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On the stability of steady finite amplitude convection

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The static state of a horizontal layer of fluid heated from below may become unstable. If the layer is infinitely large in horizontal extent, the Boussinesq equations admit many different steady solutions. A systematic method is presented here which yields the finite-amplitude steady solutions by means of successive approximations. It turns out that not every solution of the linear problem is an approximation to the non-linear problem, yet there are still an infinite number of finite amplitude solutions. A similar procedure has been applied to the stability problem for these steady finite amplitude solutions with the result that three-dimensional solutions are unstable but there is a class of two-dimensional flows which are stable. The problem has been treated for both rigid and free boundaries.

1. Introduction

When a horizontal layer of fluid is heated from below, thermal expansion causes a density gradient opposite to the direction of gravity. In cases where the temperature gradient exceeds a certain critical value the static state of the fluid becomes unstable because the buoyancy force is sufficient to overcome the dissipative effects. It is well known that the resulting cellular convective flow is not uniquely determined by the equations of motion and the boundary conditions if the layer is infinite in horizontal extent. An infinite degeneracy was first found in the early linear theories which apply only for infinitesimal flow amplitude. However, Malkus & Veronis (1958) have shown for special solutions that the degeneracy persists for finite amplitude solutions. They showed that flows with rectangular or hexagonal cell pattern are finite amplitude solutions and that their number is infinite because the ratio of the side lengths of a rectangle is a free parameter.

As important as the non-linear effects is the influence of the boundedness of the layer in horizontal extent. However, if the horizontal length of the layer is large compared to its thickness, the influence of the vertical side walls ought to be negligible at points far away from the walls.

It appears that a stability theory is needed to explain why one or the other flow is preferred. If one recalls the mathematical difficulties that arise in the stability theory of simple flow in a channel it seems at first hopeless to apply the usual stability theory here. The steady-state solutions of finite amplitude are not even known exactly; however, the flows considered have relatively small amplitude. This enables us to treat the stability equations with the aid of successive approximations similar to those Malkus & Veronis (1958) used. At the same time we generalize the method of Malkus & Veronis for the steady state by considering the whole manifold of solutions. Furthermore, we treat the problem for both rigid and free boundaries.

2. The fundamental equations

Conservation of mass, momentum, and energy are described by the equation of continuity, the Navier–Stokes equations in the Boussinesq-approximation, and the heat equation, respectively, or

$$\partial_j u_j = 0,$$

$$\partial_t u_i + u_j \partial_j u_i = -\rho_0^{-1} \partial_i p - (\rho/\rho_0) g\lambda_i + \nu \Delta u_i,$$

$$\partial_t T + u_j \partial_j T = \kappa \Delta T,$$

where we have used the summation convention and the notation

$$\partial_j = \partial/\partial x_j, \quad \partial_t = \partial/\partial t \quad (j = 1, 2, 3);$$

λ_i is the unit vector with direction opposite to the gravity acceleration vector, which is normal to the layer. All other symbols have their usual meaning. Suppose the bottom of the layer is held at the temperature T_0 and the top at the temperature T_1. Letting d be the depth of the layer we write the temperature in the form

$$T - T_0 = -\beta x_j \lambda_j + \theta,$$

where the first term with $\beta = (T_0 - T_1)/d$ describes the temperature distribution in the static state and θ is the deviation from the linear distribution. The fundamental equations must be supplemented by an equation of state which we approximate by

$$\rho = \rho_0[1 - \alpha(T - T_0)].$$

Thus we arrive at the well-known system of equations

$$\partial_i u_i = 0,$$

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i \varpi + \alpha g \theta \lambda_i + \nu \Delta u_i,$$

$$\partial_t \theta + u_j \partial_j \theta = \beta u_j \lambda_j + \kappa \Delta \theta,$$

$$\varpi = p/\rho_0 + g x_j \lambda_j - \tfrac{1}{2}\beta\alpha g x_k \lambda_k x_j \lambda_j;$$

α, g, ν, κ are assumed to be constants. To get a dimensionless form of the equations we set

$$u_i = \kappa u_i'/d, \quad \theta = \nu\kappa\theta'/\alpha g d^3, \quad t = d^2 t'/\kappa, \quad x_i = d x_i', \quad \varpi = \kappa^2 \varpi'/d^2.$$

This yields, after dropping the primes,

$$\partial_j u_j = 0,$$

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i \varpi + P\theta\lambda_i + P \Delta u_i,$$

$$\partial_t \theta + u_j \partial_j \theta = R u_j \lambda_j + \Delta \theta.$$

$P = \nu/\kappa$ is the Prandtl number and $R = \alpha g \beta d^4/\nu\kappa$ is the Rayleigh number.

Introducing the four-dimensional differential operator

$$\partial_\kappa = \begin{pmatrix} 0 \\ \partial_i \end{pmatrix},$$

the four-dimensional vector

$$v_\kappa = \begin{pmatrix} \theta \\ u_i \end{pmatrix},$$

and the matrix differential operator

$$D_{\kappa\lambda} = \begin{pmatrix} \Delta & R\lambda_\kappa \\ P\lambda_i & P\,\Delta\partial_{i\kappa} \end{pmatrix},$$

we can rewrite our equations in the form

$$\left.\begin{aligned} \partial_t v_\kappa + v_\lambda \partial_\lambda v_\kappa &= D_{\kappa\lambda} v_\lambda - \partial_\kappa \overline{\omega}, \\ \partial_\lambda v_\lambda &= 0, \end{aligned}\right\} \tag{2.1}$$

with the sum convention on Greek subscripts which run from 0 to 3. The stationary system has the form

$$\left.\begin{aligned} v_\lambda \partial_\lambda v_\kappa &= D_{\kappa\lambda} v_\lambda - \partial_\kappa \overline{\omega}, \\ \partial_\lambda v_\lambda &= 0. \end{aligned}\right\} \tag{2.2}$$

If we superpose infinitesimal disturbances $\tilde{v}_\kappa$ onto the steady functions v_κ we derive from (2.1) the stability equations

$$\left.\begin{aligned} \sigma\tilde{v}_\kappa + v_\lambda \partial_\lambda \tilde{v}_\kappa + \tilde{v}_\lambda \partial_\lambda v_\kappa &= D_{\kappa\lambda} \tilde{v}_\lambda - \partial_\kappa \tilde{\omega}, \\ \partial_\lambda \tilde{v}_\lambda &= 0, \end{aligned}\right\} \tag{2.3}$$

where we have introduced a growth rate σ by

$$\partial_t \tilde{v}_\kappa = \sigma \tilde{v}_\kappa$$

because the coefficients of the special linear system (2.3) are time-independent. v_κ is unstable if equation (2.3) has solutions for positive σ.

3. The boundary conditions

We assume that the layer is infinite in horizontal extent and require that all functions are bounded as $x^2 + y^2 \to \infty$. On the horizontal bounding surfaces the vertical component of the velocity must vanish, and since we require that the temperature has fixed values on the boundaries we have the further condition that the temperature deviation θ must vanish.

