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On the theory of the geodynamo

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Abstract

I trace the development of geodynamo theory leading from Larmor's original hypothesis (Larmor, 1919, Rep. Br. Assoc. Adv. Sci., A, 159–160) to the present. I consider a number of kinematic results, from Cowling's proof (Cowling, 1934, Mon. Not. R. Astron. Soc., 94: 39–48) that two-dimensional dynamo action is not possible, to the proofs by Backus (1958, Ann. Phys., 4: 372–447) and Herzenberg (1958, Philos. Trans. R. Soc. London, Ser. A, 250: 543–585) that three-dimensional dynamo action is possible. I next turn to various mean-field and convective models in which the fluid flow is no longer kinematically prescribed, but is itself dynamically determined. In these dynamical models, I describe the distinction between weak and strong field regimes that comes about owing to the effect of the field on the pattern of convection in a rapidly rotating system. I consider the dynamics of Taylor's constraint (Taylor, 1963, Proc. R. Soc. London, Ser. A, 274: 274–283), and demonstrate how it makes the analysis of the geophysically appropriate strong field regime particularly difficult.

1. Introduction

That the Earth possesses a magnetic field has been known for many centuries. However, it is only in the past few decades that we have begun to understand the origin of this field. Today it is generally recognized that the field is created by the 'dynamo action' of convective fluid motions in the Earth's electrically conducting core, as first suggested by Larmor (1919). In this paper I will review some of the advances we have made since then in our understanding of the geodynamo. I will not even attempt to cover all of these advances, but will restrict myself to those aspects I am most familiar with, and consider most interesting. In view of the many other review papers that have previously been written on the subject (see, e.g. Busse, 1983; Roberts, 1988; Braginsky, 1991; Soward, 1991; Roberts and Soward, 1992; Fearn, 1996), it hardly seems necessary for any one to cover all aspects of the geodynamo. This particular review is intended for readers who have some familiarity with fluid dynamics, perhaps even rotating fluid dynamics, but not necessarily magnetohydrodynamics.

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In their simplest Boussinesq form, the equations governing the geodynamo are

$$\frac{\partial}{\partial t}\boldsymbol{B} = \nabla^2 \boldsymbol{B} + \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) \tag{1}$$

$$\mathrm{Ro}\left(\frac{\partial}{\partial t} + \boldsymbol{U}\cdot\nabla\right)\boldsymbol{U} + 2\hat{\boldsymbol{k}} \times \boldsymbol{U} = -\nabla p + \mathrm{E}\nabla^2\boldsymbol{U} + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \mathrm{q}\widetilde{\mathrm{Ra}}\Theta\boldsymbol{r} \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{U}\cdot\nabla\right)\Theta = \mathrm{q}\nabla^2\Theta \tag{3}$$

together with the nondivergence conditions $\nabla \cdot \boldsymbol{B} = \nabla \cdot \boldsymbol{U} = 0$. Here $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{U}$ is the fluid flow, and Θ is the buoyancy.

This buoyancy Θ could be due to either thermal or compositional effects, and so the buoyancy Eq. (3) could be interpreted as modelling either. Of course, the associated boundary conditions would be very different for these two possibilities, and it is precisely these boundary conditions that maintain the buoyancy gradients that will ultimately drive the dynamo. In fact, it is at present believed (Braginsky and Roberts, 1995) that both thermal and compositional sources of buoyancy are important in the Earth's core, and so perhaps one should really have two separate buoyancy equations, (3a) and (3b). However, we will avoid this additional complication, and will henceforth consider only thermal sources of buoyancy. This is perhaps justified by the experiments of Cardin and Olson (1992), which suggest that mixed thermal and compositional convection is qualitatively more like pure thermal than like pure compositional convection.

The nondimensionalization of these geodynamo equations, Eqs. (1)–(3), is as follows. Length is scaled by the difference in core radii $L = r_o - r_i$, where the inner and outer core radii are $r_i = 1220$ km and $r_o = 3480$ km. Time is scaled by the magnetic diffusion time $T = L^2/\eta$, which turns out to be around 65 000 years. The fluid flow is then scaled by $U = L/T$, around 10^{-4} cm s^{-1}, so that the diffusive and advective timescales in the induction equation, Eq. (1), are comparable. It should be noted, however, that the actual amplitude of the flow can only emerge as part of the solution, and may well turn out to be different from $O(1)$. The magnetic field is scaled by $B = (\Omega\rho\mu_0\eta)^{1/2}$, where Ω is the Earth's rotation rate, ρ is density, and μ_0 is the permeability. This scaling has been chosen so that the Coriolis and Lorentz forces in the momentum equation, Eq. (2), are comparable. For reasons that should become clearer as we proceed, this is believed to be the appropriate force balance in the Earth's core. Again, however, it should be noted that the actual amplitude of the field can only emerge as part of the solution, and may be different from $O(1)$, as it is defined within the context of this nondimensionalization. However, one would hope that the amplitude that does emerge is not too different from $O(1)$, as the numerical value corresponding to this scaling comes out to around 10–20 G, which is indeed only slightly larger than the observed field at the core–mantle boundary (Bloxham et al., 1989).

The nondimensional parameters that then appear in the geodynamo equations are, first, the (modified) Rayleigh number

$$\widetilde{\mathrm{Ra}} = g\alpha\beta L^2/\Omega\kappa \tag{4}$$

where g is gravity, α is the coefficient of thermal expansion, and β is the basic radial temperature gradient that ultimately drives the convection. The Rayleigh number is thus a measure of the buoyancy force one must provide for the dynamo to operate. It should be noted that this modified Rayleigh number is measured against the Coriolis force, whereas the usual Rayleigh number is measured against the viscous force. In a rotating system, in which the Coriolis force is known to be dominant, the modified Rayleigh number is the more appropriate measure. However, it should be noted that even this modified Rayleigh number is really only appropriate in the strong field regime discussed below, in the sense that it is only in this regime that the critical Rayleigh number is a pure number, independent of the other parameters in the problem.

There are next the Rossby and Ekman numbers

$$\mathrm{Ro} = \eta/\Omega L^2 \tag{5}$$

and

$$\mathrm{E} = \nu/\Omega L^2 \tag{6}$$

The Rossby number is a measure of the rotational timescale Ω^{-1} to the diffusive timescale L^2/η, and is $O(10^{-8})$. The Ekman number is a measure of viscous to Coriolis forces, and is $O(10^{-12})$. It is the extreme smallness of these two parameters, and the corresponding dominance of rotation, that makes the geodynamo equations so difficult.

Finally, there is the Roberts number

$$\mathrm{q} = \kappa/\eta \tag{7}$$

measuring the ratio of thermal to magnetic diffusivities. This ratio is also small, perhaps $O(10^{-6})$, and this smallness too can lead to difficulties (Zhang and Jones, 1996), although these have not been explored in such detail as the difficulties related to the smallness of Ro and E.

Before discussing the details of how the geodynamo is believed to operate, it is perhaps worth while to consider briefly just why fluid flows should generate magnetic fields at all. After all, $\boldsymbol{B} \equiv 0$ seems like a perfectly sensible solution to the geodynamo equations. Why should this nonmagnetic solution be unstable? That is, why should the flow amplify some small seed field to the point where it completely alters the pattern of the original flow (as we shall see below)? There are two ways to look at the amplification of a magnetic field: the first is kinematic, based on the induction equation alone; the second is dynamic, based on the induction and momentum equations together. As both yield valuable insight into the mechanism of dynamo action, we will of course consider both, and in the process derive some useful energetic results.

Considered first from the kinematic point of view, we focus on the effect of the $\nabla \times (\boldsymbol{U} \times \boldsymbol{B})$ term in Eq. (1). If we temporarily neglect the $\nabla^2 \boldsymbol{B}$ term, the equation we obtain,

$$\frac{\partial}{\partial t}\boldsymbol{B} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) \tag{8}$$

is the same as the equation governing the evolution of material line elements in the fluid. That is, the fluid will advect magnetic field lines as material lines, frozen into the fluid. A proof of this result, known as Alfvén's frozen-flux theorem, is given in any textbook on magnetohydrodynamics, such as those by Roberts (1967), Moffatt (1978), or Parker (1979). Alternatively, one might note the close analogy to the vorticity equation,

$$\frac{\partial}{\partial t}\boldsymbol{\Omega} = \nabla \times (\boldsymbol{U} \times \boldsymbol{\Omega}) \tag{9}$$

of ordinary fluid dynamics, for which the corresponding result is that vortex lines evolve as material lines (Batchelor, 1967). Thus, just as vortex lines may be stretched and thereby amplified, so too may magnetic field lines. (The analogy between $\boldsymbol{B}$ and $\boldsymbol{\Omega}$ should not be pushed too far, however; $\boldsymbol{B}$ and $\boldsymbol{U}$ are independent quantities in Eq. (8), whereas $\boldsymbol{\Omega} = \nabla \times \boldsymbol{U}$ in Eq. (9).)

Of course, in the presence of the $\nabla^2 \boldsymbol{B}$ term, magnetic field lines will not evolve exactly as material lines; there will merely be a tendency for them to do so. The presence of this diffusive term also raises another point: the magnitude of the flow must be sufficiently large that it is stretching and thereby amplifying the field more rapidly than the diffusion is damping it. Backus (1958), Childress (1969a), and Proctor (1977b) have derived rigorous lower bounds on the magnitude of the flow necessary (but not sufficient) for dynamo action.

