Progress of Theoretical Physics, Vol. 54, No. 6, December 1975

# Einstein's Theory of Relativity and Mach's Principle

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(Received January 10, 1975)

The equations of motion of a test particle near the center of a rotating spherical shell with the mass $M$ and the radius $R$ are investigated in the framework of Einstein's theory of relativity up to the post-post-Newtonian order of approximation.

Among the forces acting on the test particle, the Coriolis and the centrifugal forces appear. In order that Mach's thought about rotation is realized, two conditions on $M/R$ must be imposed. It is shown that these two conditions are not consistent with each other.

In a previous paper[1] we investigated in the context of Einstein's theory of relativity the equations of motion of a test particle near the center of a massive rotating ring with the angular velocity $\omega$. Calculations were carried out up to the post-post-Newtonian order of approximation. Among the forces acting on the test particle, there appear the Coriolis and the centrifugal forces, and the $\omega$-dependent force directing to the rotation axis. The latter vanishes when $GM/R=3\pi/(27\pi-7)$, where $G$, $M$ and $R$ are the gravitational constant, the mass and the radius of the ring, respectively. On this condition the relative magnitude of the Coriolis force to the centrifugal force in the equations of motion agrees with the expected one from the equations of motion in a rotating reference frame.

As a more realistic model of the universe, in this paper, we shall consider the case of a spherical shell rotating with the angular velocity $\omega$. This case was first studied by Thirring[2] in the lowest order of $G$. However, in order to know whether Einstein's theory of gravity realizes Mach's thought[3] on the relativity of motion, it is necessary to calculate the equations of motion of a test particle at least to the order of $(G\omega)^2$, because the Coriolis force and the centrifugal force are proportional to $G\omega$ and $(G\omega)^2$ in the lowest order of $G$, respectively.*)

*) The necessity for the calculation of the equations of motion up to the order of $(G\omega)^2$ was stressed by A. Lausberg and R. Simon [Bull. Acad. roy. de Belg. (1971), 125; (1972), 58]. We would like to thank Dr. Lausberg who has recently informed us of these papers.

We shall discuss the equations of motion of the test particle by using the metric tensor for many-body system which was obtained previously by solving Einstein's equation up to the post-post-Newtonian order of approximation.[4] The metric tensor becomes Minkowskian at spatial infinity.

Let $F(x^1, x^2, x^3)$ be a frame of reference in which fixed fictitious stars are at rest. Consider a slowly moving test particle with the mass $m$ near the center of the spherical shell which rotates counter-clockwise around the $x^3$-axis in the frame $F$. The equations of motion of the particle are given in Einstein's theory of relativity by

$$m\ddot{x}^i = m(-\Gamma^i_{00} - 2\Gamma^i_{0j}v^j + \Gamma^0_{00}v^i - \Gamma^i_{jk}v^j v^k + 2\Gamma^0_{0j}v^i v^j + \Gamma^0_{jk}v^i v^j v^k), \qquad (1)$$

where $\Gamma^\alpha_{\beta\gamma}$ denotes the Christoffel symbol evaluated at the position of the test particle and $v^i = \dot{x}^i$.*) Now we would like to know whether these equations of motion reduce to the form

$$m\ddot{\boldsymbol{x}} = \boldsymbol{K} - m(\boldsymbol{x} \times \boldsymbol{\Sigma}) \times \boldsymbol{\Sigma} + 2m(\boldsymbol{v} \times \boldsymbol{\Sigma}), \qquad (2)$$

where $\boldsymbol{K}$ is the $\omega$-independent force and $\boldsymbol{\Sigma}$ is an angular velocity proportional to $\boldsymbol{\omega}$. The form of Eq. (2) is just the same with that of the equations of motion in a rotating reference frame in the Newtonian mechanics. Therefore the second and the third terms on the right-hand side of Eq. (2) represent the centrifugal force and the Coriolis force in the frame $F$, respectively. Neglecting the second and the third powers of $v^i$ in Eq. (1) we have

$$m\ddot{x}^i = -m\Gamma^i_{00} - m(2\Gamma^i_{0j}v^j - \Gamma^0_{00}v^i). \qquad (3)$$

The centrifugal force and the Coriolis force should be included in the first and the second terms on the right-hand side of Eq. (3), respectively.