Since we are concerned with a viscous fluid at rigid boundaries the horizontal components of the velocity must also vanish. For the so-called 'free' case absence of stress requires that the normal derivative of the horizontal velocity components vanishes. So we have two sets of boundary conditions

$$u_i = \theta = 0 \qquad \text{at rigid boundaries,}$$
$$u_j \lambda_j = \partial_l \lambda_l \epsilon_{ijk} \lambda_j u_k = \theta = 0 \quad \text{at free boundaries,}$$

for the steady state as well as for the disturbances.

4. Perturbation theory in the vicinity of $R = R_c$

We try to solve the stationary equations (2.2) and the stability equations (2.3) for small values of $R - R_c$, which means small amplitude convection.

Regarding the quadratic terms in (2.2) and the interaction terms in (2.3) as perturbations, the unperturbed equations are

$$\left.\begin{aligned} 0 &= D^{(0)}_{\kappa\lambda} v^{(1)}_\lambda - \partial_\kappa \overline{\omega}^{(1)}, \\ 0 &= \partial_\lambda v^{(1)}_\lambda, \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \sigma^{(0)}\tilde{v}^{(1)}_\kappa &= D^{(0)}_{\kappa\lambda} \tilde{v}^{(1)}_\lambda - \partial_\kappa \tilde{\overline{\omega}}^{(1)}, \\ 0 &= \partial_\lambda \tilde{v}^{(1)}_\lambda, \end{aligned}\right\} \tag{4.2}$$

where the superscript on $D^{(0)}_{\kappa\lambda}$ means that R is replaced by $R^{(0)}$. The equations are linear with constant coefficients and their solutions are well known (see, for instance, Pellew & Southwell 1940). Solutions of the non-linear equations (2.2) can be approximated by the formal expansions

$$R = R^{(0)} + \epsilon R^{(1)} + \epsilon^2 R^{(2)} + \ldots, \tag{4.3}$$

$$v_\kappa = \epsilon v^{(1)}_\kappa + \epsilon^2 v^{(2)}_\kappa + \epsilon^3 v^{(3)}_\kappa + \ldots, \tag{4.4}$$

where the amplitude ϵ is a small parameter. If we substitute these series into the non-linear system (2.2) we get a set of inhomogeneous equations which are in general not solvable. We determine the $R^{(m)}$ from certain existence conditions for the solutions of the inhomogeneous equations. Since R is an externally given parameter, equation (4.3) defines ϵ.

We substitute the series (4.3), (4.4) into the stability equations which we regard as an eigenvalue problem for the growth rate σ. Since ϵ is the perturbation parameter given by the steady non-linear solutions, we can apply the ordinary techniques of perturbation theory to the disturbance equations writing

$$\sigma = \sigma^{(0)} + \epsilon\sigma^{(1)} + \epsilon^2\sigma^{(2)} + \ldots, \tag{4.5}$$

$$\tilde{v}_\kappa = \tilde{v}^{(1)}_\kappa + \epsilon\tilde{v}^{(2)}_\kappa + \epsilon^2\tilde{v}^{(3)}_\kappa + \ldots. \tag{4.6}$$

5. The unperturbed problems

Equation (4.2) with $\sigma^{(0)} = 0$ is the same as (4.1), so we need only discuss the more general (4.2). Let v'_κ and v''_κ be any functions which satisfy $\partial_\kappa v'_\kappa = \partial_\kappa v''_\kappa = 0$ and the same boundary conditions as v_κ. We define the weighted scalar product

$$\langle v'_\kappa, v''_\kappa \rangle = P(v'_0 v''_0)_m + R^{(0)}(v'_j v''_j)_m, \tag{5.1}$$

where $(\ )_m$ means the average over the entire layer. Then for the free as well as for the rigid case the operator $D^{(0)}_{\kappa\lambda}$ has the following property of self-adjointness

$$\begin{aligned} \langle v'_\kappa, D^{(0)}_{\kappa\lambda} v''_\lambda \rangle &= R^{(0)}P[(\theta' u_j \lambda_j)_m + (\theta'' u'_j \lambda_j)_m + (u'_j \Delta u''_j)_m] + P(\theta' \Delta\theta'')_m \\ &= \langle v''_\kappa, D^{(0)}_{\kappa\lambda} v'_\lambda \rangle \end{aligned}$$

from which we can immediately conclude that $\sigma^{(0)}$ is real.

Referring to Pellew & Southwell (1940) the vertical and horizontal dependences of the solutions of the linear equations (4.2) can be separated by assuming

$$\Delta_2 \tilde{v}^{(1)}_\kappa + a^2 \tilde{v}^{(1)}_\kappa = 0,$$

where a is the wave-number, and $\Delta_2 = \Delta - \partial_j \lambda_j \partial_k \lambda_k$ is the two-dimensional Laplacian operator in the horizontal plane. The neutral curve $\sigma^{(0)} = 0$ divides the $(a^2, R^{(0)})$-plane into a stable and an unstable region. On the neutral curve there is a minimum value $R^{(0)} = R_c$ and a corresponding a_c. The first-order disturbances have the highest growth rate $\sigma^{(0)} = 0$ if their wave-number is the same as that of the steady solutions. We shall prove that these disturbances will lead to the instability of three-dimensional steady solutions. With $\sigma^{(0)} = 0$ the two systems (4.1) and (4.2) become identical and we rewrite them explicitly:

$$0 = R^{(0)} u^{(1)}_j \lambda_j + \Delta\theta^{(1)}, \tag{5.2}$$

$$0 = -\partial_i \overline{\omega}^{(1)} + P\theta^{(1)}\lambda_i + P\,\Delta u^{(1)}_i. \tag{5.3}$$

We first notice that h, the vertical component of the vorticity, vanishes, for if we take the curl of (5.3) we get

$$\Delta h = 0, \quad h \equiv \lambda_i \epsilon_{ijk} \partial_j u^{(1)}_\kappa, \tag{5.4}$$

with $h = 0$ or $\lambda_k \partial_k h = 0$ on the boundaries. By multiplying (5.4) by h, averaging over the whole layer, and integrating by parts, we see that $h \equiv 0$. (In the free case h could be a constant if the entire layer were rotating about a vertical axis, a case we do not want to discuss here.)