Considered next from the dynamic point of view, we focus on the effect of the $(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$ term in Eq. (2). One can show that this Lorentz force can be decomposed into a magnetic pressure and a magnetic tension. Again, the details may be found in any textbook on magnetohydrodynamics. As we have taken the core fluid to be incompressible, the magnetic pressure (indeed, the total pressure) has no effect on the flow. It should be noted, for example, how the pressure is completely eliminated by taking the curl of Eq. (23) (below). The magnetic tension, however, has a very considerable effect. The tension in the field lines resists this purely kinematic stretching that we have just discussed. It is precisely by doing work against this elastic tension that the flow pumps energy into the field. Fig. 1 shows schematically how a flow will tend to stretch a field line, and how the magnetic tension in the field line will tend to resist further stretching.

To demonstrate that this work done by the flow really does go into the field, we derive the energy balance associated with Eq. (1) and Eq. (2). Taking the dot product of Eq. (1) with $\boldsymbol{B}$, and of Eq. (2) with $\boldsymbol{U}$ and adding, one obtains after a few vector identities,

$$\frac{1}{2}\frac{\partial}{\partial t}(|\boldsymbol{B}|^2+\mathrm{Ro}|\boldsymbol{U}|^2)=\mathrm{q}\widetilde{\mathrm{Ra}}\Theta(\boldsymbol{U}\cdot\boldsymbol{r})+\nabla\cdot[(\boldsymbol{U}\times\boldsymbol{B})\times\boldsymbol{B}]+\nabla\cdot[\boldsymbol{B}\times(\nabla\times\boldsymbol{B})]-|\nabla\times\boldsymbol{B}|^2 \\ +\mathrm{E}\nabla\cdot[\boldsymbol{U}\times(\nabla\times\boldsymbol{U})]-\mathrm{E}|\nabla\times\boldsymbol{U}|^2-\nabla\cdot\left[\mathrm{Ro}\boldsymbol{U}|\boldsymbol{U}|^2/2+\boldsymbol{U}p\right] \tag{10}$$

The two terms we have been considering, the $\nabla\times(\boldsymbol{U}\times\boldsymbol{B})$ term in Eq. (1) and the $(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$ term in Eq. (2), thus combine to give the $\nabla\cdot[(\boldsymbol{U}\times\boldsymbol{B})\times\boldsymbol{B}]$ term in Eq. (10). And the point now is that, by virtue of the no-slip boundary condition on $\boldsymbol{U}$, this term vanishes when integrated over the entire volume of the core. That is, in doing work against the Lorentz force (the $\boldsymbol{U}\cdot[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}]$ term), one really does convert this mechanical energy into magnetic energy (the $\boldsymbol{B}\cdot[\nabla\times(\boldsymbol{U}\times\boldsymbol{B})]$ term), and thereby maintains the field against dissipation. That the interaction of the flow with the Lorentz force merely transfers energy between different forms, but has no effect on the global energy balance, has been noted previously by Childress (1969b).

When integrated over all of space, the final energy balance one then obtains is

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V(|\boldsymbol{B}|^2+\mathrm{Ro}|\boldsymbol{U}|^2)\mathrm{d}V=\mathrm{q}\widetilde{\mathrm{Ra}}\int_V\Theta(\boldsymbol{U}\cdot\boldsymbol{r})\mathrm{d}V-\int_V|\nabla\times\boldsymbol{B}|^2\mathrm{d}V-E\int_V|\nabla\times\boldsymbol{U}|^2\mathrm{d}V \tag{11}$$

The integration is over all space to include the magnetic energy in the external field. One can show, using the appropriate boundary conditions on $\boldsymbol{B}$, that the $\nabla\cdot[\boldsymbol{B}\times(\nabla\times\boldsymbol{B})]$ divergence term exactly accounts for this. Of

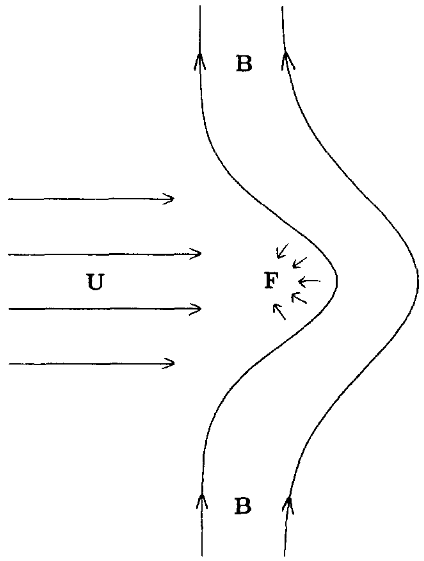

Fig. 1. A sketch of how the flow tends to stretch the field lines, and how the elastic tension in the field lines, indicated by F, tends to resist the stretching.

course, all other quantities only contribute within the volume of the core, and all other divergence terms contribute nothing, using the appropriate boundary conditions on $\boldsymbol{U}$.

This energy balance indicates that one must have some sort of buoyancy force for the dynamo to overcome its ohmic and viscous losses, and that the Rayleigh number $\tilde{\text{Ra}}$ is indeed an appropriate measure of this forcing. Also, for a dynamo driven by buoyancy forcing, the flow must have a radial component for the buoyancy force to be able to do work on it. Elsasser (1946) has shown more generally that any flow, even a kinematically prescribed one, must have a radial component if it is to act as a dynamo. Busse (1975a) has also derived a lower bound on this radial component necessary for dynamo action. Finally, it should be noted that, if $\boldsymbol{B}$ and $\boldsymbol{U}$ do indeed equilibrate at $O(1)$, the magnetic energy will be many times the kinetic energy, because $\text{Ro} \ll 1$. The geodynamo is thus not characterized by an equipartition of energy.

Having considered how magnetic fields may be amplified in general, by stretching field lines and thereby doing work against the magnetic tension, we now return to the details of how the geodynamo is believed to operate. As noted above, it is the dominance of rotation in the momentum equation, as measured by the extreme smallness of the Rossby and Ekman numbers, that makes the geodynamo equations so difficult. It is worth considering this dominance of rotation in a little more detail. In the absence of the Lorentz and buoyancy forces, the dominant balance is then the so-called geostrophic balance

$$2\hat{\boldsymbol{k}} \times \boldsymbol{U} \approx -\nabla p \tag{12}$$

Taking the curl of this equation yields the Taylor–Proudman theorem

$$\frac{\partial}{\partial z}\boldsymbol{U} \approx 0 \tag{13}$$

stating that the motion must be essentially independent of z, the coordinate parallel to the axis of rotation. In spherical geometry, the only flow that can satisfy this theorem is a purely zonal flow $U_\phi(s)$, where (z,s,ϕ) are cylindrical coordinates. Below, we will discover again and again that this geostrophic flow plays a crucial role in the dynamics of the geodynamo. However, because the purely radial buoyancy force can do no work on this purely zonal geostrophic flow, it can play no role in actually driving the geodynamo.

Therefore convection must occur as an ageostrophic flow, that is, one that breaks the constraints of the Taylor–Proudman theorem. It turns out that the buoyancy force by itself cannot break this constraint, whereas the Lorentz force can. This will turn out to have profound implications for the structure of the convection in the nonmagnetic and magnetic regimes. It is this sequence of bifurcations leading from the nonconvecting, nonmagnetic state to the convecting, magnetic state, that we consider next.

We imagine gradually increasing the Rayleigh number, and consider the sequence of bifurcations that will occur. For some sufficiently large $\tilde{\text{Ra}}_c$, the conducting state will become unstable, and will yield to a (still nonmagnetic) convecting state. As the buoyancy force by itself cannot break the rotational constraint, it is the viscous force that does. However, because the viscous force is very small, it can only do so if it acts on very short lengthscales. The result is that the azimuthal wavenumber of the convection cells is $O(\text{E}^{-1/3})$, and this critical Rayleigh number $\tilde{\text{Ra}}_c$ is then also $O(\text{E}^{-1/3})$. That is, the Rayleigh number must be very large before one can break the rotational constraint and obtain a convecting state. This onset of convection was first investigated asymptotically by Roberts (1968), Busse (1970), and Soward (1977), and much later numerically by Zhang (1992). Incidentally, once again one should be aware that until one is in the strong field regime discussed below, the details of the appropriate nondimensionalization are different from that presented here. For example, until one actually has a reasonably strong field, it makes no sense to scale time by the magnetic diffusion time.

If one continues to increase the Rayleigh number, the convection will become more and more vigorous, until at some $\tilde{\text{Ra}}_w$ it is sufficiently vigorous that it will begin to generate a magnetic field, by the stretching mechanism indicated above. Beyond $\tilde{\text{Ra}}_w$ the field will then equilibrate at some finite amplitude. If one is just slightly beyond $\tilde{\text{Ra}}_w$, the field will equilibrate at some very small amplitude, assuming for the sake of argument that this initial bifurcation at $\tilde{Ra}_w$ is supercritical. Therefore, the convection pattern will remain much the same, and one can consider the effect of the field by a weakly nonlinear perturbation analysis. This weak field regime has been considered asymptotically by Busse (1975b) and Soward (1979), and numerically by Cuong and Busse (1981).

