# The Sagnac Effect Explained Using the Special Relativity Theory

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**ABSTRACT**

The discovery by Sagnac in the 1910s that a light beam that is forced to travel in a circular path along an orbiting circular disk needs different time to make a revolution, dependent on the direction, along or against. the direction of revolution. Kelly in a paper discusses efforts being made by different scientists in order to explain the Sagnac effect.
Kelly himself succeeds in deriving a model able to explain the numerical results, thereby claiming that the Special relativity theory is not needed; a classical approach suffices.

Another author, Post, analyses the Sagnac experiment thoroughly and uses the relativistic concept of time dilatation when evaluating an expression for the different propagation time along the two directions of the rays. He thereby uses a model developed by Langevin, which results in an expression for the times for the two rays as if they had a relative velocity approximately equal to c-v and c-v with respect to the sender. He claims this to be in line with the SRT He is speaking of a 'recasting' of the Lorentz transformation into polar coordinates.

Einstein on his part basically pretends that the relative velocity of light is c, but is also hesitating, when the question of non-liner movement arises. In one connection for example he claims that the time loss for a clock being moved between two points is independent of which way the journey is being performed; it might even be 'along any polygonal line', he claims, which is problematic when regarding the results of the Sagnac experiment.

However, in this paper it is being shown that the Special Relativity Theory (SRT), too, is able to explain the Sagnac effect, thereby giving just the same results as Kelly. This is a pure matter of coincidence and if velocities increase, the similarity begins to disappear. There are problems in connections with the Kelly theory, as his model implies observers' seeing velocities higher than that of light, whereas the usage of SRT presumes the velocity of light to be the highest one can ever observe. The SRT succeeds through the usage of the concept of time dilatation, extended in a differential sense when applied to a circular orbit.

# 1. Introduction

The Special relativity theory (SRT) has continuously been questioned by the scientific community. The results by Sagnac in the 1910s, entitled the "Sagnac effect", were such that it has been difficult to use the SRT straightforwardly in order to explain the effect. The Sagnac effect means the phenomenon that light emitted on a circular disk, forced to follow a circular path back to its origin along a clock-wise and an anti-clock-wise circular orbit along the disk

## 2. The Sagnac effect, referred to by Kelly.

Kelly in a paper makes an effort to explain the Sagnac effect [1]. The following figure may be used in order to visualize the process, but Kelly has written a more detailed figure in his paper [2].

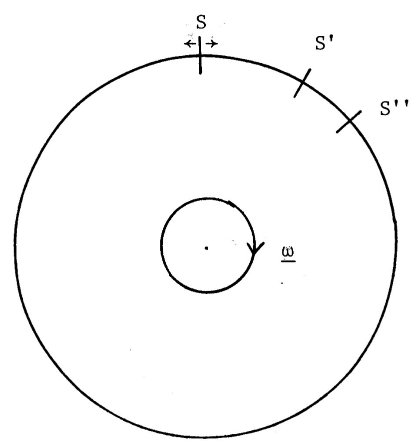


**Figure 1. A rotating disk with a sender of light S. S' denotes the position of the disk, when an anti-clock-wise signal arrives to the sender; S'' the position of the disk, when a clock-wise signal arrives.**

Kelly is performing his analysis strictly classically, i.e. he uses coordinates, velocities and time without taking into account the SRT.
And he succeeds, too. However, the velocities are low, v<<c. hence, one should be careful, not to draw to far going conclusion by a numerical result.

### 2.1. The method according to Kelly.

Kelly [2] basically analyses the travel of the light beams sent out from a place on the periphery of a rotating disk in two directions, clockwise and anticlockwise, the disk rotating clockwise according to the reference coordinate system K, according to which the disk it rotating around a fixed axis.('stationary frame'). Since the disk is rotating, while the light beams is travelling, they will arrive at two different points, S' and S'' according to his definitions. The distance from the sending position to point S' is called ds' and the travelling time is called t' (not optional, may cause confusion with respect to the common used definitions); the small r:s in his paper changed into large R:s here for convenience.

$$t' = \frac{2\pi R - ds'}{c} \qquad (1)$$

This expression may be rewritten into

$$t' = \frac{2\pi R}{c+v} \qquad (2)$$

In a similar manner Kelly attains an expression for the anticlockwise beam to travel from S to S'':

$$t'' = \frac{2\pi R}{c-v} \qquad (3)$$

The difference between the two expressions gives the time difference for the two beams respectively.

$$dt = \frac{4\pi R v}{c^2 - v^2} \qquad (4)$$

which, since v<<c, approaches

$$dt = \frac{4\pi R v}{c^2} \qquad (5)$$

Furtheron, the author claims that, since the difference is the same, independently of in which system the observer is being situated, the SRT is thereby proved false and the velocity of light must, hence, be independent of the choice of coordinate system.