The velocity thus satisfies the relations

$$\lambda_i \epsilon_{ijk} \partial_j u^{(1)}_\kappa = \partial_j u^{(1)}_j = 0. \tag{5.5}$$

By introducing the operator $\delta_i \equiv \partial_i \partial_k \lambda_k - \lambda_i \Delta$ we write the general solution of (5.5) in the form

$$u^{(1)}_i = \delta_i v^{(1)} \tag{5.6}$$

with $v^{(1)}$ an arbitrary function. (In the free case we could add a constant horizontal vector to our solution (5.6). But this would correspond to an uninteresting uniform horizontal translation.) By operating with δ_i on (5.3) we find

$$\Delta_2(\theta^{(1)} - \Delta^2 v^{(1)}) = 0,$$

or

$$\theta^{(1)} = \Delta^2 v^{(1)}.$$

Then equation (5.2) yields

$$(\Delta^3 - R^{(0)}\Delta_2)\, v^{(1)} = 0 \tag{5.7}$$

with $$v^{(1)} = \Delta^2 v^{(1)} = \begin{cases} \partial_k \lambda_k v^{(1)} = 0 \\ \text{or} \quad \partial_j \partial_k \lambda_j \lambda_k v^{(1)} = 0 \end{cases} \quad \text{on the boundaries.}$$

If a co-ordinate system with the origin in the middle of the layer and the z-axis in the direction of $\boldsymbol{\lambda}$ is introduced, the solution of (5.7) has the following form (see Pellew & Southwell 1940; Reid & Harris 1958):

$$v^{(1)} = w(\mathbf{r}) f(z)\, \theta^{(1)} = \Delta^2 v^{(1)} = w(\mathbf{r})\, g(z), \tag{5.8}$$

$$\Delta_2 w = -a^2 w, \tag{5.9}$$

$$f(z) = \sum_{n=1}^{3} A_n \cosh q_n z, \quad g(z) = \sum_{n=1}^{3} B_n \cosh q_n z, \tag{5.10}$$

where $\mathbf{r}$ is the two-dimensional position vector in the horizontal plane. In the rigid case we have for the lowest value of $R^{(0)}$:

$$\begin{aligned}
&a_c = 3{\cdot}117, \quad R_c^{(0)} = 1707{\cdot}762;\\
&q_1 = i\,.\,3{\cdot}974, \quad q_2 = 5{\cdot}195 - i\,.\,2{\cdot}126, \quad q_3 = q_2^*;\\
&A_1 = 1, \quad A_2 = -3{\cdot}076\,.\,10^{-2} + i\,.\,5{\cdot}194\,.\,10^{-2}, \quad A_3 = A_2^*;\\
&B_1 = 650{\cdot}68, \quad B_2 = 39{\cdot}277 + i\,.\,0{\cdot}433, \quad B_3 = B_2^*.
\end{aligned}$$

In the free case

$$R^{(0)} = (\pi^2 + a^2)^3/a^2, \quad a_c = \pi/\sqrt{2}, \quad R_c^{(0)} = \tfrac{27}{4}\pi^4;$$

$$q_1 = \pi i, \quad A_1 = 1, \quad A_2 = A_3 = 0, \quad B_1 = (\pi^2 + a^2)^2, \quad B_2 = B_3 = 0.$$

We write the solutions of (5.9) in the form

$$w = \sum_{\substack{n=-N\\ n\neq 0}}^{+N} C_n w_n, \quad w_n \equiv \exp(i\mathbf{k}_n . \mathbf{r}), \tag{5.11}$$

with
$$|\mathbf{k}_n|^2 = a^2.$$

In order that (5.11) be real, there must be a $\mathbf{k}_{-n} = -\mathbf{k}_n$ with $C_{-n} = C_n^*$ for each $\mathbf{k}_n$. With the normalization

$$\sum_{n=-N}^{+N} |C_n|^2 = 1,$$

the final form of the first-order solutions is

$$\left.\begin{aligned}
v_\kappa^{(1)} &= \begin{pmatrix}\Delta^2\\ \delta_i\end{pmatrix} v^{(1)}, \quad v^{(1)} = f(z) \sum_{n=-N}^{+N} C_n w_n \quad \text{for the steady motion,}\\
\tilde{v}_\kappa^{(1)} &= \begin{pmatrix}\Delta^2\\ \delta_i\end{pmatrix} \tilde{v}^{(1)}, \quad \tilde{v}^{(1)} = f(z) \sum_{n} \tilde{C}_n w_n \quad \text{for the disturbances.}
\end{aligned}\right\} \tag{5.12}$$

The last expression is a sum over the entire manifold of solutions w_n.

6. Solutions of the second order

At second order the inhomogeneous equations

$$\left.\begin{aligned}
D_{\kappa\lambda}^{(0)} v_\lambda^{(2)} - \partial_\kappa \varpi^{(2)} &= -R^{(1)} A_{\kappa\lambda} v_\lambda^{(1)} + v_\lambda^{(1)} \partial_\lambda v_\kappa^{(1)},\\
\partial_\lambda v_\lambda^{(2)} &= 0
\end{aligned}\right\} \tag{6.1}$$

must be satisfied, where $A_{\kappa\lambda}$ is the constant matrix

$$A_{\kappa\lambda} \equiv \begin{pmatrix} 0 & \lambda_\kappa \\ 0 & \\ 0 & \mathbf{0} \\ 0 & \end{pmatrix}.$$

Forming the special scalar product, defined in (5.1), with the equation (6.1) and the functions $v_\kappa^{(1)*}$ of the first order, we derive the existence condition†

$$-R^{(1)}\langle v_\kappa^{(1)*}, A_{\kappa\lambda} v_\lambda^{(1)}\rangle + \langle v_\kappa^{(1)*}, v_\lambda^{(1)}\partial_\lambda v_\kappa^{(1)}\rangle = \langle v_\kappa^{(1)*}, D_{\kappa\lambda}^{(0)} v_\lambda^{(2)}\rangle - \langle v_\kappa^{(1)*}, \partial_k \overline{\varpi}^{(2)}\rangle$$
$$= \langle v_\kappa^{(2)}, D_{\kappa\lambda}^{(0)} v_\lambda^{(1)*}\rangle = \langle v_\kappa^{(2)}, D_{\kappa\lambda}^{(0)} v_\kappa^{(1)} - \partial_\kappa \overline{\varpi}^{(1)*}\rangle = 0. \quad (6.2)$$

We have used the facts that $D_{\kappa\lambda}^{(0)}$ is self-adjoint in the defined sense and the terms

$$\langle v_\kappa^{(1)*}, \partial_\kappa \overline{\varpi}^{(2)}\rangle, \langle v_\kappa^{(2)} \partial_\kappa \overline{\varpi}^{(1)*}\rangle$$

vanish due to the boundary conditions and the continuity equation.