However, if one continues to increase the Rayleigh number, and therefore presumably increases the amplitude at which the field equilibrates, it has been conjectured by Roberts (1978) that eventually something rather dramatic will happen: eventually the field will become sufficiently large that it is the Lorentz force, rather than the viscous force, which breaks the rotational constraint. But then the lengthscale of the convection cells can increase somewhat. The resulting reduction in dissipation means the field becomes even larger, which further breaks the rotational constraint, so the lengthscale of the convection cells can increase some more, and so on. The resulting runaway growth of the field will continue until the convection cells fill the entire volume of the core, at which point the field will equilibrate at some $O(1)$ value. However, if the lengthscale of the convection cells increases, the critical Rayleigh number should also decrease. That is, once one is in this strong field regime, one should be able to reduce the Rayleigh number back to some $O(1)$ value $\tilde{Ra}_s$, and still maintain the convecting, magnetic state. The presence of an $O(1)$ field can actually facilitate convection sufficiently that one can reduce the Rayleigh number from $O(E^{-1/3})$ to $O(1)$. Indeed, Malkus (1959) suggested that the Earth generates its field precisely in order to facilitate the convection, and that the $O(1)$ amplitude of the field is precisely that amplitude which most facilitates the convection.

Fig. 2 shows a sketch of this postulated bifurcation diagram, showing the weak and strong field regimes. Again, in the weak field regime, the Lorentz force is considerably smaller than the viscous force, and so the convection occurs on small lengthscales, whereas in the strong field regime, the Lorentz force is considerably larger than the viscous force, and so the convection occurs on large lengthscales. Also, because it is easier to maintain convection on large than on small lengthscales, the strong field regime should exist for considerably smaller Rayleigh numbers than the weak field regime. The only difficulty is that one cannot attain the strong field regime by any stable sequence of bifurcations from the nonconvecting, nonmagnetic state; one must go through this runaway growth of the weak field regime instead. Needless to say, the geodynamo is believed to be in the strong field regime, and it is the impossibility of achieving this in a smooth, stable sequence of bifurcations that makes the analysis so difficult.

The initial motivation for postulating this complicated bifurcation sequence came from various studies of magnetoconvection, that is, convection in the presence of an imposed field. It was found there that if the

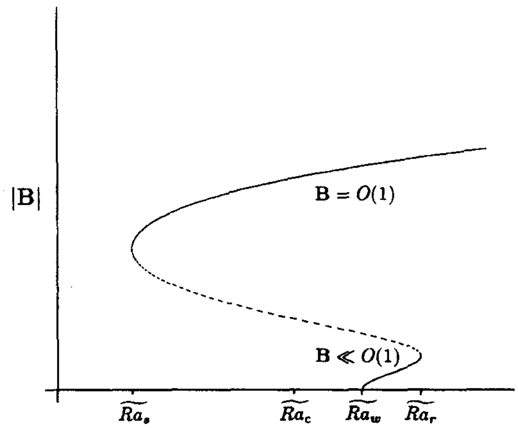

Fig. 2. Following Roberts (1978), a sketch of the conjectured bifurcation diagram, as discussed in the text. Plotted is some arbitrary measure of the field amplitude vs. the Rayleigh number $\widetilde{Ra}$. The weak field regime begins at $\widetilde{Ra} = \widetilde{Ra}_w = O(E^{-1/3})$, and ends at $\widetilde{Ra}_r$, at which point the runaway growth to the strong field regime occurs. Once established, the strong field regime exists for all $\widetilde{Ra} \geq \widetilde{Ra}_s = O(1)$.

imposed field was too weak convection occurred on small lengthscales, and only for $O(E^{-1/3})$ Rayleigh numbers, whereas if the imposed field was sufficiently strong convection occurred on large lengthscales, and for $O(1)$ Rayleigh numbers. If one then assumes that the same general pattern of convection will occur for a self-generated as for an imposed field, one obtains the postulated bifurcation diagram. These magnetoconvection studies include work by Eltayeb (1972) in plane layer geometry, and by Fearn (1979) in spherical geometry. Magnetoconvection will not be considered further here, but the reader is referred to the comprehensive review by Proctor (1994).

In the final section of this paper we will return to a discussion of two convective dynamo models. We will see that there is indeed evidence for separate weak and strong field regimes, even for self-generated rather than imposed fields. First, however, we will describe the sequence of much simpler models leading up to fully self-consistent convective models. It will turn out that even these 'simpler' models contain some rather complicated dynamics, which one must understand before one can even hope to understand the convective models.

2. Kinematic dynamos

We will begin with a discussion of various kinematic dynamo models, that is, models of the induction equation alone, with the fluid flow arbitrarily prescribed. Actually, we will begin with a kinematic anti-dynamo theorem, that of Cowling (1934): no fluid flow, however cleverly prescribed, can maintain a purely axisymmetric field. Cowling's theorem is sufficiently important that we will give a proof of it (although not Cowling's original proof, which only applies to steady fields), and in the process introduce some of the notation of the subsequent development. ?

We begin by noting that, without loss of generality, we may also take the flow to be purely axisymmetric, as any flow that had nonaxisymmetric components would immediately induce nonaxisymmetric components in the field as well. Then, if $\boldsymbol{U}$ and $\boldsymbol{B}$ are both to be purely axisymmetric, we may use $\nabla\cdot\boldsymbol{U}=\nabla\cdot\boldsymbol{B}=0$ to decompose them as

$$\boldsymbol{U}=\nabla\times(\psi\hat{\boldsymbol{e}}_\phi)+v\hat{\boldsymbol{e}}_\phi \qquad (14a)$$

$$\boldsymbol{B}=\nabla\times(A\hat{\boldsymbol{e}}_\phi)+B\hat{\boldsymbol{e}}_\phi \qquad (14b)$$

It should be noted that this decomposition is valid only for purely axisymmetric quantities; we will consider the more generally valid decomposition shortly.

Having decomposed $\boldsymbol{U}$ and $\boldsymbol{B}$ as in Eq. (14a) and Eq. (14b), the induction equation then becomes

$$\frac{\partial}{\partial t}A=D^2A+N(\psi,A) \qquad (15a)$$

$$\frac{\partial}{\partial t}B=D^2B+M(v,A)-M(B,\psi) \qquad (15b)$$

where in the notation of Proctor (1977c)

$$D^2=\nabla^2-1/s^2 \qquad (16a)$$

$$N(X,Y)=\hat{\boldsymbol{e}}_\phi\cdot\left[\nabla\times(X\hat{\boldsymbol{e}}_\phi)\times\nabla\times(Y\hat{\boldsymbol{e}}_\phi)\right] \qquad (16b)$$

$$M(X,Y)=\hat{\boldsymbol{e}}_\phi\cdot\nabla\times\left[X\hat{\boldsymbol{e}}_\phi\times\nabla\times(Y\hat{\boldsymbol{e}}_\phi)\right] \qquad (16c)$$

Here (z,s,ϕ) are again cylindrical coordinates. This might also be an appropriate place to comment briefly on the associated boundary conditions. For $\boldsymbol{U}$ we certainly want to impose the no normal flow boundary condition

$\psi = 0$, but for the kinematically prescribed flows considered here no further conditions need be imposed. For $\boldsymbol{B}$, matching to an external potential field ends up as $D^2A = 0$ and $B = 0$ in the insulating mantle. The toroidal field $B\hat{\boldsymbol{e}}_\phi$ is thus not observable at the surface of the Earth, only the poloidal field $\nabla \times (A\hat{\boldsymbol{e}}_\phi)$ is.

To demonstrate that this purely axisymmetric field cannot be maintained, we begin by multiplying Eq. (15a) by s^2A, rewriting it as

$$\frac{1}{2}\frac{\partial}{\partial t}s^2A^2 = -|\nabla(sA)|^2 + \nabla\cdot[s^2A\nabla A] - \nabla\cdot\left[s^2A^2/2\nabla\times\left(\psi\hat{\boldsymbol{e}}_\phi\right)\right] \quad (17a)$$

and integrating over the volume of the core. The divergence term involving ψ then vanishes by virtue of the boundary condition on ψ. Matching to $D^2A = 0$ in the exterior, the $\nabla\cdot[s^2A\nabla A]$ term can be shown to be negative definite. Again, this term merely accounts for the fact that there is also an external field. Thus, we have demonstrated that A must necessarily decay.

With A thus eliminated, we proceed by multiplying Eq. (15b) by $s^{-2}B$, rewriting it as

$$\frac{1}{2}\frac{\partial}{\partial t}s^{-2}B^2 = -|\nabla(s^{-1}B)|^2 + \nabla\cdot[s^{-2}B\nabla B] - \nabla\cdot\left[s^{-2}B^2/2\nabla\times\left(\psi\hat{\boldsymbol{e}}_\phi\right)\right] \quad (17b)$$

and integrating over the volume of the core. The divergence term involving ψ again vanishes by virtue of the boundary condition on ψ. Matching to $B = 0$ in the exterior, the $\nabla\cdot[s^{-2}B\nabla B]$ term is then also zero. Thus, we have demonstrated that B too must necessarily decay.

The virtue of this particular proof, by Braginsky (1965), is not only that it applies to nonsteady fields as well, but that it isolates very clearly where the difficulty lies: in the lack of any purely axisymmetric mechanism of regenerating the poloidal field in Eq. (15a). The meridional circulation $\nabla\times(\psi\hat{\boldsymbol{e}}_\phi)$ affects both A and B, as indicated in Eq. (15a, b), but does not directly amplify either, as indicated in Eq. (17a, b). There is thus no way of preventing first the poloidal and then the toroidal field from decaying. In the next section we will introduce a mechanism of regenerating the poloidal field, in terms of a parameterization of various nonaxisymmetric effects, and thereby fix this particular difficulty.