We shall rewrite the metric tensor $g_{\alpha\beta}$ as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}. \quad (\eta_{00} = -1,\ \eta_{0i} = 0,\ \eta_{ij} = \delta_{ij}) \qquad (4)$$

Using this notation, we have

$$\Gamma^i_{0j} = \tfrac{1}{2}(h_{0i,j} - h_{0j,i} + h_{ij,0}) + O(h^2), \qquad (5)$$

$$\Gamma^0_{00} = -\tfrac{1}{2}h_{00,0} + O(h^2), \qquad (6)$$

$$\Gamma^i_{00} = -\tfrac{1}{2}h_{00,i} + h_{0i,0} + \tfrac{1}{2}h_{ij}h_{00,j} + \tfrac{1}{2}h_{0i}h_{00,0} - h_{ij}h_{0j,0} + O(h^3). \qquad (7)$$

We calculated $h_{\alpha\beta}$ for many-body system up to the post-post-Newtonian order of approximation.[4] The part of the expression of $h_{\alpha\beta}$ which is necessary for our calculation will be given in the Appendix. We shall calculate $h_{\alpha\beta}$ for the spherical shell from the expression of $h_{\alpha\beta}$ for many-body system by assuming that the shell is composed of infinitely many bodies and its mass-distribution is uniform.

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*) In this paper we use the following conventions. Greek indices take the values 0, 1, 2 and 3, while Latin indices take 1, 2 and 3. Repetition of these indices implies summation. A comma in a subscript denotes a partial derivative. We use the unit of $c=1$, $c$ being the velocity of light.

Let $M$ and $R$ be the mass and the radius of the spherical shell, whose center is fixed at the origin of the coordinate system. From the expression (A·3), we get*)

$$h_{0i}=-\frac{GM\omega}{3R}(-y,\, x,\, 0)^{i}\,. \tag{8}$$

Since the test particle is near the origin of the coordinate system, the terms of the order of $(x^i)^3$ and of higher orders are neglected in the expression (8). We can show that $h_{\alpha\beta,0}$ vanishes in the case of the spherical shell rotating with a constant angular velocity. From Eqs. (5), (6) and (8) we have, in the lowest order of $G$,

$$\Gamma^{i}_{0j}=\frac{4GM\omega}{3R}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}\,, \quad \Gamma^{0}_{00}=0\,. \tag{9}$$

Then the second term on the right-hand side of Eq. (3) reduces to

$$-\frac{8GmM}{3R}(\boldsymbol{v}\times\boldsymbol{\omega}), \tag{10}$$

which means

$$\boldsymbol{\Sigma}=-\frac{4GM}{3R}\boldsymbol{\omega}\,. \tag{11}$$

We shall evaluate $\Gamma^{i}_{00}$ up to the order of $(G\omega)^2$. The expression of $h_{00}$ consists of seventeen terms. Eleven terms of them are written explicitly in the expression (A·4) and the remaining six are given in the expression (A·5). We call them in turn 1, 2, ···, 17, following the order written in the expressions (A·4) and (A·5). We shall show the result of the evaluation of these

Table I.

| Type \ No. | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| $\frac{GM}{R}$ | 2 | 0 | 0 | 0 | 2 |
| $GMR\omega^2$ | 0 | 2 | 0 | 0 | 2 |
| $\frac{G^2M^2}{R^2}$ | 0 | 0 | $-2$ | $-2$ | $-4$ |
| $\frac{GM\omega^2}{R}(x^2+y^2-2z^2)$ | 0 | $\frac{1}{5}$ | 0 | 0 | $\frac{1}{5}$ |

| Type \ No. | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $G^2M^2\omega^2$ | 0 | $-\frac{10}{3}$ | 0 | $\frac{20}{9}$ | 0 | 0 | $-\frac{8}{9}$ | 0 | $-1$ | $-\frac{2}{9}$ | 0 | $\frac{4}{45}$ | $\frac{17}{15}$ | $-2$ |
| $\frac{G^2M^2\omega^2}{R^2}(x^2+y^2)$ | 0 | 0 | $\frac{8}{9}$ | 0 | 0 | 0 | $\frac{8}{9}$ | 0 | $\frac{1}{6}$ | 0 | 0 | 0 | $-\frac{1}{6}$ | $\frac{16}{9}$ |
| $\frac{G^2M^2\omega^2}{R^2}(x^2+y^2-2z^2)$ | 0 | $-\frac{1}{3}$ | 0 | $\frac{74}{225}$ | 0 | 0 | $-\frac{8}{45}$ | 0 | $-\frac{1}{10}$ | $\frac{13}{225}$ | 0 | $-\frac{4}{525}$ | $\frac{7}{180}$ | $-\frac{1217}{6300}$ |