For further reference we show that

$$\langle v_\kappa^{(1)\prime}, v_\lambda \ \partial_\lambda v_\kappa^{(1)\prime\prime}\rangle = \langle v_\kappa^{(1)\prime}, u_j \partial_j v_\kappa^{(1)\prime\prime}\rangle = 0, \quad (6.3)$$

where $u_j = \delta_j v$ has vanishing normal component on the boundaries and the first order functions $v_\kappa^{(1)\prime}$, $v_\kappa^{(1)\prime\prime}$ may have different horizontal dependences. By partial integration we obtain

$$\langle v_\kappa^{(1)\prime}, u_j \partial_j v_\kappa^{(1)\prime\prime}\rangle = \langle v_\kappa^{(1)\prime}, (\delta_j v)\, \partial_j v_\kappa^{(1)\prime\prime}\rangle = \langle v \delta_j v_\kappa^{(1)\prime}, \partial_j v_\kappa^{(1)\prime\prime}\rangle.$$

This vanishes because, without summation over κ

$$\delta_j v_\kappa^{(1)\prime} \partial_j v_\kappa^{(1)\prime\prime} = (\delta_j v_\kappa^{(1)\prime})\, \partial_j v_\kappa^{(1)\prime\prime} - 2(\partial_k v_\kappa^{(1)\prime})\, \partial_k \partial_j \lambda_j v_\kappa^{(1)\prime\prime}$$
$$+ (\partial_k \lambda_k v_\kappa^{(1)\prime})\, \Delta v_\kappa^{(1)\prime\prime} + (\partial_j v_\kappa^{(1)\prime})\, \partial_k \lambda_k \partial_j v_\kappa^{(1)\prime\prime}$$
$$= (\delta_j v_\kappa^{(1)\prime})\, \partial_j v_\kappa^{(1)\prime\prime} - (\partial_j v_\kappa^{(1)\prime})\, \delta_j v_\kappa^{(1)\prime\prime} = 0.$$

The last two terms cancel because of the relation $\Delta_2 v_\kappa^{(1)} = -a^2 v_\kappa^{(1)}$ and because different first-order solutions have the same vertical dependence.

The second term of (6.2) is of the form (6.3) and therefore vanishes, so the coefficient of $R^{(1)}$ in (6.2) is

$$\langle v_\kappa^{(1)*}, A_{\kappa\lambda} v_\lambda^{(1)}\rangle = P(\theta^{(1)*} u_j^{(1)} \lambda_j)_m$$
$$= -P(u_j^{(1)} \Delta u_j^{(1)*})_m = P([\partial_i u_j^{(1)}]\, \partial_i u_j^{(1)*})_m \neq 0.$$

$R^{(1)}$ must therefore vanish.

The second-order equations of the stability problem become

$$D_{\kappa\lambda}^{(0)} \tilde{v}_\lambda^{(2)} - \partial_\kappa \tilde{\overline{\varpi}}^{(2)} = \sigma^{(1)} \tilde{v}_\kappa^{(1)} + v_\lambda^{(1)} \partial_\lambda \tilde{v}_\kappa^{(1)} + \tilde{v}_\lambda^{(1)}\, \partial_\lambda v_\kappa^{(1)}, \quad \partial_\lambda \tilde{v}_\lambda^{(2)} = 0$$

and their existence condition is

$$\sigma^{(1)} \langle \tilde{v}_\kappa^{(1)*}, \tilde{v}_\kappa^{(1)}\rangle + \langle \tilde{v}_\kappa^{(1)*}, \tilde{v}_\lambda^{(1)}\, \partial_\lambda v_\kappa^{(1)}\rangle + \langle \tilde{v}_\kappa^{(1)*}, v_\lambda^{(1)}\, \partial_\lambda \tilde{v}_\kappa^{(1)}\rangle = 0,$$

analogously to (6.2). The triple products have the same form as in (6.3), hence they are also zero and the existence condition is

$$\sigma^{(1)} = 0.$$

Consequently, it is found that in the second order no steady solution is preferred. This result also holds for the unsymmetrical case of one free and one rigid bounding

† We should recall that the solutions $v_\kappa^{(1)}$ are real. The asterisk is introduced only because the conclusions are also valid for complex functions $v_\kappa^{(1)}$.

surface, which contradicts Malkus & Veronis (1958) who conclude incorrectly that in the unsymmetrical case hexagons are preferred.

In order to find higher approximations for the problem we must construct a particular solution of (6.1), which we rewrite explicitly for $R^{(1)} = 0$

$$\Delta\theta^{(2)} + R^{(0)} u_j^{(2)} \lambda_j = u_j^{(1)} \partial_j \theta^{(1)}, \tag{6.4}$$

$$P\Delta u_i^{(2)} + P\theta^{(2)}\lambda_i - \partial_i \overline{\omega}^{(2)} = u_j^{(1)} \partial_j u_i^{(1)}. \tag{6.5}$$

If we take the vertical component of the curl of the equation (6.5) and use the identity

$$\epsilon_{ikl} \lambda_k \partial_l \delta_i \equiv 0$$

we get on the right-hand side

$$\begin{aligned}\epsilon_{ikl} \lambda_k \partial_l u_j^{(1)} \partial_j u_i^{(1)} &= \epsilon_{ikl}[(\lambda_k \partial_l(\partial_j \lambda_m \partial_m - \lambda_j \Delta) v^{(1)}) \partial_j(\partial_i \lambda_n \partial_n - \lambda_i \Delta) v^{(1)}] \\ &= -\epsilon_{ikl} \lambda_k (\partial_l \Delta v^{(1)}) \partial_i \lambda_j \partial_j \lambda_n \partial_n v^{(1)}. \end{aligned} \tag{6.6}$$

The rest of the terms in the square brackets are symmetrical in i, k or i, l and hence cancel because of the antisymmetry of ϵ_{ikl}. Using the properties

$$v^{(1)} = f(z)\, w(\mathbf{r}), \quad \Delta_2 w = -a^2 w$$

of the first-order functions, one easily verifies that the last term of (6.6) vanishes. Thus $\epsilon_{ikl} \lambda_k \partial_l \Delta u_i^{(2)} = 0$ and with $\partial_j u_j^{(2)} = 0$ we can derive the velocities from a potential by $u_i^{(2)} = \delta_i v^{(2)}$ in the same way as for the first-order functions. By operating with $-\delta_i$ on the equation (6.5),

$$P\Delta_2 \theta^{(2)} + P\Delta^2 u_j^{(2)} \lambda_j = -\delta_i u_j^{(1)} \partial_j u_i^{(1)},$$

and finally by eliminating $\theta^{(2)}$ or $u_j^{(2)}\lambda_j$ respectively

$$(\Delta^3 - R^{(0)} \Delta_2)\, u_j^{(2)} \lambda_j = -P^{-1} \Delta \delta_i u_j^{(1)} \partial_j u_i^{(1)} - \Delta_2 u_j^{(1)} \partial_j \theta^{(1)},$$

$$(\Delta^3 - R^{(0)} \Delta_2)\, \theta^{(2)} = P^{-1} R^{(0)} \delta_i u_j^{(1)} \partial_i u_j^{(1)} + \Delta^2 u_j^{(1)} \partial_j \theta^{(1)}.$$