Since Cowling's original paper, there has been considerable interest in strengthening and extending his theorem. (See, e.g. James et al. (1980), Hide and Palmer (1982), Ivers and James (1984), or Fearn et al. (1988) for some more of these results.) That Cowling's theorem excludes purely axisymmetric dynamos is perhaps not particularly surprising. Remembering again the analogy to the vorticity Eq. (9), for which vortex stretching is also an inherently three-dimensional process, one might almost have anticipated such a result. It is, however, unfortunate, as it greatly increases the complexity of even the simplest models.

Armed with the knowledge that purely axisymmetric dynamos must necessarily fail, Bullard and Gellman (1954) considered the more generally valid decomposition

$$\boldsymbol{U} = \nabla\times(T'\hat{\boldsymbol{r}}) + \nabla\times\nabla\times(S'\hat{\boldsymbol{r}}) \quad (18a)$$

$$\boldsymbol{B} = \nabla\times(T\hat{\boldsymbol{r}}) + \nabla\times\nabla\times(S\hat{\boldsymbol{r}}) \quad (18b)$$

They prescribed some (three-dimensional) structure for the toroidal and poloidal components T' and S' of the flow, and numerically attempted to solve the induction equation for the components T and S of the field. For appropriate choices of T' and S', they claimed to obtain dynamo action, that is, exponentially growing solutions for T and S.

However, subsequent analysis by Lilley (1970) and Gubbins (1973) demonstrated that this supposed dynamo action was an artifact of insufficient resolution. Since then, a number of workers have reconsidered numerical solutions of the kinematic dynamo problem (see, e.g. Dudley and James (1989) for some of these results). With modern computers the truncation problem is not as severe as it was for Bullard and Gellman working in the early 1950s, and the results of Dudley and James working in the late 1980s are fully resolved.

These later numerical results thus demonstrate conclusively that kinematic dynamo action is possible. It is nevertheless of interest to consider how Backus (1958) and Herzenberg (1958) overcame the truncation problem analytically, and thereby provided the first rigorous proofs that dynamo action is possible. The difficulty faced by Bullard and Gellman was that the harmonics of the field had to be truncated at some (fairly low) level, but there was no proof that the neglect of the higher harmonics was justified. Backus (1958) and Herzenberg (1958) were able to devise cleverly chosen flows for which this neglect was rigorously justifiable. Both flows consist of two parts, neither of which acts as a dynamo on its own. The dynamo action occurs when the two parts each act upon the field created by the other.

The proof that the neglect of the higher harmonics is justified is then accomplished in the following ways. Backus separates his two parts of the flow in time, with a stationary time between them. Although the field generated by one part is then rather complicated, because the higher harmonics decay more rapidly in time, the field acted upon by the other part is then very simple. In contrast, Herzenberg separates his two parts of the flow in space, with a stationary space between them. And again, although the field generated by one part is then rather complicated, because the higher harmonics also decay more rapidly in space, the field acted upon by the other part is again very simple. Thus, in both cases, one can rigorously prove that the higher harmonics of the output field of each part are negligible, because they decay away (in either time or space) before this field is used as the input field of the other part. One can then demonstrate that the effect of both parts together is simply the amplification of the lower harmonics of the fields, thereby proving dynamo action.

In view of the later numerical results, it is easy to underestimate the significance of these proofs by Backus and Herzenberg. However, at the time they were enormously important. Following the publication of Cowling's theorem, it was widely believed to be only a matter of time before a more general theorem was discovered which would disallow not only two-dimensional but three-dimensional dynamo action as well. It was thus only through these counter-examples by Backus and Herzenberg that dynamo theory was firmly established.

3. Mean-field dynamos

Having demonstrated that dynamo action is possible in principle, we next turn to a discussion of various mean-field models. As the name implies, mean-field theory is concerned with purely axisymmetric flows and fields; Cowling's theorem is circumvented by parameterizing the effects of the nonaxisymmetric flows and fields into a so-called α-effect. In a sense, mean-field theory is thus a step backward, since the details of the field generation are no longer being fully included, as they were in the kinematic models discussed in the previous section. However, precisely because mean-field models are thus simplified, it is possible to include more of the other dynamics not included in the kinematic models. In particular, it is in elucidating some of the dynamics associated with the momentum equation that mean-field theory has really proved its worth.

To derive the mean-field induction equation, one begins by explicitly separating $\boldsymbol{U}$ and $\boldsymbol{B}$ into their axisymmetric and nonaxisymmetric components, $\boldsymbol{U}=\overline{\boldsymbol{U}}+\boldsymbol{U}'$ and $\boldsymbol{B}=\overline{\boldsymbol{B}}+\boldsymbol{B}'$. The azimuthally averaged induction equation then becomes

$$\frac{\partial}{\partial t}\overline{\boldsymbol{B}}=\nabla^2\overline{\boldsymbol{B}}+\nabla\times(\overline{\boldsymbol{U}}\times\overline{\boldsymbol{B}})+\nabla\times(\overline{\boldsymbol{U}'\times\boldsymbol{B}'}) \quad (19)$$

Under suitable statistical assumptions, in particular that the small-scale turbulence is not mirror symmetric, that is, not symmetric under the reflection $s\to -s$, Steenbeck et al. (1966) showed that the $\overline{\boldsymbol{U}'\times\boldsymbol{B}'}$ term may be replaced by $\alpha\overline{\boldsymbol{B}}$, where α is just some scalar parameter. The lack of mirror symmetry is crucial, but because of the Coriolis force the small-scale turbulence will indeed lack mirror symmetry. Also, because the vertical component of the Coriolis force (the component most important for buoyancy-driven turbulence) has the opposite sign in the northern and southern hemispheres, so does α. Except for this equatorial antisymmetry, though, the spatial structure of α is not determined by the theory; one simply prescribes various forms.

With this replacement of $\overline{U' \times B'}$ by $\alpha \overline{B}$, one then obtains

$$\frac{\partial}{\partial t} B = \nabla^2 B + \nabla \times (\alpha B) + \nabla \times (U \times B) \tag{20}$$

where we have dropped the overbars on the understanding that U and B now refer to the mean components only. (For a more detailed derivation of this mean-field induction equation, see Moffatt (1978) or Krause and Rädler (1980).)

Again decomposing U and B as in Eq. (14a) and Eq. (14b), the mean-field induction equation then becomes

$$\frac{\partial}{\partial t} A = D^2 A + \alpha B + N(\psi, A) \tag{21a}$$

$$\frac{\partial}{\partial t} B = D^2 B + \hat{e}_\phi \cdot \nabla \times \left[\alpha \nabla \times \left(A \hat{e}_\phi \right) \right] + M(v, A) - M(B, \psi) \tag{21b}$$

One may now proceed as before with kinematic dynamo theory: one prescribes some structure for α, and sees if one obtains dynamo action. In fact, one need not prescribe any flow U at all; one can drive the dynamo entirely by the small-scale flow implicitly given by α, rather than by the large-scale flow explicitly given by U. Dynamos of this type are known as α^2-dynamos, in recognition of the fact that they are driven by the product of the two α terms in Eq. (21a) Eq. (21b).

Alternatively, one may prescribe some relatively strong differential rotation $v = R_\omega s \omega$ as well, where R_ω is a measure of the strength. It then turns out one needs only a relatively weak α-effect, $R_\omega^{-1} \alpha$. Rescaling as $A = R_\omega^{-1/2} A'$, $B = R_\omega^{1/2} B'$, for large R_ω one then obtains

$$\frac{\partial}{\partial t} A' = D^2 A' + \alpha B' + N(\psi, A') \tag{22a}$$

$$\frac{\partial}{\partial t} B' = D^2 B' + M(s\omega, A') - M(B', \psi) \tag{22b}$$

Dynamos of this type are known as $\alpha\omega$-dynamos, in recognition of the fact that they are driven by the product of the α term in Eq. (22a) and the ω term in Eq. (22b). Incidentally, the $\alpha\omega$-dynamo is derivable not only from the turbulence theory mentioned above, but also from Braginsky's nearly axially symmetric theory (Braginsky, 1965).

In view of the ubiquity of these α and ω effects in the literature, it is perhaps worth giving a qualitative idea of the underlying physical effects they are intended to model. The regeneration of poloidal from toroidal field via the α-effect, as in Eq. (21a) and Eq. (22a), is illustrated in Fig. 3(a). One starts with a toroidal field, and acts upon it with some small-scale helical flow. As in Fig. 1, this flow will tend to stretch the original field as indicated, thereby inducing a poloidal field. At this point this poloidal field is still small scale, as it was created by a small-scale flow, but averaging over many such events gives a large-scale field. It should be noted also that it is in this averaging that the lack of mirror symmetry of the small-scale flow is essential: if this helical flow in Fig. 3(a) could equally well have been clockwise, averaging over many such events would not yield a large-scale field. However, if the small-scale turbulence does have such a 'handedness', then one can quite plausibly regenerate the large-scale poloidal field in this manner. In the same way, one can of course also regenerate the toroidal field from the poloidal field, as in Eq. (21b). However, it is at this point that the ω-effect allows us to regenerate the large-scale toroidal field without any recourse to the small-scale turbulence. The regeneration of toroidal from poloidal field via the ω-effect, as in Eq. (22b), is illustrated in Fig. 3(b). One starts with a poloidal field, and acts upon it with some large-scale zonal flow. Again as in Fig. 1, this flow will tend to stretch the original field as indicated, thereby inducing a toroidal field, but of course this time this toroidal field is already large scale, as it was created by a large-scale flow. In this qualitative form, these α and ω effects were first introduced by Parker (1955).