*) Hereafter we use the notations $x=x^1$, $y=x^2$, $z=x^3$.

terms in Table I. From this table we get

$$h_{00}=\frac{2GM}{R}-\frac{4G^2M^2}{R^2}+2GMR\omega^2-2G^2M^2\omega^2+\frac{16G^2M^2\omega^2}{9R^2}(x^2+y^2)$$
$$+\left(\frac{GM\omega^2}{5R}-\frac{1217}{6300}\frac{G^2M^2\omega^2}{R^2}\right)(x^2+y^2-2z^2). \tag{12}$$

The third term on the right-hand side of Eq. (7) gives

$$\frac{1}{2}h_{ij}h_{00,j}=\frac{2G^2M^2\omega^2}{5R^2}(x, y, -2z)^i. \tag{13}$$

As a result we get the force

$$-m\Gamma^i_{00}=\frac{16G^2M^2m\omega^2}{9R^2}(x, y, 0)^i+\frac{GMm\omega^2}{5R}\left(1-\frac{3737}{1260}\frac{GM}{R}\right)(x, y, -2z)^i. \tag{14}$$

The first term on the right-hand side is just the centrifugal force expected from Eq. (2):

$$-m(\boldsymbol{x}\times\boldsymbol{\Sigma})\times\boldsymbol{\Sigma}, \tag{15}$$

where $\boldsymbol{\Sigma}$ is given by (11). The second term contributes to the centrifugal force and the $\omega$-dependent force directing to the $z$-axis.

Now we shall study the equation of motion of the particle in the frame $F'(x', y', z')$, which rotates counter-clockwise to $F$ around $z$-axis with a constant angular velocity $\Omega$. The shell also rotates, in the frame $F$, around the same axis with the angular velocity $\omega$. Under the coordinate transformations*)

$$x'=x\cos\Omega t+y\sin\Omega t, \quad y'=-x\sin\Omega t+y\cos\Omega t,$$
$$z'=z, \quad t'=t, \tag{16}$$

the metric tensor $g_{\alpha\beta}(x)$ is transformed as

$$g'_{\alpha\beta}(x')=\frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta}g_{\gamma\delta}(x). \tag{17}$$

We shall evaluate the metric tensor $g'_{\alpha\beta}(x')$, and calculate Eq. (3) in the frame $F'$. After a lengthy calculation, we get

$$m(\ddot{x}', \ddot{y}', \ddot{z}')^i$$
$$=2m\left(\Omega-\frac{4GM}{3R}\omega\right)(y', -x', 0)^i-m\left(\Omega-\frac{4GM}{3R}\omega\right)^2(x', y', 0)^i$$

*) We cannot apply the coordinate transformations (16) at large distances, where the time component of the metric tensor changes its sign and becomes positive. Thus we must modify the transformation laws (16). Transformation laws must satisfy at least the following two conditions: (1) At small distances they reduce to the laws (16). (2) At $(x^2+y^2)^{1/2}\to\infty$, the velocity $\Omega(x^2+y^2)^{1/2}$ does not exceed the velocity of light. As discussed in the previous paper,[1] the equations of motion (18) remain unchanged under any coordinate transformations satisfying the above two conditions.

$$+\frac{GMm\omega^2}{5R}\left(1-\frac{3737}{1260}\frac{GM}{R}\right)(x', y', -2z')^t . \tag{18}$$

The first and the second terms on the right-hand side of Eq. (18) represent the Coriolis and the centrifugal forces, respectively. The last term includes the $\omega$-dependent force directing to $z$-axis.

Newton considered that we can select an absolute rest-system with respect to rotation. Mach criticized Newton's thought. He asserted in his book[2] that it is impossible, in principle, to choose the absolute rest-system by observing the motion of a test particle. According to him, we cannot determine whether the distant matter in the universe rotates or the coordinate system does. If there exists an $\omega$-dependent force directing to $z$-axis, we can recognize the rotation of the distant matter, that is, the existence of the absolute rest-frame $F$.

Here we would like to examine whether Mach's thought is realized in Einstein's theory of relativity or not. In order that Mach's thought is realized, the following two conditions must be satisfied in Eq. (18):

(i) $$\frac{GM}{R}=\frac{1260}{3737},$$

(ii) $$\frac{GM}{R}=\frac{3}{4}.$$

The vanishing of the force directing to $z$-axis requires the condition (i). The condition (ii) is obtained from the requirement that the Coriolis and the centrifugal forces must vanish when $\Omega=\omega$. If $GM/R\neq 3/4$, there exists a coordinate system $F'$ in which both the Coriolis and the centrifugal forces disappear, but the distant matter rotates relatively to $F'$. This imples the existence of the absolute rest-frame.