If we write the second-order solutions in the form

$$v_\kappa^{(2)} = \begin{pmatrix} \theta^{(2)} \\ \delta_i v^{(2)} \end{pmatrix} = P^{-1} \begin{pmatrix} R^{(0)} \Delta_2 \\ \Delta \delta_i \end{pmatrix} p + \begin{pmatrix} \Delta^2 \\ \delta_i \end{pmatrix} q, \tag{6.7}$$

then p, q satisfy the equations

$$\left.\begin{aligned} \Delta_2(\Delta^3 - R^{(0)} \Delta_2)\, p &= \delta_i u_j^{(1)} \partial_j u_i^{(1)}, \\ (\Delta^3 - R^{(0)} \Delta_2)\, q &= u_j^{(1)} \partial_j \theta^{(1)}, \end{aligned}\right\} \tag{6.8}$$

with the boundary conditions

$$p = p'' = \begin{Bmatrix} \text{either } p''' \\ \text{or } \quad p^{\text{iv}} \end{Bmatrix} = 0$$

and

$$q = \Delta^2 q = \begin{Bmatrix} \text{either } q' \\ \text{or } \quad q'' \end{Bmatrix} = 0,$$

where the primes indicate vertical differentiation. With the form (5.12) of the first-order solutions, the inhomogeneous terms in (6.8) become

$$\left.\begin{aligned} \delta_i u_j^{(1)} \partial_j u_i^{(1)} &= -\sum_{n,m} C_n C_m w_n w_m a^4 (1 + \phi_{nm}) \\ &\qquad \times [f''' f + (1 - 2\phi_{nm}) f'' f' - 2a^2 (1 - \phi_{nm}) f' f], \\ u_j^{(1)} \partial_j \theta^{(1)} &= -\sum_{n,m} C_n C_m w_n w_m a^2 (\phi_{nm} f' g - f g'), \end{aligned}\right\} \tag{6.9}$$

with the abbreviation

$$\phi_{nm} \equiv (\mathbf{k}_n . \mathbf{k}_m)/a^2.$$

(6.8) has a solution of the form

$$q = \sum_{nm} C_n C_m F_q(\phi_{nm}, z)\, w_n w_m, \quad p = \sum_{nm} C_n C_m F_p(\phi_{nm}, z)\, w_n w_m. \tag{6.10}$$

Since f and g in (6.9) are sums of hyperbolic functions according to (5.10), we finally obtain the formulas

$$\left.\begin{aligned}
F_q(\phi_{nm}, z) &= -\sum_{\nu\mu=1}^{3} \frac{a^2}{2} q_\nu(\phi_{nm} A_\nu B_\mu - A_\mu B_\nu)\left[\frac{\sinh(q_\nu+q_\mu)z}{D_+} + \frac{\sinh(q_\nu-q_\mu)z}{D_-}\right] \\
&\qquad - \sum_{\nu=1}^{3} b_\nu(\phi_{nm})\sinh q'_\nu z, \\
D_\pm &= [(q_\nu \pm q_\mu)^2 - 2a^2(1+\phi_{nm})]^3 + 2a^2(1+\phi_{nm})\, R^{(0)}, \\
F_p(\phi_{nm}, z) &= \sum_{\nu\mu=1}^{3} \tfrac{1}{4} a^2 q_\nu A_\nu A_\mu [E_+ \sinh(q_\nu+q_\mu) z + E_- \sinh(q_\nu - q_\mu) z] \\
&\qquad - \sum_{\nu=1}^{3} d_\nu(\phi_{nm})\sinh q'_\nu z, \\
E_\pm &= (1/D_\pm)\,[q_\nu^2 \pm (1-2\phi_{nm})\, q_\nu q_\mu - 2a^2(1-\phi_{nm})].
\end{aligned}\right\} \tag{6.11}$$

The second terms are solutions of the homogeneous part of (6.8) which must be added to satisfy the boundary conditions. The complex coefficients b_ν, d_ν are uniquely determined because the homogeneous boundary-value problem has no antisymmetrical solution if $R^{(0)}$ is near $R_c^{(0)}$. It turns out that the coefficients b_ν, d_ν are zero in the free case. The case of rigid boundaries is discussed in the appendix. Since the calculation for the disturbances is quite analogous we write the second-order solution

$$\left.\begin{aligned}
v_\kappa^{(2)} &= \sum_{nm} C_n C_m \begin{pmatrix} G(\phi_{nm}, z) \\ \delta_i F(\phi_{nm}, z) \end{pmatrix} w_n w_m, \\
\tilde{v}_\kappa^{(2)} &= \sum_{nm} (\tilde{C}_n C_m + C_n \tilde{C}_m) \begin{pmatrix} G(\phi_{nm}, z) \\ \delta_i F(\phi_{nm}, z) \end{pmatrix} w_n w_m,
\end{aligned}\right\} \tag{6.12}$$

where G and F can be derived from F_p and F_q by differentiation (cf. equation (6.7)).

7. Eigenvalue perturbations to the order ϵ^3

In order to find differences in the behaviour of the various steady solutions we consider the third-order terms in the equations (2.2) and (2.3)

$$\begin{aligned}
D_{\kappa\lambda}^{(0)} v_\lambda^{(3)} - \partial_\lambda \overline{\omega}^{(3)} &= -R^{(2)} A_{\kappa\lambda} v_\lambda^{(1)} + v_\lambda^{(1)} \partial_\lambda v_\kappa^{(2)} + v_\lambda^{(2)} \partial_\lambda v_\kappa^{(1)}, \\
D_{\kappa\lambda}^{0)} \tilde{v}_\lambda^{(3)} - \partial_\lambda \tilde{\overline{\omega}}^{(3)} &= \sigma^{(2)} \tilde{v}_\kappa^{(1)} - R^{(2)} A_{\kappa\lambda} \tilde{v}_\lambda^{(1)} + v_\lambda^{(1)} \partial_\lambda \tilde{v}_\kappa^{(2)} + v_\lambda^{(2)} \partial_\lambda \tilde{v}_\kappa^{(1)} \\
&\quad + \tilde{v}_\lambda^{(1)} \partial_\lambda v_\kappa^{(2)} + \tilde{v}_\lambda^{(2)} \partial_\lambda v_\kappa^{(1)}.
\end{aligned}$$

The existence conditions require that the right-hand sides of the equations be orthogonal to all first-order solutions which are represented by

$$v_{\kappa n}^{(1)} = \begin{pmatrix} \Delta^2 \\ \delta_i \end{pmatrix} f(z)\, w_n.$$

Since $\langle v^{(1)*}_{\kappa n}, v^{(2)}_{\lambda}\partial_\lambda v^{(1)}_{\kappa}\rangle = \langle v^{(1)*}_{\kappa n}, \tilde{v}^{(2)}_{\lambda}\partial_\lambda v^{(1)}_{\kappa}\rangle = \langle v^{(1)*}_{\kappa n}, v^{(2)}_{\lambda}\partial_\lambda \tilde{v}^{(1)}_{\kappa}\rangle = 0,$

due to equation (6.3), these conditions are

$$-R^{(2)}\langle v^{(1)*}_{\kappa n}, A_{\kappa\lambda}v^{(1)}_{\lambda}\rangle + \langle v^{(1)*}_{\kappa n}, v^{(1)}_{\lambda}\partial_\lambda v^{(2)}_{\kappa}\rangle = 0, \tag{7.1}$$

$$-R^{(2)}\langle v^{(1)*}_{\kappa n}, A_{\kappa\lambda}\tilde{v}^{(1)}_{\lambda}\rangle + \sigma^{(2)}\langle v^{(1)*}_{\kappa n}, \tilde{v}^{(1)}_{\kappa}\rangle + \langle v^{(1)*}_{\kappa n}, \tilde{v}^{(1)}_{\lambda}\partial_\lambda v^{(2)}_{\kappa}\rangle + \langle v^{(1)*}_{\kappa n}, v^{(1)}_{\lambda}\partial_\lambda \tilde{v}^{(2)}_{\kappa}\rangle = 0. \tag{7.2}$$