## 3. The Sagnac effect according to Post.

Post [12] in a paper makes a unique effort to explain the Sagnac effect. He is, however, formulating the solution by defining a different ordering of the varaiables than Kelly above. He is defining the time the ray is travelling in the anticlockwise and clockwise directions respectively (according to K) as

$$\tau' = (\frac{2\pi R - \Delta s'}{c}) = \frac{\Delta s'}{\Omega R} \qquad (6)$$

and

$$\tau'' = (\frac{2\pi R + \Delta s'')}{c} = \frac{\Delta s''}{\Omega R} \qquad (7)$$

thereby using Greek sign $\Omega$ to indicate angular velocity. Realizing that the area enclosed by the circumference, $A = \pi R^2$,
by combining eq. (6) and (7) one accordingly attains the time difference for the two ways

$$(\Delta\tau)_s = \frac{4\Omega A}{c^2 - (\Omega R)^2} \qquad (8)$$

Thereafter Post assumes that it is allowed to use the time dilatation term $\gamma$ when turning from K (system in which the circular desk is observed to be orbiting around a fixed axis, the so-called stationary frame) into that attached to the beam splitter at the disk K'.

However, here one has to make an objection, namely that a system attached the rotating disk be no means can be regarded as an inertial system. Einstein namely requires that there are to be ”two co-ordinates systems, which have a uniform translatory motion relative to each other” [3]. Post bases his derivation on a concept by Langevin [13], [14], who pretends to perform a transformation between a stationary frame and a rotating one, using polar coordinates. By that same moment one leaves the SRT and does something else. However, after some work he attains expressions for the time the journeys of the two beams demand (according to the rotating frame) as follows:

$$t_1' = \frac{2\pi R}{(c - \Omega R)\gamma} \quad (9)$$

$$t_2' = \frac{2\pi R}{(c + \Omega R)\gamma} \quad (10)$$

Combining those gives the time difference between the two journeys as

$$(\Delta\tau)_m = \frac{4\pi\Omega R^2}{\gamma(c^2 - \Omega^2 R^2)} \quad (11)$$

or simpler

$$(\Delta\tau)_m = \frac{4A\Omega}{\gamma(c^2 - \Omega^2 R^2)} \quad (12)$$

which is approximately the same as Eq.(8) as long as v<<c (here $\Omega R = v$)

One may also observe that in Eq.(9) and Eq.(10) there appear expressions for the velocity in the denominator, as if the speed of light were to be different to c with respect to the beam splitter.

## 4. Explanation, using the Special Relativity Theory.

The usage of the SRT seems to be rather complicated in the case of a rotating disk, since there is no inertial system on the disk. The sender (or any other point on the disk) is continuously being accelerated; hence, the ‘standard configuration’ may not be used, since it can not be defined a coordinate system on the disk, which has a velocity that differs from the original one K, constant in time.

One method to resolve the problem would be to define an infinite amount of inertial systems K''( $\varphi$ ) along the circumference of the disk. The formula of time dilatation [4] is then used.

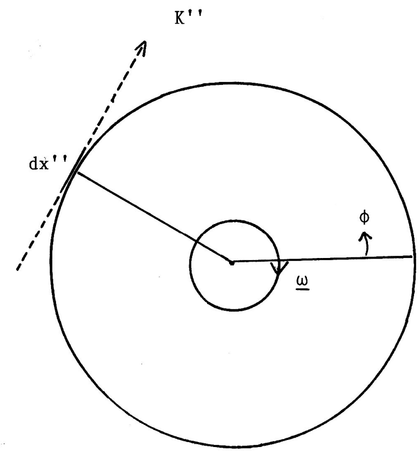


**Figure 2. A rotating disk, principally the same as that being illustrated in Fig. 1. It is here illustrated, how it is possible to define a angle dependent inertial system K''. It is thereby assumed that the circumference may be divided into infinitesimally small elements of length dx''**

**4.1. Time dilation**

On apparent obstacle when attempting to use the results of the SRT in the case of a rotating movement is that the usage of the standard configuration stipulates a constant velocity between two coordinate systems. If allowing a coordinate system to follow the circular movement of the disk, it would be impossible to use the SRT transformation laws [3]. However, if regarding a circular movement as divided into infinitesimally small displacements **dx'' (**using double primes to indicate a coordinate system on the disk), they each are to be regarded as linear and, hence, the transformation laws may be used with respect to that very short displacement from one point of a circle to the next one. That is the kernel of infinitesimal analysis. The **dx**:s refer to the system in which the disk is regarded as orbiting (i.e. K) as described by both the figures above.