Obviously the condition (ii) is not consistent with the condition (i). We conclude that Mach's thought is not realized, in the case of the rotating spherical shell*) up to the post-post-Newtonian order of approximation. This conclusion will be unchanged in higher order of approximation. We cannot deny the possibility that, for some model with artificial matter distribution, and/or for some artificial boundary conditions, the conditions corresponding to (i) and (ii) become consistent with each other. It must be emphasized, however, that Mach's thought is not satisfied automatically in Einstein's theory of relativity.

## Appendix

*——Expression for $h_{\alpha\beta}$——*

In previous papers[5] we solved Einstein's equation for the many-body system

*) We have discussed the case of the rotating ring.[1] The conclusion in that case is the same with that obtained in this paper.

up to the post-post-Newtonian order of approximation and obtained the expression for $h_{\alpha\beta}$. In this appendix we shall write down only the part of the expression which is necessary for the evaluation of the Coriolis and the centrifugal forces.

Let $m_a$ and $\boldsymbol{z}_a$ be the mass and the coordinate of $a$-th body, respectively. We have

$$h_{ij}=\delta_{ij}\left(2G\sum_a\frac{m_a}{r_a}+G\sum_a\frac{m_a}{r_a}\boldsymbol{v}_a^2\right)+h_{ij}^{TT(v)}+\cdots, \qquad (A\cdot 1)$$

where $r_a=|\boldsymbol{x}-\boldsymbol{z}_a|$, $\boldsymbol{v}_a=\dot{\boldsymbol{z}}_a$ and $h_{ij}^{TT(v)}$ is the velocity-dependent part of the transverse traceless part of $h_{ij}$. The expression for $h_{ij}^{TT(v)}$ is given by

$$h_{ij}^{TT(v)}=\frac{1}{4}G\sum_a\frac{m_a}{r_a}\{2v_a^iv_a^j+\delta_{ij}\boldsymbol{v}_a^2-5\delta_{ij}(\boldsymbol{n}_a\cdot\boldsymbol{v}_a)^2+6(v_a^in_a^j+v_a^jn_a^i)(\boldsymbol{n}_a\cdot\boldsymbol{v}_a)$$
$$-5n_a^in_a^j\boldsymbol{v}_a^2+3n_a^in_a^j(\boldsymbol{n}_a\cdot\boldsymbol{v}_a)^2\}, \qquad (A\cdot 2)$$

where $\boldsymbol{n}_a=(\boldsymbol{x}-\boldsymbol{z}_a)/r_a$. The necessary parts of $h_{0i}$ and $h_{00}$ are

$$h_{0i}=-\frac{1}{2}G\sum_a\frac{m_a}{r_a}\{7v_a^i+n_a^i(\boldsymbol{n}_a\cdot\boldsymbol{v}_a)\}+\cdots, \qquad (A\cdot 3)$$

$$h_{00}=2G\sum_a\frac{m_a}{r_a}+3G\sum_a\frac{m_a}{r_a}\boldsymbol{v}_a^2-2G^2\sum_a\sum_b\frac{m_am_b}{r_ar_b}-2G^2\sum_a\sum_{b\neq a}\frac{m_am_b}{r_ar_{ab}}$$
$$-\frac{1}{4}G^2\sum_a\frac{m_a^2}{r_a^2}\{19\boldsymbol{v}_a^2-35(\boldsymbol{n}_a\cdot\boldsymbol{v}_a)^2\}-5G^2\sum_a\sum_b\frac{m_am_b}{r_ar_b}\boldsymbol{v}_a^2$$
$$+8G^2\sum_a\sum_b\frac{m_am_b}{r_ar_b}(\boldsymbol{v}_a\cdot\boldsymbol{v}_b)+G^2\sum_a\sum_{b\neq a}\frac{m_am_b}{r_ar_{ab}}\Big\{8\boldsymbol{v}_a^2-2\boldsymbol{v}_b^2-\frac{17}{2}(\boldsymbol{v}_a\cdot\boldsymbol{v}_b)$$
$$+\frac{1}{2}(\boldsymbol{n}_{ab}\cdot\boldsymbol{v}_a)(\boldsymbol{n}_{ab}\cdot\boldsymbol{v}_b)\Big\}-\frac{1}{2}G^2\sum_a\sum_b\frac{m_am_b}{r_a^2}(\boldsymbol{n}_a\cdot\boldsymbol{v}_a)(\boldsymbol{n}_b\cdot\boldsymbol{v}_b)$$
$$+\frac{1}{2}G^2\sum_a\sum_{b\neq a}\frac{m_am_b}{r_a^2}(\boldsymbol{n}_a\cdot\boldsymbol{v}_a)(\boldsymbol{n}_{ab}\cdot\boldsymbol{v}_b)+G^2\sum_a\sum_{b\neq a}m_am_b$$
$$\times\left\{(5v_a^iv_b^j-16v_a^jv_b^i)\frac{\partial^2}{\partial z_a^i\partial z_b^j}-8v_a^iv_b^j\frac{\partial^2}{\partial z_a^i\partial z_a^j}\right\}\ln(r_a+r_b+r_{ab})$$
$$+h_{00}^{TT(v)}+\cdots, \qquad (A\cdot 4)$$