The second term in (7.1) can be transformed by partial integration

$$\begin{aligned} J \equiv \langle v^{(1)*}_{\kappa n}, v^{(1)}_{\lambda}\partial_\lambda v^{(2)}_{\kappa}\rangle &= \langle v^{(1)*}_{\kappa n}, u^{(1)}_{j}\partial_j v^{(2)}_{\kappa}\rangle = -\langle v^{(2)}_{\kappa n}, u^{(1)}_{j}\partial_j v^{(1)*}_{\kappa}\rangle \\ &= -P(\theta^{(2)}u^{(1)}_j\partial_j\theta^{(1)*}_n)_m - R^{(0)}((\delta_i v^{(2)})\,u^{(1)}_j\partial_j u^{(1)*}_{in})_m \\ &= -P(\theta^{(2)}u^{(1)}_j\partial_j\theta^{(1)*}_n)_m - R^{(0)}(v^{(2)}\delta_i u^{(1)}_j\partial_j u^{(1)*}_{in})_m. \end{aligned}$$

Substituting the representations (6.12) for the first- and second-order functions we get

$$J = \sum_{k,l,m=-N}^{+N} C_k C_l C_m L(\phi_{kl}, \phi_{mn})\,(w^*_n w_k w_l w_m)_m,$$

$$\begin{aligned} L(\phi_{kl}, \phi_{mn}) \equiv &-(G(\phi_{kl})\,(fg' + \phi_{mn}f'g)\,a^2P + R^{(0)}F(\phi_{kl}) \\ &\times[-f'''f - (1+2\phi_{mn})f''f' + 2a^2(1+\phi_{mn})f'f]\,a^4(1-\phi_{mn}))_m. \end{aligned} \tag{7.3}$$

The horizontal average is unequal to zero only in the following cases:

(i) $k = n$	(ii) $l = n$	(iii) $m = n$
$l = -m$	$k = -m$	$k = -l$
	$k \neq n$	$k \neq n$
		$k \neq -n.$

Let us define the matrix

$$T_{nm} = \begin{Bmatrix} 2L(-1,+1) + L(+1,-1) \\ \quad \text{for } m = \pm n \\ 2(L(\phi_{nm}, -\phi_{nm}) + L(-\phi_{nm}, \phi_{nm}) \\ \quad + L(-1,+1)) \quad \text{otherwise} \end{Bmatrix} \quad (n, m = -N, \ldots -1, 1, \ldots, +N)$$

which has the symmetries

$$T_{nm} = T_{-n,m} = T_{n,-m}; \quad T_{nm} = T_{mn},$$

since $\phi_{nm} = \phi_{mn}$, $-\phi_{nm} = \phi_{-nm}$.

Note further that the diagonal elements of T_{nm} are equal to each other. After dividing by C_n (7.1) can then be written

$$-KR^{(2)} + \frac{1}{2}\sum_{m=-N}^{N} T_{nm}C^*_m C_m = 0, \quad \text{with} \quad K \equiv Pa^2((f'' - a^2 f)^2)_m. \tag{7.4}$$

From the symmetries of T_{nm} we see that only N equations of (7.4) are independent.

Together with the normalization condition

$$\sum_{m=1}^{N} C^*_m C_m = \tfrac{1}{2}$$

we have a system of $(N+1)$ inhomogeneous equations, which determine the $(N+1)$ values $R^{(2)}$, $C^*_1 C_1, \ldots, C^*_N C_N$. This means that the manifold of first-order solutions is restricted by the non-linearities of the equations. For instance, in the 'regular' case, in which all angles between two neighbouring **k**-vectors are equal the solution is

$$C_1^* C_1 =, \ldots + C_N^* C_N = \frac{1}{2N}, \quad R^{(2)} = \frac{1}{2KN} \sum_{m=1}^{N} T_{1m}.$$

The question arises as to whether further constraints arise due to the existence conditions of the higher orders. This is certainly a complex problem in general. In the 'regular' case, however, there is no preferred **k**-vector so the N equations of the νth-order existence condition, $\nu = 3, 4, \ldots$, reduce to only one equation, which determines $R^{(\nu)}$. Hence there is definitely an infinite number of steady finite-amplitude solutions, the 'regular' ones, and we must find conditions for the stability of the various solutions.

We first restrict the class of disturbances to those whose coefficients $\tilde{C}_n$ are zero except for **k**-vectors for which the C_n are unequal to zero. Then (7.2) has the form

$$\left.\begin{aligned} M\sigma^{(2)}\tilde{C}_n + \sum_{m=-N}^{+N} T_{nm} C_m^* C_n \tilde{C}_m &= 0, \\ M &\equiv (Pg^2 + R^{(0)} a^4 f^2 + R^{(0)} a^2 f'^2)_m, \end{aligned}\right\} \tag{7.5}$$

where we have used relation (7.3). Since T_{nm} is symmetrical, $T_{nm} C_m^* C_n$ is Hermitian and all eigenvalues $\sigma^{(2)}$ are real. The system (7·5) has non-trivial solutions $\tilde{C}_n$ if and only if the characteristic equation

$$\det |M\sigma^{(2)}\delta_{nm} + T_{nm} C_m^* C_n| = 0$$

or

$$\det \left| \frac{M}{C_n^* C_n} \sigma^{(2)}\delta_{nm} + T_{nm} \right| = 0 \tag{7.6}$$

is satisfied. The left-hand side of (7.6) is a real polynomial in $\sigma^{(2)}$ with $2N$ real roots, and the coefficient of $(\sigma^{(2)})^{2N}$ is positive. If we subtract the $(-m)$th column from the $(+m)$th column and add the $(+n)$th row to the $(-n)$th, and use the symmetry properties of T_{nm} we get a matrix with zero elements for negative m, from which we conclude that N eigenvalues $\sigma^{(2)}$ are zero.