Time dilatation may therefore be defined [4]

$$\frac{dt'}{dt} = \lambda(v)(1 - \frac{v}{c^2}\frac{dx}{dt}) \qquad (13)$$

That expression may be integrated in order to attain the total time t' during one revolution as a function of t.

It must thereby be understood that the dt':s are relating to the element immediately before, just as in the case of iteration formulas.

When allowing to signals go in two different directions, clockwise and anti-clockwise, the first term in Eq.(1) cancels when integrating.

Since v<<c for the case of a rotating disk, e.g. the surface of the Earth, $\gamma(v)$ can be neglected and hence, the displacements would be regarded as being of equal length both for the case of K and the local K':s on the disk.

Since the sending points remain the same one according to observers on the disk, a light beam travels equal length L in both directions, $L = 2\pi R$ , and hence, the total time difference between two arrivals that are nor simultaneous is

$$t' = 4\pi R\frac{v}{c^2} \qquad (14)$$

thereby intentionally writing ‘strictly equal’, since v<<c.
This result is in limes consistent with the result attained by Kelly [5], even though he uses the variable dt. In his expression there is a term $c^2 - v^2$ in the denominator, but since v<<c for the rotating disk, the result Eq.(14) still holds.

## 5. Objections against the use of the SRT according to “Evans Field Theory”

Amador et al.[15] claim that the SRT may not be used in order to explain the Sagnac Effect, since rotational motion is involved and that is not allowed according to the SRT. Instead they stick to the General Relativity Theory (GRT), thus using a method based upon “rotating tetrad fields and a Cartan geometry. Doing so, they succeed in attaining a result

$$\Delta t = \frac{4\pi\Omega}{{\omega_1}^2 - \Omega^2} \qquad (12b)$$

The result seems to be rather agreeable
However, as follows from the proof above by Jonson, there might be a way to apply the SRT upon non-linear movement, too.

## 6. Discussion.

### 6.1 Einstein hesitating about the constancy of the velocity of light.

It is interesting that not even Einstein, founder of the Special Relativity Theory, did not believe in the constancy of light for the case of accelerating coordinate systems, he even claims that “the principle of the constancy of light must be modified” in that case [6]. That makes it easy to understood, why Kelly rejects the SRT in his attempt to explain the Sagnac effect. It would have been best, if both Einstein and Kelly had stuck to the original theory.

### 6.2. Why hesitating?

As a matter of fact, Einstein made his claim without further basing it upon any mathematical treatment. Slightly before [6] he says that K and K’ “can both with the same legitimacy be taken as at rest”. He has already earlier defined the “Principle of Constancy of the velocity of light [7], [8] according to which the velocity of light is independent of the velocity of the sender, i.e. c.

### 6.3. Solution of the problem with rotating coordinate systems.

Apparently Einstein did not solve the problem with treating non-linear coordinate transformations. Indeed, he sticks to a not allowed trick some lecturers use, when coming to a point they themselves don’t master: saying “it is at once apparent” [9] or in the German original “Man sieht sofort, dass…..[10]. He says that when he attains to gain support to the claim that it does not matter which way a light beam travels between two points A and B; it might be “any polygonal line”[9]. Or in the German original: “Man sieht sofort, dass dies Resultat auch dann noch gilt, wenn die Uhr in einer beliebigen polygonalen Linie sich von A nach B bewegt,,,”[10]. It is this reference Kelly uses, when he is refuting the validity of the Special Relativity Theory above [11]. **However this author resolves the problem above in chapter 3.1 by applying the coordinate**

**transformation to infinitesimally small displacements, each linear, thereby allowing for an extended use of the standard configuration (Lorentz transformation) with K.s and K's into non-linear transformations as in the Sagnac case.**

**6.4. Einstein's fallacy.**
When claiming that the direction of travel is irrelevant above, Einstein performs a failure, too. Eq. (13) above namely has the consequence –which is closer discussed in connection with the equation in chapter 3.1. – that the time for the travel along the circular disk is dependent on the direction, due to the different sign of the second term of the expression for the time dilatation Eq.(13).

## 7. Conclusions

The result of the discussion is that a new basis for the Special Relativity Theory has been laid, so that one might even speak of a **Updated Relativity Theory (URT).** The means by which it was possible to extend the competence of the **Special Relativity Theory (SRT)** was to use its postulates in an infinitesimal sense. Instead of speaking of a time dilation t' one ought to speak of a differential time dilation dt'. For the case of linear coordinate transformations this is equivalent to the original expression if only omitting the prime sign. For the other cases one must analyse the curvature closer.
A lesson is also given to any lecturer saying "it is at once apparent" that he ought to make a mathematical treatment first.
It has concluding become apparent that the introduction of the URT has created a new way for treating movement problems, which earlier were believed to require the usage of the GRT. Further analysis of the achievements of the GRT are therefore needed, applying the URT.

## 8. References.

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