where $r_{ab}=|\boldsymbol{z}_a-\boldsymbol{z}_b|$, $\boldsymbol{n}_{ab}=(\boldsymbol{z}_a-\boldsymbol{z}_b)/r_{ab}$ and $h_{00}^{TT(v)}$ denotes the contribution of $h_{ij}^{TT(v)}$ to $h_{00}$. The expression for $h_{00}^{TT(v)}$ is given by

$$h_{00}^{TT(v)}=\frac{1}{4}G^2\sum_a\frac{m_a^2}{r_a^2}\{17\boldsymbol{v}_a^2-35(\boldsymbol{n}_a\cdot\boldsymbol{v}_a)^2\}-\frac{1}{2}G^2\sum_a\sum_b\frac{m_am_b}{r_ar_b}\{3\boldsymbol{v}_a^2-(\boldsymbol{n}_a\cdot\boldsymbol{v}_a)^2\}$$
$$+\frac{1}{2}G^2\sum_a\sum_{b\neq a}\frac{m_am_b}{r_ar_{ab}}\{-3\boldsymbol{v}_b^2+(\boldsymbol{n}_{ab}\cdot\boldsymbol{v}_b)^2+2\boldsymbol{v}_a^2\}$$
$$-\frac{1}{4}G^2\sum_a\sum_b\frac{m_am_b}{r_a^2}[(\boldsymbol{n}_a\cdot\boldsymbol{n}_b)\{5\boldsymbol{v}_b^2-(\boldsymbol{n}_b\cdot\boldsymbol{v}_b)^2\}-14(\boldsymbol{n}_a\cdot\boldsymbol{v}_b)(\boldsymbol{n}_b\cdot\boldsymbol{v}_b)]$$

$$+\frac{1}{4}G^2\sum_a\sum_{b\neq a}\frac{m_a m_b}{r_a^2}[(\boldsymbol{n}_a\cdot\boldsymbol{n}_{ab})\{5\boldsymbol{v}_b^2-(\boldsymbol{n}_{ab}\cdot\boldsymbol{v}_b)^2\}-14(\boldsymbol{n}_a\cdot\boldsymbol{v}_b)(\boldsymbol{n}_{ab}\cdot\boldsymbol{v}_b)]$$

$$+G^2\sum_a\sum_{b\neq a}m_a m_b\left\{(v_a^i v_a^j+8v_b^i v_b^j)\frac{\partial^2}{\partial z_a^i\partial z_a^j}+8v_a^i v_a^j\frac{\partial^2}{\partial z_a^i\partial z_b^j}\right\}$$

$$\times\ln(|\boldsymbol{x}-\boldsymbol{z}_a|+|\boldsymbol{x}-\boldsymbol{z}_b|+|\boldsymbol{z}_a-\boldsymbol{z}_b|)\,. \qquad (\mathrm{A}\cdot 5)$$

The metric tensor for the rotating spherical shell is obtained from the above expressions for $h_{\alpha\beta}$ for many-body system by performing the following substitutions:

$$z_a^i\to\{R\sin\theta_a\cos(\omega t+\phi_a),\ R\sin\theta_a\sin(\omega t+\phi_a),\ R\cos\theta_a\}^i\,,$$

$$\sum_a m_a\to\frac{M}{4\pi}\int_0^\pi d\theta_a\sin\theta_a\int_0^{2\pi}d\phi_a\,.$$

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