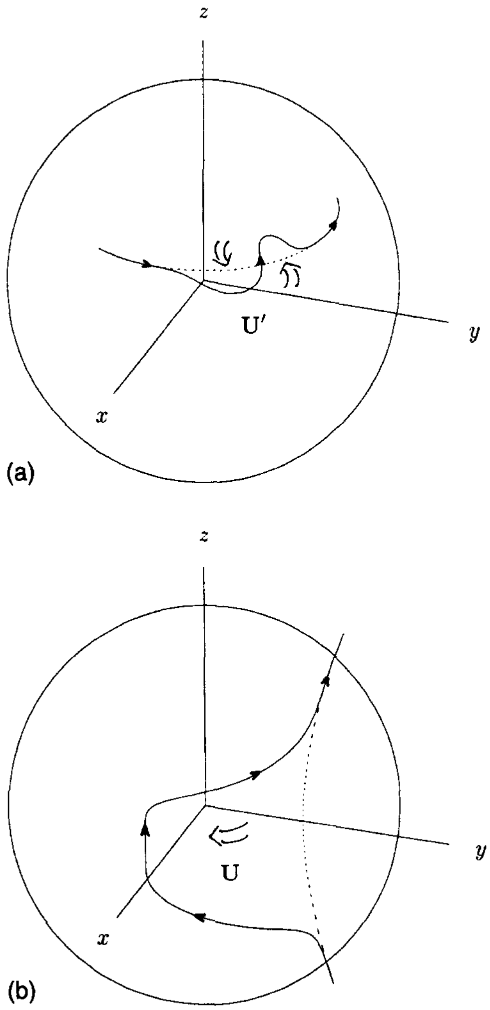

Fig. 3. A qualitative illustration of the α and ω effects: (a) shows the generation of poloidal from toroidal field via a small-scale helical flow U' (the α-effect); (b) shows the generation of toroidal from poloidal field via a large-scale zonal flow U (the ω-effect). In both cases the initial undistorted field is shown as the dashed line, and the final distorted field, including the induced field, as the continuous line. It should be noted that both (a) and (b) are essentially just the general stretching concept already illustrated in Fig. 1, merely applied to specific flows stretching specific initial fields.

Returning to our two types of dynamos Eqs. (21a, b) and (22a, b), we note first that in the α^2-dynamos the toroidal and poloidal components are of the same magnitude, whereas in the $\alpha\omega$-dynamos the toroidal component is greater than the poloidal component by this factor R_ω. As a result, α^2 and $\alpha\omega$ models are sometimes referred to as weak and strong field models, respectively. However, to avoid confusion with the weak and strong field regimes discussed in the Introduction, we will not follow that usage here. Remembering again that the toroidal component $B\hat{e}_\phi$ is not observable at the surface of the Earth, one cannot decide on the basis of observations alone whether α^2 or $\alpha\omega$ models are more appropriate to the geodynamo. Nevertheless, because of the relative ease with which a strong differential rotation may be excited, it is believed that the geodynamo is probably more $\alpha\omega$ than α^2 in character.

This is unfortunate, as α^2-dynamos are really much better behaved than $\alpha\omega$-dynamos, as we shall see. Roberts (1972) has investigated kinematic models of both types, and found that α^2 models are typically steady state, whereas $\alpha\omega$ models are typically oscillatory. It should be emphasized, though, that this is only the typical behaviour; one can choose models which do not yield this behaviour. However, considering that α is only a very crude parameterization of effects not otherwise included in these models, one should probably be cautious about accepting non-typical results, which depend crucially on some particular choice of spatial structure of α.

Considering that the Earth's field is known to reverse occasionally, one might then think that the oscillatory $\alpha\omega$-dynamos would indeed be better models than the steady-state α^2-dynamos. That, however, is not necessarily the case: although the α^2-dynamos never reverse, the $\alpha\omega$-dynamos reverse far too often. As we shall see below, the difficulty with reversals is not so much to obtain them, but rather trying not to obtain them too often.

Another feature of these kinematic models that is worth mentioning is the parity of the solutions. For the geophysically relevant symmetries for α and ω, namely α antisymmetric and ω symmetric about the equator, the solutions to both types of models decouple into distinct dipolar (with A symmetric and B antisymmetric) and quadrupolar (with A antisymmetric and B symmetric) solutions. (See, e.g. Gubbins and Zhang (1993) for a general discussion of the symmetry properties of the geodynamo equations.) Roberts (1972) found that the two parities are typically excited about equally easily, a finding reinforced by Proctor (1977a), who showed that this is to be expected as a general result. The rather strong preference for dipole symmetry observed in the Earth's field is thus another feature for which geodynamo theory does not yet have a fully satisfactory answer. This is particularly true in view of the results of Hollerbach (1991), who showed that there does not appear to be any particular preference for dipole over quadrupole solutions even in the nonlinear regime, to which we turn next.

As was noted above, it is in addressing some of the dynamics in this nonlinear regime that mean-field theory has made its greatest contribution to geodynamo theory. The linear, kinematic models we have been discussing so far are really just a foundation upon which to base the nonlinear, dynamic models. The dynamics we wish to address concerns the extension of the geostrophic balance Eq. (12) to include the Lorentz and buoyancy forces. One then obtains the so-called magnetostrophic balance

$$2\hat{\boldsymbol{k}}\times\boldsymbol{U} = -\nabla p + (\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \mathrm{q}\widetilde{\mathrm{Ra}}\Theta\boldsymbol{r} \tag{23}$$

Again taking the curl of this equation, one obtains

$$-2\frac{\partial}{\partial z}\boldsymbol{U} = \nabla\times\left[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\right] + \mathrm{q}\widetilde{\mathrm{Ra}}\nabla\times(\Theta\boldsymbol{r}) \tag{24}$$

and so one might suppose that the solution is simply

$$\boldsymbol{U} = \boldsymbol{U}_{\mathrm{M}} + \boldsymbol{U}_{\mathrm{T}} \tag{25}$$

where

$$\boldsymbol{U}_{\mathrm{M}} = -\frac{1}{2}\int^{z}\nabla\times\left[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\right]\mathrm{d}z' \tag{26}$$

$$\boldsymbol{U}_{\mathrm{T}} = -\frac{1}{2}\int^{z}\mathrm{q}\widetilde{\mathrm{Ra}}\nabla\times(\Theta\boldsymbol{r})\mathrm{d}z' \tag{27}$$

and we are temporarily neglecting the constant of integration in Eq. (25). (See also Hollerbach and Proctor (1993) for a discussion of this constant of integration.) Incidentally, it is this thermal wind $\boldsymbol{U}_{\mathrm{T}}$ that corresponds to the kinematically prescribed ω-effect introduced above; according to Eq. (27) prescribing a buoyancy force is equivalent to prescribing a differential rotation.

In fact, this magnetostrophic Eq. (23) may have no solution at all. Taylor (1963) showed that for a solution to exist, the Lorentz torque integrated on the geostrophic contours $C(s)$ must vanish,

$$\int_{C(s)}\left[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\right]_{\phi}\mathrm{d}S = 0 \tag{28}$$

for each such contour. Fig. 4 shows these geostrophic contours $C(s)$, concentric cylindrical shells parallel to the axis of rotation. It should be noted also that because these contours are axisymmetric, Taylor's constraint is

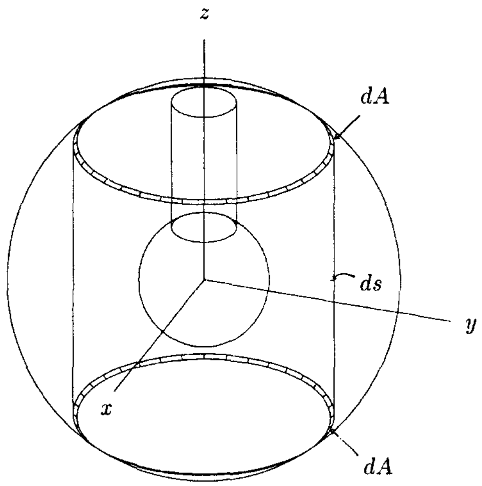

Fig. 4. The geostrophic contours $C(s)$ in Eq. (28). (Note how these contours change abruptly across the inner core tangent cylinder.) Also indicated is the relationship Eq. (31) between the area dA of the Ekman layers and the infinitesimal thickness ds of the cylindrical shell.

inherently axisymmetric, and is thus ideally suited to an analysis within the context of mean-field theory. All of the considerations discussed here will also apply to the strong field regime of the fully three-dimensional convective dynamo models discussed in the next section. They will not, of course, apply to the weak field regime, as we already know that in that regime the Lorentz force is considerably weaker than the viscous force, and so it would make no sense even to attempt to neglect the viscous force compared with the Lorentz force. In the weak field regime we already know viscosity is essential. In contrast, in the strong field regime we believe it is not essential, and we are trying to discover what that might imply. And as we have just seen, it implies Eq. (28).

From the geophysical point of view, Taylor's constraint can be thought of as a torque balance, as noted below. However, from the mathematical point of view, it can also be thought of as a solvability condition associated with the existence of the homogeneous solution $U_\phi(s)$, as noted by Fearn and Proctor (1992). Indeed, even if Taylor's constraint is satisfied, so that a solution exists, it is still only determined to within this arbitrary geostrophic flow $U_\phi(s)$, essentially the constant of integration that we conveniently neglected in Eq. (25). Taylor showed further that this geostrophic flow may be determined by requiring not only that the field satisfy Eq. (28) at some initial time, but that it also evolve so as to satisfy Eq. (28) at all later times. The 'arbitrary' geostrophic flow is thus precisely that flow which distorts the field in such a way that Taylor's constraint continues to be satisfied. However, despite an attempt by Fearn and Proctor (1987), no one has fully succeeded in following Taylor's prescription for evolving the field.

