theta$
This equation represents the x-coordinate in the spherical system, where ϕ_1 is the parametric latitude and Θ is the longitude.

(y) $\eta = \eta_1 \sin \phi_1 \cos d_1 - \cos \phi_1 \sin d_1 \cos \Theta$
This equation represents the y-coordinate in the spherical system. It involves the parametric latitude (ϕ_1), the latitude (d_1), and the longitude (Θ).

(z) $\zeta = \zeta_1 \sin \phi_1 \sin d_1 + \cos \phi_1 \cos d_1 \cos \Theta$
This equation represents the z-coordinate in the spherical system. It also involves the parametric latitude (ϕ_1), the latitude (d_1), and the longitude (Θ).

The parametric latitude (ϕ_1) is derived from the geocentric latitude (ϕ') and is used to calculate the coordinates in the spherical system.

Additionally, the relationship between ζ and η_1, ζ_1, and the derived quantities (p_1, p_2, $\sin(d_1 - d_2)$, $\cos(d_1 - d_2)$) is expressed by: $\zeta = p_2 (\zeta_1 \cos (d_1 - d_2) - \eta_1 \sin (d_1 - d_2))$. This equation provides an alternative way to calculate ζ using the derived quantities and η_1, ζ_1.

In summary, these equations allow for the transformation of geocentric rectangular coordinates (ξ, η, ζ) to spherical coordinates (ϕ_1, Θ, ζ)

APPLYING THE TRANSFORMATIONS

The given equations facilitate the transformation from geocentric rectangular coordinates (ξ, η, ζ) in the XYZ system to spherical coordinates (ϕ_1, ϴ, ζ) in a 3D space, representing latitude and longitude.
To summarize the process:

1. Start with geocentric rectangular latitude and longitude coordinates (ξ, η, ζ)

2. Calculate the parametric latitude (ϕ_1) using the geocentric latitude (ϕ').

3. Apply the flattening correction to account for the Earth's spheroid shape:
   - Compute derived quantities p_1, p_2, $\sin(d_1 - d_2)$, and $\cos(d_1 - d_2)$ using the geocentric latitude (ϕ') and Earth's ellipticity (e).
   - Use these derived quantities and ϕ_1 to calculate the coordinates (ξ, η, ζ) in the spherical system, as described in the equations mentioned previously.

4. The resulting spherical coordinates (ϕ_1, ϴ, ζ) represent the latitude and longitude in the 3D spherical coordinate system, considering the Earth's alleged spheroidal shape.

The flattening correction adjusts for the Earth's flattened shape, ensuring that the 2D latitude and longitude coordinates match up with the 3D spherical coordinates. It is incorporated into the calculation of the derived quantities p_1, p_2, $\sin(d_1 - d_2)$, $\cos(d_1 - d_2)$, which are then used to determine the spherical coordinates. By considering the flattening correction, the transformation accurately maps the geocentric rectangular latitude and longitude coordinate system to a 3D spherical latitude and longitude coordinate system.