The rest of the eigenvalues $\sigma^{(2)}$ satisfy the equation

$$\det \left| \frac{M\sigma^{(2)}\delta_{nm}}{C_n C_n^*} + 2T_{nm} \right| = 0 \quad (n, m = 1, \ldots, N),$$

or

$$\det |\sigma^{(2)}\delta_{nm} + 2C_n C_m^* T_{nm}| = 0. \tag{7.7}$$

We show in the appendix that the matrix T_{nm} has the additional property

$$T_{nm} > T_{nn} > 0 \quad (n \neq m; \text{ without summation convention}), \tag{7.8}$$

from which it follows that all two-dimensional principal minors

$$4C_n C_n^* C_m C_m^* (T_{nn} T_{mm} - T_{nm}^2)$$

of (7.7) are negative. Thus in the polynomial (7.7) the coefficient of $(\sigma^{(2)})^{N-2}$ is negative. Since all zeros of (7.7) are real, we can then conclude by the sign rule of Descartes that at least one zero $\sigma^{(2)}$ is positive, because the coefficients of $\sigma^{(2)N}$

and $\sigma^{(2)N-2}$ have opposite sign. We have thus shown that all solutions with $N > 1$ are unstable.

An exception is the case $N = 1$, the two-dimensional flow in the form of rolls, for which the polynomial (7.7) is of degree one and has only a negative root because the trace of T_{nm} is positive. In order to show that the two-dimensional flow is not unstable with respect to any other disturbance we have to drop our original restrictions on the disturbances. We go back to equation (7.2) and consider the case that the $\tilde{C}_m$ are unequal to zero for $\mathbf{k}_r$-vectors different from those of the steady motion. Then the horizontal average in (7.2) yields the diagonal elements

$$\sigma^{(2)}M + L(-\phi_{1r}, \phi_{1r}) + L(\phi_{1r}, -\phi_{1r}) - \tfrac{1}{2}L(+1, -1), \tag{7.9}$$

while the non-diagonal elements vanish so this part of the stability matrix can be discussed separately. In order that the associated determinant vanishes the expression (7.9) has to be zero. As will be shown in the appendix

$$L(-\phi_{1r}, \phi_{1r}) + L(\phi_{1r}, -\phi_{1r}) - \tfrac{1}{2}L(+1, -1) > 0 \tag{7.10}$$

and hence all $\sigma^{(2)}$-values are negative in that case, so for rolls the highest growth rate is $\sigma^{(2)} = 0$. The disturbance with this growth rate turns out to be an infinitesimal horizontal translation of the steady motion normal to the axis of the rolls. This is clearly an exact solution of the stability problem with the growth rate zero, which we see by differentiating the steady non-linear equations.

To complete the proof for the stability of small amplitude rolls we have to consider disturbances with wave-numbers $\tilde{a}$ different from the wave-number a of the rolls. Such disturbances satisfy the first-order equations with $\sigma^{(0)}$ not necessarily zero. To second order $\sigma^{(1)}$ vanishes due to the vertical symmetry of the first-order functions, if we consider the cases of symmetric boundary conditions. At third order we get non-diagonal elements in the stability matrix, if the condition

$$\tilde{\mathbf{k}}_n + \tilde{\mathbf{k}}_m - 2\mathbf{k}_1 = 0, \quad |\tilde{\mathbf{k}}| = \tilde{a}$$

is satisfied, which is possible only in the case of $\tilde{a} \geqslant a$. If we approximate the values of the matrix element by taking the limit $|\tilde{a} - a| \to 0$, we find in the case of $\tilde{a} > a$ the same matrix elements as in the case $\tilde{a} = a$. Hence in this limit $\tilde{a} - a \to +0$, the highest value of $\sigma^{(2)}$ is zero. This means that rolls are unstable for $a < a_c$, because the positive value $\sigma^{(0)}$ of disturbances with a wavelength slightly greater than a cannot be compensated by the contribution of $\sigma^{(2)}\epsilon^2$ to the growth rate σ. In the case of $\tilde{a} < a$ the non-diagonal elements vanish and in the limit $\tilde{a} - a \to -0$ the diagonal elements are of the form (7.9), which yield only negative values $\sigma^{(2)}$. Hence for $a > a_c$ the positive value $\sigma^{(0)}$ of disturbances with $\tilde{a} < a$ can be compensated for finite amplitude ϵ, while disturbances with $\tilde{a} > a$ have negative values $\sigma^{(0)}$. Thus we find that a finite-amplitude roll is stable if its horizontal scale corresponds to a value a with $a_1(\epsilon) > a > a_c$ and unstable otherwise. For free as well as for rigid boundaries figure 1 gives a qualitative picture of the stability range of finite amplitude rolls.

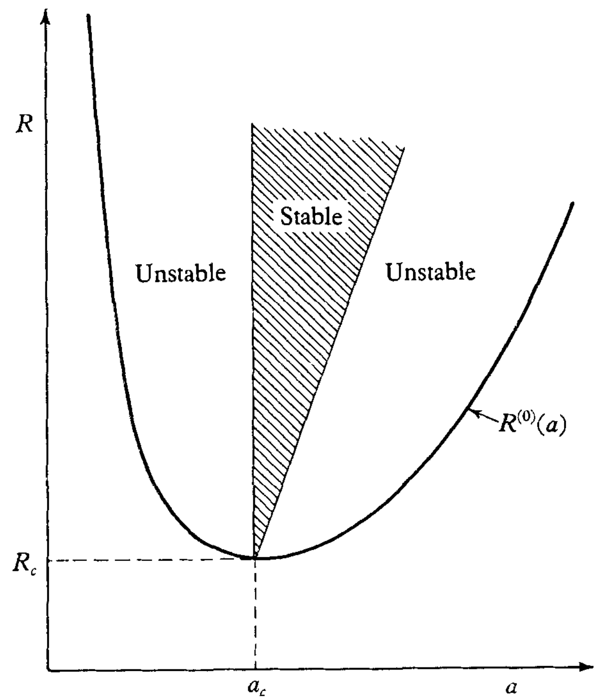

FIGURE 1. Stability range of rolls.

8. The convective heat transport

The difference between the total and the conduction heat transport through the layer is given by
$$\bar{H} = \overline{u_j\lambda_j\theta} - \overline{\partial_j\lambda_j\theta} = (u_j\lambda_j\theta)_m,$$
where the bars indicate a horizontal average. The first term unequal to zero is of the second order,
$$\bar{H} = (u_j^{(1)}\lambda_j\theta^{(1)})_m(R-R_c)/R^{(2)} + \ldots = (K/PR^{(2)})(R-R_c)+\ldots,$$
giving the initial slope of the convective heat transport curve. For the free case Malkus & Veronis (1958) calculated for rolls, rectangles and hexagons. From table 1 in the appendix one derives the following results for 'regular' solutions in the case of two rigid boundaries with $K/P = 2904{\cdot}4$:

rolls
$$C_1^*C_1 = \tfrac{1}{2};$$
$$\bar{H}/(R-R_c) = (0{\cdot}69942 - 0{\cdot}00472P^{-1} + 0{\cdot}00832P^{-2})^{-1}:$$
square cells
$$C_1^*C_1 = C_2^*C_2 = \tfrac{1}{4};$$
$$\bar{H}/(R-R_c) = (0{\cdot}77890 + 0{\cdot}03996P^{-1} + 0{\cdot}06363P^{-2})^{-1}:$$
hexagons
$$C_1^*C_1 = C_2^*C_2 = C_3^*C_3 = \tfrac{1}{6};$$
$$\bar{H}/(R-R_c) = (0{\cdot}89360 + 0{\cdot}04959P^{-1} + 0{\cdot}06787P^{-2})^{-1}.$$