Instead, following Braginsky (1975), most work has proceeded more cautiously, not adopting the magnetostrophic balance in the first place. The magnetostrophic equation, one recalls, only comes about if one sets the Rossby and Ekman numbers identically equal to zero in the momentum equation. If instead one restores viscous effects, at least in the Ekman boundary layers, an asymptotic analysis of these boundary layers then yields

$$\int_{C(s)}[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}]_\phi dS = \mathrm{E}^{1/2}\frac{4\pi s}{(1-s^2)^{1/4}}U_\phi(s) \qquad (29)$$

instead of Eq. (28). Incidentally, one should note that we are here restricting ourselves to a full sphere. Below, we will consider some of the complications introduced by having a spherical shell, with an inner core.

From the geophysical point of view, Eq. (28) and Eq. (29) are rather similar: both come about from considering the torque balance on geostrophic contours. Although the Coriolis force is one of the dominant terms in the momentum equation at any individual point, it turns out that when integrated on these geostrophic contours, its contribution to the torque balance vanishes identically. And of course the pressure-gradient and buoyancy forces also contribute nothing. Therefore, if the only other force one includes is the Lorentz force, one obtains Eq. (28), stating that the Lorentz torque also has to vanish identically, whereas if one also includes some small viscous force in the boundary layers, one obtains Eq. (29), stating that the Lorentz torque does not have to vanish identically, but may be balanced by this small viscous torque.

This torque balance on the geostrophic contours is sufficiently important that we want to consider the contributions from the various terms in a little more detail. We begin by verifying that the Coriolis force contributes nothing: its ϕ-component is just $2U_s$, and so integrating on $C(s)$ yields the net outflow through that particular cylindrical shell. However, because we have taken the core fluid to be incompressible, that net outflow must vanish identically; the fluid cannot 'pile up' either inside or outside any given shell.

Turning next to the viscous force, in the asymptotic limit the viscous drag per unit area of the Ekman layer is $\mathrm{E}\Delta U_\phi/\delta$, where δ is the layer thickness, and $\Delta U_\phi = 0 - U_\phi$ is the jump in the zonal flow across it, from zero inside to U_ϕ outside. The viscous torque is thus

$$-\mathrm{E}U_\phi \delta^{-1} s\,\mathrm{d}A \tag{30}$$

were $\mathrm{d}A$ is the area of the Ekman layers indicated in Fig. 4, and may be related by

$$\mathrm{d}A = 2 \cdot 2\pi s \frac{\mathrm{d}s}{(1-s^2)^{1/2}} \tag{31}$$

to the infinitesimal thickness $\mathrm{d}s$ of the cylindrical shell in Fig. 4. (At this point it is convenient to take a shell of nonzero rather than zero thickness. In a moment we will simply let $\mathrm{d}s \to 0$.) Finally, we apply the standard result (Batchelor, 1967) that the thickness of the Ekman layer on a spherical boundary is $\delta = \mathrm{E}^{1/2}/(1-s^2)^{1/4}$ to obtain

$$-\mathrm{E}^{1/2} \frac{4\pi s}{(1-s^2)^{1/4}} U_\phi(s) s\,\mathrm{d}s \tag{32}$$

for the viscous torque on this shell. (See also Roberts and Soward (1992) for the details of this derivation.) And of course the Lorentz torque on this infinitesimally thick cylindrical shell is just the left-hand side of Eq. (28) or Eq. (29) times $s\mathrm{d}s$, so letting $\mathrm{d}s \to 0$ yields the desired results, Eq. (28) if the viscous torque is not included and Eq. (29) if it is.

Although Eq. (28) and Eq. (29) are thus indeed similar from the geophysical point of view, from the mathematical point of view they are very different: whereas Eq. (28) is a solvability condition, Eq. (29) is not, and whereas Eq. (28) does not directly determine $U_\phi(s)$, Eq. (29) does. That is, instead of following Taylor's implicit prescription for determining the geostrophic flow, Eq. (29) gives us the explicit prescription

$$U_\phi(s) = \mathrm{E}^{-1/2} \frac{(1-s^2)^{1/4}}{4\pi s} \int_{C(s)} [(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}]_\phi \mathrm{d}S \tag{33}$$

We will see shortly that this prescription is more general than Taylor's.

We now return to the linear, kinematic models Eqs. (21a, b) and (22a, b), and consider what happens if we include the geostrophic nonlinearity Eq. (33), and perhaps also the ageostrophic nonlinearity Eq. (26). If we imagine gradually increasing the dynamo number D, we will eventually exceed some critical dynamo number D_c, beyond which we get exponentially growing solutions. (The dynamo number D is just the product of the amplitudes of the two α terms in Eq. (21a) and Eq. (21b), or of the α and ω terms in Eq. (22a) and Eq. (22b). It is thus a measure of the strength of the kinematically prescribed forcing.)

As they are not yet influenced by the dynamics of the momentum equation, these kinematic eigensolutions will generally not satisfy the constraint Eq. (28). Therefore, according to Eq. (33), they will induce an $O(1)$

$U_\phi(s)$ when $\boldsymbol{B}$ is only $O(\mathrm{E}^{1/4})$. Also, as long as this geostrophic flow is capable of equilibrating the solution at this small amplitude, the ageostrophic flow is negligible. So it is to this equilibration through the geostrophic flow that we turn to next. (See also the reviews by Jones (1991) and Fearn (1994).) Taking the field to be purely axisymmetric, as is appropriate for mean-field models, Eq. (33) simplifies to

$$U_\phi(s) = -\mathrm{E}^{-1/2}\frac{z_\mathrm{T}^{1/2}}{2s^2}\frac{\mathrm{d}}{\mathrm{d}s}[s^2T] \tag{34}$$

where

$$T = \int_{-z_\mathrm{T}}^{+z_\mathrm{T}} B\frac{\partial A}{\partial z}\mathrm{d}z \tag{35}$$

and $z_\mathrm{T} \equiv (1-s^2)^{1/2}$. According to Eqs. (21a, b) and (22a, b), the only effect of this flow on the field is on the toroidal field via the term $M(U_\phi(s), A)$, which works out to be

$$\frac{\partial}{\partial t}B = -s\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{U_\phi(s)}{s}\right)\frac{\partial A}{\partial z} \tag{36}$$

Therefore, the effect of this flow on the energy in the toroidal field is then

$$\frac{1}{2}\frac{\partial}{\partial t}B^2 = -s\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{U_\phi(s)}{s}\right)B\frac{\partial A}{\partial z} \tag{37}$$

and integrated over the sphere, one obtains after integrating by parts,

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V B^2\mathrm{d}V = -\mathrm{E}^{-1/2}\int_0^1\frac{z_\mathrm{T}^{1/2}}{s^3}\left[\frac{\mathrm{d}}{\mathrm{d}s}(s^2T)\right]^2\pi\mathrm{d}s \tag{38}$$

The effect of the geostrophic flow on the toroidal energy is thus negative-semidefinite, as noted by Childress (1969b). The importance of preserving this property in any numerical integration has been independently pointed out by Barenghi and Jones (1991) and Hollerbach and Ierley (1991). Thus, as long as Taylor's constraint is not satisfied, the geostrophic flow will always equilibrate the solution, at this $O(\mathrm{E}^{1/4})$ amplitude, by limiting the growth of the toroidal field, and thereby indirectly also limiting the growth of the poloidal field.

However, if we continue increasing the dynamo number, we can imagine that the solutions, which are now influenced by the dynamics of the momentum equation, will evolve from this viscously limited state toward a Taylor state. Malkus and Proctor (1975) conjectured that the geostrophic flow would evolve toward precisely that flow required by Taylor's prescription, and that therefore one would eventually reach some second critical dynamo number D_T, beyond which Taylor's constraint is satisfied. The mechanism by which the field might be expected to evolve toward a Taylor state is essentially as follows. In the increasingly supercritical regime the purely kinematic terms will tend to amplify more and more different field structures. Those structures which most closely satisfy Taylor's constraint will be least affected by the geostrophic flow, and will therefore be amplified the most before ultimately being equilibrated. In a sense, one can think of a competition between different field structures, in which structures satisfying Taylor's constraint more closely win out over structures satisfying it less closely. As more and more structure becomes available, one can imagine reaching a point at which Taylor's constraint is completely satisfied. However, then the geostrophic flow can no longer limit the growth of the field at all. The field will continue to grow, from an $O(\mathrm{E}^{1/4})$ to an $O(1)$ amplitude, at which point the ageostrophic flow will limit the growth of the field. (Very recently, after this paper was originally written, it was demonstrated that the field could conceivably also equilibrate at an intermediate $O(\mathrm{E}^{1/8})$ amplitude (Anufriev et al., 1995; Hollerbach, 1996a). However, the possible geophysical significance of these intermediate states is not yet known.)

Fig. 5 shows the solution sequence postulated by Malkus and Proctor. Again, in the viscously limited regime it is the geostrophic flow that equilibrates the solutions, at an $O(\mathrm{E}^{1/4})$ amplitude. In contrast, in the Taylor regime it is the ageostrophic flow that equilibrates the solutions, at an $O(1)$ amplitude; the geostrophic flow 'merely' enforces Taylor's constraint, essentially as in his original development. Also, once again, based on the numerical values associated with our basic scalings, the geodynamo is much more likely to correspond to an $O(1)$ than to an $O(\mathrm{E}^{1/4})$ amplitude (10–20 G vs. 10–20 mG).

In some respects, this distinction between the viscously limited regime and the Taylor regime is similar to the distinction between the weak and strong field regimes in Fig. 2, but the two should not be confused. The distinction between weak and strong field regimes is due to the effect of the field on the pattern of convection, which is completely neglected in mean-field theory. In a sense, mean-field theory is thus inconsistent in taking the same prescribed distributions of α and ω in the viscously limited and in the Taylor regimes, when we know that really the pattern of convection, and hence α and ω, would be very different in the two regimes. That is precisely why we ultimately want to move from mean-field models to fully self-consistent convective models. However, as indicated above, mean-field theory is ideally suited to analysing the dynamics of Taylor's constraint, all of which will also apply to the strong field regimes of convective models.

Although we speak of Taylor's constraint as being satisfied in the Taylor regime, and not in the viscously limited regime, it is not quite correct to say that the viscously limited regime is characterized by $T \neq 0$, and the Taylor regime by $T = 0$. In fact, in both regimes $T = O(\mathrm{E}^{1/2})$. The difference is how this scaling comes about. In the viscously limited regime it is trivially accomplished by having $\boldsymbol{B} = O(\mathrm{E}^{1/4})$. In the Taylor regime it is less trivially accomplished by having $\boldsymbol{B} = O(1)$, but having sufficient internal cancellation in the integration in z. In both regimes, then, the geostrophic flow is $O(1)$, and the viscous dissipation associated with it is $O(\mathrm{E}^{1/2})$. In contrast, the ohmic dissipation is also $O(\mathrm{E}^{1/2})$ in the viscously limited regime, but $O(1)$ in the Taylor regime.

The need for this increasingly exact internal cancellation in T, to within $O(\mathrm{E}^{1/2})$, then indicates just how delicate the Taylor state really is. Let us imagine a Taylor state evolving in time in such a way that this internal cancellation suddenly breaks down. This will induce a very large geostrophic flow, and according to Eq. (38) an unsustainably large drain on the toroidal energy. That is, the solution must necessarily evolve back toward a Taylor state, or otherwise end up in a viscously limited state, on a timescale that can be as rapid as $O(\mathrm{E}^{1/2})$. The potential, and sometimes actual, presence of these extremely rapid adjustments about the Taylor state makes these equations extremely stiff, and it is this stiffness, probably more than anything else, that makes them so

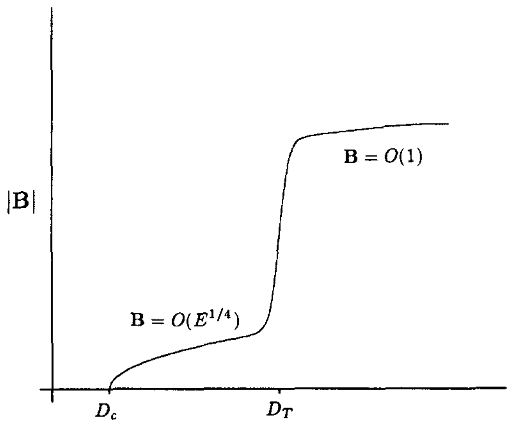

Fig. 5. Following Malkus and Proctor (1975), a sketch of the postulated sequence from the viscously limited regime to the Taylor regime. Plotted is some arbitrary measure of the field amplitude vs. the dynamo number D. The transition from the viscously limited regime to the Taylor regime occurs at D_{T}.

difficult. However, Jault (1995) has recently demonstrated that restoring inertial effects, which we have completely neglected in this development, may help in damping some of these rapid timescales.

It should also be noted that there is another alternative to the viscously limited and Taylor regimes, namely Braginsky's model-Z regime. (See, e.g. Braginsky and Roberts (1987).) In this regime A and B are also $O(1)$, but $\partial A/\partial z$ is small. The poloidal field lines thus tend to align with the z-axis, hence the name model-Z. Because the field lines tend to align in this manner, there is relatively little electromagnetic coupling between adjacent geostrophic contours (we recall that the coupling is via the magnetic tension in the field lines). As a result, the geostrophic flow can become rather large for sufficiently small E. Model-Z will not be considered further here, but the reader is referred to the extensive review by Braginsky (1994).

Having discussed the theoretical background, we turn now to the actual results. Various α^2 models have been considered, by Soward and Jones (1983) in plane layer geometry, and by Hollerbach and Ierley (1991) in spherical geometry. They both demonstrated that there is indeed a second critical dynamo number D_T at which the solutions approach the Taylor state, and beyond which the geostrophic flow alone is no longer capable of equilibrating the solutions. Also, both found that the transition from the viscously limited regime to the Taylor regime may be more complicated than that shown in Fig. 5; in particular, the solution tracks leading to the two regimes may be disconnected. Whereas Soward and Jones only considered this approach to the Taylor state through the geostrophic flow, Hollerbach and Ierley also considered the subsequent equilibration within the Taylor state through the ageostrophic flow. They demonstrated that the subsequent equilibration is indeed independent of viscosity, in complete agreement with the original hypothesis of Malkus and Proctor (1975). Finally, as indicated above, Hollerbach (1991) investigated the interaction between dipole and quadrupole solutions in these models, and came to no definite conclusions on the question of parity selection.

Various $\alpha\omega$ models have also been considered, by Barenghi and Jones (1991) and by Hollerbach et al. (1992) in spherical geometry, and by Jones and Wallace (1992) in duct geometry. Again, they all demonstrated that there is a second critical dynamo number beyond which the geostrophic flow alone is no longer capable of equilibrating the solutions. Now, however, the subsequent equilibration is no longer independent of viscosity. Although the average amplitude in some sense is, the details of the time-dependent evolution are not. We believe the very fact that these solutions are time dependent may be what is preventing them from being completely independent of viscosity. As noted above, the α^2 models are typically steady state (although subsequent bifurcations to time-dependent solutions do occur for some models). The solutions are then typically unambiguously in either the viscously limited regime or the Taylor regime. In contrast, the $\alpha\omega$ models are typically time dependent from the outset. That raises the unpleasant possibility that they may be in different regimes at different times. It is in this sense that the α^2-dynamos are much better behaved that the $\alpha\omega$-dynamos.

Finally, we will discuss briefly a third option for solving the momentum equation. Instead of following either Taylor's or Braginsky's prescription, one may solve it directly, explicitly including inertial and/or viscous terms. Again following Proctor's notation (Proctor, 1977c), the momentum equation then becomes

$$\mathrm{Ro}\left(\frac{\partial}{\partial t}v - N(\psi,v)\right) - 2\frac{\partial}{\partial z}\psi = \mathrm{E}D^2 v + N(B,A) \tag{39a}$$

$$\mathrm{Ro}\left(\frac{\partial}{\partial t}D^2\psi + M(v,v) + M\left(D^2\psi,\psi\right)\right) + 2\frac{\partial}{\partial z}v = \mathrm{E}D^4\psi + M(B,B) + M(D^2A,A) \tag{39b}$$

These equations are either parabolic if $\mathrm{Ro} \neq 0$ and $\mathrm{E} \neq 0$, or elliptic if $\mathrm{Ro} = 0$ and $\mathrm{E} \neq 0$. The mathematical difficulty is not so much in setting $\mathrm{Ro} = 0$, but in attempting to set $\mathrm{E} = 0$ as well, which would make them hyperbolic, and introduce all the difficulties of Taylor's constraint that we have just discussed. However, at any finite value of E, one may solve Eq. (39a) and Eq. (39b) directly. The price one pays, of course, is that one will never be able to reduce E sufficiently to be in the geophysically realistic regime; one can only hope to be able to reduce it sufficiently to be in the asymptotic regime at least, where according to all of our previous analysis the geophysically relevant strong field solution should ultimately become more or less independent of E.

The first two such direct numerical solutions were Proctor's finite-difference code (Proctor, 1977c) and Ierley's spectral code (Ierley, 1985), both in a full sphere. The real advantage of this approach, however, comes when one is in a spherical shell, with an inner core. It turns out that the dynamics may be very different inside and outside the inner core tangent cylinder. It should be noted, for example, how the geostrophic contours in Fig. 4 change abruptly across this tangent cylinder. Properly dealing with this discontinuity in either Taylor's or Braginsky's formulation is nontrivial, although it has been attempted by Anufriev (1994). Although Hollerbach (1994a, Hollerbach, 1996b) has demonstrated that a sufficiently strong magnetic field will ultimately suppress the shear layers that smooth out such discontinuities on the tangent cylinder in the nonmagnetic case (Stewartson, 1966), even just the potential presence of such shear layers may cause difficulties for methods that do not explicitly include viscosity. In contrast, these direct numerical methods that do explicitly include viscosity have fewer such difficulties.

Incidentally, Hollerbach and Proctor (1993) have shown that these shear layers on the tangent cylinder are considerably more severe for the nonaxisymmetric than for the axisymmetric components of the flow. Hollerbach (1994b) has demonstrated that a sufficiently strong field will suppress these nonaxisymmetric shear layers as well, but at a price: the magnetic field must adjust to satisfy a certain integral constraint on the tangent cylinder. Although superficially these constraints on the nonaxisymmetric components may seem similar to Taylor's constraint on the axisymmetric component, they are really quite different. Whereas the constraint Eq. (28) ensures that the axisymmetric component of Eq. (23) has a solution at all, the nonaxisymmetric components of Eq. (23) always have a solution. The constraint identified by Hollerbach and Proctor 'merely' ensures that it will be a sensible, nonsingular solution. The possible geophysical significance of these constraints is not yet known, however. In any case, these nonaxisymmetric asides do not apply to the mean-field models discussed in this section, but they do apply to the fully three-dimensional models discussed in the next section.