In this connexion Malkus's 'hypothesis of maximum heat transport' (1954*a*, *b*) should be mentioned. This states that, if there are several possible convective motions, the fluid prefers the motion with the highest absolute value of heat transport. We prove this hypothesis for small amplitudes by showing that the heat transport of rolls has an absolute maximum, i.e. corresponding $R^{(2)}$ has an absolute minimum. Since the diagonal elements of T_{nm} are equal to each other we derive from (7.4), using inequality (7.8) and the normalization condition,

$$KR^{(2)} = \sum_{m=1}^{N} T_{nm} C_m^* C_m = \frac{1}{N} \sum_{n,m=1}^{N} T_{nm} C_m^* C_m \geqslant \frac{1}{N} \sum_{n=1}^{N} T_{nn} \sum_{m=1}^{N} C_m^* C_m = \tfrac{1}{2} T_{nn},$$

where the equality sign is only valid for $N = 1$, the case of rolls.

9. Conclusions

We have found that not every linear steady solution is an approximation to the non-linear problem, but the degree of degeneracy of the finite-amplitude steady state is still extremely high. Exact formulae for the initial slope of the convective heat transport curve for a given cell pattern have been derived for rigid boundaries. Experiments of sufficient precision, especially with respect to the temperature boundary conditions, are not yet available to test these formulae.

Our systematic stability theory yields the result that three-dimensional convection flows are unstable with respect to infinitesimal disturbances and that there is a class of two dimensional solutions in the form of rolls that are stable. Whether or not a given finite amplitude roll is stable depends on its wave length. The stability conclusions have been obtained at third order in an expansion in terms of the steady-state amplitude, so any small change in the Boussinesq equations being used could essentially alter the stability behaviour. This seems to be the reason why it is so difficult to produce the two-dimensional convection flows in a laboratory experiment. Stability analyses for the Boussinesq approximation in which density is the only temperature-dependent property have been extended by Palm (1960), Busse (1962), Palm & Qiann (1964), and Segel (1965), to take into account slight dependence of the other material properties on temperature. The main conclusions are that the corresponding vertical asymmetry in the layer leads to the stability of the hexagonal cell pattern in a range between the critical Rayleigh number and a certain supercritical value, beyond which rolls are stable.

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Appendix

We calculate here the elements of the matrix T_{nm}. For free boundaries the sums in the formulae (6.11) reduce to one term, because the coefficients A_ν, B_ν are zero except for one ν. The boundary conditions are satisfied with vanishing coefficients b_ν, d_ν. Substituting f, g, F and G into the equation (7.3) we find

$$L(\phi_{nm}, -\phi_{nm}) = \frac{1}{8}\frac{\pi^2 a^4(\pi^2+a^2)^2(1-\phi_{nm})^2}{[4\pi^2+2a^2(1+\phi_{nm})]^3 - 2a^2(1+\phi_{nm})R^{(0)}}\{[4\pi^2+2a^2(1+\phi_{nm})]^2$$
$$\times (a^2+\pi^2)^2 P + 2a^2(1+\phi_{nm})R^{(0)}[(4\pi^2+2a^2(1+\phi_{nm}))P^{-1}+2(\pi^2+a^2)]\}.$$

This is a positive expression for all ϕ_{nm} with $-1 \leqslant \phi_{nm} < 1$, if the wave number a does not deviate very much from its critical value a_c, and it is zero for $\phi_{nm} = 1$. This means that the relation (7.10) is verified and according to the definition of T_{nm} the inequality (7.8) holds.

In the more realistic case of two rigid boundaries the calculation of the second-order functions (6.11) is more complicated. Since the form of the first-order functions depends on a in this case, we restricted the calculations to the critical value a_c of the wave-number and used the results of Reid & Harris (1958). In order that a solution of the homogeneous part of (6.8) has the vertical dependence $\sinh[q'_\nu(\phi_{nm})z]$ the q'_ν must satisfy the equation

$$[q_\nu'^2 - 2a_c^2(1+\phi_{nm})]^3 + 2a_c^2 R_c^{(0)}(1+\phi_{nm}) = 0,$$

which has the roots

$$q_\nu'^2 = 2a_c^2(1+\phi_{nm}) - \omega_\nu\{2a_c^2 R_c^{(0)}(1+\phi_{nm})\}^{\frac{1}{3}},$$

where ω_ν are the three cube roots of unity:

$$\omega_1 = 1, \quad \omega_2 = -\tfrac{1}{2}(1+\sqrt{3}\,i), \quad \omega_3 = \omega_2^*.$$

The coefficients $b_\nu(\phi_{nm})$, $d_\nu(\phi_{nm})$ are determined by the complex inhomogeneous equations

$$F_q(\phi_{nm}, \tfrac{1}{2}) = F'_q(\phi, \tfrac{1}{2}) = [d^2/dz^2 - 2a_c^2(1+\phi_{nm})]^2 F_q(\phi, z)|_{z=\frac{1}{2}} = 0$$

and $F_p(\phi, \frac{1}{2}) = F''_p(\phi, \frac{1}{2}) = F''''_p(\phi, \frac{1}{2}) = 0$, respectively. The expression

$$L(\phi_{nm}, -\phi_{nm})$$

depends in the following way on the Prandtl number P:

$$L(\phi_{nm}, -\phi_{nm}) = a^4[L_{-1}(\phi_{nm})P^{-1} + L_0(\phi_{nm})P^0 + L_1(\phi_{nm})P^1], \quad \text{say}.$$

We computed the coefficients L_ν of P^ν, which involve many complex numbers, on the electronic computer G3 of the Max–Planck–Institut für Physik und Astrophysik, München. The results are given in table 1, which shows that the relations (7.8) and (7.10) hold for rigid boundaries, too.

The unsymmetric case of one rigid, and one free boundary has been treated by Busse (1962) for the limit of large Prandtl number, with the same qualitative result for the relations (7.8) and (7.10).

$8\phi_{nm}$	$L_{-1}(\phi_{nm})$	$L_0(\phi_{nm})$	$L_1(\phi_{nm})$
−8	0·0	0·0	61,846
−7	2087·1	3125·6	51,124
−6	3630·4	4978·4	41,658
−5	4698·8	5885·7	33,391
−4	5364·3	6114·1	26,257
−3	5696·0	5873·2	20,176
−2	5759·8	5325·7	15,067
−1	5615·2	4597·8	10,846
0	5314·6	3782·2	7,433
1	4903·2	2947·4	4,754
2	4419·7	2142·4	2,736
3	3896·8	1400·4	1,316
4	3361·6	744·2	434
5	2836·6	186·5	40
6	2339·9	−265·1	83
7	1886·4	−608·3	523
8	1487·8	−843·9	1,322

TABLE 1. The coefficients $L_\nu(\phi_{nm})$ for rigid boundaries.