If one includes not just an inner core, but specifically a finitely conducting inner core, one obtains yet another dynamical constraint. The inner core is of course free to rotate about its axis, and what determines its differential rotation is a torque balance similar to Eq. (29). The torque balance on the whole of the inner core is

$$\int B_\phi B_r r \sin\theta \, dS = -E \int r \frac{\partial}{\partial r}\left(\frac{v}{r}\right) r \sin\theta \, dS \tag{40}$$

stating that the Lorentz torque may be balanced by this small viscous torque, as in Eq. (29). However, we then obtain yet another constraint very similar to Taylor's constraint, namely that the integrated Lorentz torque on the inner core must vanish with vanishing E. Braginsky (1964) imposed this constraint on an otherwise kinematic model, and found that it could have a significant effect on the solutions. After all, as was also true of Taylor's constraint, the purely kinematic eigensolutions will generally not satisfy this constraint either, so forcing them to might be expected to have a certain effect. More generally, Gubbins (1981) suggested that the dynamic adjustment to this constraint might be very similar to the dynamic adjustment to Taylor's constraint proposed by Malkus and Proctor (1975). To the extent that one can distinguish between the viscously limited and Taylor regimes in these finite E direct numerical solutions, the α^2 model of Hollerbach and Jones (1993a) would seem to support this conjecture.

The real significance of a finitely conducting inner core, however, was only revealed in the $\alpha\omega$ model of Hollerbach and Jones (1993b, Hollerbach and Jones, 1995). We recall that earlier we found that $\alpha\omega$ models typically have a very complicated time-dependent behaviour, and in particular reverse far too often to be good models of the geodynamo. However, because a finitely conducting inner core has a diffusive timescale of its own that is long compared with the shortest advective timescales in the outer core, the field in the inner core must necessarily average over these rapid timescales. Hollerbach and Jones then found that this averaging may have a stabilizing influence, preventing these very rapid timescales from dominating the whole solution. In the next section we will see that this idea of using the finitely conducting inner core to stabilize the field is equally applicable in the convective model of Glatzmaier and Roberts (1995a, Glatzmaier and Roberts, 1995b).

4. Convective dynamos

We end with a brief discussion of two convective dynamo models, and in the process demonstrate how some of the ideas developed earlier apply to these fully self-consistent models. Convective dynamos in a rotating plane layer have been considered by Childress and Soward (1972). Because of the constraints of the Taylor–Proudman theorem, at onset the convection will indeed occur on very short horizontal lengthscales. By applying a multiscale asymptotic analysis, Soward (1974) considered the weak field regime, and demonstrated the existence of a point at which it ends in runaway growth, as in Fig. 2. Fautrelle and Childress (1982) then inferred the existence of an unstable branch connecting the weak field regime to the strong field regime, again as in Fig. 2. Because the convection no longer occurs on short horizontal lengthscales in the strong field regime, these multiscale asymptotic methods no longer apply, and numerical methods are needed. St. Pierre (1993) considered the strong field regime of the Childress–Soward dynamo numerically, and provided convincing evidence that the field does indeed equilibrate at an $O(1)$ value, with the dominant balance being between Lorentz, Coriolis, and pressure-gradient forces. Furthermore, this solution is indeed a subcritical dynamo, operating at Rayleigh numbers less than that required for the initial onset of convection. Because this solution must now deal with all of these adjustments about the Taylor state, these numerical integrations are extremely difficult. Indeed, because the geostrophic flow in a plane layer is two-dimensional rather than merely one-dimensional, they are perhaps unnecessarily difficult. That is, whereas in spherical geometry the geostrophic flow depends only on the single coordinate s, in plane layer geometry it depends on the two horizontal coordinates x and y, say, making these calculations very difficult, and perhaps not even entirely applicable to the geodynamo problem. It is not known to what extent this difference in the geostrophic flow influences the general character of the solutions one obtains, but in view of the overall importance of the geostrophic flow, the effect could be considerable.

Convective dynamos in a rotating spherical shell have also been considered, by Zhang and Busse (1988, Zhang and Busse, 1989, Zhang and Busse, 1990), Glatzmaier and Roberts (1995a, Glatzmaier and Roberts, 1995b) and Jones et al. (1995). Incidentally, all of these models use the finite E direct numerical approach. Zhang and Busse followed the sequence of bifurcations leading from the onset of nonmagnetic convection to the onset of magnetic convection, and were thus restricted to the weak field regime (to the extent that one can distinguish between weak and strong field regimes at finite E). Jones et al. considered the strong field regime, but included only a single nonaxisymmetric mode. Glatzmaier and Roberts also considered the strong field regime, but included all nonaxisymmetric modes. Their model is thus a fully three-dimensional strong field dynamo. However, again, as with St. Pierre's model, the computational price one must pay to obtain such a solution is considerable, and has made an investigation of parameter space impossible so far. For example, it is not known if this solution is also a subcritical dynamo. This would be an extremely interesting result, as it would conclusively demonstrate the existence of separate weak and strong field regimes in the proper spherical shell geometry.

Finally, we comment briefly on the reversal obtained by Glatzmaier and Roberts. Following Hollerbach and Jones (1993b), they too incorporated a finitely conducting inner core, and found that it had a considerable stabilizing influence on their solution as well. In its absence, essentially all fluctuations in the outer core triggered reversals, whereas in its presence, only the largest and longest triggered a reversal. This would seem to confirm the conjecture of Hollerbach and Jones that 'a geomagnetic reversal could only occur as a result of a particularly large fluctuation, large enough and lasting long enough to reverse the field throughout the inner core as well'. On this view, then, there are fluctuations occurring in the outer core all the time, but it is only on those rare occasions when these fluctuations are coherent for a sufficiently long time (perhaps a thousand years or so) to reverse the field in the inner core that the whole field reverses. Reversals are thus essentially random events, with only the overall reversal frequency determined by the overall level of fluctuations in the outer core. At this point this is probably the best theory we have of reversals, but it too is still incomplete, as we do not yet understand all the dynamics of the fluctuations in the outer core. Also, of course, the Glatzmaier and Roberts model is still too expensive to run long enough to obtain reversal statistics, as opposed to a single reversal.

5. Conclusion

In this work I have attempted to describe the key aspects in the development of geodynamo theory, starting with Larmor's original hypothesis (Larmor, 1919) and culminating with the fully self-consistent convective geodynamo model of Glatzmaier and Roberts (1995a, Glatzmaier and Roberts, 1995b). Not surprisingly, the early work was dominated by kinematic models, in which the fluid flow is arbitrarily prescribed. In particular, following the publication of Cowling's theorem (Cowling, 1934), the hunt was on for a more general anti-dynamo theorem which would disallow any fluid flow, however cleverly prescribed, from acting as a dynamo. This endeavour came to an abrupt end with the proofs of Backus (1958) and Herzenberg (1958) that at least some flows can act as dynamos. Although these flows chosen by Backus and Herzenberg were geophysically not realistic, these proofs were none the less extremely influential, as they demonstrated for the first time the basic validity of Larmor's suggestion, that fluid flows can maintain magnetic fields. Following these proofs that dynamo action is possible in principle, the emphasis of research then gradually shifted to various models in which the fluid flow is dynamically determined. In these dynamical models of the geodynamo, the single most important aspect is almost certainly the effect of the Earth's rotation. It is the effect of a magnetic field on the pattern of convection in a rapidly rotating system that leads to distinct weak and strong field regimes. And although the weak field regime is much simpler in many respects, the geodynamo is believed to be in the strong field regime, in which the dominant balance is between Lorentz, Coriolis, and pressure-gradient forces, with the inertial and viscous forces negligible. This balance has two immediate consequences, both of which make the problem more difficult. First, because the Lorentz force can no longer be treated as a perturbation, the asymptotic methods that worked so well in the weak field regime no longer apply. Second, all of the dynamics of Taylor's constraint now come into play. In particular, these extremely rapid adjustments about the Taylor state make the equations very difficult to integrate forward in time. One could, of course, filter out these rapid timescales by setting the inertial and viscous terms identically equal to zero, and essentially following Taylor's original prescription. However, no one has yet succeeded in doing so even in a full sphere, let alone in a spherical shell. And it is not clear that one should follow this prescription, even if one could. As we saw in our discussion of the $\alpha\omega$-dynamos, these models may not always be in a Taylor state. That is, some of these dynamics that Taylor's prescription would filter out may have real geophysical significance. Therefore it is probably safest to include them, and deal with them as best one can. In these finite E direct numerical methods, one can always make the problem somewhat less severe by increasing E somewhat. Of course, one then runs the risk of no longer being in the asymptotic limit in which one has a clear distinction between weak and strong field regimes, as noted above. It is the need to reduce E sufficiently to be in this asymptotic regime that makes a numerical solution of the geodynamo so difficult. Nevertheless, I think one can confidently say that our theoretical understanding of these difficulties is sufficiently great that we will be able to overcome them, particularly as more powerful computers become available, and thereby continue to make good progress in understanding the inner workings of the geodynamo.

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